MATHEMATICS FOR THE LIBERAL ARTS



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Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

2: Mathematics for the Liberal Arts

2.1: Course Overview
2.2: Table of Contents
2.3: Sample Syllabus

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2.1: Course Overview

Course Overview

This course was originally developed for the Open Course Library project. The text used is *Math in Society*, edited by David Lippman, Pierce College Ft Steilacoom. Development of this book was supported, in part, by the Transition Math Project and the Open Course Library Project. Topics covered in the course include problem solving, voting theory, graph theory, growth models, finance, data collection and description, and probability. Student learning outcomes include:

1. Describe how mathematics can contribute to the solution of problems in the natural world or human society.
2. Employ critical thinking skills, drawing upon prior knowledge when possible, to analyze and explore new and unfamiliar problems
3. Form and communicate generalizations of patterns discovered through individual or group investigations.
4. Solve problems using algorithms or formulas
5. Model and solve problems using graphical methods
6. Communicate methods of solutions and solutions to problems for the clarity of the receiver.
7. Analyze and interpret data, including calculating numerical summaries and creating graphical representations, to propose possible implications
8. Identify multicultural perspectives of, or multicultural contributions to, at least one mathematical topic studied

Course Access

The course materials and on-line homework sets are housed in MyOpenMath. You can access the course as a guest student to see how the course is set up and view the online homework system. Should you choose to use this course, MyOpenMath can be integrated into your school's LMS for ease of access.

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If you would like to use MyOpenMath in your classroom, you can request an instructor account through the MyOpenMath homepage. Select "request an instructor account" at the bottom of the page. Give us a couple days to verify that you are an instructor at an accredited institution. MyOpenMath generates an automated email when your account is verified, but sometimes it will end up in your junk email or won't make it through the filters set by your institution. To avoid this, you can check your junk folder or ask your institution to let mail from ***do-not-reply@myopenmath.com*** through.

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2.2: Table of Contents

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CHAPTER OVERVIEW

3: Faculty Resources

3.1: Assessments

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3.1: Assessments

Homework Sets, Exercises, Quizzes, and Written Assignments for this course are all located in MyOpenMath.

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4.1: Module 1 Overview

What You'll Learn To Do: Solve problems involving percents, geometry, and estimation.

Learning Objectives

- Solve problems involving percents, proportions, and rates.
- Solve problems using basic geometry.
- Use problem solving and estimation techniques.

Learning Activities

See daily handout April 10 (Day 23)



Reading Assignments and Videos

- **Read:** Problem Solving
- **Watch:** Supplemental Videos

Homework Assignments: **See daily handout April 10 (Day 23)**

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4.2: Discuss- Introduce Yourself to the Class

Go to the **Introduction** forum. Click on the **Create Thread** button to post your introduction. Include your *first* and *last* name in the **Subject** line. Within your posting, please be sure to include the following information:

- Your interests including what you like to do outside of class.
- Why you are taking this class and what you hope to gain out of taking this class this semester.
- What your most successful mathematical experience in or out of class has been.

Respond to the introductions of **at least two** classmates. Your responses should be substantive. That is, they should contain more than just a "Hello."

This assignment is required and worth up to 20 points.

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4.3: Problem Solving

In previous math courses, you've no doubt run into the infamous "word problems." Unfortunately, these problems rarely resemble the type of problems we actually encounter in everyday life. In math books, you usually are told exactly which formula or procedure to use, and are given exactly the information you need to answer the question. In real life, problem solving requires identifying an appropriate formula or procedure, and determining what information you will need (and won't need) to answer the question.

In this chapter, we will review several basic but powerful algebraic ideas: percents, rates, and proportions. We will then focus on the problem solving process, and explore how to use these ideas to solve problems where we don't have perfect information.

Percents

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties.[1] Who is correct? How can we make sense of these numbers?

Percent literally means "per 100," or "parts per hundred." When we write 40%, this is equivalent to the fraction $\displaystyle\frac{40}{100}$ or the decimal 0.40. Notice that 80 out of 200 and 10 out of 25 are also 40%, since $\displaystyle\frac{80}{200}=\frac{10}{25}=\frac{40}{100}$.

Example 1

243 people out of 400 state that they like dogs. What percent is this?

Solution

$\displaystyle\frac{243}{400}=0.6075=\frac{60.75}{100}$. This is 60.75%.

Notice that the percent can be found from the equivalent decimal by moving the decimal point two places to the right.

Example 2

Write each as a percent:

1. $\displaystyle\frac{1}{4}$
2. 0.02
3. 2.35

Solutions

1. $\displaystyle\frac{1}{4}=0.25$ = 25%
2. 0.02 = 2%
3. 2.35 = 235%

Percents

If we have a *part* that is some *percent* of a *whole*, then $\displaystyle\text{percent}=\frac{\text{part}}{\text{whole}}$, or equivalently, $\text{part}\cdot\text{whole}=\text{percent}$.

To do the calculations, we write the percent as a decimal.

Example 3

The sales tax in a town is 9.4%. How much tax will you pay on a $140 purchase?

Solution

Here, $140 is the whole, and we want to find 9.4% *of* $140. We start by writing the percent as a decimal by moving the decimal point two places to the left (which is equivalent to dividing by 100). We can then compute: tax = 0.094(140) = $13.16 in tax.

Example 4

In the news, you hear "tuition is expected to increase by 7% next year." If tuition this year was $1200 per quarter, what will it be next year?

Solution

The tuition next year will be the current tuition plus an additional 7%, so it will be 107% of this year's tuition: $1200(1.07) = $1284.

Alternatively, we could have first calculated 7% of $1200: $1200(0.07) = $84.

Notice this is *not* the expected tuition for next year (we could only wish). Instead, this is the expected *increase*, so to calculate the expected tuition, we'll need to add this change to the previous year's tuition: $1200 + $84 = $1284.

Try It Now

A TV originally priced at $799 is on sale for 30% off. There is then a 9.2% sales tax. Find the price after including the discount and sales tax.

Example 5

The value of a car dropped from $7400 to $6800 over the last year. What percent decrease is this?

Solution

To compute the percent change, we first need to find the dollar value change: $6800 – $7400 = –$600. Often we will take the absolute value of this amount, which is called the **absolute change**: |–600| = 600.

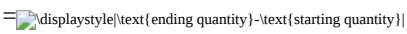
Since we are computing the decrease relative to the starting value, we compute this percent out of $7400:

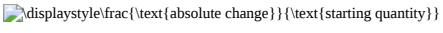


8.1% decrease. This is called a **relative change**.

Absolute and Relative Change

Given two quantities,

Absolute change = 

Relative change: 

Absolute change has the same units as the original quantity.

Relative change gives a percent change.

The starting quantity is called the **base** of the percent change.

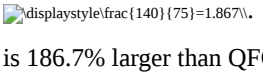
The base of a percent is very important. For example, while Nixon was president, it was argued that marijuana was a "gateway" drug, claiming that 80% of marijuana smokers went on to use harder drugs like cocaine. The problem is, this isn't true. The true claim is that 80% of harder drug users first smoked marijuana. The difference is one of base: 80% of marijuana smokers using hard drugs, vs. 80% of hard drug users having smoked marijuana. These numbers are not equivalent. As it turns out, only one in 2,400 marijuana users actually go on to use harder drugs.[2]

Example 6

There are about 75 QFC supermarkets in the United States. Albertsons has about 215 stores. Compare the size of the two companies.

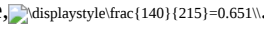
Solution

When we make comparisons, we must ask first whether an absolute or relative comparison. The absolute difference is 215 – 75 = 140. From this, we could say "Albertsons has 140 more stores than QFC." However, if you wrote this in an article or paper, that number does not mean much. The relative difference may be more meaningful. There are two different relative changes we could calculate, depending on which store we use as the base:

Using QFC as the base, 

.

This tells us Albertsons is 186.7% larger than QFC.

Using Albertsons as the base, 

.

This tells us QFC is 65.1% smaller than Albertsons.

Notice both of these are showing percent *differences*. We could also calculate the size of Albertsons relative to QFC: $\frac{215}{75}=2.867$, which tells us Albertsons is 2.867 times the size of QFC. Likewise, we could calculate the size of QFC relative to Albertsons: $\frac{75}{215}=0.349$, which tells us that QFC is 34.9% of the size of Albertsons.

Example 7

Suppose a stock drops in value by 60% one week, then increases in value the next week by 75%. Is the value higher or lower than where it started?

Solution

To answer this question, suppose the value started at $100. After one week, the value dropped by 60%: $100 – $100(0.60) = $100 – $60 = $40.

In the next week, notice that base of the percent has changed to the new value, $40. Computing the 75% increase: $40 + $40(0.75) = $40 + $30 = $70.

In the end, the stock is still $30 lower, or $\frac{\$30}{100}$ = 30% lower, valued than it started.

Try It Now

The US federal debt at the end of 2001 was $5.77 trillion, and grew to $6.20 trillion by the end of 2002. At the end of 2005 it was $7.91 trillion, and grew to $8.45 trillion by the end of 2006.[3] Calculate the absolute and relative increase for 2001–2002 and 2005–2006. Which year saw a larger increase in federal debt?

Example 8

A Seattle Times article on high school graduation rates reported "The number of schools graduating 60 percent or fewer students in four years—sometimes referred to as "dropout factories"—decreased by 17 during that time period. The number of kids attending schools with such low graduation rates was cut in half."

1. Is the "decrease by 17" number a useful comparison?
2. Considering the last sentence, can we conclude that the number of "dropout factories" was originally 34?

Solution

1. This number is hard to evaluate, since we have no basis for judging whether this is a larger or small change. If the number of "dropout factories" dropped from 20 to 3, that'd be a very significant change, but if the number dropped from 217 to 200, that'd be less of an improvement.
2. The last sentence provides relative change, which helps put the first sentence in perspective. We can estimate that the number of "dropout factories" was probably previously around 34. However, it's possible that students simply moved schools rather than the school improving, so that estimate might not be fully accurate.

Example 9

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties. Who is correct?

Solution

Without more information, it is hard for us to judge who is correct, but we can easily conclude that these two percents are talking about different things, so one does not necessarily contradict the other. Edward's claim was a percent with coalition forces as the base of the percent, while Cheney's claim was a percent with both coalition and Iraqi security forces as the base of the percent. It turns out both statistics are in fact fairly accurate.

Try It Now

In the 2012 presidential elections, one candidate argued that "the president's plan will cut $716 billion from Medicare, leading to fewer services for seniors," while the other candidate rebuts that "our plan does not cut current spending and actually expands benefits for seniors, while implementing cost saving measures." Are these claims in conflict, in agreement, or not comparable because they're talking about different things?

We'll wrap up our review of percents with a couple cautions. First, when talking about a change of quantities that are already measured in percents, we have to be careful in how we describe the change.

Example 10

A politician's support increases from 40% of voters to 50% of voters. Describe the change.

Solution

We could describe this using an absolute change: $|50\%-40\%|=10\%$. Notice that since the original quantities were percents, this change also has the units of percent. In this case, it is best to describe this as an increase of 10 **percentage points**.

In contrast, we could compute the percent change: $\displaystyle\frac{10\%}{40\%}=0.25=25\%$ increase. This is the relative change, and we'd say the politician's support has increased by 25%.

Lastly, a caution against averaging percents.

Example 11

A basketball player scores on 40% of 2-point field goal attempts, and on 30% of 3-point of field goal attempts. Find the player's overall field goal percentage.

Solution

It is very tempting to average these values, and claim the overall average is 35%, but this is likely not correct, since most players make many more 2-point attempts than 3-point attempts. We don't actually have enough information to answer the question. Suppose the player attempted 200 2-point field goals and 100 3-point field goals. Then they made 200(0.40) = 80 2-point shots and 100(0.30) = 30 3-point shots. Overall, they made 110 shots out of 300, for a $\displaystyle\frac{110}{300}=0.367=36.7\%$ overall field goal percentage.

Proportions and Rates

If you wanted to power the city of Seattle using wind power, how many windmills would you need to install? Questions like these can be answered using rates and proportions.

Rates

A rate is the ratio (fraction) of two quantities.

A **unit rate** is a rate with a denominator of one.

Example 12

Your car can drive 300 miles on a tank of 15 gallons. Express this as a rate.

Solution

Expressed as a rate, $\displaystyle\frac{300\text{ miles}}{15\text{ gallons}}$. We can divide to find a unit rate: $\displaystyle\frac{20\text{ miles}}{1\text{ gallon}}$, which we could also write as $\displaystyle 20\frac{\text{miles}}{\text{gallon}}$, or just 20 miles per gallon.

Proportion Equation

A proportion equation is an equation showing the equivalence of two rates or ratios.

Example 13

Solve the proportion $\displaystyle\frac{5}{3}=\frac{x}{6}$ for the unknown value x.

Solution

This proportion is asking us to find a fraction with denominator 6 that is equivalent to the fraction $\displaystyle\frac{5}{3}$. We can solve this by multiplying both sides of the equation by 6, giving $\displaystyle x=\frac{5}{3}\cdot6=10$.

Example 14

A map scale indicates that ½ inch on the map corresponds with 3 real miles. How many miles apart are two cities that are $\displaystyle 2\frac{1}{4}$ inches apart on the map?

Solution

We can set up a proportion by setting equal two $\frac{\text{map inches}}{\text{real miles}}$ rates, and introducing a variable, x, to represent the unknown quantity—the mile distance between the cities.

$\frac{\frac{1}{2}\text{map inch}}{3\text{ miles}}=\frac{2\frac{1}{4}\text{map inches}}{x\text{ miles}}$	Multiply both sides by x and rewriting the mixed number
$\frac{\frac{1}{2}}{3}\cdot{x}=\frac{9}{4}$	Multiply both sides by 3
$\frac{1}{2}x=\frac{27}{4}$	Multiply both sides by 2 (or divide by ½)
$x=\frac{27}{2}=13\frac{1}{2}\text{ miles}$	

Many proportion problems can also be solved using **dimensional analysis**, the process of multiplying a quantity by rates to change the units.

Example 15

Your car can drive 300 miles on a tank of 15 gallons. How far can it drive on 40 gallons?

Solution

We could certainly answer this question using a proportion: $\frac{300\text{ miles}}{15\text{ gallons}}=\frac{x\text{ miles}}{40\text{ gallons}}$.

However, we earlier found that 300 miles on 15 gallons gives a rate of 20 miles per gallon. If we multiply the given 40 gallon quantity by this rate, the *gallons* unit "cancels" and we're left with a number of miles:

$$40\text{ gallons}\cdot\frac{20\text{ miles}}{\text{gallon}}=\frac{40\text{ gallons}}{1}\cdot\frac{20\text{ miles}}{\text{gallons}}=800\text{ miles}$$

Notice if instead we were asked "how many gallons are needed to drive 50 miles?" we could answer this question by inverting the 20 mile per gallon rate so that the *miles* unit cancels and we're left with gallons:

$$50\text{ miles}\cdot\frac{1\text{ gallon}}{20\text{ miles}}=\frac{50\text{ miles}}{1}\cdot\frac{1\text{ gallon}}{20\text{ miles}}=\frac{50\text{ gallons}}{20}=2.5\text{ gallons}$$

Dimensional analysis can also be used to do unit conversions. Here are some unit conversions for reference.

Unit Conversions

Length

1 foot (ft) = 12 inches (in)	1 yard (yd) = 3 feet (ft)
1 mile = 5,280 feet	
1000 millimeters (mm) = 1 meter (m)	100 centimeters (cm) = 1 meter
1000 meters (m) = 1 kilometer (km)	2.54 centimeters (cm) = 1 inch

Weight and Mass

1 pound (lb) = 16 ounces (oz)	1 ton = 2000 pounds
1000 milligrams (mg) = 1 gram (g)	1000 grams = 1kilogram (kg)
1 kilogram = 2.2 pounds (on earth)	

Capacity

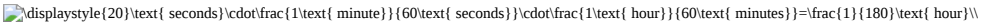
1 cup = 8 fluid ounces (fl oz)[4]	1 pint = 2 cups
1 quart = 2 pints = 4 cups	1 gallon = 4 quarts = 16 cups
1000 milliliters (ml) = 1 liter (L)	

Example 16

A bicycle is traveling at 15 miles per hour. How many feet will it cover in 20 seconds?

Solution

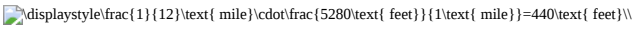
To answer this question, we need to convert 20 seconds into feet. If we know the speed of the bicycle in feet per second, this question would be simpler. Since we don't, we will need to do additional unit conversions. We will need to know that 5280 ft = 1 mile. We might start by converting the 20 seconds into hours:



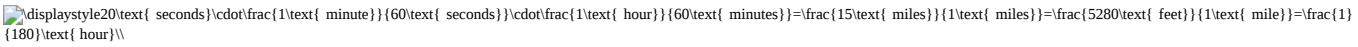
Now we can multiply by the 15 miles/hr



Now we can convert to feet



We could have also done this entire calculation in one long set of products:



Try It Now

A 1000 foot spool of bare 12-gauge copper wire weighs 19.8 pounds. How much will 18 inches of the wire weigh, in ounces?

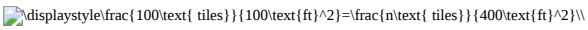
Notice that with the miles per gallon example, if we double the miles driven, we double the gas used. Likewise, with the map distance example, if the map distance doubles, the real-life distance doubles. This is a key feature of proportional relationships, and one we must confirm before assuming two things are related proportionally.

Example 17

Suppose you're tiling the floor of a 10 ft by 10 ft room, and find that 100 tiles will be needed. How many tiles will be needed to tile the floor of a 20 ft by 20 ft room?

Solution

In this case, while the width the room has doubled, the area has quadrupled. Since the number of tiles needed corresponds with the area of the floor, not the width, 400 tiles will be needed. We could find this using a proportion based on the areas of the rooms:



Other quantities just don't scale proportionally at all.

Example 18

Suppose a small company spends $1000 on an advertising campaign, and gains 100 new customers from it. How many new customers should they expect if they spend $10,000?

Solution

While it is tempting to say that they will gain 1000 new customers, it is likely that additional advertising will be less effective than the initial advertising. For example, if the company is a hot tub store, there are likely only a fixed number of people interested in buying a hot tub, so there might not even be 1000 people in the town who would be potential customers.

Sometimes when working with rates, proportions, and percents, the process can be made more challenging by the magnitude of the numbers involved. Sometimes, large numbers are just difficult to comprehend.

Example 19

Compare the 2010 U.S. military budget of $683.7 billion to other quantities.

Solution

Here we have a very large number, about $683,700,000,000 written out. Of course, imagining a billion dollars is very difficult, so it can help to compare it to other quantities.

If that amount of money was used to pay the salaries of the 1.4 million Walmart employees in the U.S., each would earn over $488,000.

There are about 300 million people in the U.S. The military budget is about $2,200 per person.

If you were to put $683.7 billion in $100 bills, and count out 1 per second, it would take 216 years to finish counting it.

Example 20

Compare the electricity consumption per capita in China to the rate in Japan.

Solution

To address this question, we will first need data. From the CIA[5] website we can find the electricity consumption in 2011 for China was 4,693,000,000,000 KWH (kilowatt-hours), or 4.693 trillion KWH, while the consumption for Japan was 859,700,000,000, or 859.7 billion KWH. To find the rate per capita (per person), we will also need the population of the two countries. From the World Bank,[6] we can find the population of China is 1,344,130,000, or 1.344 billion, and the population of Japan is 127,817,277, or 127.8 million.

Computing the consumption per capita for each country:

China: $\displaystyle\frac{4,693,000,000,000\text{KWH}}{1,344,130,000\text{ people}}$ ≈ 3491.5 KWH per person

Japan: $\displaystyle\frac{859,700,000,000\text{KWH}}{127,817,277\text{ people}}$ ≈ 6726 KWH per person

While China uses more than 5 times the electricity of Japan overall, because the population of Japan is so much smaller, it turns out Japan uses almost twice the electricity per person compared to China.

Geometry

Geometric shapes, as well as area and volumes, can often be important in problem solving.

Example 21

You are curious how tall a tree is, but don't have any way to climb it. Describe a method for determining the height.

Solution

There are several approaches we could take. We'll use one based on triangles, which requires that it's a sunny day. Suppose the tree is casting a shadow, say 15 ft long. I can then have a friend help me measure my own shadow. Suppose I am 6 ft tall, and cast a 1.5 ft shadow. Since the triangle formed by the tree and its shadow has the same angles as the triangle formed by me and my shadow, these triangles are called **similar triangles** and their sides will scale proportionally. In other words, the ratio of height to width will be the same in both triangles. Using this, we can find the height of the tree, which we'll denote by h:

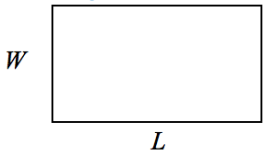
$$\frac{6\text{ft tall}}{1.5\text{ft shadow}}=\frac{h\text{ft tall}}{15\text{ft shadow}}$$

Multiplying both sides by 15, we get h = 60. The tree is about 60 ft tall.

It may be helpful to recall some formulas for areas and volumes of a few basic shapes.

Areas

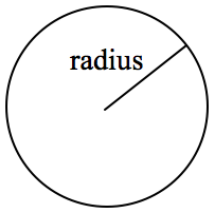
Rectangle



Area: $L \cdot W$

Perimeter: $2L + 2W$

Circle



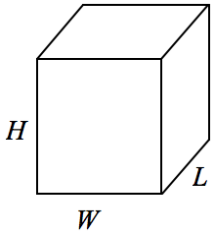
Radius: r

Area: πr^2

Circumference: $2\pi r$

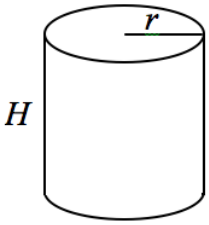
Volumes

Rectangular Box



Volume: $L \cdot W \cdot H$

Cylinder

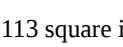


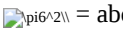
Volume: $\pi r^2 H$

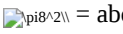
Example 22

If a 12 inch diameter pizza requires 10 ounces of dough, how much dough is needed for a 16 inch pizza?

Solution

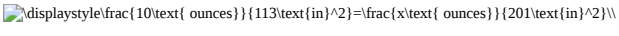
To answer this question, we need to consider how the weight of the dough will scale. The weight will be based on the volume of the dough. However, since both pizzas will be about the same thickness, the weight will scale with the area of the top of the pizza. We can find the area of each pizza using the formula for area of a circle,  :

A 12″ pizza has radius 6 inches, so the area will be  = about 113 square inches.

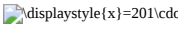
A 16″ pizza has radius 8 inches, so the area will be  = about 201 square inches.

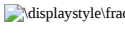
Notice that if both pizzas were 1 inch thick, the volumes would be 113 in^3 and 201 in^3 respectively, which are at the same ratio as the areas. As mentioned earlier, since the thickness is the same for both pizzas, we can safely ignore it.

We can now set up a proportion to find the weight of the dough for a 16″ pizza:



Multiply both sides by 201

 = about 17.8 ounces of dough for a 16″ pizza.

It is interesting to note that while the diameter is  = 1.33 times larger, the dough required, which scales with area, is $1.33^2 = 1.78$ times larger.

Example 23

A company makes regular and jumbo marshmallows. The regular marshmallow has 25 calories. How many calories will the jumbo marshmallow have?

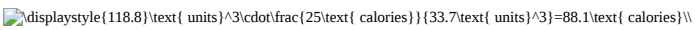
Solution

We would expect the calories to scale with volume. Since the marshmallows have cylindrical shapes, we can use that formula to find the volume. From the grid in the image, we can estimate the radius and height of each marshmallow.

The regular marshmallow appears to have a diameter of about 3.5 units, giving a radius of 1.75 units, and a height of about 3.5 units. The volume is about $\pi(1.75)^2(3.5) = 33.7$ units^3.

The jumbo marshmallow appears to have a diameter of about 5.5 units, giving a radius of 2.75 units, and a height of about 5 units. The volume is about $\pi(2.75)^2(5) = 118.8$ units^3.

We could now set up a proportion, or use rates. The regular marshmallow has 25 calories for 33.7 cubic units of volume. The jumbo marshmallow will have:



$$\displaystyle{118.8}\text{ units}^3\cdot\frac{25\text{ calories}}{33.7\text{ units}^3}=88.1\text{ calories}\\$$

It is interesting to note that while the diameter and height are about 1.5 times larger for the jumbo marshmallow, the volume and calories are about $1.5^3 = 3.375$ times larger.

Try It Now

A website says that you'll need 48 fifty-pound bags of sand to fill a sandbox that measure 8ft by 8ft by 1ft. How many bags would you need for a sandbox 6ft by 4ft by 1ft?

Problem Solving and Estimating

Finally, we will bring together the mathematical tools we've reviewed, and use them to approach more complex problems. In many problems, it is tempting to take the given information, plug it into whatever formulas you have handy, and hope that the result is what you were supposed to find. Chances are, this approach has served you well in other math classes.

This approach does not work well with real life problems. Instead, problem solving is best approached by first starting at the end: identifying exactly what you are looking for. From there, you then work backwards, asking "what information and procedures will I need to find this?" Very few interesting questions can be answered in one mathematical step; often times you will need to chain together a solution pathway, a series of steps that will allow you to answer the question.

Problem Solving Process

1. Identify the question you're trying to answer.
2. Work backwards, identifying the information you will need and the relationships you will use to answer that question.
3. Continue working backwards, creating a solution pathway.
4. If you are missing necessary information, look it up or estimate it. If you have unnecessary information, ignore it.
5. Solve the problem, following your solution pathway.

In most problems we work, we will be approximating a solution, because we will not have perfect information. We will begin with a few examples where we will be able to approximate the solution using basic knowledge from our lives.

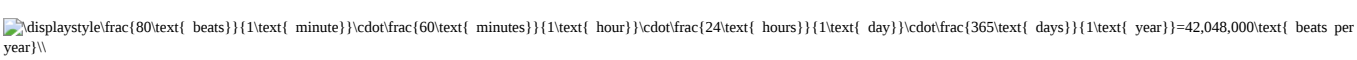
Example 24

How many times does your heart beat in a year?

Solution

This question is asking for the rate of heart beats per year. Since a year is a long time to measure heart beats for, if we knew the rate of heart beats per minute, we could scale that quantity up to a year. So the information we need to answer this question is heart beats per minute. This is something you can easily measure by counting your pulse while watching a clock for a minute.

Suppose you count 80 beats in a minute. To convert this beats per year:



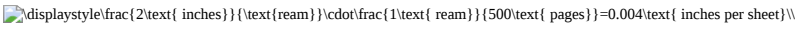
$$\displaystyle\frac{80\text{ beats}}{1\text{ minute}}\cdot\frac{60\text{ minutes}}{1\text{ hour}}\cdot\frac{24\text{ hours}}{1\text{ day}}\cdot\frac{365\text{ days}}{1\text{ year}}=42{,}048{,}000\text{ beats per year}\\$$

Example 25

How thick is a single sheet of paper? How much does it weigh?

Solution

While you might have a sheet of paper handy, trying to measure it would be tricky. Instead we might imagine a stack of paper, and then scale the thickness and weight to a single sheet. If you've ever bought paper for a printer or copier, you probably bought a ream, which contains 500 sheets. We could estimate that a ream of paper is about 2 inches thick and weighs about 5 pounds. Scaling these down,



$$\displaystyle\frac{2\text{ inches}}{\text{ream}}\cdot\frac{1\text{ ream}}{500\text{ pages}}=0.004\text{ inches per sheet}\\$$



$$\displaystyle\frac{5\text{ pounds}}{\text{ream}}\cdot\frac{1\text{ ream}}{500\text{ pages}}=0.01\text{ pounds per sheet, or }=0.16\text{ ounces per sheet.}\\$$

Example 26

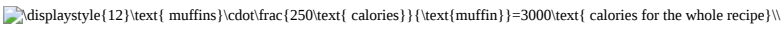
A recipe for zucchini muffins states that it yields 12 muffins, with 250 calories per muffin. You instead decide to make mini-muffins, and the recipe yields 20 muffins. If you eat 4, how many calories will you consume?

Solution

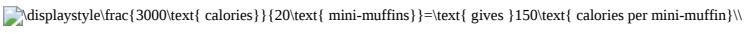
There are several possible solution pathways to answer this question. We will explore one.

To answer the question of how many calories 4 mini-muffins will contain, we would want to know the number of calories in each mini-muffin. To find the calories in each mini-muffin, we could first find the total calories for the entire recipe, then divide it by the number of mini-muffins produced. To find the total calories for the recipe, we could multiply the calories per standard muffin by the number per muffin. Notice that this produces a multi-step solution pathway. It is often easier to solve a problem in small steps, rather than trying to find a way to jump directly from the given information to the solution.

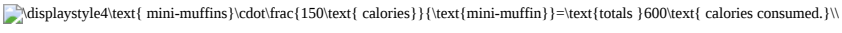
We can now execute our plan:



$$\displaystyle{12}\text{ muffins}\cdot\frac{250\text{ calories}}{\text{muffin}}=3000\text{ calories for the whole recipe}\\$$



$$\displaystyle\frac{3000\text{ calories}}{20\text{ mini-muffins}}=\text{ gives }150\text{ calories per mini-muffin}\\$$



$$\displaystyle4\text{ mini-muffins}\cdot\frac{150\text{ calories}}{\text{mini-muffin}}=\text{totals }600\text{ calories consumed.}\\$$

Example 27

You need to replace the boards on your deck. About how much will the materials cost?

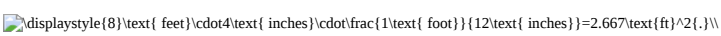
Solution

There are two approaches we could take to this problem: 1) estimate the number of boards we will need and find the cost per board, or 2) estimate the area of the deck and find the approximate cost per square foot for deck boards. We will take the latter approach.

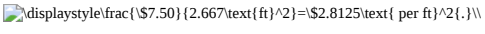
For this solution pathway, we will be able to answer the question if we know the cost per square foot for decking boards and the square footage of the deck. To find the cost per square foot for decking boards, we could compute the area of a single board, and divide it into the cost for that board. We can compute the square footage of the deck using geometric formulas. So first we need information: the dimensions of the deck, and the cost and dimensions of a single deck board.

Suppose that measuring the deck, it is rectangular, measuring 16 ft by 24 ft, for a total area of 384 ft^2.

From a visit to the local home store, you find that an 8 foot by 4 inch cedar deck board costs about $7.50. The area of this board, doing the necessary conversion from inches to feet, is:

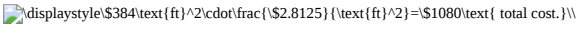


$$\displaystyle{8}\text{ feet}\cdot4\text{ inches}\cdot\frac{1\text{ foot}}{12\text{ inches}}=2.667\text{ft}^2{.}\\$$ The cost per square foot is then



$$\displaystyle\frac{\$7.50}{2.667\text{ft}^2}=\$2.8125\text{ per ft}^2{.}\\$$

This will allow us to estimate the material cost for the whole 384 ft^2 deck



$$\displaystyle\$384\text{ft}^2\cdot\frac{\$2.8125}{\text{ft}^2}=\$1080\text{ total cost.}\\$$

Of course, this cost estimate assumes that there is no waste, which is rarely the case. It is common to add at least 10% to the cost estimate to account for waste.

Example 28

Is it worth buying a Hyundai Sonata hybrid instead the regular Hyundai Sonata?

Solution

To make this decision, we must first decide what our basis for comparison will be. For the purposes of this example, we'll focus on fuel and purchase costs, but environmental impacts and maintenance costs are other factors a buyer might consider.

It might be interesting to compare the cost of gas to run both cars for a year. To determine this, we will need to know the miles per gallon both cars get, as well as the number of miles we expect to drive in a year. From that information, we can find the number of gallons required from a year. Using the price of gas per gallon, we can find the running cost.

From Hyundai's website, the 2013 Sonata will get 24 miles per gallon (mpg) in the city, and 35 mpg on the highway. The hybrid will get 35 mpg in the city, and 40 mpg on the highway.

An average driver drives about 12,000 miles a year. Suppose that you expect to drive about 75% of that in the city, so 9,000 city miles a year, and 3,000 highway miles a year.

We can then find the number of gallons each car would require for the year.

Sonata: $\displaystyle{9000}\text{ city miles}\cdot\frac{1\text{ gallon}}{24\text{ city miles}}+3000\text{ highway miles}\cdot\frac{1\text{ gallon}}{35\text{ highway miles}}=460.7\text{ gallons}$

Hybrid: $\displaystyle{9000}\text{ city miles}\cdot\frac{1\text{ gallon}}{35\text{ city miles}}+3000\text{ highway miles}\cdot\frac{1\text{ gallon}}{40\text{ highway miles}}=332.1\text{ gallons}$

If gas in your area averages about $3.50 per gallon, we can use that to find the running cost:

Sonata: $\displaystyle{460.7}\text{ gallons}\cdot\frac{\$3.50}{\text{gallon}}=\$1612.45\\$

Hybrid: $\displaystyle{332.1}\text{ gallons}\cdot\frac{\$3.50}{\text{gallon}}=\$1162.35\\$

The hybrid will save $450.10 a year. The gas costs for the hybrid are about $\displaystyle\frac{\$450.10}{\$1612.45}\\$ = 0.279 = 27.9% lower than the costs for the standard Sonata.

While both the absolute and relative comparisons are useful here, they still make it hard to answer the original question, since "is it worth it" implies there is some tradeoff for the gas savings. Indeed, the hybrid Sonata costs about $25,850, compared to the base model for the regular Sonata, at $20,895.

To better answer the "is it worth it" question, we might explore how long it will take the gas savings to make up for the additional initial cost. The hybrid costs $4965 more. With gas savings of $451.10 a year, it will take about 11 years for the gas savings to make up for the higher initial costs.

We can conclude that if you expect to own the car 11 years, the hybrid is indeed worth it. If you plan to own the car for less than 11 years, it may still be worth it, since the resale value of the hybrid may be higher, or for other non-monetary reasons. This is a case where math can help guide your decision, but it can't make it for you.

Try It Now

If traveling from Seattle, WA to Spokane WA for a three-day conference, does it make more sense to drive or fly?

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1. http://www.factcheck.org/cheney_edwards_mangle_facts.html ↵
2. http://tvtropes.org/pmwiki/pmwiki.php/Main/LiesDamnedLiesAndStatistics ↵
3. http://www.whitehouse.gov/sites/default/files/omb/budget/fy2013/assets/hist07z1.xls ↵
4. Fluid ounces are a capacity measurement for liquids. 1 fluid ounce ≈ 1 ounce (weight) for water only. ↵
5. https://www.cia.gov/library/publications/the-world-factbook/rankorder/2042rank.html ↵
6. http://data.worldbank.org/indicator/SP.POP.TOTL ↵

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4.4: Test Page

Topic A. Solving Equations and Evaluating Expressions

Objectives:

1. Use additional resources to practice algebra, as needed.
2. Learn to check answers in some way other than looking “in the back of the book.”
3. Use the distributive property and other properties to simplify algebraic expressions and check your work.
4. Solve and simplify linear equations and check solutions of equations by plugging them back in.
5. Solve equations with variables in the denominator and check solutions by plugging them in.
6. Solve equations of those two types with multiple variables for one variable, resulting in a formula for that one variable.
7. Evaluate variable expressions when given the value of each variable, using the correct order of operations.

~~~~~~

Overview

The prerequisite for these materials includes mastery of basic math skills and some algebra, as needed to achieve “college-level” on Texas’ THEA test. Many students who have learned math at this level forget it easily, so this review section of some algebra is provided. Some students in the class will go through this material quickly and others will need to go much more slowly and do more review work. Additional optional practice problems are provided on the course web pages. The next few topics in the do not use this algebra, so it isn’t necessary to master everything in this topic immediately. It is necessary to master it before moving into Topic G (Linear Equations) and later.

A major difference between this course and most other math courses involves “checking your answer by looking in the back of the book.” In standard algebra classes, students practice quite a lot of similar problems and are expected to check many of the answers by looking in the back of the book. When we use mathematics in applications in the real world, there is no “back of the book.” In this course, we will learn methods for independently checking our results to see if they are correct, or at least reasonable, without having to rely on answers that someone else gives us. Most students find this awkward at first, but when they persist in practicing the checking methods, develop more confidence in using mathematics and in their problem-solving skills. There are not easy checking methods for all the techniques covered here, so some answers are provided. Generally, however, for particular topic, if you are expected to check your answer by an independent method, an answer key is not provided.

Section 1. Use the distributive property and other properties to simplify expressions and check your work.

Distributive Property:

For any real numbers a, b, and c: $a(b + c) = ab + ac$.

For any real numbers a, b, and c: $a(b – c) = ab – ac$

The fact that two expressions are equal means that they are equal for any values of the variable. So you can check by taking a few values for the variable and making sure that those do make the two sides equal. Usually we don’t use the values 0 or 1 or 2 as the value for the variable, because that’s too simple. Also it’s best to avoid numbers that already appear in the problem.
~~~~~~

Example 1. Simplify .

Solution:	Partial Check: Use

Example 2. Simplify .

Solution:	Partial Check: Use

Example 3. Simplify .

Solution:	Partial Check: Use

Example 4. Simplify .

Solution:	Partial Check: Use

Caution: Checking your work on problems like these is not completely satisfactory, because students who make a mistake in the original solution often make a corresponding mistake in the checking process. So the fact that it checks does not provide complete confidence that the solution is correct. But it's still a good idea to check because you'll catch most of your mistakes.

For links to material with additional examples and explanations, see the course web page.

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Section 2. Solving and simplifying linear equations and checking the result.

To solve equations, write a simpler equivalent equation using the following rules. Keep writing simpler equivalent equations until you have simplified it enough that one side has the variable alone and the other has a number. That gives the solution.

1. Add or subtract the same number or expression to both sides of the equation.
2. Multiply or divide both sides of the equation by the same nonzero number or expression.
~~~~~~

You can check the solution to an equation by plugging the number into the original equation and seeing if the resulting statement is true.

For additional explanation and examples, see the course web page for links. These include links to instructional materials and a link to an equation solver, where you can enter any equation and have it find the solution. You can use that to check your work on as many problems as you have the time and energy to work.

Example 1. Solve .

Solution:	Check:

Example 2. Find a formula for *y* (that is, solve for *y* in terms of *x*):

Discussion: This is more complicated than a problem like Example 1. *However, it uses exactly the same techniques*. Problems like this arise when we begin to investigate graphing lines. Probably you have done problems like these in a previous algebra course, but maybe you haven't done many of them. We will practice many problems like these in the first few weeks of the course. Don't practice these until you can do problems like Example 1 fairly easily.

Solution:	Check by plugging in the answer for *y*:

Example 3. Solve

Discussion: This is basically like Example 1, except that the coefficients are decimals and the variable is *t* rather than *x*. In your previous algebra classes, you may not have done much work with decimal numbers, but we will use them often in this class. We will practice many problems like these in the first few weeks of the course. Don't practice these until you can do problems like Example 1 fairly easily.

Solution:	Check:

~~~~~~

Section 3. Solving equations with variables in the denominator.

Example 1: Solve

Discussion: Sometimes students learn to solve problems like this one by "cross-multiplying." That is correct. However, most math teachers prefer to think of solving this by multiplying both sides by the same thing – in this case the product of the two denominators. The result is the same. Both methods are shown below.

Solution:
~~~~~~

Solution Method: *Cross-multiply:*	Alternate Solution Method: *Multiply by common denominator*	Check:
		Here, the two sides aren't exactly equal. But that isn't surprising, because we rounded the final answer to the original problem, so we don't expect the two sides here to be exactly equal.

Notice that the quotient, which is the final answer, didn't come out even, so I had to round off. I chose to round off to four decimal places. We'll talk more specifically about how exactly answers should be given later in the course. For now, when you have to round in the final computation, always keep at least three decimal places. That will ensure that your checking will produce answers close enough that you can recognize them as being essentially the same, so your checking is useful.

Example 2: Solve

Discussion: Here the variable isn't in the denominator, but this illustrates that the basic principle of multiplying both sides by the same non-zero expression works here too.

Solution:

Solution Method: *Cross-multiply:*	Alternate Solution Method: *Multiply by common denominator*	Check:

Example 3: Solve

Solution Method: *Cross-multiply:*	Solution Method: *Multiply by common denominator*	Check:

Example 4: Find a formula for h (that is, solve for h.) .

Solution Method: *Cross-multiply:*	Alternate Solution Method: *Multiply by common denominator*	Check:
		Substitute

Example 5: Find a formula for d (that is, solve for d.) .

Solution Method: *Cross-multiply:*	Alternate Solution Method: *Multiply by common denominator*	Check:
		Substitute for d on the right.

Section 4. Evaluate variable expressions when given the value of each variable, using the correct order of operations.

Recall these rules about order of operations. Do them in this order.

1. All operations inside symbols of grouping (parentheses) from the inside out.
2. All operations of exponents or roots.
3. All multiplications and divisions, in order from left to right.
4. All additions and subtractions, in order from left to right.

The best method I know is to write the expression first with the variable, then with parentheses in place of the variable, and then with the values inserted into the parentheses.

Then you can remove any of those parentheses that aren't needed to keep the negative numbers clear and to keep the products of two numbers clear.

Then, begin to evaluate the expression according to the order of operations, doing one operation per step.

For problems in this course, the most important of these rules are to do exponents first and then to do multiplications/divisions before additions/subtractions.

Example 1. Evaluate when .

Solution:	Notice that the 7 here is in parentheses, but it is not an OPERATION in parentheses, so there is no need to think about that first rule in the order of operations.

Discussion: Later in the course, some students frequently do problems like this incorrectly because they do the addition before the multiplication. It is important to learn this rule well.

Example 2: Evaluate when .

Solution:

Discussion: Later in the course, some students have trouble remembering not to do the multiplication of 9 times 2 here first. Many of our formulas will include exponents such as this, so it is important to learn this rule well.

Checking your work: When evaluating expressions, the only clear method to check your work is to simply re-work it. That's not completely satisfactory because if you have a mistake in understanding, you're likely to make it both times.

However, in this course, most of the times we will evaluate an expression, it is in the context of a larger problem for which there is a good method of checking available.

Exercises.

After you work each of these problems, use some method of checking your answer and show that check right beside your solution. Don't forget to prepare and fill out the homework cover sheet as you do the problems.

Part I.	Part II.
1. Simplify	1. Solve
2. Simplify	2. Solve
3. Simplify	3. Simplify
4. Simplify	4. Simplify
5. Solve	5. Solve
6. Find a formula for y (that is, solve for y in terms of x):	6. Solve
7. Solve	7. Solve
8. Solve	8. Solve
9. Solve	9. Find a formula for y (that is, solve for y):
10. Solve	10. Find a formula for y (that is, solve for y):
11. : Find a formula for h (that is, solve for h.) .	11. Find a formula for k (that is, solve for k.) .
12. Find a formula for d (that is, solve for d.) .	12. Find a formula for m (that is, solve for m.) .
13. Evaluate when	13. Evaluate where
14. Evaluate when	14. Evaluate where

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4.5: Supplemental Videos



This YouTube playlist contains several videos that supplement the reading on Problem Solving.

You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

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4.6: Optional Discussion Boards

There are two optional Discussion Boards in this course:



- Go to **Discussion Board > Open Discussion** to ask questions, answer questions, post current event examples, or contribute insights related to the reading. You can also use this forum to form study groups. *You will use this board throughout the duration course. It is **not** specific to this module.*
- Go to **Discussion Board > Problem Solving Homework Discussion** to discuss issues you are having with the homework or to help your classmates with their questions. *There will be a different homework discussion board for each module. They **are** specific to each module.*

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4.7: Discuss- Problem Solving Application

Pick a unique real problem and try to solve it using the general problem solving strategies from this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the General Problem Solving Strategies Application Directions for detailed directions.

Create a new thread in the **Problem-Solving Application** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 20 points.

Grading Criteria	Points Possible
The problem: • Is it a real-life problem? • Is it challenging, not trivial? • Is it a unique problem instead of a copy of a classmate's posting?	5
The strategies: • Are one or more general problem solving strategies used? • Are the strategies correctly identified?	5
The presentation: • Is the problem explained well? • Are the problem solving strategies explained well? • Are the appropriate terms used?	4
Your responses: • Did you post at least two responses? • Did you explain how the examples helped you better understand the math in this module? • Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?	6

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4.8: Discuss- Math Editor Practice (Blackboard)

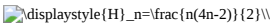
You will need to become familiar with the use of the Math Editor in Blackboard. You may choose to use the Math Editor for some of your exam questions.

Use the **Math Editor Practice** forum in the Discussion Board to complete this assignment. Click on the **Create Thread** button to begin a new thread.

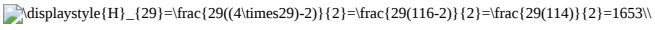
With this assignment, you will accomplish two things:

- Practice using the math editor in Blackboard. Click here for instructions: BlackboardMathEditor.doc
- Troubleshooting the Math Editor: Upgrading JAVA
- Practice presenting a problem

Let n = your age + number of the month of your birth. Determine the nth hexagonal number using the formula



Post your entire solution, including the original formula, in the discussion board **using the math editor**. Be sure to use proper fraction notation. Do not use / to represent the fraction bar. For example, if you are 25 and you were born in April, then n = 25 + 4 = 29 and the 29th hexagonal number is



This assignment is required and worth up to 10 points.

You are strongly encouraged to review & repeat this practice exercise before taking each of the exams, but no extra points will be awarded.

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4.9: Problem Solving Exercises

Exercises

1. Out of 230 racers who started the marathon, 212 completed the race, 14 gave up, and 4 were disqualified. What percentage did not complete the marathon?

2. Patrick left an $8 tip on a $50 restaurant bill. What percent tip is that?

3. Ireland has a 23% VAT (value-added tax, similar to a sales tax). How much will the VAT be on a purchase of a €250 item?

4. Employees in 2012 paid 4.2% of their gross wages towards social security (FICA tax), while employers paid another 6.2%. How much will someone earning $45,000 a year pay towards social security out of their gross wages?

5. A project on Kickstarter.com was aiming to raise $15,000 for a precision coffee press. They ended up with 714 supporters, raising 557% of their goal. How much did they raise?

6. Another project on Kickstarter for an iPad stylus raised 1,253% of their goal, raising a total of $313,490 from 7,511 supporters. What was their original goal?

7. The population of a town increased from 3,250 in 2008 to 4,300 in 2010. Find the absolute and relative (percent) increase.

8. The number of CDs sold in 2010 was 114 million, down from 147 million the previous year[6]. Find the absolute and relative (percent) decrease.

9. A company wants to decrease their energy use by 15%.
    1. If their electric bill is currently $2,200 a month, what will their bill be if they're successful?
    2. If their next bill is $1,700 a month, were they successful? Why or why not?

10. A store is hoping an advertising campaign will increase their number of customers by 30%. They currently have about 80 customers a day.
    1. How many customers will they have if their campaign is successful?
    2. If they increase to 120 customers a day, were they successful? Why or why not?

11. An article reports "attendance dropped 6% this year, to 300." What was the attendance before the drop?

12. An article reports "sales have grown by 30% this year, to $200 million." What were sales before the growth?

13. The Walden University had 47,456 students in 2010, while Kaplan University had 77,966 students. Complete the following statements:
    1. Kaplan's enrollment was ___% larger than Walden's.
    2. Walden's enrollment was ___% smaller than Kaplan's.
    3. Walden's enrollment was ___% of Kaplan's.

14. In the 2012 Olympics, Usain Bolt ran the 100m dash in 9.63 seconds. Jim Hines won the 1968 Olympic gold with a time of 9.95 seconds.
    1. Bolt's time was ___% faster than Hines'.
    2. Hine' time was ___% slower than Bolt's.
    3. Hine' time was ___% of Bolt's.

15. A store has clearance items that have been marked down by 60%. They are having a sale, advertising an additional 30% off clearance items. What percent of the original price do you end up paying?

16. Which is better: having a stock that goes up 30% on Monday than drops 30% on Tuesday, or a stock that drops 30% on Monday and goes up 30% on Tuesday? In each case, what is the net percent gain or loss?

17. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    16. "16.3% of Americans are without health insurance"[7]
    17. "only 55.9% of adults receive employer provided health insurance"[8]

18. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    1. "We mark up the wholesale price by 33% to come up with the retail price"
    2. "The store has a 25% profit margin"

19. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    1. "Every year since 1950, the number of American children gunned down has doubled."
    2. "The number of child gunshot deaths has doubled from 1950 to 1994."

20. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?[9]
    1. "75 percent of the federal health care law's taxes would be paid by those earning less than $120,000 a year"
    2. "76 percent of those who would pay the penalty [health care law's taxes] for not having insurance in 2016 would earn under $120,000"

21. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    1. "The school levy is only a 0.1% increase of the property tax rate."
    2. "This new levy is a 12% tax hike, raising our total rate to $9.33 per $1000 of value."

22. Are the values compared in this statement comparable or not comparable? "Guns have murdered more Americans here at home in recent years than have died on the battlefields of Iraq and Afghanistan. In support of the two wars, more than 6,500 American soldiers have lost their lives. During the same period, however, guns have been used to murder about 100,000 people on American soil"[10]

23. A high school currently has a 30% dropout rate. They've been tasked to decrease that rate by 20%. Find the equivalent percentage point drop.

24. A politician's support grew from 42% by 3 percentage points to 45%. What percent (relative) change is this?

25. Marcy has a 70% average in her class going into the final exam. She says “I need to get a 100% on this final so I can raise my score to 85%.” Is she correct?

26. Suppose you have one quart of water/juice mix that is 50% juice, and you add 2 quarts of juice. What percent juice is the final mix?

27. Find a unit rate: You bought 10 pounds of potatoes for $4.

28. Find a unit rate: Joel ran 1500 meters in 4 minutes, 45 seconds.

29. Solve: .

30. Solve: .

31. A crepe recipe calls for 2 eggs, 1 cup of flour, and 1 cup of milk. How much flour would you need if you use 5 eggs?

32. An 8ft length of 4 inch wide crown molding costs $14. How much will it cost to buy 40ft of crown molding?

33. Four 3-megawatt wind turbines can supply enough electricity to power 3000 homes. How many turbines would be required to power 55,000 homes?

34. A highway had a landslide, where 3,000 cubic yards of material fell on the road, requiring 200 dump truck loads to clear. On another highway, a slide left 40,000 cubic yards on the road. How many dump truck loads would be needed to clear this slide?

35. Convert 8 feet to inches.

36. Convert 6 kilograms to grams.

37. A wire costs $2 per meter. How much will 3 kilometers of wire cost?

38. Sugar contains 15 calories per teaspoon. How many calories are in 1 cup of sugar?

39. A car is driving at 100 kilometers per hour. How far does it travel in 2 seconds?

40. A chain weighs 10 pounds per foot. How many ounces will 4 inches weigh?

41. The table below gives data on three movies. Gross earnings is the amount of money the movie brings in. Compare the net earnings (money made after expenses) for the three movies.[11]

Movie	Release Date	Budget	Gross earnings
Saw	10/29/2004	$1,200,000	$103,096,345
Titanic	12/19/1997	$200,000,000	$1,842,879,955
Jurassic Park	6/11/1993	$63,000,000	$923,863,984

42. For the movies in the previous problem, which provided the best return on investment?

43. The population of the U.S. is about 309,975,000, covering a land area of 3,717,000 square miles. The population of India is about 1,184,639,000, covering a land area of 1,269,000 square miles. Compare the population densities of the two countries.

44. The GDP (Gross Domestic Product) of China was $5,739 billion in 2010, and the GDP of Sweden was $435 billion. The population of China is about 1,347 million, while the population of Sweden is about 9.5 million. Compare the GDP per capita of the two countries.

45. In June 2012, Twitter was reporting 400 million tweets per day. Each tweet can consist of up to 140 characters (letter, numbers, etc.). Create a comparison to help understand the amount of tweets in a year by imagining each character was a drop of water and comparing to filling something up.

46. The photo sharing site Flickr had 2.7 billion photos in June 2012. Create a comparison to understand this number by assuming each picture is about 2 megabytes in size, and comparing to the data stored on other media like DVDs, iPods, or flash drives.

47. Your chocolate milk mix says to use 4 scoops of mix for 2 cups of milk. After pouring in the milk, you start adding the mix, but get distracted and accidentally put in 5 scoops of mix. How can you adjust the mix if:
    1. There is still room in the cup?
    2. The cup is already full?

48. A recipe for sabayon calls for 2 egg yolks, 3 tablespoons of sugar, and ¼ cup of white wine. After cracking the eggs, you start measuring the sugar, but accidentally put in 4 tablespoons of sugar. How can you compensate?

49. The Deepwater Horizon oil spill resulted in 4.9 million barrels of oil spilling into the Gulf of Mexico. Each barrel of oil can be processed into about 19 gallons of gasoline. How many cars could this have fueled for a year? Assume an average car gets 20 miles to the gallon, and drives about 12,000 miles in a year.

50. The store is selling lemons at 2 for $1. Each yields about 2 tablespoons of juice. How much will it cost to buy enough lemons to make a 9-inch lemon pie requiring ½ cup of lemon juice?

51. A piece of paper can be made into a cylinder in two ways: by joining the short sides together, or by joining the long sides together[12]. Which cylinder would hold more? How much more?

52. Which of these glasses contains more liquid? How much more?

In the next 4 questions, estimate the values by making reasonable approximations for unknown values, or by doing some research to find reasonable values.

53. Estimate how many gallons of water you drink in a year.

54. Estimate how many times you blink in a day.

55. How much does the water in a 6-person hot tub weigh?

56. How many gallons of paint would be needed to paint a two-story house 40 ft long and 30 ft wide?

57. During the landing of the Mars Science Laboratory *Curiosity*, it was reported that the signal from the rover would take 14 minutes to reach earth. Radio signals travel at the speed of light, about 186,000 miles per second. How far was Mars from Earth when *Curiosity* landed?
58. It is estimated that a driver takes, on average, 1.5 seconds from seeing an obstacle to reacting by applying the brake or swerving. How far will a car traveling at 60 miles per hour travel (in feet) before the driver reacts to an obstacle?

59. The flash of lightning travels at the speed of light, which is about 186,000 miles per second. The sound of lightning (thunder) travels at the speed of sound, which is about 750 miles per hour.
    1. If you see a flash of lightning, then hear the thunder 4 seconds later, how far away is the lightning?
    2. Now let's generalize that result. Suppose it takes n seconds to hear the thunder after a flash of lightning. How far away is the lightning, in terms of n?

60. Sound travels about 750 miles per hour. If you stand in a parking lot near a building and sound a horn, you will hear an echo.
    1. Suppose it takes about ½ a second to hear the echo. How far away is the building[13]?
    2. Now let's generalize that result. Suppose it takes n seconds to hear the echo. How far away is the building, in terms of n?

61. It takes an air pump 5 minutes to fill a twin sized air mattress (39 by 8.75 by 75 inches). How long will it take to fill a queen sized mattress (60 by 8.75 by 80 inches)?

62. It takes your garden hose 20 seconds to fill your 2-gallon watering can. How long will it take to fill
63. An inflatable pool measuring 3 feet wide, 8 feet long, and 1 foot deep.[14]
64. A circular inflatable pool 13 feet in diameter and 3 feet deep.[15]

63. You want to put a 2″ thick layer of topsoil for a new 20'x30′ garden. The dirt store sells by the cubic yards. How many cubic yards will you need to order?

64. A box of Jell-O costs $0.50, and makes 2 cups. How much would it cost to fill a swimming pool 4 feet deep, 8 feet wide, and 12 feet long with Jell-O? (1 cubic foot is about 7.5 gallons)

65. You read online that a 15 ft by 20 ft brick patio would cost about $2,275 to have professionally installed. Estimate the cost of having a 18 by 22 ft brick patio installed.

66. I was at the store, and saw two sizes of avocados being sold. The regular size sold for $0.88 each, while the jumbo ones sold for $1.68 each. Which is the better deal?

67. The grocery store has bulk pecans on sale, which is great since you're planning on making 10 pecan pies for a wedding. Your recipe calls for 1¾ cups pecans per pie. However, in the bulk section there's only a scale available, not a measuring cup. You run over to the baking aisle and find a bag of pecans, and look at the nutrition label to gather some info. How many pounds of pecans should you buy?

68. Soda is often sold in 20 ounce bottles. The nutrition label for one of these bottles is shown to the right. A packet of sugar (the kind they have at restaurants for your coffee or tea) typically contain 4 grams of sugar in the U.S. Drinking a 20 oz soda is equivalent to eating how many packets of sugar?[16]

For the next set of questions, *first* identify the information you need to answer the question, and *then* turn to the end of the section to find that information. The details may be imprecise; answer the question the best you can with the provided information. Be sure to justify your decision.

69. You're planning on making 6 meatloafs for a party. You go to the store to buy breadcrumbs, and see they are sold by the canister. How many canisters do you need to buy?

70. Your friend wants to cover their car in bottle caps, like in this picture.[17] How many bottle caps are you going to need?

71. You need to buy some chicken for dinner tonight. You found an ad showing that the store across town has it on sale for $2.99 a pound, which is cheaper than your usual neighborhood store, which sells it for $3.79 a pound. Is it worth the extra drive?

72. I have an old gas furnace, and am considering replacing it with a new, high efficiency model. Is upgrading worth it?

73. Janine is considering buying a water filter and a reusable water bottle rather than buying bottled water. Will doing so save her money?

74. Marcus is considering going car-free to save money and be more environmentally friendly. Is this financially a good decision?

For the next set of problems, research or make educated estimates for any unknown quantities needed to answer the question.

75. You want to travel from Tacoma, WA to Chico, CA for a wedding. Compare the costs and time involved with driving, flying, and taking a train. Assume that if you fly or take the train you'll need to rent a car while you're there. Which option is best?

76. You want to paint the walls of a 6ft by 9ft storage room that has one door and one window. You want to put on two coats of paint. How many gallons and/or quarts of paint should you buy to paint the room as cheaply as possible?

77. A restaurant in New York tiled their floor with pennies[18]. Just for the materials, is this more expensive than using a more traditional material like ceramic tiles? If each penny has to be laid by hand, estimate how long it would take to lay the pennies for a 12ft by 10ft room. Considering material and labor costs, are pennies a cost-effective replacement for ceramic tiles?

78. You are considering taking up part of your back yard and turning it into a vegetable garden, to grow broccoli, tomatoes, and zucchini. Will doing so save you money, or cost you more than buying vegetables from the store?

79. Barry is trying to decide whether to keep his 1993 Honda Civic with 140,000 miles, or trade it in for a used 2008 Honda Civic. Consider gas, maintenance, and insurance costs in helping him make a decision.

80. Some people claim it costs more to eat vegetarian, while some claim it costs less. Examine your own grocery habits, and compare your current costs to the costs of switching your diet (from omnivore to vegetarian or vice versa as appropriate). Which diet is more cost effective based on your eating habits?

Info for the breadcrumbs question



Nutrition Facts

Serving Size: 1/3 cup (30g)
Servings per Container: about 14

Amount Per Serving	
Calories 110	Calories from Fat 15
	% Daily Value*
Total Fat 1.5g	2%

How much breadcrumbs does the recipe call for?

It calls for 1½ cups of breadcrumbs.

How many meatloaves does the recipe make?

It makes 1 meatloaf.

How many servings does that recipe make?

It says it serves 8.

How big is the canister?

It is cylindrical, 3.5 inches across and

7 inches tall.

What is the net weight of the contents of 1 canister?

15 ounces.

How much does a cup of breadcrumbs weigh?

I'm not sure, but maybe something from the nutritional label will help.

How much does a canister cost? $2.39

Info for bottle cap car

What kind of car is that?

A 1993 Honda Accord.

How big is that car / what are the dimensions? Here is some details from MSN autos:

Weight: 2800lb Length: 185.2 in Width: 67.1 in Height: 55.2 in

How much of the car was covered with caps?

Everything but the windows and the underside.

How big is a bottle cap?

Caps are 1 inch in diameter.

Info for chicken problem

How much chicken will you be buying?

Four pounds

How far are the two stores?

My neighborhood store is 2.2 miles away, and takes about 7 minutes. The store across town is 8.9 miles away, and takes about 25 minutes.

What kind of mileage does your car get?

It averages about 24 miles per gallon in the city.

How many gallons does your car hold?

About 14 gallons

How much is gas?

About $3.69/gallon right now.

Info for furnace problem

How efficient is the current furnace?

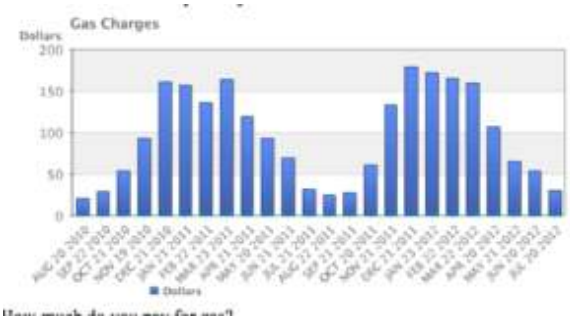
It is a 60% efficient furnace.

How efficient is the new furnace?

It is 94% efficient.

What is your gas bill?

Here is the history for 2 years:

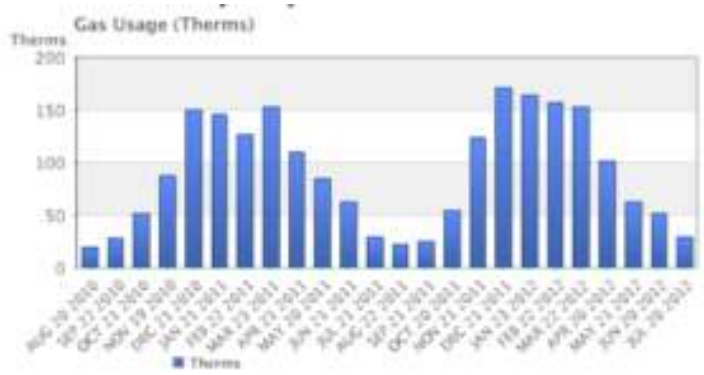


How much do you pay for gas?

There is $10.34 base charge, plus $0.39097 per Therm for a delivery charge, and $0.65195 per Therm for cost of gas.

How much gas do you use?

Here is the history for 2 years:



How much does the new furnace cost?

It will cost $7,450.

How long do you plan to live in the house?

Probably at least 15 years.

Info for water filter problem

How much water does Janine drink in a day?

She normally drinks 3 bottles a day, each 16.9 ounces.

How much does a bottle of water cost?

She buys 24-packs of 16.9 ounce bottles for $3.99.

How much does a reusable water bottle cost?

About $10.

How long does a reusable water bottle last?

Basically forever (or until you lose it).

How much does a water filter cost? How much water will they filter?

- A faucet-mounted filter costs about $28. Refill filters cost about $33 for a 3-pack. The box says each filter will filter up to 100 gallons (378 liters)
- A water filter pitcher costs about $22. Refill filters cost about $20 for a 4-pack. The box says each filter lasts for 40 gallons or 2 months
- An under-sink filter costs $130. Refill filters cost about $60 each. The filter lasts for 500 gallons.

Info for car-free problem

Where does Marcus currently drive? He:

- Drives to work 5 days a week, located 4 miles from his house.
- Drives to the store twice a week, located 7 miles from his house.
- Drives to other locations on average 5 days a week, with locations ranging from 1 mile to 20 miles.
- Drives to his parent's house 80 miles away once a month.

How will he get to these locations without a car?

- For work, he can walk when it's sunny and he gets up early enough. Otherwise he can take a bus, which takes about 20 minutes

- For the store, he can take a bus, which takes about 35 minutes.
- Some of the other locations he can bus to. Sometimes he'll be able to get a friend to pick him up. A few locations he is able to walk to. A couple locations are hard to get to by bus, but there is a ZipCar (short term car rental) location within a few blocks.
- He'll need to get a ZipCar to visit his parents.

How much does gas cost?

About $3.69/gallon.

How much does he pay for insurance and maintenance?

- He pays $95/month for insurance.
- He pays $30 every 3 months for an oil change, and has averaged about $300/year for other maintenance costs.

How much is he paying for the car?

- He's paying $220/month on his car loan right now, and has 3 years left on the loan.
- If he sold the car, he'd be able to make enough to pay off the loan.
- If he keeps the car, he's planning on trading the car in for a newer model in a couple years.

What mileage does his car get?

About 26 miles per gallon on average.

How much does a bus ride cost?

$2.50 per trip, or $90 for an unlimited monthly pass.

How much does a ZipCar rental cost?

- The "occasional driving plan": $25 application fee and $60 annual fee, with no monthly commitment. Monday-Thursday the cost is $8/hour, or $72 per day. Friday-Sunday the cost is $8/hour or $78/day. Gas, insurance, and 180 miles are included in the cost. Additional miles are $0.45/mile.
- The "extra value plan": Same as above, but with a $50 monthly commitment, getting you a 10% discount on the usage costs.

Extension: Taxes

Governments collect taxes to pay for the services they provide. In the United States, federal income taxes help fund the military, the environmental protection agency, and thousands of other programs. Property taxes help fund schools. Gasoline taxes help pay for road improvements. While very few people enjoy paying taxes, they are necessary to pay for the services we all depend upon.

Taxes can be computed in a variety of ways, but are typically computed as a percentage of a sale, of one's income, or of one's assets.

Example 1

The sales tax rate in a city is 9.3%. How much sales tax will you pay on a $140 purchase?

The sales tax will be 9.3% of $140. To compute this, we multiply $140 by the percent written as a decimal: $140(0.093) = $13.02.

When taxes are not given as a fixed percentage rate, sometimes it is necessary to calculate the **effective rate.**

Effective rate

The effective tax rate is the equivalent percent rate of the tax paid out of the dollar amount the tax is based on.

Example 2

Joan paid $3,200 in property taxes on her house valued at $215,000 last year. What is the effective tax rate?

We can compute the equivalent percentage: 3200/215000 = 0.01488, or about 1.49% effective rate.

Taxes are often referred to as progressive, regressive, or flat.

Tax categories

A **flat tax**, or proportional tax, charges a constant percentage rate.

A **progressive tax** increases the percent rate as the base amount increases.

A **regressive tax** decreases the percent rate as the base amount increases.

Example 3

The United States federal income tax on earned wages is an example of a progressive tax. People with a higher wage income pay a higher percent tax on their income.

For a single person in 2011, adjusted gross income (income after deductions) under $8,500 was taxed at 10%. Income over $8,500 but under $34,500 was taxed at 15%.

A person earning $10,000 would pay 10% on the portion of their income under $8,500, and 15% on the income over $8,500, so they'd pay:

8500(0.10) = 850 10% of $8500
1500(0.15) = 225 15% of the remaining $1500 of income
Total tax: = $1075

The effective tax rate paid is 1075/10000 = 10.75%

A person earning $30,000 would also pay 10% on the portion of their income under $8,500, and 15% on the income over $8,500, so they'd pay:

8500(0.10) = 850 10% of $8500
21500(0.15) = 3225 15% of the remaining $21500 of income
Total tax: = $4075

The effective tax rate paid is 4075/30000 = 13.58%.

Notice that the effective rate has increased with income, showing this is a progressive tax.

Example 4

A gasoline tax is a flat tax when considered in terms of consumption, a tax of, say, $0.30 per gallon is proportional to the amount of gasoline purchased. Someone buying 10 gallons of gas at $4 a gallon would pay $3 in tax, which is $3/$40 = 7.5%. Someone buying 30 gallons of gas at $4 a gallon would pay $9 in tax, which is $9/$120 = 7.5%, the same effective rate.

However, in terms of income, a gasoline tax is often considered a regressive tax. It is likely that someone earning $30,000 a year and someone earning $60,000 a year will drive about the same amount. If both pay $60 in gasoline taxes over a year, the person earning $30,000 has paid 0.2% of their income, while the person earning $60,000 has paid 0.1% of their income in gas taxes.

Try it Now 1

A sales tax is a fixed percentage tax on a person's purchases. Is this a flat, progressive, or regressive tax?

Try it Now Answers

1. While sales tax is a flat percentage rate, it is often considered a regressive tax for the same reasons as the gasoline tax.

Income Taxation

Many people have proposed various revisions to the income tax collection in the United States. Some, for example, have claimed that a flat tax would be fairer. Others call for revisions to how different types of income are taxed, since currently investment income is taxed at a different rate than wage income.

The following two projects will allow you to explore some of these ideas and draw your own conclusions.

Project 1: Flat tax, Modified Flat Tax, and Progressive Tax.

Imagine the country is made up of 100 households. The federal government needs to collect $800,000 in income taxes to be able to function. The population consists of 6 groups:

Group A: 20 households that earn $12,000 each

Group B: 20 households that earn $29,000 each

Group C: 20 households that earn $50,000 each

Group D: 20 households that earn $79,000 each

Group E: 15 households that earn $129,000 each

Group F: 5 households that earn $295,000 each

This scenario is roughly proportional to the actual United States population and tax needs. We are going to determine new income tax rates.

The first proposal we'll consider is a flat tax – one where every income group is taxed at the same percentage tax rate.

1) Determine the total income for the population (all 100 people together)

2) Determine what flat tax rate would be necessary to collect enough money.

The second proposal we'll consider is a modified flat-tax plan, where everyone only pays taxes on any income over $20,000. So, everyone in group A will pay no taxes. Everyone in group B will pay taxes only on $9,000.

3) Determine the total *taxable* income for the whole population

4) Determine what flat tax rate would be necessary to collect enough money in this modified system

5) Complete this table for both the plans

		Flat Tax Plan		**Modified Flat Tax Plan**	
Group	**Income per household**	**Income tax per household**	**Income after taxes**	**Income tax per household**	**Income after taxes**
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				

F	$295,000				

The third proposal we'll consider is a progressive tax, where lower income groups are taxed at a lower percent rate, and higher income groups are taxed at a higher percent rate. For simplicity, we're going to assume that a household is taxed at the same rate on *all* their income.

6) Set progressive tax rates for each income group to bring in enough money. There is no one right answer here – just make sure you bring in enough money!

Group	Income per household	Tax rate (%)	Income tax per household	Total tax collected for all households	Income after taxes per household
A	$12,000				
B	$29,000				
C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				
				This better total to $800,000	

7) Discretionary income is the income people have left over after paying for necessities like rent, food, transportation, etc. The cost of basic expenses does increase with income, since housing and car costs are higher, however usually not proportionally. For each income group, estimate their essential expenses, and calculate their discretionary income. Then compute the effective tax rate for each plan relative to discretionary income rather than income.

Group	Income per household	Discretionary Income (estimated)	Effective rate, flat	Effective rate, modified	Effective rate, progressive
A	$12,000				
B	$29,000				

C	$50,000				
D	$79,000				
E	$129,000				
F	$295,000				

8) Which plan seems the most fair to you? Which plan seems the least fair to you? Why?

Project 2: Calculating Taxes.

Visit www.irs.gov, and download the most recent version of forms 1040, and schedules A, B, C, and D.

Scenario 1: Calculate the taxes for someone who earned $60,000 in standard wage income (W-2 income), has no dependents, and takes the standard deduction.

Scenario 2: Calculate the taxes for someone who earned $20,000 in standard wage income, $40,000 in qualified dividends, has no dependents, and takes the standard deduction. (Qualified dividends are earnings on certain investments such as stocks.)

Scenario 3: Calculate the taxes for someone who earned $60,000 in small business income, has no dependents, and takes the standard deduction.

Based on these three scenarios, what are your impressions of how the income tax system treats these different forms of income (wage, dividends, and business income)?

Scenario 4: To get a more realistic sense for calculating taxes, you'll need to consider itemized deductions. Calculate the income taxes for someone with the income and expenses listed below.

Married with 2 children, filing jointly

Wage income: $50,000 combined

Paid sales tax in Washington State

Property taxes paid: $3200

Home mortgage interest paid: $4800

Charitable gifts: $1200

[1] http://www.factcheck.org/cheney_edwards_mangle_facts.html

[2] http://tvtropes.org/pmwiki/pmwiki.php/Main/LiesDamnedLiesAndStatistics

[3] http://www.whitehouse.gov/sites/defa...s/hist07z1.xls

[4] https://www.cia.gov/library/publicat.../2042rank.html

[5] http://data.worldbank.org/indicator/SP.POP.TOTL

[6] http://www.cnn.com/2010/SHOWBIZ/Musi...les/index.html

[7] http://www.cnn.com/2012/06/27/politi...are/index.html

[8] http://www.politico.com/news/stories/0712/78134.html

[9] http://factcheck.org/2012/07/twistin...th-care-taxes/

[10] http://www.northjersey.com/news/opin...tml?c=y&page=2

[11] http://www.the-numbers.com/movies/records/budgets.php

[12] http://vimeo.com/42501010

[13] http://vimeo.com/40377128

[14] http://www.youtube.com/watch?v=DIkwefReHZc

[15] http://www.youtube.com/watch?v=p9SABH7Yg9M

[16] http://www.youtube.com/watch?v=62JMfv0tf3Q

[17] Photo credit: http://www.flickr.com/photos/swayze/, CC-BY

[18] http://www.notcot.com/archives/2009/...-of-pennie.php

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CHAPTER OVERVIEW

5: Growth Models

5.1: Linear (Algebraic) Growth
5.2: Exponential (Geometric) Growth
5.3: Solve Exponentials for Time- Logarithms
5.4: Logistic Growth
5.5: Exercises

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5.1: Linear (Algebraic) Growth

Marco is a collector of antique soda bottles. His collection currently contains 437 bottles. Every year, he budgets enough money to buy 32 new bottles. Can we determine how many bottles he will have in 5 years, and how long it will take for his collection to reach 1000 bottles?

While both of these questions you could probably solve without an equation or formal mathematics, we are going to formalize our approach to this problem to provide a means to answer more complicated questions.

Suppose that Pn represents the number, or population, of bottles Marco has after n years. So $P0$ would represent the number of bottles now, $P1$ would represent the number of bottles after 1 year, $P2$ would represent the number of bottles after 2 years, and so on. We could describe how Marco's bottle collection is changing using:

$P0 = 437$

$Pn = Pn\text{-}1 + 32$

This is called a **recursive relationship**. A recursive relationship is a formula which relates the next value in a sequence to the previous values. Here, the number of bottles in year n can be found by adding 32 to the number of bottles in the previous year, $Pn\text{-}1$. Using this relationship, we could calculate:

$P1 = P0 + 32 = 437 + 32 = 469$

$P2 = P1 + 32 = 469 + 32 = 501$

$P3 = P2 + 32 = 501 + 32 = 533$

$P4 = P3 + 32 = 533 + 32 = 565$

$P5 = P4 + 32 = 565 + 32 = 597$

We have answered the question of how many bottles Marco will have in 5 years. However, solving how long it will take for his collection to reach 1000 bottles would require a lot more calculations.

While recursive relationships are excellent for describing simply and cleanly *how* a quantity is changing, they are not convenient for making predictions or solving problems that stretch far into the future. For that, a closed or explicit form for the relationship is preferred. An **explicit equation** allows us to calculate Pn directly, without needing to know $Pn\text{-}1$. While you may already be able to guess the explicit equation, let us derive it from the recursive formula. We can do so by selectively not simplifying as we go:

$P1 = 437 + 32 = 437 + 1(32)$

$P2 = P1 + 32 = 437 + 32 + 32 = 437 + 2(32)$

$P3 = P2 + 32 = (437 + 2(32)) + 32 = 437 + 3(32)$

$P4 = P3 + 32 = (437 + 3(32)) + 32 = 437 + 4(32)$

You can probably see the pattern now, and generalize that

$Pn = 437 + n(32) = 437 + 32n$

Using this equation, we can calculate how many bottles he'll have after 5 years:

$P5 = 437 + 32(5) = 437 + 160 = 597$

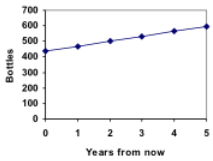
We can now also solve for when the collection will reach 1000 bottles by substituting in 1000 for Pn and solving for n

$1000 = 437 + 32n$

$563 = 32n$

$n = 563/32 = 17.59$

So Marco will reach 1000 bottles in 18 years.



In the previous example, Marco's collection grew by the *same number* of bottles every year. This constant change is the defining characteristic of linear growth. Plotting the values we calculated for Marco's collection, we can see the values form a straight line, the shape of linear growth.

Linear Growth

If a quantity starts at size *P0* and grows by *d* every time period, then the quantity after *n* time periods can be determined using either of these relations:

Recursive form:

$Pn = Pn\text{-}1 + d$

Explicit form:

$Pn = P0 + d\,n$

In this equation, *d* represents the **common difference** – the amount that the population changes each time *n* increases by 1

Connection to prior learning – slope and intercept

You may recognize the common difference, *d*, in our linear equation as *slope*. In fact, the entire explicit equation should look familiar – it is the same linear equation you learned in algebra, probably stated as $y = mx + b$.

In the standard algebraic equation $y = mx + b$, *b* was the *y*-intercept, or the *y* value when *x* was zero. In the form of the equation we're using, we are using *P0* to represent that initial amount.

In the $y = mx + b$ equation, recall that *m* was the slope. You might remember this as "rise over run", or the change in *y* divided by the change in *x*. Either way, it represents the same thing as the common difference, *d*, we are using – the amount the output *Pn* changes when the input *n* increases by 1.

The equations $y = mx + b$ and $Pn = P0 + d\,n$ mean the same thing and can be used the same ways, we're just writing it somewhat differently.

Example 1

The population of elk in a national forest was measured to be 12,000 in 2003, and was measured again to be 15,000 in 2007. If the population continues to grow linearly at this rate, what will the elk population be in 2014?

To begin, we need to define how we're going to measure *n*. Remember that *P0* is the population when $n = 0$, so we probably don't want to literally use the year 0. Since we already know the population in 2003, let us define $n = 0$ to be the year 2003. Then

$P0 = 12{,}000$.

Next we need to find *d*. Remember *d* is the growth per time period, in this case growth per year. Between the two measurements, the population grew by 15,000-12,000 = 3,000, but it took 2007-2003 = 4 years to grow that much. To find the growth per year, we can divide: 3000 elk / 4 years = 750 elk in 1 year.

Alternatively, you can use the slope formula from algebra to determine the common difference, noting that the population is the output of the formula, and time is the input.

$d=slope=\frac{\text{changeinoutput}}{\text{changeininput}}=\frac{15,000-12,000}{2007-2003}=\frac{3000}{4}=750$

We can now write our equation in whichever form is preferred.

Recursive form:

$P0 = 12{,}000$

Pn = *Pn-1* + 750

Explicit form:

Pn = 12,000 + 750*n*

To answer the question, we need to first note that the year 2014 will be *n* = 11, since 2014 is 11 years after 2003. The explicit form will be easier to use for this calculation:

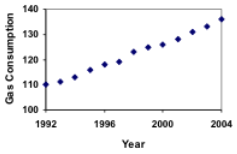
P11 = 12,000 + 750(11) = 20,250 elk

Example 2

Gasoline consumption in the US has been increasing steadily. Consumption data from 1992 to 2004 is shown below[1]. Find a model for this data, and use it to predict consumption in 2016. If the trend continues, when will consumption reach 200 billion gallons?

Year	’92	’93	’94	’95	’96	’97	’98	’99	’00	’01	’02	’03	’04
Consumption (billion of gallons)	110	111	113	116	118	119	123	125	126	128	131	133	136

Plotting this data, it appears to have an approximately linear relationship:



While there are more advanced statistical techniques that can be used to find an equation to model the data, to get an idea of what is happening, we can find an equation by using two pieces of the data – perhaps the data from 1993 and 2003.

Letting *n* = 0 correspond with 1993 would give *P0* = 111 billion gallons.

To find *d*, we need to know how much the gas consumption increased each year, on average. From 1993 to 2003 the gas consumption increased from 111 billion gallons to 133 billion gallons, a total change of 133 – 111 = 22 billion gallons, over 10 years. This gives us an average change of 22 billion gallons / 10 year = 2.2 billion gallons per year.

Equivalently,


billion gallons per year

We can now write our equation in whichever form is preferred.

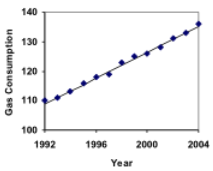
Recursive form:

P0 = 111

Pn = *Pn-1* + 2.2

Explicit form:

Pn = 111 + 2.2*n*



Calculating values using the explicit form and plotting them with the original data shows how well our model fits the data.

We can now use our model to make predictions about the future, assuming that the previous trend continues unchanged. To predict the gasoline consumption in 2016:

n = 23 (2016 – 1993 = 23 years later)

$P23$ = 111 + 2.2(23) = 161.6

Our model predicts that the US will consume 161.6 billion gallons of gasoline in 2016 if the current trend continues.

To find when the consumption will reach 200 billion gallons, we would set Pn = 200, and solve for n:

Pn = 200	Replace Pn with our model
111 + 2.2n = 200	Subtract 111 from both sides
2.2n = 89	Divide both sides by 2.2

n = 40.4545

This tells us that consumption will reach 200 billion about 40 years after 1993, which would be in the year 2033.

Example 3

The cost, in dollars, of a gym membership for n months can be described by the explicit equation Pn = 70 + 30n. What does this equation tell us?

The value for $P0$ in this equation is 70, so the initial starting cost is $70. This tells us that there must be an initiation or start-up fee of $70 to join the gym.

The value for d in the equation is 30, so the cost increases by $30 each month. This tells us that the monthly membership fee for the gym is $30 a month.

Try it Now 1

The number of stay-at-home fathers in Canada has been growing steadily[2]. While the trend is not perfectly linear, it is fairly linear. Use the data from 1976 and 2010 to find an explicit formula for the number of stay-at-home fathers, then use it to predict the number if 2020.

When good models go bad

When using mathematical models to predict future behavior, it is important to keep in mind that very few trends will continue indefinitely.

Example 4

Suppose a four year old boy is currently 39 inches tall, and you are told to expect him to grow 2.5 inches a year.

We can set up a growth model, with n = 0 corresponding to 4 years old.

Recursive form:

$P0$ = 39

Pn = Pn-1 + 2.5

Explicit form:

Pn = 39 + 2.5n

So at 6 years old, we would expect him to be

$P2$ = 39 + 2.5(2) = 44 inches tall

Any mathematical model will break down eventually. Certainly, we shouldn't expect this boy to continue to grow at the same rate all his life. If he did, at age 50 he would be

$P46$ = 39 + 2.5(46) = 154 inches tall = 12.8 feet tall!

When using any mathematical model, we have to consider which inputs are reasonable to use. Whenever we **extrapolate**, or make predictions into the future, we are assuming the model will continue to be valid.

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1. http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html ↵
2. http://www.fira.ca/article.php?id=140 ↵

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5.2: Exponential (Geometric) Growth

Suppose that every year, only 10% of the fish in a lake have surviving offspring. If there were 100 fish in the lake last year, there would now be 110 fish. If there were 1000 fish in the lake last year, there would now be 1100 fish. Absent any inhibiting factors, populations of people and animals tend to grow by a percent of the existing population each year.

Suppose our lake began with 1000 fish, and 10% of the fish have surviving offspring each year. Since we start with 1000 fish, $P0$ = 1000. How do we calculate $P1$? The new population will be the old population, plus an additional 10%. Symbolically:

$P1 = P0 + 0.10P0$

Notice this could be condensed to a shorter form by factoring:

$P1 = P0 + 0.10P0 = 1P0 + 0.10P0 = (1+ 0.10)P0 = 1.10P0$

While 10% is the **growth rate**, 1.10 is the **growth multiplier**. Notice that 1.10 can be thought of as "the original 100% plus an additional 10%"

For our fish population,

$P1 = 1.10(1000) = 1100$

We could then calculate the population in later years:

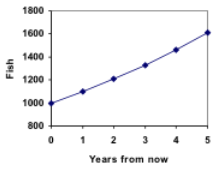
$P2 = 1.10P1 = 1.10(1100) = 1210$

$P3 = 1.10P2 = 1.10(1210) = 1331$

Notice that in the first year, the population grew by 100 fish, in the second year, the population grew by 110 fish, and in the third year the population grew by 121 fish.

While there is a constant *percentage* growth, the actual increase in number of fish is increasing each year.

Graphing these values we see that this growth doesn't quite appear linear.



To get a better picture of how this percentage-based growth affects things, we need an explicit form, so we can quickly calculate values further out in the future.

Like we did for the linear model, we will start building from the recursive equation:

$P1 = 1.10P0 = 1.10(1000)$

$P2 = 1.10P1 = 1.10(1.10(1000)) = 1.102(1000)$

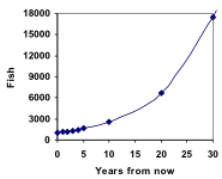
$P3 = 1.10P2 = 1.10(1.102(1000)) = 1.103(1000)$

$P4 = 1.10P3 = 1.10(1.103(1000)) = 1.104(1000)$

Observing a pattern, we can generalize the explicit form to be:

$Pn = 1.10n(1000)$, or equivalently, $Pn = 1000(1.10n)$

From this, we can quickly calculate the number of fish in 10, 20, or 30 years:



$P10 = 1.1010(1000) = 2594$

$P20 = 1.1020(1000) = 6727$

$P30 = 1.1030(1000) = 17449$

Adding these values to our graph reveals a shape that is definitely not linear. If our fish population had been growing linearly, by 100 fish each year, the population would have only reached 4000 in 30 years compared to almost 18000 with this percent-based growth, called **exponential growth.**

In exponential growth, the population grows proportional to the size of the population, so as the population gets larger, the same percent growth will yield a larger numeric growth.

Exponential Growth

If a quantity starts at size $P0$ and grows by $R\%$ (written as a decimal, r) every time period, then the quantity after n time periods can be determined using either of these relations:

Recursive form:

$Pn = (1+r)\ Pn-1$

Explicit form:

$Pn = (1+r)nP0$ or equivalently, $Pn = P0\ (1+r)n$

We call r the **growth rate**.

The term $(1+r)$ is called the **growth multiplier**, or common ratio.

Example 5

Between 2007 and 2008, Olympia, WA grew almost 3% to a population of 245 thousand people. If this growth rate was to continue, what would the population of Olympia be in 2014?

As we did before, we first need to define what year will correspond to $n = 0$. Since we know the population in 2008, it would make sense to have 2008 correspond to $n = 0$, so $P0 = 245{,}000$. The year 2014 would then be $n = 6$.

We know the growth rate is 3%, giving $r = 0.03$.

Using the explicit form:

$P6 = (1+0.03)6\ (245{,}000) = 1.19405(245{,}000) = 292{,}542.25$

The model predicts that in 2014, Olympia would have a population of about 293 thousand people.

Evaluating exponents on the calculator

To evaluate expressions like (1.03)6, it will be easier to use a calculator than multiply 1.03 by itself six times. Most scientific calculators have a button for exponents. It is typically either labeled like:

^ , y^x , or x^y .

To evaluate 1.036 we'd type 1.03 ^ 6, or 1.03 yx 6. Try it out – you should get an answer around 1.1940523.

Try it Now 2

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, what will India's population grow to by 2020?

Example 6

A friend is using the equation $Pn = 4600(1.072)n$ to predict the annual tuition at a local college. She says the formula is based on years after 2010. What does this equation tell us?

In the equation, $P0 = 4600$, which is the starting value of the tuition when $n = 0$. This tells us that the tuition in 2010 was $4,600.

The growth multiplier is 1.072, so the growth rate is 0.072, or 7.2%. This tells us that the tuition is expected to grow by 7.2% each year.

Putting this together, we could say that the tuition in 2010 was $4,600, and is expected to grow by 7.2% each year.

Example 7

In 1990, the residential energy use in the US was responsible for 962 million metric tons of carbon dioxide emissions. By the year 2000, that number had risen to 1182 million metric tons[1]. If the emissions grow exponentially and continue at the same rate, what will the emissions grow to by 2050?

Similar to before, we will correspond $n = 0$ with 1990, as that is the year for the first piece of data we have. That will make $P0 = 962$ (million metric tons of CO2). In this problem, we are not given the growth rate, but instead are given that $P10 = 1182$.

When $n = 10$, the explicit equation looks like:

$P10 = (1+r)10\ P0$

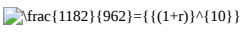
We know the value for $P0$, so we can put that into the equation:

$P10 = (1+r)10\ 962$

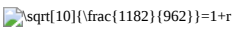
We also know that $P10 = 1182$, so substituting that in, we get

$1182 = (1+r)10\ 962$

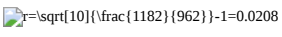
We can now solve this equation for the growth rate, r. Start by dividing by 962.



Take the 10th root of both sides



Subtract 1 from both sides



= 2.08%

So if the emissions are growing exponentially, they are growing by about 2.08% per year. We can now predict the emissions in 2050 by finding $P60$

$P60 = (1+0.0208)60\ 962 = 3308.4$ million metric tons of CO2 in 2050

Rounding

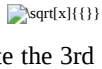
As a note on rounding, notice that if we had rounded the growth rate to 2.1%, our calculation for the emissions in 2050 would have been 3347. Rounding to 2% would have changed our result to 3156. A very small difference in the growth rates gets magnified greatly in exponential growth. For this reason, it is recommended to round the growth rate as little as possible.

If you need to round, **keep at least three significant digits** – numbers after any leading zeros. So 0.4162 could be reasonably rounded to 0.416. A growth rate of 0.001027 could be reasonably rounded to 0.00103.

Evaluating roots on the calculator

In the previous example, we had to calculate the 10th root of a number. This is different than taking the basic square root, √. Many scientific calculators have a button for general roots. It is typically labeled like:



, 

, or 

To evaluate the 3rd root of 8, for example, we'd either type 3 

8, or 8 

3, depending on the calculator. Try it on yours to see which to use – you should get an answer of 2.

If your calculator does not have a general root button, all is not lost. You can instead use the property of exponents which states that . So, to compute the 3rd root of 8, you could use your calculator's exponent key to evaluate . To do this, type:

8 y^x (1 ÷ 3)

The parentheses tell the calculator to divide 1/3 before doing the exponent.

Try it Now 3

The number of users on a social networking site was 45 thousand in February when they officially went public, and grew to 60 thousand by October. If the site is growing exponentially, and growth continues at the same rate, how many users should then expect two years after they went public?

Example 8

Looking back at the last example, for the sake of comparison, what would the carbon emissions be in 2050 if emissions grow linearly at the same rate?

Again we will get $n = 0$ correspond with 1990, giving $P0 = 962$. To find d, we could take the same approach as earlier, noting that the emissions increased by 220 million metric tons in 10 years, giving a common difference of 22 million metric tons each year.

Alternatively, we could use an approach similar to that which we used to find the exponential equation. When $n = 10$, the explicit linear equation looks like:

$P10 = P0 + 10d$

We know the value for $P0$, so we can put that into the equation:

$P10 = 962 + 10d$

Since we know that $P10 = 1182$, substituting that in we get

$1182 = 962 + 10d$

We can now solve this equation for the common difference, d.

$1182 – 962 = 10d$

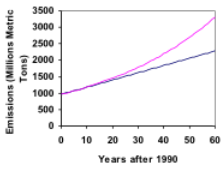
$220 = 10d$

$d = 22$

This tells us that if the emissions are changing linearly, they are growing by 22 million metric tons each year. Predicting the emissions in 2050,

$P60 = 962 + 22(60) = 2282$ million metric tons.

You will notice that this number is substantially smaller than the prediction from the exponential growth model. Calculating and plotting more values helps illustrate the differences.



So how do we know which growth model to use when working with data? There are two approaches which should be used together whenever possible:

- 1) Find more than two pieces of data. Plot the values, and look for a trend. Does the data appear to be changing like a line, or do the values appear to be curving upwards?
- 2) Consider the factors contributing to the data. Are they things you would expect to change linearly or exponentially? For example, in the case of carbon emissions, we could expect that, absent other factors, they would be tied closely to population values, which tend to change exponentially.

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1. http://www.eia.doe.gov/oiaf/1605/ggrpt/carbon.html ↵

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5.3: Solve Exponentials for Time- Logarithms

Earlier, we found that since Olympia, WA had a population of 245 thousand in 2008 and had been growing at 3% per year, the population could be modeled by the equation

$Pn = (1+0.03)n\ (245{,}000)$, or equivalently, $Pn = 245{,}000(1.03)n$.

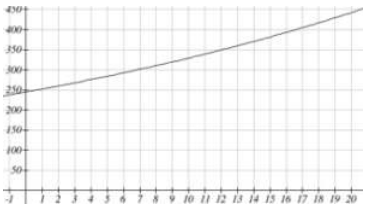
Using this equation, we were able to predict the population in the future.

Suppose we wanted to know when the population of Olympia would reach 400 thousand. Since we are looking for the year n when the population will be 400 thousand, we would need to solve the equation

$400{,}000 = 245{,}000(1.03)n$ dividing both sides by 245,000 gives

$1.6327 = 1.03n$

One approach to this problem would be to create a table of values, or to use technology to draw a graph to estimate the solution.



From the graph, we can estimate that the solution will be around 16 to 17 years after 2008 (2024 to 2025). This is pretty good, but we'd really like to have an algebraic tool to answer this question. To do that, we need to introduce a new function that will undo exponentials, similar to how a square root undoes a square. For exponentials, the function we need is called a logarithm. It is the inverse of the exponential, meaning it undoes the exponential. While there is a whole family of logarithms with different bases, we will focus on the common log, which is based on the exponential $10x$.

Common Logarithm

The common logarithm, written $\log(x)$, undoes the exponential $10x$

This means that $\log(10x) = x$, and likewise $10\log(x) = x$

This also means the statement $10a = b$ is equivalent to the statement $\log(b) = a$

$\log(x)$ is read as "log of x", and means "the logarithm of the value x". It is important to note that this is *not* multiplication – the log doesn't mean anything by itself, just like √ doesn't mean anything by itself; it has to be applied to a number.

Example 9

Evaluate each of the following

a) $\log(100)$ b) $\log(1000)$ c) $\log(10000)$ d) $\log(1/100)$ e) $\log(1)$

a) $\log(100)$ can be written as $\log(10^2)$. Since the log undoes the exponential, $\log(10^2) = 2$

b) $\log(1000) = \log(10^3) = 3$

c) $\log(10000) = \log(10^4) = 4$

d) Recall that

- \log\left(\frac{1}{100}\right)=\log\left({{10}^{-2}}\right)=-2

e) Recall that $x0 = 1$. $\log(1) = \log(100) = 0$

It is helpful to note that from the first three parts of the previous example that the number we're taking the log of has to get *10 times bigger* for the log to increase in value by 1.

Of course, most numbers cannot be written as a nice simple power of 10. For those numbers, we can evaluate the log using a scientific calculator with a log button.

Example 10

Evaluate log(300)

Using a calculator, log(300) is approximately 2.477121

With an equation, just like we can add a number to both sides, multiply both sides by a number, or square both sides, we can also take the logarithm of both sides of the equation and end up with an equivalent equation. This will allow us to solve some simple equations.

Example 11

a) Solve $10x = 1000$ b) Solve $10x = 3$ c) Solve $2(10x) = 8$

a) Taking the log of both sides gives $\log(10x) = \log(1000)$

Since the log undoes the exponential, $\log(10x) = x$. Similarly $\log(1000) = \log(103) = 3$.

The equation simplifies then to $x = 3$.

b) Taking the log of both sides gives $\log(10x) = \log(3)$.

On the left side, $\log(10x) = x$, so $x = \log(3)$. We can approximate this value with a calculator. $x \approx 0.477$

c) Here we would first want to isolate the exponential by dividing both sides of the equation by 2, giving

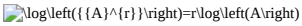
$10x = 4$.

Now we can take the log of both sides, giving $\log(10x) = \log(4)$, which simplifies to

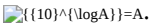
$x = \log(4) \approx 0.602$

This approach allows us to solve exponential equations with powers of 10, but what about problems like $2 = 1.03n$ from earlier, which have a base of 1.03? For that, we need the exponent property for logs.

Properties of Logs: Exponent Property



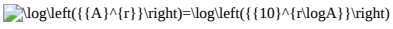
To show why this is true, we offer a proof.

Since the logarithm and exponential undo each other,  .

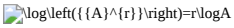
So

Utilizing the exponential rule that states

,

So then 

Again utilizing the property that the log undoes the exponential on the right side yields the result



Example 12

Rewrite log(25) using the exponent property for logs

log(25) = log(52) = 2log(5)

This property will finally allow us to answer our original question.

Solving exponential equations with logarithms

1. Isolate the exponential. In other words, get it by itself on one side of the equation. This usually involves dividing by a number multiplying it.
2. Take the log of both sides of the equation.
3. Use the exponent property of logs to rewrite the exponential with the variable exponent multiplying the logarithm.
4. Divide as needed to solve for the variable.

Example 13

If Olympia is growing according to the equation, $P_n = 245(1.03)^n$, where n is years after 2008, and the population is measured in thousands. Find when the population will be 400 thousand.

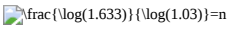
We need to solve the equation

$400 = 245(1.03)^n$ Begin by dividing both sides by 245 to isolate the exponential

$1.633 = 1.03n$ Now take the log of both sides

$\log(1.633) = \log(1.03n)$ Use the exponent property of logs on the right side

$\log(1.633) = n \log(1.03)$ Now we can divide by log(1.03)



We can approximate this value on a calculator

$n \approx 16.591$

Alternatively, after applying the exponent property of logs on the right side, we could have evaluated the logarithms to decimal approximations and completed our calculations using those approximations, as you'll see in the next example. While the final answer may come out slightly differently, as long as we keep enough significant values during calculation, our answer will be close enough for most purposes.

Example 14

Polluted water is passed through a series of filters. Each filter removes 90% of the remaining impurities from the water. If you have 10 million particles of pollutant per gallon originally, how many filters would the water need to be passed through to reduce the pollutant to 500 particles per gallon?

In this problem, our "population" is the number of particles of pollutant per gallon. The initial pollutant is 10 million particles per gallon, so $P0$ = 10,000,000. Instead of changing with time, the pollutant changes with the number of filters, so n will represent the number of filters the water passes through.

Also, since the amount of pollutant is *decreasing* with each filter instead of increasing, our "growth" rate will be negative, indicating that the population is decreasing instead of increasing, so r = -0.90.

We can then write the explicit equation for the pollutant:

$Pn = 10{,}000{,}000(1 - 0.90)n = 10{,}000{,}000(0.10)n$

To solve the question of how many filters are needed to lower the pollutant to 500 particles per gallon, we can set Pn equal to 500, and solve for n.

$500 = 10{,}000{,}000(0.10)n$ Divide both sides by 10,000,000

$0.00005 = 0.10n$ Take the log of both sides

$\log(0.00005) = \log(0.10n)$ Use the exponent property of logs on the right side

$\log(0.00005) = n \log(0.10)$ Evaluate the logarithms to a decimal approximation

$-4.301 = n(-1)$ Divide by -1, the value multiplying n

$4.301 = n$

It would take about 4.301 filters. Of course, since we probably can't install 0.3 filters, we would need to use 5 filters to bring the pollutant below the desired level.

Try it Now 4

India had a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, when will India's population reach 1.2 billion?

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5.4: Logistic Growth

In our basic exponential growth scenario, we had a recursive equation of the form

$Pn = Pn\text{-}1 + r\ Pn\text{-}1$

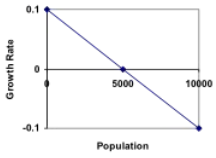
In a confined environment, however, the growth rate may not remain constant. In a lake, for example, there is some *maximum sustainable population* of fish, also called a **carrying capacity**.

Carrying Capacity

The **carrying capacity**, or **maximum sustainable population**, is the largest population that an environment can support.

For our fish, the carrying capacity is the largest population that the resources in the lake can sustain. If the population in the lake is far below the carrying capacity, then we would expect the population to grow essentially exponentially. However, as the population approaches the carrying capacity, there will be a scarcity of food and space available, and the growth rate will decrease. If the population exceeds the carrying capacity, there won't be enough resources to sustain all the fish and there will be a negative growth rate, causing the population to decrease back to the carrying capacity.

If the carrying capacity was 5000, the growth rate might vary something like that in the graph shown. Note that this is a linear equation with intercept at 0.1 and slope  , so we could write an equation for this adjusted growth rate as:



$r_{adjusted} = 0.1-\frac{0.1}{5000}P=0.1\left(1-\frac{P}{5000}\right)$

Substituting this in to our original exponential growth model for r gives

Logistic Growth

If a population is growing in a constrained environment with carrying capacity K, and absent constraint would grow exponentially with growth rate r, then the population behavior can be described by the logistic growth model:

Unlike linear and exponential growth, logistic growth behaves differently if the populations grow steadily throughout the year or if they have one breeding time per year. The recursive formula provided above models generational growth, where there is one breeding time per year (or, at least a finite number); there is no explicit formula for this type of logistic growth.

Example 15

A forest is currently home to a population of 200 rabbits. The forest is estimated to be able to sustain a population of 2000 rabbits. Absent any restrictions, the rabbits would grow by 50% per year. Predict the future population using the logistic growth model.

Modeling this with a logistic growth model, $r = 0.50$, $K = 2000$, and $P0 = 200$. Calculating the next year:

$${{P}_{1}}={{P}_{0}}+0.50\left(1-\frac{{{P}_{0}}}{2000}\right){{P}_{0}}=200+0.50\left(1-\frac{200}{2000}\right)200=290$$

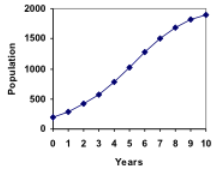
We can use this to calculate the following year:

$${{P}_{2}}={{P}_{1}}+0.50\left(1-\frac{{{P}_{1}}}{2000}\right){{P}_{1}}=290+0.50\left(1-\frac{290}{2000}\right)290\approx414$$

A calculator was used to compute several more values:

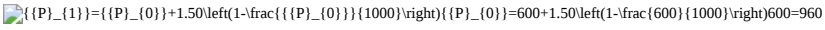
n	0	1	2	3	4	5	6	7	8	9	10
P_n	200	290	414	578	784	1022	1272	1503	1690	1821	1902

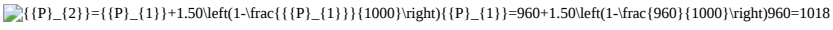
Plotting these values, we can see that the population starts to increase faster and the graph curves upwards during the first few years, like exponential growth, but then the growth slows down as the population approaches the carrying capacity.



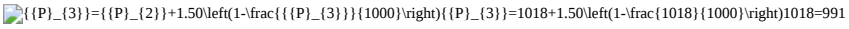
Example 16

On an island that can support a population of 1000 lizards, there is currently a population of 600. These lizards have a lot of offspring and not a lot of natural predators, so have very high growth rate, around 150%. Calculating out the next couple generations:





Interestingly, even though the factor that limits the growth rate slowed the growth a lot, the population still overshot the carrying capacity. We would expect the population to decline the next year.



Calculating out a few more years and plotting the results, we see the population wavers above and below the carrying capacity, but eventually settles down, leaving a steady population near the carrying capacity.

Try it Now 5

A field currently contains 20 mint plants. Absent constraints, the number of plants would increase by 70% each year, but the field can only support a maximum population of 300 plants. Use the logistic model to predict the population in the next three years.

Example 17

On a neighboring island to the one from the previous example, there is another population of lizards, but the growth rate is even higher – about 205%.

Calculating out several generations and plotting the results, we get a surprise: the population seems to be oscillating between two values, a pattern called a 2-cycle.

While it would be tempting to treat this only as a strange side effect of mathematics, this has actually been observed in nature. Researchers from the University of California observed a stable 2-cycle in a lizard population in California.[1]

Taking this even further, we get more and more extreme behaviors as the growth rate increases higher. It is possible to get stable 4-cycles, 8-cycles, and higher. Quickly, though, the behavior approaches chaos (remember the movie Jurassic Park?).

Try it Now Answers

1. Letting $n = 0$ correspond with 1976, then $P0 = 20{,}610$.

From 1976 to 2010 the number of stay-at-home fathers increased by

$53{,}555 – 20{,}610 = 32{,}945$

This happened over 34 years, giving a common different d of $32{,}945 / 34 = 969$.

$Pn = 20{,}610 + 969n$

Predicting for 2020, we use $n = 44$

$P44 = 20{,}610 + 969(44) = 63{,}246$ stay-at-home fathers in 2020.

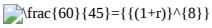
2. Using $n = 0$ corresponding with 2008,

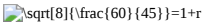
$P12 = (1+0.0134)12\ (1.14)$ = about 1.337 billion people in 2020

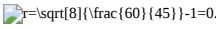
3. Here we will measure n in months rather than years, with $n = 0$ corresponding to the February when they went public. This gives $P0 = 45$ thousand. October is 8 months later, so $P8 = 60$.

$P8 = (1+r)8\ P0$

60 = (1+*r*)8 45





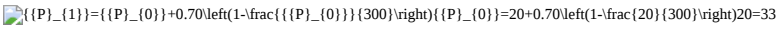
 , or 3.66%

The general explicit equation is *Pn* = (1.0366)*n* 45

Predicting 24 months (2 years) after they went public:

P24 = (1.0366)24 45 = 106.63 thousand users.

4. 1.14(1.0134)*n* = 1.2. *n* = 3.853, which is during 2011

5. 

*P*2 = 54

*P*3 = 85

---

1. http://users.rcn.com/jkimball.ma.ultranet/BiologyPages/P/Populations2.html ↵

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5.5: Exercises

Skills

1. Marko currently has 20 tulips in his yard. Each year he plants 5 more.
   a. Write a recursive formula for the number of tulips Marko has
   b. Write an explicit formula for the number of tulips Marko has
2. Pam is a Disc Jockey. Every week she buys 3 new albums to keep her collection current. She currently owns 450 albums.
   a. Write a recursive formula for the number of albums Pam has
   b. Write an explicit formula for the number of albums Pam has
3. A store's sales (in thousands of dollars) grow according to the recursive rule $P_n=P_{n-1}+15$, with initial population $P_0=40$.
   a. Calculate P_1 and P_2
   b. Find an explicit formula for P_n
   c. Use your formula to predict the store's sales in 10 years
   d. When will the store's sales exceed $100,000?
4. The number of houses in a town has been growing according to the recursive rule $P_n=P_{n-1}+30$, with initial population $P_0=200$.
   a. Calculate P_1 and P_2
   b. Find an explicit formula for P_n
   c. Use your formula to predict the number of houses in 10 years
   d. When will the number of houses reach 400 houses?
5. A population of beetles is growing according to a linear growth model. The initial population (week 0) was $P_0=3$, and the population after 8 weeks is $P_8=67$.
   a. Find an explicit formula for the beetle population in week n
   b. After how many weeks will the beetle population reach 187?
6. The number of streetlights in a town is growing linearly. Four months ago ($n = 0$) there were 130 lights. Now ($n = 4$) there are 146 lights. If this trend continues,
   a. Find an explicit formula for the number of lights in month n
   b. How many months will it take to reach 200 lights?
7. Tacoma's population in 2000 was about 200 thousand, and had been growing by about 9% each year.
   a. Write a recursive formula for the population of Tacoma
   b. Write an explicit formula for the population of Tacoma
   c. If this trend continues, what will Tacoma's population be in 2016?
   d. When does this model predict Tacoma's population to exceed 400 thousand?
8. Portland's population in 2007 was about 568 thousand, and had been growing by about 1.1% each year.
   a. Write a recursive formula for the population of Portland
   b. Write an explicit formula for the population of Portland
   c. If this trend continues, what will Portland's population be in 2016?
   d. If this trend continues, when will Portland's population reach 700 thousand?
9. Diseases tend to spread according to the exponential growth model. In the early days of AIDS, the growth rate was around 190%. In 1983, about 1700 people in the U.S. died of AIDS. If the trend had continued unchecked, how many people would have died from AIDS in 2005?
10. The population of the world in 1987 was 5 billion and the annual growth rate was estimated at 2 percent per year. Assuming that the world population follows an exponential growth model, find the projected world population in 2015.
11. A bacteria culture is started with 300 bacteria. After 4 hours, the population has grown to 500 bacteria. If the population grows exponentially,
    a. Write a recursive formula for the number of bacteria
    b. Write an explicit formula for the number of bacteria
    c. If this trend continues, how many bacteria will there be in 1 day?
    d. How long does it take for the culture to triple in size?

12. A native wolf species has been reintroduced into a national forest. Originally 200 wolves were transplanted. After 3 years, the population had grown to 270 wolves. If the population grows exponentially,
    a. Write a recursive formula for the number of wolves
    b. Write an explicit formula for the number of wolves
    c. If this trend continues, how many wolves will there be in 10 years?
    d. If this trend continues, how long will it take the population to grow to 1000 wolves?
13. One hundred trout are seeded into a lake. Absent constraint, their population will grow by 70% a year. The lake can sustain a maximum of 2000 trout. Using the logistic growth model,
    a. Write a recursive formula for the number of trout
    b. Calculate the number of trout after 1 year and after 2 years.
14. Ten blackberry plants started growing in my yard. Absent constraint, blackberries will spread by 200% a month. My yard can only sustain about 50 plants. Using the logistic growth model,
    a. Write a recursive formula for the number of blackberry plants in my yard
    b. Calculate the number of plants after 1, 2, and 3 months
15. In 1968, the U.S. minimum wage was $1.60 per hour. In 1976, the minimum wage was $2.30 per hour. Assume the minimum wage grows according to an exponential model where n represents the time in years after 1960.
    a. Find an explicit formula for the minimum wage.
    b. What does the model predict for the minimum wage in 1960?
    c. If the minimum wage was $5.15 in 1996, is this above, below or equal to what the model predicts?

Concepts

16. The population of a small town can be described by the equation $Pn = 4000 + 70n$, where n is the number of years after 2005. Explain in words what this equation tells us about how the population is changing.
17. The population of a small town can be described by the equation $Pn = 4000(1.04)n$, where n is the number of years after 2005. Explain in words what this equation tells us about how the population is changing.

Exploration

Most of the examples in the text examined growing quantities, but linear and exponential equations can also describe decreasing quantities, as the next few problems will explore.

18. A new truck costs $32,000. The car's value will depreciate over time, which means it will lose value. For tax purposes, depreciation is usually calculated linearly. If the truck is worth $24,500 after three years, write an explicit formula for the value of the car after n years.
19. Inflation causes things to cost more, and for our money to buy less (hence your grandparents saying, "In my day, you could buy a cup of coffee for a nickel"). Suppose inflation decreases the value of money by 5% each year. In other words, if you have $1 this year, next year it will only buy you $0.95 worth of stuff. How much will $100 buy you in 20 years?
20. Suppose that you have a bowl of 500 M&M candies, and each day you eat ¼ of the candies you have. Is the number of candies left changing linearly or exponentially? Write an equation to model the number of candies left after n days.
21. A warm object in a cooler room will decrease in temperature exponentially, approaching the room temperature according to the formula where Tn is the temperature after n minutes, r is the rate at which temperature is changing, a is a constant, and Tr is the temperature of the room. Forensic investigators can use this to predict the time of death of a homicide victim. Suppose that when a body was discovered ($n = 0$) it was 85 degrees. After 20 minutes, the temperature was measured again to be 80 degrees. The body was in a 70 degree room.
    a. Use the given information with the formula provided to find a formula for the temperature of the body.
    b. When did the victim die, if the body started at 98.6 degrees?
22. Recursive equations can be very handy for modeling complicated situations for which explicit equations would be hard to interpret. As an example, consider a lake in which 2000 fish currently reside. The fish population grows by 10% each year, but every year 100 fish are harvested from the lake by people fishing.
    a. Write a recursive equation for the number of fish in the lake after n years.
    b. Calculate the population after 1 and 2 years. Does the population appear to be increasing or decreasing?

c. What is the maximum number of fish that could be harvested each year without causing the fish population to decrease in the long run?

23. The number of Starbucks stores grew after first opened. The number of stores from 1990-2007, as reported on their corporate website[1], is shown below.
    a. Carefully plot the data. Does is appear to be changing linearly or exponentially?
    b. Try finding an equation to model the data by picking two points to work from. How well does the equation model the data?
    c. Try using an equation of the form , where *k* is a constant, to model the data. This type of model is called a Power model. Compare your results to the results from part b. *Note: to use this model, you will need to have 1990 correspond with n = 1 rather than n = 0.*

Year	Number of Starbucks stores		Year	Number of Starbucks stores
1990	84		1999	2498
1991	116		2000	3501
1992	165		2001	4709
1993	272		2002	5886
1994	425		2003	7225
1995	677		2004	8569
1996	1015		2005	10241
1997	1412		2006	12440
1998	1886		2007	15756

24. Thomas Malthus was an economist who put forth the principle that population grows based on an exponential growth model, while food and resources grow based on a linear growth model. Based on this, Malthus predicted that eventually demand for food and resources would out outgrow supply, with doom-and-gloom consequences. Do some research about Malthus to answer these questions.
    a. What societal changes did Malthus propose to avoid the doom-and-gloom outcome he was predicting?
    b. Why do you think his predictions did not occur?
    c. What are the similarities and differences between Malthus's theory and the logistic growth model?

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1. http://www.starbucks.com/aboutus/Company_Timeline.pdf retrieved May 2009 ↵

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CHAPTER OVERVIEW

6: Module 2- Geometry

6.1: Module 2 Overview
6.2: Perimeter and Area
6.3: Triangles
6.4: Volume
6.5: Surface Area
6.6: Graph Theory
6.7: Supplemental Videos
6.8: Topology, Tiling, and Non-Euclidean Geometry
6.9: Discuss- Application of Geometry

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6.1: Module 2 Overview

What You'll Learn To Do: Understand and use geometry to solve problems.

Learning Objectives

- Apply formulas to solve perimeter, area, surface area, and volume problems.
- Solve problems using similar triangles and the Pythagorean Theorem.
- Distinguish Euclidean and non-Euclidean geometry.
- Describe the study of topology and its importance.
- Describe the study of networks and its importance.
- Recognize and create tilings.
- Recognize when geometry is applicable to real-life situations, solve real-life problems, and communicate real-life problems and solutions to others.

Learning Activities

 The learning activities for this module include:

Reading Assignments

- **Read:** Perimeter and Area
- **Read:** Triangles
- **Read:** Volume
- **Read:** Surface Area
- **Read:** Graph Theory
- **Watch:** Graph Theory Videos
- **Read:** Topology, Tiling, Non-Euclidean Geometry

Homework Assignments

- **Submit:** Perimeter and Area (12 points)
- **Submit:** Triangles (4 points)
- **Submit:** Surface Area and Volume (14 points)
- **Submit:** Graph Theory (2 points)
- **Submit:** Topology, Tiling, and Non-Euclidean Geometry (7 points)
- **Discuss:** Application of Geometry (20 points)
- **Complete:** Exam 1

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6.2: Perimeter and Area

Perimeter

Perimeter is a one-dimensional measurement that is taken around the outside of a closed geometric shape. Let's start our discussion of the concept of perimeter with an example.

Guided Example

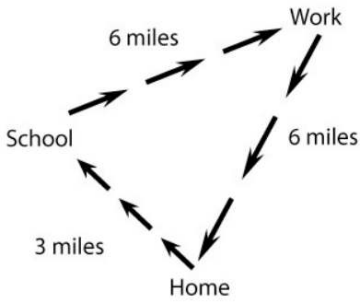


Figure 1.

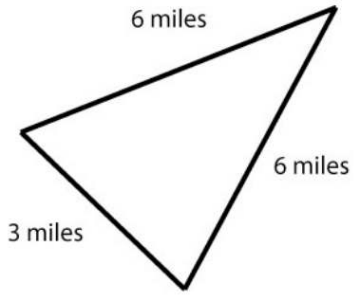


Figure 2.

Joseph does not own a car so must ride the bus or walk everywhere he goes. On Mondays, he must get to school, to work, and back home again. His route is pictured in figure 1.

The obvious question to ask in this situation is, "how many miles does Joseph travel on Mondays"? To compute, we each distance: $3 + 6 + 6 = 15$.

Joseph travels 15 miles on Mondays.

Another way to work with this situation is to draw a shape that represents Joseph's travel route and is labeled with the distance from one spot to another.

Notice that the shape made by Joseph's route is that of a closed geometric figure with three sides (a triangle) (see figure 2). What we can ask about this shape is, "what is the *perimeter* of the triangle"?

Perimeter means "distance around a closed figure or shape" and to compute we add each length: $3 + 6 + 6 = 15$

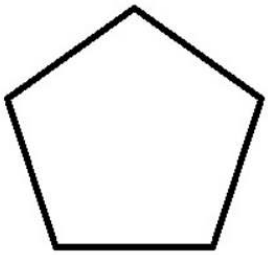
Our conclusion is the same as above: Joseph travels 15 miles on Mondays.

However, what we did was model the situation with a geometric shape and then apply a specific geometric concept (*perimeter*) to computer how far Joseph traveled.

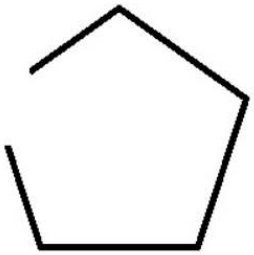
Notes on Perimeter

- Perimeter is a one-dimensional measurement that represents the distance around a closed geometric figure or shape (no gaps).
- To find perimeter, add the lengths of each side of the shape.
- If there are units, include units in your final result. Units will always be of single dimension (i.e. feet, inches, yards, centimeters, etc…)

To compute perimeter, our shapes must be closed. Figure 3 shows the difference between a closed figure and an open figure.



Closed figure:
There is an inside and an outside to the shape. To get from inside to outside, you must cross the boundary of the shape



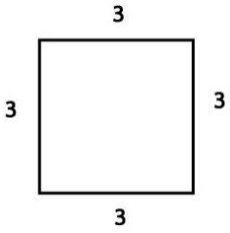
Open figure Not closed:
There isn't an inside or outside. Even a portion that seems enclosed can be reached without crossing the boundary of the shape.

Figure 3.

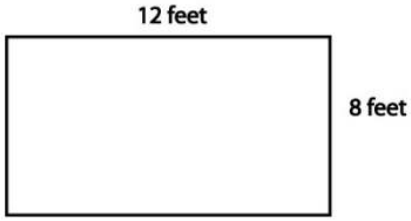
Example 1

Find the perimeter for each of the shapes below.

1. Add the lengths of each side.



2. Sometimes you have to make assumptions if lengths are not labeled.

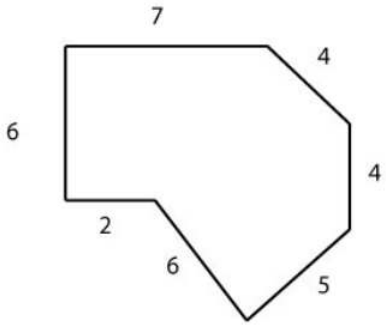


Solutions

1. 12 units
2. 40 feet

Example 2

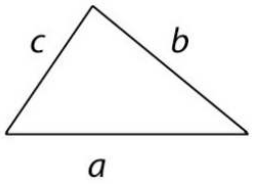
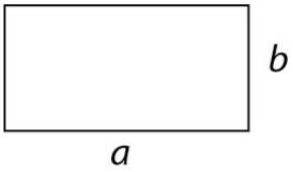
How do we find the perimeter of this more complicated shape?



Solution

Just keep adding those side lengths. 6 + 7 + 4 + 4 + 5 + 6 + 2 = 34 units

If you look closely at the shapes in the previous examples, you might notice some ways to write each perimeter as a more explicit formula. See if the results from what we have done so far match the formulas below.

Shape		Perimeter
Triangle with side lengths, *a*, *b*, *c*:		$P=a+b+c\\$
Square with side length *a*:		$P=a+a+a+a\\$ $P=4a\\$
Rectangle with side lengths *a*, *b*:		$P=a+b+a+b\\$ $P=a+a+b+b\\$ $P=2a+2b\\$

Circumference

You may realize that we have not yet discussed the distance around a very important geometric shape: a circle! The distance around a circle has a special name called the **circumference**. To find the circumference of a circle, we use this formula: C = 2πr

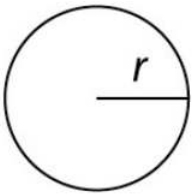


Figure 4.

In this formula, π is pronounced “pi,” and is defined as the circumference of a circle divided by its diameter: $\displaystyle\pi=\frac{C}{d}\\$. We usually replace π with the approximation 3.14. The letter *r* represents the **radius** of the circle.

Let’s see where the formula for circumference comes from. Figure 4 shows a generic circle with radius *r*.

Notes about $C = 2\pi r$

Remember that in the formula, when computing the circumference $C = 2\pi r$, we multiply as follows **usually** substituting 3.14 in place of π:

$C = 2 \times 3.14 \times r$

Often, the use of ( ) will help make the different parts of the formula easier to see:

$C = (2)\times(3.14)\times(r)$

Origins of $C = 2\pi r$

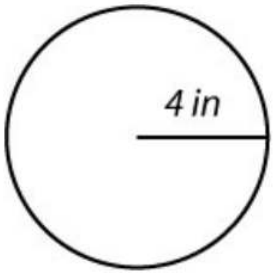
As mentioned earlier, the special number π is defined as the ratio of a circle’s circumference to its diameter. We can write this in equation form as: $\displaystyle\frac{C}{d}=\pi\\$

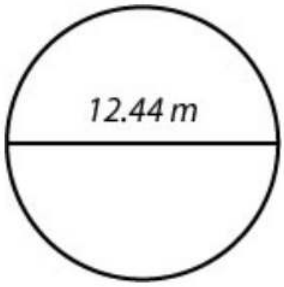
We know from our previous work that to identify the unknown, C, we can move d to the other side of the equation by writing $C = \pi d$. The diameter is all the way across the circle's middle so the diameter is twice the radius. We can update C in terms of the radius as $C = \pi(2r)$. With a little final rearranging of the order our parts are written in, we can say that $C = 2\pi r$.

Let's use the formula to find the circumference of a few circles.

Example 3

Find the circumference of each of the following circles. Leave your answers first in exact form and then in rounded form (to the hundredths place). (Note that when a radius is given, its value is centered above a radius segment. When a diameter is given, its value is centered above a diameter segment.)

1. 

2. 

Solutions

1. Exact 8π in; rounded from exact answer 25.13 in; rounded using 3.14 for π 25.12 in
2. Exact 12.44π m; rounded from exact answer 39.08 m; rounded using 3.14 for π 39.06 m

Exact Form vs. Rounded Form

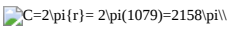
- π is a number in exact form. It is not rounded.
- 3.14 is a rounded form approximation for π

Why does it matter which form we use? It matters because when we round, we introduce error into our final result. For this class, that error is usually acceptable. However, you will find in other subjects such as physics or chemistry, that level of accuracy is a concept of great importance. Let's see an example of the difference in forms.

Example 4

The radius of the moon is about 1079 miles. What is the circumference? Let's solve this using both the exact form and the rounded form:

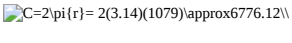
Exact Solution



To round **from** the exact solution, use the π button on your calculator to get



Rounded Solution



Notice that our final results are different. That difference is the error created by using 3.14 as an initial approximation for π. When doing homework and tests, read the directions carefully on each problem to see which form to use.

Example 5

Find the circumference or perimeter given in each described situation below. Include a drawing of the shape with the included information. Use the examples to help determine what shapes to draw. Show all work. As in the examples, if units are included then units should be present in your final result. Round to tenths unless indicated otherwise.

1. Find the perimeter of a square with side length 2.17 feet.
2. Find the perimeter of a rectangle with sides of length 4.2 and 3.8.
3. Find the perimeter of a triangle with sides of length 2, 5, 7.
4. Find the circumference of a circle with radius 6 inches. Present answer in exact form and also compute using 3.14 for π. Present rounded form to the nearest tenth.
5. Find the circumference of a circle with diameter 14.8 inches. Present answer in exact form and also compute using 3.14 for π. Present rounded form to the nearest tenth.

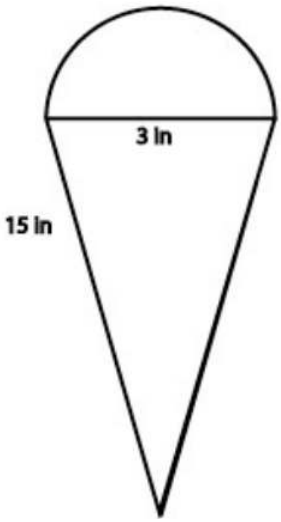
Solutions

1. 8.68 feet
2. 16
3. 14
4. Exact 12π in, Rounded 37.7 in
5. Exact 14.8π in, Rounded 46.5 in

Example 6

Finding the Distance around Non-Standard Shapes

The basic formulas for perimeter of straight-line shapes and the circumference of a circle will help us find the distance around more complicated figures. Find the distance around the following shape. Round final answer to tenths and use 3.14 for π.



Solution

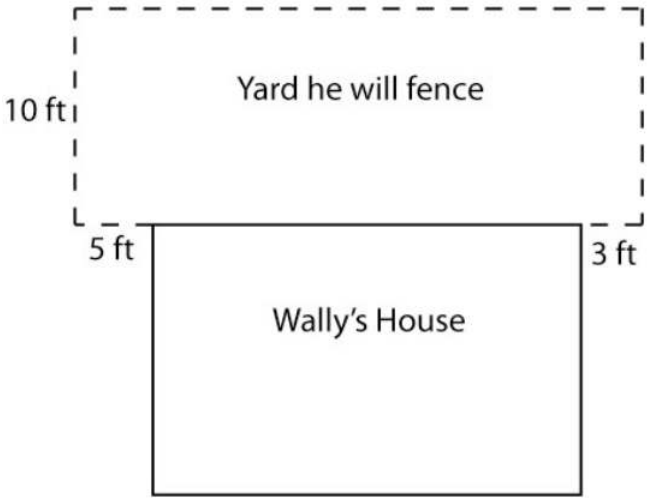
34.7 in

Example 7

Applications of Perimeter and Circumference

Our knowledge of basic geometric shapes can be applied to solve “real-life” problems.

Wally wants to add a fence to the back of his house to make some room for his children to play safely (see diagram below). He began measuring his yard but got distracted and forgot to finish measuring before he went to the store. If he remembers that the back wall of his house is 15 yards long, does he have enough information to buy the fencing he needs? If so, how many feet should he buy?



Solution

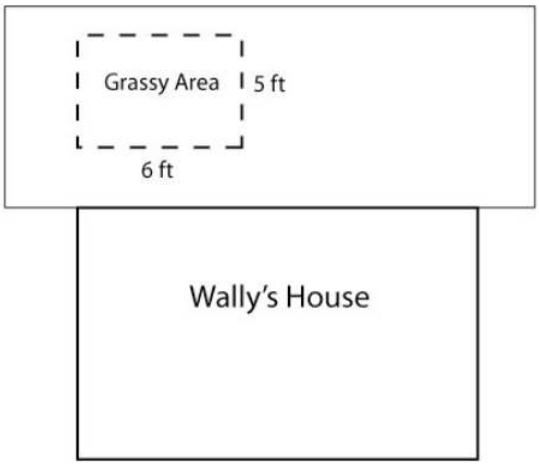
81 feet

Area

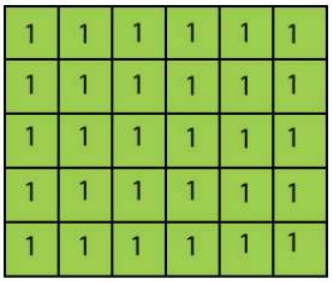
Let's take another look at Wally's backyard from Example 7 in order to introduce the next concept: area.

Guided Example

Wally successfully fenced his yard but now wants to add some landscaping and create a grassy area as shown below.



He heads down to the local lawn store and finds out that in order to determine how much sod he needs, he must figure out the square footage of the area he wants to add grass to. On his way home, he realizes that if he divides the grassy area into sections that are 1 foot by 1 foot and then counts them, he can determine the square footage. Here is the information Wally drew up when he got home.



6 top squares, 5 side squares
Each square is
1 ft x 1 ft = 1 square foot

Total unit squares = 30

Grassy Area = 30 square feet

Wally correctly determined the area of the rectangular grassy section to be 30 square feet.

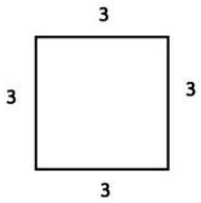
Notes on Area

- **Area** is a two-dimensional measurement that represents the amount of space inside a two-dimensional shape.
- To find the area, count the number of unit squares inside the shape.
- If there are units, include units in your final result. Units will always be two-dimensional (i.e. square feet, square yards, square miles, etc…)

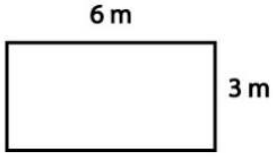
Example 8

Find the area for each of the shapes below.

1. Remember to count the unit squares inside the shape.

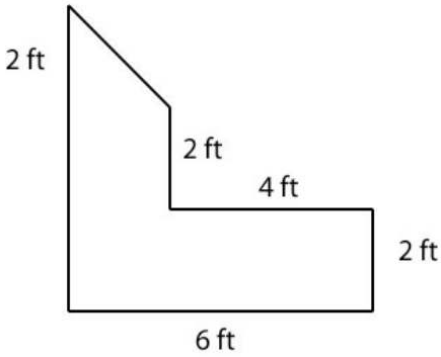


2. Is there a pattern here that would make our work easier?



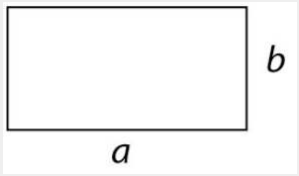
Example 9

How do we find the area for shapes that are more complicated? Break up the areas into shapes that we recognize and add the area values together.

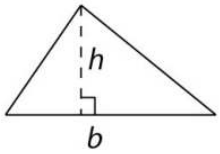
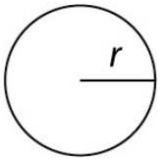
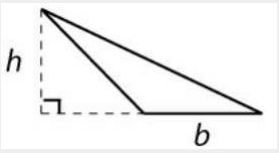


If you look closely at the shapes in the previous examples, you might notice some ways to write each area as a more explicit formula. See if the results from what we have done so far match the formulas below.

Shape		Area
Square with side length a	 a	$A=a\cdot{a}\\$ $A=a^2\\$

Shape		Area
Rectangle with side lengths a, b		$A=a\cdot{b}\\$ (You will also see this as $A=\text{length}\cdot\text{width}\\$)

The area formulas for the shapes below are more complicated to derive so the formulas are listed for you in the table.

Shape	Shape
Triangle with height h and base b  $\displaystyle{A}=\frac{1}{2}bh=\frac{bh}{2}\\$ Read as “one-half base times height” Note that h is the straight-line distance from top of the triangle directly to the other side. The small box next to h indicates this. In math terms the box indicates a 90º (right) angle.	Circle with radius r  $A=\pi{r}^2\\$ Read as “pi times radius squared”
	If your triangle is as pictured at left, then the height is drawn and measured outside the triangle. The area formula is the same.

Example 10

Find the area for each described situation. Create a drawing of the shape with the included information. Show all work. As in the examples, if units are included then units should be present in your final result. Use 3.14 for π and round answers to tenths as needed.

1. Find the area of a rectangle whose length is 12.9 meters and height is one-third that amount.
2. Find the area of a triangle with base $\displaystyle{24}\frac{1}{2}\\$ inches and height 7 inches.
3. Find the area of a circle with radius $\displaystyle{2}\frac{1}{3}\\$ inches. Present answer in exact form and also compute rounded form using 3.14 for π. Present rounded form to the nearest tenth.

Solutions

1. 55.5 m^2 or 55.5 square meters (rounded)
2. 85.8 in^2 or 85.8 square inches (rounded)
3. Exact 49/9 π in^2, Rounded 17.1 in^2

Example 11

Find the area given each described situation. Include a drawing of the shape with the included information. Show all work. As in the examples, if units are included then units should be present in your final result. Round answers to tenths unless otherwise indicated.

1. Find the area of a square with side length 4.2 feet.
2. Find the area of a rectangle with sides of length 4.2 and 3.8.
3. Find the area of a triangle with height 7 inches and base 12 inches.

4. Find the area of a circle with radius 6 inches. Present answer in exact form and also compute using 3.14 for π. Present rounded form to the nearest tenth.

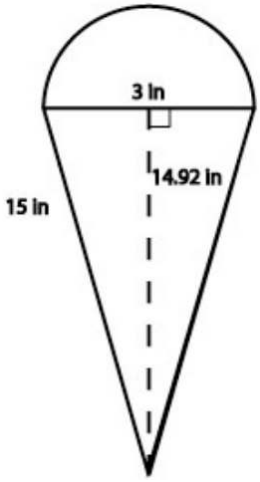
Solutions

1. 17.64 ft^2 or 17.64 square feet
2. 16.0
3. 42 in^2 or 42 square inches
4. Exact 36π in^2 or 36π square inches, Rounded using 3.14 for π 113.0 in^2 or 113.0 square inches

Example 12

Finding the Area of Non-Standard Shapes

The basic formulas for area will help us find the area of more complicated figures. This is the same problem we found the perimeter for earlier. Find the area of the given shape. Compute using 3.14 for π and round to the nearest tenth.



Solution

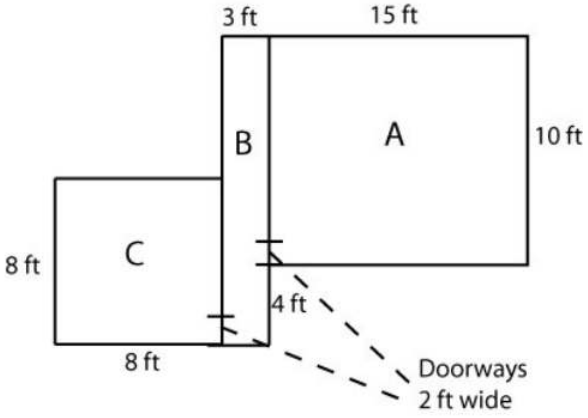
Rounded using 3.14 for π 25.9 in^2

Example 13

Applications of Area and Perimeter

We can combine our knowledge of area/perimeter to solve problems such as this one.

Wally is still fixing up his house and has a flooring project to complete. He wants to buy enough bamboo flooring to cover the floor space in rooms A, C and hallway B and enough bamboo edging for baseboards in all the spaces as well. How many square feet of flooring and how many feet of baseboards should he buy?



Solution

256 ft^2 flooring, 108 ft edging

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6.3: Triangles

Similar Triangles

Let’s begin our discussion of similar triangles with an example.

Guided Example

Mary was out in the yard one day and had her two daughters with her. She was doing some renovations and wanted to know how tall the house was. She noticed a shadow 3 feet long when her daughter was standing 12 feet from the house and used it to set up figure 1.

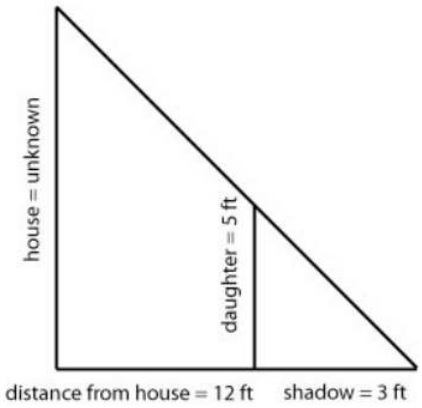


Figure 1.

We can take that drawing and separate the two triangles as follows allowing us to focus on the numbers and the shapes.

These triangles are what are called **similar triangles**. They have the same angles and sides in *proportion* to each other. We can use that information to determine the height of the house as seen in figure 2.

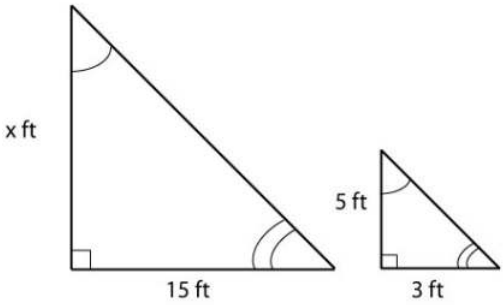
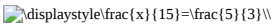


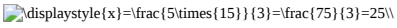
Figure 2.

To determine the height of the house, we set up the following proportion:



$$\displaystyle\frac{x}{15}=\frac{5}{3}\\$$

Then, we solve for the unknown x by using cross products as we have done before:

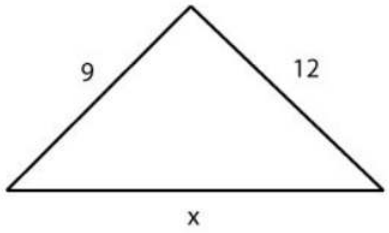


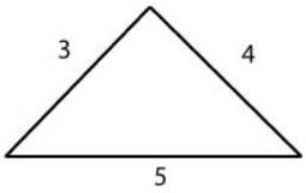
$$\displaystyle{x}=\frac{5\times{15}}{3}=\frac{75}{3}=25\\$$

Therefore, we can conclude that the house is 25 feet high.

Example 1

Use the Similar Triangles process to determine the length of the missing side. Set up the proportions in as many ways as possible and show the results are all the same.



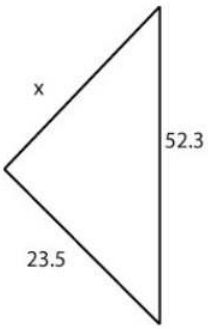


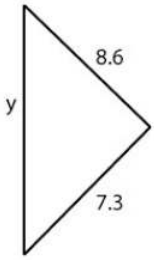
Solution

15 units

Example 2

Use the similar triangles process to determine the length of the missing sides. You may need to redraw your triangles to set up the proportions correctly.



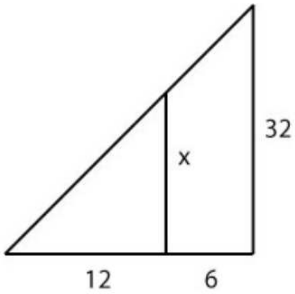


Solution

$x = 27.7$ (rounded), $y = 16.2$ (rounded)

Example 3

Given the similar triangles below, find the missing lengths. Round to tenths as needed. Feel free to redraw the triangles so you can see the proportional sides.



Solution

21.3 (rounded)

Example 4

Applications of Similar Triangles

Mary (from the application that started this topic), decides to use what she knows about the height of the roof to measure the height of her second daughter. If her second daughter casts a shadow that is 1.5 feet long when she is 13.5 feet from the house, what is the height of the second daughter? Draw an accurate diagram and use similar triangles to solve.

Solution

2.5 ft

Square Roots

Before we get to our last topic in this lesson—the Pythagorean Theorem,—we need to know a little bit about square roots.

- The **square root** of a number is that number which, when multiplied times itself, gives the original number. On your calculator, look for \sqrt\text{ }\\ to compute square roots. \sqrt{16}=4\\ because $4 \times 4 = 16$
- A **perfect square** is a number whose square root is a whole number. The square root of a non-perfect square is a decimal value. \sqrt{16}\\ is a perfect square.\sqrt{19}\\ is **not** a perfect square.
- To obtain a decimal value for non-perfect square roots on your calculator, you may need to change the settings under your MODE button. Check your owner's manual for help if needed. \sqrt{19}=4.36\\ (rounded to hundredths)

Example 5

Find the square root of each of the following. Round to two decimal places if needed. Indicate those that are perfect squares and explain why.

1. \sqrt{169}\\
2. \sqrt{31}\\
3. \sqrt{9}\\

Solutions

1. 13 (perfect square)
2. 5.57 (not a perfect square)
3. 9 (perfect square)

Example 6

Find the square root of each of the following. Round to two decimal places if needed. Indicate those that are perfect squares and explain why.

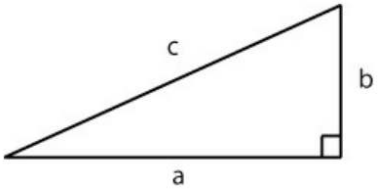
1. \sqrt{225}\\
2. \sqrt{17}\\
3. \sqrt{324}\\

Solutions

1. 15 (perfect square)
2. 4.12 (not a perfect square)
3. 18 (perfect square)

The Pythagorean Theorem

The mathematician Pythagoras proved the Pythagorean theorem. The theorem states that given any right triangle with sides a, b, and c as below, the following relationship is always true: $a^2 + b^2 = c^2$



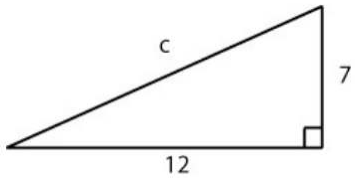
$a^2 + b^2 = c^2$

Notes about the Pythagorean theorem:

- The triangle must be a **right** triangle (contains a 90º angle).
- The side c is called the **hypotenuse** and *always* sits across from the right angle.
- The lengths a and b are interchangeable in the theorem, but c cannot be interchanged with a or b. In other words, the location of c is very important and cannot be changed.

Example 7

Use the Pythagorean theorem to find the missing sides length for the triangle given below. Round to the tenths place.

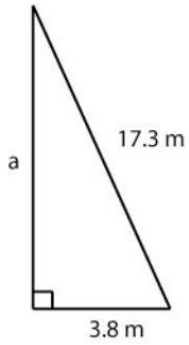


Solution

13.9 (rounded)

Example 8

Use the Pythagorean theorem to find the missing sides length for the triangle given below. Round to the tenths place.

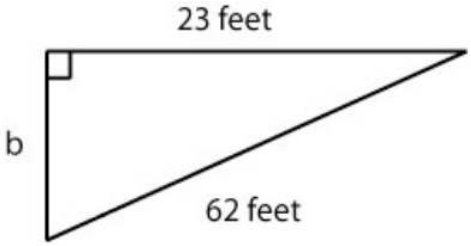


Solution

16.9 m (rounded)

Example 9

Use the Pythagorean theorem to find the missing sides length for the triangle given below. Round to hundredths.



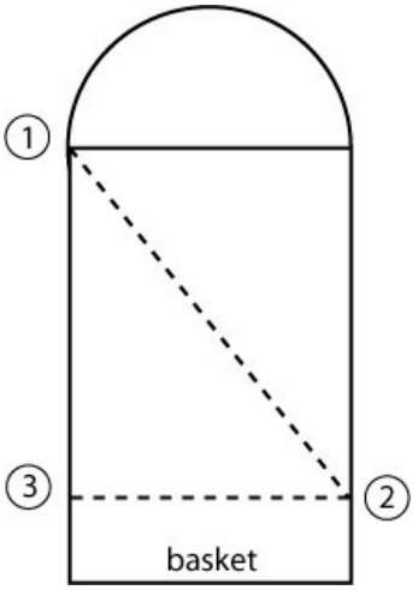
Solution

57.58 ft (rounded)

Example 10

Applications of the Pythagorean Theorem

In NBA Basketball, the width of the free-throw line is 12 feet.[1] A player stands at one exact corner of the free throw line (Player 1) and wants to throw a pass to his open teammate across the lane and close to the basket (Player 2). If his other teammate (Player 3—heavily guarded) is directly down the lane from him 16 feet, how far is his pass to the open teammate? Fill in the diagram below and use it to help you solve the problem.



Solution

20 ft

1. http://www.sportsknowhow.com ↵

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6.4: Volume

Let's revisit our friend Wally from "Area and Perimeter" and use another aspect of his yard to introduce the concept of **volume**. Wally is a swimmer and wants to install a lap pool in his backyard. Because he has some extra space, he is going to build a pool that is 25 yards long, 2 yards wide, and 2 yards deep. How many cubic yards of water must be used to fill the pool (assuming right to the top).

Much as we did with area (counting unit squares), with **volume** we will be counting unit cubes. What is the volume of a unit cube? Let's look at figure 1:

Finding the Volume of a Cube

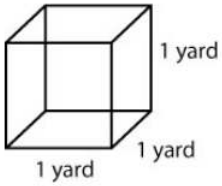


Figure 1. Volume = 1 yd × 1yd × yd = 1 cubic yard

The shape at left is a cube (all sides are equal length). In particular, because all sides are of length 1, this cube is called a unit cube.

We know the area of the base from our previous work (1 yd × 1 yd or 1 square yard). We are going to take that area and extend it vertically through a height of 1 yard so our volume becomes

Volume = 1 yd × 1yd × yd = 1 cubic yard

How does this help Wally? Well, if he can count the number of unit cubes in his pool, he can determine the volume of water needed to fill the pool.

Guided Example

Filling Wally's Pool

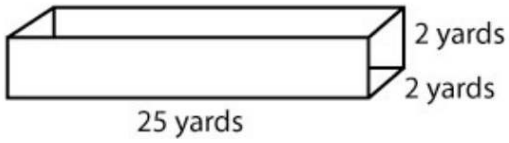
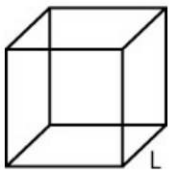
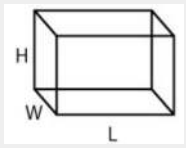


Figure 2. Volume = Length × Width × Height **or** V = LWH

If we fill the pool with unit cubes, we can fill 25 unit cubes along the length, 2 along the width and 2 along the height. That would give us 25 × 2 × 2 = 100 unit cubes or:

Volume = 25 yd × 2 yd × 2 yd = 100 cubic yd

Explicit formulas for the types of rectangular solids used in the previous section are as follows:

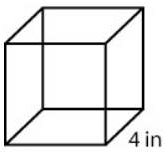
Shape		Volume
Cub of side length L		$V=L\times{L}\times{L}\\$ $V=L^3\\$
Box with sides of length L, W, H		$V=L\times{W}\times{H}\\$

Notes on Volume

- **Volume** is a three-dimensional measurement that represents the amount of space inside a closed three-dimensional shape.
- To find volume, count the number of unit cubes inside a given shape.
- If there are units, include units in your final result. Units will always be three-dimensional (i.e. cubic feet, cubic yards, cubic miles, etc…)

Example 1

Find the volume of each shape below.

1. 

2. A box with sides of length 2 ft, 3 ft, $\displaystyle{2}\frac{1}{2}$ ft.

Solutions

1. 64 in^3
2. 15 ft^3

Volume of A Circular Cylinder

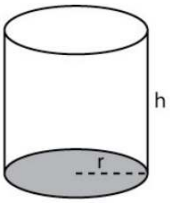


Figure 3.

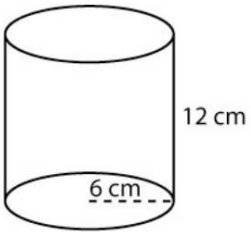
Can we use what we know about the area of a circle to formulate the volume of a can (also called a **cylinder**)? Take a look at figure 3.

The base circle is shaded. If we take the area of that circle (A = $A = \pi r^2$) and extend it up through the height h, then our volume for the can would be:

$V = \pi r^2 h$.

Example 2

Find the volume of the cylinder shown below.



Solution

432π cm^3

Volume of Other Shapes

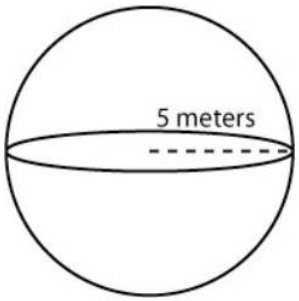
The chart below shows the volumes of some other basic geometric shapes.

Shape	Volume
Sphere	$\displaystyle{V}=\frac{4}{3}\pi{r}^3$

Shape	Volume
Cone	$\displaystyle{V}=\frac{1}{3}\pi{r}^2h\\$
Pyramid	$\displaystyle{V}=\frac{1}{3}LWH\\$

Example 3

Find the volume of a sphere with radius 5 meters.



Solution

$\displaystyle\frac{500}{3}\pi\text{ cm}^3\\$

Example 4

Determine the volume of each of the following. Include a drawing of the shape with the included information. Show all work. As in the examples, if units are included then units should be present in your final result. Use 3.14 for π and round answers to tenths as needed.

1. Find the volume of a cube with side 3.25 meters.
2. Find the volume of a box with sides of length 4 feet by $\displaystyle{2}\frac{1}{2}\\$ feet by 6 feet.
3. Find the volume of a can with radius 4.62 cm and height 10 cm.
4. Find the volume of a sphere with diameter 12 yards.

Solution

1. 34.3 m^3 or 34.3 cubic meters
2. 60 ft^3 or 60 cubic feet
3. 670.2 cm^3 or 670.2 cubic centimeters
4. 904.3 yd^3 or 904.3 cubic yards

Example 5

Applications of Volume

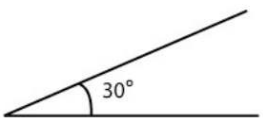
If you drank sodas from 5 cans each of diameter 4 inches and height 5 inches, how many cubic inches of soda did you drink? Use 3.14 for π and round to tenths.

Solution

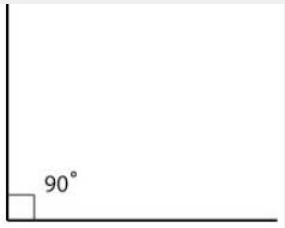
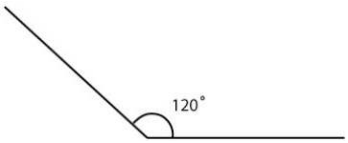
314.0 in^3 (rounded)

Angles

Angles, often measured in *degrees*, measure the amount of rotation or "arc" between intersecting line segments. We need to have some sense of what an angle is before moving on to the next topic. See some examples and terminology below.



- This angle measures 30º
- This angle measure is less than 90º
- Angles less than 90º are called **acute angles**

	• This angle measures 90º • Angles that measure exactly 90º are called **right angles** • The small box in the angle corner denotes a right angle
	• This angle measures 120º • This angle measure is more than 90º • Angles that measure more than 90º but less 180º than are called **obtuse angles**
	• This angle measures 180º • Angles that measure exactly 180º are called **straight angles**

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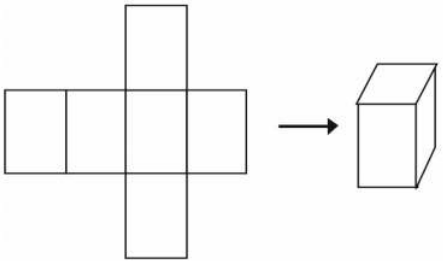
6.5: Surface Area

Learning Objectives

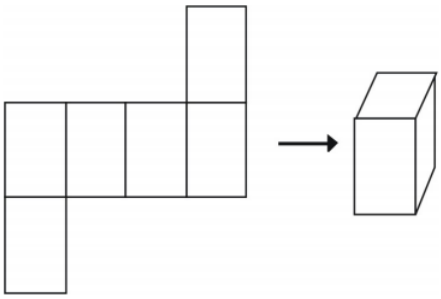
- Understand nets.
- Use nets to represent prisms.
- Find the surface area of a prism.
- Find the surface area of cylinders.
- Find the surface area of a sphere.

Nets

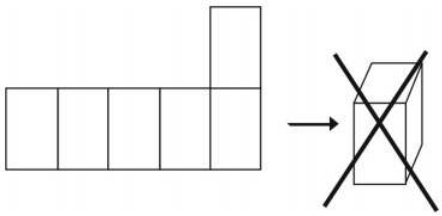
One final way to represent a solid is to use a net. If you cut out a net you can fold it into a model of a figure. Nets can also be used to analyze a single solid. Here is an example of a net for a cube.



There is more than one way to make a net for a single figure.

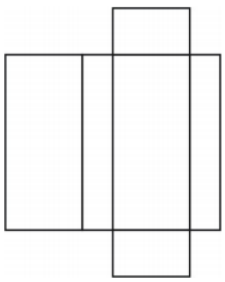


However, not all arrangements will create a cube.

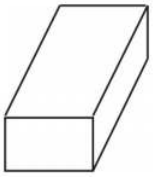


Example 1

What kind of figure does the net create? Draw the figure.

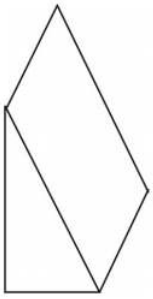


The net creates a box-shaped rectangular prism as shown below.

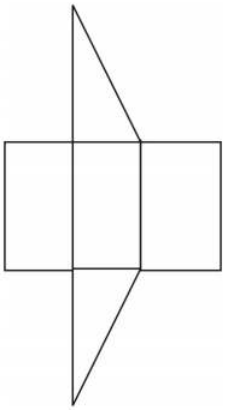


Example 2

What kind of net can you draw to represent the figure shown? Draw the net.

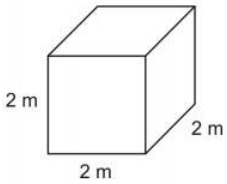


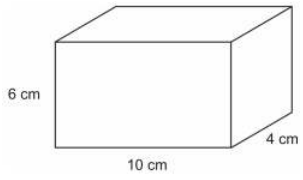
A net for the prism is shown. Other nets are possible.



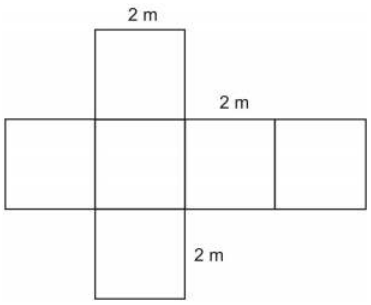
Review Exercises

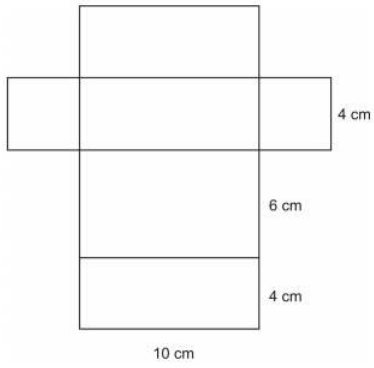
Draw a net for each of the following:

1. 

2. 

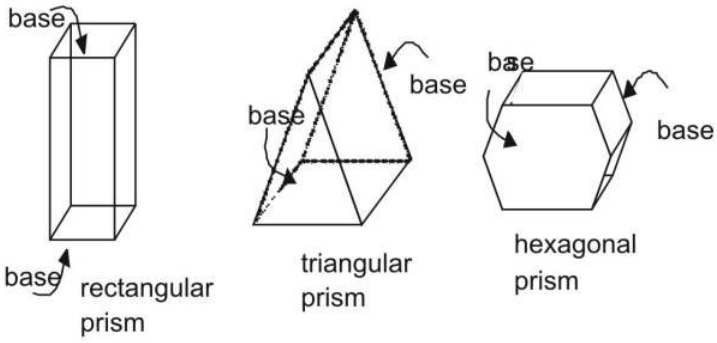
Answers

1. 

2. 

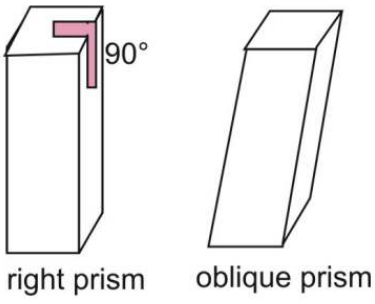
Prisms

A prism is a three-dimensional figure with a pair of parallel and congruent ends, or bases. The sides of a prism are parallelograms. Prisms are identified by their bases.



Surface Area of a Prism Using Nets

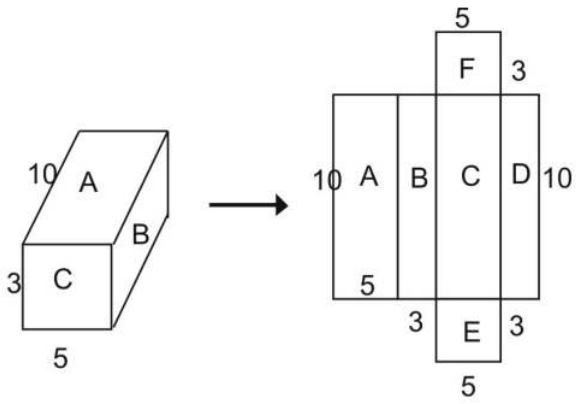
The prisms above are **right prisms**. In a right prism, the lateral sides are perpendicular to the bases of prism. Compare a right prism to an **oblique prism**, in which sides and bases are not perpendicular.



Two postulates that apply to area are the Area Congruence Postulate and the Area Addition Postulate.

Area Congruence Postulate: If two polygons (or plane figures) are congruent, then their areas are congruent.
Area Addition Postulate: The surface area of a three-dimensional figure is the sum of the areas of all of its non-overlapping parts.

You can use a net and the Area Addition Postulate to find the surface area of a right prism.



From the net, you can see that that the surface area of the entire prism equals the sum of the figures that make up the net:

Total surface area = area A + area B + area C + area D + area E + area F

Using the formula for the area of a rectangle, you can see that the area of rectangle A is:

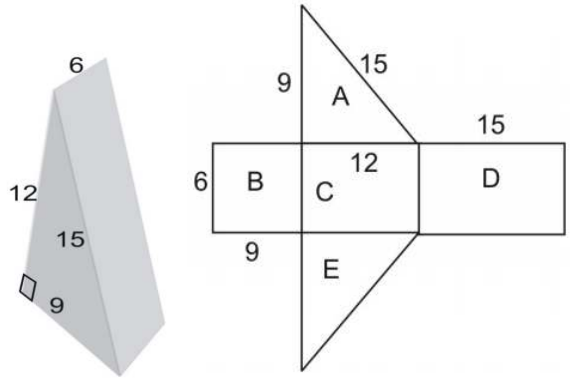
$A = l \cdot w$
$A = 10 \cdot 5 = 50$ square units

Similarly, the areas of the other rectangles are inserted back into the equation above.

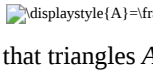
Total surface area = area A + area B + area C + area D + area E + area F
Total surface area = $(10 \cdot 5) + (10 \cdot 3) + (10 \cdot 5) + (10 \cdot 3) + (5 \cdot 3) + (5 \cdot 3)$
Total surface area = $50 + 30 + 50 + 30 + 15 + 15$
Total surface area = 190 square units

Example 3

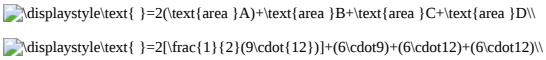
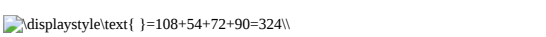
Use a net to find the surface area of the prism.



The area of the net is equal to the surface area of the figure. To find the area of the triangle, we use the formula:

 where h is the height of the triangle and b is its base.

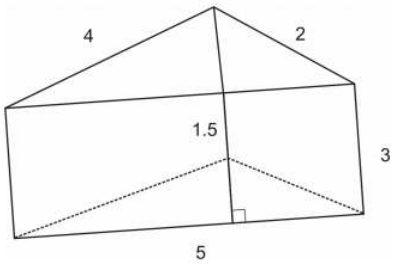
Note that triangles A and E are congruent so we can multiply the area of triangle A by 2.

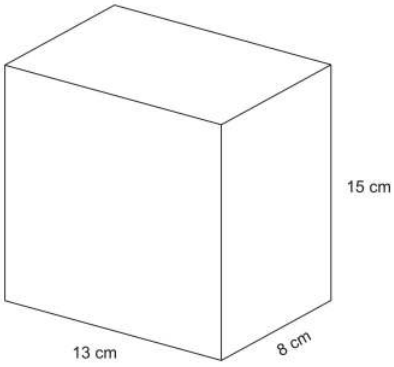





Thus, the surface area is 324 square units.

Review Exercises

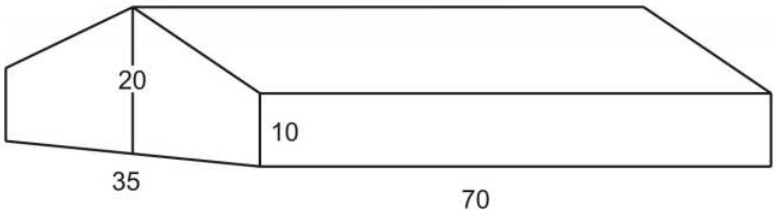
For each of the following find the surface are a using the method of nets and the perimeter

1. 

2. 

3. The base of a prism is a right triangle whose legs are 3 and 4 and show height is 20. What is the total area of the prism?
4. A right hexagonal prism is 24 inches tall and has bases that are regular hexagons measuring 8 inches on a side. What is the total surface area?
5. What is the volume of the prism in problem #4?

In the following questions, a barn is shaped like a pentagonal prism with dimensions shown in feet:



1. How many square feet (excluding the roof) are there on the surface of the barn to be painted?
2. If a gallon of paint covers 250 square feet, how many gallons of paint are needed to paint the barn?
3. A cardboard box is a perfect cube with an edge measuring 17 inches. How many cubic feet can it hold?
4. A swimming pool is 16 feet wide, 32 feet long and is uniformly 4 feet deep. How many cubic feet of water can it hold?
5. A cereal box has length 25 cm, width 9 cm and height 30 cm. How much cereal can it hold?

Answers

Nets and perimeter:

1. 40.5 in^2
2. 838 cm^2
3. 252 square units
4. 1484.6 square units
5. 3990.7 cubic inches

The barn:

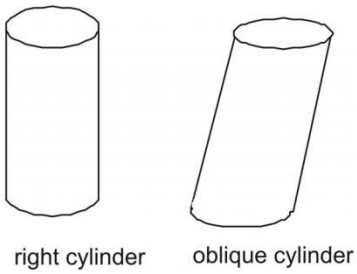
1. 2450 square feet
2. 10 gallons of paint
3. 2.85 cubic feet (be careful here. The units in the problem are given in inches but the question asks for feet.)

4. 2048 cubic feet
5. 6750 cm^3

Cylinders

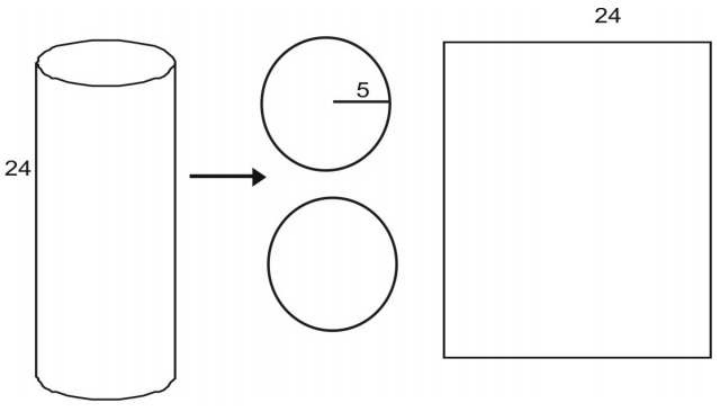
A **cylinder** is a three-dimensional figure with a pair of parallel and congruent circular ends, or **bases**. A cylinder has a single curved side that forms a rectangle when laid out flat.

As with prisms, cylinders can be *right* or *oblique*. The side of a right cylinder is perpendicular to its circular bases. The side of an oblique cylinder is not perpendicular to its bases.

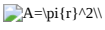


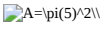
Surface Area of a Cylinder Using Nets

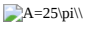
You can deconstruct a cylinder into a net.

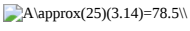


The area of each base is given by the area of a circle:

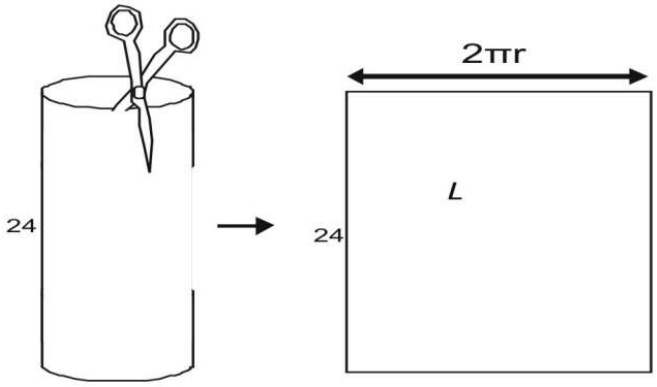








The area of the rectangular lateral area L is given by the product of a width and height. The height is given as 24. You can see that the width of the area is equal to the circumference of the circular base.



To find the width, imagine taking a can-like cylinder apart with a scissors. When you cut the lateral area, you see that it is equal to the circumference of the can's top. The circumference of a circle is given by $C = 2\pi r$, the lateral area, L, is

$$L=2\pi rh\\$$

$$L=2\pi(5)(24)\\$$

$$L=240\pi\\$$

$$L\approx(240)(3.14)=753.6\\$$

Now we can find the area of the entire cylinder using A = (area of two bases) + (area of lateral side).

$$A=2(75.36)+753.6\\$$

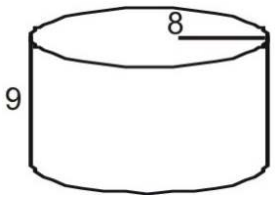
$$A=904.32\\$$

You can see that the formula we used to find the total surface area can be used for any right cylinder.

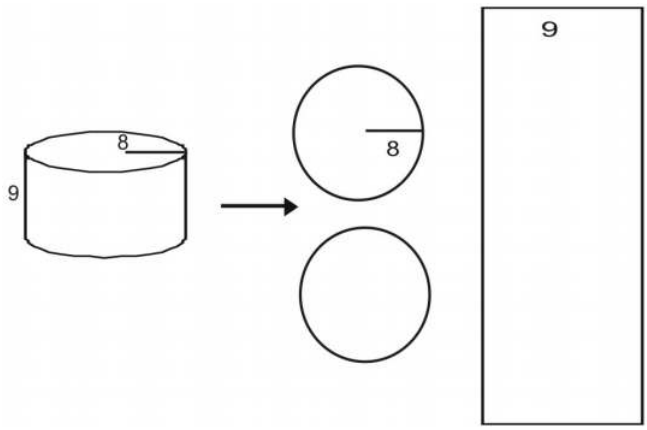
Area of a Right Cylinder: The surface area of a right cylinder, with radius and height h is given by $A = 2B + L$, where B is the area of each base of the cylinder and L is the lateral area of the cylinder.

Example 4

Use a net to find the surface area of the cylinder.



First draw and label a net for the figure.



Calculate the area of each base.

$$A=\pi{r}^2\\$$

$$A=\pi(8)^2\\$$

$$A=64\pi\\$$

$$A\approx(64)(3.14)=200.96\\$$

Calculate L.

$$L=2{\pi}rh\\$$

$$L=2{\pi}(8)(9)\\$$

$$L=144\pi\\$$

$$L\approx(144)(3.14)=452.16\\$$

Find the area of the entire cylinder.

$$A=2(200.96)+452.16\\$$

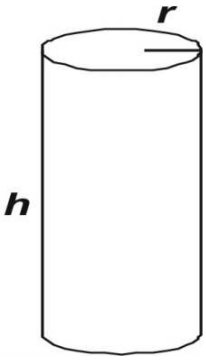
$$A=854.08\\$$

Thus, the total surface area is approximately 854.08 square units

Surface Area of a Cylinder Using a Formula

You have seen how to use nets to find the total surface area of a cylinder. The postulate can be broken down to create a general formula for all right cylinders.

$A = 2B + L$



Notice that the base, B, of any cylinder is: $B = \pi r^2$

The lateral area, L, for any cylinder is:

$$L=\text{width of lateral area}\cdot\text{height of cylinder}\\$$

$$L=\text{circumference of base}\cdot\text{height of cylinder}\\$$

$$L=2\pi{r}\cdot{h}\\$$

Putting the two equations together we get:

Factoring out a from the equation gives:

The Surface Area of a Right Cylinder: A right cylinder with radius r and height h can be expressed as:

$A = 2\pi r^2 + 2\pi rh$

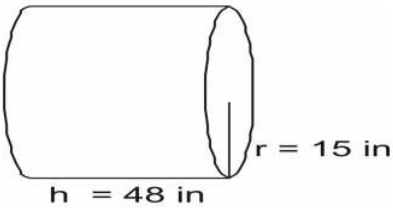
or:

$A = 2\pi r(r + h)$

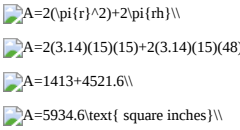
You can use the formulas to find the area of any right cylinder.

Example 5

Use the formula to find the surface area of the cylinder.

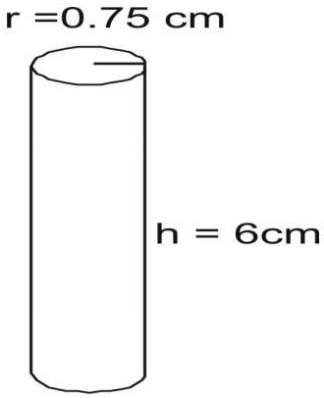


Write the formula and substitute in the values and solve.

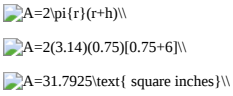


Example 6

Find the surface area of the cylinder.



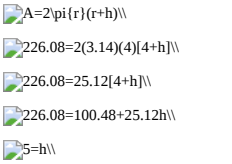
Write the formula and substitute in the values and solve.



Example 7

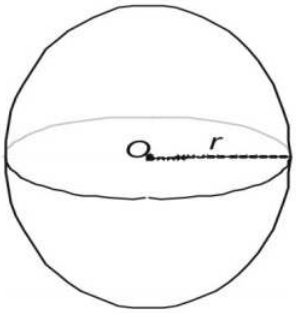
Find the height of a cylinder that has radius 4cm and surface area of 226.08 sq cm.

Write the formula with the given information and solve for *h*.



Spheres

A sphere is a three-dimensional figure that has the shape of a ball.



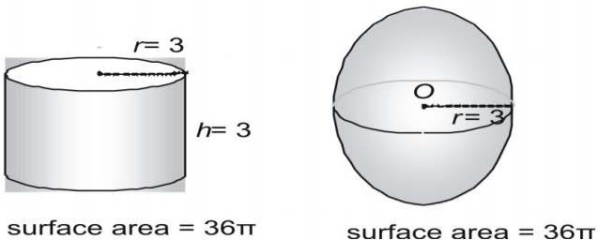
Spheres can be characterized in three ways.

- A sphere is the set of all points that lie a fixed distance r from a single center point O.
- A sphere is the surface that results when a circle is rotated about any of its diameters.

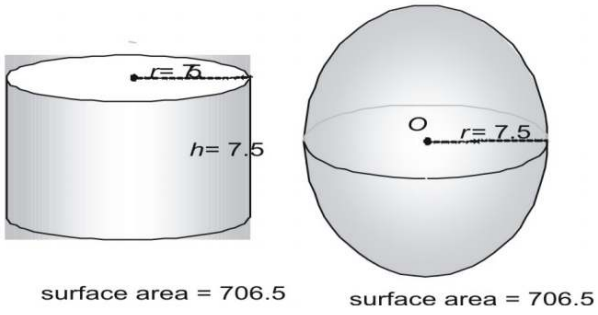


Surface Area of a Sphere

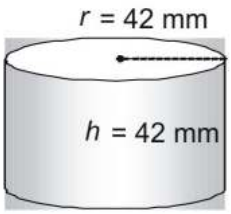
You can infer the formula for the surface area of a sphere by taking measurements of spheres and cylinders. Here we show a sphere with a radius of 3 and a right cylinder with both a radius and a height of 3 and express the area in terms of π.



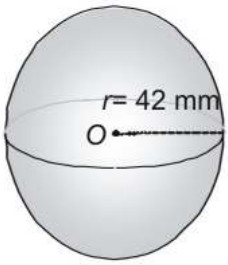
Now try a larger pair, expressing the surface area in decimal form.



Look at a third pair.



surface area = 22,155.84 sq mm



surface area = 22,155.84 sq mm

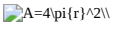
Is it a coincidence that a sphere and a cylinder whose radius and height are equal to the radius of the sphere have the exact same surface area? Not at all! In fact, the ancient Greeks used a method that showed that the following formula can be used to find the surface area of any sphere (or any cylinder in which).

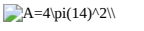
The Surface Area of a Sphere is given by: $A = 4\pi r^2$

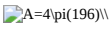
Example 8

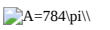
Find the surface area of a sphere with a radius of 14 feet.

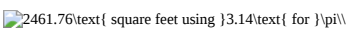
Use the formula.





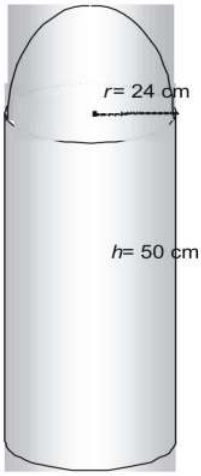




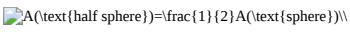


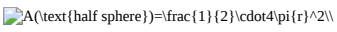
Example 9

Find the surface area of the following figure in terms of π.

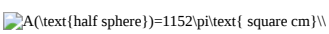


The figure is made of one half sphere or hemisphere, and one cylinder without its top.

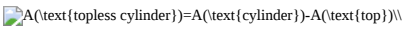


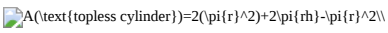


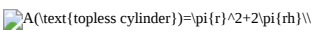




Now find the area of the cylinder without its top.







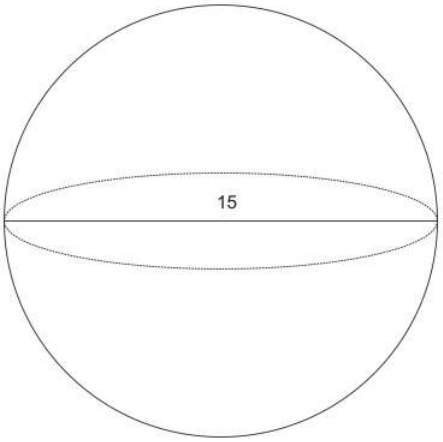
A(\text{topless cylinder})=\pi(576)+2\pi(24)(50)\\

A(\text{topless cylinder})=2976\pi\text{ square cm}\\

Thus, the total surface area is 1152\pi+2976\pi=4128\pi\\

Review Exercises

1. Find the radius of the sphere that has a volume of 335 cm^3.
2. Determine the surface area and volume of this shape:



3. The radius of a sphere is 4. Find its volume and total surface area.
4. A sphere has a radius of 5. A right cylinder, having the same radius has the same volume. Find the height and total surface area of the cylinder.
5. Sphere: volume = 296 cm^3. Find the Diameter.
6. Sphere: surface area is 179 in^2. Find the Radius.
7. Tennis balls with a diameter of 3.5 inches are sold in cans of three. The can is a cylinder. Assume the balls touch the can on the sides, top and bottom. What is the volume of the space not occupied by the tennis balls?
8. A sphere has surface area of 36π in2. Find its volume.
9. A giant scoop, operated by a crane, is in the shape of a hemisphere of radius = 21 inches. The scoop is filled with melted hot steel. When the steel is poured into a cylindrical storage tank that has a radius of 28 inches, the melted steel will rise to a height of how many inches?

Answers

Note that these problems use π not 3.14.

1. 1. Radius = 4.31 cm
2. Surface area = 706.86 cm2
   Volume = 1767.15 cm3
3. Volume = 268.08 units3
   Surface area = 201.06 units2
4. Height = 20/3 units total surface area = 366.52 units2
5. Diameter = 8.27cm
6. Radius = 3.77 inches
7. Volume of cylinder = 32.16π in3 volume of tennis balls = 21.44π in3
   Volume of space not occupied by tennis balls = 33.68 in3
8. Volume = 113.10 in3
9. Height of molten steel in cylinder will be 7.88 inches

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6.6: Graph Theory

In the modern world, planning efficient routes is essential for business and industry, with applications as varied as product distribution, laying new fiber optic lines for broadband internet, and suggesting new friends within social network websites like Facebook.

This field of mathematics started nearly 300 years ago as a look into a mathematical puzzle (we'll look at it in a bit). The field has exploded in importance in the last century, both because of the growing complexity of business in a global economy and because of the computational power that computers have provided us.

Drawing Graphs

Example 1

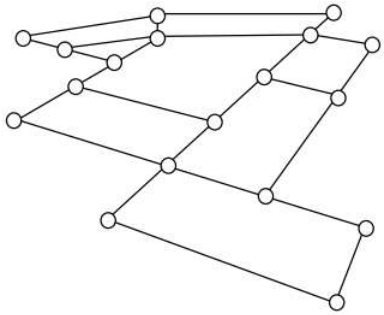
Here is a portion of a housing development from Missoula, Montana. As part of her job, the development's lawn inspector has to walk down every street in the development making sure homeowners' landscaping conforms to the community requirements.



Naturally, she wants to minimize the amount of walking she has to do. Is it possible for her to walk down every street in this development without having to do any backtracking? While you might be able to answer that question just by looking at the picture for a while, it would be ideal to be able to answer the question for any picture regardless of its complexity.

To do that, we first need to simplify the picture into a form that is easier to work with. We can do that by drawing a simple line for each street. Where streets intersect, we will place a dot.



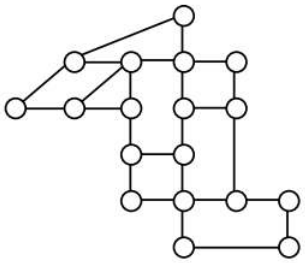


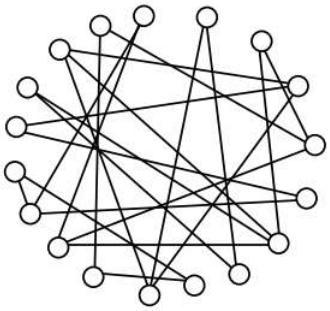
This type of simplified picture is called a **graph**.

Graphs, Vertices, and Edges

A **graph** consists of a set of dots, called **vertices**, and a set of **edges** connecting pairs of vertices.

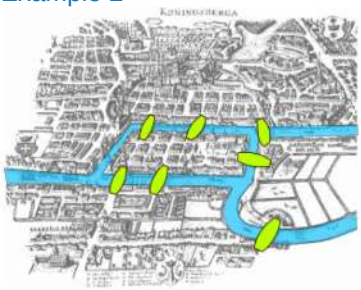
While we drew our original graph to correspond with the picture we had, there is nothing particularly important about the layout when we analyze a graph. Both of the graphs below are equivalent to the one drawn above.





You probably already noticed that we are using the term *graph* differently than you may have used the term in the past to describe the graph of a mathematical function.

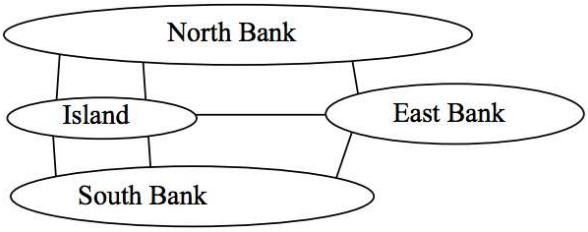
Example 2



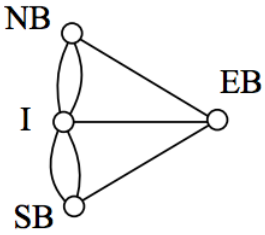
Back in the 18th century in the Prussian city of Königsberg, a river ran through the city and seven bridges crossed the forks of the river. The river and the bridges are highlighted in the picture to the right.[1]

As a weekend amusement, townsfolk would see if they could find a route that would take them across every bridge once and return them to where they started.

Leonard Euler (pronounced OY-lur), one of the most prolific mathematicians ever, looked at this problem in 1735, laying the foundation for graph theory as a field in mathematics. To analyze this problem, Euler introduced edges representing the bridges:



Since the size of each land mass it is not relevant to the question of bridge crossings, each can be shrunk down to a vertex representing the location:



Notice that in this graph there are *two* edges connecting the north bank and island, corresponding to the two bridges in the original drawing. Depending upon the interpretation of edges and vertices appropriate to a scenario, it is entirely possible and reasonable to have more than one edge connecting two vertices.

While we haven't answered the actual question yet of whether or not there is a route which crosses every bridge once and returns to the starting location, the graph provides the foundation for exploring this question.

Definitions

While we loosely defined some terminology earlier, we now will try to be more specific.

Vertex

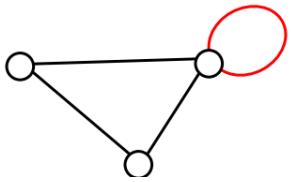
A vertex is a dot in the graph that could represent an intersection of streets, a land mass, or a general location, like “work” or “school”. Vertices are often connected by edges. Note that vertices only occur when a dot is explicitly placed, not whenever two edges cross. Imagine a freeway overpass—the freeway and side street cross, but it is not possible to change from the side street to the freeway at that point, so there is no intersection and no vertex would be placed.

Edges

Edges connect pairs of vertices. An edge can represent a physical connection between locations, like a street, or simply that a route connecting the two locations exists, like an airline flight.

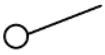
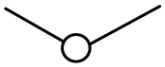
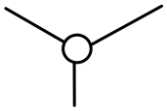
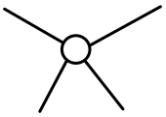
Loop

A loop is a special type of edge that connects a vertex to itself. Loops are not used much in street network graphs.



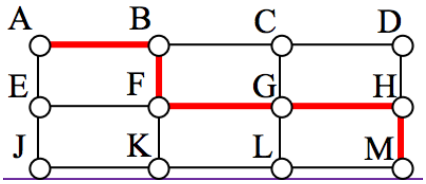
Degree of a vertex

The degree of a vertex is the number of edges meeting at that vertex. It is possible for a vertex to have a degree of zero or larger.

Degree 0	Degree 1	Degree 2	Degree 3	Degree 4
				

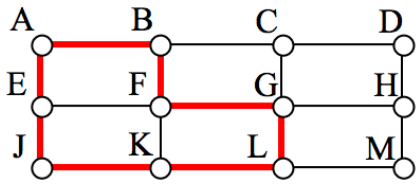
Path

A path is a sequence of vertices using the edges. Usually we are interested in a path between two vertices. For example, a path from vertex A to vertex M is shown below. It is one of many possible paths in this graph.



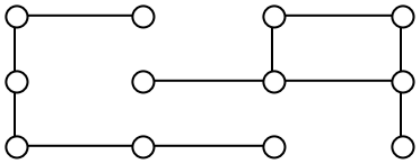
Circuit

A circuit is a path that begins and ends at the same vertex. A circuit starting and ending at vertex A is shown below.



Connected

A graph is connected if there is a path from any vertex to any other vertex. Every graph drawn so far has been connected. The graph below is **disconnected**; there is no way to get from the vertices on the left to the vertices on the right.



Weights

Depending upon the problem being solved, sometimes weights are assigned to the edges. The weights could represent the distance between two locations, the travel time, or the travel cost. It is important to note that the distance between vertices in a graph does not necessarily correspond to the weight of an edge.

Euler Circuits and the Chinese Postman Problem

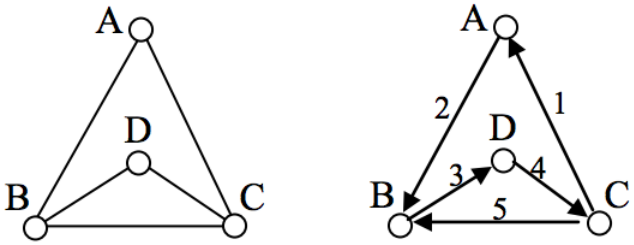
In the first section, we created a graph of the Königsberg bridges and asked whether it was possible to walk across every bridge once. Because Euler first studied this question, these types of paths are named after him.

Euler Path

An **Euler path** is a path that uses every edge in a graph with no repeats. Being a path, it does not have to return to the starting vertex.

Example 3

In the graph shown below, there are several Euler paths. One such path is CABDCB. The path is shown in arrows to the right, with the order of edges numbered.

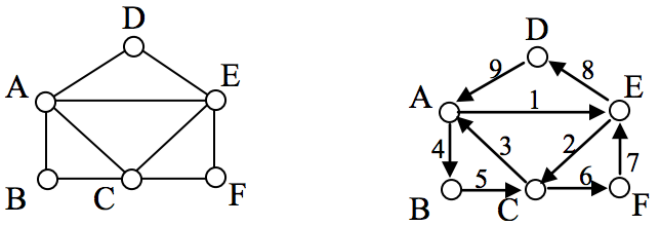


Euler Circuit

An **Euler circuit** is a circuit that uses every edge in a graph with no repeats. Being a circuit, it must start and end at the same vertex.

Example 4

The graph below has several possible Euler circuits. Here's a couple, starting and ending at vertex A: ADEACEFCBA and AECABCFEDA. The second is shown in arrows.



Look back at the example used for Euler paths—does that graph have an Euler circuit? A few tries will tell you no; that graph does not have an Euler circuit. When we were working with shortest paths, we were interested in the optimal path. With Euler paths and circuits, we're primarily interested in whether an Euler path or circuit *exists*.

Why do we care if an Euler circuit exists? Think back to our housing development lawn inspector from the beginning of the chapter. The lawn inspector is interested in walking as little as possible. The ideal situation would be a circuit that covers every street with no repeats. That's an Euler circuit! Luckily, Euler solved the question of whether or not an Euler path or circuit will exist.

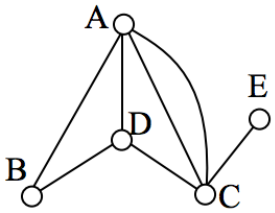
Euler's Path and Circuit Theorems

A graph will contain an Euler path if it contains at most two vertices of odd degree.

A graph will contain an Euler circuit if all vertices have even degree

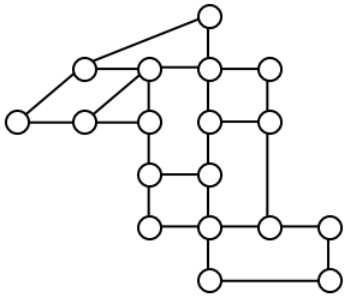
Example 5

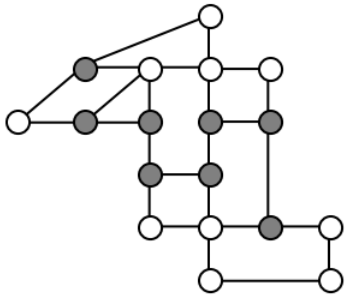
In the graph below, vertices A and C have degree 4, since there are 4 edges leading into each vertex. B is degree 2, D is degree 3, and E is degree 1. This graph contains two vertices with odd degree (D and E) and three vertices with even degree (A, B, and C), so Euler's theorems tell us this graph has an Euler path, but not an Euler circuit.



Example 6

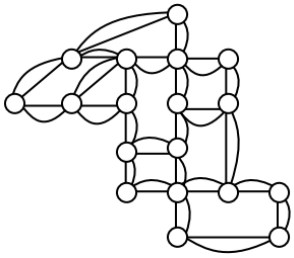
Is there an Euler circuit on the housing development lawn inspector graph we created earlier in the chapter? All the highlighted vertices have odd degree. Since there are more than two vertices with odd degree, there are no Euler paths or Euler circuits on this graph. Unfortunately our lawn inspector will need to do some backtracking.





Example 7

When it snows in the same housing development, the snowplow has to plow both sides of every street. For simplicity, we'll assume the plow is out early enough that it can ignore traffic laws and drive down either side of the street in either direction. This can be visualized in the graph by drawing two edges for each street, representing the two sides of the street.



Notice that every vertex in this graph has even degree, so this graph does have an Euler circuit.

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1. Bogdan Giuşcă. http://en.wikipedia.org/wiki/File:Konigsberg_bridges.png↵

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6.7: Supplemental Videos



This YouTube playlist contains several videos that supplement the reading on Graph Theory.

You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

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6.8: Topology, Tiling, and Non-Euclidean Geometry

Topology

Topology is a branch of mathematics studying spaces, in which “connectiveness” of objects is a main focus. Robert Bruner has written a more detailed description: “What is Topology?”

For this course, we will determine whether two objects are topologically equivalent by comparing their genus, or number of holes. Please read and work through the lesson starting from page 1. Be sure to click the link at the bottom right hand corner of each page to get to the next page in the sequence (there are five pages in total). On the last page, you’ll find topics that can be used in the **Application of Geometry Discussion Board**.

This is a link to a *deformation*, showing how a coffee mug can be transformed into a donut; thus they are topologically equivalent.

Tiling

Please read through the following three pages for a mini-lesson on tiling:

- What is a Tiling?
- Tilings with Just a Few Shapes
- Monomorphic and K-Morphic Tilings

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6.9: Discuss- Application of Geometry

Pick a real problem and try to solve it using the geometry you learned in this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the Geometry Application Directions for detailed directions.

Use the **Geometry Application** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 20 points.

Grading Criteria	Points Possible
The problem: • Is it a real-life problem? • Is it challenging, not trivial? • Is it a unique problem instead of a copy of a classmate's posting?	5
The strategies: • Are one or more general problem solving strategies used? • Are the strategies correctly identified?	5
The presentation: • Is the problem explained well? • Are the problem solving strategies explained well? • Are the appropriate terms used?	4
Your responses: • Did you post at least two responses? • Did you explain how the examples helped you better understand the math in this module? • Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?	6

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CHAPTER OVERVIEW

7: Finance

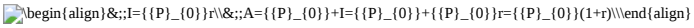
7.1: Simple Interest
7.2: Compound Interest
7.3: Annuities
7.4: Payout Annuities
7.5: Loans
7.6: Remaining Loan Balance
7.7: Which Equation to Use?
7.8: Solving for Time
7.9: Exercises

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7.1: Simple Interest

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed $100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $100(0.05) = $5. The total amount you would repay would be $105, the original principal plus the interest.

Simple One-time Interest



$$\begin{aligned} I &= P_0 r \\ A &= P_0 + I = P_0 + P_0 r = P_0(1+r) \end{aligned}$$

I is the interest

A is the end amount: principal plus interest

P0 is the principal (starting amount)

r is the interest rate (in decimal form. Example: 5% = 0.05)

Example 1

A friend asks to borrow $300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

$P_0 = \$300$ the principal

$r = 0.03$ 3% rate

$I = \$300(0.03) = \$9$. You will earn $9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

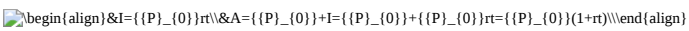
Example 2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a $1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Each year, you would earn 5% interest: $1000(0.05) = $50 in interest. So over the course of five years, you would earn a total of $250 in interest. When the bond matures, you would receive back the $1,000 you originally paid, leaving you with a total of $1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time



$$\begin{aligned} I &= P_0 rt \\ A &= P_0 + I = P_0 + P_0 rt = P_0(1+rt) \end{aligned}$$

I is the interest

A is the end amount: principal plus interest

P0 is the principal (starting amount)

r is the interest rate in decimal form

t is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

APR – Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.

A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 3

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

$P0 = \$1000$ the principal

$r = 0.02$ 2% rate per half-year

$t = 8$ 4 years = 8 half-years

$I = \$1000(0.02)(8) = \$160$. You will earn $160 interest total over the four years.

Try it Now 1

A loan company charges $30 interest for a one month loan of $500. Find the annual interest rate they are charging.

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7.2: Compound Interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit $1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3\%}{12}$ = 0.25% per month.

In the first month,

P0 = $1000

r = 0.0025 (0.25%)

I = $1000 (0.0025) = $2.50

A = $1000 + $2.50 = $1002.50

In the first month, we will earn $2.50 in interest, raising our account balance to $1002.50.

In the second month,

P0 = $1002.50

I = $1002.50 (0.0025) = $2.51 (rounded)

A = $1002.50 + $2.51 = $1005.01

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original $1000 we deposited, but we also earned interest on the $2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:

Month	Starting balance	Interest earned	Ending Balance
1	1000.00	2.50	1002.50
2	1002.50	2.51	1005.01
3	1005.01	2.51	1007.52
4	1007.52	2.52	1010.04
5	1010.04	2.53	1012.57
6	1012.57	2.53	1015.10
7	1015.10	2.54	1017.64
8	1017.64	2.54	1020.18
9	1020.18	2.55	1022.73
10	1022.73	2.56	1025.29
11	1025.29	2.56	1027.85
12	1027.85	2.57	1030.42

To find an equation to represent this, if *Pm* represents the amount of money after *m* months, then we could write the recursive equation:

$P0$ = \$1000

$Pm = (1+0.0025)Pm\text{-}1$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$P0$ = \$1000

$P1 = 1.0025P0 = 1.0025\ (1000)$

$P2 = 1.0025P1 = 1.0025\ (1.0025\ (1000)) = 1.0025\ 2(1000)$

$P3 = 1.0025P2 = 1.0025\ (1.00252(1000)) = 1.00253(1000)$

$P4 = 1.0025P3 = 1.0025\ (1.00253(1000)) = 1.00254(1000)$

Observing a pattern, we could conclude

$Pm = (1.0025)m(\$1000)$

Notice that the \$1000 in the equation was $P0$, the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year.

Generalizing our result, we could write

In this formula:

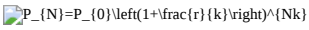
m is the number of compounding periods (months in our example)

r is the annual interest rate

k is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If N is the number of years, then $m = N\,k$. Making this change gives us the standard formula for compound interest.

Compound Interest



PN is the balance in the account after N years.

$P0$ is the starting balance of the account (also called initial deposit, or principal)

r is the annual interest rate in decimal form

k is the number of compounding periods in one year.

If the compounding is done annually (once a year), $k = 1$.

If the compounding is done quarterly, $k = 4$.

If the compounding is done monthly, $k = 12$.

If the compounding is done daily, $k = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

Example 4

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

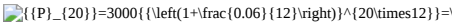
In this example,

$P0$ = \$3000 the initial deposit

$r = 0.06$ 6% annual rate

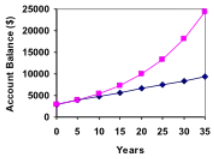
$k = 12$ 12 months in 1 year

$N = 20$ since we're looking for how much we'll have after 20 years

So  (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

Years	Simple Interest ($15 per month)	6% compounded monthly = 0.5% each month.
5	$3900	$4046.55
10	$4800	$5458.19
15	$5700	$7362.28
20	$6600	$9930.61
25	$7500	$13394.91
30	$8400	$18067.73
35	$9300	$24370.65



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Evaluating exponents on the calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5 \cdot 5 \cdot 5 = 125$. But when we need to calculate something like 1.005^{240}, it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

^ , yx , or xy .

To evaluate 1.005^{240} we'd type 1.005 ^ 240, or 1.005 yx 240. Try it out – you should get something around 3.3102044758.

Example 5

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

In this example,

We're looking for *P0*.

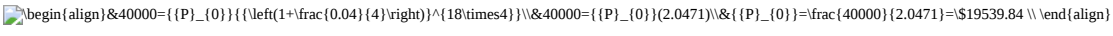
$r = 0.04$ 4%

$k = 4$ 4 quarters in 1 year

$N = 18$ Since we know the balance in 18 years

$P18 = \$40{,}000$ The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for *P0*.



So you would need to deposit $19,539.84 now to have $40,000 in 18 years.

Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better.

Example 6

To see why not over-rounding is so important, suppose you were investing $1000 at 5% interest compounded monthly for 30 years.

$P0 = \$1000$ the initial deposit

$r = 0.05$ 5%

$k = 12$ 12 months in 1 year

$N = 30$ since we're looking for the amount after 30 years

If we first compute r/k, we find 0.05/12 = 0.00416666666667

Here is the effect of rounding this to different values:

***r/k* rounded to:**	**Gives *P30* to be:**	**Error**
0.004	$4208.59	$259.15
0.0042	$4521.45	$53.71
0.00417	$4473.09	$5.35
0.004167	$4468.28	$0.54
0.0041667	$4467.80	$0.06
no rounding	$4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough – $5 off of $4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

We can quickly calculate 12×30 = 360, giving

.

Now we can use the calculator.

Type this	**Calculator shows**
0.05 ÷ 12 = .	0.00416666666667
+ 1 = .	1.00416666666667
yx 360 = .	4.46774431400613
× 1000 = .	4467.74431400613

Using your calculator continued

The previous steps were assuming you have a “one operation at a time” calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

$1000 \times (1 + 0.05 \div 12) y^x 360 = .$

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7.3: Annuities

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

For a savings annuity, we simply need to add a deposit, *d*, to the account with each compounding period:

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

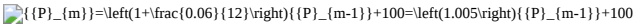
Suppose we will deposit $100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

r = 0.06 (6%)

k = 12 (12 compounds/deposits per year)

d = $100 (our deposit per month)

Writing out the recursive equation gives

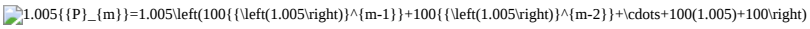


Assuming we start with an empty account, we can begin using this relationship:

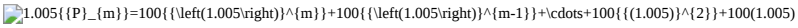
Continuing this pattern, after *m* deposits, we'd have saved:

In other words, after *m* months, the first deposit will have earned compound interest for *m*-1 months. The second deposit will have earned interest for *m*-2 months. Last months deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

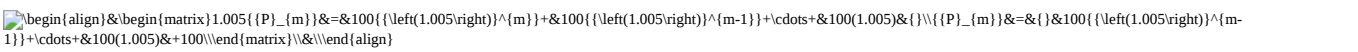
This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:



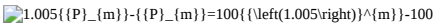
Distributing on the right side of the equation gives



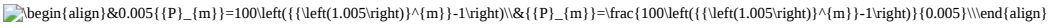
Now we'll line this up with like terms from our original equation, and subtract each side



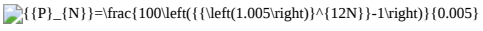
Almost all the terms cancel on the right hand side when we subtract, leaving



Solving for *Pm*

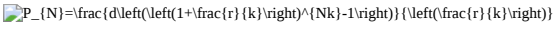


Replacing *m* months with 12*N*, where *N* is measured in years, gives



Recall 0.005 was *r/k* and 100 was the deposit *d*. 12 was *k*, the number of deposit each year. Generalizing this result, we get the saving annuity formula.

Annuity Formula



PN is the balance in the account after *N* years.

d is the regular deposit (the amount you deposit each year, each month, etc.)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, $k = 12$.

If you make your deposits every year, use yearly compounding, $k = 1$.

If you make your deposits every quarter, use quarterly compounding, $k = 4$.

Etc.

When do you use this

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Example 7

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit $100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

In this example,

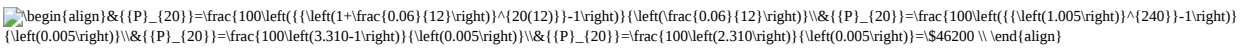
$d = \$100$ the monthly deposit

$r = 0.06$ 6% annual rate

$k = 12$ since we're doing monthly deposits, we'll compound monthly

$N = 20$ we want the amount after 20 years

Putting this into the equation:



$$\begin{aligned}&{{P}_{20}}=\frac{100\left({\left(1+\frac{0.06}{12}\right)}^{20(12)}-1\right)}{\left(\frac{0.06}{12}\right)}\\&{{P}_{20}}=\frac{100\left({\left(1.005\right)}^{240}-1\right)}{\left(0.005\right)}\\&{{P}_{20}}=\frac{100\left(3.310-1\right)}{\left(0.005\right)}\\&{{P}_{20}}=\frac{100\left(2.310\right)}{\left(0.005\right)}=\$46200 \\ \end{aligned}$$

The account will grow to $46,200 after 20 years.

Notice that you deposited into the account a total of $24,000 ($100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $46,200 – $24,000 = $22,200.

Example 8

You want to have $200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

In this example,

We're looking for *d*.

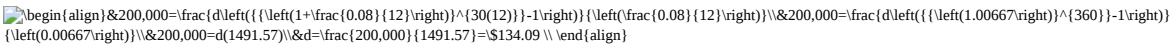
$r = 0.08$ 8% annual rate

$k = 12$ since we're depositing monthly

$N = 30$ 30 years

P30 = $200,000 The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for *d*.



So you would need to deposit $134.09 each month to have $200,000 in 30 years if your account earns 8% interest

Try it Now 2

A more conservative investment account pays 3% interest. If you deposit $5 a day into this account, how much will you have after 10 years? How much is from interest?

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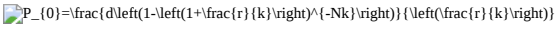
7.4: Payout Annuities

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved $500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula



P0 is the balance in the account at the beginning (starting amount, or principal).

d is the regular withdrawal (the amount you take out each year, each month, etc.)

r is the annual interest rate (in decimal form. Example: 5% = 0.05)

k is the number of compounding periods in one year.

N is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Example 9

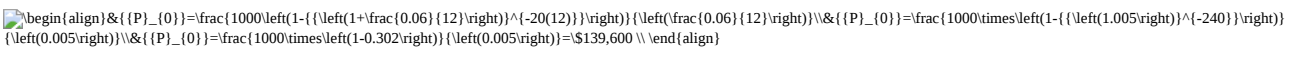
After retiring, you want to be able to take $1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

In this example,

d = $1000	the monthly withdrawal
r = 0.06	6% annual rate
k = 12	since we're doing monthly withdrawals, we'll compound monthly
N = 20	since were taking withdrawals for 20 years

We're looking for *P0*; how much money needs to be in the account at the beginning.

Putting this into the equation:



You will need to have $139,600 in your account when you retire.

Notice that you withdrew a total of $240,000 ($1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is $240,000 – $139,600 = $100,400 in interest.

Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this (-) , some with +/- . The button is often near the = key or the decimal point.

If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005-240 you'd type something like: 1.005 ^ (-) 240

If your calculator only shows one value at a time, then usually you hit the (-) key after a number to negate it, so you'd hit: 1.005 yx 240 (-) =

Give it a try – you should get 1.005-240 = 0.302096

Example 10

You know you will have $500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

In this example,

We're looking for *d*.

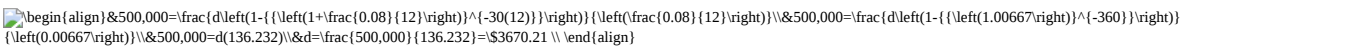
$r = 0.08$ 8% annual rate

$k = 12$ since we're withdrawing monthly

$N = 30$ 30 years

$P0 = \$500{,}000$ we are beginning with $500,000

In this case, we're going to have to set up the equation, and solve for *d*.



$$\begin{aligned}&500{,}000=\frac{d\left(1-{\left(1+\frac{0.08}{12}\right)}^{-30(12)}\right)}{\left(\frac{0.08}{12}\right)}\\&500{,}000=\frac{d\left(1-{\left(1.00667\right)}^{-360}\right)}{\left(0.00667\right)}\\&500{,}000=d(136.232)\\&d=\frac{500{,}000}{136.232}=\$3670.21\end{aligned}$$

You would be able to withdraw $3,670.21 each month for 30 years.

Try it Now 3

A donor gives $100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

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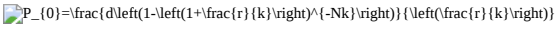
7.5: Loans

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had $10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests $10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loans Formula



P0 is the balance in the account at the beginning (the principal, or amount of the loan).

d is your loan payment (your monthly payment, annual payment, etc)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

N is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Loans: Many payments

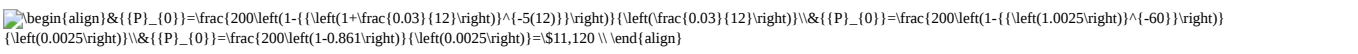
Example 11

You can afford $200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with $200 per month?

In this example,

d = $200	the monthly loan payment
r = 0.03	3% annual rate
k = 12	since we're doing monthly payments, we'll compound monthly
N = 5	since we're making monthly payments for 5 years

We're looking for *P0*, the starting amount of the loan.



You can afford a $11,120 loan.

You will pay a total of $12,000 ($200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying $12,000-$11,120 = $880 interest total.

Example 12

You want to take out a $140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

In this example,

We're looking for *d*.

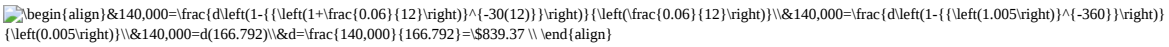
$r = 0.06$ 6% annual rate

$k = 12$ since we're paying monthly

$N = 30$ 30 years

$P0 = \$140{,}000$ the starting loan amount

In this case, we're going to have to set up the equation, and solve for *d*.



$$\begin{aligned}&140{,}000=\frac{d\left(1-{\left(1+\frac{0.06}{12}\right)}^{-30(12)}\right)}{\left(\frac{0.06}{12}\right)}\\&140{,}000=\frac{d\left(1-{\left(1.005\right)}^{-360}\right)}{\left(0.005\right)}\\&140{,}000=d(166.792)\\&d=\frac{140{,}000}{166.792}=\$839.37\end{aligned}$$

You will make payments of $839.37 per month for 30 years.

You're paying a total of $302,173.20 to the loan company: $839.37 per month for 360 months. You are paying a total of $302,173.20 – $140,000 = $162,173.20 in interest over the life of the loan.

Try it Now 4

Janine bought $3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

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7.6: Remaining Loan Balance

With loans, it is often desirable to determine what the remaining loan balance will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were $1,000 a month, after a year you will *not* have paid off $12,000 of the loan balance.

To determine the remaining loan balance, we can think "how much loan will these loan payments be able to pay off in the remaining time on the loan?"

Example 13

If a mortgage at a 6% interest rate has payments of $1,000 a month, how much will the loan balance be 10 years from the end the loan?

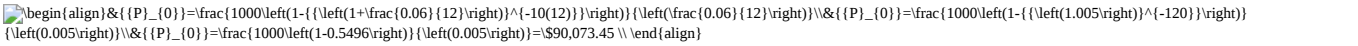
To determine this, we are looking for the amount of the loan that can be paid off by $1,000 a month payments in 10 years. In other words, we're looking for P0 when

d = $1,000 the monthly loan payment

r = 0.06 6% annual rate

k = 12 since we're doing monthly payments, we'll compound monthly

N = 10 since we're making monthly payments for 10 more years



$$\begin{aligned}&{{P}_{0}}=\frac{1000\left(1-{{\left(1+\frac{0.06}{12}\right)}^{-10(12)}}\right)}{\left(\frac{0.06}{12}\right)}\\&{{P}_{0}}=\frac{1000\left(1-{{\left(1.005\right)}^{-120}}\right)}{\left(0.005\right)}\\&{{P}_{0}}=\frac{1000\left(1-0.5496\right)}{\left(0.005\right)}=\$90,073.45 \\ \end{aligned}$$

The loan balance with 10 years remaining on the loan will be $90,073.45

Often times answering remaining balance questions requires two steps:

1) Calculating the monthly payments on the loan

2) Calculating the remaining loan balance based on the *remaining time* on the loan

Example 14

A couple purchases a home with a $180,000 mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

First we will calculate their monthly payments.

We're looking for d.

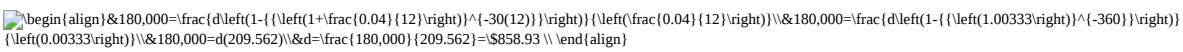
r = 0.04 4% annual rate

k = 12 since they're paying monthly

N = 30 30 years

$P0$ = $180,000 the starting loan amount

We set up the equation and solve for d.



$$\begin{aligned}&180,000=\frac{d\left(1-{{\left(1+\frac{0.04}{12}\right)}^{-30(12)}}\right)}{\left(\frac{0.04}{12}\right)}\\&180,000=\frac{d\left(1-{{\left(1.00333\right)}^{-360}}\right)}{\left(0.00333\right)}\\&180,000=d(209.562)\\&d=\frac{180,000}{209.562}=\$858.93 \\ \end{aligned}$$

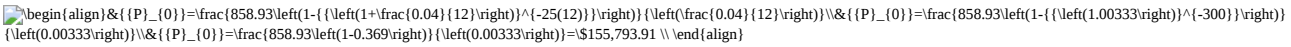
Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

d = $858.93 the monthly loan payment we calculated above

r = 0.04 4% annual rate

k = 12 since they're doing monthly payments

$N = 25$ since they'd be making monthly payments for 25 more years



$$\begin{aligned}&{{P}_{0}}=\frac{858.93\left(1-{{\left(1+\frac{0.04}{12}\right)}^{-25(12)}}\right)}{\left(\frac{0.04}{12}\right)}\\&{{P}_{0}}=\frac{858.93\left(1-{{\left(1.00333\right)}^{-300}}\right)}{\left(0.00333\right)}\\&{{P}_{0}}=\frac{858.93\left(1-0.369\right)}{\left(0.00333\right)}=\$155,793.91 \\ \end{aligned}$$

The loan balance after 5 years, with 25 years remaining on the loan, will be $155,793.91

Over that 5 years, the couple has paid off $180,000 – $155,793.91 = $24,206.09 of the loan balance. They have paid a total of $858.93 a month for 5 years (60 months), for a total of $51,535.80, so $51,535.80 – $24,206.09 = $27,329.71 of what they have paid so far has been interest.

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7.7: Which Equation to Use?

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan", "amortize" (the fancy word for loans), "finance (a car)", or "mortgage" (a home loan). Look for these words. If they're there, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annually/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly) then you're looking at a basic Annuity problem. Basic annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future.

If you're pulling money out of the account on a regular basis, then you're looking at a Payout Annuity problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

Try it Now 5

For each of the following scenarios, determine if it is a compound interest problem, a savings annuity problem, a payout annuity problem, or a loans problem. Then solve each problem.

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7.8: Solving for Time

Often we are interested in how long it will take to accumulate money or how long we'd need to extend a loan to bring payments down to a reasonable level.

Note: This section assumes you've covered solving exponential equations using logarithms, either in prior classes or in the growth models chapter.

Example 15

If you invest $2000 at 6% compounded monthly, how long will it take the account to double in value?

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

$P0$ = $2000 the initial deposit

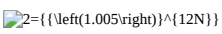
r = 0.06 6% annual rate

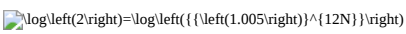
k = 12 12 months in 1 year

So our general equation is

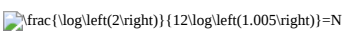
. We also know that we want our ending amount to be double of $2000, which is $4000, so we're looking for N so that PN = 4000. To solve this, we set our equation for PN equal to 4000.


Divide both sides by 2000


To solve for the exponent, take the log of both sides


Use the exponent property of logs on the right side


Now we can divide both sides by 12log(1.005)


Approximating this to a decimal

N = 11.581

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example if you rounded log(2) to 0.301 and log(1.005) to 0.00217, then your final answer would have been about 11.577 years.

Example 16

If you invest $100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to $10,000?

This is a savings annuity problem since we are making regular deposits into the account.

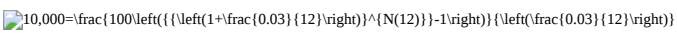
d = $100 the monthly deposit

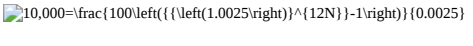
r = 0.03 3% annual rate

k = 12 since we're doing monthly deposits, we'll compound monthly

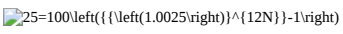
We don't know N, but we want PN to be $10,000.

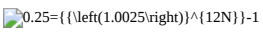
Putting this into the equation:


Simplifying the fractions a bit

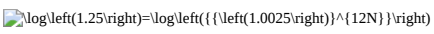


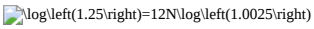
We want to isolate the exponential term, 1.002512N, so multiply both sides by 0.0025

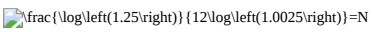

Divide both sides by 100


Add 1 to both sides


Now take the log of both sides


Use the exponent property of logs


Divide by 12log(1.0025)


Approximating to a decimal

N = 7.447 years

It will take about 7.447 years to grow the account to $10,000.

Try it Now 6

Joel is considering putting a $1,000 laptop purchase on his credit card, which has an interest rate of 12% compounded monthly. How long will it take him to pay off the purchase if he makes payments of $30 a month?

Try it Now Answers

1.

I = $30 of interest

$P0$ = $500 principal

r = unknown

t = 1 month

Using $I = P0rt$, we get $30 = 500 \cdot r \cdot 1$. Solving, we get $r = 0.06$, or 6%. Since the time was monthly, this is the monthly interest. The annual rate would be 12 times this: 72% interest.

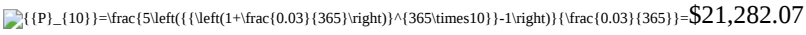
2.

d = $5 the daily deposit

r = 0.03 3% annual rate

k = 365 since we're doing daily deposits, we'll compound daily

N = 10 we want the amount after 10 years



We would have deposited a total of $5·365·10 = $18,250, so $3,032.07 is from interest

3.

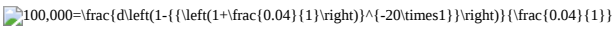
d = unknown

r = 0.04 4% annual rate

k = 1 since we're doing annual scholarships

N = 20 20 years

$P0$ = 100,000 we're starting with $100,000



Solving for d gives $7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that, $100,000(0.04) = $4,000 a year would have been available.

Try it Now Answers continued

4.

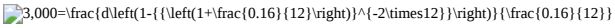
d = unknown

r = 0.16 16% annual rate

k = 12 since we're making monthly payments

N = 2 2 years to repay

$P0$ = 3,000 we're starting with a $3,000 loan



Solving for *d* gives $146.89 as monthly payments.

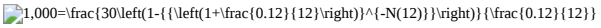
In total, she will pay $3,525.36 to the store, meaning she will pay $525.36 in interest over the two years.

5.

6.

d = $30	The monthly payments
r = 0.12	12% annual rate
k = 12	since we're making monthly payments
$P0$ = 1,000	we're starting with a $1,000 loan

We are solving for *N*, the time to pay off the loan



Solving for *N* gives 3.396. It will take about 3.4 years to pay off the purchase.

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7.9: Exercises

Skills

1. A friend lends you $200 for a week, which you agree to repay with 5% one-time interest. How much will you have to repay?
2. Suppose you obtain a $3,000 T-note with a 3% annual rate, paid quarterly, with maturity in 5 years. How much interest will you earn?
3. A T-bill is a type of bond that is sold at a discount over the face value. For example, suppose you buy a 13-week T-bill with a face value of $10,000 for $9,800. This means that in 13 weeks, the government will give you the face value, earning you $200. What annual interest rate have you earned?
4. Suppose you are looking to buy a $5000 face value 26-week T-bill. If you want to earn at least 1% annual interest, what is the most you should pay for the T-bill?
5. You deposit $300 in an account earning 5% interest compounded annually. How much will you have in the account in 10 years?
6. How much will $1000 deposited in an account earning 7% interest compounded annually be worth in 20 years?
7. You deposit $2000 in an account earning 3% interest compounded monthly.
    a. How much will you have in the account in 20 years?
    b. How much interest will you earn?
8. You deposit $10,000 in an account earning 4% interest compounded monthly.
    a. How much will you have in the account in 25 years?
    b. How much interest will you earn?
9. How much would you need to deposit in an account now in order to have $6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.
10. How much would you need to deposit in an account now in order to have $20,000 in the account in 4 years? Assume the account earns 5% interest.
11. You deposit $200 each month into an account earning 3% interest compounded monthly.
    a. How much will you have in the account in 30 years?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
12. You deposit $1000 each year into an account earning 8% compounded annually.
    a. How much will you have in the account in 10 years?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
13. Jose has determined he needs to have $800,000 for retirement in 30 years. His account earns 6% interest.
    a. How much would you need to deposit in the account each month?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
14. You wish to have $3000 in 2 years to buy a fancy new stereo system. How much should you deposit each quarter into an account paying 8% compounded quarterly?
15. You want to be able to withdraw $30,000 each year for 25 years. Your account earns 8% interest.
    a. How much do you need in your account at the beginning
    b. How much total money will you pull out of the account?
    c. How much of that money is interest?
16. How much money will I need to have at retirement so I can withdraw $60,000 a year for 20 years from an account earning 8% compounded annually?
    a. How much do you need in your account at the beginning
    b. How much total money will you pull out of the account?
    c. How much of that money is interest?
17. You have $500,000 saved for retirement. Your account earns 6% interest. How much will you be able to pull out each month, if you want to be able to take withdrawals for 20 years?

18. Loren already knows that he will have $500,000 when he retires. If he sets up a payout annuity for 30 years in an account paying 10% interest, how much could the annuity provide each month?
19. You can afford a $700 per month mortgage payment. You've found a 30 year loan at 5% interest.
    a. How big of a loan can you afford?
    b. How much total money will you pay the loan company?
    c. How much of that money is interest?
20. Marie can afford a $250 per month car payment. She's found a 5 year loan at 7% interest.
    a. How expensive of a car can she afford?
    b. How much total money will she pay the loan company?
    c. How much of that money is interest?
21. You want to buy a $25,000 car. The company is offering a 2% interest rate for 48 months (4 years). What will your monthly payments be?
22. You decide finance a $12,000 car at 3% compounded monthly for 4 years. What will your monthly payments be? How much interest will you pay over the life of the loan?
23. You want to buy a $200,000 home. You plan to pay 10% as a down payment, and take out a 30 year loan for the rest.
    a. How much is the loan amount going to be?
    b. What will your monthly payments be if the interest rate is 5%?
    c. What will your monthly payments be if the interest rate is 6%?
24. Lynn bought a $300,000 house, paying 10% down, and financing the rest at 6% interest for 30 years.
    a. Find her monthly payments.
    b. How much interest will she pay over the life of the loan?
25. Emile bought a car for $24,000 three years ago. The loan had a 5 year term at 3% interest rate. How much does he still owe on the car?
26. A friend bought a house 15 years ago, taking out a $120,000 mortgage at 6% for 30 years. How much does she still owe on the mortgage?
27. Pat deposits $6,000 into an account earning 4% compounded monthly. How long will it take the account to grow to $10,000?
28. Kay is saving $200 a month into an account earning 5% interest. How long will it take her to save $20,000?
29. James has $3,000 in credit card debt, which charges 14% interest. How long will it take to pay off the card if he makes the minimum payment of $60 a month?
30. Chris has saved $200,000 for retirement, and it is in an account earning 6% interest. If she withdraws $3,000 a month, how long will the money last?

Concepts

31. Suppose you invest $50 a month for 5 years into an account earning 8% compounded monthly. After 5 years, you leave the money, without making additional deposits, in the account for another 25 years. How much will you have in the end?
32. Suppose you put off making investments for the first 5 years, and instead made deposits of $50 a month for 25 years into an account earning 8% compounded monthly. How much will you have in the end?
33. Mike plans to make contributions to his retirement account for 15 years. After the last contribution, he will start withdrawing $10,000 a quarter for 10 years. Assuming Mike's account earns 8% compounded quarterly, how large must his quarterly contributions be during the first 15 years, in order to accomplish his goal?
34. Kendra wants to be able to make withdrawals of $60,000 a year for 30 years after retiring in 35 years. How much will she have to save each year up until retirement if her account earns 7% interest?
35. You have $2,000 to invest, and want it to grow to $3,000 in two years. What interest rate would you need to find to make this possible?
36. You have $5,000 to invest, and want it to grow to $20,000 in ten years. What interest rate would you need to find to make this possible?
37. You plan to save $600 a month for the next 30 years for retirement. What interest rate would you need to have $1,000,000 at retirement?
38. You really want to buy a used car for $11,000, but can only afford $200 a month. What interest rate would you need to find to be able to afford the car, assuming the loan is for 60 months?

Exploration

39. Pay day loans are short term loans that you take out against future paychecks: The company advances you money against a future paycheck. Either visit a pay day loan company, or look one up online. Be forewarned that many companies do not make their fees obvious, so you might need to do some digging or look at several companies.
    a. Explain the general method by which the loan works.
    b. We will assume that we need to borrow $500 and that we will pay back the loan in 14 days. Determine the total amount that you would need to pay back and the effective loan rate. The effective loan rate is the percentage of the original loan amount that you pay back. It is not the same as the APR (annual rate) that is probably published.
    c. If you cannot pay back the loan after 14 days, you will need to get an extension for another 14 days. Determine the fees for an extension, determine the total amount you will be paying for the now 28 day loan, and compute the effective loan rate.
40. Suppose that 10 years ago you bought a home for $110,000, paying 10% as a down payment, and financing the rest at 9% interest for 30 years.
    a. Let's consider your existing mortgage:
        i. How much money did you pay as your down payment?
        ii. How much money was your mortgage (loan) for?
        iii. What is your current monthly payment?
        iv. How much total interest will you pay over the life of the loan?
    b. This year, you check your loan balance. Only part of your payments have been going to pay down the loan; the rest has been going towards interest. You see that you still have $88,536 left to pay on your loan. Your house is now valued at $150,000.
        i. How much of the loan have you paid off? (i.e., how much have you reduced the loan balance by? Keep in mind that interest is charged each month – it's not part of the loan balance.)
        ii. How much money have you paid to the loan company so far?
        iii. How much interest have you paid so far?
        iv. How much equity do you have in your home (equity is value minus remaining debt)
    c. Since interest rates have dropped, you consider refinancing your mortgage at a lower 6% rate.
        i. If you took out a new 30 year mortgage at 6% for your remaining loan balance, what would your new monthly payments be?
        ii. How much interest will you pay over the life of the new loan?
    d. Notice that if you refinance, you are going to be making payments on your home for another 30 years. In addition to the 10 years you've already been paying, that's 40 years total.
        i. How much will you save each month because of the lower monthly payment?
        ii. How much total interest will you be paying (you need to consider the amount from 2c and 3b)
        iii. Does it make sense to refinance? (there isn't a correct answer to this question. Just give your opinion and your reason)

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CHAPTER OVERVIEW

8: Topic F- Using a Calculator

8.1: F1.01 Beginning and Example 3
8.2: F1.02- Examples 4-5
8.3: F1.03- Example 6
8.4: F1.04- Exercises

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8.1: F1.01 Beginning and Example 3

Topic F. Using a Calculator

Objectives:

1. Understand that different calculators require a somewhat different order of entering numbers and operations. Be able to use the correct order on YOUR calculator, and recognize which other calculators require a different order.
2. Perform basic operations and square root
3. Use exponents—not just squares and cubes
4. Use the constant pi.
5. Learn the difference between entering a negative number and the operation of subtraction.
6. Learn to put in parentheses when needed.
7. Read the output when it includes scientific notation.
8. When and how much should you round the results?
9. Checking your work with a calculator. (Estimation)

Discussion: Types of calculators

Generally speaking, the various brands of calculators all work in pretty similar ways, so the differences are in the type of calculator, not the brand.

We might think of four categories of calculators typically used by students:

- Basic calculators. These add, subtract, multiply, divide, and a few other things. They cost $2 to $8.
- Scientific calculators with a one-line display. These cost $5 to $15.
- Scientific calculators with a two-line display. These cost $8 to $20.
- Graphing calculators. These cost $40 to $150.

For these materials, the basic calculators don't have enough capability and the graphing calculators have much more capability than needed and are more expensive than needed. Either of the two types of scientific calculators is acceptable. However, we enter various operations into them in different ways, so you'll find it easier to pick one type and always use it. If you already have a scientific calculator, you can use it for this course. Most students find that they prefer a calculator with a two-line display. Those enable you to see what you entered and the result at the same time, which is not possible on the one-line-display calculators.

Example 1. Perform basic operations and square root.

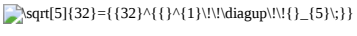
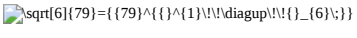
Try all of these problems on your calculator to make sure that you understand what to enter to obtain the correct answer:

a. 
b. 
c. 
d. 
e. 

Solution: You know how to do all of these operations without a calculator. Do them and then make sure you can get the same answer with your calculator.

The order in which you enter the numbers and operations is different on different calculators. Practice with the calculator you will use in this course. Make a note in the margin here about anything you must remember in order to enter these into your calculator.

Example 2. Use exponents—not just squares and cubes.

a. Find
b. Find
c. Find 
d. Find 

Solution: Most scientific calculators have a square and a cube key (powers of 2 and 3.) But we will need to compute other powers. The exponent key on most calculators is denoted by one of these symbols:

, , or ^.

a. By hand (or in your head) find that {{2}^{4}=16}.
   Find the exponent key on your calculator and make sure that you can use it correctly.
   Practice by evaluating .
b. Then use the same method to find . (That answer should be 2.157669.)
c. You'll need parentheses around the fraction in the exponent. \sqrt[5]{32}={32}^{{}^{1}\!\!\diagup\!\!{}_{5}\;}=2
d. Again, use parentheses around the fraction in the exponent. \sqrt[6]{79}={79}^{{}^{1}\!\!\diagup\!\!{}_{6}\;}=2.071434

Example 3. Use the constant \pi.

a. Find \pi.
b. b. Find 2\pi
c. c. Evaluate the area of a circle with radius 2: A=\pi{r}^{2}.

Solution:

a. This is the Greek letter "pi" which denotes a number which is the ratio of the circumference of a circle to its diameter and is approximately 3.14. It is used in many geometry formulas involving round objects. Often we need to use it to a greater accuracy than two decimal places. Most scientific calculators have a key for . Have you noticed that your scientific calculator has two different values for most keys? One, called the main value, is labeled on the key itself and the other, called the secondary value, is usually labeled right above it. To get that secondary value, you must press a specific other key on the calculator first. That is the "2nd" key or maybe "Shift" or "Inv". Almost always it is the top left key of the calculator keyboard. On many calculators, the \pi key is a secondary key value, so you'll need to punch that top left key first. When you find that key, you'll be able to see that \pi=3.14159..... The different calculators have different numbers of decimal places.
b. To find 2\pion your calculator, you must learn to use multiplication and the second key in the correct order. You can check your work, of course, by multiplying 2 times 3.14159… by hand to see if your calculator is giving you the correct answer.
c. Notice how to enter these into your calculator so that the operations are done correctly. On one-line-display calculators, you may need to square the radius before multiplying by \pi.

A=\pi{r}^{2}=\pi\cdot{2}^{2}=\text{12.56637061}

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8.2: F1.02- Examples 4-5

Example 4. Negative numbers and the operation of subtraction.

a. 8-6
b. -6+8

Solution: When we do subtraction problems by hand or work with negative numbers, we usually write the negative sign and the subtraction symbol in exactly the same way, so we think of them as the same. But calculators treat them differently. Use your calculator to find 8-6.

Now try to find -6+8. Using the subtraction symbol won't work here. The key you need is probably labeled +/-\,\,\,\,or\,\,\left(-\right). Find that key and experiment with it until you see how to use your calculator to find -6+8.

Most students go through entire algebra courses and never need the negative number key because they handle all the sign parts of the problem mentally. For example, we'd just say -6+8=8-6 and then use the calculator for this resulting problem. But when we learn about trigonometry later in the course, we will need to be able to fully handle negative numbers, so we'll need to use this key.

Discussion: Order of operations.

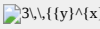
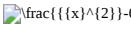
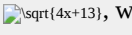
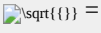
There are three types of mathematical expressions which we write, by hand, without parentheses, but which need parentheses when entering them into a calculator or spreadsheet.

Evaluate each expression	Enter this	Why?
, where x=3	3 ^ (2*3) = or 3\,\,{{y}^{x}} (2*3)=	In the original expression, the placement of the symbols indicates that the exponent is 2 times x. But when we have to just enter symbols one after the other – on the same line – we have to use parentheses to clarify what is in the exponent.
\frac{{{x}^{2}}-6x}{4x+2} , where x=3	(3^2-6*3)/(4*3+2) = or (32-6*3)/(4*3+2) =	In the original expression, the placement of the symbols indicates that the entire numerator is divided by the entire denominator. But when we have to just enter symbols one after the other – on the same line – we have to use parentheses to clarify what is to be divided by what.
\sqrt{4x+13}, where	(4*3+13) = OR 4*3+13 = \sqrt{{}} =	In the original expression, the fact that the expression was completely under the square root symbol made it clear. In the calculator, we have to use parentheses to say that.

Example 5. For each of the expressions above, evaluate it by hand and then evaluate it with your calculator. By hand

, where x=3	\begin{align}&\,\,\,\,{{3}^{2\cdot3}}= {{3}^{6}}=3\cdot3\cdot3\cdot3\cdot3\cdot3\\&=9\cdot3\cdot3\cdot3\cdot3=27\cdot3\cdot3\cdot3=81\cdot3\cdot3=243\cdot3=729\\\end{align}
\frac{{{x}^{2}}-6x}{4x+2}, where x=3	\frac{{{3}^{2}}-6\cdot3}{4\cdot3+2}=\frac{9-18}{12+2}=\frac{-9}{14} Now use your calculator to find this is –0.642857
\sqrt{4x+13}, where x=3	\sqrt{4x+13}=\sqrt{4\cdot3+13}=\sqrt{25}=5

Plug these into your calculator using the expressions above, with parentheses

Evaluate each expression	Enter this
, where 	3 ^ (2*3) = 729 OR  729
 , where 	(3^2-6*3)/(4*3+2) = –0.642857 OR (32-6*3)/(4*3+2) = –0.642857
 , where 	 (4*3+13) = 5 OR 4*3+13 =  = 5

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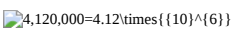
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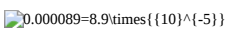
8.3: F1.03- Example 6

Example 6. Scientific notation. Multiply 8,000,000 by 60,000.

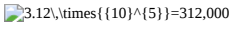
Solution: When you do this on your calculator, you'll get a strange-looking answer with a E in it. You must learn to interpret that answer. That's the way calculators give scientific notation.

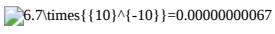
In some algebra classes you learned scientific notation as a shorter way of writing some very large or very small numbers.



and


When using a calculator or spreadsheet we might easily obtain a number that is too big for the display and must be expressed in scientific notation. But calculator displays usually use shorthand for this. On most calculators and spreadsheets, we'll have 3.12 E 05 or 6.7 E–10. These mean

3.12 E 05 = 

6.7 E–10 = 

On your calculator, multiply 8,000,000 by 60,000. What do you get? How would you write it in scientific notation? How would you write it in regular notation? (Answer: 480,000,000,000,= 

)

Review: Additional review of scientific notation is available from the course website.

Going further: Scientific notation is also used to convey the precision of measured values clearly and concisely. We will discuss that in later Topics in this course.

Discussion. When and how much should you round the results of a calculator computation?

Calculators keep more accuracy in calculations than we will probably want to do in our hand calculations. Typically, that is about 12 decimal places for the inexpensive scientific calculators. When computing, it is tempting for students to use the calculator to do each individual operation and then write down that result correct to about three decimal places and then do the next individual operation. This is not considered good practice because if it is done for several steps, then quite a bit of accuracy can be lost.

Good practice in using a calculator is to **do all the calculations in the problem by keeping the intermediate results in the calculator and only round at the end to report the final answer**. That enables us to keep as much accuracy in our result as our original data had and not to introduce inaccuracy as a result of our computation.

However, when formulas are particularly long or complicated, you may need to write down intermediate results in order to better understand the techniques. While that is acceptable to help you make progress, it is important for you to understand that you are losing accuracy. If there is anything important depending on your computed result, you should learn to keep all the computed values in the calculator to produce the final answer.

Discussion. Checking your work

When learning to use the keys on a calculator, use problems that you can easily do by hand or even mentally, so that you can easily check to see whether the calculator has given the correct answer. In fact, using such problems to experiment is one of the main ways people learn to do new things on their calculators. After all, what percentage of time do most of us spend reading the calculator manual? In fact, how many of us can even remember where we put the calculator manual?

Notice that most of the examples so far used simple numbers so that it would be easy for you to do the calculation by hand and check your calculator work. That is always how you should approach learning to use a key on your calculator that you have never used before.

When using the calculator to do messy problems that we wouldn't want to do by hand, it is also important to do some checking. (It is very easy to punch in the wrong numbers, decimals, or operations.) In such cases, you should estimate what the answer will be and then be sure that your final answer on the calculator is reasonably close to your estimate. If it is not, you'll need to determine whether your estimate was wrong or the something went wrong with the calculation in the calculator.

Example 7: A sofa is priced at $887 in the furniture store. The sales tax is 8.25%.

a. Estimate the total amount you'll have to pay with both the cost and the tax.
b. Use your calculator to compute the total amount you'll have to pay.
c. Is your calculator answer close to your estimate?

Solution.

a. Since $887 is not too far from to $1000, let's estimate the cost of the sofa as $1000. 8.25% is pretty close to 8%, so let's compute 8% of $1000.Since 8% of $100 is $8, then 8% of $1000 must be ten times that, so it's about $80. Thus, sales tax amount is pretty close to $80.So how much is the whole bill? It's about $80 more than the cost of the sofa itself. Here we could continue to use the estimate of the sofa as $1000, but it's really closer to $900.

   It's pretty easy to add $900 and $80, so we estimate the total is about $980.

b. Using a calculator, we compute 887 + 0.0825*887 = 960.18.
c. Yes, the calculator answer is pretty close to my estimate, so it seems likely that it is correct.

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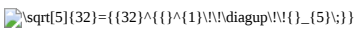
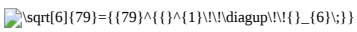
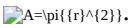
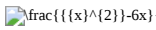
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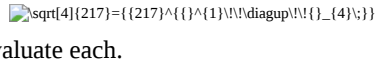
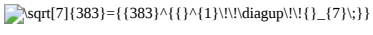
Part I.

As you work through these, make a page of notes of anything you need to remember about how to use your calculator.

1. Evaluate each.
   a. 
   b. 
   c. 
   d. 
   e. 
2. a. Find
   b. Find
   c. Find 
   d. Find 
3. a. Find  .
   b. Find 
   c. Evaluate the area of a circle with radius 2:  .
4. Use a calculator to evaluate each.
   a. 
   b. 
5. Evaluate each of these expressions.
   a. , where 
   b.  , where 
6. Evaluate this expression:  , where 
7. Use a calculator to multiply 8,000,000 by 60,000.
8. A sofa is priced at $887 in the furniture store. The sales tax is 8.25%.
   a. Estimate the total amount you'll have to pay with both the cost and the tax.
   b. Use your calculator to compute the total amount you'll have to pay.
   c. Is your calculator answer close to your estimate?

Part II.

For any problems for which the answers are not whole numbers, round to three decimal places. That will enable you to check the problems without the results being too confusing. For the algebra problems, use the calculator as needed, including in checking the results.

9. Evaluate using your calculator's exponent key and then check it by multiplying by hand.
10. Evaluate using your calculator's exponent key and then check it by multiplying by hand.
11. Evaluate each.
    a.
    b. 
12. Evaluate each.
    a.
    b. 
13. Use the  key on your calculator when evaluating  . Check your work by estimating as 3 and working it in your head.
14. Use the  key on your calculator when evaluating  . Check your work by estimating  as 3 and working it in your head.
15. Use your calculator to evaluate  and then use the negative number key as part of evaluating  . Check your work by doing it by hand.

16. Use your calculator to evaluate $24-35$ and then use the negative number key as part of evaluating $-35+24$. Check your work by doing it by hand.
17. Evaluate each of these expressions:
    a. , where $x=3$
    b. $\frac{{x}^{2}-6x}{4x+2}$, where $x=3$
18. Evaluate each of these expressions:
    a. , where $x=4$
    b. $\frac{3x+15}{5{x}^{2}-11}$, where $x=4$
19. Use your calculator to multiply 2,700,000 by 1,138.3 . Write what your calculator shows and then the answer in standard number notation, not scientific notation. Check your work by estimating the first number as about 3,000,000 and the second number as about 1000. Multiply those by hand and see if this estimate is close to your calculator's answer.
20. Use your calculator to multiply 9,701.3 by 81,200,000. Write what your calculator shows and then the answer in standard number notation, not scientific notation. Check your work by estimating the first number as about 10,000 and the second number as about 80,000,000. Multiply those by hand and see if this estimate is close to your calculator's answer.
21. The price of a car is $24,871 and the tax due on this is 9.3%. Find the total of the price and tax. First estimate it, explaining how you are estimating, and then use a calculator to compute it.
22. The price of a car is $40,373 and the tax due on this is 4.6%. Find the total of the price and tax. First estimate it, explaining how you are estimating, and then use a calculator to compute it.

Work the following problems and check your work as we learned in Topic A. Use a calculator to do the arithmetic easily in the solution of the problem and in checking your answer.

23. Simplify $-12(x-2)$	29. Solve $\frac{12}{m}=\frac{5}{2}$	35. Solve for y: $y-7=3(x-5)$
24. Simplify $7(3m-8)$	30. Solve $\frac{8}{17}=\frac{t}{9}$	36. Solve for *y*: $y-9=13(x-7)$
25. Solve $32y-15=81$	31. Solve $\frac{18}{11}=\frac{38}{w}$	37. 37. Solve for *y*: $y-2=0.12(x-10)$
26. Solve $19=3-7t$	32. Solve $\frac{1.12}{7.2}=\frac{v}{96}$	38. Solve for *y*: $y-10=-1.42(x-20)$
27. Solve $6(3m-11)=32$	33. Solve $\frac{m}{72}=0.6482$	39. 39. Solve for *y*: $y-1.12=-0.063(x-12.72)$
28. Solve $9(2q-7)=12$	34. Solve $\frac{18}{r}=0.938$	

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CHAPTER OVERVIEW

9: Describing Data

9.1: Presenting Categorical Data Graphically
9.2: Presenting Quantitative Data Graphically
9.3: Measures of Central Tendency
9.4: Measures of Variation
9.5: Exercises

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9.1: Presenting Categorical Data Graphically

Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. We usually begin working with categorical data by summarizing the data into a **frequency table.**

Frequency Table

A frequency table is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

Example 1

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total-loss collisions. The data is summarized in the frequency table below.

Color	Frequency
Blue	25
Green	52
Red	41
White	36
Black	39
Grey	23

Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph called a bar graph. In this section we will work with bar graphs that display categorical data; the next section will be devoted to bar graphs that display quantitative data.

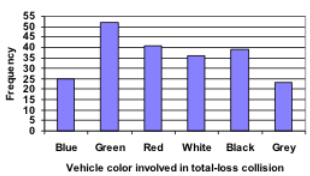
Bar graph

A **bar graph** is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

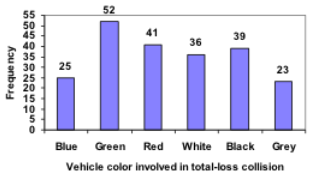
To construct a bar graph, we need to draw a vertical axis and a horizontal axis. The vertical direction will have a scale and measure the frequency of each category; the horizontal axis has no scale in this instance. The construction of a bar chart is most easily described by use of an example.

Example 2

Using our car data from above, note the highest frequency is 52, so our vertical axis needs to go from 0 to 52, but we might as well use 0 to 55, so that we can put a hash mark every 5 units:



Notice that the height of each bar is determined by the frequency of the corresponding color. The horizontal gridlines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs using graph paper, so the gridlines will already be in place, or using technology. Instead of gridlines, we might also list the frequencies at the top of each bar, like this:



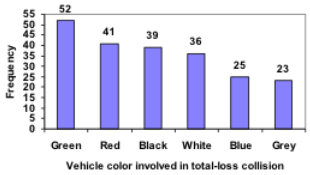
In this case, our chart might benefit from being reordered from largest to smallest frequency values. This arrangement can make it easier to compare similar values in the chart, even without gridlines. When we arrange the categories in decreasing frequency order like this, it is called a **Pareto chart**.

Pareto chart

A **Pareto chart** is a bar graph ordered from highest to lowest frequency

Example 3

Transforming our bar graph from earlier into a Pareto chart, we get:

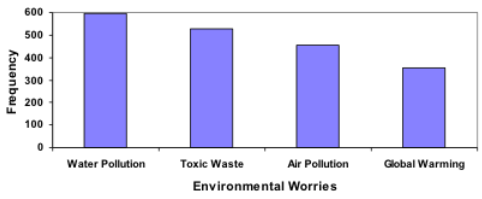


Example 4

In a survey[1], adults were asked whether they personally worried about a variety of environmental concerns. The numbers (out of 1012 surveyed) who indicated that they worried “a great deal” about some selected concerns are summarized below.

Environmental Issue	Frequency
Pollution of drinking water	597
Contamination of soil and water by toxic waste	526
Air pollution	455
Global warming	354

This data could be shown graphically in a bar graph:



To show relative sizes, it is common to use a pie chart.

Pie Chart

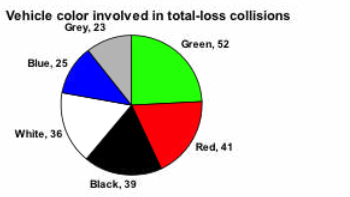
A **pie chart** is a circle with wedges cut of varying sizes marked out like slices of pie or pizza. The relative sizes of the wedges correspond to the relative frequencies of the categories.

Example 5

For our vehicle color data, a pie chart might look like this:



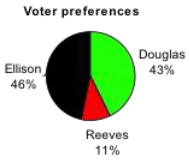
Pie charts can often benefit from including frequencies or relative frequencies (percents) in the chart next to the pie slices. Often having the category names next to the pie slices also makes the chart clearer.



Example 6

The pie chart to the right shows the percentage of voters supporting each candidate running for a local senate seat.

If there are 20,000 voters in the district, the pie chart shows that about 11% of those, about 2,200 voters, support Reeves.



Pie charts look nice, but are harder to draw by hand than bar charts since to draw them accurately we would need to compute the angle each wedge cuts out of the circle, then measure the angle with a protractor. Computers are much better suited to drawing pie charts. Common software programs like Microsoft Word or Excel, OpenOffice.org Write or Calc, or Google Docs are able to create bar graphs, pie charts, and other graph types. There are also numerous online tools that can create graphs.[2]

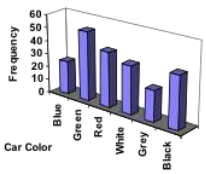
Try it Now 1

Create a bar graph and a pie chart to illustrate the grades on a history exam below.

A: 12 students, B: 19 students, C: 14 students, D: 4 students, F: 5 students

Don’t get fancy with graphs! People sometimes add features to graphs that don’t help to convey their information. For example, 3-dimensional bar charts like the one shown below are usually not as effective as their two-dimensional counterparts.

Here is another way that fanciness can lead to trouble. Instead of plain bars, it is tempting to substitute meaningful images. This type of graph is called a **pictogram**.



Pictogram

A **pictogram** is a statistical graphic in which the size of the picture is intended to represent the frequencies or size of the values being represented.

Example 7



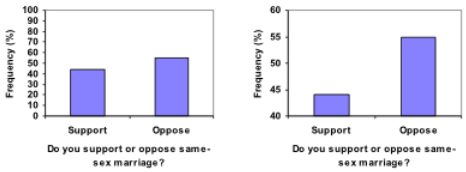
A labor union might produce the graph to the right to show the difference between the average manager salary and the average worker salary.

Looking at the picture, it would be reasonable to guess that the manager salaries is 4 times as large as the worker salaries – the area of the bag looks about 4 times as large. However, the manager salaries are in fact only twice as large as worker salaries, which were reflected in the picture by making the manager bag twice as tall.

Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the vertical axis, representing the least number of cases that could have occurred in a category. Normally, this number should be zero.

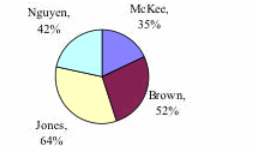
Example 8

Compare the two graphs below showing support for same-sex marriage rights from a poll taken in December 2008[3]. The difference in the vertical scale on the first graph suggests a different story than the true differences in percentages; the second graph makes it look like twice as many people oppose marriage rights as support it.



Try it Now 2

A poll was taken asking people if they agreed with the positions of the 4 candidates for a county office. Does the pie chart present a good representation of this data? Explain.



1. Gallup Poll. March 5-8, 2009. http://www.pollingreport.com/enviro.htm↵
2. For example: http://nces.ed.gov/nceskids/createAgraph/ or http://docs.google.com↵
3. CNN/Opinion Research Corporation Poll. Dec 19-21, 2008, from http://www.pollingreport.com/civil.htm↵

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9.2: Presenting Quantitative Data Graphically

Quantitative, or numerical, data can also be summarized into frequency tables.

Example 9

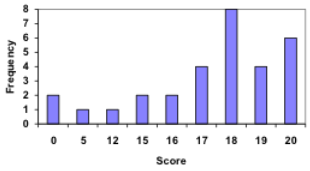
A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are:

19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

These scores could be summarized into a frequency table by grouping like values:

Score	Frequency
0	2
5	1
12	1
15	2
16	2
17	4
18	8
19	4
20	6

Using this table, it would be possible to create a standard bar chart from this summary, like we did for categorical data:



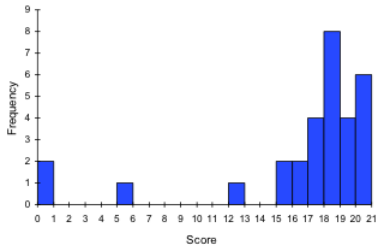
However, since the scores are numerical values, this chart doesn't really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be more correct to treat the horizontal axis as a number line. This type of graph is called a **histogram**.

Histogram

A histogram is like a bar graph, but where the horizontal axis is a number line

Example 10

For the values above, a histogram would look like:



Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. Some people choose to have bars start at ½ values to avoid this ambiguity.

Unfortunately, not a lot of common software packages can correctly graph a histogram. About the best you can do in Excel or Word is a bar graph with no gap between the bars and spacing added to simulate a numerical horizontal axis.

If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table, and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals**.

Class Intervals

Class intervals are groupings of the data. In general, we define class intervals so that:

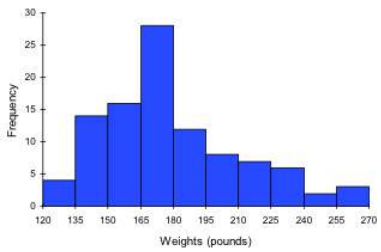
- Each interval is equal in size. For example, if the first class contains values from 120-129, the second class should include values from 130-139.
- We have somewhere between 5 and 20 classes, typically, depending upon the number of data we're working with.

Example 11

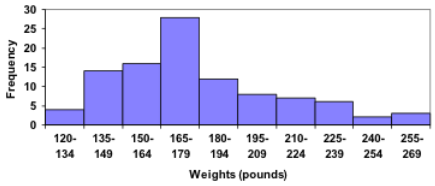
Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a total span of 263-121 = 142. We could create 7 intervals with a width of around 20, 14 intervals with a width of around 10, or somewhere in between. Often time we have to experiment with a few possibilities to find something that represents the data well. Let us try using an interval width of 15. We could start at 121, or at 120 since it is a nice round number.

Interval	Frequency
120 – 134	4
135 – 149	14
150 – 164	16
165 – 179	28
180 – 194	12
195 – 209	8
210 – 224	7
225 – 239	6
240 – 254	2
255 – 269	3

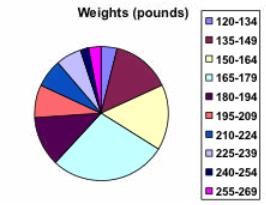
A histogram of this data would look like:



In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.



Other graph types such as pie charts are possible for quantitative data. The usefulness of different graph types will vary depending upon the number of intervals and the type of data being represented. For example, a pie chart of our weight data is difficult to read because of the quantity of intervals we used.



Try it Now 3

The total cost of textbooks for the term was collected from 36 students. Create a histogram for this data.

$140 $160 $160 $165 $180 $220 $235 $240 $250 $260 $280 $285

$285 $285 $290 $300 $300 $305 $310 $310 $315 $315 $320 $320

$330 $340 $345 $350 $355 $360 $360 $380 $395 $420 $460 $460

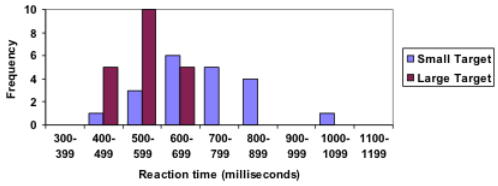
When collecting data to compare two groups, it is desirable to create a graph that compares quantities.

Example 12

The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.

Interval (milliseconds)	Frequency small target	Frequency large target
300-399	0	0
400-499	1	5
500-599	3	10
600-699	6	5
700-799	5	0
800-899	4	0
900-999	0	0
1000-1099	1	0
1100-1199	0	0

One option to represent this data would be a comparative histogram or bar chart, in which bars for the small target group and large target group are placed next to each other.

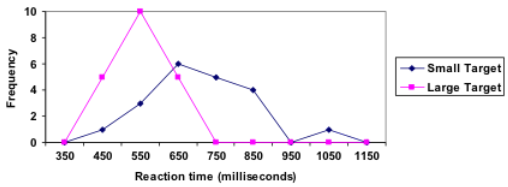


Frequency polygon

An alternative representation is a **frequency polygon**. A frequency polygon starts out like a histogram, but instead of drawing a bar, a point is placed in the midpoint of each interval at height equal to the frequency. Typically the points are connected with straight lines to emphasize the distribution of the data.

Example 13

This graph makes it easier to see that reaction times were generally shorter for the larger target, and that the reaction times for the smaller target were more spread out.



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9.3: Measures of Central Tendency

It is often desirable to use a few numbers to summarize a distribution. One important aspect of a distribution is where its center is located. Measures of central tendency are discussed first. A second aspect of a distribution is how spread out it is. In other words, how much the data in the distribution vary from one another. The second section describes measures of variability.

Let's begin by trying to find the most "typical" value of a data set.

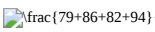
Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average".

Mean

The **mean** of a set of data is the sum of the data values divided by the number of values.

Example 14

Marci's exam scores for her last math class were: 79, 86, 82, 94. The mean of these values would be:

 . Typically we round means to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3.

Example 15

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20

20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get 634/31 = 20.4516. It would be appropriate to round this to 20.5.

It would be most correct for us to report that "The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes," but it is not uncommon to see the more casual word "average" used in place of "mean."

Try it Now 4

The price of a jar of peanut butter at 5 stores was: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the mean price.

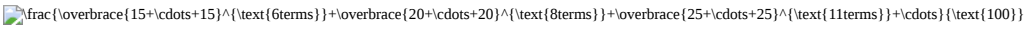
Example 16

The one hundred families in a particular neighborhood are asked their annual household income, to the nearest $5 thousand dollars. The results are summarized in a frequency table below.

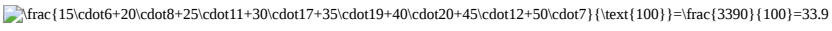
Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12

50	7

Calculating the mean by hand could get tricky if we try to type in all 100 values:



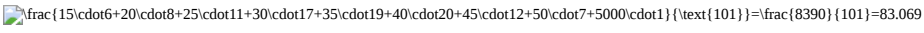
We could calculate this more easily by noticing that adding 15 to itself six times is the same as = 90. Using this simplification, we get



The mean household income of our sample is 33.9 thousand dollars ($33,900).

Example 17

Extending off the last example, suppose a new family moves into the neighborhood example that has a household income of $5 million ($5000 thousand). Adding this to our sample, our mean is now:



While 83.1 thousand dollars ($83,069) is the correct mean household income, it no longer represents a “typical” value.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



If we graph our household data, the $5 million data value is so far out to the right that the mean has to adjust up to keep things in balance



For this reason, when working with data that have **outliers** – values far outside the primary grouping – it is common to use a different measure of center, the **median**.

Median

The **median** of a set of data is the value in the middle when the data is in order

To find the median, begin by listing the data in order from smallest to largest, or largest to smallest.

If the number of data values, *N*, is odd, then the median is the middle data value. This value can be found by rounding *N*/2 up to the next whole number.

If the number of data values is even, there is no one middle value, so we find the mean of the two middle values (values *N*/2 and *N*/2 + 1)

Example 18

Returning to the football touchdown data, we would start by listing the data in order. Luckily, it was already in decreasing order, so we can work with it without needing to reorder it first.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20

20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Since there are 31 data values, an odd number, the median will be the middle number, the 16th data value (31/2 = 15.5, round up to 16, leaving 15 values below and 15 above). The 16th data value is 20, so the median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Example 19

Find the median of these quiz scores: 5 10 8 6 4 8 2 5 7 7

We start by listing the data in order: 2 4 5 5 6 7 7 8 8 10

Since there are 10 data values, an even number, there is no one middle number. So we find the mean of the two middle numbers, 6 and 7, and get (6+7)/2 = 6.5.

The median quiz score was 6.5.

Try it Now 5

The price of a jar of peanut butter at 5 stores were: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the median price.

Example 20

Let us return now to our original household income data

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Here we have 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values – the 50th and 51st data values. To find these, we start counting up from the bottom:

There are 6 data values of $15, so	Values 1 to 6 are $15 thousand
The next 8 data values are $20, so	Values 7 to (6+8)=14 are $20 thousand
The next 11 data values are $25, so	Values 15 to (14+11)=25 are $25 thousand
The next 17 data values are $30, so	Values 26 to (25+17)=42 are $30 thousand
The next 19 data values are $35, so	Values 43 to (42+19)=61 are $35 thousand

From this we can tell that values 50 and 51 will be $35 thousand, and the mean of these two values is $35 thousand. The median income in this neighborhood is $35 thousand.

Example 21

If we add in the new neighbor with a $5 million household income, then there will be 101 data values, and the 51st value will be the median. As we discovered in the last example, the 51st value is $35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

In addition to the mean and the median, there is one other common measurement of the "typical" value of a data set: the **mode**.

Mode

The **mode** is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

Example 22

In our vehicle color survey, we collected the data

For this data, Green is the mode, since it is the data value that occurred the most frequently.

It is possible for a data set to have more than one mode if several categories have the same frequency, or no modes if each every category occurs only once.

Try it Now 6

Reviewers were asked to rate a product on a scale of 1 to 5. Find

a. The mean rating

b. The median rating

c. The mode rating

Rating	Frequency
1	4
2	8
3	7
4	3
5	1

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9.4: Measures of Variation

Consider these three sets of quiz scores:

Section A: 5 5 5 5 5 5 5 5 5 5

Section B: 0 0 0 0 0 10 10 10 10 10

Section C: 4 4 4 5 5 5 5 6 6 6

All three of these sets of data have a mean of 5 and median of 5, yet the sets of scores are clearly quite different. In section A, everyone had the same score; in section B half the class got no points and the other half got a perfect score, assuming this was a 10-point quiz. Section C was not as consistent as section A, but not as widely varied as section B.

In addition to the mean and median, which are measures of the "typical" or "middle" value, we also need a measure of how "spread out" or varied each data set is.

There are several ways to measure this "spread" of the data. The first is the simplest and is called the **range**.

Range

The range is the difference between the maximum value and the minimum value of the data set.

Example 23

Using the quiz scores from above,

For section A, the range is 0 since both maximum and minimum are 5 and 5 – 5 = 0

For section B, the range is 10 since 10 – 0 = 10

For section C, the range is 2 since 6 – 4 = 2

In the last example, the range seems to be revealing how spread out the data is. However, suppose we add a fourth section, Section D, with scores 0 5 5 5 5 5 5 5 5 10.

This section also has a mean and median of 5. The range is 10, yet this data set is quite different than Section B. To better illuminate the differences, we'll have to turn to more sophisticated measures of variation.

Standard deviation

The standard deviation is a measure of variation based on measuring how far each data value deviates, or is different, from the mean. A few important characteristics:

- Standard deviation is always positive. Standard deviation will be zero if all the data values are equal, and will get larger as the data spreads out.
- Standard deviation has the same units as the original data.
- Standard deviation, like the mean, can be highly influenced by outliers.

Using the data from section D, we could compute for each data value the difference between the data value and the mean:

data value	deviation: data value – mean
0	0-5 = -5
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0

5	5-5 = 0
5	5-5 = 0
10	10-5 = 5

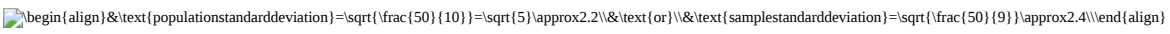
We would like to get an idea of the "average" deviation from the mean, but if we find the average of the values in the second column the negative and positive values cancel each other out (this will always happen), so to prevent this we square every value in the second column:

data value	deviation: data value – mean	deviation squared
0	0-5 = -5	(-5)2 = 25
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
5	5-5 = 0	02 = 0
10	10-5 = 5	(5)2 = 25

We then add the squared deviations up to get 25 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 25 = 50. Ordinarily we would then divide by the number of scores, *n*, (in this case, 10) to find the mean of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by $n - 1$ (in this case, $10 - 1 = 9$).[1]

So in our example, we would have 50/10 = 5 if section D represents a population and 50/9 = about 5.56 if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

Variance can be a useful statistical concept, but note that the units of variance in this instance would be points-squared since we squared all of the deviations. What are points-squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:



If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data, as we have done above.

To compute standard deviation:

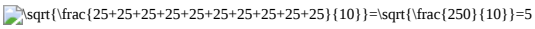
1. Find the deviation of each data from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by *n*, the number of data values, if the data represents a whole population; divide by $n - 1$ if the data is from a sample.
5. Compute the square root of the result.

Example 24

Computing the standard deviation for Section B above, we first calculate that the mean is 5. Using a table can help keep track of your computations for the standard deviation:

data value	deviation: data value – mean	deviation squared
0	0-5 = -5	(-5)2 = 25
0	0-5 = -5	(-5)2 = 25
0	0-5 = -5	(-5)2 = 25
0	0-5 = -5	(-5)2 = 25
0	0-5 = -5	(-5)2 = 25
10	10-5 = 5	(5)2 = 25
10	10-5 = 5	(5)2 = 25
10	10-5 = 5	(5)2 = 25
10	10-5 = 5	(5)2 = 25
10	10-5 = 5	(5)2 = 25

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:



Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out.

For comparison, the standard deviations of all four sections are:

Section A: 5 5 5 5 5 5 5 5 5 5	Standard deviation: 0
Section B: 0 0 0 0 0 10 10 10 10 10	Standard deviation: 5
Section C: 4 4 4 5 5 5 5 6 6 6	Standard deviation: 0.8
Section D: 0 5 5 5 5 5 5 5 5 10	Standard deviation: 2.2

Try it Now 7

The price of a jar of peanut butter at 5 stores were: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the standard deviation of the prices.

Where standard deviation is a measure of variation based on the mean, **quartiles** are based on the median.

Quartiles

Quartiles are values that divide the data in quarters.

The first quartile (Q1) is the value so that 25% of the data values are below it; the third quartile (Q3) is the value so that 75% of the data values are below it. You may have guessed that the second quartile is the same as the median, since the median is the value so that 50% of the data values are below it.

This divides the data into quarters; 25% of the data is between the minimum and Q1, 25% is between Q1 and the median, 25% is between the median and Q3, and 25% is between Q3 and the maximum value

While quartiles are not a 1-number summary of variation like standard deviation, the quartiles are used with the median, minimum, and maximum values to form a **5 number summary** of the data.

Five number summary

The five number summary takes this form:

Minimum, Q1, Median, Q3, Maximum

To find the first quartile, we need to find the data value so that 25% of the data is below it. If n is the number of data values, we compute a locator by finding 25% of n. If this locator is a decimal value, we round up, and find the data value in that position. If the locator is a whole number, we find the mean of the data value in that position and the next data value. This is identical to the process we used to find the median, except we use 25% of the data values rather than half the data values as the locator.

To find the first quartile, Q1

Begin by ordering the data from smallest to largest

Compute the locator: $L = 0.25n$

If L is a decimal value:

Round up to $L+$

Use the data value in the $L+$th position

If L is a whole number:

Find the mean of the data values in the Lth and $L+1$th positions.

To find the third quartile, Q3

Use the same procedure as for Q1, but with locator: $L = 0.75n$

Examples should help make this clearer.

Example 25

Suppose we have measured 9 females and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70 72

To find the first quartile we first compute the locator: 25% of 9 is $L = 0.25(9) = 2.25$. Since this value is not a whole number, we round up to 3. The first quartile will be the third data value: 62 inches.

To find the third quartile, we again compute the locator: 75% of 9 is $0.75(9) = 6.75$. Since this value is not a whole number, we round up to 7. The third quartile will be the seventh data value: 69 inches.

Example 26

Suppose we had measured 8 females and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70

To find the first quartile we first compute the locator: 25% of 8 is $L = 0.25(8) = 2$. Since this value *is* a whole number, we will find the mean of the 2nd and 3rd data values: (60+62)/2 = 61, so the first quartile is 61 inches.

The third quartile is computed similarly, using 75% instead of 25%. $L = 0.75(8) = 6$. This is a whole number, so we will find the mean of the 6th and 7th data values: (67+69)/2 = 68, so Q3 is 68.

Note that the median could be computed the same way, using 50%.

The 5-number summary combines the first and third quartile with the minimum, median, and maximum values.

Example 27

For the 9 female sample, the median is 66, the minimum is 59, and the maximum is 72. The 5 number summary is: 59, 62, 66, 69, 72.

For the 8 female sample, the median is 65, the minimum is 59, and the maximum is 70, so the 5 number summary would be: 59, 61, 65, 68, 70.

Example 28

Returning to our quiz score data. In each case, the first quartile locator is 0.25(10) = 2.5, so the first quartile will be the 3rd data value, and the third quartile will be the 8th data value. Creating the five-number summaries:

Section and data	5-number summary
Section A: 5 5 5 5 5 5 5 5 5 5	5, 5, 5, 5, 5
Section B: 0 0 0 0 0 10 10 10 10 10	0, 0, 5, 10, 10
Section C: 4 4 4 5 5 5 6 6 6	4, 4, 5, 6, 6
Section D: 0 5 5 5 5 5 5 5 5 10	0, 5, 5, 5, 10

Of course, with a relatively small data set, finding a five-number summary is a bit silly, since the summary contains almost as many values as the original data.

Try it Now 8

The total cost of textbooks for the term was collected from 36 students. Find the 5 number summary of this data.

$140 $160 $160 $165 $180 $220 $235 $240 $250 $260 $280 $285

$285 $285 $290 $300 $300 $305 $310 $310 $315 $315 $320 $320

$330 $340 $345 $350 $355 $360 $360 $380 $395 $420 $460 $460

Example 29

Returning to the household income data from earlier, create the five-number summary.

Income (thousands of dollars)	**Frequency**
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

By adding the frequencies, we can see there are 100 data values represented in the table. In Example 20, we found the median was $35 thousand. We can see in the table that the minimum income is $15 thousand, and the maximum is $50 thousand.

To find Q1, we calculate the locator: $L = 0.25(100) = 25$. This is a whole number, so Q1 will be the mean of the 25th and 26th data values.

Counting up in the data as we did before,

There are 6 data values of $15, so Values 1 to 6 are $15 thousand

The next 8 data values are $20, so Values 7 to (6+8)=14 are $20 thousand

The next 11 data values are $25, so Values 15 to (14+11)=25 are $25 thousand

The next 17 data values are $30, so Values 26 to (25+17)=42 are $30 thousand

The 25th data value is $25 thousand, and the 26th data value is $30 thousand, so Q1 will be the mean of these: (25 + 30)/2 = $27.5 thousand.

To find Q3, we calculate the locator: $L = 0.75(100) = 75$. This is a whole number, so Q3 will be the mean of the 75th and 76th data values. Continuing our counting from earlier,

The next 19 data values are $35, so Values 43 to (42+19)=61 are $35 thousand

The next 20 data values are $40, so Values 61 to (61+20)=81 are $40 thousand

Both the 75th and 76th data values lie in this group, so Q3 will be $40 thousand.

Putting these values together into a five-number summary, we get: 15, 27.5, 35, 40, 50

Note that the 5 number summary divides the data into four intervals, each of which will contain about 25% of the data. In the previous example, that means about 25% of households have income between $40 thousand and $50 thousand.

For visualizing data, there is a graphical representation of a 5-number summary called a **box plot**, or box and whisker graph.

Box plot

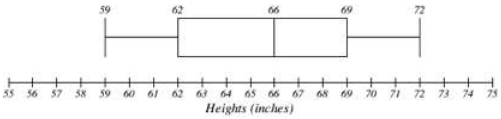
A **box plot** is a graphical representation of a five-number summary.

To create a box plot, a number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. "Whiskers" are extended out to the minimum and maximum values.

Example 30

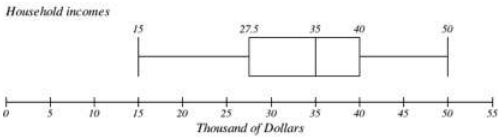
The box plot below is based on the 9 female height data with 5 number summary:

59, 62, 66, 69, 72.



Example 31

The box plot below is based on the household income data with 5 number summary:



15, 27.5, 35, 40, 50

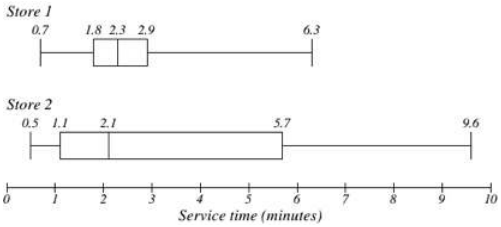
Try it Now 9

Create a boxplot based on the textbook price data from the last Try it Now.

Box plots are particularly useful for comparing data from two populations.

Example 32

The box plot of service times for two fast-food restaurants is shown below.



While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data.

At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry? That depends upon your opinions about luck – 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes.

Example 33

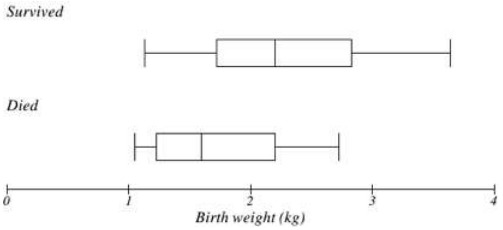
The boxplot below is based on the birth weights of infants with severe idiopathic respiratory distress syndrome (SIRDS)[2]. The boxplot is separated to show the birth weights of infants who survived and those that did not.

Comparing the two groups, the boxplot reveals that the birth weights of the infants that died appear to be, overall, smaller than the weights of infants that survived. In fact, we can see that the median birth weight of infants that survived is the same as the third quartile of the infants that died.

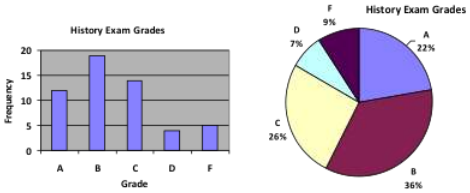
Similarly, we can see that the first quartile of the survivors is larger than the median weight of those that died, meaning that over 75% of the survivors had a birth weight larger than the median birth weight of those that died.

Looking at the maximum value for those that died and the third quartile of the survivors, we can see that over 25% of the survivors had birth weights higher than the heaviest infant that died.

The box plot gives us a quick, albeit informal, way to determine that birth weight is quite likely linked to survival of infants with SIRDS.



Try it Now Answers



1.
2. While the pie chart accurately depicts the relative size of the people agreeing with each candidate, the chart is confusing, since usually percents on a pie chart represent the percentage of the pie the slice represents.
3. Using a class intervals of size 55, we can group our data into six intervals:

Cost interval	Frequency
$140-194	5
$195-249	3
$250-304	9
$305-359	12
$360-414	4
$415-469	3

We can use the frequency distribution to generate the histogram

4. Adding the prices and dividing by 5 we get the mean price: $3.682
5. First we put the data in order: $3.29, $3.59, $3.75, $3.79, $3.99. Since there are an odd number of data, the median will be the middle value, $3.75.
6. There are 23 ratings.
   a. The mean is $\frac{1\cdot4+2\cdot8+3\cdot7+4\cdot3+5\cdot1}{23} \approx 2.5$

b. There are 23 data values, so the median will be the 12th data value. Ratings of 1 are the first 4 values, while a rating of 2 are the next 8 values, so the 12th value will be a rating of 2. The median is 2.
c. The mode is the most frequent rating. The mode rating is 2.

7. Earlier we found the mean of the data was $3.682.

data value	deviation: data value – mean	deviation squared
3.29	3.29 – 3.682 = -0.391	0.153664
3.59	3.59 – 3.682 = -0.092	0.008464
3.79	3.79 – 3.682 = 0.108	0.011664
3.75	3.75 – 3.682 = 0.068	0.004624
3.99	3.99 – 3.682 = 0.308	0.094864

This data is from a sample, so we will add the squared deviations, divide by 4, the number of data values minus 1, and compute the square root:  \sqrt{\frac{0.153664+0.008464+0.011664+0.004624+0.094864}{4}}\approx$0.261

8. The data is already in order, so we don't need to sort it first.The minimum value is $140 and the maximum is $460.There are 36 data values so n = 36. n/2 = 18, which is a whole number, so the median is the mean of the 18th and 19th data values, $305 and $310. The median is $307.50.

   To find the first quartile, we calculate the locator, L = 0.25(36) = 9. Since this is a whole number, we know Q1 is the mean of the 9th and 10th data values, $250 and $260. Q1 = $255.

   To find the third quartile, we calculate the locator, L = 0.75(36) = 27. Since this is a whole number, we know Q3 is the mean of the 27th and 28th data values, $345 and $350. Q3 = $347.50.

   The 5 number summary of this data is: $140, $255, $307.50, $347.50, $460

9. Boxplot of textbook costs

---

1. The reason we do this is highly technical, but we can see how it might be useful by considering the case of a small sample from a population that contains an outlier, which would increase the average deviation: the outlier very likely won't be included in the sample, so the mean deviation of the sample would underestimate the mean deviation of the population; thus we divide by a slightly smaller number to get a slightly bigger average deviation. ↵
2. van Vliet, P.K. and Gupta, J.M. (1973) Sodium bicarbonate in idiopathic respiratory distress syndrome. Arch. Disease in Childhood, 48, 249–255. As quoted on http://openlearn.open.ac.uk/mod/oucontent/view.php?id=398296&section=1.1.3↵

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9.5: Exercises

Skills

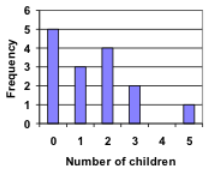
1. The table below shows scores on a Math test.
   a. Complete the frequency table for the Math test scores
   b. Construct a histogram of the data
   c. Construct a pie chart of the data

80	50	50	90	70	70	100	60	70	80	70	50
90	100	80	70	30	80	80	70	100	60	60	50

2. A group of adults where asked how many cars they had in their household
   a. Complete the frequency table for the car number data
   b. Construct a histogram of the data
   c. Construct a pie chart of the data

1	4	2	2	1	2	3	3	1	4	2	2
1	2	1	3	2	2	1	2	1	1	1	2

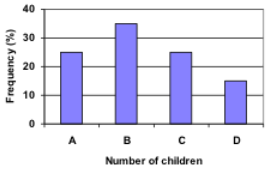
3. A group of adults were asked how many children they have in their families. The bar graph to the right shows the number of adults who indicated each number of children.
   a. How many adults where questioned?
   b. What percentage of the adults questioned had 0 children?



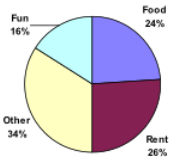
4. Jasmine was interested in how many days it would take an order from Netflix to arrive at her door. The graph below shows the data she collected.
   a. How many movies did she order?
   b. What percentage of the movies arrived in one day?



5. The bar graph below shows the *percentage* of students who received each letter grade on their last English paper. The class contains 20 students. What number of students earned an A on their paper?



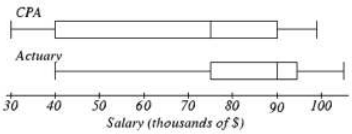
6. Kori categorized her spending for this month into four categories: Rent, Food, Fun, and Other. The percents she spent in each category are pictured here. If she spent a total of $2600 this month, how much did she spend on rent?



7. A group of diners were asked how much they would pay for a meal. Their responses were: $7.50, $8.25, $9.00, $8.00, $7.25, $7.50, $8.00, $7.00.
    1. Find the mean
    2. Find the median
    3. Write the 5-number summary for this data
8. You recorded the time in seconds it took for 8 participants to solve a puzzle. The times were: 15.2, 18.8, 19.3, 19.7, 20.2, 21.8, 22.1, 29.4.
    1. Find the mean
    2. Find the median
    3. Write the 5-number summary for this data
9. Refer back to the histogram from question #3.
    1. Compute the mean number of children for the group surveyed
    2. Compute the median number of children for the group surveyed
    3. Write the 5-number summary for this data
    4. Create box plot
10. Refer back to the histogram from question #4.
    1. Computer the mean number of shipping days
    2. Compute the median number of shipping days
    3. Write the 5-number summary for this data
    4. Create box plot

Concepts

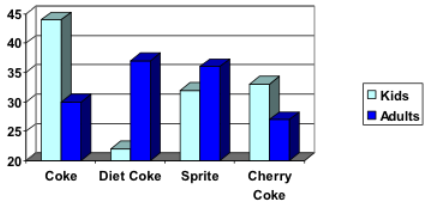
11. The box plot below shows salaries for Actuaries and CPAs. Kendra makes the median salary for an Actuary. Kelsey makes the first quartile salary for a CPA. Who makes more money? How much more?



12. Referring to the boxplot above, what percentage of actuaries makes more than the median salary of a CPA?

Exploration

13. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 40 AIDS patients from the start of treatment until their deaths. The following data (in months) are collected.Researcher 1: 3; 4; 11; 15; 16; 17; 22; 44; 37; 16; 14; 24; 25; 15; 26; 27; 33; 29; 35; 44; 13; 21; 22; 10; 12; 8; 40; 32; 26; 27; 31; 34; 29; 17; 8; 24; 18; 47; 33; 34Researcher 2: 3; 14; 11; 5; 16; 17; 28; 41; 31; 18; 14; 14; 26; 25; 21; 22; 31; 2; 35; 44; 23; 21; 21; 16; 12; 18; 41; 22; 16; 25; 33; 34; 29; 13; 18; 24; 23; 42; 33; 29
    a. Create comparative histograms of the data
    b. Create comparative boxplots of the data
14. A graph appears below showing the number of adults and children who prefer each type of soda. There were 130 adults and kids surveyed. Discuss some ways in which the graph below could be improved



15. Make up three data sets with 5 numbers each that have:
    a. the same mean but different standard deviations.
    b. the same mean but different medians.
    c. the same median but different means.
16. A sample of 30 distance scores measured in yards has a mean of 7, a variance of 16, and a standard deviation of 4.
    a. You want to convert all your distances from yards to feet, so you multiply each score in the sample by 3. What are the new mean, median, variance, and standard deviation?
    b. You then decide that you only want to look at the distance past a certain point. Thus, after multiplying the original scores by 3, you decide to subtract 4 feet from each of the scores. Now what are the new mean, median, variance, and standard deviation?
17. In your class, design a poll on a topic of interest to you and give it to the class.
    a. Summarize the data, computing the mean and five-number summary.
    b. Create a graphical representation of the data.
    c. Write several sentences about the topic, using your computed statistics as evidence in your writing.

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CHAPTER OVERVIEW

10: Sets

10.1: Basics
10.2: Union, Intersection, and Complement
10.3: Venn Diagrams
10.4: Cardinality
10.5: Exercises

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10.1: Basics

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1

Some examples of sets defined by describing the contents:

a) The set of all even numbers

b) The set of all books written about travel to Chile

Some examples of sets defined by listing the elements of the set:

a) {1, 3, 9, 12}

b) {red, orange, yellow, green, blue, indigo, purple}

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}.

Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol $\in$ means "is an element of".

A set that contains no elements, { }, is called the **empty set** and is notated $\varnothing$

Example 2

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we'd write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A, but may not contain all the elements of A.

If B is a subset of A, we write $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set – it contains fewer elements.

If B is a proper subset of A, we write $B \subset A$

Example 3

Consider these three sets

A = the set of all even numbers $B = \{2, 4, 6\}$ $C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of B is also an even number, so is an element of A.

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.

It is also true that $B \subset C$.

C is not a subset of A, since C contains an element, 3, that is not contained in A

Example 4

Suppose a set contains the plays *Much Ado About Nothing*, *MacBeth*, and *A Midsummer's Night Dream*. What is a larger set this might be a subset of?

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try it Now 1

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

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10.2: Union, Intersection, and Complement

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets).

The union is notated $A \cup B$.

More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)

The **intersection** of two sets contains only the elements that are in both sets.

The intersection is notated $A \cap B$.

More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$

The **complement** of a set A contains everything that is *not* in the set A.

The complement is notated A', or Ac, or sometimes $\sim A$.

Example 5

Consider the sets: A = {red, green, blue} B = {red, yellow, orange}

C = {red, orange, yellow, green, blue, purple}

a) Find $A \cup B$

The union contains all the elements in either set: $A \cup B$ = {red, green, blue, yellow, orange}

Notice we only list red once.

b) Find $A \cap B$

The intersection contains all the elements in both sets: $A \cap B$ = {red}

c) Find $Ac \cap C$

Here we're looking for all the elements that are *not* in set A and are also in C.

$Ac \cap C$ = {orange, yellow, purple}

Try it Now 2

Using the sets from the previous example, find $A \cup C$ and $Bc \cap A$

Notice that in the example above, it would be hard to just ask for Ac, since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so Ac contains all the elements in the universal set that are not in A.

Example 6

a) If we were discussing searching for books, the universal set might be all the books in the library.

b) If we were grouping your Facebook friends, the universal set would be all your Facebook friends.

c) If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 7

Suppose the universal set is U = all whole numbers from 1 to 9. If A = {1, 2, 4}, then

Ac = {3, 5, 6, 7, 8, 9}.

As we saw earlier with the expression $Ac \bigcap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an order of operations.

Example 8

Suppose H = {cat, dog, rabbit, mouse}, F = {dog, cow, duck, pig, rabbit}

W = {duck, rabbit, deer, frog, mouse}

a) Find $(H \bigcap F) \bigcup W$

We start with the intersection: $H \bigcap F$ = {dog, rabbit}

Now we union that result with W: $(H \bigcap F) \bigcup W$ = {dog, duck, rabbit, deer, frog, mouse}

b) Find $H \bigcap (F \bigcup W)$

We start with the union: $F \bigcup W$ = {dog, cow, rabbit, duck, pig, deer, frog, mouse}

Now we intersect that result with H: $H \bigcap (F \bigcup W)$ = {dog, rabbit, mouse}

c) Find $(H \bigcap F)c \bigcap W$

We start with the intersection: $H \bigcap F$ = {dog, rabbit}

Now we want to find the elements of W that are *not* in $H \bigcap F$

$(H \bigcap F)c \bigcap W$ = {duck, deer, frog, mouse}

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10.3: Venn Diagrams

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations now called **Venn Diagrams**.

Venn Diagram

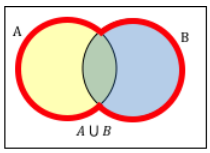
A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

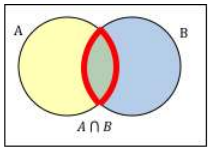
Example 9

Create Venn diagrams to illustrate $A \bigcup B$, $A \bigcap B$, and $Ac \bigcap B$

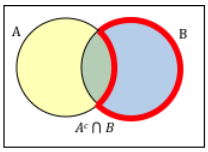
$A \bigcup B$ contains all elements in *either* set.



$A \bigcap B$ contains only those elements in both sets – in the overlap of the circles.



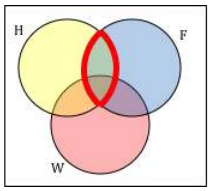
Ac will contain all elements *not* in the set A. $A^c \bigcap B$ will contain the elements in set *B* that are not in set *A*.



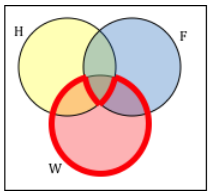
Example 10

Use a Venn diagram to illustrate $(H \bigcap F)^c \bigcap W$

We'll start by identifying everything in the set $H \bigcap F$



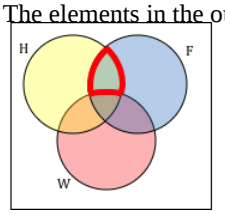
Now, $(H \bigcap F)c \bigcap W$ will contain everything *not* in the set identified above that is also in set *W*.



Example 11

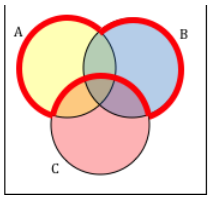
Create an expression to represent the outlined part of the Venn diagram shown.

The elements in the outlined set *are* in sets H and F, but are not in set W. So we could represent this set as $H \cap F \cap W^c$



Try it Now 3

Create an expression to represent the outlined portion of the Venn diagram shown



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10.4: Cardinality

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality

The number of elements in a set is the cardinality of that set.

The cardinality of the set A is often notated as $|A|$ or $n(A)$

Example 12

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.

What is the cardinality of B? $A \bigcup B$, $A \bigcap B$?

The cardinality of B is 4, since there are 4 elements in the set.

The cardinality of $A \bigcup B$ is 7, since $A \bigcup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.

The cardinality of $A \bigcap B$ is 3, since $A \bigcap B = \{2, 4, 6\}$, which contains 3 elements.

Example 13

What is the cardinality of P = the set of English names for the months of the year?

The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

Example 14

A survey asks 200 people "What beverage do you drink in the morning", and offers choices:

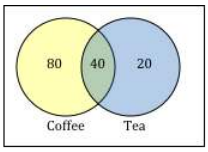
- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

200 – 20 – 80 – 40 = 60 people who drink neither.



Example 15

A survey asks: Which online services have you used in the last month:

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \bigcup T$ is not simply 70% + 40%, since that would count those who use both services twice. To find the cardinality of $F \bigcup T$, we can add the cardinality of F and the cardinality of T, then subtract those in intersection that we've counted twice. In symbols,

$n(F \bigcup T) = n(F) + n(T) – n(F \bigcap T)$

$n(F \bigcup T) = 70\% + 40\% – 20\% = 90\%$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \bigcup T)c$. Since the universal set contains 100% of people and the cardinality of $F \bigcup T = 90\%$, the cardinality of $(F \bigcup T)c$ must be the other 10%.

The previous example illustrated two important properties

Cardinality properties

$n(A \bigcup B) = n(A) + n(B) – n(A \bigcap B)$

$n(Ac) = n(U) – n(A)$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

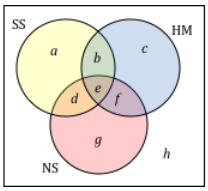
$n(A \bigcap B) = n(A) + n(B) – n(A \bigcup B)$

Example 16

Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course	26 were taking a HM course
19 were taking a NS course	9 were taking SS and HM
7 were taking SS and NS	10 were taking HM and NS
3 were taking all three	7 were taking none

How many students are only taking a SS course?



It might help to look at a Venn diagram.

From the given data, we know that there are

3 students in region e and

7 students in region h.

Since 7 students were taking a SS and NS course, we know that $n(d) + n(e) = 7$. Since we know there are 3 students in region 3, there must be

$7 – 3 = 4$ students in region d.

Similarly, since there are 10 students taking HM and NS, which includes regions e and f, there must be

$10 – 3 = 7$ students in region f.

Since 9 students were taking SS and HM, there must be $9 – 3 = 6$ students in region b.

Now, we know that 21 students were taking a SS course. This includes students from regions a, b, d, and e. Since we know the number of students in all but region a, we can determine that $21 – 6 – 4 – 3 = 8$ students are in region a.

8 students are taking only a SS course.

Try it Now 4

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs

44 believed in ghosts

25 believed in Bigfoot

10 believed in UFOs and ghosts

8 believed in ghosts and Bigfoot

5 believed in UFOs and Bigfoot

2 believed in all three

How many people surveyed believed in at least one of these things?

Try it Now Answers

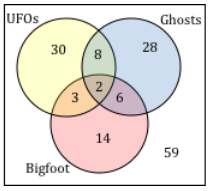
1. There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

2. $A \bigcup C$ = {red, orange, yellow, green, blue purple}

$Bc \bigcap A$ = {green, blue}

3. $A \bigcup B \bigcap Cc$

4. Starting with the intersection of all three circles, we work our way out. Since 10 people believe in UFOs and Ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and Ghosts. We work our way out, filling in all the regions. Once we have, we can add up all those regions, getting 91 people in the union of all three sets. This leaves 150 – 91 = 59 who believe in none.



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10.5: Exercises

1. List out the elements of the set “The letters of the word Mississippi”
2. List out the elements of the set “Months of the year”
3. Write a verbal description of the set {3, 6, 9}
4. Write a verbal description of the set {a, i, e, o, u}
5. Is {1, 3, 5} a subset of the set of odd integers?
6. Is {A, B, C} a subset of the set of letters of the alphabet?

For problems 7-12, consider the sets below, and indicate if each statement is true or false.

7. A = {1, 2, 3, 4, 5} B = {1, 3, 5} C = {4, 6} U = {numbers from 0 to 10}
8. $3 \in B$ 8. $5 \in C$
9. $B \subset A$
10. $C \subset A$
11. $C \subset B$
12. $C \subset D$

Using the sets from above, and treating U as the Universal set, find each of the following:

13. $A \cup B$
14. $A \cup C$
15. $A \cap C$
16. $B \cap C$
17. Ac
18. Bc

Let D = {b, a, c, k}, E = {t, a, s, k}, F = {b, a, t, h}. Using these sets, find the following:

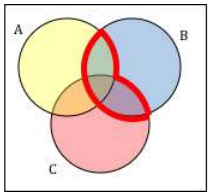
19. $Dc \cap E$
20. $Fc \cap D$
21. $(D \cap E) \cup F$
22. $D \cap (E \cup F)$
23. $(F \cap E)c \cap D$
24. $(D \cup E)c \cap F$

Create a Venn diagram to illustrate each of the following:

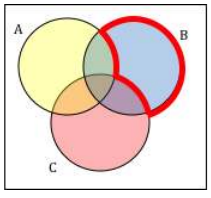
25. $(F \cap E) \cup D$
26. $(D \cup E)c \cap F$
27. $(F\,c \cap E\,c) \cap D$
28. $(D \cup E) \cup F$

Write an expression for the shaded region.

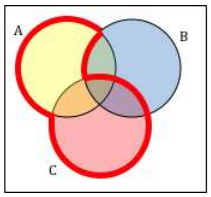
29.



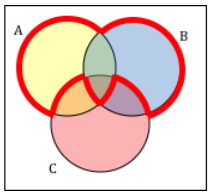
30.



31.



32.



Let A = {1, 2, 3, 4, 5} B = {1, 3, 5} C = {4, 6}. Find the cardinality of the given set.

33. n(*A*)
34. n(*B*)
35. n($A \cup C$)
36. n($A \cap C$)

The Venn diagram here shows the cardinality of each set. Use this in 37-40 to find the cardinality of given set.

37. n($A \cap C$)
38. n($B \cup C$)
39. n($A \cap B \cap Cc$)
40. n($A \cap Bc \cap C$)
41. If n(G) = 20, n(H) = 30, n($G \cap H$) = 5, find n($G \cup H$)
42. If n(G) = 5, n(H) = 8, n($G \cap H$) = 4, find n($G \cup H$)
43. A survey was given asking whether they watch movies at home from Netflix, Redbox, or a video store. Use the results to determine how many people use Redbox.
    52 only use Netflix 62 only use Redbox
    24 only use a video store 16 use only a video store and Redbox
    48 use only Netflix and Redbox 30 use only a video store and Netflix
    10 use all three 25 use none of these
44. A survey asked buyers whether color, size, or brand influenced their choice of cell phone. The results are below. How many people were influenced by brand?
    5 only said color 8 only said size
    16 only said brand 20 said only color and size
    42 said only color and brand 53 said only size and brand
    102 said all three 20 said none of these
45. Use the given information to complete a Venn diagram, then determine: a) how many students have seen exactly one of these movies, and b) how many had seen only *Star Wars*.
    18 had seen *The Matrix* (*M*) 24 had seen *Star Wars (SW)*
    20 had seen *Lord of the Rings (LotR)* 10 had seen *M* and *SW*
    14 had seen *LotR* and *SW* 12 had seen *M* and *LotR*
    6 had seen all three
46. A survey asked people what alternative transportation modes they use. Using the data to complete a Venn diagram, then determine: a) what percent of people only ride the bus, and b) how many people don't use any alternate transportation.
    30% use the bus 20% ride a bicycle
    25% walk 5% use the bus and ride a bicycle
    10% ride a bicycle and walk 12% use the bus and walk
    2% use all three

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CHAPTER OVERVIEW

11: Statistics

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11.2: Populations and Samples
11.3: Categorizing Data
11.4: Sampling Methods
11.5: How to Mess Things Up Before You Start
11.6: Experiments
11.7: Describing Data
11.8: Understanding Normal Distribution
11.9: The Empirical Rule
11.10: Z-Scores
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11.12: Regression and Correlation
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11.1: Module Overview

Like most people, you probably feel that it is important to "take control of your life." But what does this mean? Partly it means being able to properly evaluate the data and claims that bombard you every day. If you cannot distinguish good from faulty reasoning, then you are vulnerable to manipulation and to decisions that are not in your best interest. Statistics provides tools that you need in order to react intelligently to information you hear or read. In this sense, Statistics is one of the most important things that you can study.

To be more specific, here are some claims that we have heard on several occasions. (We are *not* saying that each one of these claims is true!)

- 4 out of 5 dentists recommend Dentyne.
- Almost 85% of lung cancers in men and 45% in women are tobacco-related.
- Condoms are effective 94% of the time.
- Native Americans are significantly more likely to be hit crossing the streets than are people of other ethnicities.
- People tend to be more persuasive when they look others directly in the eye and speak loudly and quickly.
- Women make 75 cents to every dollar a man makes when they work the same job.
- A surprising new study shows that eating egg whites can increase one's life span.
- People predict that it is very unlikely there will ever be another baseball player with a batting average over 400.
- There is an 80% chance that in a room full of 30 people that at least two people will share the same birthday.
- 79.48% of all statistics are made up on the spot.

All of these claims are statistical in character. We suspect that some of them sound familiar; if not, we bet that you have heard other claims like them. Notice how diverse the examples are; they come from psychology, health, law, sports, business, etc. Indeed, data and data-interpretation show up in discourse from virtually every facet of contemporary life.

Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading, and push you into decisions that you might find cause to regret. For these reasons, learning about statistics is a long step towards taking control of your life. (It is not, of course, the only step needed for this purpose.) These chapters will help you learn statistical essentials. It will make you into an intelligent consumer of statistical claims.

You can take the first step right away. To be an intelligent consumer of statistics, your first reflex must be to question the statistics that you encounter. The British Prime Minister Benjamin Disraeli famously said, "There are three kinds of lies—lies, damned lies, and statistics." This quote reminds us why it is so important to understand statistics. So let us invite you to reform your statistical habits from now on. No longer will you blindly accept numbers or findings. Instead, you will begin to think about the numbers, their sources, and most importantly, the procedures used to generate them.

We have put the emphasis on defending ourselves against fraudulent claims wrapped up as statistics. Just as important as detecting the deceptive use of statistics is the appreciation of the proper use of statistics. You must also learn to recognize statistical evidence that supports a stated conclusion. When a research team is testing a new treatment for a disease, statistics allows them to conclude based on a relatively small trial that there is good evidence their drug is effective. Statistics allowed prosecutors in the 1950s and 60s to demonstrate racial bias existed in jury panels. Statistics are all around you, sometimes used well, sometimes not. We must learn how to distinguish the two cases.

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11.2: Populations and Samples

Before we begin gathering and analyzing data we need to characterize the **population** we are studying. If we want to study the amount of money spent on textbooks by a typical first-year college student, our population might be all first-year students at your college. Or it might be:

- All first-year community college students in the state of Washington.
- All first-year students at public colleges and universities in the state of Washington.
- All first-year students at all colleges and universities in the state of Washington.
- All first-year students at all colleges and universities in the entire United States.
- And so on.

Population

The **population** of a study is the group the collected data is intended to describe.

Sometimes the intended population is called the **target population**, since if we design our study badly, the collected data might not actually be representative of the intended population.

Why is it important to specify the population? We might get different answers to our question as we vary the population we are studying. First-year students at the University of Washington might take slightly more diverse courses than those at your college, and some of these courses may require less popular textbooks that cost more; or, on the other hand, the University Bookstore might have a larger pool of used textbooks, reducing the cost of these books to the students. Whichever the case (and it is likely that some combination of these and other factors are in play), the data we gather from your college will probably not be the same as that from the University of Washington. Particularly when conveying our results to others, we want to be clear about the population we are describing with our data.

Example 1

A newspaper website contains a poll asking people their opinion on a recent news article.

What is the population?

While the target (intended) population may have been all people, the real population of the survey is readers of the website.

If we were able to gather data on every member of our population, say the average (we will define “average” more carefully in a subsequent section) amount of money spent on textbooks by each first-year student at your college during the 2009-2010 academic year, the resulting number would be called a **parameter**.

Parameter

A **parameter** is a value (average, percentage, etc.) calculated using all the data from a population

We seldom see parameters, however, since surveying an entire population is usually very time-consuming and expensive, unless the population is very small or we already have the data collected.

Census

A survey of an entire population is called a **census**.

You are probably familiar with two common censuses: the official government Census that attempts to count the population of the U.S. every ten years, and voting, which asks the opinion of all eligible voters in a district. The first of these demonstrates one additional problem with a census: the difficulty in finding and getting participation from everyone in a large population, which can bias, or skew, the results.

There are occasionally times when a census is appropriate, usually when the population is fairly small. For example, if the manager of Starbucks wanted to know the average number of hours her employees worked last week, she should be able to pull up payroll records or ask each employee directly.

Since surveying an entire population is often impractical, we usually select a **sample** to study;

Sample

A **sample** is a smaller subset of the entire population, ideally one that is fairly representative of the whole population.

We will discuss sampling methods in greater detail in a later section. For now, let us assume that samples are chosen in an appropriate manner. If we survey a sample, say 100 first-year students at your college, and find the average amount of money spent by these students on textbooks, the resulting number is called a **statistic**.

Statistic

A **statistic** is a value (average, percentage, etc.) calculated using the data from a sample.

Example 2

A researcher wanted to know how citizens of Tacoma felt about a voter initiative. To study this, she goes to the Tacoma Mall and randomly selects 500 shoppers and asks them their opinion. 60% indicate they are supportive of the initiative. What is the sample and population? Is the 60% value a parameter or a statistic?

The sample is the 500 shoppers questioned. The population is less clear. While the intended population of this survey was Tacoma citizens, the effective population was mall shoppers. There is no reason to assume that the 500 shoppers questioned would be representative of all Tacoma citizens.

The 60% value was based on the sample, so it is a statistic.

Try it Now 1

To determine the average length of trout in a lake, researchers catch 20 fish and measure them. What is the sample and population in this study?

Try it Now 2

A college reports that the average age of their students is 28 years old. Is this a statistic or a parameter?

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11.3: Categorizing Data

Once we have gathered data, we might wish to classify it. Roughly speaking, data can be classified as categorical data or quantitative data.

Quantitative and categorical data

Categorical (qualitative) data are pieces of information that allow us to classify the objects under investigation into various categories.

Quantitative data are responses that are numerical in nature and with which we can perform meaningful arithmetic calculations.

Example 3

We might conduct a survey to determine the name of the favorite movie that each person in a math class saw in a movie theater.

When we conduct such a survey, the responses would look like: *Finding Nemo*, *The Hulk*, or *Terminator 3: Rise of the Machines*. We might count the number of people who give each answer, but the answers themselves do not have any numerical values: we cannot perform computations with an answer like "*Finding Nemo*." This would be categorical data.

Example 4

A survey could ask the number of movies you have seen in a movie theater in the past 12 months (0, 1, 2, 3, 4, …)

This would be quantitative data.

Other examples of quantitative data would be the running time of the movie you saw most recently (104 minutes, 137 minutes, 104 minutes, …) or the amount of money you paid for a movie ticket the last time you went to a movie theater ($5.50, $7.75, $9, …).

Sometimes, determining whether or not data is categorical or quantitative can be a bit trickier.

Example 5

Suppose we gather respondents' ZIP codes in a survey to track their geographical location.

ZIP codes are numbers, but we can't do any meaningful mathematical calculations with them (it doesn't make sense to say that 98036 is "twice" 49018 — that's like saying that Lynnwood, WA is "twice" Battle Creek, MI, which doesn't make sense at all), so ZIP codes are really categorical data.

Example 6

A survey about the movie you most recently attended includes the question "How would you rate the movie you just saw?" with these possible answers:

1 – it was awful
2 – it was just OK
3 – I liked it
4 – it was great
5 – best movie ever!

Again, there are numbers associated with the responses, but we can't really do any calculations with them: a movie that rates a 4 is not necessarily twice as good as a movie that rates a 2, whatever that means; if two people see the movie and one of them thinks it stinks and the other thinks it's the best ever it doesn't necessarily make sense to say that "on average they liked it."

As we study movie-going habits and preferences, we shouldn't forget to specify the population under consideration. If we survey 3-7 year-olds the runaway favorite might be *Finding Nemo*. 13-17 year-olds might prefer *Terminator 3*. And 33-37 year-olds might prefer…well, *Finding Nemo*.

Try it Now 3

Classify each measurement as categorical or quantitative

a. Eye color of a group of people

b. Daily high temperature of a city over several weeks

c. Annual income

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11.4: Sampling Methods

As we mentioned in a previous section, the first thing we should do before conducting a survey is to identify the population that we want to study. Suppose we are hired by a politician to determine the amount of support he has among the electorate should he decide to run for another term. What population should we study? Every person in the district? Not every person is eligible to vote, and regardless of how strongly someone likes or dislikes the candidate, they don't have much to do with him being re-elected if they are not able to vote.

What about eligible voters in the district? That might be better, but if someone is eligible to vote but does not register by the deadline, they won't have any say in the election either. What about registered voters? Many people are registered but choose not to vote. What about "likely voters?"

This is the criteria used in much political polling, but it is sometimes difficult to define a "likely voter." Is it someone who voted in the last election? In the last general election? In the last presidential election? Should we consider someone who just turned 18 a "likely voter?" They weren't eligible to vote in the past, so how do we judge the likelihood that they will vote in the next election?

In November 1998, former professional wrestler Jesse "The Body" Ventura was elected governor of Minnesota. Up until right before the election, most polls showed he had little chance of winning. There were several contributing factors to the polls not reflecting the actual intent of the electorate:

- Ventura was running on a third-party ticket and most polling methods are better suited to a two-candidate race.
- Many respondents to polls may have been embarrassed to tell pollsters that they were planning to vote for a professional wrestler.
- The mere fact that the polls showed Ventura had little chance of winning might have prompted some people to vote for him in protest to send a message to the major-party candidates.

But one of the major contributing factors was that Ventura recruited a substantial amount of support from young people, particularly college students, who had never voted before and who registered specifically to vote in the gubernatorial election. The polls did not deem these young people likely voters (since in most cases young people have a lower rate of voter registration and a turnout rate for elections) and so the polling samples were subject to **sampling bias**: they omitted a portion of the electorate that was weighted in favor of the winning candidate.

Sampling bias

A sampling method is biased if every member of the population doesn't have equal likelihood of being in the sample.

So even identifying the population can be a difficult job, but once we have identified the population, how do we choose an appropriate sample? Remember, although we would prefer to survey all members of the population, this is usually impractical unless the population is very small, so we choose a sample. There are many ways to sample a population, but there is one goal we need to keep in mind: we would like the sample to be *representative of the population*.

Returning to our hypothetical job as a political pollster, we would not anticipate very accurate results if we drew all of our samples from among the customers at a Starbucks, nor would we expect that a sample drawn entirely from the membership list of the local Elks club would provide a useful picture of district-wide support for our candidate.

One way to ensure that the sample has a reasonable chance of mirroring the population is to employ *randomness*. The most basic random method is simple random sampling.

Simple random sample

A **random sample** is one in which each member of the population has an equal probability of being chosen. A **simple random sample** is one in which every member of the population and any group of members has an equal probability of being chosen.

Example 7

If we could somehow identify all likely voters in the state, put each of their names on a piece of paper, toss the slips into a (very large) hat and draw 1000 slips out of the hat, we would have a simple random sample.

In practice, computers are better suited for this sort of endeavor than millions of slips of paper and extremely large headgear.

It is always possible, however, that even a random sample might end up not being totally representative of the population. If we repeatedly take samples of 1000 people from among the population of likely voters in the state of Washington, some of these

samples might tend to have a slightly higher percentage of Democrats (or Republicans) than does the general population; some samples might include more older people and some samples might include more younger people; etc. In most cases, this **sampling variability** is not significant.

Sampling variability

The natural variation of samples is called **sampling variability**.

This is unavoidable and expected in random sampling, and in most cases is not an issue.

To help account for variability, pollsters might instead use a **stratified sample**.

Stratified sampling

In **stratified sampling**, a population is divided into a number of subgroups (or strata). Random samples are then taken from each subgroup with sample sizes proportional to the size of the subgroup in the population.

Example 8

Suppose in a particular state that previous data indicated that the electorate was comprised of 39% Democrats, 37% Republicans and 24% independents. In a sample of 1000 people, they would then expect to get about 390 Democrats, 370 Republicans and 240 independents. To accomplish this, they could randomly select 390 people from among those voters known to be Democrats, 370 from those known to be Republicans, and 240 from those with no party affiliation.

Stratified sampling can also be used to select a sample with people in desired age groups, a specified mix ratio of males and females, etc. A variation on this technique is called **quota sampling**.

Quota sampling

Quota sampling is a variation on stratified sampling, wherein samples are collected in each subgroup until the desired quota is met.

Example 9

Suppose the pollsters call people at random, but once they have met their quota of 390 Democrats, they only gather people who do not identify themselves as a Democrat.

You may have had the experience of being called by a telephone pollster who started by asking you your age, income, etc. and then thanked you for your time and hung up before asking any “real” questions. Most likely, they already had contacted enough people in your demographic group and were looking for people who were older or younger, richer or poorer, etc. Quota sampling is usually a bit easier than stratified sampling, but also does not ensure the same level of randomness.

Another sampling method is **cluster sampling**, in which the population is divided into groups, and one or more groups are randomly selected to be in the sample.

Cluster sampling

In **cluster sampling**, the population is divided into subgroups (clusters), and a set of subgroups are selected to be in the sample

Example 10

If the college wanted to survey students, since students are already divided into classes, they could randomly select 10 classes and give the survey to all the students in those classes. This would be cluster sampling.

Other sampling methods include **systematic sampling**.

Systematic sampling

In **systematic sampling**, every *nth* member of the population is selected to be in the sample.

Example 11

To select a sample using systematic sampling, a pollster calls every 100th name in the phone book.

Systematic sampling is not as random as a simple random sample (if your name is Albert Aardvark and your sister Alexis Aardvark is right after you in the phone book, there is no way you could both end up in the sample) but it can yield acceptable samples.

Perhaps the worst types of sampling methods are **convenience samples** and **voluntary response samples**.

Convenience sampling and voluntary response sampling

Convenience sampling is samples chosen by selecting whoever is convenient.

Voluntary response sampling is allowing the sample to volunteer.

Example 12

A pollster stands on a street corner and interviews the first 100 people who agree to speak to him. This is a convenience sample.

Example 13

A website has a survey asking readers to give their opinion on a tax proposal. This is a self-selected sample, or voluntary response sample, in which respondents volunteer to participate.

Usually voluntary response samples are skewed towards people who have a particularly strong opinion about the subject of the survey or who just have way too much time on their hands and enjoy taking surveys.

Try it Now 4

In each case, indicate what sampling method was used

a. Every 4th person in the class was selected

b. A sample was selected to contain 25 men and 35 women

c. Viewers of a new show are asked to vote on the show's website

d. A website randomly selects 50 of their customers to send a satisfaction survey to

e. To survey voters in a town, a polling company randomly selects 10 city blocks, and interviews everyone who lives on those blocks.

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11.5: How to Mess Things Up Before You Start

There are number of ways that a study can be ruined before you even start collecting data. The first we have already explored – **sampling** or **selection bias**, which is when the sample is not representative of the population. One example of this is **voluntary response bias**, which is bias introduced by only collecting data from those who volunteer to participate. This is not the only potential source of bias.

Sources of bias

Sampling bias – when the sample is not representative of the population

Voluntary response bias – the sampling bias that often occurs when the sample is volunteers

Self-interest study – bias that can occur when the researchers have an interest in the outcome

Response bias – when the responder gives inaccurate responses for any reason

Perceived lack of anonymity – when the responder fears giving an honest answer might negatively affect them

Loaded questions – when the question wording influences the responses

Non-response bias – when people refusing to participate in the study can influence the validity of the outcome

Example 14

Consider a recent study which found that chewing gum may raise math grades in teenagers[1]. This study was conducted by the Wrigley Science Institute, a branch of the Wrigley chewing gum company. This is an example of a **self-interest study**; one in which the researches have a vested interest in the outcome of the study. While this does not necessarily ensure that the study was biased, it certainly suggests that we should subject the study to extra scrutiny.

Example 15

A survey asks people "when was the last time you visited your doctor?" This might suffer from **response bias**, since many people might not remember exactly when they last saw a doctor and give inaccurate responses.

Sources of response bias may be innocent, such as bad memory, or as intentional as pressuring by the pollster. Consider, for example, how many voting initiative petitions people sign without even reading them.

Example 16

A survey asks participants a question about their interactions with members of other races. Here, a **perceived lack of anonymity** could influence the outcome. The respondent might not want to be perceived as racist even if they are, and give an untruthful answer.

Example 17

An employer puts out a survey asking their employees if they have a drug abuse problem and need treatment help. Here, answering truthfully might have consequences; responses might not be accurate if the employees do not feel their responses are anonymous or fear retribution from their employer.

Example 18

A survey asks "do you support funding research of alternative energy sources to reduce our reliance on high-polluting fossil fuels?" This is an example of a **loaded** or **leading question** – questions whose wording leads the respondent towards an answer.

Loaded questions can occur intentionally by pollsters with an agenda, or accidentally through poor question wording. Also a concern is **question order**, where the order of questions changes the results. A psychology researcher provides an example[2]:

"My favorite finding is this: we did a study where we asked students, 'How satisfied are you with your life? How often do you have a date?' The two answers were not statistically related – you would conclude that there is no relationship between dating frequency and life satisfaction. But when we reversed the order and asked, 'How often do you have a date? How satisfied are you with your life?' the statistical relationship was a strong one. You would now conclude that there is nothing as important in a student's life as dating frequency."

Example 19

A telephone poll asks the question “Do you often have time to relax and read a book?”, and 50% of the people called refused to answer the survey. It is unlikely that the results will be representative of the entire population. This is an example of **non-response bias**, introduced by people refusing to participate in a study or dropping out of an experiment. When people refuse to participate, we can no longer be so certain that our sample is representative of the population.

Try it Now 5

In each situation, identify a potential source of bias

a. A survey asks how many sexual partners a person has had in the last year

b. A radio station asks readers to phone in their choice in a daily poll.

c. A substitute teacher wants to know how students in the class did on their last test. The teacher asks the 10 students sitting in the front row to state their latest test score.

d. High school students are asked if they have consumed alcohol in the last two weeks.

e. The Beef Council releases a study stating that consuming red meat poses little cardiovascular risk.

f. A poll asks “Do you support a new transportation tax, or would you prefer to see our public transportation system fall apart?”

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1. Reuters. http://news.yahoo.com/s/nm/20090423/od_uk_nm/oukoe_uk_gum_learning. Retrieved 4/27/09 ↵
2. Swartz, Norbert. http://www.umich.edu/~newsinfo/MT/01/Fal01/mt6f01.html. Retrieved 3/31/2009 ↵

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11.6: Experiments

So far, we have primarily discussed **observational studies** – studies in which conclusions would be drawn from observations of a sample or the population. In some cases these observations might be unsolicited, such as studying the percentage of cars that turn right at a red light even when there is a "no turn on red" sign. In other cases the observations are solicited, like in a survey or a poll.

In contrast, it is common to use **experiments** when exploring how subjects react to an outside influence. In an experiment, some kind of **treatment** is applied to the subjects and the results are measured and recorded.

Observational studies and experiments

An **observational study** is a study based on observations or measurements

An **experiment** is a study in which the effects of a **treatment** are measured

Here are some examples of experiments:

Example 20

a. A pharmaceutical company tests a new medicine for treating Alzheimer's disease by administering the drug to 50 elderly patients with recent diagnoses. The treatment here is the new drug.

b. A gym tests out a new weight loss program by enlisting 30 volunteers to try out the program. The treatment here is the new program.

c. You test a new kitchen cleaner by buying a bottle and cleaning your kitchen. The new cleaner is the treatment.

d. A psychology researcher explores the effect of music on temperament by measuring people's temperament while listening to different types of music. The music is the treatment.

Try it Now 6

Is each scenario describing an observational study or an experiment?

a. The weights of 30 randomly selected people are measured

b. Subjects are asked to do 20 jumping jacks, and then their heart rates are measured

c. Twenty coffee drinkers and twenty tea drinkers are given a concentration test

When conducting experiments, it is essential to isolate the treatment being tested.

Example 21

Suppose a middle school (junior high) finds that their students are not scoring well on the state's standardized math test. They decide to run an experiment to see if an alternate curriculum would improve scores. To run the test, they hire a math specialist to come in and teach a class using the new curriculum. To their delight, they see an improvement in test scores.

The difficulty with this scenario is that it is not clear whether the curriculum is responsible for the improvement, or whether the improvement is due to a math specialist teaching the class. This is called **confounding** – when it is not clear which factor or factors caused the observed effect. Confounding is the downfall of many experiments, though sometimes it is hidden.

Confounding

Confounding occurs when there are two potential variables that could have caused the outcome and it is not possible to determine which actually caused the result.

Example 22

A drug company study about a weight loss pill might report that people lost an average of 8 pounds while using their new drug. However, in the fine print you find a statement saying that participants were encouraged to also diet and exercise. It is not clear in this case whether the weight loss is due to the pill, to diet and exercise, or a combination of both. In this case confounding has occurred.

Example 23

Researchers conduct an experiment to determine whether students will perform better on an arithmetic test if they listen to music during the test. They first give the student a test without music, then give a similar test while the student listens to music. In this case, the student might perform better on the second test, regardless of the music, simply because it was the second test and they were warmed up.

There are a number of measures that can be introduced to help reduce the likelihood of confounding. The primary measure is to use a **control group**.

Control group

When using a control group, the participants are divided into two or more groups, typically a **control group** and a treatment group. The treatment group receives the treatment being tested; the control group does not receive the treatment.

Ideally, the groups are otherwise as similar as possible, isolating the treatment as the only potential source of difference between the groups. For this reason, the method of dividing groups is important. Some researchers attempt to ensure that the groups have similar characteristics (same number of females, same number of people over 50, etc.), but it is nearly impossible to control for every characteristic. Because of this, random assignment is very commonly used.

Example 24

To determine if a two day prep course would help high school students improve their scores on the SAT test, a group of students was randomly divided into two subgroups. The first group, the treatment group, was given a two day prep course. The second group, the control group, was not given the prep course. Afterwards, both groups were given the SAT.

Example 25

A company testing a new plant food grows two crops of plants in adjacent fields, the treatment group receiving the new plant food and the control group not. The crop yield would then be compared. By growing them at the same time in adjacent fields, they are controlling for weather and other confounding factors.

Sometimes not giving the control group anything does not completely control for confounding variables. For example, suppose a medicine study is testing a new headache pill by giving the treatment group the pill and the control group nothing. If the treatment group showed improvement, we would not know whether it was due to the medicine in the pill, or a response to have taken any pill. This is called a **placebo effect**.

Placebo effect

The **placebo effect** is when the effectiveness of a treatment is influenced by the patient's perception of how effective they think the treatment will be, so a result might be seen even if the treatment is ineffectual.

Example 26

A study found that when doing painful dental tooth extractions, patients told they were receiving a strong painkiller while actually receiving a saltwater injection found as much pain relief as patients receiving a dose of morphine.[1]

To control for the placebo effect, a **placebo**, or dummy treatment, is often given to the control group. This way, both groups are truly identical except for the specific treatment given.

Placebo and Placebo controlled experiments

A **placebo** is a dummy treatment given to control for the placebo effect.

An experiment that gives the control group a placebo is called a **placebo controlled experiment**.

Example 27

a. In a study for a new medicine that is dispensed in a pill form, a sugar pill could be used as a placebo.

b. In a study on the effect of alcohol on memory, a non-alcoholic beer might be given to the control group as a placebo.

c. In a study of a frozen meal diet plan, the treatment group would receive the diet food, and the control could be given standard frozen meals stripped of their original packaging.

In some cases, it is more appropriate to compare to a conventional treatment than a placebo. For example, in a cancer research study, it would not be ethical to deny any treatment to the control group or to give a placebo treatment. In this case, the currently acceptable medicine would be given to the second group, called a **comparison group** in this case. In our SAT test example, the

non-treatment group would most likely be encouraged to study on their own, rather than be asked to not study at all, to provide a meaningful comparison.

When using a placebo, it would defeat the purpose if the participant knew they were receiving the placebo.

Blind studies

A **blind study** is one in which the participant does not know whether or not they are receiving the treatment or a placebo.

A **double-blind study** is one in which those interacting with the participants don't know who is in the treatment group and who is in the control group.

Example 28

In a study about anti-depression medicine, you would not want the psychological evaluator to know whether the patient is in the treatment or control group either, as it might influence their evaluation, so the experiment should be conducted as a double-blind study.

It should be noted that not every experiment needs a control group.

Example 29

If a researcher is testing whether a new fabric can withstand fire, she simply needs to torch multiple samples of the fabric – there is no need for a control group.

Try it Now 7

To test a new lie detector, two groups of subjects are given the new test. One group is asked to answer all the questions truthfully, and the second group is asked to lie on one set of questions. The person administering the lie detector test does not know what group each subject is in.

Does this experiment have a control group? Is it blind, double-blind, or neither?

Try it Now Answers

1. The sample is the 20 fish caught. The population is all fish in the lake. The sample may be somewhat unrepresentative of the population since not all fish may be large enough to catch the bait.

2. This is a parameter, since the college would have access to data on all students (the population)

3. a. Categorical. b. Quantitative c. Quantitative

4. a. Systematic

b. Stratified or Quota

c. Voluntary response

d. Simple random

e. Cluster

5. a. Response bias – historically, men are likely to over-report, and women are likely to under-report to this question.

b. Voluntary response bias – the sample is self-selected

c. Sampling bias – the sample may not be representative of the whole class

d. Lack of anonymity

e. Self-interest study

f. Loaded question

6. a. Observational study

b. Experiment; the treatment is the jumping jacks

c. Experiment; the treatments are coffee and tea

7. The truth-telling group could be considered the control group, but really both groups are treatment groups here, since it is important for the lie detector to be able to correctly identify lies, and also not identify truth telling as lying. This study is blind, since the person running the test does not know what group each subject is in.

1. Levine JD, Gordon NC, Smith R, Fields HL. (1981) Analgesic responses to morphine and placebo in individuals with postoperative pain. Pain. 10:379-89. ↵

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11.7: Describing Data

Once we have collected data from surveys or experiments, we need to summarize and present the data in a way that will be meaningful to the reader. We will begin with graphical presentations of data then explore numerical summaries of data.

Presenting Categorical Data Graphically

Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. We usually begin working with categorical data by summarizing the data into a **frequency table.**

Frequency Table

A frequency table is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

Example 1

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total-loss collisions. The data is summarized in the frequency table below.

Color	Frequency
Blue	25
Green	52
Red	41
White	36
Black	39
Grey	23

Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph called a bar graph. In this section we will work with bar graphs that display categorical data; the next section will be devoted to bar graphs that display quantitative data.

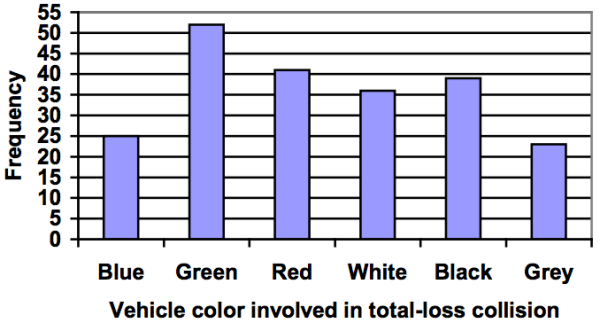
Bar Graph

A **bar graph** is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

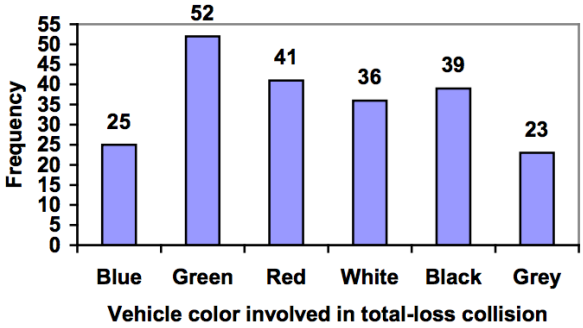
To construct a bar graph, we need to draw a vertical axis and a horizontal axis. The vertical direction will have a scale and measure the frequency of each category; the horizontal axis has no scale in this instance. The construction of a bar chart is most easily described by use of an example.

Example 2

Using our car data from above, note the highest frequency is 52, so our vertical axis needs to go from 0 to 52, but we might as well use 0 to 55, so that we can put a hash mark every 5 units:



Notice that the height of each bar is determined by the frequency of the corresponding color. The horizontal gridlines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs using graph paper, so the gridlines will already be in place, or using technology. Instead of gridlines, we might also list the frequencies at the top of each bar, like this:



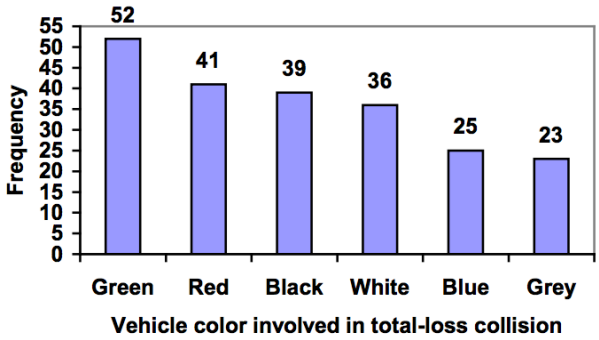
In this case, our chart might benefit from being reordered from largest to smallest frequency values. This arrangement can make it easier to compare similar values in the chart, even without gridlines. When we arrange the categories in decreasing frequency order like this, it is called a **Pareto chart**.

Pareto Chart

A **Pareto chart** is a bar graph ordered from highest to lowest frequency

Example 3

Transforming our bar graph from earlier into a Pareto chart, we get:



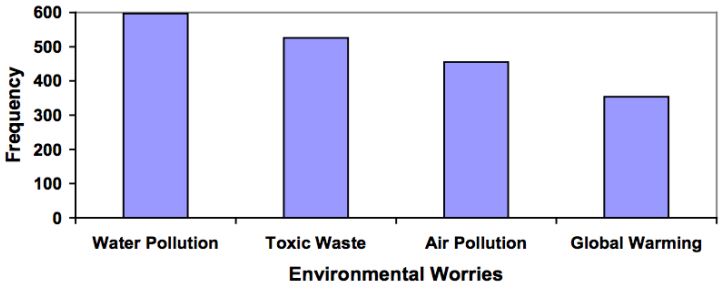
Example 4

In a survey,[1] adults were asked whether they personally worried about a variety of environmental concerns. The numbers (out of 1012 surveyed) who indicated that they worried “a great deal” about some selected concerns are summarized below.

Environmental Issue	Frequency
Pollution of drinking water	597
Contamination of soil and water by toxic waste	526

Environmental Issue	Frequency
Air pollution	455
Global warming	354

This data could be shown graphically in a bar graph:



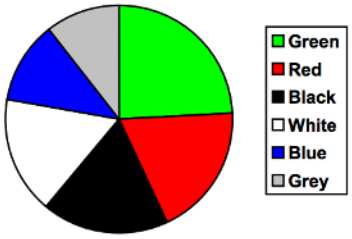
To show relative sizes, it is common to use a pie chart.

Pie Chart

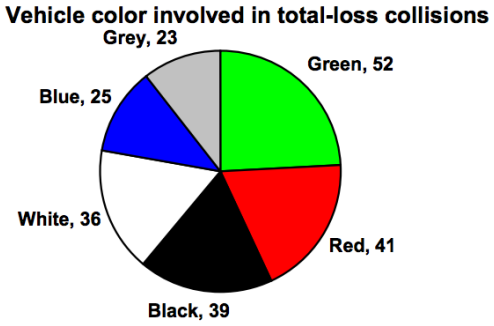
A **pie chart** is a circle with wedges cut of varying sizes marked out like slices of pie or pizza. The relative sizes of the wedges correspond to the relative frequencies of the categories.

Example 5

For our vehicle color data, a pie chart might look like this:



Pie charts can often benefit from including frequencies or relative frequencies (percents) in the chart next to the pie slices. Often having the category names next to the pie slices also makes the chart clearer.



Example 6

The pie chart to the right shows the percentage of voters supporting each candidate running for a local senate seat.

If there are 20,000 voters in the district, the pie chart shows that about 11% of those, about 2,200 voters, support Reeves.

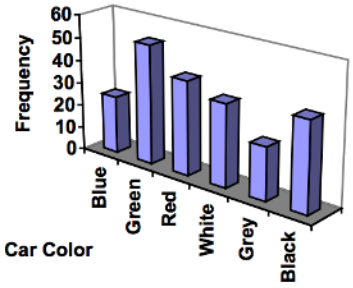
Pie charts look nice, but are harder to draw by hand than bar charts since to draw them accurately we would need to compute the angle each wedge cuts out of the circle, then measure the angle with a protractor. Computers are much better suited to drawing pie charts. Common software programs like Microsoft Word or Excel, OpenOffice.org Write or Calc, or Google Docs are able to create bar graphs, pie charts, and other graph types. There are also numerous online tools that can create graphs.[2]

Try It Now

Create a bar graph and a pie chart to illustrate the grades on a history exam below.

A: 12 students, B: 19 students, C: 14 students, D: 4 students, F: 5 students

Don't get fancy with graphs! People sometimes add features to graphs that don't help to convey their information. For example, 3-dimensional bar charts like the one shown below are usually not as effective as their two-dimensional counterparts.



Here is another way that fanciness can lead to trouble. Instead of plain bars, it is tempting to substitute meaningful images. This type of graph is called a **pictogram**.

Pictogram

A **pictogram** is a statistical graphic in which the size of the picture is intended to represent the frequencies or size of the values being represented.

Example 7



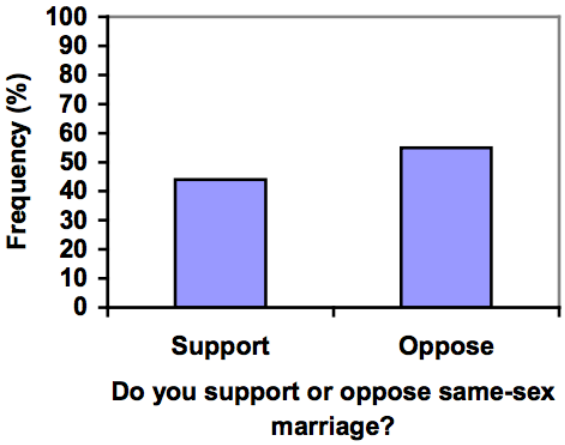
A labor union might produce the graph to the right to show the difference between the average manager salary and the average worker salary.

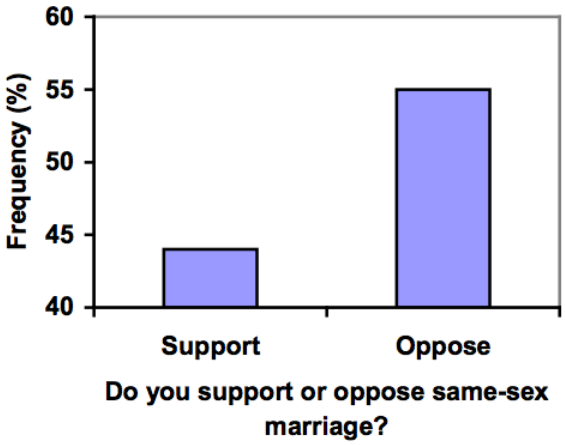
Looking at the picture, it would be reasonable to guess that the manager salaries is 4 times as large as the worker salaries—the area of the bag looks about 4 times as large. However, the manager salaries are in fact only twice as large as worker salaries, which were reflected in the picture by making the manager bag twice as tall.

Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the vertical axis, representing the least number of cases that could have occurred in a category. Normally, this number should be zero.

Example 8

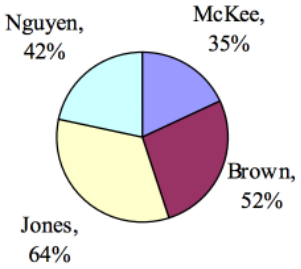
Compare the two graphs below showing support for same-sex marriage rights from a poll taken in December 2008.[3] The difference in the vertical scale on the first graph suggests a different story than the true differences in percentages; the second graph makes it look like twice as many people oppose marriage rights as support it.





Try It Now

A poll was taken asking people if they agreed with the positions of the 4 candidates for a county office. Does the pie chart present a good representation of this data? Explain.



Presenting Quantitative Data Graphically

Quantitative, or numerical, data can also be summarized into frequency tables.

Example 9

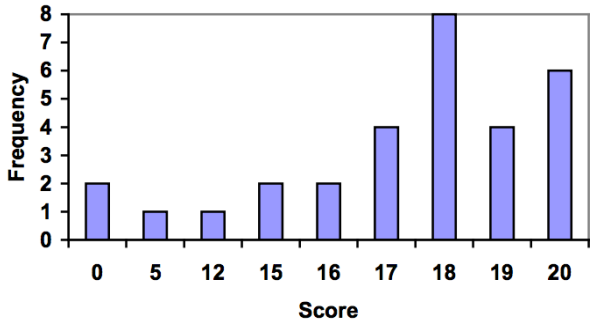
A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are:

19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

These scores could be summarized into a frequency table by grouping like values:

Score	Frequency
0	2
5	1
12	1
15	2
16	2
17	4
18	8
19	4
20	6

Using this table, it would be possible to create a standard bar chart from this summary, like we did for categorical data:



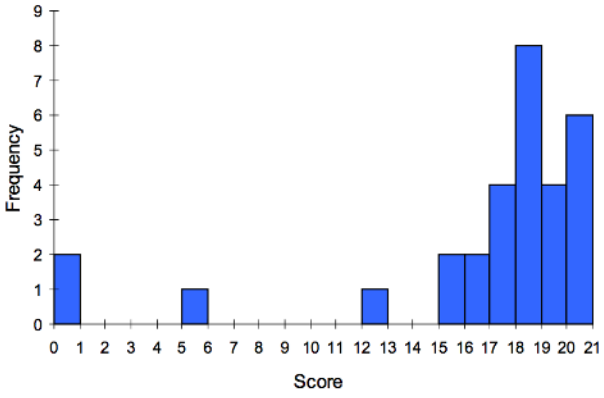
However, since the scores are numerical values, this chart doesn't really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be more correct to treat the horizontal axis as a number line. This type of graph is called a **histogram**.

Histogram

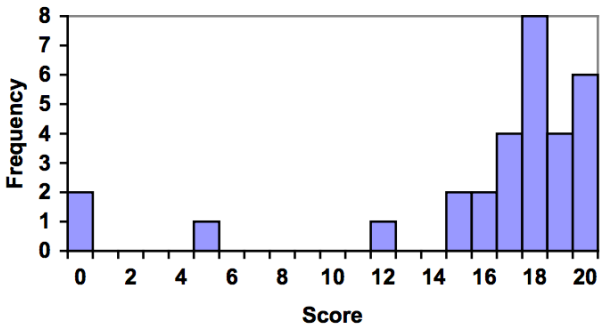
A histogram is like a bar graph, but where the horizontal axis is a number line

Example 10

For the values above, a histogram would look like:



Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. Some people choose to have bars start at ½ values to avoid this ambiguity.



Unfortunately, not a lot of common software packages can correctly graph a histogram. About the best you can do in Excel or Word is a bar graph with no gap between the bars and spacing added to simulate a numerical horizontal axis.

If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table, and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals**.

Class Intervals

Class intervals are groupings of the data. In general, we define class intervals so that:

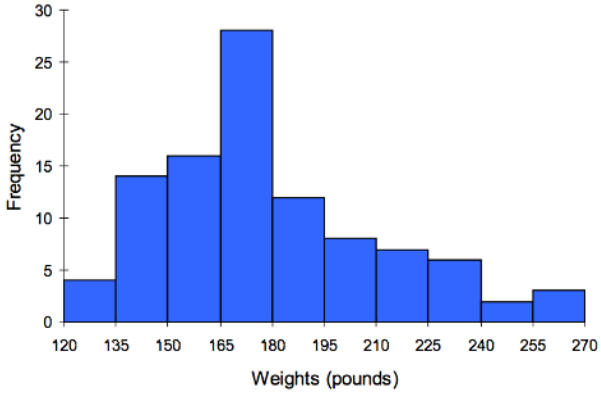
- Each interval is equal in size. For example, if the first class contains values from 120 to 129, the second class should include values from 130 to 139.
- We have somewhere between 5 and 20 classes, typically, depending upon the number of data we're working with.

Example 11

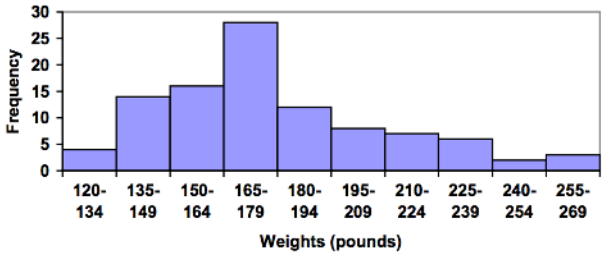
Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a total span of 263-121 = 142. We could create 7 intervals with a width of around 20, 14 intervals with a width of around 10, or somewhere in between. Often time we have to experiment with a few possibilities to find something that represents the data well. Let us try using an interval width of 15. We could start at 121, or at 120 since it is a nice round number.

Interval	Frequency
120–134	4
135–149	14
150–164	16
165–179	28
180–194	12
195–209	8
210–224	7
225–239	6
240–254	2
255–269	3

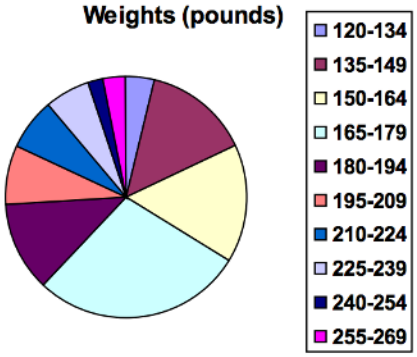
A histogram of this data would look like:



In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.



Other graph types such as pie charts are possible for quantitative data. The usefulness of different graph types will vary depending upon the number of intervals and the type of data being represented. For example, a pie chart of our weight data is difficult to read because of the quantity of intervals we used.



Try It Now

The total cost of textbooks for the term was collected from 36 students. Create a histogram for this data.

$140	$160	$160	$165	$180	$220	$235	$240	$250	$260	$280	$285
$285	$285	$290	$300	$300	$305	$310	$310	$315	$315	$320	$320
$330	$340	$345	$350	$355	$360	$360	$380	$395	$420	$460	$460

When collecting data to compare two groups, it is desirable to create a graph that compares quantities.

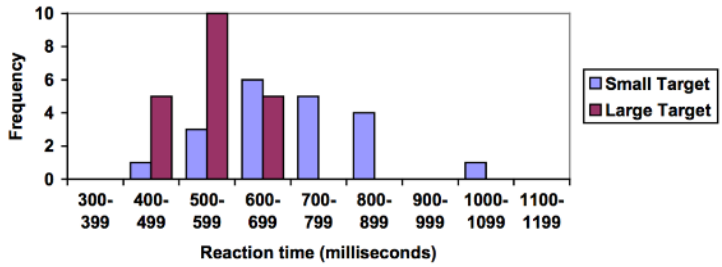
Example 12

The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.

Interval (milliseconds)	Frequency: small target	Frequency: large target
300–399	0	0
400–499	1	5
500–599	3	10
600–699	6	5
700–799	5	0
800–899	4	0
900–999	0	0
1000–1099	1	0

Interval (milliseconds)	Frequency: small target	Frequency: large target
1100–1199	0	0

One option to represent this data would be a comparative histogram or bar chart, in which bars for the small target group and large target group are placed next to each other.

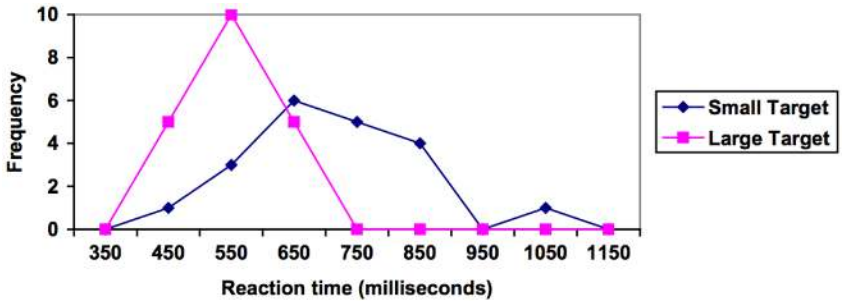


Frequency Polygon

An alternative representation is a **frequency polygon**. A frequency polygon starts out like a histogram, but instead of drawing a bar, a point is placed in the midpoint of each interval at height equal to the frequency. Typically the points are connected with straight lines to emphasize the distribution of the data.

Example 13

This graph makes it easier to see that reaction times were generally shorter for the larger target, and that the reaction times for the smaller target were more spread out.



Numerical Summaries of Data

It is often desirable to use a few numbers to summarize a distribution. One important aspect of a distribution is where its center is located. Measures of central tendency are discussed first. A second aspect of a distribution is how spread out it is. In other words, how much the data in the distribution vary from one another. The second section describes measures of variability.

Measures of Central Tendency

Let’s begin by trying to find the most “typical” value of a data set.

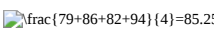
Note that we just used the word “typical” although in many cases you might think of using the word “average.” We need to be careful with the word “average” as it means different things to different people in different contexts. One of the most common uses of the word “average” is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. “Arithmetic mean” sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word “average”.

Mean

The **mean** of a set of data is the sum of the data values divided by the number of values.

Example 14

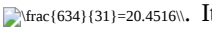
Marci's exam scores for her last math class were: 79, 86, 82, 94. The mean of these values would be:

 . Typically we round means to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3.

Example 15

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20
20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get  . It would be appropriate to round this to 20.5.

It would be most correct for us to report that "The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes," but it is not uncommon to see the more casual word "average" used in place of "mean."

Try It Now

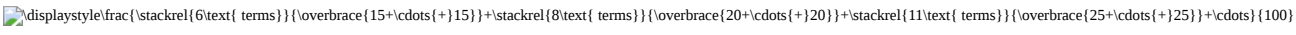
The price of a jar of peanut butter at 5 stores was: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the mean price.

Example 16

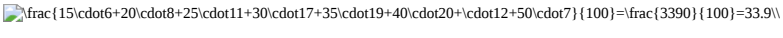
The one hundred families in a particular neighborhood are asked their annual household income, to the nearest $5 thousand dollars. The results are summarized in a frequency table below.

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Calculating the mean by hand could get tricky if we try to type in all 100 values:



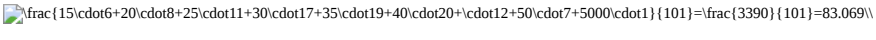
We could calculate this more easily by noticing that adding 15 to itself six times is the same as 15 × 6 = 90. Using this simplification, we get



The mean household income of our sample is 33.9 thousand dollars ($33,900).

Example 17

Extending off the last example, suppose a new family moves into the neighborhood example that has a household income of $5 million ($5000 thousand). Adding this to our sample, our mean is now:

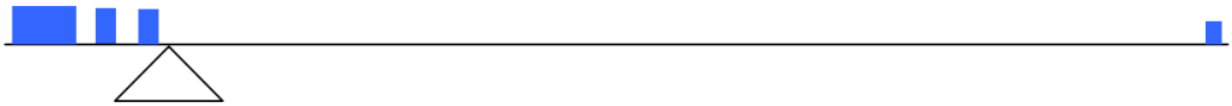


While 83.1 thousand dollars ($83,069) is the correct mean household income, it no longer represents a "typical" value.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



If we graph our household data, the $5 million data value is so far out to the right that the mean has to adjust up to keep things in balance



For this reason, when working with data that have **outliers**—values far outside the primary grouping—it is common to use a different measure of center, the **median**.

Median

The **median** of a set of data is the value in the middle when the data is in order.

To find the median, begin by listing the data in order from smallest to largest, or largest to smallest.

If the number of data values, *N*, is odd, then the median is the middle data value. This value can be found by rounding *N*/2 up to the next whole number.

If the number of data values is even, there is no one middle value, so we find the mean of the two middle values (values *N*/2 and *N*/2 + 1)

Example 18

Returning to the football touchdown data, we would start by listing the data in order. Luckily, it was already in decreasing order, so we can work with it without needing to reorder it first.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20
20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Since there are 31 data values, an odd number, the median will be the middle number, the 16th data value $\frac{31}{2}=15.5\\$, round up to 16, leaving 15 values below and 15 above). The 16th data value is 20, so the median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Example 19

Find the median of these quiz scores: 5 10 8 6 4 8 2 5 7 7

We start by listing the data in order: 2 4 5 5 6 7 7 8 8 10

Since there are 10 data values, an even number, there is no one middle number. So we find the mean of the two middle numbers, 6 and 7, and get $\frac{6+7}{2}=6.5\\$.

The median quiz score was 6.5.

Try It Now

The price of a jar of peanut butter at 5 stores were: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the median price.

Example 20

Let us return now to our original household income data:

Income (thousands of dollars)	Frequency
15	6
20	8

Income (thousands of dollars)	Frequency
25	11
30	17
35	19
40	20
45	12
50	7

Here we have 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values: the 50th and 51st data values. To find these, we start counting up from the bottom:

There are 6 data values of $15, so values 1 to 6 are $15 thousand
The next 8 data values are $20, so values 7 to (6+8)=14 are $20 thousand
The next 11 data values are $25, so values 15 to (14+11)=25 are $25 thousand
The next 17 data values are $30, so values 26 to (25+17)=42 are $30 thousand
The next 19 data values are $35, so values 43 to (42+19)=61 are $35 thousand

From this we can tell that values 50 and 51 will be $35 thousand, and the mean of these two values is $35 thousand. The median income in this neighborhood is $35 thousand.

Example 21

If we add in the new neighbor with a $5 million household income, then there will be 101 data values, and the 51st value will be the median. As we discovered in the last example, the 51st value is $35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

In addition to the mean and the median, there is one other common measurement of the "typical" value of a data set: the **mode**.

Mode

The **mode** is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

Example 22

In our vehicle color survey, we collected the data

Color	Frequency
Blue	3
Green	5
Red	4
White	3
Black	2
Grey	3

For this data, Green is the mode, since it is the data value that occurred the most frequently.

It is possible for a data set to have more than one mode if several categories have the same frequency, or no modes if each every category occurs only once.

Try It Now

Reviewers were asked to rate a product on a scale of 1 to 5. Find

1. The mean rating
2. The median rating
3. The mode rating

Rating	Frequency
1	4
2	8
3	7
4	3
5	1

Measures of Variation

Consider these three sets of quiz scores:

Section A: 5 5 5 5 5 5 5 5 5 5
Section B: 0 0 0 0 0 10 10 10 10 10
Section C: 4 4 4 5 5 5 6 6 6

All three of these sets of data have a mean of 5 and median of 5, yet the sets of scores are clearly quite different. In section A, everyone had the same score; in section B half the class got no points and the other half got a perfect score, assuming this was a 10-point quiz. Section C was not as consistent as section A, but not as widely varied as section B.

In addition to the mean and median, which are measures of the "typical" or "middle" value, we also need a measure of how "spread out" or varied each data set is.

There are several ways to measure this "spread" of the data. The first is the simplest and is called the **range**.

Range

The range is the difference between the maximum value and the minimum value of the data set.

Example 23

Using the quiz scores from above,

- For section A, the range is 0 since both maximum and minimum are 5 and 5 – 5 = 0
- For section B, the range is 10 since 10 – 0 = 10
- For section C, the range is 2 since 6 – 4 = 2

In the last example, the range seems to be revealing how spread out the data is. However, suppose we add a fourth section, Section D, with scores 0 5 5 5 5 5 5 5 5 10.

This section also has a mean and median of 5. The range is 10, yet this data set is quite different than Section B. To better illuminate the differences, we'll have to turn to more sophisticated measures of variation.

Standard Deviation

The standard deviation is a measure of variation based on measuring how far each data value deviates, or is different, from the mean. A few important characteristics:

- Standard deviation is always positive. Standard deviation will be zero if all the data values are equal, and will get larger as the data spreads out.

- Standard deviation has the same units as the original data.
- Standard deviation, like the mean, can be highly influenced by outliers.

Using the data from section D, we could compute for each data value the difference between the data value and the mean:

data value	deviation: data value − mean
0	0 − 5 = −5
5	5 − 5 = 0
5	5 − 5 = 0
5	5 − 5 = 0
5	5 − 5 = 0
5	5 − 5 = 0
5	5 − 5 = 0
5	5 − 5 = 0
5	5 − 5 = 0
10	10 − 5 = 5

We would like to get an idea of the "average" deviation from the mean, but if we find the average of the values in the second column the negative and positive values cancel each other out (this will always happen), so to prevent this we square every value in the second column:

data value	deviation: data value − mean	deviation squared
0	0−5 = −5	$(-5)^2 = 25$
5	5 − 5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5 − 5 = 0	$0^2 = 0$
5	5 − 5 = 0	$0^2 = 0$
5	5 − 5 = 0	$0^2 = 0$
5	5 − 5 = 0	$0^2 = 0$
5	5 − 5 = 0	$0^2 = 0$
5	5 − 5 = 0	$0^2 = 0$
10	10 − 5 = 5	$(5)^2 = 25$

We then add the squared deviations up to get 25 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 25 = 50. Ordinarily we would then divide by the number of scores, *n*, (in this case, 10) to find the mean of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by $n - 1$ (in this case, 10 − 1 = 9).[4]

So in our example, we would have 50/10 = 5 if section D represents a population and 50/9 = about 5.56 if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

Variance can be a useful statistical concept, but note that the units of variance in this instance would be points-squared since we squared all of the deviations. What are points-squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:

population standard deviation $=\sqrt{\frac{50}{10}}=\sqrt{5}\approx{2.2}\\$

or

population standard deviation $=\sqrt{\frac{50}{9}}\approx{2.4}\\$

If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data, as we have done above.

To compute standard deviation:

1. Find the deviation of each data from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by n, the number of data values, if the data represents a whole population; divide by $n - 1$ if the data is from a sample.
5. Compute the square root of the result.

Example 24

Computing the standard deviation for Section B above, we first calculate that the mean is 5. Using a table can help keep track of your computations for the standard deviation:

data value	deviation: data value − mean	deviation squared
0	0 − 5 = −5	$(-5)^2 = 25$
0	0 − 5 = −5	$(-5)^2 = 25$
0	0 − 5 = −5	$(-5)^2 = 25$
0	0 − 5 = −5	$(-5)^2 = 25$
0	0 − 5 = −5	$(-5)^2 = 25$
10	10 − 5 = −5	$(5)^2 = 25$
10	10 − 5 = −5	$(5)^2 = 25$
10	10 − 5 = −5	$(5)^2 = 25$
10	10 − 5 = −5	$(5)^2 = 25$
10	10 − 5 = −5	$(5)^2 = 25$

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:

$\displaystyle\sqrt{\frac{25+25+25+25+25+25+25+25+25+25}{10}}=\sqrt{\frac{250}{10}}=5\\$

Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out.

For comparison, the standard deviations of all four sections are:

Section A: 5 5 5 5 5 5 5 5 5 5	Standard deviation: 0
Section B: 0 0 0 0 0 10 10 10 10 10	Standard deviation: 5
Section C: 4 4 4 5 5 5 5 6 6 6	Standard deviation: 0.8
Section D: 0 5 5 5 5 5 5 5 5 10	Standard deviation: 2.2

Try It Now

The price of a jar of peanut butter at 5 stores were: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the standard deviation of the prices.

Where standard deviation is a measure of variation based on the mean, **quartiles** are based on the median.

Quartiles

Quartiles are values that divide the data in quarters.

The first quartile (Q_1) is the value so that 25% of the data values are below it; the third quartile (Q_3) is the value so that 75% of the data values are below it. You may have guessed that the second quartile is the same as the median, since the median is the value so that 50% of the data values are below it.

This divides the data into quarters; 25% of the data is between the minimum and Q_1, 25% is between Q_1 and the median, 25% is between the median and Q_3, and 25% is between Q_3 and the maximum value

While quartiles are not a 1-number summary of variation like standard deviation, the quartiles are used with the median, minimum, and maximum values to form a **5 number summary** of the data.

Five number summary

The five number summary takes this form:

Minimum, Q_1, Median, Q_3, Maximum

To find the first quartile, we need to find the data value so that 25% of the data is below it. If n is the number of data values, we compute a locator by finding 25% of n. If this locator is a decimal value, we round up, and find the data value in that position. If the locator is a whole number, we find the mean of the data value in that position and the next data value. This is identical to the process we used to find the median, except we use 25% of the data values rather than half the data values as the locator.

To find the first quartile, Q_1

Begin by ordering the data from smallest to largest

Compute the locator: $L = 0.25n$

If L is a decimal value:

- Round up to L^+
- Use the data value in the L^+th position

If L is a whole number:

- Find the mean of the data values in the Lth and $L + 1$th positions.

To find the third quartile, Q_3

Use the same procedure as for Q_1L = $0.75n$

Examples should help make this clearer.

Example 25

Suppose we have measured 9 females and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70 72

To find the first quartile we first compute the locator: 25% of 9 is $L = 0.25(9) = 2.25$. Since this value is not a whole number, we round up to 3. The first quartile will be the third data value: 62 inches.

To find the third quartile, we again compute the locator: 75% of 9 is 0.75(9) = 6.75. Since this value is not a whole number, we round up to 7. The third quartile will be the seventh data value: 69 inches.

Example 26

Suppose we had measured 8 females and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70

To find the first quartile we first compute the locator: 25% of 8 is $L = 0.25(8) = 2$. Since this value *is* a whole number, we will find the mean of the 2nd and 3rd data values: (60+62)/2 = 61, so the first quartile is 61 inches.

The third quartile is computed similarly, using 75% instead of 25%. $L = 0.75(8) = 6$. This is a whole number, so we will find the mean of the 6th and 7th data values: (67+69)/2 = 68, so Q_3 is 68.

Note that the median could be computed the same way, using 50%.

The 5-number summary combines the first and third quartile with the minimum, median, and maximum values.

Example 27

For the 9 female sample, the median is 66, the minimum is 59, and the maximum is 72. The 5 number summary is: 59, 62, 66, 69, 72.

For the 8 female sample, the median is 65, the minimum is 59, and the maximum is 70, so the 5 number summary would be: 59, 61, 65, 68, 70.

Example 28

Returning to our quiz score data. In each case, the first quartile locator is 0.25(10) = 2.5, so the first quartile will be the 3rd data value, and the third quartile will be the 8th data value. Creating the five-number summaries:

Section and data	5-number summary
Section A: 5 5 5 5 5 5 5 5 5 5	5, 5, 5, 5, 5
Section B: 0 0 0 0 0 10 10 10 10 10	0, 0, 5, 10, 10
Section C: 4 4 4 5 5 5 6 6 6	4, 4, 5, 6, 6
Section D: 0 5 5 5 5 5 5 5 5 10	0, 5, 5, 5, 10

Of course, with a relatively small data set, finding a five-number summary is a bit silly, since the summary contains almost as many values as the original data.

Try It Now

The total cost of textbooks for the term was collected from 36 students. Find the 5 number summary of this data.

$140	$160	$160	$165	$180	$220	$235	$240	$250	$260	$280	$285
$285	$285	$290	$300	$300	$305	$310	$310	$315	$315	$320	$320
$330	$340	$345	$350	$355	$360	$360	$380	$395	$420	$460	$460

Example 29

Returning to the household income data from earlier, create the five-number summary.

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

By adding the frequencies, we can see there are 100 data values represented in the table. In Example 20, we found the median was $35 thousand. We can see in the table that the minimum income is $15 thousand, and the maximum is $50 thousand.

To find Q_1, we calculate the locator: $L = 0.25(100) = 25$. This is a whole number, so Q_1 will be the mean of the 25th and 26th data values.

Counting up in the data as we did before,

There are 6 data values of $15, so values 1 to 6 are $15 thousand

The next 8 data values are $20, so values 7 to (6+8)=14 are $20 thousand

The next 11 data values are $25, so values 15 to (14+11)=25 are $25 thousand

The next 17 data values are $30, so values 26 to (25+17)=42 are $30 thousand

The 25th data value is $25 thousand, and the 26th data value is $30 thousand, so Q_1 will be the mean of these: (25 + 30)/2 = $27.5 thousand.

To find Q_3, we calculate the locator: $L = 0.75(100) = 75$. This is a whole number, so Q_3 will be the mean of the 75th and 76th data values. Continuing our counting from earlier,

The next 19 data values are $35, so values 43 to (42+19)=61 are $35 thousand

The next 20 data values are $40, so values 61 to (61+20)=81 are $40 thousand

Both the 75th and 76th data values lie in this group, so Q_3 will be $40 thousand.

Putting these values together into a five-number summary, we get: 15, 27.5, 35, 40, 50

Note that the 5 number summary divides the data into four intervals, each of which will contain about 25% of the data. In the previous example, that means about 25% of households have income between $40 thousand and $50 thousand.

For visualizing data, there is a graphical representation of a 5-number summary called a **box plot**, or box and whisker graph.

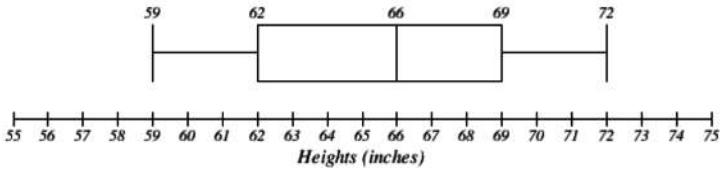
Box Plot

A **box plot** is a graphical representation of a five-number summary.

To create a box plot, a number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. “Whiskers” are extended out to the minimum and maximum values.

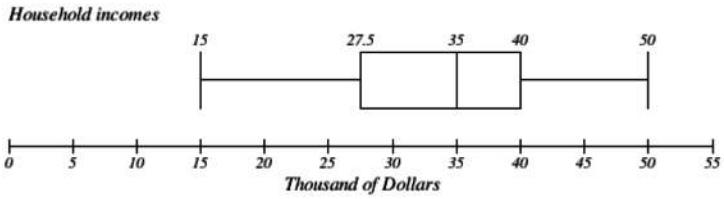
Example 30

The box plot below is based on the 9 female height data with 5 number summary: 59, 62, 66, 69, 72.



Example 31

The box plot below is based on the household income data with 5 number summary: 15, 27.5, 35, 40, 50



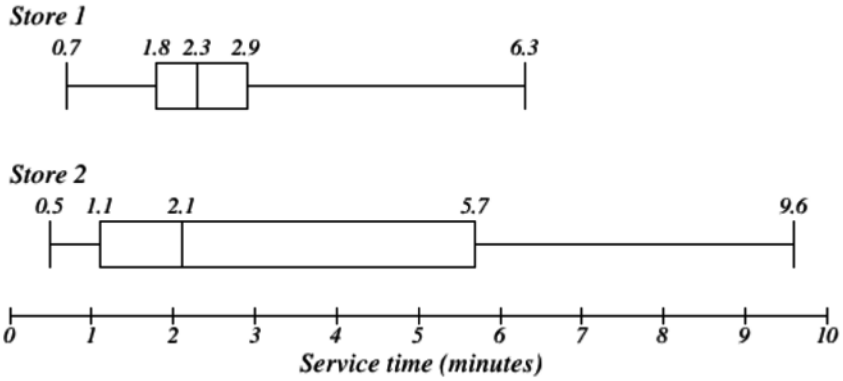
Try It Now

Create a boxplot based on the textbook price data from the last Try it Now.

Box plots are particularly useful for comparing data from two populations.

Example 32

The box plot of service times for two fast-food restaurants is shown below.



While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data.

At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry? That depends upon your opinions about luck: 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes.

Example 33

The boxplot below is based on the birth weights of infants with severe idiopathic respiratory distress syndrome (SIRDS).[5] The boxplot is separated to show the birth weights of infants who survived and those that did not.

Comparing the two groups, the boxplot reveals that the birth weights of the infants that died appear to be, overall, smaller than the weights of infants that survived. In fact, we can see that the median birth weight of infants that survived is the same as the third quartile of the infants that died.

Similarly, we can see that the first quartile of the survivors is larger than the median weight of those that died, meaning that over 75% of the survivors had a birth weight larger than the median birth weight of those that died.

Looking at the maximum value for those that died and the third quartile of the survivors, we can see that over 25% of the survivors had birth weights higher than the heaviest infant that died.

The box plot gives us a quick, albeit informal, way to determine that birth weight is quite likely linked to survival of infants with SIRDS.

---

1. Gallup Poll. March 5–8, 2009. http://www.pollingreport.com/enviro.htm↵
2. For example: http://nces.ed.gov/nceskids/createAgraph/ or http://docs.google.com↵
3. CNN/Opinion Research Corporation Poll. Dec 19-21, 2008, from http://www.pollingreport.com/civil.htm↵
4. The reason we do this is highly technical, but we can see how it might be useful by considering the case of a small sample from a population that contains an outlier, which would increase the average deviation: the outlier very likely won't be included in the sample, so the mean deviation of the sample would underestimate the mean deviation of the population; thus we divide by a slightly smaller number to get a slightly bigger average deviation. ↵
5. van Vliet, P.K. and Gupta, J.M. (1973) Sodium bicarbonate in idiopathic respiratory distress syndrome. *Arch. Disease in Childhood*, 48, 249–255. As quoted on http://openlearn.open.ac.uk/mod/oucontent/view.php?id=398296&section=1.1.3↵

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11.8: Understanding Normal Distribution

Objective

Here you will learn about the Normal Distribution. You will learn what it is and why it is important, and you will begin to develop an intuition for the rarity of a value in a set by comparing it to the mean and standard deviation of the data.

If you knew that the prices of t-shirts sold in an online shopping site were **normally distributed**, and had a mean cost of $10, with a standard deviation of $1.50, how could that information benefit you as you are looking at various t-shirt prices on the site? How could you use what you know if you were looking to make a profit by purchasing unusually inexpensive shirts to resell at prices that are more common?



Watch This: Spread of a Normal Distribution

Spread of a Normal Distribution

A Vimeo element has been excluded from this version of the text. You can view it online here: http://pb.libretexts.org/mala/?p=154

Guidance

A **distribution** is an evaluation of the way that points in a data set are clustered or spread across their **range** of values. A **normal distribution** is a very specific symmetrical distribution that indicates, among other things, that exactly $\frac{1}{2}$ of the data is below the mean, and $\frac{1}{2}$ is above, that approximately 68% of the data is within 1, approximately 96% of the data is within 2, and approximately 99.7% is within 3 **standard deviations** of the mean.

There are a number of reasons that it is important to become familiar with the normal distribution, as you will discover throughout this chapter. Examples of values associated with normal distribution:

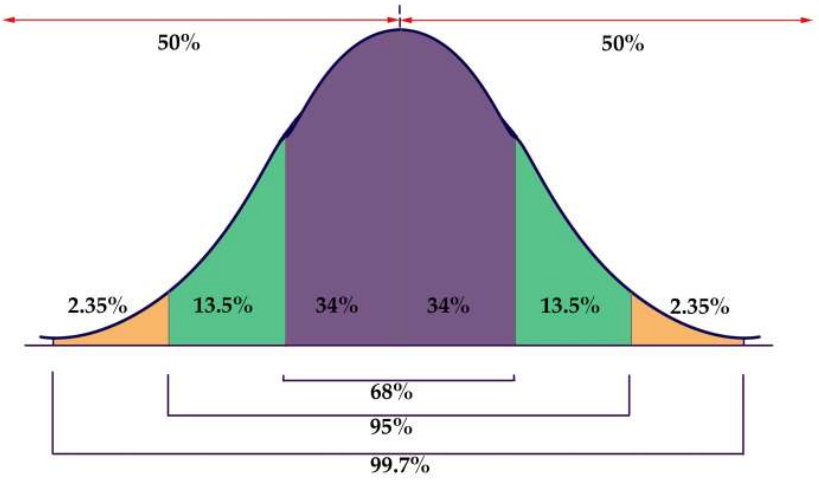
- Physical characteristics such as height, weight, arm or leg length, etc.
- The percentile rankings of standardized testing such as the ACT and SAT
- The volume of water produced by a river on a monthly or yearly basis
- The velocity of molecules in an ideal gas

Knowing that the values in a set are exactly or approximately normally distributed allows you to get a feel for how common a particular value might be in that set. Because the values of a normal distribution are predictably clustered around the mean, you can estimate in short order the rarity of a given value in the set. In our upcoming lesson on the Empirical Rule, you will see that it is worth memorizing that normally distributed data has the characteristics mentioned above:

- 50% of all data points are above the mean and 50% are below
- Apx 68% of all data points are within 1 standard deviation of the mean

- Apx 95% of all data points are within 2 standard deviations of the mean
- Apx 99.7% of all data points are within 3 standard deviations of the mean

In this lesson, we will be practicing a 'rough estimate' of the probability that a value within a given range will occur in a particular set of data, just to develop an intuition of the use of a normal distribution. In subsequent lessons, we will become more specific with our estimates. The image below will be used in greater detail in the lesson on the Empirical Rule, but you may use it as a reference for this lesson also.



Example 1

Human height is commonly considered an approximately normally distributed measure. If the mean height of a male adult in the United States is 5′ 1″, with a standard deviation of 1.5″, how common are men with heights greater than 6′ 2″?

Solution

Since each standard deviation of this normally distributed data is 1.5″, and 6′ 2″ is 400 above the mean for the population, 6′ 2″ is nearly 3 standard deviations above the mean. That tells us that men taller than 6′ 2″ are quite rare in this population.

Example 2

If the fuel mileage of a particular model of car is normally distributed, with a mean of 26 mpg and a standard deviation of 2 mpg, how common are cars with a fuel efficiency of 24 to 25 mpg?



Solution

We know that apx 68% of the cars in the population have an efficiency of between 24 and 28 mpg, since that would be 1 SD below and 1 SD above the mean. That suggests that apx 34% have an efficiency of 24 to 26 mpg, so we can say that it is uncommon to see a car with an efficiency between 24 and 25 mpg, but not extremely so.

Example 3

If the maximum jumping height of US high school high jumpers is normally distributed with a mean of 5′ 11.5″ and a SD of 2.2″, how unusual is it to see a high school jumper clear 6′ 3″?

Solution

If the mean is 5′ 11.5″, then 1 SD above is 6′ 1.7″ and 2 SDs is 6′ 3.9″. That means that less than 2.5% of jumpers 6′ 3.9″, so it would be pretty uncommon to see a high-school competitor exceed 6′ 3″.

Intro Problem Revisited

*If you knew that the prices of t-shirts sold in an online shopping site were **normally distributed**, and had a mean cost of $10, with a standard deviation of $1.50, how could that information benefit you as you are looking at various t-shirt styles and designs on the site? How could you use what you know if you were looking to make a profit by purchasing unusually inexpensive shirts to resell at prices that are more common?*

By knowing the mean and SD of the shirt prices, and knowing that they are normally distributed, you can estimate right away if a shirt is priced at a point significantly below the norm. For instance, with this data, we can estimate that a shirt priced at $7.00 is less expensive than apx 97.5% of all shirts on the site, and could likely be resold at a profit (assuming there is not something wrong the shirt that is not obvious from the listing).

Vocabulary

Distribution: an arrangement of values of a variable showing their observed or theoretical frequency of occurrence.

Range of values of a distribution: is the difference between the least and greatest values.

Normal distribution: a very specific distribution that is symmetric about its mean. Half the values of the random variable are below the mean and half are above the mean. Approximately 68% of the data is within 1 standard deviation of the mean; aproximately 96% is within 2 SDs, and 99.7% within 3 SDs.

Standard deviation: a measure of how spread out the data is from the mean. To determine if a data value is far from the mean, determine how many standard deviations it is from the mean. The SD is calculated as the square root of the variance.

Guided Practice

Assume the data to be normally distributed, and describe the rarity of an event using the following scale:

- 0% to < 1% probability = very rare
- 1% to < 5% = rare
- 5% to < 34% = uncommon
- 34% to < 50% = common
- 50% to 100% = likely

Questions

1. If the mean (μ) of the data is 75, and the standard deviation (σ) is 5, how common is a value between 70 and 75?
2. If the μ is .02 and the σ is .005, how common is a value between .005 and .01?
3. If the μ is 1280 and the σ is 70, how common is a value between 1210 and 1350?
4. If the mean defect rate at a cellphone production plant is .1%, with a standard deviation of .03%, would it seem reasonable for a quality assurance manager to be concerned about 3 defective phones in a single 1000 unit run?

Solutions

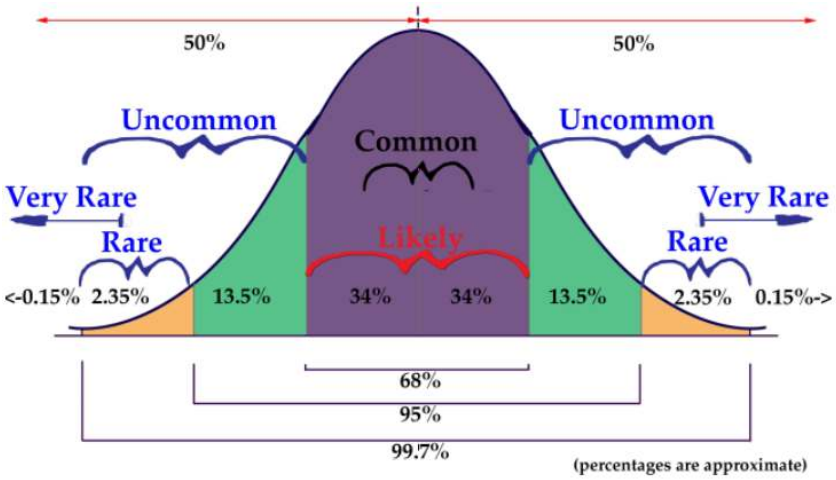
1. A value of 70 is only 1 standard deviation below the mean, so a value between 70 and 75 would be expected approximately 34% of the time, so it would be common.
2. A value of .01 is 2 SDs below the mean, and .005 is 3 SDs below, so we would expect there to be about a 2.5% probability of a value occurring in that range. A value between 0.005 and 0.01 would be rare.
3. 1210 is 1 SD below the mean, and 1350 is 1 SD above the mean, so we would expect approximately 68% of the data to be in that range, meaning that it is likely that a value in that range would occur.
4. .1% translates into 1 per thousand, with a standard deviation of 3 per ten thousand. That means that 3 defects in the same thousand is nearly 7 SDs above the mean, well into the very rare category. While it is not impossible for random chance to result in such a value, it would certainly be prudent for the manager to investigate.

Practice Questions

Assume all sets/populations to be approximately normally distributed, and describe the rarity of an event using the following scale:

- 0% to< 1% probability = very rare
- 1% to < 5% = rare
- 5% to < 34% = uncommon
- 34% to < 50% = common
- 50% to 100% = likely.

You may reference the image below:



1. Scores on a certain standardized test have a mean of 500, and a standard deviation of 100. How common is a score between 600 and 700?
2. Considering a full-grown show-quality male Siberian Husky has a mean weight of 52.5 lbs, with SD of 7.5 lbs, how common are male huskies in the 37.5–45 lbs range?
3. A population μ = 125, and σ = 25, how common are values in the 100 – 150 range?
4. Population μ = 0.0025 and σ = 0.0005, how common are values between 0.0025 and 0.0030?
5. A 12 oz can of soda has a mean volume of 12 oz, with a standard deviation of .25 oz. How common are cans with between 11 and 11.5 oz of soda?
6. μ = 0.0025 and σ = 0.0005, how common are values between 0.0045 and 0.005?
7. If a population μ = 1130 and σ = 5, how common are values between 0 and 1100?
8. Assuming population μ = 1130 and σ = 5, how common are values between 1125 and 1135?
9. The American Robin Redbreast has a mean weight of 77 g, with a standard deviation of 6 g. How common are Robins in the 59 g−71 g range?
10. Population μ = frac{3}{5}\\ and σ = \frac{1}{10}\\, how common are values between frac{2}{5}\\ and 1?
11. Population μ = 0.25% and σ = 0.05%, how common are values between 0.35% and 0.45%?
12. Population μ = 156.5 and σ = 0.25, how common are values between 155 and 156?

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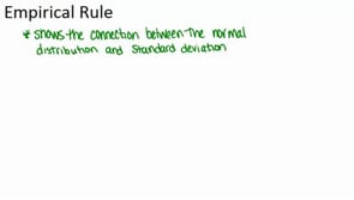
11.9: The Empirical Rule

Objective

Here you will learn how to use the Empirical Rule to estimate the probability of an event.

If the price per pound of USDA Choice Beef is normally distributed with a mean of $4.85/lb and a standard deviation of $0.35/lb, what is the estimated probability that a randomly chosen sample (from a randomly chosen market) will be between $5.20 and $5.55 per pound?

Watch This: Empirical Rule

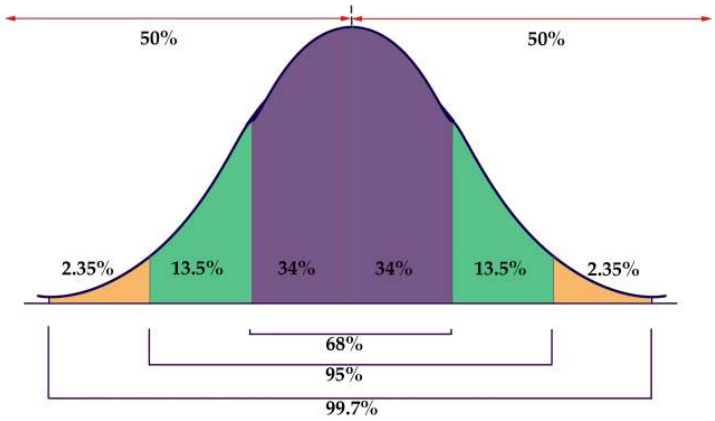


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Guidance

This reading on the **Empirical Rule** is an extension of the previous reading "Understanding the Normal Distribution." In the prior reading, the goal was to develop an intuition of the interaction between decreased probability and increased distance from the mean. In this reading, we will practice applying the Empirical Rule to estimate the specific probability of occurrence of a sample based on the range of the sample, measured in standard deviations.

The graphic below is a representation of the Empirical Rule:



The graphic is a rather concise summary of the **vital statistics** of a Normal Distribution. Note how the graph resembles a bell? Now you know why the normal distribution is also called a " bell curve."

- 50% of the data is above, and 50% below, the mean of the data
- Approximately 68% of the data occurs within 1 SD of the mean
- Approximately 95% occurs within 2 SD's of the mean
- Approximately 99.7% of the data occurs within 3 SDs of the mean

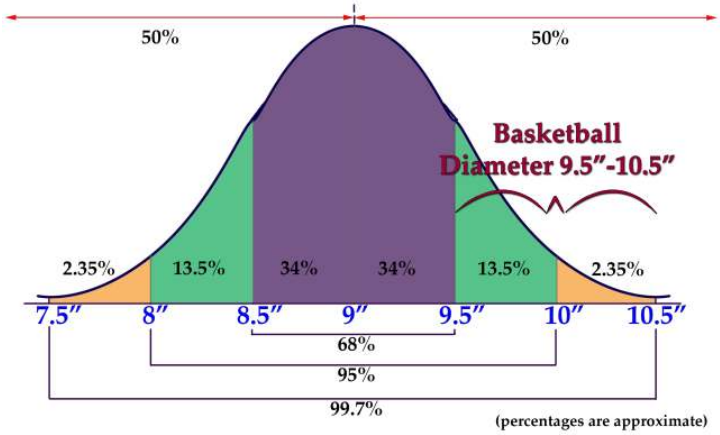
It is due to the probabilities associated with 1, 2, and 3 SDs that the Empirical Rule is also known as the **68−95−99.7 rule**.

Example 1

If the diameter of a basketball is normally distributed, with a mean (μ) of 9″, and a standard deviation (σ) of 0.5″, what is the probability that a randomly chosen basketball will have a diameter between 9.5″ and 10.5″?

Solution

Since the σ = 0.5″ and the μ = 9″, we are evaluating the probability that a randomly chosen ball will have a diameter between 1 and 3 standard deviations above the mean. The graphic below shows the portion of the normal distribution included between 1 and 3 SDs:



The percentage of the data spanning the 2nd and 3rd SDs is 13.5% + 2.35% = 15.85%

The probability that a randomly chosen basketball will have a diameter between 9.5 and 10.5 inches is 15.85%.

Example 2

If the depth of the snow in my yard is normally distributed, with μ = 2.5″ and σ = .25″, what is the probability that a randomly chosen location will have a snow depth between 2.25 and 2.75 inches?



Solution

2.25 inches is μ − 1σ, and 2.75 inches is μ + 1σ, so the area encompassed approximately represents 34% + 34% = 68%.

The probability that a randomly chosen location will have a depth between 2.25 and 2.75 inches is 68%.

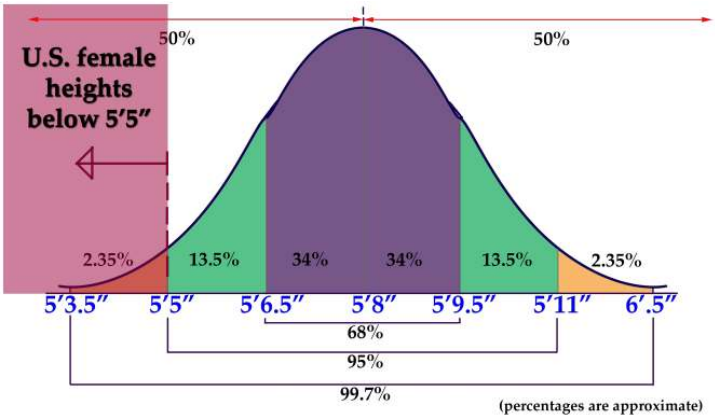
Example 3

If the height of women in the United States is normally distributed with μ = 5′ 8″ and σ = 1.5″, what is the probability that a randomly chosen woman in the United States is shorter than 5′ 5″?

Solution

This one is slightly different, since we aren't looking for the probability of a limited range of values. We want to evaluate the probability of a value occurring anywhere below 5′ 5″. Since the domain of a normal distribution is infinite, we can't actually state the probability of the portion of the distribution on "that end" because it has no "end"! What we need to do is add up the probabilities that we do know and subtract them from 100% to get the remainder.

Here is that normal distribution graphic again, with the height data inserted:



Recall that a normal distribution always has 50% of the data on each side of the mean. That indicates that 50% of US females are taller than 5′ 8″, and gives us a solid starting point to calculate from. There is another 34% between 5′ 6.5″ and 5′ 8″ and a final 13.5% between 5′ 5″ and 5′ 6.5″. Ultimately that totals: 50% + 34% + 13.5% = 97.5%. Since 97.5% of US females are 5′ 5″ or taller, that leaves 2.5% that are less than 5′ 5″ tall.

Intro Problem Revisited

If the price per pound of USDA Choice Beef is normally distributed with a mean of $4.85/lb and a standard deviation of $0.35/lb, what is the estimated probability that a randomly chosen sample (from a randomly chosen market) will be between $5.20 and $5.55 per pound?

$5.20 is μ + 1σ, and $5.55 is μ + 2σ, so the probability of a value occurring in that range is approximately 13.5%.

Vocabulary

Normal distribution: a common, but specific, distribution of data with a set of characteristics detailed in the lesson above.

Empirical Rule: a name for the way in which the normal distribution divides data by standard deviations: 68% within 1 SD, 95% within 2 SDs and 99.7 within 3 SDs of the mean

68-95-99.7 rule: another name for the Empirical Rule

Bell curve: the shape of a normal distribution

Guided Practice

1. A normally distributed data set has μ = 10 and σ = 2.5, what is the probability of randomly selecting a value greater than 17.5 from the set?
2. A normally distributed data set has μ = .05 and σ = .01, what is the probability of randomly choosing a value between .05 and .07 from the set?
3. A normally distributed data set has μ = 514 and an unknown standard deviation, what is the probability that a randomly selected value will be less than 514?

Solutions

1. If μ = 10 and σ = 2.5, then 17.5 = μ + 3σ. Since we are looking for all data above that point, we need to subtract the probability that a value will occur below that value from 100%: The probability that a value will be less than 10 is 50%, since 10 is the mean. There is another 34% between 10 and 12.5, another 13.5% between 12.5 and 15, and a final 2.35% between 15 and 17.5. 100% −50% −34% −13.5% −2.35% = 0.15% probability of a value greater than 17.5
2. 0.05 is the mean, and 0.07 is 2 standard deviations above the mean, so the probability of a value in that range is 34% + 13.5% = 47.5%
3. 514 is the mean, so the probability of a value less than that is 50%.

Practice Questions

Assume all distributions to be normal or approximately normal, and calculate percentages using the 68−95−99.7 rule.

1. Given mean 63 and standard deviation of 168, find the approximate percentage of the distribution that lies between −105 and 567.
2. Approximately what percent of a normal distribution is between 2 standard deviations and 3 standard deviations from the mean?
3. Given standard deviation of 74 and mean of 124, approximately what percentage of the values are greater than 198?
4. Given $\sigma = 39$ and $\mu = 101$, approximately what percentage of the values are less than 23?
5. Given mean 92 and standard deviation 189, find the approximate percentage of the distribution that lies between −286 and 470.
6. Approximately what percent of a normal distribution lies between $\mu + 1\sigma$ and $\mu + 2\sigma$?
7. Given standard deviation of 113 and mean 81, approximately what percentage of the values are less than −145?
8. Given mean 23 and standard deviation 157, find the approximate percentage of the distribution that lies between 23 and 337.
9. Given $\sigma = 3$ and $\mu = 84$, approximately what percentage of the values are greater than 90?
10. Approximately what percent of a normal distribution is between μ and $\mu+1\sigma$?
11. Given mean 118 and standard deviation 145, find the approximate percentage of the distribution that lies between −27 and 118.
12. Given standard deviation of 81 and mean 67, approximately what percentage of values are greater than 310?
13. Approximately what percent of a normal distribution is less than 2 standard deviations from the mean?
14. Given $\mu + 1\sigma = 247$ and $\mu + 2\sigma = 428$, find the approximate percentage of the distribution that lies between 66 and 428.
15. Given $\mu - 1\sigma = -131$ and $\mu + 1\sigma = 233$, approximately what percentage of the values are greater than −495?

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11.10: Z-Scores

Objectives

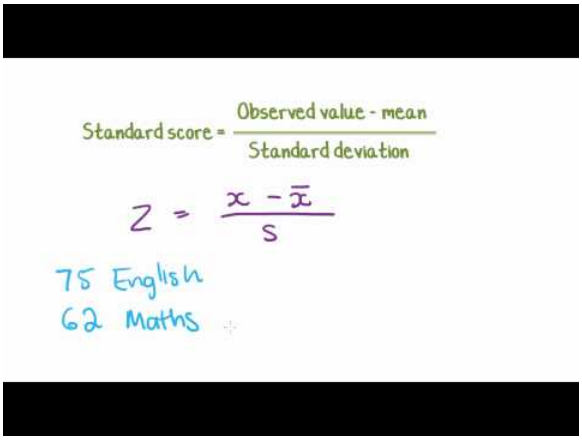
1. Here you will learn how z-scores can be used to evaluate how extreme a given value is in a particular set or population.
2. Here you will learn to evaluate z-scores as they relate to probability.
3. Here you will learn to calculate the probability of a z-score between two others.

Part I

Using the Empirical Rule can give you a good idea of the probability of occurrence of a value that happens to be exactly one, two or three to either side of the mean, but how do you compare the probabilities of values that are in between standard deviations?

Watch This: Maths Tutorial: Z scores

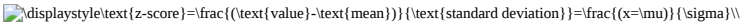
The British video below is very clear and easy to follow. It is worth noting, particularly for US students, that the instructor uses the notation *x* rather than μ for mean, and pronounces *z* as "zed."



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Z-scores are related to the Empirical Rule from the standpoint of being a method of evaluating how extreme a particular value is in a given set. **You can think of a z-score as the number of standard deviations there are between a given value and the mean of the set.** While the Empirical Rule allows you to associate the first three standard deviations with the percentage of data that each SD includes, the z-score allows you to state (as accurately as you like), just how many SDs a given value is above or below the mean.

Conceptually, the z-score calculation is just what you might expect, given that you are calculating the number of SDs between a value and the mean. You calculate the z-score by first calculating the difference between your value and the mean, and then dividing that amount by the standard deviation of the set. The formula looks like this:



In this lesson, we will practice calculating the z-score for various values. In the next lesson, we will learn how to associate the z-score of a value with the probability that the value will occur.

Example 1

What is the z-score of a value of 27, given a set mean of 24, and a standard deviation of 2?

Solution

To find the z-score we need to divide the difference between the value, 27, and the mean, 24, by the standard deviation of the set, 2.

$$\text{z-score} = \frac{27-\mu}{\sigma} = \frac{27-24}{2} = \frac{3}{2}$$

$$\text{z-score of } 27 = +1.5$$

This indicates that 27 is 1.5 standard deviations above the mean.

Example 2

What is the z-score of a value of 104.5, in a set with μ = 125 and σ = 6.2?

Solution

Find the difference between the given value and the mean, then divide it by the standard deviation.

$$\text{z-score} = \frac{104.5 - \mu}{\sigma} = \frac{104.5 - 125}{6.2} = \frac{-20.5}{6.2}$$

$$\text{z-score of } 104005 = -3.306$$

Note that the z-score is negative, since the measured value, 104.5, is less than (below) the mean, 125.

Example 3

Find the value represented by a z-score of 2.403, given μ = 63 and σ = 4.25.

Solution

This one requires that we solve for a missing value rather than for a missing z-score, so we just need to fill in our formula with what we know and solve for the missing value:

$$\text{z-score} = \frac{x - \mu}{\sigma}$$

$$2.403 = \frac{x - 63}{4.25}$$

$$10.213 = x - 63$$

$$73.213 = x$$

73.213 has a z-score of 2.403

Intro Problem Revisited

Using the Empirical Rule can give you a good idea of the probability of occurrence of a value that happens to be right on one of the first three standard deviations to either side of the mean, but how do you compare the probabilities of values that are in between standard deviations?

The z-score of a value is the count of the number of standard deviations between the value and the mean of the set. You can find it by subtracting the value from the mean, and dividing the result by the standard deviation.

Vocabulary

The z-score of a value is the number of standard deviations between the value and the mean of the set.

Guided Practice



1. What is the z-score of the price of a pair of skis that cost $247, if the mean ski price is $279, with a standard deviation of $16?
2. What is the z-score of a 5-scoop ice cream cone if the mean number of scoops is 3, with a standard deviation of 1 scoop?

3. What is the z-score of the weight of a cow that tips the scales at 825 lbs, if the mean weight for cows of her type is 1150 lbs, with a standard deviation of 77 lbs?
4. What is the z-score of a measured value of 0.0034, given μ = 0.0041 and σ = 0.0008?

Solutions

1. First find the difference between the measured value and the mean, then divide that difference by the standard deviation:

2. $\frac{247-279}{16} = \frac{-32}{16}$
   z-score = −2
3. This one is easy: The difference between 5 scoops and 3 scoops is +2, and we divide that by the standard deviation of 1, so the z-score is +2.
4. First find the difference between the measured value and the mean, then divide that difference by the standard deviation:
   $\frac{825\text{ lbs}-1150\text{ lbs}}{771\text{ lbs}} = \frac{-325}{77}$
   z-score = −4.2407
5. First find the difference between the measured value and the mean, then divide that difference by the standard deviation:
   $\frac{0.0034-0.0041}{0.0008} = \frac{-0.0007}{0.0008}$
   z-score = −0.875

Part II

Knowing the z-score of a given value is great, but what can you do with it? How does a z-score relate to probability? What is the probability of occurrence of a z-score less than +2.47?

Watch This: The Normal Distribution

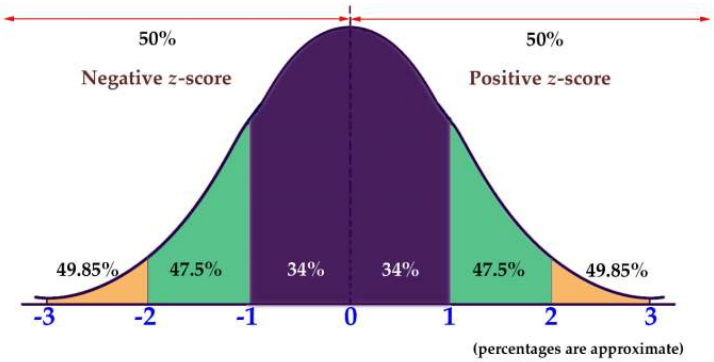
The video below provides a demonstration of how to use a z-score probability reference table, as we do in this lesson. The table he uses in the video is slightly different, but the concept is the same.



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Since z-scores are a measure of the number of SDs between a value and the mean, they can be used to calculate probability by comparing the location of the z-score to the area under a normal curve either to the left or right. The area can be calculated using calculus, but we will just use a table to look up the area.

I believe that the concept of comparing z-scores to probability is most easily understood with a graphic like the one we used in the lesson on the Empirical Rule, so I included one below. Be sure to review the examples to see how the scores work.



Like the graphic we viewed in the Empirical Rule lesson, this one only provides probability percentages for integer values of z-scores (standard deviations). In order to find the values for z-scores that aren't integers, you can use a table like the one below. To find the value associated with a given z-score, you find the first decimal of your z-score on the left or right side and then the 2nd decimal of your z-score across the top or bottom of the table. Where they intersect you will find the decimal expression of the percentage of values that are less than your sample (see example 4).

Table 1 (scroll to see all values)

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	Z
0.0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359	0.0
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	0.1
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	0.2
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517	0.3
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879	0.4
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224	0.5
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	0.6
0.7	0.758	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	0.7
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	0.8
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389	0.9
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	1.0
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883	1.1
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015	1.2
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	1.3
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1.4
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1.5
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1.6
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1.7
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1.8
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767	1.9

Table 1 (scroll to see all values)

2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	2.0
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857	2.1
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989	2.2
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	2.3
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	2.4
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	2.5
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.996	0.9961	0.9962	0.9963	0.9964	2.6
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.997	0.9971	0.9972	0.9973	0.9974	2.7
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.998	0.9981	2.8
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	2.9
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.999	0.999	3.0
3.1	0.999	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993	3.1
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	3.2
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	3.3
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	3.4
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	3.5
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.6
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.7
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.8
3.9	1	1	1	1	1	1	1	1	1	1	3.9
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	Z

Z-score tables like the one above describe the probability that a given value, or any value less than it, will occur in a given set. This particular table assumes you are looking to find the probability associated with a positive z-score. You may have additional work to do if the z-score is negative.

- To find the percentage of values greater than a negative Z score, just look up the matching positive Z score value.
- To find the percentage of values less than a negative z-score, subtract the chart value from 1.
- To find the percentage of values greater than a positive z-score, subtract the chart value from 1.

Example 4

What is the probability that a value with a z-score less than 2.47 will occur in a normal distribution?

Solution

Scroll up to the table above and find “2.4” on the left or right side. Now move across the table to “0.07” on the top or bottom, and record the value in the cell: **0.9932**. That tells us that **99.32% of values in the set are at or below a z-score of 2.47**.

Example 5

What is the probability that a value with a z-score greater than 1.53 will occur in a normal distribution?

Solution

Scroll up to the table of z-score probabilities again and find the intersection between 1.5 on the left or right and 3 on the top or bottom, record the value in the cell: **0.937**.

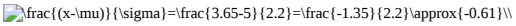
That decimal lets us know that 93.7% of values in the set are *below* the z-score of 1.53. To find the percentage that is above that value, we subtract 0.937 from 1.0 (or 93.7% from 100%), to get **0.063** or **6.3%**.

Example 6

What is the probability of a random selection being less than 3.65, given a normal distribution with $\mu = 5$ and $\sigma = 2.2$?

Solution

This question requires us to first find the z-score for the value 3.65, then calculate the percentage of values below that z-score from a reference.

1. Find the z-score for 3.65, using the z-score formula: 
2. Now we can scroll up to our z-score reference above and find the intersection of 0.6 and 0.01, which should be .7291.
3. Since this is a negative z-score, and we want the percentage of values below it, we subtract that decimal from 1.0 (reference the three steps highlighted by bullet points below the chart if you didn't recall this), to get 1 − .7291 = .2709

There is approximately a 27.09% probability that a value less than 3.65 would occur from a random selection of a normal distribution with mean 5 and standard deviation 2. 2.

Concept Problem Revisited

Knowing the z-score of a given value is great, but what can you do with it? How does a z-score relate to probability? What is the probability of occurrence of a z-score less than 2.47?

A z-score lets you calculate the probability that a randomly selected value will be greater or less than a particular value in a set.

To find the probability of a z-score below +2.47, using a reference such as the table in the lesson above:

1. Find 2.4 on the left or right side
2. Move across to 0.07 on the top or bottom.
3. The cell you arrive at says: 0.9932, which means that apx 99.32% of the values in a normal distribution will occur below a z-score of 2.47.

Vocabulary

Z-score table: a table that associates the various common z-scores between 0 and 3.99 with the decimal probability of being less than or equal to that z-score.

Guided Practice

1. What is the probability of occurrence of a value with z-score greater than 1.24?
2. What is the probability of $z < -.23$?
3. What is $P(Z < 2.13)$?

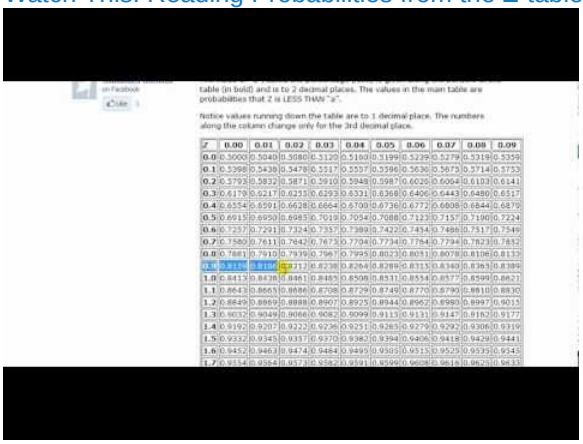
Solutions

1. Since this is a positive z-score, we can use the value for $z = 1.24$ directly from the table, and just express it as a percentage: 0.8925 or 89.25%
2. This is a negative z-score, and we want the percentage of values greater than it, so we need to subtract the value for $z = +0.23$ from 1: 1 − 0.591 = .409 or 40.9%
3. This is a positive z-score, and we need the percentage of values below it, so we can use the percentage associated with $z = +2.13$ directly from the table: 0.9834 or 98.34%

Part III

Do z-score probabilities always need to be calculated as the chance of a value either above or below a given score? How would you calculate the probability of a z-score between −0.08 and +1.92?

Watch This: Reading Probabilities from the Z-table



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To calculate the probability of getting a value with a z-score between two other z-scores, you can either use a reference table to look up the value for both scores and subtract them to find the difference, or you can use technology. In this lesson, which is an extension of Z-scores and Z-scores II, we will practice both methods.

Historically, it has been very common to use a z-score probability table like the one below to look up the probability associated with a given z-score:

Table 2 (scroll to see all values)

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	Z
0.0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359	0.0
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	0.1
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	0.2
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517	0.3
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879	0.4
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224	0.5
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	0.6
0.7	0.758	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	0.7
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	0.8
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389	0.9
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	1.0
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883	1.1
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015	1.2
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	1.3

Table 2 (scroll to see all values)

1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1.4
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1.5
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1.6
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1.7
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1.8
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767	1.9
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	2.0
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857	2.1
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989	2.2
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	2.3
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	2.4
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	2.5
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.996	0.9961	0.9962	0.9963	0.9964	2.6
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.997	0.9971	0.9972	0.9973	0.9974	2.7
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.998	0.9981	2.8
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	2.9
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.999	0.999	3.0
3.1	0.999	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993	3.1
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	3.2
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	3.3
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	3.4
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	3.5
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.6
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.7
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.8
3.9	1	1	1	1	1	1	1	1	1	1	3.9
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	Z

Since the proliferation of the Internet, however, you can also use a free online calculator such as one of these three:

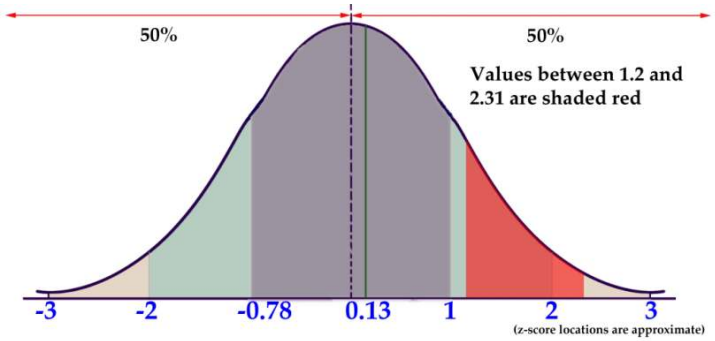
- z-Score Calculator
- z-Score to p-Value Calculator
- z-Score Calculator

Example 7

What is the probability associated with a z-score between 1.2 and 2.31?

Solution

To evaluate the probability of a value occurring within a given range, you need to find the probability of both the upper and lower values in the range, and subtract to find the difference.



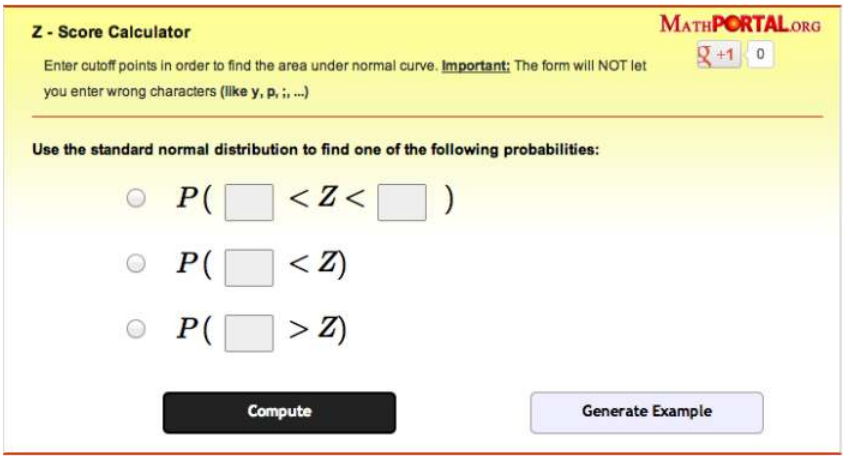
- First find z = 1.2 on the z-score probability reference above: .8849. Remember that value represents the percentage of values below 1.2.
- Next, find and record the value associated with z = 2.31: .9896
- Since approximately 88.49% of all values are below z = 1.2 and approximately 98.96% of all values are below z = 2.31, there are 98.96% −88.49% = 10.47% of values between.

Example 8

What is the probability that a value with a z-score between -1.32 and +1.49 will occur in a normal distribution?

Solution

Let's use the online calculator at mathportal.org for this one. When you open the page, you should see a window like this:



All you need to do is select the radio button to the left of the first type of probability, input "−1.32" into the first box, and 1.49 into the second. When you click **Compute**, you should get the result $P(-1.32 < Z < 1.49) = 0.8385$

Which tells us that there is approximately and 83.85% probability that a value with a z-score between 1.32 and 1.49 will occur in a normal distribution.

Notice that the calculator also details the steps involved with finding the answer:

1. Estimate the probability using a graph, so you have an idea of what your answer should be.
2. Find the probability of $z < 1.49$, using a reference. (0.9319)
3. Find the probability of $z < -1.32$, again, using a reference. (0.0934)
4. Subtract the values: 0.9319 − 0.0934 = 0.8385 or 83.85%

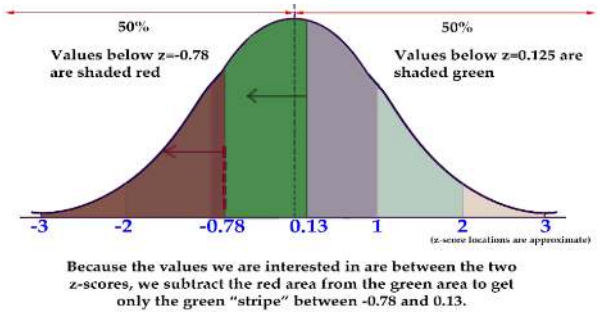
Example 9

What is the probability that a random selection will be between 8.45 and 10.25, if it is from a normal distribution with μ = 10 and σ = 2?

Solution

This question requires us to first find the z-scores for the value 8.45 and 10.25, then calculate the percentage of value between them by using values from a z-score reference and finding the difference.

1. Find the z-score for 8.45, using the z-score formula: $\frac{(x-\mu)}{\sigma}\\$
   $\frac{8.45-10}{2}=\frac{-1.55}{2}\approx{-0.78}\\$
2. Find the z-score for 10.25 the same way:
   $\frac{10.25-10}{2}=\frac{0.25}{2}\approx{.13}\\$
3. Now find the percentages for each, using a reference (don't forget we want the probability of values less than our negative score and less than our positive score, so we can find the values between):
   $P(Z < -0.78) = .2177$ or 21.77%
   $P(Z < .13) = .5517$ or 55.17%
4. At this point, let's sketch the graph to get an idea what we are looking for:



Because the values we are interested in are between the two z-scores, we subtract the red area from the green area to get only the green "stripe" between -0.78 and 0.13.

5. Finally, subtract the values to find the difference:
   .5517 − .2177 = .3340 *or about* 33.4%
   There is approximately a 33.4% probability that a value between 8.45 and 10.25 would result from a random selection of a normal distribution with mean 10 and standard deviation 2.

Concept Problem Revisited

Do z-score probabilities always need to be calculated as the chance of a value either above or below a given score? How would you calculate the probability of a z-score between −0.08 and +1.92?

After this lesson, you should know without question that z-score probabilities do not need to assume only probabilities above or below a given value, the probability between values can also be calculated.

The probability of a z-score below −0.08 is 46.81%, and the probability of a z-score below 1.92 is 97.26%, so the probability between them is 97.26%−46.81% = 50.45%.

Vocabulary

z-score: a measure of how many standard deviations there are between a data value and the mean.

z-score probability table: a table that associates z-scores to area under the normal curve. The table may be used to associate a Z-score with a percent probability.

Guided Practice

1. What is the probability of a z-score between −0.93 and 2.11?
2. What is $P(1.39 < Z < 2.03)$?
3. What is $P(-2.11 < Z < 2.11)$?

Solutions

1. Using the z-score probability table above, we can see that the probability of a value below −0.93 is .1762, and the probability of a value below 2.11 is .9826. Therefore, the probability of a value between them is .9826−.1762 = .8064 or 80.64%

2. Using the z-score probability table, we see that the probability of a value below z = 1.39 is .9177, and a value below z = 2.03 is .9788. That means that the probability of a value between them is .9788 − .9177 = .0611 or 6.11%
3. Using the online calculator at mathportal.org, we select the top calculation with the associated radio button to the left of it, enter "−2.11" in the first box, and "2.11" in the second box. Click **Compute** to get " .9652," and convert to a percentage. The probability of a z-score between −2.11 and +2.11 is about 96.52%.

Practice Questions

1. Given a distribution with a mean of 70 and standard deviation of 62, find a value with a z-score of −1.82.
2. What does a z-score of 3.4 mean?
3. Given a distribution with a mean of 60 and standard deviation of 98, find the z-score of 120.76.
4. Given a distribution with a mean of 60 and standard deviation of 21, find a value with a z-score of 2.19.
5. Find the z-score of 187.37, given a distribution with a mean of 185 and standard deviation of 1.
6. What does a z-score of −3.8 mean?
7. Find the z-score of 125.18, given a distribution with a mean of 101 and standard deviation of 62.
8. Given a distribution with a mean of 117 and standard deviation of 42, find a value with a z-score of −0.94.
9. Given a distribution with a mean of 126 and standard deviation of 100, find a value with a z-score of −0.75.
10. Find the z-score of 264.16, given $\mu = 188$ and $\sigma = 64$.
11. Find a value with a z-score of −0.2, given $\mu = 145$ and $\sigma = 56$.
12. Find the z-score of 89.79 given $\mu = 10$ and $\sigma = 79$.

Find the probabilities, use the table from the lesson above.

1. What is the probability of a z-score less than +2.02?
2. What is the probability of a z-score greater than +2.02?
3. What is the probability of a z-score less than −1.97?
4. What is the probability of a z-score greater than −1.97?
5. What is the probability of a z-score less than +0.09?
6. What is the probability of a z-score less than −0.02?
7. What is $P(Z < 1.71)$?
8. What is $P(Z > 2.22)$?
9. What is $P(Z < -1.19)$?
10. What is $P(Z > -2.71)$?
11. What is $P(Z < 3.71)$?
12. What is the probability of the random occurrence of a value greater than 56 from a normally distributed population with mean 62 and standard deviation 4.5?
13. What is the probability of a value of 329 or greater, assuming a normally distributed set with mean 290 and standard deviation 32?
14. What is the probability of getting a value below 1.2 from the random output of a normally distributed set with $\mu = 2.6$ and $\sigma = .9$?

Find the probabilities, use the table from the lesson or an online resource.

1. What is the probability of a z-score between +1.99 and +2.02?
2. What is the probability of a z-score between −1.99 and +2.02?
3. What is the probability of a z-score between −1.20 and −1.97?
4. What is the probability of a z-score between +2.33 and −0.97?
5. What is the probability of a z-score greater than +0.09?
6. What is the probability of a z-score greater than −0.02?
7. What is $P(1.42 < Z < 2.01)$?
8. What is $P(1.77 < Z < 2.22)$?
9. What is $P(-2.33 < Z < -1.19)$?
10. What is $P(-3.01 < Z < -0.71)$?
11. What is $P(2.66 < Z < 3.71)$?
12. What is the probability of the random occurrence of a value between 56 and 61 from a normally distributed population with mean 62 and standard deviation 4.5?

13. What is the probability of a value between 301 and 329, assuming a normally distributed set with mean 290 and standard deviation 32?
14. What is the probability of getting

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11.11: Supplemental Videos



This YouTube playlist contains several videos that supplement the reading in this unit. You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

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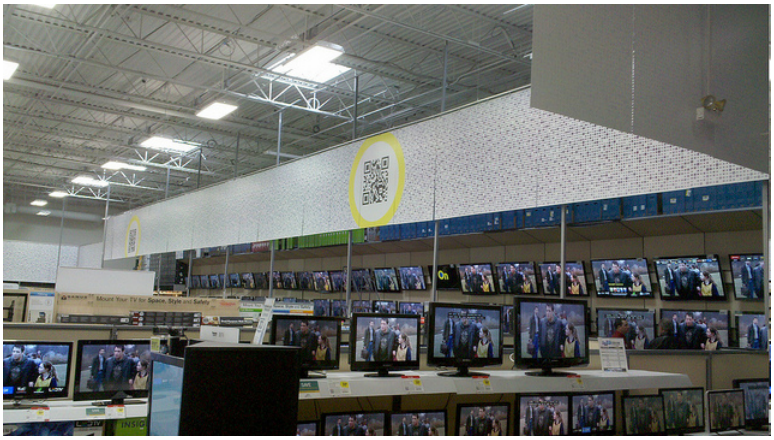
11.12: Regression and Correlation

Objective

- Here you will learn about pairs of variables that are related in a linear fashion, including those with values occurring in a slightly random manner.
- Here you will learn to calculate the linear correlation coefficient, and how to use it to describe the relationship between an explanatory and response variable.

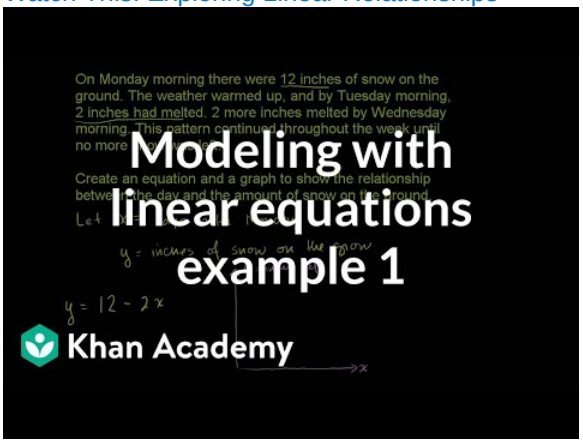
Linear Relationships

Imagine walking through the electronics section of your local department store. On the wall are examples of dozens of television sets, from little 1900 units made to sit on a kitchen counter to 72″+ monsters meant to be the centerpiece of a home theatre. Looking at the prices, you note without surprise that the 72″ model is more expensive than the 19″, and a 42″ model is priced in between. It seems rather clear that as the TV gets larger, the price goes up. Does that mean increased screen size causes increased price?



Look to the end of the section for the answer.

Watch This: Exploring Linear Relationships



A YouTube element has been excluded from this version of the text. You can view it online here: http://pb.libretexts.org/mala/?p=160

When two quantities are compared, it is not uncommon to note a relationship between them that indicates both quantities increase and decrease at the same time, or that one increases as the other decreases. If both quantities are plotted on coordinate axes, the data points show a general or definite linear trend.

If the points actually form a clearly defined line, the variables may be an example of a **deterministic** relationship. A deterministic relationship indicates that the value of one variable can be reliably and accurately determined by the manipulation of the other

variable. An example might be inches and centimeters: one inch is the same as 2.54 centimeters. If you know how many inches long something is, you can reliably and accurately calculate the number of centimeters long the same item is.

As you likely recall from Algebra, the **slope** describes the angle of the line created by plotting points from a linear relationship, and the point where the **explanatory variable** has a value of zero is called the y-intercept (commonly denoted b).

Often, particularly in research situations when one or both variables are measured, the plotted values are generally linear, but do not line up precisely. When two variables seem to show a linear relationship, but the values display some amount of randomness, we commonly visually describe the relationship with a **scatter plot**. As you will see throughout this chapter, the strength of the linear relationship of the variables can be described through mathematics.

Example 1

Given the equation $y = 2.3x + 5$:

1. Create an x–y table to describe the values of at least four points
2. What is the slope of the line?
3. What is the y-intercept?

Solution

1. Pick a value for x, substitute the chosen value for x in the equation, and calculate y:

Table 1

x	calculation	y
1	$y = 2.3(1) + 5$	7.3
2	$y = 2.3(2) + 5$	9.6
0	$y = 2.3(0) + 5$	5
−1	$y = 2.3(-1) + 5$	2.7

   The equation in the problem is in $y = mx + b$ form (also known as **slope-intercept form**), where b is the y-value when $x = 0$, and m is the slope of the line.

2. $m = 2.3$
3. $b = 5$

Example 2

Given the equation $y = -3x + 3.9$:

1. Create an x–y table to describe the values of at least four points
2. What is the slope of the line?
3. What is the y-intercept?

Solution

1. Pick a value for x, substitute the chosen value for x in the equation, and calculate y:

Table 2

x	calculation	y
1	$y = -3(1) + 3.9$	6.9
2	$y = -3(2) + 3.9$	9.9
0	$y = -3(0) + 3.9$	3.9
−1	$y = -3(-1) + 3.9$	0.9

The equation in the problem is in $y = mx + b$ form (also known as **slope-intercept form**), where b is the y-value when $x = 0$, and m is the slope of the line.

2. $m = -3$
3. $b = 3.9$

Example 3

Given the equation $y = -2.8x - 9.1$:

1. Create an x–y table to describe the values of at least four points.
2. What is the slope of the line?
3. What is the y-intercept?

Solution

1. Pick a value for x, substitute the chosen value for x in the equation, and calculate y:

Table 3

x	calculation	y
1	$y = -2.8(1) - 9.1$	−11.9
2	$y = -2.8(2) - 9.1$	−14.7
0	$y = -2.8(0) - 9.1$	−9.1
−1	$y = -2.8(-1) - 9.1$	6.3

The equation in the problem is in $y = mx + b$ form (also known as **slope-intercept form**), where b is the y-value when $x = 0$, and m is the slope of the line.

2. $m = -2.8$
3. $b = -9.1$

Intro Problem Revisited

Imagine walking through the electronics section of your local department store. On the wall are examples of dozens of television sets, from little 1900 units made to sit on a kitchen counter to 72"+ monsters meant to be the centerpiece of a home theatre. Looking at the prices, you note without surprise that the 72" model is more expensive than the 19", and a 42" model is priced in between. It seems rather clear that as the TV gets larger, the price goes up. Does that mean increased screen size causes increased price?

No, it does not. This is an example of the difficulty associated with examining linear relationships. **Correlation does not imply causation.** Just because a pair of variables exhibit a relationship, linear or otherwise, does not mean that one variable causes changes in the other variable.

Vocabulary

Cartesian Graph: a “plus-shaped” graph, with the *explanatory variable* (the input value) plotted horizontally on the x-axis, and the *response variable* (the output value) plotted vertically on the y-axis.

Deterministic linear relationship: a relationship that plots a reliably straight and accurate single line.

Slope of a line: (commonly denoted m) describes the angle of a plotted line on a graph.

Scatter plot: a graph of individual points on an x–y graph.

Guided Practice

1. If a linear graph exhibits a positive slope, what can you predict will happen to the response variable as the explanatory variable increases?
2. If a linear graph has no slope, what does that mean?

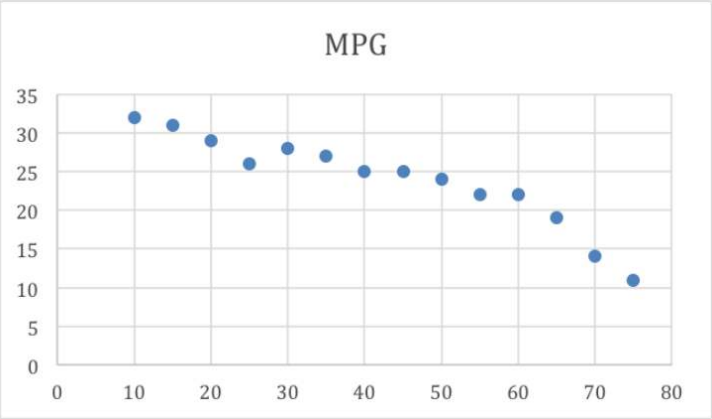
3. Given the linear equation $2y = 5.2x + 7$:
    1. What is the slope?
    2. What is the y-interept?
    3. What happens to y as x increases?
4. Given the equation $y = 2x^2 + 4$:
    1. Is this a linear equation? Why or why not?
    2. Does this equation represent a relationship?

Solutions

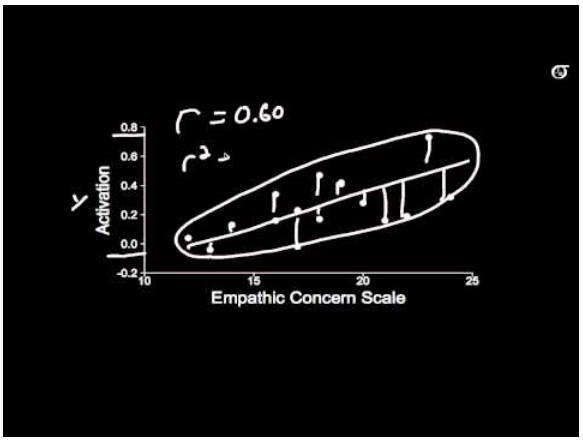
1. A positive slope indicates that the variables increase and decrease together.
2. A line with no slope is a horizontal line, since the only defined variable is the output. No matter what value is given for the explanatory variable, the response is the same.
3. Answers:
    1. The slope, m, is 5.2.
    2. The y-intercept, b, is 7.
    3. Since this line has a positive slope, y increases as x increases.
4. Answers:
    1. No, the explanatory variable is squared, this graph would form a parabola.
    2. Yes! It is just not a **linear** relationship.

Linear Correlation Coefficient

Suppose you have noted that your car seems to use more gas when you drive fast than when you drive more slowly. You decide to see how strong the relationship is, so you do some research, collect the data, and plot the data on the graph below, where the explanatory variable x is mph, and the response variable y is mpg. How can you describe how strong the correlation is without the graph?



Watch This: The Correlation Coefficient and Coefficient of Determination



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The linear correlation coefficient (sometimes called Pearson's Correlation Coefficient), commonly denoted *r*, is a measure of the strength of the linear relationship between two variables. The value of r has the following properties:

- *r* is always a value between −1 and +1
- The further an *r* value is from zero, the stronger the relationship between the two variables.
- The sign of *r* indicates the nature of the relationship: A positive *r* indicates a positive relationship, and a negative *r* indicates a negative relationship.

Generally speaking, you may think of the values of r in the following manner:

- If $|r|$ is between 0.85 and 1, there is a strong correlation.
- If $|r|$ is between 0.5 and 0.85, there is a moderate correlation.
- If $|r|$ is between 0.1 and 0.5, there is a weak correlation.
- If $|r|$ is less than 0.1, there is no apparent correlation.

Naturally, *r*-value can be calculated, but the formula is a bit beyond the scope of this course. Fortunately, there are many excellent and free online calculators for determining the r-value of a set of data. In this reading, I will be using the one at easycalculation.com, but a search for "correlation calculator online" will yield the most current options.

At the risk of overloading you with new terms, there is one more that I think it is worth learning in this reading, the **coefficient of determination**. The coefficient of determination is very simple to calculate if you know the correlation coefficient, since it is just r^2. The reason I mention it is that the coefficient of determination can be interpreted as the percentage of variation of the *y* variable that can be attributed to the relationship. In other words, a value of $r^2 = .63$ can be interpreted as "63% of the changes between one *y* value and another can be attributed to *y*'s relationship with *x*."

Example 4

Elaina is curious about the relationship between the weight of a dog and the amount of food it eats. Specifically, she wonders if heavier dogs eat more food, or if age and size factor in. She works at the Humane Society, and does some research. After some calculation, she determines that dog weight and food weight exhibit an *r*-value of 0.73.



What can Elaina say about the relationship, based on her research? What percentage of the increases in food intake can she attribute to weight, according to her research?

Solution

The calculated *r*-value of 0.73 tells us that Elaina's data demonstrates a moderate to strong correlation between the variables.

Since the coefficient of determination tells us the percentage of changes in the output variable that can be attributed to the input variable, we need to calculate r^2:

$r^2 = (0.73)^2 = .5329$

Approximately 53% of increases in food intake can be attributed to the linear relationship between food intake and the weight of the dog, suggesting that other factors, perhaps age and size, are also involved.

Example 5

Tuscany wonders if barrel racing times are related to the age of the horse. Specifically, she wonders if older horses take longer to complete a barrel racing run. As a member of the Pony Club, she does some research, and determines that horse age to barrel run time exhibits an *r*-value of 0.52.

What can Tuscany say about horse age vs barrel race time, according to her research?

Solution

Tuscany's research suggests that there is a moderate to weak correlation between horse age and barrel run time. In other words, the research suggests that $(0.52)^2 = .27 = 27\%$ of the differences between barrel run times could be attributable to the linear relationship between barrel run time and the age of the horse.

Example 6

Sayber has collected the following data regarding player score vs age in his favorite online game. He suspects that increased age is not a good indicator of gaming ability. What are the linear correlation coefficient and coefficient of determination values of his data, and how do they support or not support Sayber's hypothesis?

Table 4

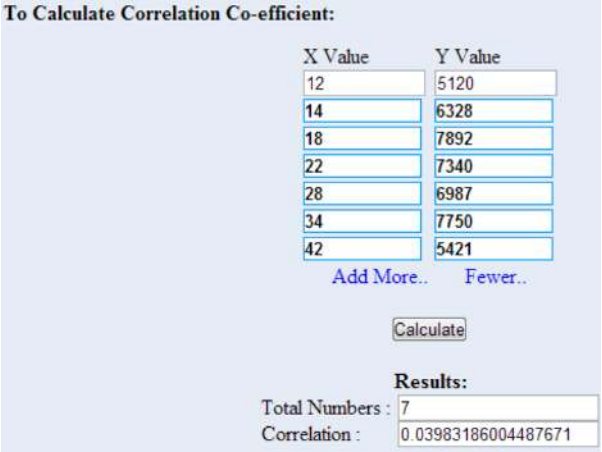
Age	Avg. Player Score
12	5,120
14	6,328
18	7,892
22	7,340

Table 4	
28	6,987
34	7,750
42	5,421

Solution

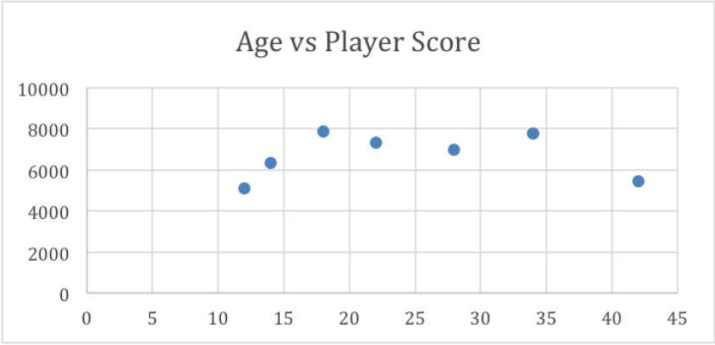
Let's use the online calculator at easycalculation.com for this one.

I entered the explanatory (Age) and response (Player Score) values into the calculator:



The linear correlation coefficient of approximately 0.04 suggests that there is no appreciable linear correlation. The coefficient of determination of 0.0016 suggests that perhaps 0.16% (practically none) of the variability of the player score is dependent on age.

Looking at the scores, however, something seems a miss with our findings. The scores suggest that age has no bearing on player score, but look at the graph of the same data:

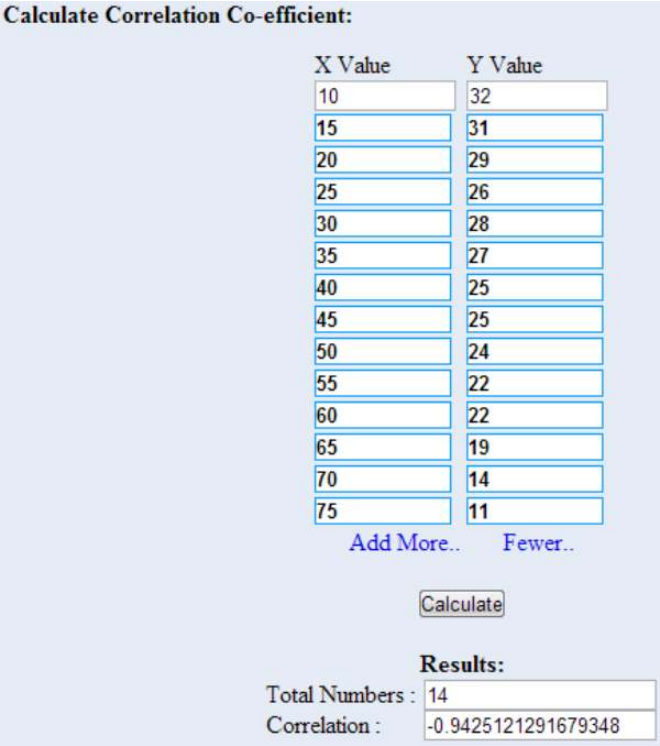


The graph suggests that the youngest and oldest polled players score less than players in late teens to mid-thirties, which seems reasonable.

This is an important example of the weakness of using just one indicator of the relationship between two variables. As I noted early in the reading, the r-value is only an indicator of linear correlation, it says nothing at all about other kinds of variable relationships. It is always a good idea to review your data in different ways to evaluate your initial conclusions.

Intro Problem Revisited

Suppose you have noted that your car seems to use more gas when you drive fast than when you drive more slowly. You decide to see how strong the relationship is, so you do some research, collect the data, and plot the data on the graph below, where the explanatory variable x is mph, and the response variable y is mpg. How can you describe how strong the correlation is without the graph?



After the reading above, we know that the r-value or r^2-value of the relationship between MPG and MPH would describe the strength of the linear relationship in a single value.

By taking the data points detailed on the graph (in practice, of course, I would have had them in table format already, since I would have needed them to build the graph in the first place), and entering them into a free linear coefficient calculator online, I get an r-value of -0.943, indicating a strong negative relationship. This also translates into an r^2-value of $(-0.943)^2 = 0.89$, indicating that the research suggests that approximately 89% of the decrease in MPG from left to right across the graph can be attributed to the increase in MPH.

Vocabulary

Linear correlation coefficient or ***r*-value of a relationship:** describes the strength of the linear relationship.

Coefficient of determination or ***r*^{2}-value of a relationship:** indicates the approximate percentage of variation in the response variable that can be attributed to the linear relationship between the response and explanatory variables, according to the data presented.

Guided Practice

1. What can you say about the strength of a linear relationship with an r-value of -0.87?
2. What can you say about the level of negative correlation of a relationship if you know the coefficient of determination is 0.82?
3. How much of the variability of y is attributable to x in a relationship with an r-value of 0.76?

Solutions

1. An $|r|$ of > 0.85 indicates a strong linear relationship. The fact that r is negative indicates that as x increases, y decreases.
2. Nothing! The coefficient of determination is r^2, and therefore always positive. We know that $|r|=\sqrt{.82}\approx{.91}$, so this is a strong linear correlation, but we have no idea if it is positive or negative.
3. The coefficient of determination describes the variation in y attributable to x, so we need to find r^2: (0.76)2 = .5776. Approximately 57.76% of the change in y-values can be attributed to the change in x.

Practice Questions

For the following questions, find the x and y intercepts of the given equations.

1. $-x + 4y = 8$

2. $3x + 5y = 15$
3. $-3x + 4y = 36$
4. $-8x + 5y = 40$
5. $5x - 6y = -30$
6. $-9x - 3y = -54$
7. $-x + 5y = -10$
8. $-3x + 8y = -72$

For the following questions, graph the line.

1. $x + 3y = 2$
2. $m = -4$, $b =$ \frac{4}{3}\\
3. x-intercept $= -1$, y-intercept $= 2$
4. $y = -4x + 2$
5. $m = -1$, $b =$ \frac{1}{2}\\
6. $x + 2y = 5$
7. $-3x + 2y = -3$

For the following questions, describe the relationship based on the r-value.

1. $r = 0$
2. $r = 0.91$
3. $r = -0.49$
4. $r = 0.05$
5. $r = 1$

For the following questions, describe the relationship based on the coefficient of determination:

1. $r^2 = 0.82$
2. $r^2 = 0.15$
3. $r^2 = 0.47$
4. $r^2 = 1$
5. $r^2 = 0$

The following questions refer to the data in the following table:

Table 5

x	y
5	70
7	69
13	58
22	47
36	36
38	25
45	14

1. What is the linear correlation coefficient of the data?
2. What does r tell you about the relationship?
3. What is the r^2 value of the data?
4. What does the coefficient of determination tell you about this relationship?
5. What would a graph of the data look like?

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11.13: Exercises

Skills

1. A political scientist surveys 28 of the current 106 representatives in a state's congress. Of them, 14 said they were supporting a new education bill, 12 said there were not supporting the bill, and 2 were undecided.
   a. What is the population of this survey?
   b. What is the size of the population?
   c. What is the size of the sample?
   d. Give the sample statistic for the proportion of voters surveyed who said they were supporting the education bill.
   e. Based on this sample, we might expect how many of the representatives to support the education bill?
2. The city of Raleigh has 9500 registered voters. There are two candidates for city council in an upcoming election: Brown and Feliz. The day before the election, a telephone poll of 350 randomly selected registered voters was conducted. 112 said they'd vote for Brown, 207 said they'd vote for Feliz, and 31 were undecided.
   a. What is the population of this survey?
   b. What is the size of the population?
   c. What is the size of the sample?
   d. Give the sample statistic for the proportion of voters surveyed who said they'd vote for Brown.
   e. Based on this sample, we might expect how many of the 9500 voters to vote for Brown?
3. Identify the most relevant source of bias in this situation: A survey asks the following: Should the mall prohibit loud and annoying rock music in clothing stores catering to teenagers?
4. Identify the most relevant source of bias in this situation: To determine opinions on voter support for a downtown renovation project, a surveyor randomly questions people working in downtown businesses.
5. Identify the most relevant source of bias in this situation: A survey asks people to report their actual income and the income they reported on their IRS tax form.
6. Identify the most relevant source of bias in this situation: A survey randomly calls people from the phone book and asks them to answer a long series of questions.
7. Identify the most relevant source of bias in this situation: A survey asks the following: Should the death penalty be permitted if innocent people might die?
8. Identify the most relevant source of bias in this situation: A study seeks to investigate whether a new pain medication is safe to market to the public. They test by randomly selecting 300 men from a set of volunteers.
9. In a study, you ask the subjects their age in years. Is this data qualitative or quantitative?
10. In a study, you ask the subjects their gender. Is this data qualitative or quantitative?
11. Does this describe an observational study or an experiment: The temperature on randomly selected days throughout the year was measured.
12. Does this describe an observational study or an experiment? A group of students are told to listen to music while taking a test and their results are compared to a group not listening to music.
13. In a study, the sample is chosen by separating all cars by size, and selecting 10 of each size grouping. What is the sampling method?
14. In a study, the sample is chosen by writing everyone's name on a playing card, shuffling the deck, then choosing the top 20 cards. What is the sampling method?
15. A team of researchers is testing the effectiveness of a new HPV vaccine. They randomly divide the subjects into two groups. Group 1 receives new HPV vaccine, and Group 2 receives the existing HPV vaccine. The patients in the study do not know which group they are in.
    a. Which is the treatment group?
    b. Which is the control group (if there is one)?
    c. Is this study blind, double-blind, or neither?
    d. Is this best described as an experiment, a controlled experiment, or a placebo controlled experiment?
16. For the clinical trials of a weight loss drug containing *Garcinia cambogia t*he subjects were randomly divided into two groups. The first received an inert pill along with an exercise and diet plan, while the second received the test medicine along with the same exercise and diet plan. The patients do not know which group they are in, nor do the fitness and nutrition advisors.

a. Which is the treatment group?
b. Which is the control group (if there is one)?
c. Is this study blind, double-blind, or neither?
d. Is this best described as an experiment, a controlled experiment, or a placebo controlled experiment?

Concepts

17. A teacher wishes to know whether the males in his/her class have more conservative attitudes than the females. A questionnaire is distributed assessing attitudes.
    a. Is this a sampling or a census?
    b. Is this an observational study or an experiment?
    c. Are there any possible sources of bias in this study?
18. A study is conducted to determine whether people learn better with spaced or massed practice. Subjects volunteer from an introductory psychology class. At the beginning of the semester 12 subjects volunteer and are assigned to the massed-practice group. At the end of the semester 12 subjects volunteer and are assigned to the spaced-practice condition.
    a. Is this a sampling or a census?
    b. Is this an observational study or an experiment?
    c. This study involves two kinds of non-random sampling: (1) Subjects are not randomly sampled from some specified population and (2) Subjects are not randomly assigned to groups. Which problem is more serious? What affect on the results does each have?
19. A farmer believes that playing Barry Manilow songs to his peas will increase their yield. Describe a controlled experiment the farmer could use to test his theory.
20. A sports psychologist believes that people are more likely to be extroverted as adults if they played team sports as children. Describe two possible studies to test this theory. Design one as an observational study and the other as an experiment. Which is more practical?

Exploration

21. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 50 AIDS patients from the start of treatment until their deaths.
    a. What is the population of this study?
    b. List two reasons why the data may differ.
    c. Can you tell if one researcher is correct and the other one is incorrect? Why?
    d. Would you expect the data to be identical? Why or why not?
    e. If the first researcher collected her data by randomly selecting 40 states, then selecting 1 person from each of those states. What sampling method is that?
    f. If the second researcher collected his data by choosing 40 patients he knew. What sampling method would that researcher have used? What concerns would you have about this data set, based upon the data collection method?
22. Find a newspaper or magazine article, or the online equivalent, describing the results of a recent study (the results of a poll are not sufficient). Give a summary of the study's findings, then analyze whether the article provided enough information to determine the validity of the conclusions. If not, produce a list of things that are missing from the article that would help you determine the validity of the study. Look for the things discussed in the text: population, sample, randomness, blind, control, placebos, etc.

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CHAPTER OVERVIEW

12: Module 3- Set Theory

12.1: Module 3 Overview
12.2: Set Theory
12.3: Supplemental Videos
12.4: Verbal, Roster, and Set-Builder Notation for a Set
12.5: Consider a Set
12.6: Helpful Links
12.7: Discuss- Application of Set Theory

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12.1: Module 3 Overview

What You'll Learn To Do: Describe set theory and perform operations.

Learning Objectives

- Describe memberships of sets, including the empty set, using proper notation, and decide whether given items are members and determine the cardinality of a given set.
- Describe the relations between sets regarding membership, equality, subset, and proper subset, using proper notation.
- Perform the operations of union, intersection, complement, and difference on sets using proper notation.
- Be able to draw and interpret Venn diagrams of set relations and operations and use Venn diagrams to solve problems.
- Recognize when set theory is applicable to real-life situations, solve real-life problems, and communicate real-life problems and solutions to others.

Learning Activities

 The learning activities for this module include:

Reading Assignments and Videos

- **Read:** Set Theory
- **Read:** Supplemental Videos for Set Theory
- **Read:** Verbal, Roster, and Set-Builder Notation for a Set
- **Read:** Consider a Set
- **Review:** Helpful Links

Homework Assignments

- **Submit:** Set Theory Homework #1 (16 points)
- **Submit:** Set Theory Homework #2 (14 points)
- **Discuss:** Application of Set Theory (20 points)

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12.2: Set Theory

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze survey data.

Basics

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1

Some examples of sets defined by describing the contents:

1. The set of all even numbers
2. The set of all books written about travel to Chile

Answers

Some examples of sets defined by listing the elements of the set:

1. {1, 3, 9, 12}
2. {red, orange, yellow, green, blue, indigo, purple}

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}.

Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol $\in$ means “is an element of”.

A set that contains no elements, { }, is called the **empty set** and is notated $\varnothing$

Example 2

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we’d write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris’s collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A, but may not contain all the elements of A.

If B is a subset of A, we write $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set—it contains fewer elements.

If B is a proper subset of A, we write $B \subset A$

Example 3

Consider these three sets:

A = the set of all even numbers
$B = \{2, 4, 6\}$
$C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of B is also an even number, so is an element of A.

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.

It is also true that $B \subset C$.

C is not a subset of A, since C contains an element, 3, that is not contained in A

Example 4

Suppose a set contains the plays "Much Ado About Nothing," "MacBeth," and "A Midsummer's Night Dream." What is a larger set this might be a subset of?

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try It Now

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Union, Intersection, and Complement

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets). The union is notated $A \bigcup B$. More formally, $x \in A \bigcup B$ if $x \in A$ or $x \in B$ (or both)

The **intersection** of two sets contains only the elements that are in both sets. The intersection is notated $A \bigcap B$. More formally, $x \in A \bigcap B$ if $x \in A$ and $x \in B$.

The **complement** of a set A contains everything that is *not* in the set A. The complement is notated A', or A^c, *or sometimes* $\sim A$.

Example 5

Consider the sets:

$A = \{\text{red, green, blue}\}$
$B = \{\text{red, yellow, orange}\}$
$C = \{\text{red, orange, yellow, green, blue, purple}\}$

Find the following:

1. Find $A \bigcup B$
2. Find $A \bigcap B$
3. Find $A^c \bigcap C$

Answers

1. The union contains all the elements in either set: $A \bigcup B = \{\text{red, green, blue, yellow, orange}\}$ Notice we only list red once.
2. The intersection contains all the elements in both sets: $A \bigcap B = \{\text{red}\}$
3. Here we're looking for all the elements that are *not* in set A and are also in C. $A^c \bigcap C = \{\text{orange, yellow, purple}\}$

Try It Now

Using the sets from the previous example, find $A \bigcup C$ and $B^c \bigcap A$

Notice that in the example above, it would be hard to just ask for A^c, since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so A^c contains all the elements in the universal set that are not in A.

Example 6

1. If we were discussing searching for books, the universal set might be all the books in the library.
2. If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
3. If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 7

Suppose the universal set is U = all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then $A^c = \{3, 5, 6, 7, 8, 9\}$.

As we saw earlier with the expression $A^c \bigcap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an order of operations.

Example 8

Suppose H = {cat, dog, rabbit, mouse}, F = {dog, cow, duck, pig, rabbit}, and W = {duck, rabbit, deer, frog, mouse}

1. Find $(H \bigcap F) \bigcup W$
2. Find $H \bigcap (F \bigcup W)$
3. Find $(H \bigcap F)^c \bigcap W$

Solutions

1. We start with the intersection: $H \bigcap F$ = {dog, rabbit}. Now we union that result with W: $(H \bigcap F) \bigcup W$ = {dog, duck, rabbit, deer, frog, mouse}
2. We start with the union: $F \bigcup W$ = {dog, cow, rabbit, duck, pig, deer, frog, mouse}. Now we intersect that result with H: $H \bigcap (F \bigcup W)$ = {dog, rabbit, mouse}
3. We start with the intersection: $H \bigcap F$ = {dog, rabbit}. Now we want to find the elements of W that are *not* in $H \bigcap F$. $(H \bigcap F)^c \bigcap W$ = {duck, deer, frog, mouse}

Venn Diagrams

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the eighteenth century. These illustrations now called **Venn Diagrams**.

Venn Diagram

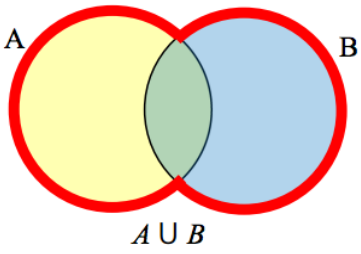
A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

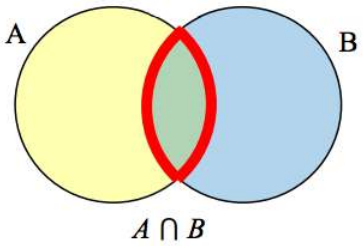
Example 9

Create Venn diagrams to illustrate $A \bigcup B$, $A \bigcap B$, and $A^c \bigcap B$

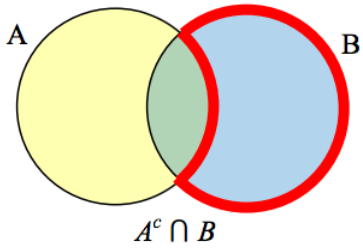
$A \bigcup B$ contains all elements in *either* set.



$A \bigcap B$ contains only those elements in both sets—in the overlap of the circles.



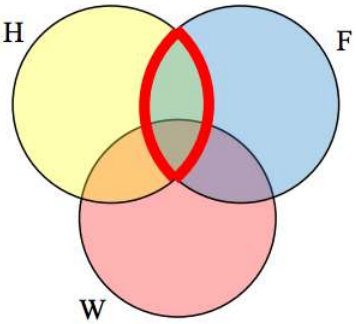
A^c will contain all elements *not* in the set A. $A^c \cap B$ will contain the elements in set B that are not in set A.



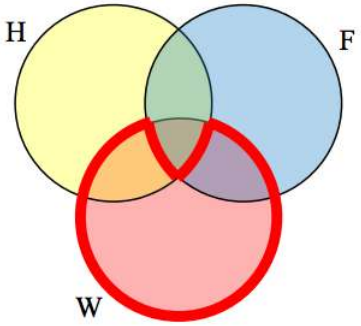
Example 10

Use a Venn diagram to illustrate $(H \cap F)^c \cap W$

We'll start by identifying everything in the set $H \cap F$

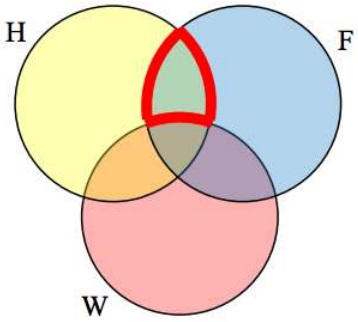


Now, $(H \cap F)^c \cap W$ will contain everything *not* in the set identified above that is also in set W.



Example 11

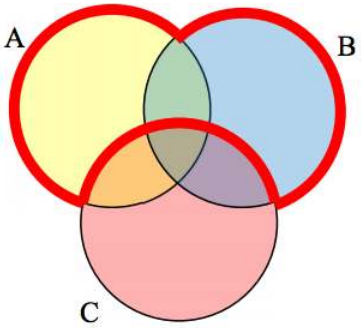
Create an expression to represent the outlined part of the Venn diagram shown.



The elements in the outlined set *are* in sets H and F, but are not in set W. So we could represent this set as $H \cap F \cap W^c$

Try It Now

Create an expression to represent the outlined portion of the Venn diagram shown



Cardinality

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality

The number of elements in a set is the cardinality of that set.

The cardinality of the set A is often notated as $|A|$ or $n(A)$

Example 12

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.

What is the cardinality of B? $A \cup B$, $A \cap B$?

Answers

The cardinality of B is 4, since there are 4 elements in the set.

The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.

The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

Example 13

What is the cardinality of P = the set of English names for the months of the year?

Answers

The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

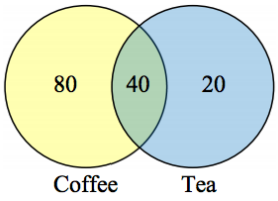
Example 14

A survey asks 200 people "What beverage do you drink in the morning", and offers choices:

- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

Answers



This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

200 – 20 – 80 – 40 = 60 people who drink neither.

Example 15

A survey asks: "Which online services have you used in the last month?"

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Answers

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \bigcup T$ is not simply 70% + 40%, since that would count those who use both services twice. To find the cardinality of $F \bigcup T$, we can add the cardinality of F and the cardinality of T, then subtract those in intersection that we've counted twice. In symbols,

$$n(F \bigcup T) = n(F) + n(T) – n(F \bigcap T)$$
$$n(F \bigcup T) = 70\% + 40\% – 20\% = 90\%$$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \bigcup T)c$. Since the universal set contains 100% of people and the cardinality of $F \bigcup T = 90\%$, the cardinality of $(F \bigcup T)c$ must be the other 10%.

The previous example illustrated two important properties

Cardinality properties

$n(A \bigcup B) = n(A) + n(B) – n(A \bigcap B)$

$n(Ac) = n(U) – n(A)$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$n(A \bigcap B) = n(A) + n(B) – n(A \bigcup B)$

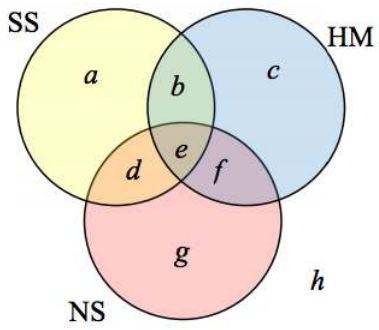
Example 16

Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course	26 were taking a HM course
19 were taking a NS course	9 were taking SS and HM
7 were taking SS and NS	10 were taking HM and NS
3 were taking all three	7 were taking none

How many students are only taking a SS course?

Answers



It might help to look at a Venn diagram. From the given data, we know that there are 3 students in region *e* and 7 students in region *h*.

Since 7 students were taking a SS and NS course, we know that n(*d*) + n(*e*) = 7. Since we know there are 3 students in region 3, there must be 7 – 3 = 4 students in region *d*.

Similarly, since there are 10 students taking HM and NS, which includes regions *e* and *f*, there must be 10 – 3 = 7 students in region *f*.

Since 9 students were taking SS and HM, there must be 9 – 3 = 6 students in region *b*.

Now, we know that 21 students were taking a SS course. This includes students from regions *a*, *b*, *d*, and *e*. Since we know the number of students in all but region *a*, we can determine that 21 – 6 – 4 – 3 = 8 students are in region *a*.

8 students are taking only a SS course.

Try It Now

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs	44 believed in ghosts
25 believed in Bigfoot	10 believed in UFOs and ghosts
8 believed in ghosts and Bigfoot	5 believed in UFOs and Bigfoot
2 believed in all three	

How many people surveyed believed in at least one of these things?

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12.3: Supplemental Videos



This YouTube playlist contains several videos that supplement the reading on Set Theory.

You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

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12.4: Verbal, Roster, and Set-Builder Notation for a Set



This site includes a written description and a video describing the three ways to name a set:

- By name or verbal description
- By roster (list or listing) form
- By set-builder notation. Note that the colon : may be replaced by a vertical line |

Click on the link below to view the article “Verbal, Roster, and Set-builder Notation for a Set” developed by Minnesota State University–Moorhead.

- Verbal, Roster, and Set-builder Notation for a Set

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12.5: Consider a Set

Consider this set: $A = \{a, b, c, d\}$.

As defined in the first reading assignment, a subset of A is another set that contains only elements from the set A, but many not contain all the elements of A. A **proper subset** is a subset that is not identical to the original set—it contains fewer elements.

We can list all of the subsets of A:

$\{\}$ (or Ø), $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$, $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, $\{a,b,c,d\}$

You can see that there are 16 subsets, 15 of which are proper subsets.

Listing the sets is fine if you have only a few elements. However, if we were to list all of the subsets of a set containing many elements, it would be quite tedious. Instead, let's consider each element of the set separately.

In our example, there are four elements. For the first element, a, either it's in the set or it's not. Thus there are 2 choices for that first element. Similarly, there are two choices for b—either it's in the set or it's not. Using just those two elements, we see that the subsets are as follows:

$\{\}$—both elements are not in the set
$\{a\}$—a is in; b is not in the set
$\{b\}$—a is not in the set; b is in
$\{a,b\}$—a is in; b is in

Two choices for a times the two for b gives us $2^2 = 4$ subsets. You can draw a tree diagram to see this as well.

Now let's include c. Again, either c is included or it isn't, which gives us two choices. The outcomes are $\{\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$. Note that there are $2^3 = 8$ subsets.

Including all four elements, there are $2^4 = 16$ subsets. 15 of those subsets are proper, 1 subset, namely $\{a,b,c,d\}$, is not.

In general, if you have n elements in your set, then there are 2^n subsets and $2^n - 1$ proper subsets.

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12.6: Helpful Links



These pages on Purplemath.com will provide further background and understanding of both Set Notation and Venn Diagrams. They provide instruction and examples.

- Set Notation
- Venn Diagrams (You'll notice the words "Page 1 of 4" at the top of this page. Be sure to look at all four pages of the Venn Diagram section.)

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12.7: Discuss- Application of Set Theory

Pick a real problem and try to solve it using the general problem solving strategies from this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the Sets Application Directions for detailed directions.

Use the **Set Theory Application** forum in the Discussion Board to complete this assignment.

This assignment is required and worth up to 20 points.

Grading Criteria	Points Possible
The problem: • Is it a real-life problem? • Is it challenging, not trivial? • Is it a unique problem instead of a copy of a classmate's posting?	5
The strategies: • Are one or more general problem solving strategies used? • Are the strategies correctly identified?	5
The presentation: • Is the problem explained well? • Are the problem solving strategies explained well? • Are the appropriate terms used?	4
Your responses: • Did you post at least two responses? • Did you explain how the examples helped you better understand the math in this module? • Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?	6

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CHAPTER OVERVIEW

13: Module 4- Logic

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13.3: Truth Tables and Analyzing Arguments- Examples
13.4: Truth Tables- Conjunction and Disjunction
13.5: Truth Tables- Implication
13.6: Analyzing Arguments with Truth Tables
13.7: Negating “all,” “some,” or “no” statements
13.8: Discuss- Truth Table Practice
13.9: Discuss- Logic Application

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13.1: Module 4 Overview

What You'll Learn To Do: Identify, make, and interpret logic.

Learning Objectives

- Identify, make, and interpret logical statements in English and symbolic notation.
- Express and interpret connectives (including negation) in English and symbolic notation.
- Express and interpret logical quantifiers, and their negations, in English.
- Express and interpret the propositions of conditional, converse, inverse, and contrapositive in English and symbolic notation.
- Construct and interpret truth tables to evaluate the truth value of a compound statement, determine whether or not two statements are logically equivalent, and determine the validity of an argument.
- Recognize when logic is applicable to real-life situations, solve real-life problems, and communicate real-life problems and solutions to others.

Learning Activities

 The learning activities for this module include:

Reading Assignments and Videos

- **Read:** Logic
- **Read:** Truth Tables and Analyzing Arguments: Examples
- **Watch:** Truth Tables: Conjunction and Disjunction
- **Watch:** Truth Tables: Implication
- **Watch:** Analyzing Arguments with Truth Tables
- **Read:** Negating "all," "some," or "no" statements

Homework Assignments

- **Submit:** Logic Homework Assignment #1 (16 points)
- **Submit:** Logic Homework Assignment #2 (14 points)
- **Discuss:** Truth Table Practice (10 points)
- **Discuss:** Logic Application (20 points)
- **Complete:** Exam 2

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13.2: Logic

Logic is a systematic way of thinking that allows us to deduce new information from old information and to parse the meanings of sentences. You use logic informally in everyday life and certainly also in doing mathematics. For example, suppose you are working with a certain circle, call it "Circle X," and you have available the following two pieces of information.

1. Circle X has radius equal to 3.
2. If any circle has radius r, then its area is πr^2 square units.

You have no trouble putting these two facts together to get:

3. Circle X has area 9π square units.

In doing this you are using logic to combine existing information to produce new information. Since a major objective in mathematics is to deduce new information, logic must play a fundamental role. This chapter is intended to give you a sufficient mastery of logic.

It is important to realize that logic is a process of deducing information correctly, *not* just deducing correct information. For example, suppose we were mistaken and Circle X actually had a radius of 4, not 3. Let's look at our exact same argument again.

1. Circle X has radius equal to 3.
2. If any circle has radius r, then its area is πr^2 square units.
3. Circle X has area 9π square units.

The sentence "*Circle X has radius equal to 3.*" is now untrue, and so is our conclusion "*Circle X has area 9π square units.*" But the logic is perfectly correct; the information was combined correctly, even if some of it was false. This distinction between correct logic and correct information is significant because it is often important to follow the consequences of an incorrect assumption. Ideally, we want both our logic *and* our information to be correct, but the point is that they are different things.

In proving theorems, we apply logic to information that is considered obviously true (such as "*Any two points determine exactly one line.*") or is already known to be true (e.g., the Pythagorean theorem). If our logic is correct, then anything we deduce from such information will also be true (or at least as true as the "obviously true" information we began with).

Statements

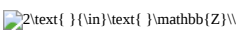
The study of logic begins with statements. A **statement** is a sentence or a mathematical expression that is either definitely true or definitely false. You can think of statements as pieces of information that are either correct or incorrect. Thus statements are pieces of information that we might apply logic to in order to produce other pieces of information (which are also statements).

Example 1

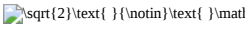
Here are some examples of statements. They are all true.

If a circle has radius r, then its area is πr^2 square units.

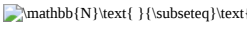
Every even number is divisible by 2.



(2 is an element of the set of integers (or more simply, 2 is an integer).)



(The square root of 2 is not an integer.)



(The set of natural numbers is a subset of the set of integers.)

The set {0,1,2} has three elements.

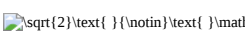
Some right triangles are isosceles.

Example 2

Here are some additional statements. They are all false.

All right triangles are isosceles.

5 = 2



(The square root of 2 is not a real number.)

\mathbb{Z}\subseteq\mathbb{N}\\ (The set of integers is a subset of the set of natural numbers.)

{0,1,2}\cap\mathbb{N}=\varnothing\\ (The intersection of the set {0,1,2} and the natural numbers is the empty set.)

Example 3

Here we pair sentences or expressions that are not statements with similar expressions that are statements.

NOT Statements	Statements
Add 5 to both sides.	Adding 5 to both sides of $x - 5 = 37$ gives $x = 42$.
\mathbb{Z}\\ (The set of integers)	42\text{ }{\in}\text{ }\mathbb{Z}\\ (42 is an element of the set of integers.)
42	42 is not a number.
What is the solution of $2x = 84$?	The solution of $2x = 84$ is 42.

Example 4

We will often use the letters P, Q, R, and S to stand for specific statements. When more letters are needed we can use subscripts. Here are more statements, designated with letters. You decide which of them are true and which are false.

P : For every integer $n > 1$, the number $2^n - 1$ is prime.
Q : Every polynomial of degree n has at most n roots.
R : The function $f(x) = x^2$ is continuous.
S_1 : \mathbb{N}\subseteq\varnothing\\
S_2 : {0,-1,-2}\cap\mathbb{N}=\varnothing\\

Designating statements with letters (as was done above) is a very useful shorthand. In discussing a particular statement, such as "The function $f(x) = x^2$ is continuous," it is convenient to just refer to it as R to avoid having to write or say it many times.

Statements can contain variables. Here is an example.

P : If an integer x is a multiple of 6, then x is even.

This is a sentence that is true. (All multiples of 6 are even, so no matter which multiple of 6 the integer x happens to be, it is even.) Since the sentence P is definitely true, it is a statement. When a sentence or statement P contains a variable such as x, we sometimes denote it as $P(x)$ to indicate that it is saying something about x. Thus the above statement can be denoted as

$P(x)$: If an integer x is a multiple of 6, then x is even.

A statement or sentence involving two variables might be denoted $P(x, y)$, and so on.

It is quite possible for a sentence containing variables to not be a statement. Consider the following example.

$Q(x)$: The integer x is even.

Is this a statement? Whether it is true or false depends on just which integer x is. It is true if $x = 4$ and false if $x = 7$, etc. But without any stipulations on the value of x it is impossible to say whether $Q(x)$ is true or false. Since it is neither definitely true nor definitely false, $Q(x)$ cannot be a statement. A sentence such as this, whose truth depends on the value of one or more variables, is called an **open sentence**. The variables in an open sentence (or statement) can represent any type of entity, not just numbers. Here is an open sentence where the variables are functions:

$R(f, g)$: The function f is the derivative of the function g.

This open sentence is true if $f(x) = 2x$ and $g(x) = x^2$. It is false if $f(x) = x^3$ and $g(x) = x^2$, etc. We point out that a sentence such as $R(f, g)$ (that involves variables) can be denoted either as $R(f, g)$ or just R. We use the expression $R(f, g)$ when we want to emphasize that the sentence involves variables.

We will have more to say about open sentences later, but for now let's return to statements.

Statements are everywhere in mathematics. Any result or theorem that has been proved true is a statement. The quadratic formula and the Pythagorean theorem are both statements:

P : The solutions of the equation $\displaystyle{a}x^2+bx+c=0\text{ are }x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\$.
Q : If a right triangle has legs of lengths a and b and hypotenuse of length $a^2+b^2={c}^2\\$..

Here is a very famous statement, so famous, in fact, that it has a name. It is called **Fermat's last theorem** after Pierre Fermat, a seventeenth-century French mathematician who scribbled it in the margin of a notebook.

R : For all numbers a, b, c, $n \in \mathbb{N}\\$ with $n > 2$, it is the case that $a^n+b^n\neq{c}^n\\$.

Fermat believed this statement was true. He noted that he could prove it was true, except his notebook's margin was too narrow to contain his proof. It is doubtful that he really had a correct proof in mind, for after his death generations of brilliant mathematicians tried unsuccessfully to prove that his statement was true (or false). Finally, in 1993, Andrew Wiles of Princeton University announced that he had devised a proof. Wiles had worked on the problem for over seven years, and his proof runs through hundreds of pages. The moral of this story is that some true statements are not obviously true.

Here is another statement famous enough to be named. It was first posed in the eighteenth century by the German mathematician Christian Goldbach, and thus is called the Goldbach conjecture:

S : Every even integer greater than 2 is a sum of two prime numbers.

You must agree that S is either true or false. It appears to be true, because when you examine even numbers that are bigger than 2, they seem to be sums of two primes: 4 = 2+2, 6 = 3+3, 8 = 3+5, 10 = 5+5, 12 = 5+7, 100 = 17+83 and so on. But that's not to say there isn't some large even number that's not the sum of two primes. If such a number exists, then S is false. The thing is, in the over 260 years since Goldbach first posed this problem, no one has been able to determine whether it's true or false. But since it is clearly either true or false, S is a statement.

This book is about the methods that can be used to prove that S (or any other statement) is true or false. To prove that a statement is true, we start with obvious statements (or other statements that have been proven true) and use logic to deduce more and more complex statements until finally we obtain a statement such as S. Of course some statements are more difficult to prove than others, and S appears to be notoriously difficult; we will concentrate on statements that are easier to prove.

But the point is this: In proving that statements are true, we use logic to help us understand statements and to combine pieces of information to produce new pieces of information. In the next several sections we explore some standard ways that statements can be combined to form new statements, or broken down into simpler statements.

And, Or, Not

The word "and" can be used to combine two statements to form a new statement. Consider for example the following sentence.

R_1 : The number 2 is even **and** the number 3 is odd.

We recognize this as a true statement, based on our common-sense understanding of the meaning of the word "and." Notice that R_1 is made up of two simpler statements:

P : The number 2 is even.
Q : The number 3 is odd.

These are joined together by the word "and" to form the more complex statement R_1. The statement R_1 asserts that P and Q are both true. Since both P and Q are in fact true, the statement R_1 is also true.

Had one or both of P and Q been false, then R_1 would be false. For instance, each of the following statements is false.

R_2 : The number 1 is even **and** the number 3 is odd.
R_3 : The number 2 is even **and** the number 4 is odd.
R_4 : The number 3 is even **and** the number 2 is odd.

From these examples we see that any two statements P and Q can be combined to form a new statement "P**and**Q." In the spirit of using letters to denote statements, we now introduce the special symbol $\wedge$ to stand for the word "and." Thus if P and Q are statements, $P \wedge Q$ stands for the statement "P**and**Q." The statement $P \wedge Q$ is true if both P and Q are true; otherwise it is false. This is summarized in the following table, called a **truth table**.

P	Q	$P \wedge Q$
T	T	T

P	Q	$P \wedge Q$
T	F	F
F	T	F
F	F	F

In this table, T stands for "True," and F stands for "False." (T and F are called **truth values**.) Each line lists one of the four possible combinations or truth values for P and Q, and the column headed by $P \wedge Q$ tells whether the statement $P \wedge Q$ is true or false in each case.

Statements can also be combined using the word "or." Consider the following four statements.

S_1 : The number 2 is even or the number 3 is odd.
S_2 : The number 1 is even or the number 3 is odd.
S_3 : The number 2 is even or the number 4 is odd.
S_4 : The number 3 is even or the number 2 is odd.

In mathematics, the assertion "P**or**Q" is always understood to mean that one or both of P and Q is true. Thus statements S_1, S_2, S_3 are all true, while S_4 is false. The symbol $\vee$ is used to stand for the word "or." So if P and Q are statements, $P \vee Q$ represents the statement "P**or**Q." Here is the truth table.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

It is important to be aware that the meaning of "or" expressed in the above table differs from the way it is sometimes used in everyday conversation. For example, suppose a university official makes the following threat:

You pay your tuition **or** you will be withdrawn from school.

You understand that this means that either you pay your tuition or you will be withdrawn from school, *but not both*. In mathematics we never use the word "or" in such a sense. For us "or" means exactly what is stated in the table for $\vee$. Thus $P \vee Q$ being true means *one* or *both* of P and Q is true. If we ever need to express the fact that exactly one of P and Q is true, we use one of the following constructions:

P**or**Q, **but not both.**

Either$P$**or**Q.

If the university official were a mathematician, he might have qualified his statement in one of the following ways.

Pay your tuition **or** you will be withdrawn from school, **but not both**.
Either you pay your tuition **or** you will be withdrawn from school.

To conclude this section, we mention another way of obtaining new statements from old ones. Given any statement P, we can form the new statement "**It is not true that**P." For example, consider the following statement.

The number 2 is even.

This statement is true. Now change it by inserting the words "It is not true that" at the beginning:

It is not true that the number 2 is even.

This new statement is false.

For another example, starting with the false statement "$2\in\varnothing\\$" we get the true statement "It is not true that $2\in\varnothing\\$."

We use the symbol $\sim$ to stand for the words "It's not true that," so $\sim P$ means "**It's not true that***P*." We often read $\sim P$ simply as "not *P*." Unlike $\wedge$ and $\vee$, which combine two statements, the symbol $\sim$ just alters a single statement. Thus its truth table has just two lines, one for each possible truth value of *P*.

P	$\sim P$
T	F
F	T

The statement $\sim P$ is called the **negation** of *P*. The negation of a specific statement can be expressed in numerous ways. Consider

P : The number 2 is even.

Here are several ways of expressing its negation.

$\sim P$: It's not true that the number 2 is even.
$\sim P$: It is false that the number 2 is even.
$\sim P$: The number 2 is not even.

In this section we've learned how to combine or modify statements with the operations $\wedge$, $\vee$ and $\sim$. Of course we can also apply these operations to open sentences or a mixture of open sentences and statements. For example, (*x* is an even integer)$\wedge$(3 is an odd integer) is an open sentence that is a combination of an open sentence and a statement.

Conditional Statements

There is yet another way to combine two statements. Suppose we have in mind a specific integer *a*. Consider the following statement about *a*.

R : If the integer a is a multiple of 6, then a is divisible by 2.

We immediately spot this as a true statement based on our knowledge of integers and the meanings of the words "if" and "then." If integer a is a multiple of 6, then a is even, so therefore a is divisible by 2. Notice that *R* is built up from two simpler statements:

P : The integer a is a multiple of 6.
Q : The integer a is divisible by 2.
R : If P, then Q.

In general, given any two statements *P* and *Q* whatsoever, we can form the new statement "*If P, then Q*." This is written symbolically as $P \Rightarrow Q$ which we read as "*If P, then Q*," or "*P implies Q*." Like $\wedge$ and $\vee$, the symbol $\Rightarrow$ has a very specific meaning. When we assert that the statement $P \Rightarrow Q$ is true, we mean that *if P* is true *then Q* must also be true. (In other words we mean that the condition *P* being true forces *Q* to be true.) A statement of form $P \Rightarrow Q$ is called a **conditional** statement because it means *Q* will be true *under the condition* that *P* is true.

You can think of $P \Rightarrow Q$ as being a promise that whenever *P* is true, *Q* will be true also. There is only one way this promise can be broken (i.e. be false) and that is if *P* is true but *Q* is false. Thus the truth table for the promise $P \Rightarrow Q$ is as follows:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Perhaps you are bothered by the fact that $P \Rightarrow Q$ is true in the last two lines of this table. Here's an example to convince you that the table is correct. Suppose your professor makes the following promise:

If you pass the final exam, **then** you will pass the course.

Your professor is making the promise

(You pass the exam) ⇒ (You pass the course).

Under what circumstances did she lie? There are four possible scenarios, depending on whether or not you passed the exam and whether or not you passed the course. These scenarios are tallied in the following table.

You pass exam	You pass course	(You pass exam) ⇒ (You pass course)
T	T	T
T	F	F
F	T	T
F	F	T

The first line describes the scenario where you pass the exam and you pass the course. Clearly the professor kept her promise, so we put a *T* in the third column to indicate that she told the truth. In the second line, you passed the exam, but your professor gave you a failing grade in the course. In this case she broke her promise, and the *F* in the third column indicates that what she said was untrue.

Now consider the third row. In this scenario you failed the exam but still passed the course. How could that happen? Maybe your professor felt sorry for you. But that doesn't make her a liar. Her only promise was that if you passed the exam then you would pass the course. She did not say passing the exam was the **only way** to pass the course. Since she didn't lie, then she told the truth, so there is a *T* in the third column.

Finally look at the fourth row. In that scenario you failed the exam and you failed the course. Your professor did not lie; she did exactly what she said she would do. Hence the *T* in the third column.

In mathematics, whenever we encounter the construction "*If P, then Q*" it means exactly what the truth table for ⇒ expresses. But of course there are other grammatical constructions that also mean $P \Rightarrow Q$. Here is a summary of the main ones.

{P}\Rightarrow{Q}\begin{cases}{\text{If }P\text{, then }Q.}\\{Q\text{ if }P.}\\{Q\text{ whenever }P.}\\{Q\text{, provided that }P.}\\{\text{Whenever }P\text{, then also }Q.}\\{P\text{ is a sufficient condition for }Q.}\\{\text{For }Q\text{, it is sufficient that }P.}\\{Q\text{ is a necessary condition for }P.}\\{\text{For }P\text{, it is necessary that }Q.}\\{P\text{ only if }Q.}\end{cases}\\

These can all be used in the place of (and mean exactly the same thing as) "*If P, then Q*." You should analyze the meaning of each one and convince yourself that it captures the meaning of $P \Rightarrow Q$. For example, $P \Rightarrow Q$ means the condition of *P* being true is enough (i.e., sufficient) to make *Q* true; hence "*P is a sufficient condition for Q*."

The wording can be tricky. Often an everyday situation involving a conditional statement can help clarify it. For example, consider your professor's promise:

(You pass the exam) ⇒ (You pass the course)

This means that your passing the exam is a sufficient (though perhaps not necessary) condition for your passing the course. Thus your professor might just as well have phrased her promise in one of the following ways.

Passing the exam is a sufficient condition for passing the course.
For you to pass the course, it is sufficient that you pass the exam.

However, when we want to say "*If P, then Q*" in everyday conversation, we do not normally express this as "*Q is a necessary condition for P*" or "*P only if Q*." But such constructions are not uncommon in mathematics. To understand why they make sense, notice that $P \Rightarrow Q$ being true means that it's impossible that *P* is true but *Q* is false, so in order for *P* to be true it is necessary that *Q* is true; hence "*Q is a necessary condition for P*." And this means that *P* can only be true if *Q* is true, i.e., "*P only if Q*."

Biconditional Statements

It is important to understand that $P \Rightarrow Q$ is not the same as $Q \Rightarrow P$. To see why, suppose that a is some integer and consider the statements

(a is a multiple of 6) ⇒ (a is divisible by 2),
(a is divisible by 2) ⇒ (a is a multiple of 6).

The first statement asserts that if a is a multiple of 6 then a is divisible by 2. This is clearly true, for any multiple of 6 is even and therefore divisible by 2. The second statement asserts that if a is divisible by 2 then it is a multiple of 6. This is not necessarily true, for a = 4 (for instance) is divisible by 2, yet not a multiple of 6. Therefore the meanings of $P \Rightarrow Q$ and $Q \Rightarrow P$ are in general quite different. The conditional statement $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$, so a conditional statement and its converse express entirely different things.

However, the **contrapositive** of $P \Rightarrow Q$, $\sim Q \Rightarrow \sim P$, is equivalent to $P \Rightarrow Q$. Similarly, the **inverse** of $P \Rightarrow Q$, which is $\sim P \Rightarrow \sim Q$, is equivalent to the converse $Q \Rightarrow P$. In "Truth Tables for Statements," we will learn how to show these equivalences using a truth table.

But sometimes, if P and Q are just the right statements, it can happen that $P \Rightarrow Q$ and $Q \Rightarrow P$ are both necessarily true. For example, consider the statements

(a is even) ⇒ (a is divisible by 2),
(a is divisible by 2) ⇒ (a is even).

No matter what value a has, both of these statements are true. Since both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true, it follows that $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is true.

We now introduce a new symbol ⇔ to express the meaning of the statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. The expression $P \Leftrightarrow Q$ is understood to have exactly the same meaning as $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. According to the previous section, $Q \Rightarrow P$ is read as "*P if Q*," and $P \Rightarrow Q$ can be read as "*P only if Q*." Therefore we pronounce $P \Leftrightarrow Q$ as "*P if and only if Q*." For example, given an integer a, we have the true statement

(a is even) ⇔ (a is divisible by 2),

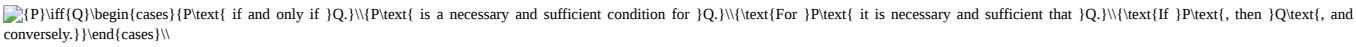
which we can read as "*Integer a is even if and only if a is divisible by 2.*"

The truth table for ⇔ is shown below. Notice that in the first and last rows, both P ⇒ Q and Q ⇒ P are true (according to the truth table for ⇒), so (P ⇒ Q) ∧ (Q ⇒ P) is true, and hence P ⇔ Q is true. However, in the middle two rows one of P ⇒ Q or Q ⇒ P is false, so (P ⇒ Q)∧(Q ⇒ P) is false, making P ⇔ Q false.

P	Q	P ⇔ Q
T	T	T
T	F	F
F	T	F
F	F	T

Compare the statement R : (a is even) ⇔ (a is divisible by 2) with this truth table. If a is even then the two statements on either side of ⇔ are true, so according to the table R is true. If a is odd then the two statements on either side of ⇔ are false, and again according to the table R is true. Thus R is true no matter what value a has. In general, $P \Leftrightarrow Q$ being true means P and Q are both true or both false.

Not surprisingly, there are many ways of saying $P \Leftrightarrow Q$ in English. The following constructions all mean $P \Leftrightarrow Q$:


{P}\iff{Q}\begin{cases}{P\text{ if and only if }Q.}\\{P\text{ is a necessary and sufficient condition for }Q.}\\{\text{For }P\text{ it is necessary and sufficient that }Q.}\\{\text{If }P\text{, then }Q\text{, and conversely.}}\end{cases}\

The first three of these just combine constructions from the previous section to express that $P \Rightarrow Q$ and $Q \Rightarrow P$. In the last one, the words "*and conversely*" mean that in addition to "*If P, then Q*" being true, the converse statement "*If Q, then P*" is also true.

Truth Tables for Statements

You should now know the truth tables for ∧, ∨, ∼, ⇒ and ⇔. They should be *internalized* as well as memorized. You must understand the symbols thoroughly, for we now combine them to form more complex statements.

For example, suppose we want to convey that one or the other of P and Q is true but they are not both true. No single symbol expresses this, but we could combine them as

$$(P \vee Q) \wedge \sim (P \wedge Q),$$

which literally means:

P or Q is true, and it is not the case that both P and Q are true.

This statement will be true or false depending on the truth values of P and Q. In fact we can make a truth table for the entire statement. Begin as usual by listing the possible true/false combinations of P and Q on four lines. The statement $(P \vee Q) \wedge \sim (P \wedge Q)$ contains the individual statements $(P \vee Q)$ and $(P \wedge Q)$, so we next tally their truth values in the third and fourth columns. The fifth column lists values for $\sim (P \wedge Q)$, and these are just the opposites of the corresponding entries in the fourth column. Finally, combining the third and fifth columns with $\wedge$, we get the values for $(P \vee Q) \wedge \sim (P \wedge Q)$ in the sixth column.

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$\sim (P \wedge Q)$	$(P \vee Q) \wedge \sim (P \wedge Q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

This truth table tells us that $(P \vee Q) \wedge \sim (P \wedge Q)$ is true precisely when one but not both of P and Q are true, so it has the meaning we intended. (Notice that the middle three columns of our truth table are just "helper columns" and are not necessary parts of the table. In writing truth tables, you may choose to omit such columns if you are confident about your work.)

For another example, consider the following familiar statement concerning two real numbers x and y:

The product xy equals zero if and only if $x = 0$ or $y = 0$.

This can be modeled as $(xy = 0) \Leftrightarrow (x = 0 \vee y = 0)$. If we introduce letters P, Q, and R for the statements $xy = 0$, $x = 0$ and $y = 0$, it becomes $P \Leftrightarrow (Q \vee R)$. Notice that the parentheses are necessary here, for without them we wouldn't know whether to read the statement as $P \Leftrightarrow (Q \vee R)$ or $(P \Leftrightarrow Q) \vee R$.

Making a truth table for $P \Leftrightarrow (Q \vee R)$ entails a line for each T/F combination for the three statements P, Q, and R. The eight possible combinations are tallied in the first three columns of the following table.

P	Q	R	$Q \vee R$	$P \Leftrightarrow (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

We fill in the fourth column using our knowledge of the truth table for $\vee$. Finally the fifth column is filled in by combining the first and fourth columns with our understanding of the truth table for $\Leftrightarrow$. The resulting table gives the true/false values of $P \Leftrightarrow (Q \vee R)$ for all values of P, Q, and R.

Notice that when we plug in various values for x and y, the statements $P : xy = 0$, $Q : x = 0$ and $R : y = 0$ have various truth values, but the statement $P \Leftrightarrow (Q \vee R)$ is always true. For example, if $x = 2$ and $y = 3$, then P, Q, and R are all false. This scenario is described in the last row of the table, and there we see that $P \Leftrightarrow (Q \vee R)$ is true. Likewise if $x = 0$ and $y = 7$, then P and Q are true and R is false, a scenario described in the second line of the table, where again $P \Leftrightarrow (Q \vee R)$ is true. There is a simple reason why $P \Leftrightarrow (Q \vee R)$ is true for any values of x and y: It is that $P \Leftrightarrow (Q \vee R)$ represents $(xy = 0) \Leftrightarrow (x = 0 \vee y = 0)$, which is a *true mathematical statement*. It is absolutely impossible for it to be false.

This may make you wonder about the lines in the table where $P \Leftrightarrow (Q \vee R)$ is false. Why are they there? The reason is that $P \Leftrightarrow (Q \vee R)$ can also represent a false statement. To see how, imagine that at the end of the semester your professor makes the following promise.

> You pass the class if and only if you get an "A" on the final or you get a "B" on the final.

This promise has the form $P \Leftrightarrow (Q \vee R)$, so its truth values are tabulated in the above table. Imagine it turned out that you got an "A" on the exam but failed the course. Then surely your professor lied to you. In fact, P is false, Q is true and R is false. This scenario is reflected in the sixth line of the table, and indeed $P \Leftrightarrow (Q \vee R)$ is false (i.e., it is a lie).

The moral of this example is that people can lie, but true mathematical statements **never** lie.

We close this section with a word about the use of parentheses. The symbol $\sim$ is analogous to the minus sign in algebra. It negates the expression it precedes. Thus $\sim P \vee Q$ means $(\sim P) \vee Q$, not $\sim(P \vee Q)$. In $\sim(P \vee Q)$, the value of the entire expression $P \vee Q$ is negated.

Logical Equivalence

In contemplating the truth table for $P \Leftrightarrow Q$, you probably noticed that $P \Leftrightarrow Q$ is true exactly when P and Q are both true or both false. In other words, $P \Leftrightarrow Q$ is true precisely when at least one of the statements $P \wedge Q$ or $\sim P \wedge \sim Q$ is true. This may tempt us to say that $P \Leftrightarrow Q$ means the same thing as $(P \wedge Q) \vee (\sim P \wedge \sim Q)$.

To see if this is really so, we can write truth tables for $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$. In doing this, it is more efficient to put these two statements into the same table, as follows. (This table has helper columns for the intermediate expressions $\sim P$, $\sim Q$, $(P \wedge Q)$, and $(\sim P \wedge \sim Q)$.)

P	Q	$\sim P$	$\sim Q$	$(P \wedge Q)$	$(\sim P \wedge \sim Q)$	$(P \wedge Q) \vee (\sim P \wedge \sim Q)$	$P \Leftrightarrow Q$
T	T	F	F	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	F	F	F	F	F
F	F	T	T	F	T	T	T

The table shows that $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ have the same truth value, no matter the values P and Q. It is as if $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ are algebraic expressions that are equal no matter what is "plugged into" variables P and Q. We express this state of affairs by writing

$$P \Leftrightarrow Q = (P \wedge Q) \vee (\sim P \wedge \sim Q)$$

and saying that $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ are **logically equivalent**.

In general, two statements are **logically equivalent** if their truth values match up line-for-line in a truth table.

Logical equivalence is important because it can give us different (and potentially useful) ways of looking at the same thing. As an example, the following table shows that $P \Rightarrow Q$ is logically equivalent to $(\sim Q) \Rightarrow (\sim P)$.

P	Q	$\sim P$	$\sim Q$	$(\sim Q) \Rightarrow (\sim P)$	$P \Rightarrow Q$
T	T	F	F	T	T
T	F	F	T	F	F

P	Q	$\sim P$	$\sim Q$	$(\sim Q) \Rightarrow (\sim P)$	$P \Rightarrow Q$
F	T	T	F	T	T
F	F	T	T	T	T

The fact that $P \Rightarrow Q = (\sim Q) \Rightarrow (\sim P)$ is useful because so many theorems have the form $P \Rightarrow Q$. As we will see in Chapter 5, proving such a theorem may be easier if we express it in the logically equivalent form $(\sim Q) \Rightarrow (\sim P)$.

There are two pairs of logically equivalent statements that come up again and again throughout this book and beyond. They are prevalent enough to be dignified by a special name: **DeMorgan's laws.**

Fact: DeMorgan's Laws

1. $\sim(P \wedge Q) = (\sim P) \vee (\sim Q)$
2. $\sim(P \vee Q) = (\sim P) \wedge (\sim Q)$

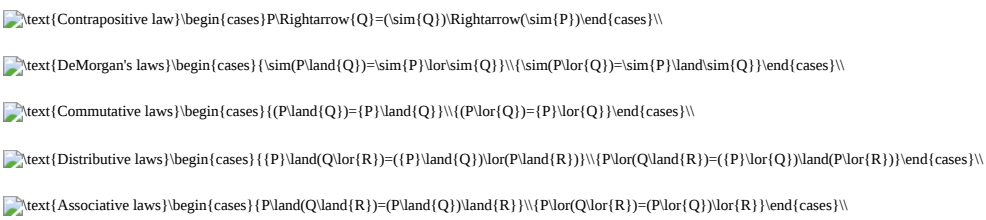
The first of DeMorgan's laws is verified by the following table. You are asked to verify the second in one of the exercises.

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$(\sim P) \vee (\sim Q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

DeMorgan's laws are actually very natural and intuitive. Consider the statement $\sim(P \wedge Q)$, which we can interpret as meaning that it is not the case that both P and Q are true. If it is not the case that both P and Q are true, then at least one of P or Q is false, in which case $(\sim P) \vee (\sim Q)$ is true. Thus $\sim(P \wedge Q)$ means the same thing as $(\sim P) \vee (\sim Q)$.

DeMorgan's laws can be very useful. Suppose we happen to know that some statement having form $\sim(P \vee Q)$ is true. The second of DeMorgan's laws tells us that $(\sim Q) \wedge (\sim P)$ is also true, hence $\sim P$ and $\sim Q$ are both true as well. Being able to quickly obtain such additional pieces of information can be extremely useful.

Here is a summary of some significant logical equivalences. Those that are not immediately obvious can be verified with a truth table.



Notice how the distributive law $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$ has the same structure as the distributive law $p(q + r) = p \cdot q + p \cdot r$ from algebra. Concerning the associative laws, the fact that $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$ means that the position of the parentheses is irrelevant, and we can write this as $P \wedge Q \wedge R$ without ambiguity. Similarly, we may drop the parentheses in an expression such as $P \vee (Q \vee R)$.

But parentheses are essential when there is a mix of $\wedge$ and $\vee$, as in $P \vee (Q \wedge R)$. Indeed, $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge R$ are **not** logically equivalent.

Negating Statements

Given a statement R, the statement $\sim R$ is called the **negation** of R. If R is a complex statement, then it is often the case that its negation $\sim R$ can be written in a simpler or more useful form. The process of finding this form is called **negating** R. In proving theorems it is often necessary to negate certain statements. We now investigate how to do this.

We have already examined part of this topic. **DeMorgan's laws**

$\sim (P \wedge Q) = (\sim P) \vee (\sim Q)$
$\sim (P \vee Q) = (\sim P) \wedge (\sim Q)$

(from "Logical Equivalence") can be viewed as rules that tell us how to negate the statements $P \wedge Q$ and $P \vee Q$. Here are some examples that illustrate how DeMorgan's laws are used to negate statements involving "and" or "or."

Example 5

Consider negating the following statement.

R : You can solve it by factoring or with the quadratic formula.

Now, R means (You can solve it by factoring) $\vee$ (You can solve it with Q.F.), which we will denote as $P \vee Q$. The negation of this is

$\sim(P \vee Q) = (\sim P) \wedge (\sim Q)$.

Therefore, in words, the negation of R is

$\sim R$: You can't solve it by factoring and you can't solve it with the quadratic formula.

Maybe you can find $\sim R$ without invoking DeMorgan's laws. That is good; you have internalized DeMorgan's laws and are using them unconsciously.

Example 6

We will negate the following sentence.

R : The numbers x and y are both odd.

This statement means (x is odd) $\wedge$ (y is odd), so its negation is

$\sim$[(x is odd) $\wedge$ (y is odd)] = $\sim$(x is odd) $\vee$ $\sim$(y is odd)
(x is odd) $\wedge$ (y is odd) = (x is even) $\vee$ (y is even).

Therefore the negation of R can be expressed in the following ways:

$\sim$ R : The number x is even or the number y is even.
$\sim$ R : At least one of x and y is even.

Now let's move on to a slightly different kind of problem. It's often necessary to find the negations of quantified statements. For example, consider $\sim(\forall x \in$ \mathbb{N}\\, $P(x))$. Reading this in words, we have the following:

It is not the case that $P(x)$ is true for all natural numbers x.

This means $P(x)$ is false for at least one x. In symbols, this is $\exists\, x \in$ \mathbb{N}\\, $\sim P(x)$. Thus $\sim$ ($\forall x \in$ \mathbb{N}\\, $P(x)$) = $\exists\, x \in$ \mathbb{N}\\, $\sim P(x)$. Similarly, you can reason out that $\sim$ ($\exists\, x \in$ \mathbb{N}\\, $P(x)$) = $\forall x \in$ \mathbb{N}\\, $\sim P(x)$. In general:

$\sim (\forall x \in S,\, P(x)) = \exists\, x \in S,\, \sim P(x)$
$\sim (\exists\, x \in S,\, P(x)) = \forall x \in S,\, \sim P(x)$

Logical Inference

Suppose we know that a statement of form $P \Rightarrow Q$ is true. This tells us that whenever P is true, Q will also be true. By itself, $P \Rightarrow Q$ being true does not tell us that either P or Q is true (they could both be false, or P could be false and Q true). However if in addition we happen to know that P is true then it must be that Q is true. This is called a **logical inference**: Given two true statements we can infer that a third statement is true. In this instance true statements $P \Rightarrow Q$ and P are "added together" to get Q. This is described below with $P \Rightarrow Q$ and P stacked one atop the other with a line separating them from Q. The intended meaning is that $P \Rightarrow Q$ combined with P produces Q.

$$\frac{\begin{array}{l}P \Rightarrow Q \\ P\end{array}}{Q} \qquad \frac{\begin{array}{l}P \Rightarrow Q \\ \sim Q\end{array}}{\sim P} \qquad \frac{\begin{array}{l}P \vee Q \\ \sim P\end{array}}{Q}$$

Two other logical inferences are listed above. In each case you should convince yourself (based on your knowledge of the relevant truth tables) that the truth of the statements above the line forces the statement below the line to be true.

Following are some additional useful logical inferences. The first expresses the obvious fact that if P and Q are both true then the statement $P \wedge Q$ will be true. On the other hand, $P \wedge Q$ being true forces P (also Q) to be true. Finally, if P is true, then $P \vee Q$ must be true, no matter what statement Q is.

$$\frac{\begin{array}{l}P\\Q\end{array}}{P \wedge Q} \qquad \frac{P \wedge Q}{P} \qquad \frac{P}{P \vee Q}$$

These inferences are so intuitively obvious that they scarcely need to be mentioned. However, they represent certain patterns of reasoning that we will frequently apply to sentences in proofs, so we should be cognizant of the fact that we are using them.

The first two statements in each case are called “premises” and the final statement is the “conclusion.” We combine premises with $\wedge$ (“and”). The premises together imply the conclusion. Thus, the first argument would have $((P \Rightarrow Q) \wedge P) \Rightarrow Q$ as its symbolic statement.

An Important Note

It is important to be aware of the reasons that we study logic. There are three very significant reasons. First, the truth tables we studied tell us the exact meanings of the words such as “and,” “or,” “not,” and so on. For instance, whenever we use or read the “If…, then” construction in a mathematical context, logic tells us exactly what is meant. Second, the rules of inference provide a system in which we can produce new information (statements) from known information. Finally, logical rules such as DeMorgan’s laws help us correctly change certain statements into (potentially more useful) statements with the same meaning. Thus logic helps us understand the meanings of statements and it also produces new meaningful statements.

Logic is the glue that holds strings of statements together and pins down the exact meaning of certain key phrases such as the “If…, then” or “For all” constructions. Logic is the common language that all mathematicians use, so we must have a firm grip on it in order to write and understand mathematics.

But despite its fundamental role, logic’s place is in the background of what we do, not the forefront. From here on, the beautiful symbols $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\sim$, $\forall$ and $\exists$ are rarely written. But we are aware of their meanings constantly. When reading or writing a sentence involving mathematics we parse it with these symbols, either mentally or on scratch paper, so as to understand the true and unambiguous meaning.

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13.3: Truth Tables and Analyzing Arguments- Examples

Truth Tables

Because complex Boolean statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the complex statement true and false

Truth Table

A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example 1

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise."

This is a complex statement made of two simpler conditions: "is a sectional," and "has a chaise." For simplicity, let's use *S* to designate "is a sectional," and *C* to designate "has a chaise." The condition *S* is true if the couch is a sectional.

A truth table for this would look like this:

S	*C*	*S* or *C*
T	T	T
T	F	T
F	T	T
F	F	F

In the table, T is used for true, and F for false. In the first row, if *S* is true and *C* is also true, then the complex statement "*S* or *C*" is true. This would be a sectional that also has a chaise, which meets our desire.

Remember also that *or* in logic is not exclusive; if the couch has both features, it does meet the condition.

To shorthand our notation further, we're going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

Symbols

The symbol $\bigwedge$ is used for *and*: *A* and *B* is notated $A \bigwedge B$.

The symbol $\bigvee$ is used for *or*: *A* or *B* is notated $A \bigvee B$

The symbol ~ is used for *not*: not *A* is notated ~*A*

You can remember the first two symbols by relating them to the shapes for the union and intersection. $A \bigwedge B$ would be the elements that exist in both sets, in $A \bigcap B$. Likewise, $A \bigvee B$ would be the elements that exist in either set, in $A \bigcup B$.

In the previous example, the truth table was really just summarizing what we already know about how the *or* statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

Basic Truth Tables

A	B	$A \bigwedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A	B	$A \bigvee B$
T	T	T
T	F	T
F	T	T
F	F	F

A	~A
T	F
F	T

Truth tables really become useful when analyzing more complex Boolean statements.

Example 2

Create a truth table for the statement $A \bigwedge \sim(B \bigvee C)$

It helps to work from the inside out when creating truth tables, and create tables for intermediate operations. We start by listing all the possible truth value combinations for *A*, *B*, and *C*. Notice how the first column contains 4 Ts followed by 4 Fs, the second column contains 2 Ts, 2 Fs, then repeats, and the last column alternates. This pattern ensures that all combinations are considered. Along with those initial values, we'll list the truth values for the innermost expression, $B \bigvee C$.

A	*B*	*C*	$B \bigvee C$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Next we can find the negation of $B \bigvee C$, working off the $B \bigvee C$ column we just created.

A	*B*	*C*	$B \bigvee C$	$\sim(B \bigvee C)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	F
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F

F	F	T	T	F
F	F	F	F	T

Finally, we find the values of $A and \sim(B \bigvee C)$

A	B	C	$B \bigvee C$	$\sim(B \bigvee C)$	$A \bigwedge \sim(B \bigvee C)$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	F

It turns out that this complex expression is only true in one case: if A is true, B is false, and C is false.

When we discussed conditions earlier, we discussed the type where we take an action based on the value of the condition. We are now going to talk about a more general version of a conditional, sometimes called an **implication**.

Implications

Implications are logical conditional sentences stating that a statement *p*, called the antecedent, implies a consequence *q*.

Implications are commonly written as $p \rightarrow q$

Implications are similar to the conditional statements we looked at earlier; p → q is typically written as "if p then q," or "p therefore q." The difference between implications and conditionals is that conditionals we discussed earlier suggest an action—if the condition is true, then we take some action as a result. Implications are a logical statement that suggest that the consequence must logically follow if the antecedent is true.

Example 3

The English statement "If it is raining, then there are clouds is the sky" is a logical implication. It is a valid argument because if the antecedent "it is raining" is true, then the consequence "there are clouds in the sky" must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining. If the antecedent is false, then the implication becomes irrelevant.

Example 4

A friend tells you that "if you upload that picture to Facebook, you'll lose your job." There are four possible outcomes:

1. You upload the picture and keep your job
2. You upload the picture and lose your job
3. You don't upload the picture and keep your job
4. You don't upload the picture and lose your job

There is only one possible case where your friend was lying—the first option where you upload the picture and keep your job. In the last two cases, your friend didn't say anything about what would happen if you didn't upload the picture, so you can't conclude their statement is invalid, even if you didn't upload the picture and still lost your job.

In traditional logic, an implication is considered valid (true) as long as there are no cases in which the antecedent is true and the consequence is false. It is important to keep in mind that symbolic logic cannot capture all the intricacies of the English language.

Truth Values for Implications

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 5

Construct a truth table for the statement $(m \bigwedge \sim p) \to r$

We start by constructing a truth table for the antecedent.

m	p	$\sim p$	$m \bigwedge \sim p$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Now we can build the truth table for the implication

m	p	$\sim p$	$m \bigwedge \sim p$	r	$(m \bigwedge \sim p) \to r$
T	T	F	F	T	T
T	F	T	T	T	T
F	T	F	F	T	T
F	F	T	F	T	T
T	T	F	F	F	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	F	F	T

In this case, when m is true, p is false, and r is false, then the antecedent $m \bigwedge \sim p$ will be true but the consequence false, resulting in a invalid implication; every other case gives a valid implication.

For any implication, there are three related statements, the converse, the inverse, and the contrapositive.

Related Statements

The original implication is “if p then q”: $p \to q$

The converse is “if q then p”: $q \to p$

The inverse is “if not p then not q”: $\sim p \to \sim q$

The contrapositive is “if not q then not p”: $\sim q \to \sim p$

Example 6

Consider again the valid implication “If it is raining, then there are clouds in the sky.”

The converse would be “If there are clouds in the sky, it is raining.” This is certainly not always true.

The inverse would be “If it is not raining, then there are not clouds in the sky.” Likewise, this is not always true.

The contrapositive would be “If there are not clouds in the sky, then it is not raining.” This statement is valid, and is equivalent to the original implication.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

		Implication	Converse	Inverse	Contrapositive
p	q	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalence

A conditional statement and its contrapositive are logically equivalent.

The converse and inverse of a statement are logically equivalent.

Arguments

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

Argument types

An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.

A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

Example 7

The argument “when I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store” is an inductive argument.

The premises are:

> I forgot my purse last week
> I forgot my purse today

The conclusion is:

> I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

Example 8

The argument “every day for the past year, a plane flies over my house at 2pm. A plane will fly over my house every day at 2pm” is a stronger inductive argument, since it is based on a larger set of evidence.

Evaluating inductive arguments

An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest it may be true.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. The commonly known scientific theories, like Newton's theory of gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence. For gravity, this happened when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes them easier to evaluate.

Evaluating deductive arguments

A deductive argument is considered valid if all the premises are true, and the conclusion follows logically from those premises. In other words, the premises are true, and the conclusion follows necessarily from those premises.

Example 9

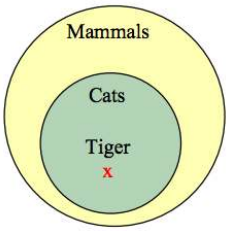
The argument "All cats are mammals and a tiger is a cat, so a tiger is a mammal" is a valid deductive argument.

The premises are:

All cats are mammals
A tiger is a cat

The conclusion is:

A tiger is a mammal



Both the premises are true. To see that the premises must logically lead to the conclusion, one approach would be use a Venn diagram. From the first premise, we can conclude that the set of cats is a subset of the set of mammals. From the second premise, we are told that a tiger lies within the set of cats. From that, we can see in the Venn diagram that the tiger also lies inside the set of mammals, so the conclusion is valid.

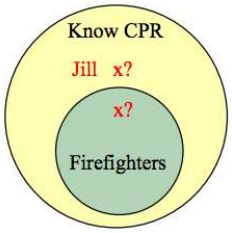
Analyzing Arguments with Venn Diagrams[1]

To analyze an argument with a Venn diagram

1. Draw a Venn diagram based on the premises of the argument
2. If the premises are insufficient to determine what determine the location of an element, indicate that.
3. The argument is valid if it is clear that the conclusion must be true

Example 10

Premise:	All firefighters know CPR
Premise:	Jill knows CPR
Conclusion:	Jill is a firefighter



From the first premise, we know that firefighters all lie inside the set of those who know CPR. From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know if she also is a member of the smaller subset that is firefighters.

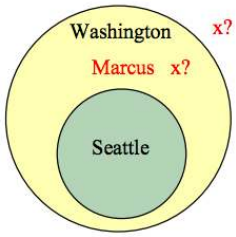
Since the conclusion does not necessarily follow from the premises, this is an invalid argument, regardless of whether Jill actually is a firefighter.

It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are only concerned with whether the premises are enough to prove the conclusion.

In addition to these categorical style premises of the form "all ___," "some ____," and "no ____," it is also common to see premises that are implications.

Example 11

Premise: If you live in Seattle, you live in Washington.
Premise: Marcus does not live in Seattle
Conclusion: Marcus does not live in Washington



From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether or not Marcus lives in Washington or not. This is an invalid argument.

Example 12

Consider the argument "You are a married man, so you must have a wife."

This is an invalid argument, since there are, at least in parts of the world, men who are married to other men, so the premise not insufficient to imply the conclusion.

Some arguments are better analyzed using truth tables.

Example 13

Consider the argument:

Premise: If you bought bread, then you went to the store
Premise: You bought bread
Conclusion: You went to the store

While this example is hopefully fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then look at the implication that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

We'll get B represent "you bought bread" and S represent "you went to the store". Then the argument becomes:

Premise: $B \to S$
Premise: B
Conclusion: S

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that $[(B \to S) \bigwedge B] \to S$?

B	S	$B \to S$	$(B \to S) \bigwedge B$	$[(B \to S) \bigwedge B] \to S$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the truth table for $[(B \to S) \bigwedge B] \to S$ is always true, this is a valid argument.

Analyzing arguments using truth tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises with and to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for that statement. If it is always true, then the argument is valid.

Example 14

Premise: If I go to the mall, then I'll buy new jeans
Premise: If I buy new jeans, I'll buy a shirt to go with it
Conclusion: If I got to the mall, I'll buy a shirt.

Let M = I go to the mall, J = I buy jeans, and S = I buy a shirt.

The premises and conclusion can be stated as:

Premise: $M \to J$
Premise: $J \to S$
Conclusion: $M \to S$

We can construct a truth table for $[(M \to J) \bigwedge (J \to S)] \to (M \to S)$

M	J	S	$M \to J$	$J \to S$	$(M \to J) \bigwedge (J \to S)$	$M \to S$	$[(M \to J) \bigwedge (J \to S)] \to (M \to S)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the truth table, we can see this is a valid argument.

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1. Technically, these are Euler circles or Euler diagrams, not Venn diagrams, but for the sake of simplicity we'll continue to call them Venn diagrams. ↵

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13.4: Truth Tables- Conjunction and Disjunction

This video explores the example “It is snowing OR I am wearing my hat,” and “It is snowing AND I am wearing my hat.”

https://youtu.be/8rNPcI_08Ec

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13.5: Truth Tables- Implication

This video explores the example “If it is snowing, then I am wearing my hat.”

https://youtu.be/2tIbh7l6Gi4

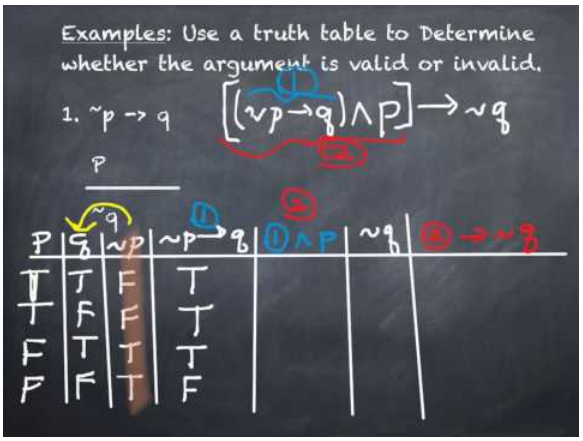
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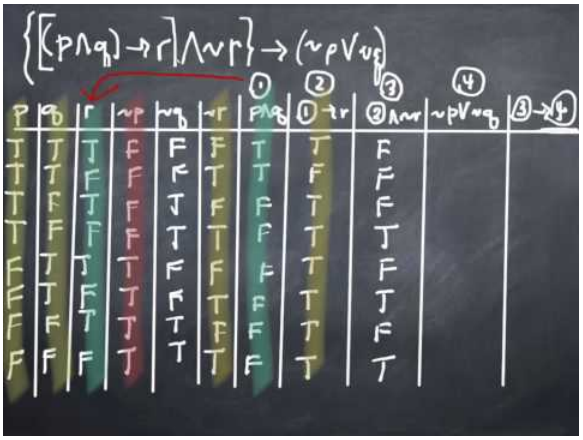
13.6: Analyzing Arguments with Truth Tables

Part I



A YouTube element has been excluded from this version of the text. You can view it online here: http://pb.libretexts.org/mala/?p=194

Part II



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13.7: Negating “all,” “some,” or “no” statements



This reading provides information about negating “all,” “some,” or “no” statements. Click on the link below to view the page “Negation—NOT—Compound Statements and ALL/SOME” created by Donna Roberts of the Oswego City School District Regents Exam Prep Center.

- Negation—NOT—Compound Statements and ALL/SOME

For more information, look at page 229 of Logic and Sets, staring with “Statements involving the universal quantifiers *all*, *no*, *non* and *every*, or the existential quantifiers *some* and *there exists at least one* have to be negated in a different way.”

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13.8: Discuss- Truth Table Practice

You will need to become familiar with creating a Truth Table in Blackboard. You may be asked to create a Truth Table for some of your exam questions. See Creating a Truth Table for instructions.

Create a new thread in the **Truth Table Practice** forum in the **Discussion Board** to submit your assignment.

This assignment is required and worth up to 10 points.

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13.9: Discuss- Logic Application

Pick a real problem and try to solve it using what you learned about logic in this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the Logic Application Directions for detailed directions.

Create a new thread in the **Logic Application** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 20 points.

Grading Criteria	Points Possible
The problem: • Is it a real-life problem? • Is it challenging, not trivial? • Is it a unique problem instead of a copy of a classmate's posting?	5
The strategies: • Are one or more general problem solving strategies used? • Are the strategies correctly identified?	5
The presentation: • Is the problem explained well? • Are the problem solving strategies explained well? • Are the appropriate terms used?	4
Your responses: • Did you post at least two responses? • Did you explain how the examples helped you better understand the math in this module? • Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?	6

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CHAPTER OVERVIEW

14: Module 5- Numeration Systems

14.1: Module 5 Overview
14.2: Numeration
14.3: Supplemental Videos
14.4: Binary, Octal, and Hexadecimal
14.5: Discuss- Numeration Application

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14.1: Module 5 Overview

What You'll Learn To Do: Understand numbers in other bases and recognize when use of other bases is applicable to real-life situations.

Learning Objectives

- Describe historical examples of simple grouping, multiplicative grouping, and positional numeration systems in the development of modern mathematics.
- Write and interpret numbers in other bases.
- Convert directly between binary, octal, decimal, and hexadecimal bases.
- Recognize when use of other bases is applicable to real-life situations, solve real-life problems, and communicate real-life problems and solutions to others.

Learning Activities

 The learning activities for this module include:

Reading Assignments and Videos

- **Read:** Numeration
- **Watch:** Supplemental Videos
- **Read:** Binary, Octal, and Hexadecimal

Homework Assignments

- **Submit:** Numeration Homework #1 (16 points)
- **Submit:** Numeration Homework #2 (16 points)
- **Discuss:** Numeration Application (20 points)

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14.2: Numeration

Historical Counting Systems Introduction and Basic Number and Counting Systems

Introduction

As we begin our journey through the history of mathematics, one question to be asked is “Where do we start?” Depending on how you view mathematics or numbers, you could choose any of a number of launching points from which to begin. Howard Eves suggests the following list of possibilities.[1]

Where to start the study of the history of mathematics…

- At the first logical geometric “proofs” traditionally credited to Thales of Miletus (600 BCE).
- With the formulation of methods of measurement made by the Egyptians and Mesopotamians/Babylonians.
- Where prehistoric peoples made efforts to organize the concepts of size, shape, and number.
- In pre-human times in the very simple number sense and pattern recognition that can be displayed by certain animals, birds, etc.
- Even before that in the amazing relationships of numbers and shapes found in plants.
- With the spiral nebulae, the natural course of planets, and other universe phenomena.

We can choose no starting point at all and instead agree that mathematics has *always* existed and has simply been waiting in the wings for humans to discover. Each of these positions can be defended to some degree and which one you adopt (if any) largely depends on your philosophical ideas about mathematics and numbers.

Nevertheless, we need a starting point. Without passing judgment on the validity of any of these particular possibilities, we will choose as our starting point the emergence of the idea of number and the process of counting as our launching pad. This is done primarily as a practical matter given the nature of this course. In the following chapter, we will try to focus on two main ideas. The first will be an examination of basic number and counting systems and the symbols that we use for numbers. We will look at our own modern (Western) number system as well those of a couple of selected civilizations to see the differences and diversity that is possible when humans start counting. The second idea we will look at will be base systems. By comparing our own base-ten (decimal) system with other bases, we will quickly become aware that the system that we are so used to, when slightly changed, will challenge our notions about numbers and what symbols for those numbers actually mean.

Recognition of More vs. Less

The idea of number and the process of counting goes back far beyond history began to be recorded. There is some archeological evidence that suggests that humans were counting as far back as 50,000 years ago.[2] However, we do not really know how this process started or developed over time. The best we can do is to make a good guess as to how things progressed. It is probably not hard to believe that even the earliest humans had some sense of *more* and *less*. Even some small animals have been shown to have such a sense. For example, one naturalist tells of how he would secretly remove one egg each day from a plover’s nest. The mother was diligent in laying an extra egg every day to make up for the missing egg. Some research has shown that hens can be trained to distinguish between even and odd numbers of pieces of food.[3] With these sorts of findings in mind, it is not hard to conceive that early humans had (at least) a similar sense of more and less. However, our conjectures about how and when these ideas emerged among humans are simply that; educated guesses based on our own assumptions of what might or could have been.

The Need for Simple Counting

As societies and humankind evolved, simply having a sense of more or less, even or odd, etc., would prove to be insufficient to meet the needs of everyday living. As tribes and groups formed, it became important to be able to know how many members were in the group, and perhaps how many were in the enemy’s camp. Certainly it was important for them to know if the flock of sheep or other possessed animals were increasing or decreasing in size. “Just how many of them do we have, anyway?” is a question that we do not have a hard time imagining them asking themselves (or each other).

In order to count items such as animals, it is often conjectured that one of the earliest methods of doing so would be with “tally sticks.” These are objects used to track the numbers of items to be counted. With this method, each “stick” (or pebble, or whatever counting device being used) represents one animal or object. This method uses the idea of **one to one correspondence**. In a one to one correspondence, items that are being counted are uniquely linked with some counting tool.

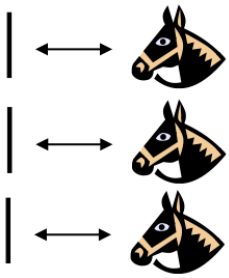


Figure 1.

In the picture to the right, you see each stick corresponding to one horse. By examining the collection of sticks in hand one knows how many animals should be present. You can imagine the usefulness of such a system, at least for smaller numbers of items to keep track of. If a herder wanted to “count off” his animals to make sure they were all present, he could mentally (or methodically) assign each stick to one animal and continue to do so until he was satisfied that all were accounted for.

Of course, in our modern system, we have replaced the sticks with more abstract objects. In particular, the top stick is replaced with our symbol “1,” the second stick gets replaced by a “2” and the third stick is represented by the symbol “3,” but we are getting ahead of ourselves here. These modern symbols took many centuries to emerge.

Another possible way of employing the “tally stick” counting method is by making marks or cutting notches into pieces of wood, or even tying knots in string (as we shall see later). In 1937, Karl Absolom discovered a wolf bone that goes back possibly 30,000 years. It is believed to be a counting device.[4] Another example of this kind of tool is the Ishango Bone, discovered in 1960 at Ishango, and shown below.[5] It is reported to be between six and nine thousand years old and shows what appear to be markings used to do counting of some sort.

The markings on rows (a) and (b) each add up to 60. Row (b) contains the prime numbers between 10 and 20. Row (c) seems to illustrate for the method of doubling and multiplication used by the Egyptians. It is believed that this may also represent a lunar phase counter.

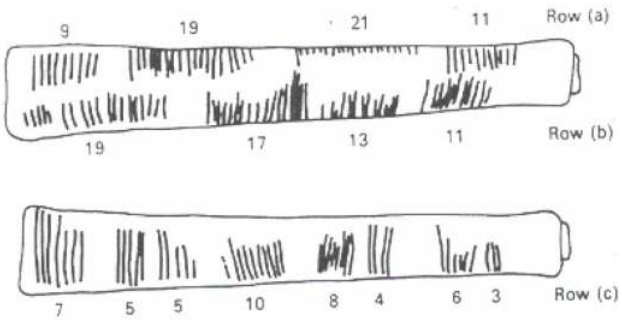


Figure 2.

Spoken Words

As methods for counting developed, and as language progressed as well, it is natural to expect that spoken words for numbers would appear. Unfortunately, the developments of these words, especially those corresponding to the numbers from one through ten, are not easy to trace. Past ten, however, we do see some patterns:

- Eleven comes from “ein lifon,” meaning “one left over.”
- Twelve comes from “twe lif,” meaning “two left over.”
- Thirteen comes from “Three and ten” as do fourteen through nineteen.
- Twenty appears to come from “twe-tig” which means “two tens.”
- Hundred probably comes from a term meaning “ten times.”

Written Numbers

When we speak of “written” numbers, we have to be careful because this could mean a variety of things. It is important to keep in mind that modern paper is only a little more than 100 years old, so “writing” in times past often took on forms that might look quite

unfamiliar to us today.

As we saw earlier, some might consider wooden sticks with notches carved in them as writing as these are means of recording information on a medium that can be “read” by others. Of course, the symbols used (simple notches) certainly did not leave a lot of flexibility for communicating a wide variety of ideas or information.

Other mediums on which “writing” may have taken place include carvings in stone or clay tablets, rag paper made by hand (twelfth century in Europe, but earlier in China), papyrus (invented by the Egyptians and used up until the Greeks), and parchments from animal skins. And these are just a few of the many possibilities.

These are just a few examples of early methods of counting and simple symbols for representing numbers. Extensive books, articles and research have been done on this topic and could provide enough information to fill this entire course if we allowed it to. The range and diversity of creative thought that has been used in the past to describe numbers and to count objects and people is staggering. Unfortunately, we don’t have time to examine them all, but it is fun and interesting to look at one system in more detail to see just how ingenious people have been.

The Number and Counting System of the Inca Civilization

Background

There is generally a lack of books and research material concerning the historical foundations of the Americas. Most of the “important” information available concentrates on the eastern hemisphere, with Europe as the central focus. The reasons for this may be twofold: first, it is thought that there was a lack of specialized mathematics in the American regions; second, many of the secrets of ancient mathematics in the Americas have been closely guarded.[6] The Peruvian system does not seem to be an exception here. Two researchers, Leland Locke and Erland Nordenskiold, have carried out research that has attempted to discover what mathematical knowledge was known by the Incas and how they used the Peruvian quipu, a counting system using cords and knots, in their mathematics. These researchers have come to certain beliefs about the quipu that we will summarize here.

Counting Boards

It should be noted that the Incas did not have a complicated system of computation. Where other peoples in the regions, such as the Mayans, were doing computations related to their rituals and calendars, the Incas seem to have been more concerned with the simpler task of record-keeping. To do this, they used what are called the “quipu” to record quantities of items. (We will describe them in more detail in a moment.) However, they first often needed to do computations whose results would be recorded on quipu. To do these computations, they would sometimes use a counting board constructed with a slab of stone. In the slab were cut rectangular and square compartments so that an octagonal (eight-sided) region was left in the middle. Two opposite corner rectangles were raised. Another two sections were mounted on the original surface of the slab so that there were actually three levels available. In the figure shown, the darkest shaded corner regions represent the highest, third level. The lighter shaded regions surrounding the corners are the second highest levels, while the clear white rectangles are the compartments cut into the stone slab.

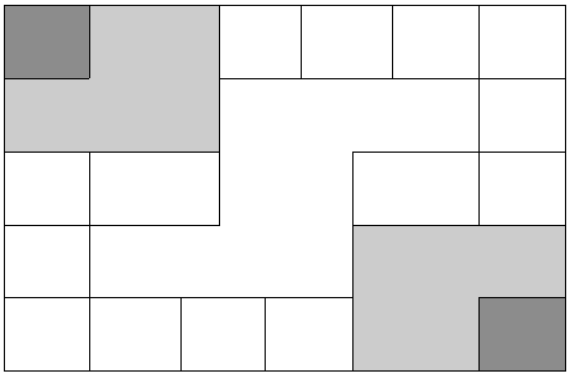


Figure 3.

Pebbles were used to keep accounts and their positions within the various levels and compartments gave totals. For example, a pebble in a smaller (white) compartment represented one unit. Note that there are 12 such squares around the outer edge of the figure. If a pebble was put into one of the two (white) larger, rectangular compartments, its value was doubled. When a pebble was put in the octagonal region in the middle of the slab, its value was tripled. If a pebble was placed on the second (shaded) level, its

value was multiplied by six. And finally, if a pebble was found on one of the two highest corner levels, its value was multiplied by twelve. Different objects could be counted at the same time by representing different objects by different colored pebbles.

Example 1

Suppose you have the following counting board with two different kind of pebbles places as illustrated. Let the solid black pebble represent a dog and the striped pebble represent a cat. How many dogs are being represented?

Solution

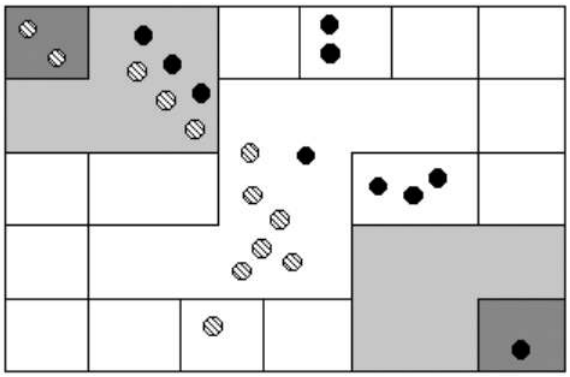


Figure 4.

There are two black pebbles in the outer square regions…these represent 2 dogs. There are three black pebbles in the larger (white) rectangular compartments. These represent 6 dogs. There is one black pebble in the middle region…this represents 3 dogs. There are three black pebbles on the second level…these represent 18 dogs. Finally, there is one black pebble on the highest corner level…this represents 12 dogs. We then have a total of 2+6+3+18+12 = 41 dogs.

Try It Now

How many cats are represented on this board?

The Quipu

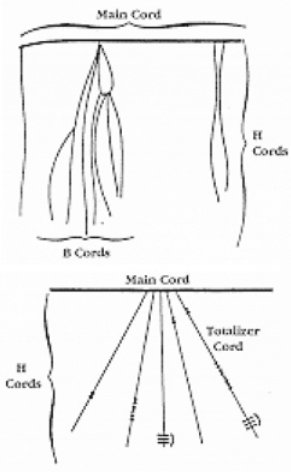


Figure 5.

This kind of board was good for doing quick computations, but it did not provide a good way to keep a permanent recording of quantities or computations. For this purpose, they used the quipu. The quipu is a collection of cords with knots in them. These cords and knots are carefully arranged so that the position and type of cord or knot gives specific information on how to decipher the cord.

A quipu is made up of a main cord which has other cords (branches) tied to it. See pictures to the right.[7]

Locke called the branches H cords. They are attached to the main cord. B cords, in turn, were attached to the H cords. Most of these cords would have knots on them. Rarely are knots found on the main cord, however, and tend to be mainly on the H and B cords. A quipu might also have a "totalizer" cord that summarizes all of the information on the cord group in one place.

Locke points out that there are three types of knots, each representing a different value, depending on the kind of knot used and its position on the cord. The Incas, like us, had a decimal (base-ten) system, so each kind of knot had a specific decimal value. The Single knot, pictured in the middle of figure 6[8] was used to denote tens, hundreds, thousands, and ten thousands. They would be on the upper levels of the H cords. The figure-eight knot on the end was used to denote the integer "one." Every other integer from 2 to 9 was represented with a long knot, shown on the left of the figure. (Sometimes long knots were used to represents tens and hundreds.) Note that the long knot has several turns in it…the number of turns indicates which integer is being represented. The units (ones) were placed closest to the bottom of the cord, then tens right above them, then the hundreds, and so on.

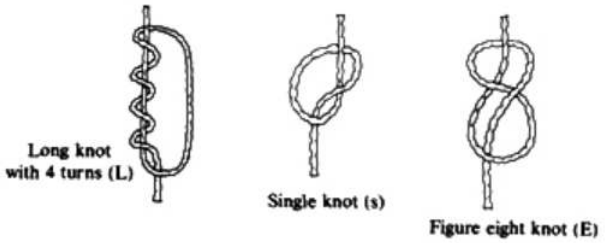


Figure 6

In order to make reading these pictures easier, we will adopt a convention that is consistent. For the long knot with turns in it (representing the numbers 2 through 9), we will use the following notation:

The four horizontal bars represent four turns and the curved arc on the right links the four turns together. This would represent the number 4.

We will represent the single knot with a large dot (·) and we will represent the figure eight knot with a sideways eight (∞).

Example 2

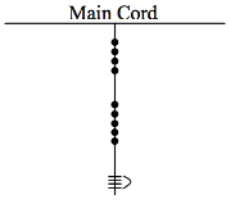


Figure 7.

What number is represented on the cord shown in figure 7?

Solution

On the cord, we see a long knot with four turns in it…this represents four in the ones place. Then 5 single knots appear in the tens position immediately above that, which represents 5 tens, or 50. Finally, 4 single knots are tied in the hundreds, representing four 4 hundreds, or 400. Thus, the total shown on this cord is 454.

Try It Now

What numbers are represented on each of the four cords hanging from the main cord?

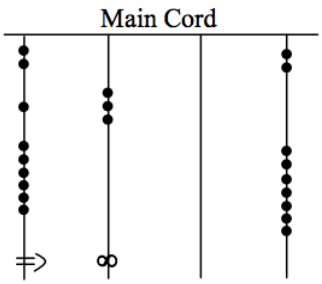


Figure 8.

The colors of the cords had meaning and could distinguish one object from another. One color could represent llamas, while a different color might represent sheep, for example. When all the colors available were exhausted, they would have to be re-used. Because of this, the ability to read the quipu became a complicated task and specially trained individuals did this job. They were called Quipucamayoc, which means keeper of the quipus. They would build, guard, and decipher quipus.

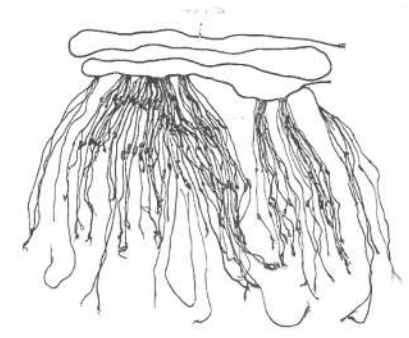


Figure 9.

As you can see from this photograph of an actual quipu (figure 9), they could get quite complex.

There were various purposes for the quipu. Some believe that they were used to keep an account of their traditions and history, using knots to record history rather than some other formal system of writing. One writer has even suggested that the quipu replaced writing as it formed a role in the Incan postal system.[9] Another proposed use of the quipu is as a translation tool. After the conquest of the Incas by the Spaniards and subsequent "conversion" to Catholicism, an Inca supposedly could use the quipu to confess their sins to a priest. Yet another proposed use of the quipu was to record numbers related to magic and astronomy, although this is not a widely accepted interpretation.

The mysteries of the quipu have not been fully explored yet. Recently, Ascher and Ascher have published a book, *The Code of the Quipu: A Study in Media, Mathematics, and Culture, which is* "an extensive elaboration of the logical-numerical system of the quipu."[10] For more information on the quipu, you may want to check out "Khipus: a unique Huarochiri legacy."

We are so used to seeing the symbols 1, 2, 3, 4, etc. that it may be somewhat surprising to see such a creative and innovative way to compute and record numbers. Unfortunately, as we proceed through our mathematical education in grade and high school, we receive very little information about the wide range of number systems that have existed and which still exist all over the world. That's not to say our own system is not important or efficient. The fact that it has survived for hundreds of years and shows no sign of going away any time soon suggests that we may have finally found a system that works well and may not need further improvement, but only time will tell that whether or not that conjecture is valid or not. We now turn to a brief historical look at how our current system developed over history.

The Hindu—Arabic Number System

The Evolution of a System

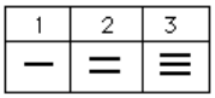
Our own number system, composed of the ten symbols {0,1,2,3,4,5,6,7,8,9} is called the *Hindu-Arabic system*. This is a base-ten (decimal) system since place values increase by powers of ten. Furthermore, this system is positional, which means that the position of a symbol has bearing on the value of that symbol within the number. For example, the position of the symbol 3 in the number 435,681 gives it a value much greater than the value of the symbol 8 in that same number. We'll explore base systems more thoroughly later. The development of these ten symbols and their use in a positional system comes to us primarily from India.[11]



Figure 10. Al-Biruni

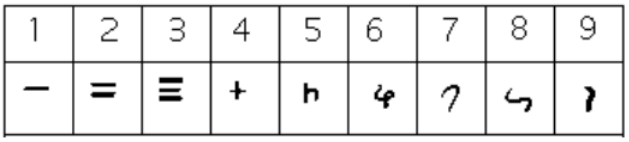
It was not until the fifteenth century that the symbols that we are familiar with today first took form in Europe. However, the history of these numbers and their development goes back hundreds of years. One important source of information on this topic is the writer al-Biruni, whose picture is shown in figure 10.[12] Al-Biruni, who was born in modern day Uzbekistan, had visited India on several occasions and made comments on the Indian number system. When we look at the origins of the numbers that al-Biruni encountered, we have to go back to the third century BCE. to explore their origins. It is then that the Brahmi numerals were being used.

The Brahmi numerals were more complicated than those used in our own modern system. They had separate symbols for the numbers 1 through 9, as well as distinct symbols for 10, 100, 1000,…, also for 20, 30, 40,…, and others for 200, 300, 400, …, 900. The Brahmi symbols for 1, 2, and 3 are shown below.[13]

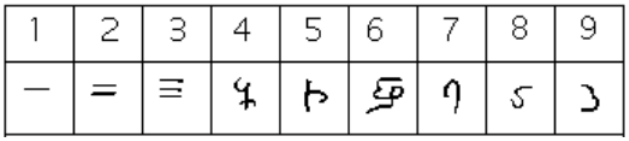


Brahmi one, two, three

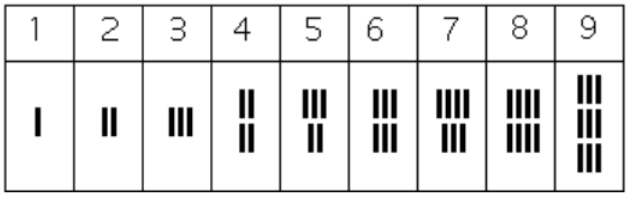
These numerals were used all the way up to the fourth century CE, with variations through time and geographic location. For example, in the first century CE, one particular set of Brahmi numerals took on the following form:[14]



From the fourth century on, you can actually trace several different paths that the Brahmi numerals took to get to different points and incarnations. One of those paths led to our current numeral system, and went through what are called the Gupta numerals. The Gupta numerals were prominent during a time ruled by the Gupta dynasty and were spread throughout that empire as they conquered lands during the fourth through sixth centuries. They have the following form:[15]



How the numbers got to their Gupta form is open to considerable debate. Many possible hypotheses have been offered, most of which boil down to two basic types.[16] The first type of hypothesis states that the numerals came from the initial letters of the names of the numbers. This is not uncommon . . . the Greek numerals developed in this manner. The second type of hypothesis states that they were derived from some earlier number system. However, there are other hypotheses that are offered, one of which is by the researcher Ifrah. His theory is that there were originally nine numerals, each represented by a corresponding number of vertical lines. One possibility is this:[17]



Because these symbols would have taken a lot of time to write, they eventually evolved into cursive symbols that could be written more quickly. If we compare these to the Gupta numerals above, we can try to see how that evolutionary process might have taken place, but our imagination would be just about all we would have to depend upon since we do not know exactly how the process unfolded.

The Gupta numerals eventually evolved into another form of numerals called the Nagari numerals, and these continued to evolve until the eleventh century, at which time they looked like this:[18]

1	2	3	4	5	6	7	8	9	0
९	२	३	४	५	६	७	८	९	०

Note that by this time, the symbol for 0 has appeared! The Mayans in the Americas had a symbol for zero long before this, however, as we shall see later in the chapter.

These numerals were adopted by the Arabs, most likely in the eighth century during Islamic incursions into the northern part of India.[19] It is believed that the Arabs were instrumental in spreading them to other parts of the world, including Spain (see below).

Other examples of variations up to the eleventh century include:[20]



Figure 11. Devangari, eighth century



Figure 12. West Arab Gobar, tenth century



Figure 13. Spain, 976 BCE

Finally, figure 14[21] shows various forms of these numerals as they developed and eventually converged to the fifteenth century in Europe.

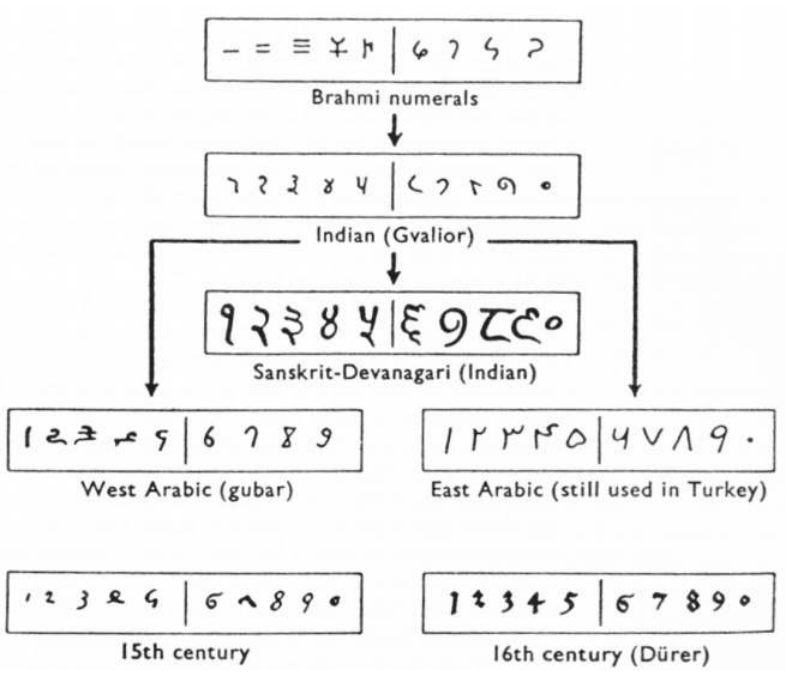


Figure 14.

The Positional System

More important than the form of the number symbols is the development of the place value system. Although it is in slight dispute, the earliest known document in which the Indian system displays a positional system dates back to 346 CE. However, some evidence suggests that they may have actually developed a positional system as far back as the first century CE.

The Indians were not the first to use a positional system. The Babylonians (as we will see in Chapter 3) used a positional system with 60 as their base. However, there is not much evidence that the Babylonian system had much impact on later numeral systems,

except with the Greeks. Also, the Chinese had a base-10 system, probably derived from the use of a counting board.[22] Some believe that the positional system used in India was derived from the Chinese system.

Wherever it may have originated, it appears that around 600 CE, the Indians abandoned the use of symbols for numbers higher than nine and began to use our familiar system where the position of the symbol determines its overall value.[23] Numerous documents from the seventh century demonstrate the use of this positional system.

Interestingly, the earliest dated inscriptions using the system with a symbol for zero come from Cambodia. In 683, the 605th year of the Saka era is written with three digits and a dot in the middle. The 608th year uses three digits with a modern 0 in the middle.[24] The dot as a symbol for zero also appears in a Chinese work (*Chiu-chih li*). The author of this document gives a strikingly clear description of how the Indian system works:

Using the [Indian] numerals, multiplication and division are carried out. Each numeral is written in one stroke. When a number is counted to ten, it is advanced into the higher place. In each vacant place a dot is always put. Thus the numeral is always denoted in each place. Accordingly there can be no error in determining the place. With the numerals, calculations is easy.[25]

Transmission to Europe

It is not completely known how the system got transmitted to Europe. Traders and travelers of the Mediterranean coast may have carried it there. It is found in a tenth-century Spanish manuscript and may have been introduced to Spain by the Arabs, who invaded the region in 711 CE and were there until 1492.

In many societies, a division formed between those who used numbers and calculation for practical, every day business and those who used them for ritualistic purposes or for state business.[26] The former might often use older systems while the latter were inclined to use the newer, more elite written numbers. Competition between the two groups arose and continued for quite some time.



Figure 15.

In a fourteenth century manuscript of Boethius' *The Consolations of Philosophy*, there appears a well-known drawing of two mathematicians. One is a merchant and is using an abacus (the "abacist"). The other is a Pythagorean philosopher (the "algorist") using his "sacred" numbers. They are in a competition that is being judged by the goddess of number. By 1500 CE, however, the newer symbols and system had won out and has persevered until today. The Seattle Times recently reported that the Hindu-Arabic numeral system has been included in the book *The Greatest Inventions of the Past 2000 Years*.[27]

One question to answer is *why* the Indians would develop such a positional notation. Unfortunately, an answer to that question is not currently known. Some suggest that the system has its origins with the Chinese counting boards. These boards were portable and it is thought that Chinese travelers who passed through India took their boards with them and ignited an idea in Indian mathematics.[28] Others, such as G. G. Joseph propose that it is the Indian fascination with very large numbers that drove them to develop a system whereby these kinds of big numbers could easily be written down. In this theory, the system developed entirely within the Indian mathematical framework without considerable influence from other civilizations.

The Development and Use of Different Number Bases

Introduction and Basics

During the previous discussions, we have been referring to positional base systems. In this section of the chapter, we will explore exactly what a base system is and what it means if a system is "positional." We will do so by first looking at our own familiar, base-ten system and then deepen our exploration by looking at other possible base systems. In the next part of this section, we will journey back to Mayan civilization and look at their unique base system, which is based on the number 20 rather than the number 10.

A base system is a structure within which we count. The easiest way to describe a base system is to think about our own base-ten system. The base-ten system, which we call the "decimal" system, requires a total of ten different symbols/digits to write any number. They are, of course, 0, 1, 2, . . . , 9.

The decimal system is also an example of a *positional* base system, which simply means that the position of a digit gives its place value. Not all civilizations had a positional system even though they did have a base with which they worked.

In our base-ten system, a number like 5,783,216 has meaning to us because we are familiar with the system and its places. As we know, there are six ones, since there is a 6 in the ones place. Likewise, there are seven "hundred thousands," since the 7 resides in that place. Each digit has a value that is explicitly determined by its position within the number. We make a distinction between digit, which is just a symbol such as 5, and a number, which is made up of one or more digits. We can take this number and assign each of its digits a value. One way to do this is with a table, which follows:

5,000,000	$= 5 \times 1{,}000{,}000$	$= 5 \times 10^6$	Five million
+700,000	$= 7 \times 100{,}000$	$= 7 \times 10^5$	Seven hundred thousand
+80,000	$= 8 \times 10{,}000$	$= 8 \times 10^4$	Eighty thousand
+3,000	$= 3 \times 1000$	$= 3 \times 10^3$	Three thousand
+200	$= 2 \times 100$	$= 2 \times 10^2$	Two hundred
+10	$= 1 \times 10$	$= 1 \times 10^1$	Ten
+6	$= 6 \times 1$	$= 6 \times 10^0$	Six
5,783,216	Five million, seven hundred eighty-three thousand, two hundred sixteen		

From the third column in the table we can see that each place is simply a multiple of ten. Of course, this makes sense given that our base is ten. The digits that are multiplying each place simply tell us how many of that place we have. We are restricted to having at most 9 in any one place before we have to "carry" over to the next place. We cannot, for example, have 11 in the hundreds place. Instead, we would carry 1 to the thousands place and retain 1 in the hundreds place. This comes as no surprise to us since we readily see that 11 hundreds is the same as one thousand, one hundred. Carrying is a pretty typical occurrence in a base system.

However, base-ten is not the only option we have. Practically any positive integer greater than or equal to 2 can be used as a base for a number system. Such systems can work just like the decimal system except the number of symbols will be different and each position will depend on the base itself.

Other Bases

For example, let's suppose we adopt a base-five system. The only modern digits we would need for this system are 0,1,2,3 and 4. What are the place values in such a system? To answer that, we start with the ones place, as most base systems do. However, if we were to count in this system, we could only get to four (4) before we had to jump up to the next place. Our base is 5, after all! What is that next place that we would jump to? It would not be tens, since we are no longer in base-ten. We're in a different numerical world. As the base-ten system progresses from 10^0 to 10^1, so the base-five system moves from 5^0 to $5^1 = 5$. Thus, we move from the ones to the fives.

After the fives, we would move to the 5^2 place, or the twenty fives. Note that in base-ten, we would have gone from the tens to the hundreds, which is, of course, 10^2.

Let's take an example and build a table. Consider the number 30412 in base five. We will write this as 30412_5, where the subscript 5 is not part of the number but indicates the base we're using. First off, note that this is NOT the number "thirty thousand, four

hundred twelve." We must be careful not to impose the base-ten system on this number. Here's what our table might look like. We will use it to convert this number to our more familiar base-ten system.

	Base 5	This column coverts to base-ten	In Base-Ten
	3×5^4	$= 3 \times 625$	$= 1875$
+	0×5^3	$= 0 \times 125$	$= 0$
+	4×5^2	$= 4 \times 25$	$= 100$
+	1×5^1	$= 1 \times 5$	$= 5$
+	2×5^0	$= 2 \times 1$	$= 2$
		Total	1982

As you can see, the number 30412_5 is equivalent to 1,982 in base-ten. We will say $30412_5 = 1982_{10}$. All of this may seem strange to you, but that's only because you are so used to the only system that you've ever seen.

Example 3

Convert 6234_7 to a base 10 number.

Solution

We first note that we are given a base-7 number that we are to convert. Thus, our places will start at the ones (7^0), and then move up to the 7s, 49s (7^2), etc. Here's the breakdown:

	Base 7	Convert	Base 10
	$= 6 \times 7^3$	$= 6 \times 343$	$= 2058$
+	$= 2 \times 7^2$	$= 2 \times 49$	$= 98$
+	$= 3 \times 7$	$= 3 \times 7$	$= 21$
+	$= 4 \times 1$	$= 4 \times 1$	$= 4$
		Total	2181

Thus $6234_7 = 2181_{10}$

Try It Now

Convert 41065_7 to a base 10 number.

Converting from Base 10 to Other Bases

Converting from an unfamiliar base to the familiar decimal system is not that difficult once you get the hang of it. It's only a matter of identifying each place and then multiplying each digit by the appropriate power. However, going the other direction can be a little trickier. Suppose you have a base-ten number and you want to convert to base-five. Let's start with some simple examples before we get to a more complicated one.

Example 4

Convert twelve to a base-five number.

Solution

We can probably easily see that we can rewrite this number as follows:

$$12 = (2 \times 5) + (2 \times 1)$$

Hence, we have two fives and 2 ones. Hence, in base-five we would write twelve as 22_5. Thus, $12_{10} = 22_5$.

Example 5

Convert sixty-nine to a base-five number.

Solution

We can see now that we have more than 25, so we rewrite sixty-nine as follows:

$$69 = (2 \times 25) + (3 \times 5) + (4 \times 1)$$

Here, we have two twenty-fives, 3 fives, and 4 ones. Hence, in base five we have 234. Thus, $69_{10} = 234_5$.

Example 6

Convert the base-seven number 3261_7 to base 10.

Solution

The powers of 7 are:

$7^0 = 1$
$7^1 = 7$
$7^2 = 49$
$7^3 = 343$
Etc…

$3261_7 = (3 \times 343) + (2 \times 49) + (6 \times 7) + (1 \times 1) = 1170_{10}$.

Thus $3261_7 = 1170_{10}$.

Try It Now

Convert 143 to base 5

Try It Now

Convert the base-three number 21021_3 to base 10.

In general, when converting from base -ten to some other base, it is often helpful to determine the highest power of the base that will divide into the given number at least once.

In the last example, $5^2 = 25$ is the largest power of five that is present in 69, so that was our starting point. If we had moved to $5^3 = 125$, then 125 would not divide into 69 at least once.

Converting from Base 10 to Base *b*

1. Find the highest power of the base *b* that will divide into the given number at least once and then divide.
2. Write down the whole number part, then use the remainder from division in the next step.
3. Repeat step two, dividing by the next highest power of the base *b*, writing down the whole number part (including 0), and using the remainder in the next step.
4. Continue until the remainder is smaller than the base. This last remainder will be in the “ones” place.
5. Collect all your whole number parts to get your number in base *b* notation.

Example 7

Convert the base-ten number 348 to base-five.

Solution

The powers of five are:

$5^0 = 1$
$5^1 = 5$
$5^2 = 25$
$5^3 = 125$
$5^4 = 625$
Etc…

Since 348 is smaller than 625, but bigger than 125, we see that $5^3 = 125$ is the highest power of five present in 348. So we divide 125 into 348 to see how many of them there are:

$348 \div 125 = 2$ with remainder 98

We write down the whole part, 2, and continue with the remainder. There are 98 left over, so we see how many 25s (the next smallest power of five) there are in the remainder:

$98 \div 25 = 3$ with remainder 23

We write down the whole part, 2, and continue with the remainder. There are 23 left over, so we look at the next place, the 5s:

$23 \div 5 = 4$ with remainder 3

This leaves us with 3, which is less than our base, so this number will be in the "ones" place. We are ready to assemble our base-five number:

$348 = (2 \times 5^3) + (3 \times 5^2) + (4 \times 5^1) + (3 \times 1)$

Hence, our base-five number is 2343. We'll say that $348_{10} = 2343_5$.

Example 8

Convert the base-ten number 4,509 to base-seven.

Solution

The powers of 7 are:

$7^0 = 1$
$7^1 = 7$
$7^2 = 49$
$7^3 = 343$
$7^4 = 2401$
$7^5 = 16807$
Etc…

The highest power of 7 that will divide into 4,509 is $7^4 = 2401$. With division, we see that it will go in 1 time with a remainder of 2108. So we have 1 in the 7^4 place.

The next power down is $7^3 = 343$, which goes into 2108 six times with a new remainder of 50. So we have 6 in the 7^3 place.

The next power down is $7^2 = 49$, which goes into 50 once with a new remainder of 1. So there is a 1 in the 7^2 place.

The next power down is 7^1 but there was only a remainder of 1, so that means there is a 0 in the 7s place and we still have 1 as a remainder.

That, of course, means that we have 1 in the ones place.

Putting all of this together means that $4{,}509_{10} = 16101_7$.

$4{,}509 \div 7^4 = 1$ R 21082108 $\div 7^3 = 6$ R 50

$50 \div 7^2 = 1$ R 1

$1 \div 7^1 = 1$

$4{,}509_{10} = 16101_7$.

Try It Now

Convert 657_{10} to a base 4 number.

Try It Now

Convert 8377_{10} to a base 8 number.

Another Method For Converting From Base 10 to Other Bases

As you read the solution to this last example and attempted the "Try it Now" problems, you may have had to repeatedly stop and think about what was going on. The fact that you are probably struggling to follow the explanation and reproduce the process yourself is mostly due to the fact that the non-decimal systems are so unfamiliar to you. In fact, the only system that you are probably comfortable with is the decimal system.

As budding mathematicians, you should always be asking questions like "How could I simplify this process?" In general, that is one of the main things that mathematicians do: they look for ways to take complicated situations and make them easier or more familiar. In this section we will attempt to do that.

To do so, we will start by looking at our own decimal system. What we do may seem obvious and maybe even intuitive but that's the point. We want to find a process that we readily recognize works and makes sense to us in a familiar system and then use it to extend our results to a different, unfamiliar system.

Let's start with the decimal number, 4863_{10}. We will convert this number to base 10. Yeah, I know it's already in base 10, but if you carefully follow what we're doing, you'll see it makes things work out very nicely with other bases later on. We first note that the highest power of 10 that will divide into 4863 at least once is $10^3 = 1000$. *In general, this is the first step in our new process; we find the highest power that a given base that will divide at least once into our given number.*

We now divide 1000 into 4863:

$$4863 \div 1000 = 4.863$$

This says that there are four thousands in 4863 (obviously). However, it also says that there are 0.863 thousands in 4863. This fractional part is our remainder and will be converted to lower powers of our base (10). If we take that decimal and multiply by 10 (since that's the base we're in) we get the following:

$$0.863 \times 10 = 8.63$$

Why multiply by 10 at this point? We need to recognize here that 0.863 thousands is the same as 8.63 hundreds. Think about that until it sinks in.

$$(0.863)(1000) = 863$$
$$(8.63)(100) = 863$$

These two statements are equivalent. So, what we are really doing here by multiplying by 10 is rephrasing or converting from one place (thousands) to the next place down (hundreds).

$$0.863 \times 10 \Rightarrow 8.63$$
$$(\text{Parts of Thousands}) \times 10 \Rightarrow \text{Hundreds}$$

What we have now is 8 hundreds and a remainder of 0.63 hundreds, which is the same as 6.3 tens. We can do this again with the 0.63 that remains after this first step.

$$0.63 \times 10 \Rightarrow 6.3$$
$$\text{Hundreds} \times 10 \Rightarrow \text{Tens}$$

So we have six tens and 0.3 tens, which is the same as 3 ones, our last place value.

Now here's the punch line. Let's put all of the together in one place:

$$4863 \div 1000 = \textcircled{4}.863$$
$$0.863 \times 10 = \textcircled{8}.63$$
$$0.63 \times 10 = \textcircled{6}.3$$
$$0.3 \times 10 = \textcircled{3}.0$$

Converting from Base 10 to Base *b*: Another methodNote that in each step, the remainder is carried down to the next step and multiplied by 10, the base. Also, at each step, the whole number part, which is circled, gives the digit that belongs in that particular place. What is amazing is that this works for any base! So, to convert from a base 10 number to some other base, *b*, we have the following steps we can follow:

Converting from Base 10 to Base *b*: Another method

1. Find the highest power of the base *b* that will divide into the given number at least once and then divide.
2. Keep the whole number part, and multiply the fractional part by the base *b*.
3. Repeat step two, keeping the whole number part (including 0), carrying the fractional part to the next step until only a whole number result is obtained.
4. Collect all your whole number parts to get your number in base *b* notation.

We will illustrate this procedure with some examples.

Example 9

Convert the base 10 number, 348_{10}, to base 5.

Solution

This is actually a conversion that we have done in a previous example. The powers of five are:

$5^0 = 1$
$5^1 = 5$
$5^2 = 25$
$5^3 = 125$
$5^4 = 625$
Etc…

The highest power of five that will go into 348 at least once is 5^3.

We divide by 125 and then proceed.

$348 \div 5^3 = ②.784$

$0.784 \times 5 = ③.92$

$0.92 \times 5 = ④0.6$

$0.6 \times 5 = ③.0$

By keeping all the whole number parts, from top bottom, gives 2343 as our base 5 number. Thus, $2343_5 = 348_{10}$.

We can compare our result with what we saw earlier, or simply check with our calculator, and find that these two numbers really are equivalent to each other.

Example 10

Convert the base 10 number, 3007_{10}, to base 5.

Solution

The highest power of 5 that divides at least once into 3007 is $5^4 = 625$. Thus, we have:

$3007 \div 625 = ④.8112$
$0.8112 \times 5 = ④.056$
$0.056 \times 5 = ⓪.28$
$0.28 \times 5 = ①0.4$
$0.4 \times 5 = ②0.0$

This gives us that $3007_{10} = 44012_5$. Notice that in the third line that multiplying by 5 gave us 0 for our whole number part. We don't discard that! The zero tells us that a zero in that place. That is, there are no 5^2s in this number.

This last example shows the importance of using a calculator in certain situations and taking care to avoid clearing the calculator's memory or display until you get to the very end of the process.

Example 11

Convert the base 10 number, 63201_{10}, to base 7.

Solution

The powers of 7 are:

$7^0 = 1$
$7^1 = 7$
$7^2 = 49$
$7^3 = 343$
$7^4 = 2401$
$7^5 = 16807$
etc…

The highest power of 7 that will divide at least once into 63201 is 7^5. When we do the initial division on a calculator, we get the following:

$63201 \div 7^5 = 3.760397453$

The decimal part actually fills up the calculators display and we don't know if it terminates at some point or perhaps even repeats down the road. So if we clear our calculator at this point, we will introduce error that is likely to keep this process from ever ending. To avoid this problem, we leave the result in the calculator and simply subtract 3 from this to get the fractional part all by itself. **Do not round off!** Subtraction and then multiplication by seven gives:

$63201 \div 7^5 = f③.760397453$
$0.760397453 \times 7 = \ldots⑤.322782174$
$0.322782174 \times 7 =, ②.259475219$
$0.259475219 \times 7 =\hat{}\ ①.816326531$
$0.816326531 \times 7 = ⑤….714285714$
$0.714285714 \times 7 = \ldots⑤.000000000$

Yes, believe it or not, that last product is exactly 5, *as long as you don't clear anything out on your calculator*. This gives us our final result: $63201_{10} = 352155_7$.

If we round, even to two decimal places in each step, clearing our calculator out at each step along the way, we will get a series of numbers that do not terminate, but begin repeating themselves endlessly. (Try it!) We end up with something that doesn't make any sense, at least not in this context. So be careful to use your calculator cautiously on these conversion problems.

Also, remember that if your first division is by 7^5, then you expect to have 6 digits in the final answer, corresponding to the places for 7^5, 7^4, and so on down to 7^0. If you find yourself with more than 6 digits due to rounding errors, you know something went wrong.

Try It Now

Convert the base 10 number, 9352_{10}, to base 5.

Try It Now

Convert the base 10 number, 1500, to base 3.

Be careful not to clear your calculator on this one. Also, if you're not careful in each step, you may not get all of the digits you're looking for, so move slowly and with caution.

The Mayan Numeral System

Background

As you might imagine, the development of a base system is an important step in making the counting process more efficient. Our own base-ten system probably arose from the fact that we have 10 fingers (including thumbs) on two hands. This is a natural development. However, other civilizations have had a variety of bases other than ten. For example, the Natives of Queensland used a base-two system, counting as follows: "one, two, two and one, two two's, much." Some Modern South American Tribes have a

base-five system counting in this way: “one, two, three, four, hand, hand and one, hand and two,” and so on. The Babylonians used a base-sixty (sexigesimal) system. In this chapter, we wrap up with a specific example of a civilization that actually used a base system other than 10.



1. Chichen Itza
2. Uxmal
3. Tulum
4. Palenque
5. Bonampak, Yaxchilan
6. Tikal
7. Altun Ha
8. Copán

The Mayan civilization is generally dated from 1500 BCE to 1700 CE. The Yucatan Peninsula (see figure 16[29]) in Mexico was the scene for the development of one of the most advanced civilizations of the ancient world. The Mayans had a sophisticated ritual system that was overseen by a priestly class. This class of priests developed a philosophy with time as divine and eternal.[30] The calendar, and calculations related to it, were thus very important to the ritual life of the priestly class, and hence the Mayan people. In fact, much of what we know about this culture comes from their calendar records and astronomy data. Another important source of information on the Mayans is the writings of Father Diego de Landa, who went to Mexico as a missionary in 1549.



There were two numeral systems developed by the Mayans—one for the common people and one for the priests. Not only did these two systems use different symbols, they also used different base systems. For the priests, the number system was governed by ritual. The days of the year were thought to be gods, so the formal symbols for the days were decorated heads,[31] like the sample to the left[32] Since the basic calendar was based on 360 days, the priestly numeral system used a mixed base system employing multiples of 20 and 360. This makes for a confusing system, the details of which we will skip.

Powers	Base-Ten Value	Place Name
20^7	12,800,000,000	Hablat
20^6	64,000,000	Alau
20^5	3,200,000	Kinchil
20^4	160,000	Cabal
20^3	8,000	Pic
20^2	400	*Bak*
20^1	20	Kal
20^0	1	Hun

The Mayan Number System

Instead, we will focus on the numeration system of the “common” people, which used a more consistent base system. As we stated earlier, the Mayans used a base-20 system, called the “vigesimal” system. Like our system, it is positional, meaning that the position of a numeric symbol indicates its place value. In the following table you can see the place value in its vertical format.[33]

Number	Vertical Form	Number	Vertical Form
0		10	
1		11	
2		12	
3		13	
4		14	
5		15	
6		16	
7		17	
8		18	
9		19	

In order to write numbers down, there were only three symbols needed in this system. A horizontal bar represented the quantity 5, a dot represented the quantity 1, and a special symbol (thought to be a shell) represented zero. The Mayan system may have been the first to make use of zero as a placeholder/number. The first 20 numbers are shown in the table to the right.[34]

Unlike our system, where the ones place starts on the right and then moves to the left, the Mayan systems places the ones on the **bottom** of a vertical orientation and moves up as the place value increases.

When numbers are written in vertical form, there should never be more than four dots in a single place. When writing Mayan numbers, every group of five dots becomes one bar. Also, there should never be more than three bars in a single place…four bars would be converted to one dot in the next place up. It’s the same as 10 getting converted to a 1 in the next place up when we carry during addition.

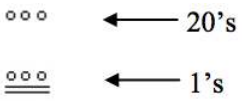
Example 12

What is the value of this number, which is shown in vertical form?



Solution

Starting from the bottom, we have the ones place. There are two bars and three dots in this place. Since each bar is worth 5, we have 13 ones when we count the three dots in the ones place. Looking to the place value above it (the twenties places), we see there are three dots so we have three twenties.



Hence we can write this number in base-ten as:

$(3 \times 20^1) + (13 \times 20^0) = (3 \times 20^1) + (13 \times 1) = 60 + 13 = 73$

Example 13

What is the value of the following Mayan number?



Solution

This number has 11 in the ones place, zero in the 20s place, and 18 in the $20^2 = 400$s place. Hence, the value of this number in base-ten is:

$18 \times 400 + 0 \times 20 + 11 \times 1 = 7211$.

Try It Now

Convert the Mayan number below to base 10.



Example 14

Convert the base 10 number 3575_{10} to Mayan numerals.

This problem is done in two stages. First we need to convert to a base 20 number. We will do so using the method provided in the last section of the text. The second step is to convert that number to Mayan symbols.

The highest power of 20 that will divide into 3575 is $20^2 = 400$, so we start by dividing that and then proceed from there:

$3575 \div 400 = 8.9375$
$0.9375 \times 20 = 18.75$
$0.75 \times 20 = 15.0$

This means that $3575_{10} = 8,18,15_{20}$

The second step is to convert this to Mayan notation. This number indicates that we have 15 in the ones position. That's three bars at the bottom of the number. We also have 18 in the 20s place, so that's three bars and three dots in the second position. Finally, we have 8 in the 400s place, so that's one bar and three dots on the top. We get the following:



Note that in the previous example a new notation was used when we wrote $8,18,15_{20}$. The commas between the three numbers 8, 18, and 15 are now separating place values for us so that we can keep them separate from each other. This use of the comma is slightly different than how they're used in the decimal system. When we write a number in base 10, such as 7,567,323, the commas are used primarily as an aide to read the number easily but they do not separate single place values from each other. We will need this notation whenever the base we use is larger than 10.

Writing numbers with bases bigger than 10

When the base of a number is larger than 10, separate each "digit" with a comma to make the separation of digits clear.

For example, in base 20, to write the number corresponding to $17 \times 20^2 + 6 \times 20^1 + 13 \times 20^0$, we'd write $17,6,13_{20}$.

Try It Now

Convert the base 10 number 10553_{10} to Mayan numerals.

Try It Now

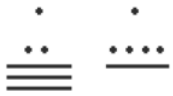
Convert the base 10 number 5617_{10} to Mayan numerals.

Adding Mayan Numbers

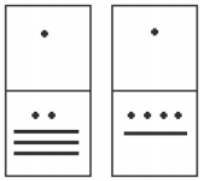
When adding Mayan numbers together, we'll adopt a scheme that the Mayans probably did not use but which will make life a little easier for us.

Example 15

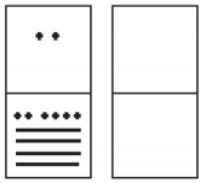
Add, in Mayan, the numbers 37 and 29:[35]



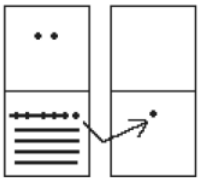
First draw a box around each of the vertical places. This will help keep the place values from being mixed up.



Next, put all of the symbols from both numbers into a single set of places (boxes), and to the right of this new number draw a set of empty boxes where you will place the final sum:

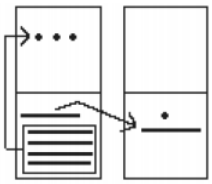


You are now ready to start carrying. Begin with the place that has the lowest value, just as you do with Arabic numbers. Start at the bottom place, where each dot is worth 1. There are six dots, but a maximum of four are allowed in any one place; once you get to five dots, you must convert to a bar. Since five dots make one bar, we draw a bar through five of the dots, leaving us with one dot which is under the four-dot limit. Put this dot into the bottom place of the empty set of boxes you just drew:

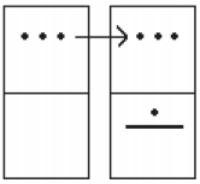


Now look at the bars in the bottom place. There are five, and the maximum number the place can hold is three. *Four bars are equal to one dot in the next highest place*.

Whenever we have four bars in a single place we will automatically convert that to a *dot* in the next place up. We draw a circle around four of the bars and an arrow up to the dots' section of the higher place. At the end of that arrow, draw a new dot. That dot represents 20 just the same as the other dots in that place. Not counting the circled bars in the bottom place, there is one bar left. One bar is under the three-bar limit; put it under the dot in the set of empty places to the right.



Now there are only three dots in the next highest place, so draw them in the corresponding empty box.



We can see here that we have 3 twenties (60), and 6 ones, for a total of 66. We check and note that 37 + 29 = 66, so we have done this addition correctly. Is it easier to just do it in base-ten? Probably, but that's only because it's more familiar to you. Your task here is to try to learn a new base system and how addition can be done in slightly different ways than what you have seen in the past. Note, however, that the concept of carrying is still used, just as it is in our own addition algorithm.

Try It Now

Try adding 174 and 78 in Mayan by first converting to Mayan numbers and then working entirely within that system. Do not add in base-ten (decimal) until the very end when you *check* your work.

Conclusion

In this reading, we have briefly sketched the development of numbers and our counting system, with the emphasis on the "brief" part. There are numerous sources of information and research that fill many volumes of books on this topic. Unfortunately, we cannot begin to come close to covering all of the information that is out there.

We have only scratched the surface of the wealth of research and information that exists on the development of numbers and counting throughout human history. What is important to note is that the system that we use every day is a product of thousands of years of progress and development. It represents contributions by many civilizations and cultures. It does not come down to us from the sky, a gift from the gods. It is not the creation of a textbook publisher. It is indeed as human as we are, as is the rest of mathematics. Behind every symbol, formula and rule there is a human face to be found, or at least sought.

Furthermore, we hope that you now have a basic appreciation for just how interesting and diverse number systems can get. Also, we're pretty sure that you have also begun to recognize that we take our own number system for granted so much that when we try to adapt to other systems or bases, we find ourselves truly having to concentrate and think about what is going on.

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34. http://www.vpds.wsu.edu/fair_95/gym/UM001.html ↵
35. http://forum.swarthmore.edu/k12/mayan.math/mayan2.html ↵

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14.3: Supplemental Videos



This YouTube playlist contains several videos that supplement the reading on Historical Counting.

You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

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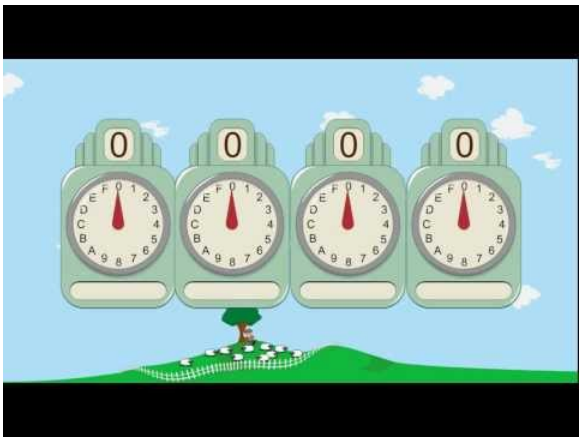
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14.4: Binary, Octal, and Hexadecimal

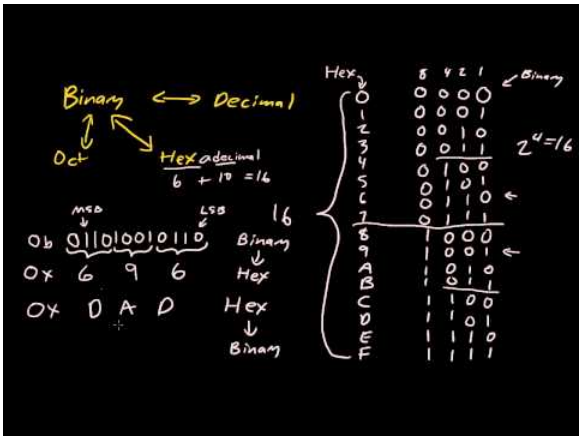
In modern computing and digital electronics, the most commonly used bases are decimal (base 10), binary (base 2), octal (base 8), and hexadecimal (base 16). If we are converting between two bases other than decimal, we typically have to convert the number to base 10 first, and then convert that number to the second base. However, we can easily convert directly from binary to octal, and vice versa, and from binary to hexadecimal, and vice versa.

This video gives a basic introduction to these conversions:



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For another description, this one is more like a math lecture:



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For further clarification, recall that the numbers 0 through 7 can be represented by up to three digits in base two. In base eight, these numbers are represented by a single digit.

Base 2 (binary) number	Base 10 (decimal) equivalent	Base 8 (octal) number
000	0	0
001	1	1
010	2	2
011	3	3
100	4	4

Base 2 (binary) number	Base 10 (decimal) equivalent	Base 8 (octal) number
101	5	5
110	6	6
111	7	7

Now when we get to the number 8, we need four digits in base 2 and two digits in base 8. In fact, the numbers 8 through 63 can be represented by two digits in base 8. We need four, five, or six digits in base 2 to represent these same numbers:

Base 2 number	Base 10 equivalent	Base 8 number
1000	8	$10 = 1 \times 8 + 0 \times 1$
1001	9	$11 = 1 \times 8 + 1 \times 1$
1010	10	$12 = 1 \times 8 + 2 \times 1$
…	…	…
111100	60	$74 = 7 \times 8 + 4 \times 1$
111101	61	$75 = 7 \times 8 + 5 \times 1$
111110	62	$76 = 7 \times 8 + 6 \times 1$
111111	63	$77 = 7 \times 8 + 7 \times 1$

The number 64 in base 8 is represented by $100_8 = 1 \times 8^2 + 0 \times 8^1 + 0 \times 8^0 = 1 \times 64 + 0 \times 8 + 0 \times 1$. In base 2, this would be 1000000_2. Do you see a pattern here? For a single digit in base 8, we need up to three digits in base 2. For two digits in base 8, we need 4, 5, or 6 digits in base 2. For three digits in base 8, we need 7, 8, or 9 digits in base 2. For each additional digit in base 8, we need up to three spaces to represent it in base 2. **Here's a way to remember this: $2^3 = 8$, so we need three spaces.**

A couple of examples would help here.

1. Convert the number 6157_8 to base 2. We split each digit in base 8 to three digits in base 2, using the three digit base 2 equivalent, so $6_8 = 110_2$, $1_8 = 001_2$, etc.
2. Convert the number 10111011001010_2 to base 8. Split this number into sets of three, **starting with the right-most digit**, then convert each set of three to its equivalent in base 8.

For hexadecimal (base 16), we need up to four digits in binary to represent each single digit. Remember this by recalling that $2^4 = 16$, so we need four digits.

You may want to print out copies of these worksheets to help you with your conversions between binary and octal or hexadecimal:

- Converting from Binary to Octal
- Converting from Binary to Hexadecimal

If you would like to quiz yourself on converting the numbers 0 through 255 to binary, octal, and hexadecimal (and between those bases), here's a link to the representations of those numbers: Binary, Octal, and Hexadecimal Numbers.

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- Pre-Algebra 3 - Decimal, Binary, Octal & Hexadecimal. **Authored by**: MyWhyU. **Located at**: https://youtu.be/5sS7w-CMHkU. **License**: *All Rights Reserved*. **License Terms**: Standard YouTube License
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14.5: Discuss- Numeration Application

Pick a real problem and try to solve it using the numeration system solving strategies from this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the Numeration Application Directions for detailed directions.

Create a new thread in the **Numeration Application** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 20 points.

Grading Criteria	Points Possible
The problem: • Is it a real-life problem? • Is it challenging, not trivial? • Is it a unique problem instead of a copy of a classmate's posting?	5
The strategies: • Are one or more general problem solving strategies used? • Are the strategies correctly identified?	5
The presentation: • Is the problem explained well? • Are the problem solving strategies explained well? • Are the appropriate terms used?	4
Your responses: • Did you post at least two responses? • Did you explain how the examples helped you better understand the math in this module? • Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?	6

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CHAPTER OVERVIEW

15: Module 6- Consumer Math

15.1: Module 6 Overview
15.2: Consumer Math
15.3: Supplemental Videos
15.4: Average Daily Balance
15.5: How an Amortization Schedule is Calculated
15.6: Discuss- Consumer Math Application
15.7: Discuss- Final Reflection

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15.1: Module 6 Overview

What You'll Learn To Do: Understand consumer math and perform computations in daily life.

Learning Objectives

- Define interest rate
- Describe the difference between simple and compound interest
- Calculate simple and compound interest
- Describe the difference between present and future values
- Calculate present and future values
- Calculate annuities and payout annuities
- Define finance charge
- Calculate finance charges using the average daily balance method
- Describe a fixed-rate mortgage
- Prepare an amortization schedule

Learning Activities


The learning activities for this module include:

Reading Assignments and Videos

- **Read:** Consumer Math
- **Watch:** Supplemental Videos
- **Read:** Average Daily Balance
- **Read:** How an Amortization Schedule is Calculated

Homework Assignments

- **Submit:** Consumer Math Homework #1 (22 points)
- **Submit:** Consumer Math Homework #2 (12 points)
- **Discuss:** Consumer Math Application (20 points)
- **Discuss:** Final Reflection (40 points)
- **Complete:** Exam 3

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15.2: Consumer Math

We have to work with money every day. While balancing your checkbook or calculating your monthly expenditures on espresso requires only arithmetic, when we start saving, planning for retirement, or need a loan, we need more mathematics.

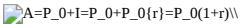
Simple Interest

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed $100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $100(0.05) = $5. The total amount you would repay would be $105, the original principal plus the interest.

Simple One-Time Interest



$$I=P_0\{r\}\\$$



$$A=P_0+I=P_0+P_0\{r\}=P_0(1+r)\\$$

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate (in decimal form. Example: 5% = 0.05)

Example 1

A friend asks to borrow $300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

Solution

P_0 = $300	the principal
r = 0.03	3% rate
I = $300(0.03) = $9	You will earn $9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Example 2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a $1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Solution

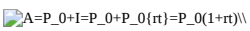
Each year, you would earn 5% interest: $1000(0.05) = $50 in interest. So over the course of five years, you would earn a total of $250 in interest. When the bond matures, you would receive back the $1,000 you originally paid, leaving you with a total of $1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time



$$I=P_0\{rt\}\\$$



$$A=P_0+I=P_0+P_0\{rt\}=P_0(1+rt)\\$$

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate in decimal form

- t is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

APR—Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)**—the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.

A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 3

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Solution

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

$P_0 = \$1000$	the principal
$r = 0.02$	2% rate per half-year
$t = 8$	4 years = 8 half-years
$I = \$1000(0.02)(8) = \$160$	You will earn $160 interest total over the four years.

Try It Now

A loan company charges $30 interest for a one month loan of $500. Find the annual interest rate they are charging.

Compound Interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit $1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR)—the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\displaystyle\frac{3\%}{12}$ = 0.25% per month.

In the first month,

$P_0 = \$1000$
$r = 0.0025$ (0.25%)
$I = \$1000\ (0.0025) = \$2.50$
$A = \$1000 + \$2.50 = \$1002.50$

In the first month, we will earn $2.50 in interest, raising our account balance to $1002.50.

In the second month,

$P_0 = \$1002.50$
$I = \$1002.50\ (0.0025) = \$2.51$ (rounded)
$A = \$1002.50 + \$2.51 = \$1005.01$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original $1000 we deposited, but we also earned interest on the $2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:

Month	Starting balance	Interest earned	Ending Balance
1	1000.00	2.50	1002.50
2	1002.50	2.51	1005.01
3	1005.01	2.51	1007.52
4	1007.52	2.52	1010.04
5	1010.04	2.53	1012.57
6	1012.57	2.53	1015.10
7	1015.10	2.54	1017.64
8	1017.64	2.54	1020.18
9	1020.18	2.55	1022.73
10	1022.73	2.56	1025.29
11	1025.29	2.56	1027.85
12	1027.85	2.57	1030.42

To find an equation to represent this, if P_m represents the amount of money after m months, then we could write the recursive equation:

$P_0 = \$1000$
$P_m = (1 + 0.0025)P_{m-1}$

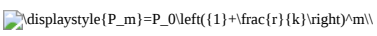
You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$P_0 = \$1000$
$P_1 = 1.0025P_0 = 1.0025\ (1000)$
$P_2 = 1.0025P_1 = 1.0025\ (1.0025\ (1000)) = 1.0025^2(1000)$
$P_3 = 1.0025P_2 = 1.0025\ (1.0025^2(1000)) = 1.0025^3(1000)$
$P_4 = 1.0025P_3 = 1.0025\ (1.0025^3(1000)) = 1.0025^4(1000)$

Observing a pattern, we could conclude

$P_m = (1.0025)^m(\$1000)$

Notice that the \$1000 in the equation was P_0, the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year. Generalizing our result, we could write

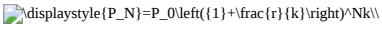


In this formula:

m is the number of compounding periods (months in our example)
r is the annual interest rate
k is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If N is the number of years, then $m = Nk$. Making this change gives us the standard formula for compound interest.

Compound Interest



- P_N is the balance in the account after N years.

- P_0 is the starting balance of the account (also called initial deposit, or principal)
- r is the annual interest rate in decimal form
- k is the number of compounding periods in one year.

If the compounding is done annually (once a year), $k = 1$.

If the compounding is done quarterly, $k = 4$.

If the compounding is done monthly, $k = 12$.

If the compounding is done daily, $k = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account **once** and let it sit there earning interest.

Example 4

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit $3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

Solution

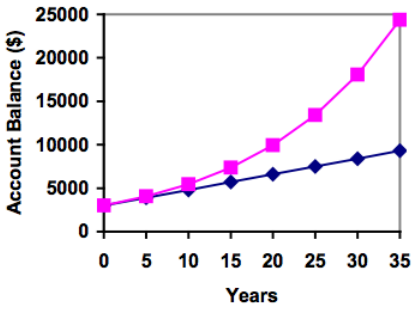
In this example,

$P_0 = \$3000$	the initial deposit
$r = 0.06$	6% annual rate
$k = 12$	12 months in 1 year
$N = 20$	since we're looking for how much we'll have after 20 years

So $\displaystyle P_{20}=3000\left(1+\frac{0.06}{12}\right)^{20\cdot 12}=\$9930.61$ (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

Years	Simple Interest ($15 per month)	6% compounded monthly = 0.5% each month.
5	$3900	$4046.55
10	$4800	$5458.19
15	$5700	$7362.28
20	$6600	$9930.61
25	$7500	$13394.91
30	$8400	$18067.73
35	$9300	$24370.65



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Evaluating Exponents on the Calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5 \cdot 5 \cdot 5 = 125$. But when we need to calculate something like 1.005^{240}, it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

^, y^x, or x^y.

To evaluate 1.005^{240} we'd type 1.005 ^ 240, or 1.005 y^x 240. Try it out—you should get something around 3.3102044758.

Example 5

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

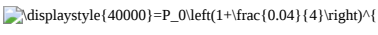
Solution

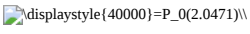
In this example,

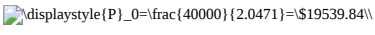
We're looking for P_0.

$r = 0.04$	4%
$k = 4$	4 quarters in 1 year
$N = 18$	Since we know the balance in 18 years
$P_{18} = \$40{,}000$	The amount we have in 18 years.

In this case, we're going to have to set up the equation, and solve for P_0.







So you would need to deposit $19,539.84 now to have $40,000 in 18 years.

Rounding

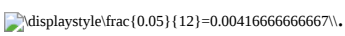
It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but **keeping more digits is always better**.

Example 6

To see why not over-rounding is so important, suppose you were investing $1000 at 5% interest compounded monthly for 30 years.

$P_0 = \$1000$	the initial deposit
$r = 0.05$	5%
$k = 12$	12 months in 1 year
$N = 30$	since we're looking for the amount after 30 years

Solution

If we first compute  , we find  .

Here is the effect of rounding this to different values:

$\displaystyle\frac{r}{k}$ rounded to	Gives P_{30} to be	Error
0.004	$4208.59	$259.15
0.0042	$4521.45	$53.71
0.00417	$4473.09	$5.35
0.004167	$4468.28	$0.54
0.0041667	$4467.80	$0.06
no rounding	$4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough—$5 off of $4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

Using Your Calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$$\displaystyle P_{30}=1000\left(1+\frac{0.05}{12}\right)^{12\cdot{30}}$$

We can quickly calculate 12 × 30 = 360, giving $\displaystyle P_{30}=1000\left(1+\frac{0.05}{12}\right)^{360}$.

Now we can use the calculator.

Type This	Calculator Shows
0.05 ÷ 12 =	0.00416666666667
+ 1 =	1.00416666666667
y^x 360 =	4.46774431400613
× 1000 =	4467.74431400613

Using Your Calculator Continued

The previous steps were assuming you have a "one operation at a time" calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 × (1 + 0.05 ÷ 12) y^x 360 = .

Annuities

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$\displaystyle P_m=\left(1+\frac{r}{k}\right)P_{m-1}$$

For a savings annuity, we simply need to add a deposit, d, to the account with each compounding period:

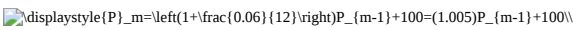
$$\displaystyle P_m=\left(1+\frac{r}{k}\right)P_{m-1}+d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit $100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

- r = 0.06 (6%)
- k = 12 (12 compounds/deposits per year)
- d = $100 (our deposit per month)

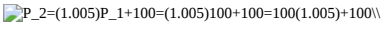
Writing out the recursive equation gives

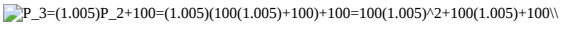


Assuming we start with an empty account, we can begin using this relationship:

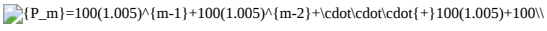






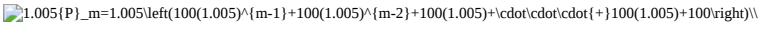


Continuing this pattern, after m deposits, we'd have saved:

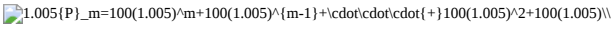


In other words, after m months, the first deposit will have earned compound interest for $m - 1$ months. The second deposit will have earned interest for $m - 2$ months. Last months deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

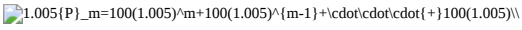
This equation leaves a lot to be desired, though—it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:



Distributing on the right side of the equation gives

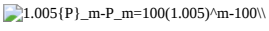


Now we'll line this up with like terms from our original equation, and subtract each side

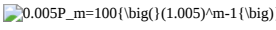


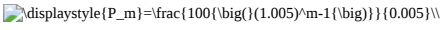


Almost all the terms cancel on the right hand side when we subtract, leaving

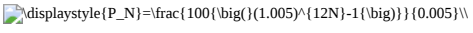


Solving for *Pm*



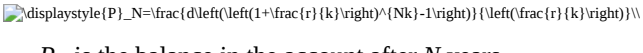


Replacing m months with $12N$, where N is measured in years, gives



Recall 0.005 was  and 100 was the deposit d. 12 was k, the number of deposit each year. Generalizing this result, we get the saving annuity formula.

Annuity Formula



- P_N is the balance in the account after N years.
- d is the regular deposit (the amount you deposit each year, each month, etc.)
- r is the annual interest rate in decimal form.
- k is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, k = 12.

If you make your deposits every year, use yearly compounding, $k = 1$.

If you make your deposits every quarter, use quarterly compounding, $k = 4$.

Etc.

When Do You Use This?

Annuities assume that you put money in the account **on a regular schedule (every month, year, quarter, etc.)** and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: **One** deposit

Annuity: **Many** deposits.

Example 7

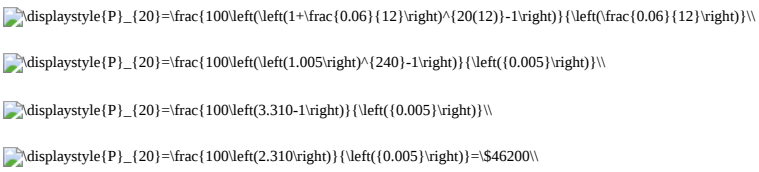
A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit $100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

Solution

In this example,

d = $100	the monthly deposit
r = 0.06	6% annual rate
k = 12	since we're doing monthly deposits, we'll compound monthly
N = 20	we want the amount after 20 years

Putting this into the equation:



The account will grow to $46,200 after 20 years.

Notice that you deposited into the account a total of $24,000 ($100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $46,200 − $24,000 = $22,200.

Example 8

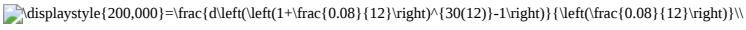
You want to have $200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

Solution

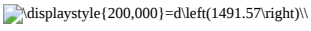
In this example, we're looking for d.

r = 0.08	8% annual rate
k = 12	since we're depositing monthly
N = 30	30 years
P_{30} = $200,000	The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for d.









So you would need to deposit $134.09 each month to have $200,000 in 30 years if your account earns 8% interest

Try It Now

A more conservative investment account pays 3% interest. If you deposit $5 a day into this account, how much will you have after 10 years? How much is from interest?

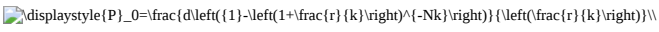
Payout Annuities

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved $500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula



- P_0 is the balance in the account at the beginning (starting amount, or principal).
- d is the regular withdrawal (the amount you take out each year, each month, etc.)
- r is the annual interest rate (in decimal form. Example: 5% = 0.05)
- k is the number of compounding periods in one year.
- N is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When Do You Use This

Payout annuities assume that you **take** money from the account **on a regular schedule** (**every month, year, quarter, etc.**) and let the rest sit there earning interest.

- Compound interest: **One** deposit
- Annuity: **Many** deposits.
- Payout Annuity: **Many withdrawals**

Example 9

After retiring, you want to be able to take $1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

Solution

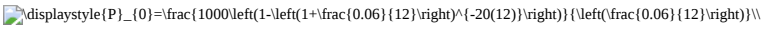
In this example,

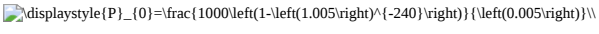
d = $1000	the monthly withdrawal
r = 0.06	6% annual rate
k = 12	since we're doing monthly withdrawals, we'll compound monthly

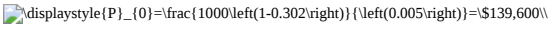
$N = 20$	since were taking withdrawals for 20 years

We're looking for P_0; how much money needs to be in the account at the beginning.

Putting this into the equation:







You will need to have $139,600 in your account when you retire.

Notice that you withdrew a total of $240,000 ($1000 a month for 240 months). The difference between what you pulled out and what you started with is the **interest earned**. In this case it is $240,000 − $139,600 = $100,400 in interest.

Evaluating Negative Exponents on Your Calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this **(-)**, some with +/-. The button is often near the = key or the decimal point.

If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005^{-240} you'd type something like: 1.005 ^ **(-)** 240

If your calculator only shows one value at a time, then usually you hit the **(-)** key after a number to negate it, so you'd hit: 1.005 y^x 240 **(-)** =

Give it a try—you should get $1.005^{-240} = 0.302096$

Example 10

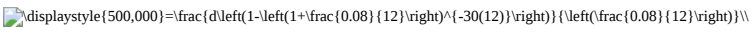
You know you will have $500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

Solution

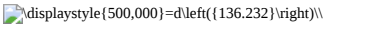
In this example, we're looking for d.

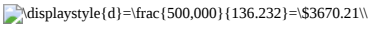
$r = 0.08$	8% annual rate
$k = 12$	since we're withdrawing monthly
$N = 30$	30 years
$P_0 = \$500{,}000$	we are beginning with $500,000

In this case, we're going to have to set up the equation, and solve for d.









You would be able to withdraw $3,670.21 each month for 30 years.

Try It Now

A donor gives $100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

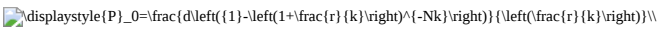
Loans

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had $10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests $10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loans Formula



- P_0 is the balance in the account at the beginning (the principal, or amount of the loan).
- d is your loan payment (your monthly payment, annual payment, etc)
- r is the annual interest rate in decimal form.
- k is the number of compounding periods in one year.
- N is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this

The loan formula assumes that you make loan payments **on a regular schedule** (**every month, year, quarter, etc.**) and are paying interest on the loan.

- Compound interest: **One** deposit
- Annuity: **Many** deposits.
- Payout Annuity: **Many withdrawals**
- Loans: **Many payments**

Example 11

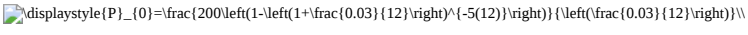
You can afford $200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with $200 per month?

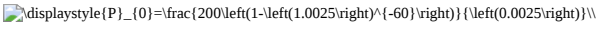
Solution

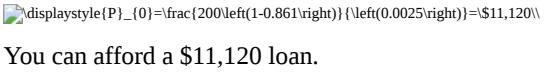
In this example,

d = $200	the monthly loan payment
r = 0.03	3% annual rate
k = 12	since we're doing monthly payments, we'll compound monthly
N = 5	since we're making monthly payments for 5 years

We're looking for P_0, the starting amount of the loan.







You can afford a $11,120 loan.

You will pay a total of $12,000 ($200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the **interest paid**. In this case, you're paying $12,000 − $11,120 = $880 interest total.

Example 12

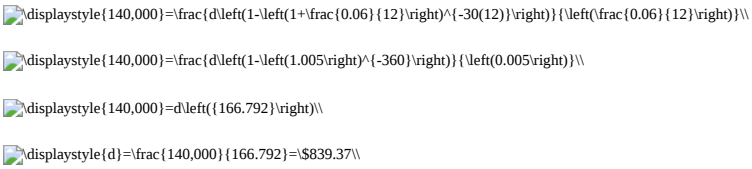
You want to take out a $140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

Solution

In this example, we're looking for d.

$r = 0.06$	6% annual rate
$k = 12$	since we're paying monthly
$N = 30$	30 years
$P_0 = \$140{,}000$	the starting loan amount

In this case, we're going to have to set up the equation, and solve for d.



You will make payments of $839.37 per month for 30 years.

You're paying a total of $302,173.20 to the loan company: $839.37 per month for 360 months. You are paying a total of $302,173.20 − $140,000 = $162,173.20 in interest over the life of the loan.

Try It Now

Janine bought $3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

Remaining Loan Balance

With loans, it is often desirable to determine what the remaining loan balance will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were $1,000 a month, after a year you will *not* have paid off $12,000 of the loan balance.

To determine the remaining loan balance, we can think "how much loan will these loan payments be able to pay off in the remaining time on the loan?"

Example 13

If a mortgage at a 6% interest rate has payments of $1,000 a month, how much will the loan balance be 10 years from the end the loan?

Solution

To determine this, we are looking for the amount of the loan that can be paid off by $1,000 a month payments in 10 years. In other words, we're looking for P_0 when

$d = \$1{,}000$	the monthly loan payment
$r = 0.06$	6% annual rate
$k = 12$	since we're doing monthly payments, we'll compound monthly

$N = 10$	since we're making monthly payments for 10 more years

$$P_0=\frac{1000\left(1-\left(1+\frac{0.06}{12}\right)^{-10(12)}\right)}{\left(\frac{0.06}{12}\right)}$$

$$P_0=\frac{1000\left(1-\left(1.005\right)^{-120}\right)}{\left(0.005\right)}$$

$$P_0=\frac{200\left(1-0.5496\right)}{\left(0.005\right)}=\$90,073.45$$

The loan balance with 10 years remaining on the loan will be $90,073.45

Often times answering remaining balance questions requires two steps:

1. Calculating the monthly payments on the loan
2. Calculating the remaining loan balance based on the *remaining time* on the loan

Example 14

A couple purchases a home with a $180,000 mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

Solution

First we will calculate their monthly payments. We're looking for d.

$r = 0.04$	4% annual rate
$k = 12$	since they're paying monthly
$N = 30$	30 years
$P_0 = \$180,000$	the starting loan amount

We set up the equation and solve for d.

$$180,000=\frac{d\left(1-\left(1+\frac{0.04}{12}\right)^{-30(12)}\right)}{\left(\frac{0.04}{12}\right)}$$

$$180,000=\frac{d\left(1-\left(1.00333\right)^{-360}\right)}{\left(.00333\right)}$$

$$180,000=d\left(209.562\right)$$

$$d=\frac{180,000}{209.562}=\$858.93$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

$d = \$858.93$	the monthly loan payment we calculated above
$r = 0.04$	4% annual rate
$k = 12$	since they're doing monthly payments
$N = 25$	since they'd be making monthly payments for 25 more years

$$P_0=\frac{858.93\left(1-\left(1+\frac{0.04}{12}\right)^{-25(12)}\right)}{\left(\frac{0.04}{12}\right)}$$

$$P_0=\frac{858.93\left(1-\left(1.00333\right)^{-300}\right)}{\left(0.00333\right)}$$

$$P_0=\frac{858.93\left(1-0.369\right)}{\left(0.00333\right)}=\$162,758.26$$

The loan balance after 5 years, with 25 years remaining on the loan, will be $162,758.26.

Over that 5 years, the couple has paid off $180,000 − $162.758.26 = $17,241.74 of the loan balance. They have paid a total of $858.93 a month for 5 years (60 months), for a total of $51,535.80, so $51,535.80 − $17,241.74 = $34,294.06 of what they have paid so far has been interest.

Which Equation to Use?

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan," "amortize" (the fancy word for loans), "finance (a car)," or "mortgage" (a home loan). Look for these words. If they're there, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annually/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly) then you're looking at a basic Annuity problem. Basic annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future.

If you're pulling money out of the account on a regular basis, then you're looking at a Payout Annuity problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

Try It Now

For each of the following scenarios, determine if it is a compound interest problem, a savings annuity problem, a payout annuity problem, or a loans problem. Then solve each problem.

1. Marcy received an inheritance of $20,000, and invested it at 6% interest. She is going to use it for college, withdrawing money for tuition and expenses each quarter. How much can she take out each quarter if she has 3 years of school left?
2. Paul wants to buy a new car. Rather than take out a loan, he decides to save $200 a month in an account earning 3% interest compounded monthly. How much will he have saved up after 3 years?
3. Keisha is managing investments for a non-profit company. They want to invest some money in an account earning 5% interest compounded annually with the goal to have $30,000 in the account in 6 years. How much should Keisha deposit into the account?
4. Miao is going to finance new office equipment at a 2% rate over a 4 year term. If she can afford monthly payments of $100, how much new equipment can she buy?
5. How much would you need to save every month in an account earning 4% interest to have $5,000 saved up in two years?

Solving for Time

Often we are interested in how long it will take to accumulate money or how long we'd need to extend a loan to bring payments down to a reasonable level.

Note: This section assumes you've covered solving exponential equations using logarithms, either in prior classes or in the growth models chapter.

Example 15

If you invest $2000 at 6% compounded monthly, how long will it take the account to double in value?

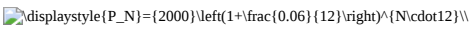
Solution

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

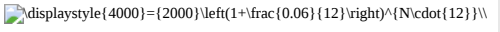
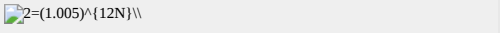
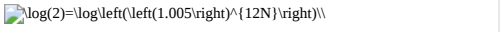
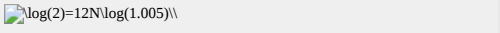
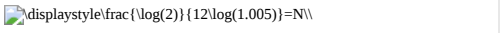
$P_0 = \$2000$	the initial deposit
$r = 0.06$	6% annual rate

$k = 12$	12 months in 1 year

So our general equation is:



We also know that we want our ending amount to be double of $2000, which is $4000, so we're looking for N so that $P_N = 4000$. To solve this, we set our equation for P_N equal to 4000.

	Divide both sides by 2000
	To solve for the exponent, take the log of both sides
	Use the exponent property of logs on the right side
	Now we can divide both sides by 12log(1.005)
	Approximating this to a decimal
$N = 11.581$	

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example if you rounded log(2) to 0.301 and log(1.005) to 0.00217, then your final answer would have been about 11.577 years.

Example 16

If you invest $100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to $10,000?

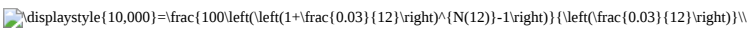
Solution

This is a savings annuity problem since we are making regular deposits into the account.

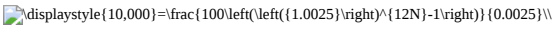
$d = \$100$	the monthly deposit
$r = 0.03$	3% annual rate
$k = 12$	since we're doing monthly deposits, we'll compound monthly

We don't know N, but we want PN to be $10,000.

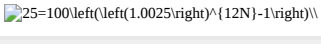
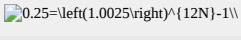
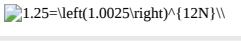
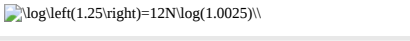
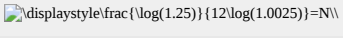
Putting this into the equation:



Simplifying the fractions a bit



We want to isolate the exponential term, 1.0025^{12N}, so multiply both sides by 0.0025

	Divide both sides by 100
	Add 1 to both sides
	Now take the log of both sides
	Use the exponent property of logs
	Divide by 12log(1.0025)
	Approximating to a decimal

$N = 7.447$ years	

It will take about 7.447 years to grow the account to $10,000.

Try It Now

Joel is considering putting a $1,000 laptop purchase on his credit card, which has an interest rate of 12% compounded monthly. How long will it take him to pay off the purchase if he makes payments of $30 a month?

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15.3: Supplemental Videos



This YouTube playlist contains several videos that supplement the reading on Finance.

You are not required to watch all of these videos, but I recommend watching the videos for any concepts you may be struggling with.

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15.4: Average Daily Balance



Click on the link below to download a document: “Credit Cards and Installment Buying” by Erie Community College. This reading shows an example of an Average Daily Balance Method of Determining Interest Rate using credit cards and installment buying to illustrate its points.

- Credit Cards and Installment Buying

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15.5: How an Amortization Schedule is Calculated



Click on the link below to view the website “How is an Amortization Schedule Calculated?” by MyAmortizationChart.com. This website provides a brief overlook Amortization Schedules.

- How is an Amortization Schedule Calculated?

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15.6: Discuss- Consumer Math Application

Pick a real problem and try to solve it using the consumer math solving strategies from this module. Present the problem and the solution to the rest of the class. View the problems posted by your classmates and respond to at least two. Read the Consumer Math Application Directions for detailed directions.

Create a new thread in the **Consumer Math Application** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 20 points.

Grading Criteria	Points Possible
The problem: • Is it a real-life problem? • Is it challenging, not trivial? • Is it a unique problem instead of a copy of a classmate's posting?	5
The strategies: • Are one or more general problem solving strategies used? • Are the strategies correctly identified?	5
The presentation: • Is the problem explained well? • Are the problem solving strategies explained well? • Are the appropriate terms used?	4
Your responses: • Did you post at least two responses? • Did you explain how the examples helped you better understand the math in this module? • Did you ask questions for clarification or make suggestions on how to change or improve the original application posting or any other follow-up postings?	6

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15.7: Discuss- Final Reflection

Reflect on what you have learned in this course, which will help you better understand and remember it. Read the Final Reflection Directions for detailed directions. Create a new thread in the **Final Reflection** forum in the **Discussion Board** to complete this assignment.

This assignment is required and worth up to 40 points.

Grading Criteria	Points Possible
The presentation: • Is it a substantive, two or three paragraph essay? • Is it clear and well-written with proper grammar and correct spelling?	15
The content: • For Option 1: ◦ Does it apply to material covered in class? ◦ Is the posting unique to the postings of other students and to your own application postings? • For Option 2: ◦ Is (are) the selected question(s) answered in depth? ◦ Are the facts accurate?	25

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CHAPTER OVERVIEW

16: Topic G- Linear Equations – Algebra and Spreadsheets

16.1: G1.01- Intro and Slope
16.2: G1.02- Intercepts and Example 3
16.3: G1.03- Examples 4-7
16.4: G1.04- Examples 8-14
16.5: G1.05- Exploring by Graphing Part I
16.6: G1.06- Exploring by Graphing Part II
16.7: G1.07- Section 4
16.8: G1.08- Exercises

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16.1: G1.01- Intro and Slope

Learning Objectives

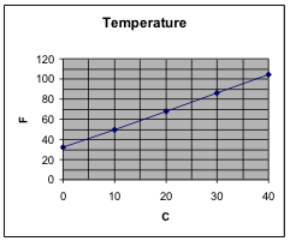
1. Understand slope of a line in two ways (graphically by rise/run, algebraic formula in *x*'s and *y*'s.)
2. Understand the *y*-intercept of a line in two ways (graphically as the intercept of the y-axis, algebraic formula in *x*'s and *y*'s.)
3. Algebraically, find the slope when given two points.
4. Write the formula for the line in standard slope-intercept form and notice which number is the slope and which number is the *y*-intercept. Interpret the slope and intercept.
5. Find the formula for a line when given the slope and one point on the line.
6. Find the formula for a line when given two points on the line.
7. Find the inverse formula, when needed, by interchanging the input and output variables. Notice the relationship between the slopes of the formula and the inverse formula.
8. Use a spreadsheet to explore the effects on the graph of changing the values of the slope and intercept.

Summary of algebraic facts and interpretations:

Slope: The **slope of a line** is $m=\frac{rise}{run}=\frac{{{y}_{2}}-{{y}_{1}}}{{{x}_{2}}-{{x}_{1}}}$, where the two points $({{x}_{1}},{{y}_{1}})$ and $({{x}_{2}},{{y}_{2}})$ are on the line.	**Slope-intercept form**: If **the formula is** $y=mx+b$ where *y* and *x* are the variables and *m* and *b* are numbers, then the graph of this formula is a straight line with slope *m* and y-intercept *b*. This can also be written as $y=b+mx$	**Point-slope form**: If the slope of a line is *m* and one point on the line is $({{x}_{0}},{{y}_{0}})$, then the **equation of the line** can be written as $y-{{y}_{0}}=m(x-{{x}_{0}})$ which gives the formula: $y=m(x-{{x}_{0}})+{{y}_{0}}$
Interpret the slope: *m* is the amount we expect *y* to increase when *x* increases by 1.	**Interpret the intercept:** *b* is the value of *y* when the value of *x* is zero.	

We want to investigate the relationship between temperature Centigrade and temperature Fahrenheit in order to take temperatures from a news broadcast in Canada (in Centigrade) and convert them to temperatures that are easy for US citizens to understand (in Fahrenheit.) We already know the formula for that and we know how to find points, plot them, and draw a graph. (See Topic C: Formulas.)

Here, the temperature Centigrade is the input value and the temperature Fahrenheit is the output value. The formula is $F=1.8\cdot{C}+32$ and following is a table of values and a graph.



C	F
0	32
10	50
20	68

30	86
40	104

Slope

Notice that the graph appears to be a straight line. That suggests that this is a linear relationship. A relationship is linear if the same increase in the input always leads to the same increase in the output. We can look at the table of values and see that, for every increase in C of 10°, there is an increase of 18° in F.

In general, for a straight line with input variable x and output variable y, the **slope** is defined to be slope = $m=\frac{rise}{run}=\frac{{{y}_{2}}-{{y}_{1}}}{{{x}_{2}}-{{x}_{1}}}$, where the two points $({{x}_{1}},{{y}_{1}})$ and $({{x}_{2}},{{y}_{2}})$are on the line.

In the temperature example, we notice that the variable C plays the role of x and the variable F plays the role of y. So we can compute the slope of the line using the two points (0,32) and (10,50). Call the first point $({{x}_{1}},{{y}_{1}})$ and the second point $({{x}_{2}},{{y}_{2}})$.

Slope = $m=\frac{rise}{run}=\frac{{{y}_{2}}-{{y}_{1}}}{{{x}_{2}}-{{x}_{1}}}=\frac{50-32}{10-0}=\frac{18\,\!\degrees\,\!F}{10\,\!\degrees\,\!C}=1.8$ degrees F per degree C.

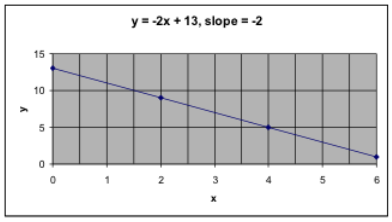
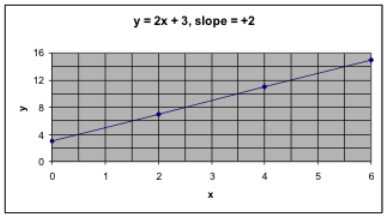
For this to be a linear relationship, the slope must be the same no matter which two points on the graph you use to compute it. Let's try one more pair of points. So we can compute the slope of the line using the two points (20,68) and (40,104). Call the first point $({{x}_{1}},{{y}_{1}})$ and the second point $({{x}_{2}},{{y}_{2}})$.

Slope = $m=\frac{rise}{run}=\frac{{{y}_{2}}-{{y}_{1}}}{{{x}_{2}}-{{x}_{1}}}=\frac{104-68}{40-20}=\frac{36\,\!\degrees\,\!F}{20\,\!\degrees\,\!C}=1.8$ degrees F per degree C.

There are several more pairs of points from our table of values that you could easily try. Try some of those and see if you get the same slope each time.

Negative slope and positive slope.

Recall from algebra class that lines which are slanted down have negative slope and those slanted up have positive slope. One important method of checking your work in computing the slope is to be sure that your graphs and algebra give results that are consistent with that.

y = 2x + 13		y = 2x + 3	

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16.2: G1.02- Intercepts and Example 3

y-intercept

In algebra class, we learned that the y-intercept of a graph is the value on the *y*-axis when $x = 0$. That is easy to find using the formula for the line, because we only have to plug in $x = 0$ and find y.

Since $F=1.8\cdot{C}+32$, when we plug in $C = 0$, then $F=1.8\cdot{C}+32=1.8\cdot0+32=32$

When we look at the graph for this formula, go to $C = 0$, and find the point on the graph above that, the *y*-value appears to be about a little bit above 30. To find a more precise estimate from the graph, we'd need a bigger graph. This is consistent with the value of 32 that we found algebraically. Understanding both of these provides a way of checking your work on either of them.

The formula for the line

Notice that both the slope and the *y*-intercept appear in the formula for the line. In algebra class, you learned that

1. Any equation in which x and y appear only to the first power is a linear equation.
2. The equation of a line can be written in a variety of ways, but if you take any of those and "solve for y" you'll get a formula like $y=mx+b$, where y is the output variable, x is the input variable, and m and b are numbers. Then the slope is the coefficient of the x-variable and the *y*-intercept is the constant. So m is the slope and b is the *y*-intercept.

For the temperature example, since F is the output value and C is the input value, and $F=1.8\cdot{C}+32$, then we see from the formula that the slope is 1.8 and the *y*-intercept is 32.

Interpret the slope and intercept

1. The *y*-intercept is the value for y when
2. When the x-value increases by 1 unit, then the *y*-value increases by the value of the slope.

So, for the temperature problem, we found that if the temperature was 0° C, then it is 32° F. And we also found that, if the temperature C increases by 1° C, then the temperature F increases by 1.8° F.

Example 1. For this formula: $R=-6.3+8.2\cdot{d}$, tell

1. Which variable is the output variable?
2. Which variable is the input variable?
3. Is it linear relationship?
4. If it is a linear relationship, what is the slope?
5. If it is a linear relationship, what is the *y*-intercept?

[reveal-answer q="892655"]Show Answer[/reveal-answer]
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1. The output variable is R.
2. the input variable is d.
3. yes, it is a linear relationship because both variables appear only to the first power.
4. the slope is 8.2 because that is the number multiplied by the input variable.
5. the y-intercept is -6.3.

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Example 2. For this formula: $R=-6.3+8.2\cdot{d}$, find three points that fit this and use them to sketch a graph.

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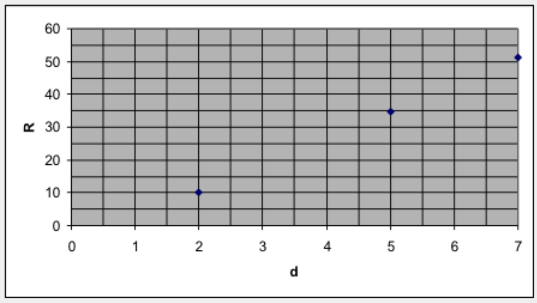
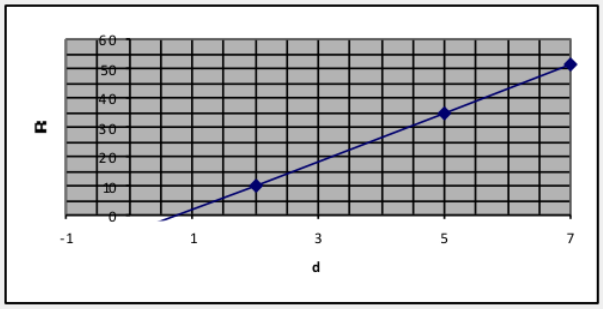
To find a point that fits this relationship, choose any value for the input variable. (Generally we choose a value that is easy to compute with and to graph, such as a small whole number.) Let's begin by choosing $d=2$. Then compute the output value.

$$\begin{align}&R=-6.3+8.2\cdot{d}\\&R=-6.3+8.2\cdot(2)\\&R=-6.3+16.4\\&R=10.1\\\end{align}$$

Now choose two other values for d and use them to compute the output values. In the interests of saving space, we will omit the details.

Choose $d=5$ and find that $R=-6.3+8.2\cdot5=34.7$

Choose $d=7$ and find that $R=-6.3+8.2\cdot7=51.1$

Graph with unconnected points	Graph with points connected by a line
	

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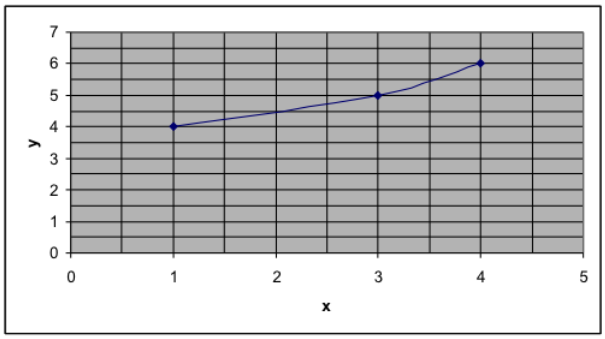
Example 3. Determine whether these three points lie on a straight line by computing slopes: (3,5) and (1,4) and (4,6).

[reveal-answer q="964443"]Show Answer[/reveal-answer]
[hidden-answer a="964443"]Find the slope of the line through (3,5) and (1,4). $m=\frac{rise}{run}=\frac{{{y}_{2}}-{{y}_{1}}}{{{x}_{2}}-{{x}_{1}}}=\frac{5-4}{3-1}=\frac{1}{2}=0.5$

Find the slope of the line through (3,5) and (4,6). $m=\frac{rise}{run}=\frac{{{y}_{2}}-{{y}_{1}}}{{{x}_{2}}-{{x}_{1}}}=\frac{5-6}{3-4}=\frac{-1}{-1}=1.0$

Since these two slopes aren't equal, these three points are not on a straight line.

Check: Check this by graphing the three points to see if they appear to lie on a straight line. In the following graph, we have drawn the line between the points and it is clear that it is not the same straight line between all three points.



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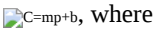
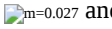
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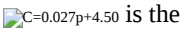
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16.2: G1.02- Intercepts and Example 3 is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

16.3: G1.03- Examples 4-7

Example 4. A manager is considering the cost *C* of printing a book based on the number of pages *p*. He is told that the formula for predicting the cost is linear based on the number of pages and that the *y*-intercept is $4.50 and the slope is $0.027. Find the formula to predict the cost from the number of pages.

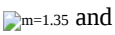
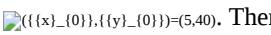
[reveal-answer q="494645"]Show Answer[/reveal-answer]
[hidden-answer a="494645"]Since it is linear,  , where  and  .

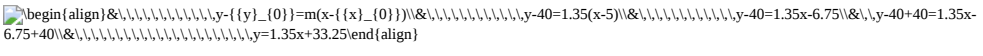
Thus  is the formula.

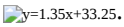
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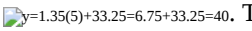
Example 5. Find the formula for the line with slope 1.35 which has the point (5,40) on it.

[reveal-answer q="337108"]Show Answer[/reveal-answer]
[hidden-answer a="337108"]

Let  and  . Then



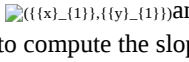
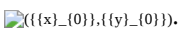
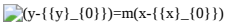
Thus the equation of the line is  .

Check: We can look at the formula and read the slope as 1.35, so that checks. To see whether the given point is on it, evaluate the formula at  and see whether it gives the correct *y*-value.  . This checks!

[/hidden-answer]

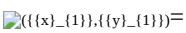
We can use this same idea to **compute the formula for a line if we have two points on it** because we can first use the two points to find the slope. Here's an outline:

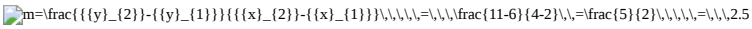
Find the formula for of a line through two points.

1. Choose the appropriate variable to be the output variable and call it *y*. Then call the input variable *x*.
2. Write two points as  and .
3. Use the two points to compute the slope. Call it *m*.
4. Pick one of the points (either is fine) and call it  .
5. Plug those values into this equation. 
6. Solve for *y*. That gives the equation of the line.
7. If different letters are needed besides *x* and *y* for the input and output variables, replace the *x* and *y* in the formula with those different letters.

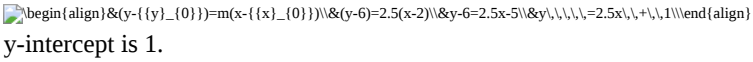
Example 6: Find the formula for the line through (2,6) and (4,11). Identify the slope and y-intercept.

[reveal-answer q="946927"]Show Answer[/reveal-answer]
[hidden-answer a="946927"]

Let  = (2,6) and = (4,11).



Now, to use the point-slope form, I will use = (2,6) to get

 So the equation of the line is , the slope is 2.5, the y-intercept is 1.

Check: Use the formula we derived to determine the output value for  and see if the given point is on the line. . This tells us that (2,6) is on the line.

Use the formula we derived to determine the output value for  and see if the second given point is on the line.  . This tells us that (4,11) is on the line. So this checks.

[/hidden-answer]

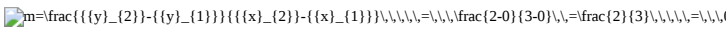
Example 7. We have been told that the amount of oatmeal needed for oatmeal cookies is linearly related to the amount of flour needed. Also, we know that if we use 3 cups of flour, we need 2 cups of oatmeal. And, of course, if we use 0 cups of flour, we will use 0 cups of oatmeal.

1. Find the formula to predict the oatmeal needed (called M) from the flour needed (F.)
2. Interpret the slope.
3. Interpret the y-intercept.

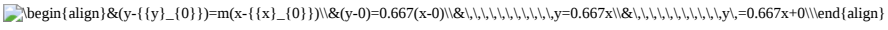
[reveal-answer q="922900″]Show Answer[/reveal-answer]
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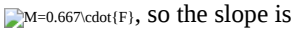
a. Here, the output variable (*y*) is M and the input variable (*x*) is F.

So the two points given are (0,0) and (3, 2).

Using the slope formula,  cups of oatmeal per one cup of flour.

Then, to use the point-slope form of the line, we'll use and the first point, which is (0,0).



So the formula is  , so the slope is 0.667 and the y-intercept is 0.

1. Interpret the slope: For every 1 additional cup of flour, we should use 0.667 additional cups of oatmeal.
2. Interpret the intercept: If we use 0 cups of flour, then we will use 0 cups of oatmeal.

[/hidden-answer]

This last example illustrates that sometimes the intercept is not a number that would be realistic in the situation that the problem describes. But it does have a meaning in the algebraic formula.

For a linear formula, the slope is always a number that is meaningful.

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16.4: G1.04- Examples 8-14

Additional examples:

Example 8: Find the slope and intercept of the line $y=-2.32x+7.89$

[reveal-answer q="429582"]Show Answer[/reveal-answer]
[hidden-answer a="429582"]The slope is -2.32 and the y-intercept is 7.89.[/hidden-answer]

Example 9: Find the slope and intercept of the line $y=84.4+9.2x$

[reveal-answer q="185443"]Show Answer[/reveal-answer]
[hidden-answer a="185443"]The slope is 9.2 and the y-intercept is 84.4[/hidden-answer]

Example 10: Find the slope and intercept of the line $y=1127-93x$

[reveal-answer q="191120"]Show Answer[/reveal-answer]
[hidden-answer a="191120"]The slope is -93 and the y-intercept is 1127[/hidden-answer]

Example 11: Find the slope and the intercept of the line $y=2178x-114$

[reveal-answer q="383183"]Show Answer[/reveal-answer]
[hidden-answer a="383183"]The slope is 2178 and the y-intercept is -114.[/hidden-answer]

Example 12 Find the formula for the line with slope -6.2 through the point (87.2, 112.7)

[reveal-answer q="695808"]Show Answer[/reveal-answer]
[hidden-answer a="695808"]$\begin{align}&y-{{y}_{0}}=m(x-{{x}_{0}})\\&y-112.7=-6.2(x-87.2)\\&y-112.7=-6.2x+540.64\\&y-112.7+112.7=-6.2x+540.64+112.7\\&y=-6.2x+653.34\end{align}$
[/hidden-answer]

Example 13: Find the formula for the line through (8,5) and (13,17)

[reveal-answer q="272460"]Show Answer[/reveal-answer]
[hidden-answer a="272460"]

$m=\frac{17-5}{13-8}=\frac{12}{5}=2.4$

Then, using this slope and the second point: $\begin{align}&y-{{y}_{0}}=m(x-{{x}_{0}})\\&y-17=2.4(x-13)\\&y-17=2.4x-31.2\\&y-17+17=2.4x-31.2+17\\&y=2.4x-14.2\end{align}$

[/hidden-answer]

Example 14: Find the formula for the line through (83.8, 79.9) and (232.7, 63.4)

[reveal-answer q="867398"]Show Answer[/reveal-answer]
[hidden-answer a="867398"]

$m=\frac{79.9-63.4}{83.8-232.7}=\frac{16.5}{-148.9}=-0.11$

Then, using this slope and the first point: $\begin{align}&y-{{y}_{0}}=m(x-{{x}_{0}})\\&y-79.9=-0.11(x-83.8)\\&y-79.9=-0.11x+9.218\\&y-79.9+79.9=-0.11x+9.218+79.9\\&y=-0.11x+89.118\\\end{align}$

[/hidden-answer]

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16.5: G1.05- Exploring by Graphing Part I

Section 2. Exploration by graphing.

We can also find lines just by graphing. That may be useful itself in some situations even without finding the actual equation of the line. And, if we have already found the equation of the line, we can use a graph to check our algebra.

We have considered two types of problems in this topic.

The first is finding a line when we have the slope and one point on the line. To graph that by hand

1. Make a set of axes.
2. Graph the point.
3. Rewrite the slope as $\frac{\text{rise}}{\text{run}}$ and use that count over and up (or down) to make another point on the graph.
4. Connect those points by a line.

The second case is finding a line when we have two points on the graph. To graph that by hand

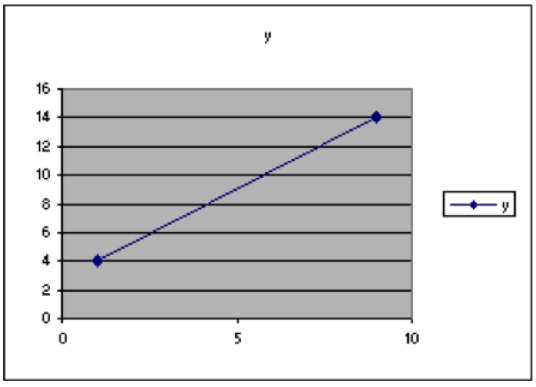
1. Make a set of axes.
2. Graph one point.
3. Graph the other point.
4. Connect those points by a line.

In either of these cases, after we have the graph of a line, we can pick any point on that line and know it is supposed to fit the equation of the line. So we can plug in the values for x and y into the algebraic formula we found. If the graph is correct and the equation is correct, then the point will fit the equation.

Using a spreadsheet to graph lines allows us to try various possibilities quickly and easily. The interesting case here is when we have two points and we want to explore various possible equations of a line.

Example 15. Use a spreadsheet to graph the line through the points (1,9) and (4,14).

1. Put the label x in cell A1 and enter the x-values: A2 should have 1 and A3 should have 4.
2. Put the label y in cell B1 and enter the y-values: B2 should have 9 and B3 should have 14.
3. Then select the data, with the labels. Highlight it. Then choose **Insert > Chart** and XY Scatter. Choose one of the options that connects the data with curves. Then click on “Next”. In this box, check the “Columns” box to indicate that the data is arranged with the data values in columns. This is a VERY IMPORTANT step when you are only dealing with two points. This step should not be necessary when you have more than two points.



	A	B	C	D	E	F	G
1	x	y					
2	1	4					
3	9	14					
4							

5							
6							
7							
8							
9							
10							
11							
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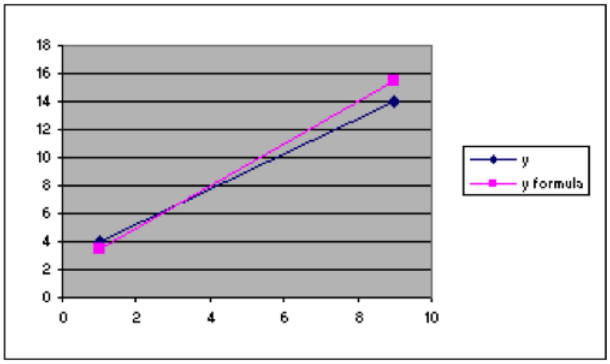
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16.6: G1.06- Exploring by Graphing Part II

Example 16. Continue with the previous spreadsheet to explore to find an equation of the line through the points (1,9) and (4,14).

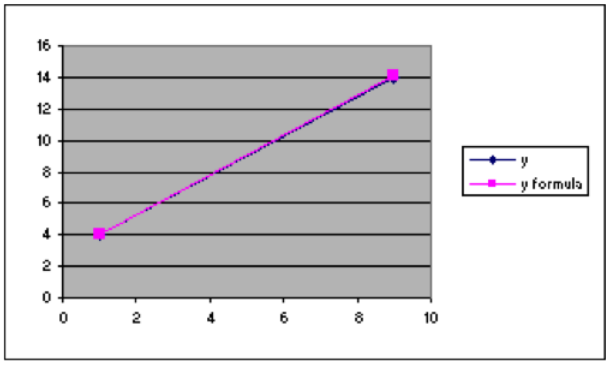
1. Starting with the previous spreadsheet, look at the points and guess what the intercept and slope might be. I guess that the intercept might be 2 and the slope might be 1.5. So put those guesses in Column G. In G2 put 2 and in H2 put the label "intercept." In G3 put 1.5 and in H3 put the label "intercept."
2. In column C, put the label "y formula" in cell C1. Then, in cell C2, put the formula for a line, , but put it in the appropriate form to use the values for the slope and intercept in column G and the *x* values in column A. That is, in cell C2 put =$G$3*A2+$G$2.



	A	B	C	D	E	F	G	H
1	x	y	y formula					
2	1	4	3.5				2	intercept
3	9	14	15.5				1.5	slope
4								
5								
6								
7								
8								
9								
10								
11								
12								
13								
14								

3. Notice that the line for the formula is close, but not correct. So adjust the values for the slope and intercept until it looks right. First, it looks as if the intercept is actually a little higher than the formula indicates. So try 3 for that. And then it looks as if the slope is a little lower than the formula indicates. So try 1.4 for that. That looks better, but still not correct.
4. The intercept of 3 is still too large. Try 2.8. The slope of 1.4 is still too large. Try 1.3. This looks better, but it could still be improved upon.
5. Leave the intercept at 2.8 and try a slope of 1.25. That looks pretty good on the graph. The numbers in the formula column show that it is not quite correct, but this is about as good as we'd expect to get from just fitting by eye. We can use algebra to

find more precise values.



	A	B	C	D	E	F	G	H
1	x	y	y formula					
2	1	4	4.05				2.8	intercept
3	9	14	14.05				1.25	slope
4								
5								
6								
7								
8								
9								
10								
11								
12								
13								
14								
15								

Using algebra here, as illustrated in Examples 5, 6, 11, and 12, we find that the equation of the line through the points (1,4) and (9,14) is $y=1.25x+2.75$. The formula we have found just by looking at the graph and approximating is very close to the correct formula.

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16.7: G1.07- Section 4

OptionalSection 4: Adding formatting to the spreadsheet used to explore parameter values.

When you are just creating a spreadsheet to use for a few minutes, it is easy to remember what you put where and what it meant. But it is also convenient to save some worksheets and use them again later. When you do that, it is good to add some features that make it easier to remember what is where. In this course, we will add several features to such spreadsheets.

1. We will make the cells in which we change numerical values green, so that it is easy to see at a glance what to change.
2. We will make the cells that are computed from those values we change yellow, so that it is easy to look at what numbers changed as a result of what we entered.
3. We will add a line that shows the formula with the numerical values we are using.

Because the colors don't show in these printed materials, the example printed here in not completely satisfactory.

Methods

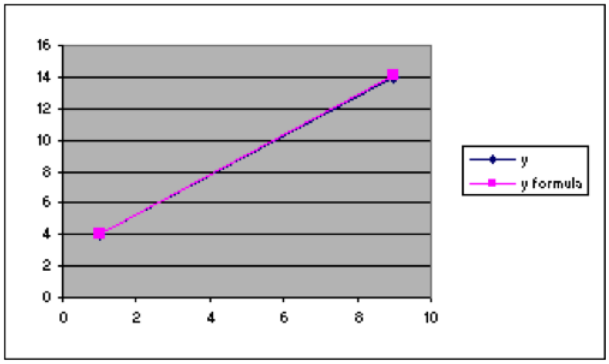
Color:

To make cells a given color, choose the cells (highlight them) and, from the menu, choose Format > Cells and then choose the "Patterns" tab. From there, choose the color you want. In this course, we will use green for input and yellow for computed values. (We will also use blue for cells where there is a "check." That has not yet been illustrated in the examples.)

Formula:

To show the formula in the spreadsheet for Example 16 above in cell G1, use Excel's CONCATENATE function in the following way.

In G1 enter =CONCATENATE("y=",G3," x + ",G2)



	A	B	C	D	E	F	G	H
1	x	y	y formula				y=2.8 x + 1.25	
2	1	4	4.05				2.8	intercept
3	9	14	14.05				1.25	slope
4								
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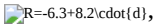
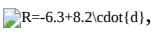
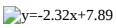
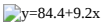
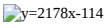
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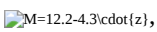
16.8: G1.08- Exercises

Part I

1. For this formula:  , tell
   a. Which variable is the output variable?
   b. Which variable is the input variable?
   c. Is it linear relationship?
   d. If it is a linear relationship, what is the slope?
   e. If it is a linear relationship, what is the *y*-intercept?
2. For this formula:  , find a three points that fit this and use them to sketch a graph.
3. A manager is considering the cost *C* of printing a book based on the number of pages *p*. He is told that the formula for predicting the cost is linear based on the number of pages and that the *y*-intercept is $4.50 and the slope is $0.027. Find the formula to predict the cost from the number of pages.
4. Find the formula for the line with slope 1.35 which has the point (5,40) on it.
5. Find the formula for the line through (2,6) and (4,11).
   Identify the slope and y-intercept.
6. We have been told that the amount of oatmeal needed for oatmeal cookies is linearly related to the amount of flour needed. Also, we know that if we use 3 cups of flour, we need 2 cups of oatmeal. And, of course, if we use 0 cups of flour, we will use 0 cups of oatmeal.
   a. Find the formula to predict the oatmeal needed (called M) from the flour needed (F.)
   b. Interpret the slope.
   c. Interpret the y-intercept.
7. Find the slope and intercept of the line 
8. Find the slope and intercept of the line 
9. Find the slope and intercept of the line 
10. Find the slope and the intercept of the line 
11. Find the formula for the line with slope -6.2 through the point (87.2, 112.7)
12. Find the formula for the line through (8,5) and (13,17)
13. Find the formula for the line through (83.8, 79.9) and (232.7, 63.4)
14. Use a spreadsheet to approximate the equation of the line through (1,4) and (9,14).

Part II

Write your formulas for lines in point-slope form. You may make the requested graphs by hand on graph paper or using a spreadsheet.

15. For this formula:  , tell
    a. Which variable is the output variable?
    b. Which variable is the input variable?
    c. Is it linear relationship?
    d. If it is a linear relationship, what is the slope?
    e. If it is a linear relationship, what is the *y*-intercept?
16. A manager has proposed that the rent on apartments in one location will be linear based on the number of square feet in the apartment. The *y*-intercept will be $160 and the slope will be $1.50. Find the formula to predict the rent from the number of square feet in the apartment.
17. Janet's recipe for split-pea soup has a linear relationship between the amount of carrots and the amount of onions. When she uses 1.5 cups of carrots, she uses 1 cup of onions. When she uses 3 cups of carrots, she uses 2 cups of onions.
    a. Find the formula to predict the onions needed (called *N*) from the carrots needed (*C*.)
    b. Interpret the slope.
    c. Interpret the *y*-intercept.

18. Find the formula for the line through the point (2,8) with slope 2. Check this by looking at the formula to see that the slope coefficient is 2 and then also plugging in the point (2,8) to confirm that it really does fit the equation.
19. Find the formula for the line through the point (13.3, 117.8) with slope 2.7. Check this by looking at the formula to see that the slope coefficient is 2.7 and then also plugging in the point (13.3, 117.8) to confirm that it really does fit the equation.
20. Find the formula for line through the points (4,12) and (18,37). Check this by plugging in the point that you DID NOT use in the last step to find the intercept, and be sure that it fits the equation.
21. Find a formula for the line through the points (82.7, 227.2) and (105.3, 304.2). Check this by plugging in the point that you DID NOT use in the last step to find the intercept, and be sure that it fits the equation.
22. Find a formula for the line through the points (4,28) and (9,13). Check this by plugging in the point that you DID NOT use in the last step to find the intercept, and be sure that it fits the equation.
23. Find a formula for the line through the points (58.2,102.7) and (75.8, 47.5). Check this by plugging in the point that you DID NOT use in the last step to find the intercept, and be sure that it fits the equation.
24. Use a spreadsheet to approximate the formula for the line through the points (4,12) and (18,37). Use this to check your work in the previous problem (number 20) where you found the formula algebraically.
25. Use a spreadsheet to approximate the formula for the line through the points (82.7, 227.2) and (105.3, 304.2). Use this to check your work in the previous problem (problem 21) where you found the formula algebraically.
26. Use a spreadsheet to approximate the formula for the line through the points (4,28) and (9,13). Use this to check your work in the previous problem (problem 22) where you found the formula algebraically.
27. Use a spreadsheet to approximate the formula for the line through the points (58.2,102.7) and (75.8,47.5). Use this to check your work in the previous problem (problem 23) where you found the formula algebraically.
28. Find the formula for the line between (12,4) and (37,18). Do this by either of the two methods in Section 3. Compare the slopes of this line and the line you found in the problem 20, where you had the input and output variable switched. Are they reciprocals? (Hint: multiply them together and see if you get 1. If so, they are reciprocals.)
29. Find a formula for the line through the points (28,4) and (13,9). Do this by either of the two methods in Section 3. Compare the slopes of this line and the line you found in problem 22, where you had the input and output variable switched. Are they reciprocals? (Hint: multiply them together and see if you get 1. If so, they are reciprocals.)

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CHAPTER OVERVIEW

17: Topic K- Finance

17.1: K1.01- Simple and Compound Interest
17.2: K1.02- Savings Plans
17.3: K1.03- Installment Loans
17.4: K1.04- Exercises

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17.1: K1.01- Simple and Compound Interest

Finance: Simple and Compound Interest

Questions Explored:

- What is the difference between simple and compound interest?
- I want to deposit a lump sum of money into an account earning interest and leave it there. What will I have in the future?
- What lump sum should I deposit into an account earning interest in order to have a certain future value?
- What happens to my credit card balance if I don't pay any of it for a period of time?

Terms:

- **Principal** – initial amount of the investment or loan
- **Interest** – amount paid to you by the lender for investing with them, or, the fee you pay for borrowing money
- **Simple interest** – interest paid only on the original principal
- **Compound interest** – interest paid on both the original principal and any interest that has been added to the original principal
- **Annual percentage rate (APR)** – % of interest earned or owed each year; may need to be divided up for smaller time periods (i.e. monthly, quarterly, etc.), APR does not take compounding into account
- **Compounding period** – period at the end of which interest is computed
  - o Annually = once a year
  - o Semiannually = twice a year
  - o Quarterly = four times a year
  - o Monthly = twelve times a year
  - o Daily = 365 times a year (was 360 in the past before computers were readily available to make math easier for the banker)
- **Savings accounts** – accounts into which you deposit money
  - o Currently savings accounts have a very low interest rate
  - o Standard savings accounts, money market accounts, and CDs (certificate of deposit) are a few different types of savings accounts with different interest rates and withdrawal restrictions.
- **FDIC** – (Federal Deposit Insurance Corporation) guarantees safety of bank deposits currently up to $250,000 per depositor per bank (as of 2016)
- **Bonds** – when you purchase a bond, the bond issuer is in debt to the bond holder and pays the bond holder interest on the bond and/or repays the principal later to the holder
  - o The bond holder is the lender and has loaned the bond issuer money, who is now the debtor.
  - o There are many types of bonds: Treasury bonds, corporate bonds, municipal bonds, etc.
  - o Bonds may have a fixed or variable interest rate.
  - o Interest may be simple or compound.
  - o Bonds may or may not be inflation-linked.
  - o Bonds may or may not have tax advantages
  - o Bonds may be low or high risk
- **Credit card** – system of payment that allows someone to purchase goods or services with the promise that the money will be repaid
- **Annual percentage yield** (APY or effective annual yield or effective yield or yield) – actual percentage by which a balance increases in one year, slightly different than the APR since it takes compounding into account

Rules

Simple interest: The amount of interest earned is the same percentage of the original principal every year.

Compound interest: The amount of interest earned in each time period is computed on the accumulated amount of money in the account at the beginning of that time period.

Annual Percentage Yield: The annual percentage yield of an investment is computed by finding the relative change from the initial balance to the balance at the end of the same year.

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17.2: K1.02- Savings Plans

Finance: Savings and Payout Annuities

Questions Explored

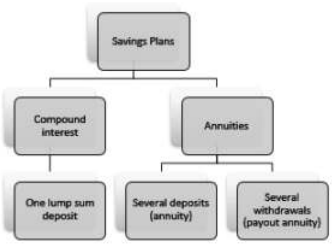
- I would like to make regular deposits into an account at regular intervals for a number of years. How much will I have in the end?
- How much do I need to deposit at regular intervals into an account in order to have a certain amount of money?
- I'd like to be able to withdraw a certain amount every year indefinitely after I retire. How much would I need to deposit at regular intervals in order to do that?

Terms

- **Savings annuity**—an account into which regularly scheduled deposits are made
- **Payout annuity**—an account with money in it where money is withdrawn regularly, eventually ending up empty

- **401(k)**—one example of an employer-sponsored savings annuity
- **IRA (individual retirement account)** – you can open one of these and make regular deposits into the account for years
  - o There are several types of IRAs. Two examples are below. You'd talk to a financial planner about what type would be best for you based on your expected future tax bracket and the type of lifestyle you'll lead now and in the future.
    - Traditional IRA – deposit now, pay taxes later
    - Roth IRA – pay taxes on deposits now, withdraw tax-free

For an annuity: Like compound interest, the amount in an annuity in each time period is computed on the amount of money in the annuity at the beginning of that time period. The amount present at the beginning of each period must be modified to reflect deposits or withdrawals since the previous time period began.



Different Types of Annuities

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17.3: K1.03- Installment Loans

Finance: Savings and Payout Annuities

Questions Explored

- I would like to make regular deposits into an account at regular intervals for a number of years. How much will I have in the end?
- How much do I need to deposit at regular intervals into an account in order to have a certain amount of money?
- I'd like to be able to withdraw a certain amount every year indefinitely after I retire. How much would I need to deposit at regular intervals in order to do that?

Terms

- **Savings annuity**—an account into which regularly scheduled deposits are made
- **Payout annuity**—an account with money in it where money is withdrawn regularly, eventually ending up empty

- **401(k)**—one example of an employer-sponsored savings annuity
- **IRA (individual retirement account)** – you can open one of these and make regular deposits into the account for years
    - o There are several types of IRAs. Two examples are below. You'd talk to a financial planner about what type would be best for you based on your expected future tax bracket and the type of lifestyle you'll lead now and in the future.
        - Traditional IRA – deposit now, pay taxes later
        - Roth IRA – pay taxes on deposits now, withdraw tax-free

For an annuity: Like compound interest, the amount in an annuity in each time period is computed on the amount of money in the annuity at the beginning of that time period. The amount present at the beginning of each period must be modified to reflect deposits or withdrawals since the previous time period began.

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17.4: K1.04- Exercises

Exercises

S kills

1. A friend lends you $200 for a week, which you agree to repay with 5% one-time interest. How much will you have to repay?

2. Suppose you obtain a $3,000 T-note with a 3% annual rate, paid quarterly, with maturity in 5 years. How much interest will you earn?

3. A T-bill is a type of bond that is sold at a discount over the face value. For example, suppose you buy a 13-week T-bill with a face value of $10,000 for $9,800. This means that in 13 weeks, the government will give you the face value, earning you $200. What annual interest rate have you earned?

4. Suppose you are looking to buy a $5000 face value 26-week T-bill. If you want to earn at least 1% annual interest, what is the most you should pay for the T-bill?

5. You deposit $300 in an account earning 5% interest compounded annually. How much will you have in the account in 10 years?

6. How much will $1000 deposited in an account earning 7% interest compounded annually be worth in 20 years?

7. You deposit $2000 in an account earning 3% interest compounded monthly.

a. How much will you have in the account in 20 years?

b. How much interest will you earn?

8. You deposit $10,000 in an account earning 4% interest compounded monthly.

a. How much will you have in the account in 25 years?

b. How much interest will you earn?

9. How much would you need to deposit in an account now in order to have $6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.

10. How much would you need to deposit in an account now in order to have $20,000 in the account in 4 years? Assume the account earns 5% interest.

11. You deposit $200 each month into an account earning 3% interest compounded monthly.

a. How much will you have in the account in 30 years?

b. How much total money will you put into the account?

c. How much total interest will you earn?

12. You deposit $1000 each year into an account earning 8% compounded annually.

a. How much will you have in the account in 10 years?

b. How much total money will you put into the account?

c. How much total interest will you earn?

13. Jose has determined he needs to have $800,000 for retirement in 30 years. His account earns 6% interest.

a. How much would you need to deposit in the account each month?

b. How much total money will you put into the account?

c. How much total interest will you earn?

14. You wish to have $3000 in 2 years to buy a fancy new stereo system. How much should you deposit each quarter into an account paying 8% compounded quarterly?

15. You want to be able to withdraw $30,000 each year for 25 years. Your account earns 8% interest.

a. How much do you need in your account at the beginning

b. How much total money will you pull out of the account?

c. How much of that money is interest?

16. How much money will I need to have at retirement so I can withdraw $60,000 a year for 20 years from an account earning 8% compounded annually?

a. How much do you need in your account at the beginning

b. How much total money will you pull out of the account?

c. How much of that money is interest?

17. You have $500,000 saved for retirement. Your account earns 6% interest. How much will you be able to pull out each month, if you want to be able to take withdrawals for 20 years?

18. Loren already knows that he will have $500,000 when he retires. If he sets up a payout annuity for 30 years in an account paying 10% interest, how much could the annuity provide each month?

19. You can afford a $700 per month mortgage payment. You've found a 30 year loan at 5% interest.

a. How big of a loan can you afford?

b. How much total money will you pay the loan company?

c. How much of that money is interest?

20. Marie can afford a $250 per month car payment. She's found a 5 year loan at 7% interest.

a. How expensive of a car can she afford?

b. How much total money will she pay the loan company?

c. How much of that money is interest?

21. You want to buy a $25,000 car. The company is offering a 2% interest rate for 48 months (4 years). What will your monthly payments be?

22. You decide finance a $12,000 car at 3% compounded monthly for 4 years. What will your monthly payments be? How much interest will you pay over the life of the loan?

23. You want to buy a $200,000 home. You plan to pay 10% as a down payment, and take out a 30 year loan for the rest.

a. How much is the loan amount going to be?

b. What will your monthly payments be if the interest rate is 5%?

c. What will your monthly payments be if the interest rate is 6%?

24. Lynn bought a $300,000 house, paying 10% down, and financing the rest at 6% interest for 30 years.

a. Find her monthly payments.

b. How much interest will she pay over the life of the loan?

25. Emile bought a car for $24,000 three years ago. The loan had a 5 year term at 3% interest rate. How much does he still owe on the car?

26. A friend bought a house 15 years ago, taking out a $120,000 mortgage at 6% for 30 years. How much does she still owe on the mortgage?

27. Pat deposits $6,000 into an account earning 4% compounded monthly. How long will it take the account to grow to $10,000?

28. Kay is saving $200 a month into an account earning 5% interest. How long will it take her to save $20,000?

29. James has $3,000 in credit card debt, which charges 14% interest. How long will it take to pay off the card if he makes the minimum payment of $60 a month?

30. Chris has saved $200,000 for retirement, and it is in an account earning 6% interest. If she withdraws $3,000 a month, how long will the money last?

Concepts

31. Suppose you invest $50 a month for 5 years into an account earning 8% compounded monthly. After 5 years, you leave the money, without making additional deposits, in the account for another 25 years. How much will you have in the end?

32. Suppose you put off making investments for the first 5 years, and instead made deposits of $50 a month for 25 years into an account earning 8% compounded monthly. How much will you have in the end?

33. Mike plans to make contributions to his retirement account for 15 years. After the last contribution, he will start withdrawing $10,000 a quarter for 10 years. Assuming Mike's account earns 8% compounded quarterly, how large must his quarterly contributions be during the first 15 years, in order to accomplish his goal?

34. Kendra wants to be able to make withdrawals of $60,000 a year for 30 years after retiring in 35 years. How much will she have to save each year up until retirement if her account earns 7% interest?

35. You have $2,000 to invest, and want it to grow to $3,000 in two years. What interest rate would you need to find to make this possible?

36. You have $5,000 to invest, and want it to grow to $20,000 in ten years. What interest rate would you need to find to make this possible?

37. You plan to save $600 a month for the next 30 years for retirement. What interest rate would you need to have $1,000,000 at retirement?

38. You really want to buy a used car for $11,000, but can only afford $200 a month. What interest rate would you need to find to be able to afford the car, assuming the loan is for 60 months?

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CHAPTER OVERVIEW

18: Topic L- Automated Fitting of Models, Comparative Goodness of Fit, and Outliers

18.1: L1.02- Section 1
18.2: L1.03- Section 2
18.3: L1.04- Section 3
18.4: L1.05- Section 4
18.5: L1.06- Section 5
18.6: L1.07- Section 6
18.7: L1.08- Section 7
18.8: L1.09- Section 8
18.9: L1.10- Exercises
18.10: L1.01- Overview

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18.1: L1.02- Section 1

Section 1: Choosing a numerical indicator of how well a model fits a dataset

In an earlier topic we computed a numerical indicator of the noise of a measurement process, the standard deviation around the average measurement. If we had a similar indicator of the scatter of the data around a model, we could use it to guide our search for the best set of model parameters. Even better, we might be able to get a computer program to help in the search since computers are good at working with numbers. Our original method of judging a model, visually comparing the data and model graphs, is easy for people but beyond the current capabilities of computers.

While scatter indicators of this kind as often referred to as measuring “goodness of fit”, keep in mind that they are showing the amount of deviation, so smaller is better (these are actually badness-of-fit indicators). We will find the best model by adjusting the parameters to make the indicator as close to zero as possible.

We want to minimize both positive and negative deviations, since here we care about how close the data is to the model, but not whether it is higher or lower. In this situation, the most productive way to treat positive and negative deviations the same is to *square* each deviation (i.e., multiply it by itself, a process that produces positive results for both positive and negative numbers). We will then minimize some indicator based on these squared deviations.

In spreadsheets that arranged similarly to Models.xls (input data in column A, output data in column B, model prediction in column C, and data-model deviation in column D), it is natural to put these squared deviations in column E. This can be done by putting the formula “=D3^2” into cell E3, then spreading that formula down all the data rows.

Example 1: *Add a squared-deviations row to the US-population spreadsheet made in an earlier topic*

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1	*x*	*Y*	*y*	Residual	***Squared***	*Exponential model: y = a * (1+r)^x*			
2	Year-1780	Population	Model	values	***Deviations***		y = 2.8 * (1.03^x)		
3	0	2.8	2.80	0.0000	***0.000000***		**2.8**	**a:** Initial value at x=0	
4	10	3.9	3.76	0.1370	***0.018778***		**0.03**	**r:** Growth rate	
5	20	5.3	5.06	0.2429	***0.058995***				
6	30	7.2	6.80	0.4037	***0.162945***				
7	40	9.6	9.13	0.4663	***0.217430***				
8	50	12.9	12.27	0.6251	***0.390704***				
9	60	17.1	16.50	0.6035	***0.364226***				
10	70	23.2	22.17	1.0301	***1.061103***				
11	80	31.4	29.79	1.6055	***2.577651***				
12	90	39.8	40.04	-0.2413	***0.058230***				

Column E now contains the formulas

“=D3^2” in E3

“=D4^2” in E4

“=D5^2” in E5

et cetera, for each data row

What number will be used to represent the squared-deviation values?

If we are going to make decisions about parameter settings by looking at a numerical goodness-of-fit indicator, that indicator must be a single number so we can decide which settings are better by choosing those that give the lowest indicator value. What indicator should be used to summarize this column of squared deviations? One obvious possibility, which turns out to be the most generally useful, is the *sum* of the squared deviations. This can be computed by applying the spreadsheet SUM function to column E, as by putting the formula "=SUM(E3:E12)" into some empty cell such as H8.

An alternative indicator that is useful in a few situations is simply the largest squared deviation (which comes from the largest deviation, whether positive or negative). This can be computed with the spreadsheet MAX function, using the formula "=MAX(E3:E12)" in this case. Other indicators are sometimes constructed that take into account the economic cost in a particular situation of different deviation sizes and/or directions. But the sum of the squared deviations is by far the most commonly used indicator for fitting models to data, and we will use it in this course except where another indicator is specifically asked for.

Example 2: Improve the earlier population model by using the sum-of-squared-deviations indicator.

The exponential-model parameter settings $a = 2.8$ and $r = 0.03$ that were found earlier fit the data pretty well, but of course other nearby settings of the parameters, such as would change the graph so little we could not tell by looking whether one of these models is slightly better than the other. But the numerical sum-of-squared-deviations calculation in H8 provides a way to choose among nearby model settings to get the very best fit.

[a] Here is the worksheet result from the initial model, which was chosen by setting the first parameter of the model to 2.8 (the data value at $x = 0$), then adjusting the growth rate until the graphs were close.

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1	*x*	*y*	*y*	Residual	Squared	*Exponential model: y = a * (1+r)^x*			
2	Year-1780	Population	Model	values	deviations		y = 2.8 * (1.03^x)		
3	0	2.8	2.80	0.0000	0.000000		**2.8**	a: Initial value at x=0	
4	10	3.9	3.76	0.1370	0.018778		**0.03**	r: Growth rate	
5	20	5.3	5.06	0.2429	0.058995				
6	30	7.2	6.80	0.4037	0.162945				
7	40	9.6	9.13	0.4663	0.217430		***Goodness of fit for these settings***		
8	50	12.9	12.27	0.6251	0.390704		Sum of sq.	**4.910062**	
9	60	17.1	16.50	0.6035	0.364226				
10	70	23.2	22.17	1.0301	1.061103				
11	80	31.4	29.79	1.6055	2.577651				
12	90	39.8	40.04	-0.2413	0.058230				

So for a model of this data, the sum of squared deviations equals about 4.91.

[b] If the initial value for the model is increased by 0.1 million (to 2.9 million), the sum of squared deviations drops from about 4.91 to 3.24, a noticeable improvement. So the model can be considered slightly better.

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1	*x*	*y*	*y*	Residual	Squared	*Exponential model: y = a * (1+r)^x*			
2	Year-1780	Population	Model	values	deviations		y = 2.9 * (1.03^x)		
3	0	2.8	2.90	-0.10	0.0100		**2.9**	**a:** Initial value at x=0	

4	10	3.9	3.90	0.00	0.0000		**0.03**	**r:** Growth rate
5	20	5.3	5.24	0.06	0.0039			
6	30	7.2	7.04	0.16	0.0259			
7	40	9.6	9.46	0.14	0.0196		***Goodness of fit for these settings***	
8	50	12.9	12.71	0.19	0.0348		Sum of sq.	**3.237818**
9	60	17.1	17.09	0.01	0.0002			
10	70	23.2	22.96	0.24	0.0568			
11	80	31.4	30.86	0.54	0.2931			
12	90	39.8	41.47	-1.67	2.7934			

Further small adjustments to G3 might reduce the standard deviation further, but it would be very tedious to try all the settings needed to find the very best fit, especially since we would need to explore values for the growth rate in G4 as well. Additionally, since a change in one of the parameters will change the effect of the other parameter, several rounds of adjustment would be needed. This process would be even more difficult for models that have three or more parameters.

But tedious tasks are what computers do best. Most spreadsheets have a built-in routine that will systematically adjust parameters to a result you ask for (in this case, the minimum value for the goodness-of-fit indicator). We will use such a built-in routine, Solver, as a way to automate the model-fitting process.

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18.2: L1.03- Section 2

Section 2: Using Solver in the Tool menu to find the best-fit parameters for a model

The “Solver” add-in can be used with an appropriately-formatted spreadsheet to tell the computer to do automatically what you did by hand in the earlier model-fitting exercises – adjust the parameters until the model is a good as it can be. Solver uses the numerical value of the sum-of-squared-deviations indicator (computed in cell H8 in the example spreadsheet) to decide on the best parameters.

When you use Solver for modeling, you will set up a modeling worksheet exactly as the previous example, then use Solver to tell the computer to minimize the cell containing the goodness-of-fit indicator by adjusting the cells containing the parameters. The software then systematically searches for the parameter settings that result in the smallest standard deviation.

The Solver routine does not know anything about modeling — it simply changes the numbers in the cells you identify (G3 and G4 in this case), searching for values that produce the result you ask for (minimization) in the target cell that you identify (H8). This procedure finds the best model because of the way the worksheet is set up, with the sequence of connections where the parameters in column G are used in the model formulas in column C, which determine the data-model deviations in column D and thus the squared deviations in column E, which are added to form the sum of squared deviations indicator.

Since Solver will default to the last values used, once you start using it you will often find that the settings are already correct, so that you can just press “Solve” to get the best-fit model parameters for the current set of data. If the computer does not find a solution when fitting a model, this usually means that you forgot to set the “Min” option, so that the program is instead trying to find a maximum, an impossible task in this case because there is no limit to how far away the model can get from the data.

Usually Solver will find a correct solution regardless of the initial values of the parameters. However, it will be more dependable if you start with the parameters having values that put the model graph in the same general region as the data points. Otherwise the formula may have a value so large or small that the computer cannot handle it correctly. In any case, it is important to look at the graph of the fitted values to make sure Solver worked correctly—after the computer does the tedious part, it’s your turn to do the thinking that is needed.

Example 3: Use Solver to find the formula of the best-fit exponential model for the worksheet modified in Example 1 to include a sum-of-square deviations value.

1. Select the cell containing the sum-of-squared-deviations value (cell H8).
2. Select Solver from the Tools menu (if it is not there, select Add-Ins and add it). The dialog box should show the selected cell (e.g., “H8”) in the Set Target Cell section at the top. If not, enter it.
3. Choose the “Min” (for Minimum) option in the second line, since we want Solver to find the smallest possible value for the indicator in H8.
4. Enter “G3,G4” into the “By Changing Cells” section on the fourth line. This tells Solver to adjust the cells that we are using as model parameters when it is trying to minimize H8.
5. Press the “Solve” button at top right.
6. *Check the graph* (or the numbers in columns B and C) to see if the model now is close to the data, with some on each side. If it is not, you have made a mistake and must repeat the process correctly.
7. If the graph is acceptable, press “OK” to accept the solution that is found. The parameters (and thus the “Model” values in the column C) are now set to the best-fit values to a very high accuracy.

[reveal-answer q=”126306″]Show Answer[/reveal-answer]
[hidden-answer a=”126306″]

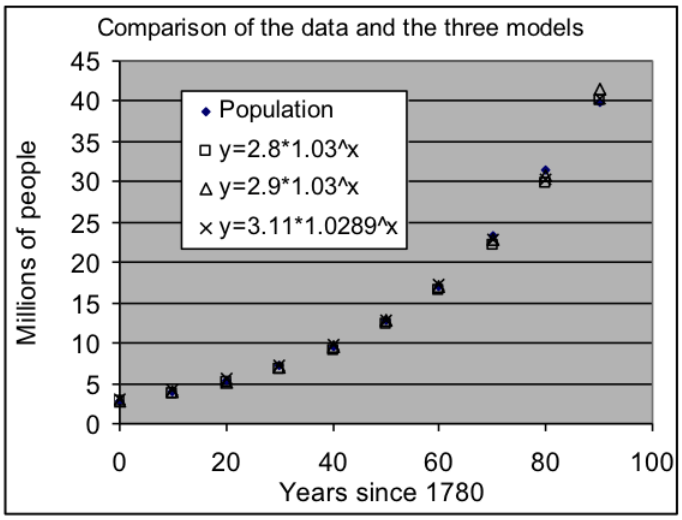
The best-fit exponential model formula for this data is  (with the parameters rounded off to 3 digits to match the data). The sum of squared deviations for this model is about 1.72, which is only 53% of the 3.24 value found by hand in the previous example, and only 35% of the value for the parameter setting we originally found by visual inspection.

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1	*x*	*data y*	*model y*	Data-Model	Squared	*Exponential model: y = a * (1+r)^x*			
2	Year-1780	Population	Prediction	deviation	deviation		y = 3.108069 * (1.028937^x)		

3	0	2.8	3.108101	-0.308101	0.094926		**3.108069**	***a:*** Initial value at x=0
4	10	3.9	4.134104	-0.234104	0.054804		**0.028937**	***r:*** Growth rate
5	20	5.3	5.498797	-0.198797	0.039520			
6	30	7.2	7.313983	-0.113983	0.012992			
7	40	9.6	9.728374	-0.128374	0.016480		***Goodness of fit for these settings***	
8	50	12.9	12.93977	-0.039770	0.001582	Sum of sq. dev.	**1.721981**	
9	60	17.1	17.21127	-0.111268	0.012381			
10	70	23.2	22.89281	0.307187	0.094364			
11	80	31.4	30.44987	0.950128	0.902744			
12	90	39.8	40.50156	-0.701561	0.492188			

Note that the best-fit model has a starting value that is a bit higher than in the earlier models (3.11 instead of 2.8 or 2.9), but this is balanced by the growth rate being slightly lower (0.0289 instead of 0.03).

Also note that the graphs of all three models are quite close to the data, so that any could be considered a good model, and it would not be feasible to choose between them visually.



Example 4: Use Solver to find the best-fit linear model for the sediment depth data from the topic on linear modeling.

Solution approach:

[a] Use a spreadsheet as before, but now add the squared deviations into column E and compute their sum with a formula in G11, then use Solver to find the best-fit parameter settings.

[b] Write the linear equation with those parameters used as intercept and slope values.

[reveal-answer q="715980"]Show Answer[/reveal-answer]
[hidden-answer a="715980"]

The best-fit linear model to the sediment data is y=1.759x+12.657**.**

	A	*B*	*C*	*D*	*E*	*F*	*G*
1	x	y data	y model	Residual	Squared		*Linear model: y = m * x + b*

2	Days	Depth	Prediction	deviation	deviation		*y = 1.759286 x + 12.65714*	
3	10	29.9	30.25	-0.35	0.1225		**12.65714**	***m:* Intercept**
4	20	48.0	47.84	0.16	0.0247		**1.759286**	***b:* Slope**
5	30	60.5	65.44	-4.94	24.3613			
6	40	88.6	83.03	5.57	31.0408			
7	50	102.9	100.62	2.28	5.1919		***Goodness of fit for these settings***	
8	60	114.1	118.21	-4.11	16.9273		Sum of sq.	**120.8929**
9	70	141.1	135.81	5.29	28.0143			
10	80	149.5	153.40	-3.90	15.2100			

[/hidden-answer]

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18.3: L1.04- Section 3

Section 3: Using the sum of squared deviations to compute the *standard deviation* for a model

While the sum-of-squared-deviations indicator works very well as the target for Solver to minimize, it does not directly provide information about the typical deviation size because several terms are added together, and each term in the sum is squared. It would be useful to have a standardized goodness-of-fit indicator that did not depend on the number of deviations and was expressed in the same units as the data, model, and deviations. This would also be valid for comparing the quality of models with different numbers of parameters that are fit to the same dataset.

We can compute such a standardized indicator by computing an average squared deviation, then taking the square root of that value. The only difference from a regular average is that the divisor is made smaller by the number of parameters (i.e., the divisor is $n-2$ instead of n for a linear or exponential model, where n is the number of squared deviations in the sum).

The number computed in this way is the *standard deviation of the data from the model*. This is the same kind of indicator as was computed to measure the noise in repeated measurements — in that case the STDEV function adjusted the divisor to be $n-1$ because the deviation is from a single-parameter model, the average. For more complex models we have to make an explicit adjustment rather than using the STDEV function. Since we are already computing the sum of squared deviations, this is not difficult.

For the 8-value sediment data, for example, this computation could be accomplished by the formula "=SQRT(H8/6)", since H8 already has been set to the sum of the 8 squared deviations from E3 to E10. For the 10-value population data, the formula to use would be "=SQRT(H8/8)". For the earlier 7-value quadratic basketball-path data, the formula would be "=SQRT(H8/4)", because a quadratic model has three parameters.

The adjustment to the divisor for the average is needed because if there are only two data points for a linear model, for example, then the model can go through them exactly even when there would be scatter if more points from the same measurement process were added. It is only when there are more points than the number of parameters that you know anything about what scatter can be expected around the model predictions.

Example 5: Modify the worksheets from Examples 3 and 4 so they also compute standard deviation.

Solution approaches (any of these will give correct answers):

[A] **Dataset-specific-formulas** — Place the formulas "=SQRT(H8/8)" or "=SQRT(H8/6)", respectively for Examples 3 and 4, into an empty cell in the worksheet, such as H9. The disadvantage of this approach is that you are likely to forget to change the formula (which is not visible) when the worksheet is re-used with a different number of data points.

[B] **Visible count totals** — Set H10 to the number of squared deviations summed in forming H8 (this will be the number of data points unless you have excluded some as outliers). Then set H11 to the number of model parameters (e.g., 2 for linear & exponential, 3 for quadratic). Finally, set H12 to the formula "=SQRT(H8/(H10-H11))". If you re-use this spreadsheet with a different dataset, all you have to change is the deviation count in H10 to the appropriate setting. The formula in H12 will not need to be changed.

[C] **Automatic count totals** — Put the formula "=COUNT(E3:E99)" into cell H10. This will evaluate to the number of cells in column E that have numeric values, indicating the presence of a deviation. Then follow approach B (set H11 to the number of parameters, and set H12 to the formula "=SQRT(H8/(H11-H12))". *(Warning: approach C will cause errors if there are any extraneous numbers in column E.)*

Whichever approach is used, the respective standard deviation values, rounded to the usual two significant digits, are **σ = 0.46 million people** for Example 3 and **σ = 4.49 mm** for Example 4.

Here is what the census spreadsheet would look like after using approach B or C:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1	*x*	*data y*	*model y*	Data-Model	Squared	*Exponential model: y = a * (1+r)^x*			
2	Year-1780	Population	Prediction	deviation	deviation		y = 3.108069 * (1.028937^x)		

3	0	2.8	3.108101	-0.308101	0.094926		**3.108069**	***a:*** Initial value at x=0
4	10	3.9	4.134104	-0.234104	0.054804		**0.028937**	***r:*** Growth rate
5	20	5.3	5.498797	-0.198797	0.039520			
6	30	7.2	7.313983	-0.113983	0.012992			
7	40	9.6	9.728374	-0.128374	0.016480		***Goodness of fit for these settings***	
8	50	12.9	12.93977	-0.039770	0.001582	Sum of sq. dev.	**1.721981**	
9	60	17.1	17.21127	-0.111268	0.012381			
10	70	23.2	22.89281	0.307187	0.094364	# of dev. summed	10	
11	80	31.4	30.44987	0.950128	0.902744	# of parameters	2	
12	90	39.8	40.50156	-0.701561	0.492188	standard deviation	**0.463948**	

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18.4: L1.05- Section 4

Section 4: Using Solver to choose the best kind of model

Sometimes data has a pattern for which more than one kind of model is a potential match. While visual inspection of the best-fit graphs will usually show which one is best, you can also choose between candidates by seeing which model type has the smallest best-fit standard deviation.

Example 6: Does a quadratic model fit the population data better than the model in Example 3? Why?

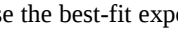
Solution approach:

[a] Use Solver with the Quadratic Model worksheet in Models.xls to fit the same data.

[b] Add the computation of standard deviation to the worksheet.

[c] Compare the standard deviation σ for the quadratic model with that of the exponential model.

[reveal-answer q="249938″]Show Answer[/reveal-answer]
[hidden-answer a="249938″]

An exponential model is a better fit to this data than a quadratic model, because the best-fit exponential model 

has **σ = 0.46 million**, while the best-fit quadratic model 

has **σ = 0.85 million**, which is almost twice as large.

[/hidden-answer]

Can you think of another reason that a quadratic model is not likely to be appropriate for this data? (Hints: Where is the vertex of the parabola? What does this imply about the quadratic model's prediction for years before 1780, the first year given in this dataset?)

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18.5: L1.06- Section 5

Section 5: Identifying and Removing Outliers in Data

Some datasets include a few points that do not fit into the overall pattern. These points are called “outliers”, and require special handling if a model is to be fit to the dataset. In general, the approach to modeling data with outliers is to exclude them from the fitting process but report them separately along with the model formula.

Sometimes outliers are simply mistakes in the data. Measurement is not a perfect process, and errors in a measurement device or in writing down the measurement results can lead to bad values in a dataset. One benefit of looking at the overall pattern of the data is that it usually will reveal any substantial errors of this kind.

On the other hand, some outliers are accurate measurements but report anomalies, situations that are different from the typical situation in which the other measurements were made. Measurements of Sunday pedestrian traffic in downtown Austin, for example, would show an outlier each spring due to the Capital 10K race.

It is important to report outliers, so that people depending on measurements similar to those that produced the data are alerted to the possibility of large deviations from the general trend of the data. Even if the outlier is a mistake, it is an indication that users should watch out for similar mistakes. If the outlier is an anomaly, it is possible that it is the most important part of the data. For example, we would want the people who design bridges to design for the maximum load and not the typical load.

When using Models.xls to find best-fit models, we can exclude outliers from the fitting process by erasing the content of the column E cell for each row that contains outlier data. This means that the deviation associated with the outlier value is not counted in computing the standard deviation, and thus does not influence the fitting process.

Example 7: Using the dataset shown to the right:

a. Fit and report on a linear model using all of the data
b. Fit and report on a linear model with the outlier excluded.

Solution approach:

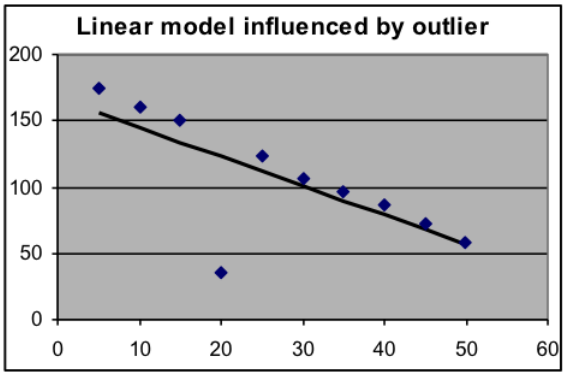
i. Copy the data to a Linear Model worksheet in Models.xls.
ii. Spread columns C, D, and E down to match the data, as usual.
iii. Make a graph of the data and model together, as usual.
iv. Use Solver to find the best-fit parameters and standard deviation.
v. Erase cell E6, since the graph shows the data in row 6 is an outlier.
vi. Use Solver again, to find results with the outlier excluded.

Remaining fuel in engine tank	
minutes	*liters*
5	174.4
10	160.1
15	149.7
20	35.3
25	123.8
30	105.4
35	96.7
40	87.0
45	72.1
50	57.6

[reveal-answer q="559929"]Show Answer[/reveal-answer]
[hidden-answer a="559929"]

i. With all the data, the best-fit linear model is $y=-2.20x+166.8$, $\sigma = 33.16$
ii. Without the outlier at ($x = 20$, $y = 35.3$), the best-fit linear model is $y=-2.56x+186.8$, $\sigma = 1.91$

[/hidden-answer]



Notice that removal of the outlier makes a great difference in the size of the standard deviation, since the way standard deviations are computed emphasizes any large deviations.

Is this particular outlier a mistake or an anomaly? You can't tell from the numbers, since any type of outlier consists of big deviations. But in this case the caption for the data indicates a process that logically must change smoothly. Thus this outlier is a mistake. Examination of the data suggests that the "35.3" *y* value for the outlier should have been about 100 higher, so perhaps an actual measurement of "135.3" was copied incorrectly.

Example 8: For the U.S. airline-passenger data provided to the right below:

a. Report the best-fit linear model formula and its standard deviation, using all the data points provided.
b. Examine the graph and list of deviations in column D to identify any data points which are outliers from the data trend from 1990 to 2000.
c. Report the best-fit line and σ when the outliers are excluded.
d. In this case, are the outlier points mistakes or anomalies?
e. How many 2006 passengers would there have been (to the nearest million) if US air traffic had continued its 1990-2000 trend?
f. Has the airline industry fully recovered its 1990-2000 trend? Why?

US Airline Traffic	
Years since 1990	Passengers (millions)
0	465.6
1	452.2
2	473.3
3	487.2
4	528.4
5	547.4
6	581.2
7	598.9
8	612.9

9	635.4
10	665.5
11	622.1
12	612.9
13	646.5
14	702.9
15	738.6
16	744.6

[reveal-answer q="437311"]Show Answer[/reveal-answer]
[hidden-answer a="437311"]

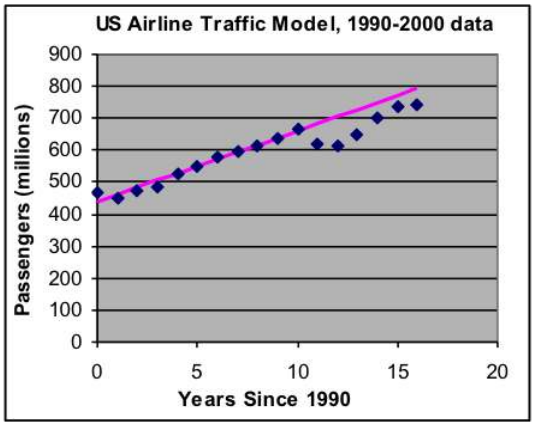
a. The all-data model is , σ = 25.1 million passengers.
b. The points for the years 2001, 2002, and 2003 are clearly outliers. The points for 2004, 2005, and 2006 are also low but are closer to the 1990-2000 trend; borderline cases of this kind are a matter of judgment, and either analysis is correct as long as you describe what decisions you have made.
c. The best-fit linear model for the 1990-2000 data is  , with σ = 12.4 million passengers.
d. These outliers are anomalies, which reflect a real change in conditions.
e. The trend for 2006 was 793 million (cell C19 of the 1990-2000 model).
f. Air business has not fully recovered, because actual 2006 air traffic of 745 million was 48 million passengers below the pre-2001 trend, and all actual values since 2001 have been lower than the pre-2001 trend.

[/hidden-answer]

Worksheet showing the result of fitting the data:

	A	*B*	*C*	*D*	*E*	*F*	*G* *H*	
1	X	y data	y model	Residual	Squared		Linear Model: y = m * x + b	
2	Year-1990	Passengers	prediction	deviation	deviation		***y = 22.06643 x + 439.4858***	
3	0	465.6	439.49	26.11	681.9503		439.4858	***b:*** **Intercept**
4	1	452.2	461.55	-9.35	87.4647		22.06643	***m:*** **Slope**
5	2	473.3	483.62	-10.32	106.4753			
6	3	487.2	505.69	-18.49	341.6996			
7	4	528.4	527.75	0.65	0.4205		**Goodness of fit for these settings**	
8	5	547.4	549.82	-2.42	5.8466	sum of sq. dev.	1380.672	
9	6	581.2	571.88	9.32	86.7801			
10	7	598.9	593.95	4.95	24.4941	# of dev. used	11	
11	8	612.9	616.02	-3.12	9.7174	# of parameters	2	
12	9	635.4	638.08	-2.68	7.2023	Standard deviation	12.3858	

13	10	665.5	660.15	5.35	28.6210			
14	11	622.1	682.22	-60.12				
15	12	612.9	704.28	-91.38				
16	13	646.5	726.35	-79.85				
17	14	702.9	748.42	-45.52				
18	15	738.6	770.48	-31.88				
19	16	744.6	792.55	-47.95				
20								



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18.6: L1.07- Section 6

Section 6: Distinguishing between linear and gently-curved data by examining deviation values

While many data relationships are linear, it is also not unusual to have relationships that are close to linear, but have a slight curve in the pattern, too small to be obvious in the graph, due to some small additional nonlinear effect. Linear-modeling spreadsheets often make it possible to detect such a situation by looking at how positive and negative deviation values are distributed (in column C in Models.xls) when the best-fit linear model has been found.

If such data is nonlinear and also low-noise, most deviations around the middle third of the input-value range will have the same sign, with the deviations at each end of the input-value range mostly having the opposite sign. This is similar to the obvious visual effect that fitting a line to strongly parabolic data would produce, but can work even when the deviations are small, as long as they are larger than the noise.

Another way of looking for this effect is to do a scatter plot of the input values and the deviations (for spreadsheets like Models.xls, this is columns A and D, with columns B and C left out). If such a *residual graph* of the deviations from the best-fit linear model displays a curved line rather than random noise, this is a sign that the data is significantly nonlinear. However, the automatic-scaling feature of spreadsheet graphs means that any outliers in the data will compress the scale so that most deviations look flat—this is why looking at the distribution of the signs of the deviations can be less confusing.

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18.7: L1.08- Section 7

Section 7: Adapting a modeling spreadsheet to use any modeling formula that is specified

The only important differences between the various templates in Models.xls are the formula that is put into cell C3, and the labels that are put in column H next to the parameters that start at row 3 in column G. This means that a spreadsheet can easily be adapted to any specified formula.

These are the formulas that are the same in every modeling spreadsheet that uses the Models.xls layout:

- "=B3-C3" in cell D3 (this computes the deviation from the model)
- "=D3^2" in cell E3 (the computed the squared deviation)
- "=SUM(E3:E100)" in cell H8 (this is the sum of squared deviations, which Solver will minimize)

Cell C3 will have a different formula for each type of model. These formulas will use "A3" to refer to *x* (this will automatically change to A4, A5, etc. as the formula in C3 is spread down to the other data rows. For the parameters of the model, the formula in C3 will use absolute references to cells in column G: $G$3 for the first parameter, $G$4 for the second parameter, etc. For clarity, each parameter should have a label put beside it in column H, but the labels do not affect Solver or the fitting process.

To make a spreadsheet apply to a specific model, the following three steps are essential (in addition to the standard formulas that are entered into columns D and E, and into cell H8:

1. **Identify the parameters in the model we intend to use.**
2. **Decide which parameter cell in column G (e.g., G3) corresponds to each parameter.**
3. **Enter an appropriate spreadsheet formula for the model into cell C3**
   (and spread it down beside all the data rows, along with the formulas in D3 and E3).

It will be easier to use the new model if we also take two additional steps:

4. **Put labels in column H next to each of the chosen parameters in column G.**
5. **Describe the kind of modeling formula either in G1 or on the tab of the worksheet.**

Once these modifications have been made, the worksheet is used in the same way as the other templates:

- Make a graph of the data and model together, so that you can see if a good fit is found.
- Set the parameters to initial values that ensure the model and data are in the same region.
- Use the Solver tool to minimize H8 (the sum of squared deviations).

Often a new model formula is based on a combination or adaptation of models you already have used. For example, we earlier used an exponential model to fit data showing the cooling of a liquid to a known room temperature. What if the room temperature had not been known? We could have found it from the data by adding it as a parameter to the exponential model, which would now be an exponential-plus-baseline model. The model suitable for that is shown in the example below.

Example 10: Fit the model $y = a\,(1+ r)x + b$ to this dataset (a, r, and b are parameters)

[reveal-answer q="967417″]Show Answer[/reveal-answer]
[hidden-answer a="967417″]

Modify a spreadsheet to use this formula and the standard modeling components:

a. copy the data to columns A & B of a sheet, with the numbers starting in row 3.
b. assign cells G3, G4, and G5 to parameters *a*, *r*, and *b*, respectively.
c. set cell C3 to the formula "=$G$3*(1+$G$4)^A3+$G$5".
d. set cells D3 to "=B3-C3" and E3 to "=D3^2", as usual.
e. spread the formulas in C3, D3, and E3 down beside all the data rows.
f. set cell H8 to "=SUM(E3:E99)" to compute the sum of squared deviations.
g. tell Solver to minimize cell H8 by changing G3, G4, and G5.

The resulting parameter settings are a = 54.2, r = 0.13, and b = 75. These show that the liquid started at 54.2 degrees above the room temperature of 75 degrees, and cooled at the rate of 13% of the difference per minute.

Minutes	Degrees

0	127.2
1	116.4
2	107.7
3	100.8
4	95.3
5	91.0
6	87.7
7	85.3
8	82.9
9	81.3
10	79.9
11	78.9

[/hidden-answer]

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18.8: L1.09- Section 8

Section 8: Using alternative goodness-of-fit indicators

The standard deviation (or the sum of squared deviations on which it is based) is an example of what is called a “cost” function, a computed value that an adjustment or decision process tries to make as small as possible. While standard deviation is by far the most commonly used cost function in modeling processes, there are other possible ways to measure a model’s quality that can be more appropriate in some circumstances. Usually these different cost functions are designed to reflect differences in economic costs of different errors (such as when an overestimate has more expensive consequences than an underestimate), to limit the effect of big deviations on the fitting process (so that a single bad value will not change the model vary much), or to allow for cases where the importance of a deviation depends on the size of the deviation relative to the data value (which is often true for exponential models).

If you use an alternative goodness-of-fit indicator to find a “best-fit” model, make an explicit statement about what indicator was minimized. This is needed both so that people can decide how to use the model, and so that they would be able to reproduce your results from the same data.

Using the maximum deviation as an alternative goodness-of-fit indicator

Standard deviation is a good description of the size of the typical deviation between a model and its data. But in some situations you may be more concerned about the *maximum* difference, rather than about the size of the typical difference. This can happen when you want to be able to identify a deviation amount that people can confidently expect will not be exceeded (although in that case you will usually need to allow some extra margin based on the typical random noise).

If you want to minimize the greatest difference between the population model and any individual population value, you would use the indicator “=MAX(E3:E12)” (placed in a spare cell such as H9) rather than “=SUM(E3:E12)”. If Solver is asked to minimize a cell with this formula (e.g., H9), it will adjust G3 and G4 (and any other parameters) in ways that may make the standard deviation higher, but will make the maximum distance above or below the data as small as is possible for the chosen kind of model.

Example 11: Find the exponential model that minimizes the maximum deviation from the census data.

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1	*x*	*y data*	*y model*	Residual	Squared	*Exponential model: y = a * (1+r)^x*			
2	Year-1780	Population	Prediction	deviation	deviations		MODEL PARAMETERS		
3	0	2.8	3.420962	-0.620962	0.385594		**3.420962**	**a:** Initial value at x=0	
4	10	3.9	4.501465	-0.601465	0.361761		**0.0278937**	**r:** Growth rate	
5	20	5.3	5.923243	-0.623243	0.388432				
6	30	7.2	7.794087	-0.594087	0.352939				
7	40	9.6	10.25583	-0.655832	0.430116		***Goodness of fit for these settings***		
8	50	12.9	13.49511	-0.595114	0.354161		Sum of sq dev	3.592537	
9	60	17.1	17.75752	-0.657517	0.432328		**Max sq dev**	**0.432328**	
10	70	23.2	23.36619	-0.166189	0.027619				
11	80	31.4	30.74635	0.653651	0.427259				
12	90	39.8	40.45752	-0.657517	0.432328				

Note that the best-fit parameters using this indicator are slightly different from those that arose from minimizing the sum of squared deviations: **a** is now 3.42 rather than 3.11, and **r** is 2.79% instead of 2.89%. This would not make much visible difference in the graph (since the slightly-higher starting value is mostly balanced by the slightly-lower growth rate), but the maximum squared deviation has been cut about in half, to 0.43 rather than 0.95. On the other hand, the sum of squared deviations is more than twice as large (3.59 rather than 1.72), resulting in a standard deviation that is 44% larger. Such trade-offs are typical with different cost functions—the model will be better for what you focus on, but not quite as good for other purposes.

Relative standard deviation

A *relative deviation* is the data-minus-model difference divided by the value of the model. This ratio can be expressed as a decimal fraction or a percentage – a few percent would be typical of most situations. Relative deviations can be useful when the model will not have zero values (in which case the division would not work) and there is a natural zero to the data scale (so that the ratio has corresponds to some real relationship).

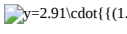
If we wish to use relative deviations in a worksheet like Models.xls, we replace the formula "=B3-C3" in cell D3 (and similarly other column-D cells) by the formula "=(B3-C3)/C3". Nothing else in the worksheet need be changed. Now Solver will minimize the sum of squared relative deviations, since that is now what is being computed in H8. The relative standard deviation can be computed from that sum in same way as before, with the formula "=SQRT(H8/(10-2))" in this case, which has 10 data points and 2 parameters.

Example 12: Fit an exponential model to the population data, minimizing relative standard deviation; state the best-fit formula and the relative standard deviation.

Solution approach:

a. Make a copy of the Exponential Model worksheet in Models.xls.
b. Set cell D3 to the formula "=(B3-C3)/C3" to make column D relative deviations
c. Copy the data to the worksheet and find the best-fit exponential model using Solver.
d. Set H12 (or some other cell) to "=SQRT(H8/(10-2))", to compute standard deviation
e. Put appropriate labels in cells G8 and G10 to explain the contents of H8 and H10.
f. Report the resulting formula and relative standard deviation.

[reveal-answer q="977672″]Show Answer[/reveal-answer]
[hidden-answer a="977672″]

The best-fit exponential model for this data when minimizing relative standard deviation is  , whose relative standard deviation from the data is 0.0231, or 2.31%.

[/hidden-answer]

The worksheet for the above example should look similar to this:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1	*x*	*y data*	*y model*	Residual	Squared	*Exponential model: y = a * (1+r)^x*			
2	Year-1780	Population	Prediction	deviation	deviations		MODEL PARAMETERS		
3	0	2.8	2.907038	-0.03682	0.001356		**2.907038**	**a:** Initial value at x=0	
4	10	3.9	3.906720	-0.00172	2.96E-06		**0.029997**	**r:** Growth rate	
5	20	5.3	5.250177	0.00949	9.01E-05				
6	30	7.2	7.055627	0.02046	0.000419				
7	40	9.6	9.481941	0.01245	0.000155		***Goodness of fit for these settings***		
8	50	12.9	12.74263	0.01235	0.000153	Sum of sq rel dev		0.004275	
9	60	17.1	17.12461	-0.00144	2.06E-06				
10	70	23.2	23.01348	0.00811	6.57E-05	Relative std dev		0.023117	

11	80	31.4	30.92743	0.01528	0.000233			
12	90	39.8	41.56287	-0.04242	0.001799			

The reason that these model parameters are slightly different from those that Solver found when minimizing the absolute standard deviation is that using relative standard deviation means that deviations from the smaller values count more in the fitting process than they did before. This is because a deviation of 0.1 million is about 3.5% of the 2.8 million for 1780, while that same 0.1 deviation is only 0.25% of the 39.8 million population 90 years later.

Exponential models are good possible uses for relative standard deviation since such models are based on constant-percentage relative growth or decay, and never have a zero value that would make it impossible to compute the ratio. However, absolute standard deviation is still the most usual fitting method even for this kind of model, and should be used unless you have a specific reason to use some other cost function.

Procedure for using Solver with a template from the Models.xls spreadsheet:

i. Set up an appropriate worksheet:
   a. Make a copy of the template for the desired model, and put the data into columns A and B.
   b. Spread the formulas in C3, D3, and E3 down to the same number of rows as the data.
   c. Make a graph with a scatter plot that shows both the data and the model.
   d. Set the model parameters to a rough estimate, so that the two graphs are nearby.

ii. Select Solver from the Tools menu and fill out the dialog box it displays:
   a. Enter the name of the cell containing the standard deviation in the “Set Target Cell” section at the top of the Solver dialog box (this the bottom computed cell in column G).
   b. Choose the “Min” (for Minimum) option in the second line, since we want Solver to find the smallest possible value for the standard deviation.
   c. Designate the parameter cells (“G3,G4” for models with two parameters, “G3:G5” for models with three parameters, etc.) in the “By Changing Cells” section on the fourth line.
   d. Press the “Solve” button at top right.

iii. *Check the graph* to see if the model now is close to the data, with some on each side. If it is not, you have made a mistake (probably in step [ii]) and must repeat the process correctly. If the model goes through the data but is the wrong shape, you are probably using an inappropriate model.

iv. If the graph is acceptable, press “OK” to accept the solution that is found. The parameters (and thus the “Model” values in the column C) are now set to the best-fit values to a very high accuracy.

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18.9: L1.10- Exercises

Part I

Reproduce the results in Examples 1–12.

Part II—Work the assigned problems

[13] A bank balance earning a constant rate of compound interest has these values: $1550 after 5 years, $2002 after 10 years, $2585 after 15 years, $3339 after 20 years, and $4313 after 25 years. What was the original deposit amount (that is, the balance after 0 years), and what annual interest rate was applied?

[14] The activity of a radioactive substance is measured on the same day each year for several years, with these results: 5.7 Curies after 1 year, 3.8 Curies after 2 year, 2.6 Curies after 3 years, and 1.7 Curies after 4 years. What is the decay rate of the substance? What will the activity be after 10 years?

Problems 15–24 have the same instructions, applied to different datasets. Copy and paste the datasets from the course web site copy of this topic into the Models.xls spreadsheet, rather than retyping them.

For each of the datasets listed below

a. Display the dataset and visually determine which of the models discussed in this topic is most suitable for this data.
b. Identify which points, if any, are outliers for this dataset.
c. Fit an appropriate model to the dataset, omitting the outliers (if any are present).
d. Report the best-fit model parameters and their standard deviation for this data.

[15] Dataset A

x	*y*
5	457.4
10	250.9
15	138.7
20	76.2
25	41.4
30	22.9
35	12.6
40	78.0
45	4.5
50	1.8
55	1.4
60	0.8

[16] Dataset B

x	*y*
0	172
1	195
2	216
3	230
4	244
5	256
6	261
7	266
8	264
9	262
10	255
11	247

[17] Dataset C

x	*y*
0	314.27
0.5	297.66
1	282.01
1.5	267.46
2	249.19
2.5	235.10
3	218.96
3.5	20.06
4	184.62
4.5	166.80
5	145.85
5.5	131.63

[18] Dataset D

x	*y*
1	239.7
2	296.6
3	386.6
4	469.9
5	597.6
6	777.3
7	952.2
8	1180.0
9	1424.4
10	1682.6
11	1980.3
12	2309.7

[19] Dataset E

x	*y*
1992	45,619
1993	49,529
1994	53,405
1995	57,228
1996	60,877
1997	65,003
1998	68,849
1999	72,399
2000	76,529
2001	80,448
2002	84,030
2003	88,027

[20] Dataset F

x	y
0	-68
1	-66.1
2	-62.6
3	-56.3
4	-47.3
5	-35.4
6	-23.4
7	-1.0
8	11.4
9	34.3
10	61.3
11	91.8
12	119.3
13	151.8
14	188.5
15	225.8
16	266.4
17	309.1
18	352.1
19	402.4
20	451.2

[21] Dataset G

x	y
0	10.65
1	8.46
2	7.10
3	5.60
4	4.74
5	3.90
6	3.09
7	2.62
8	2.00
9	1.55
10	1.47
11	0.98
12	0.76
13	0.97
14	0.62
15	0.49
16	0.42
17	0.29
18	0.27
19	0.21
20	0.20

[22] Dataset H

x	y
5.7	740.79
20.0	722.19
44.4	690.51
61.0	668.94
76.8	648.33
117.2	595.89
125.9	684.50
133.7	574.36
151.9	550.79
176.8	518.33
191.5	499.26
205.8	480.61
225.0	455.74
237.9	438.93
250.4	422.71
276.2	389.13
298.2	360.53
321.4	330.37
343.9	201.12
348.7	294.86
379.0	255.51

[23] Dataset I

x	y
0	50.9
1	158.9
2	63.9
3	71.5
4	77.7
5	83.6
6	87.9
7	92.9
8	94.4
9	94.1
10	98.2
11	99.3
12	100.1
13	98.4
14	100
15	96.5
16	92.8
17	90.6
18	88
19	83.4
20	77.5

[24] Dataset J

x	y
232.27	448.127
87.53	634.918
307.27	346.181
98.05	620.371
312.34	342.395
277.44	387.947
462.19	147.666
211.58	471.990
145.38	558.238
309.27	346.403
449.25	164.451
196.22	491.841
335.40	312.652
187.02	505.495
131.27	577.226
354.55	287.090
336.17	310.246
369.39	266.686
124.17	586.563
214.03	467.349
386.05	246.012

[25] Scientists have found that the total energy requirements of animals increase somewhat more slowly than body size. For example, a 1.2-pound mongoose requires 47 kilocalories per day, a 10-pound fox requires 240, a 22-pound bobcat requires 440, a 100-pound wolf requires 1350, a 300-pound lion requires 3100, a 400-pound tiger requires 3850, and a 700-pound polar bear requires 5900.

a. What are the best-fit parameters to this data for a “power” model? [The general formula for a power model is $y = a*x\text{^}b$.]
b. What is the standard deviation of the data from the best-fit model?
c. Does this data support the idea that a power model is appropriate for predicting the energy requirements of animals?
d. What daily energy requirement can be expected for a 45-pound lynx?

[26] The frequency of earthquakes varies by their size, with stronger ones being less frequent. In a recent one-year reporting period, the number of earthquakes detected at a particular facility was: 302,417 magnitude-2 quakes, 36,288 magnitude-3 quakes, 4,354 magnitude-4 quakes, 525 magnitude-5 quakes, and 60 magnitude-6 quakes.

a. What are the best-fit parameters to this data for an exponential model, where the magnitude is the input parameter and the earthquake count is the output variable?

b. What is the standard deviation of the data from the best-fit model?
c. Does the data support the idea that this relationship is exponential?
d. How many magnitude-7 earthquakes does this model predict this facility will detect each year?

[27] For Dataset C, find the **inverse model** (that is, the model when the *x* and *y* columns are swapped).

[28] For Dataset J, find the **inverse model** (that is, the model when the *x* and *y* columns are swapped).

[29] For Dataset G, use Solver to find the best-fit parameters if the spreadsheet is modified to minimize the relative standard deviation. Compare these parameters to those found in Exercise 21.

Exercise 30 **Dataset K**	
x	y
0	0.192
0.25	0.171
0.5	0.152
0.75	0.140
1	0.129
1.25	0.117
1.5	0.112
1.75	0.104
2	0.098
2.25	0.088
2.5	0.087
2.75	0.081
3	0.075
3.25	0.073
3.5	0.069
3.75	0.068
4	0.061
4.25	0.061
4.5	0.058
4.75	0.057
5	0.055

Exercise 31 **Dataset L**	
x	y
10	263
20	378
30	453
40	525
50	585
60	646
70	693
80	744
90	789
100	827
110	871
120	908
130	943
140	985
150	1013
160	1052
170	1081

Exercise 32 **Dataset M**	
x	*y*
0	8
2	193
4	364
6	529
8	657
10	722
12	725
14	678
16	531
18	426
20	235
22	61
24	-162
26	-335
28	-467
30	-637
32	-693
34	-721
36	-670
38	-570
40	-461
42	-303
44	-61

Exercise 33 **Dataset N**	
x	y
-3.0	0
-2.8	0
-2.6	0
-2.4	1
-2.2	2
-2.0	4
-1.8	11
-1.6	17
-1.4	35
-1.2	60
-1.0	97
-0.8	94
-0.6	187
-0.4	205
-0.2	236
0.0	237
0.2	228
0.4	208
0.6	182
0.8	118
1.0	82
1.2	51
1.4	25

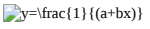
Exercise 34 **Dataset O**	
x	*y*
-7	0.0
-6	0.2
-5	0.9
-4	1.2
-3	3.0
-2	5.0
-1	8.1
0	12.2
1	16.6
2	19.7
3	21.6
4	23.2
5	23.8
6	24.2
7	24.3

Exercise 30-34 Instructions

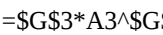
The formulas supplied below (in both algebraic form and as the spreadsheet formula for C3) will fit the corresponding dataset well as soon as the best settings are found for the parameters *a* and *b*.

For each formula supplied, make a worksheet that uses it as a model. Then put the specified dataset into the worksheet and use the Solver tool to find the *a* and *b* values that fit the dataset best.

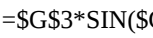
For **Dataset K**, use formula


=1/($G$3+$G$4*A3)
[Start with G3=5 & G4=1]

For **Dataset L**, use formula

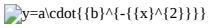

=$G$3*A3^$G$4
[Start with G3=1 & G4=1]

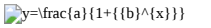
For **Dataset M**, use formula


=$G$3*SIN($G$4*A3)
[Start with G3=1000 & G4=1]

46	98
48	300
50	468
52	579
54	688
56	735
58	713
60	620
62	498
64	325
66	134
68	-57
70	-248
72	-437
74	-574
76	-683
78	-724
80	-743
82	-645
84	-513
86	-345
88	-182
90	13
92	216
94	405
96	540
98	660

1.6	15
1.8	9
2.0	1
2.2	2
2.4	0
2.6	0
2.8	0
3.0	0

For **Dataset N**, use formula

=$G$3*($G$4^-(A3^2))
[Start with G3=100 & G4=2]

For **Dataset O**, use formula

=$G$3/(1+$G$4^A3)
[Start with G3=10 & G4=1]

[Optional: for each model, choose names for the *a* and *b* parameters that are suggestive of the effect that parameter has on the model.]

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18.10: L1.01- Overview

Automated Fitting of Models, Comparative Goodness of Fit, and Outliers

Objectives:

1. Be able to adapt a modeling worksheet to compute a goodness-of-fit indicator from the deviation values between the data and the model.
2. Be able to use the Solver add-in with to automatically find the model parameters which minimize the indicator, and thus fit the data as well as possible as measured by that indicator.
3. Derive the standard deviation around the model from the sum of squared deviations, the number of deviation used in the sum, and the number of parameters in the model.
4. Be able to use alternative goodness-of-fit measures such as maximum deviation.
5. Select between different kinds of model by comparing best-fit standard deviations.
6. Be able to use relative deviation to compute an alternative form of standard deviation
7. Be able to identify data points that are outliers to the model implied by most of the data points, and to remove identified outliers from the model-fitting process when appropriate.
8. Be able to make and use a modeling spreadsheet to fit any specified function to a dataset.

Overview—getting the computer to do the tedious part so that models can be applied more easily

Splitting the work of modeling. The modeling process has two parts, one requiring thought and the other just requiring patience. The thoughtful part is deciding what model makes sense to use, looking at the graphs to see if a better model is needed of if some data points should be handled specially, and interpreting what the final parameter values imply about the situation in which the data was taken. The second part of fitting is the tedious process of adjusting each parameter in turn until you find the best parameter values for the chosen kind of model.

Tedious tasks are what computers do best. In this topic you will learn to use Solver tool that can makes the computer quickly find the best-fit parameter values for a model. Choosing which *kind* of model to fit is still your job, but automation will give you more time to do it. Using an automated fitting process will also make it easier to use models that have more parameters, permitting models that are more realistic and can solve a wider range of problems.

What is the "best" fit? So far, we have relied on a person looking at the graph to decide when the parameter settings for a model gave a good fit. People are good at this kind of visual assessment, but computers are not — they are much better at dealing with numbers. So to automate the fitting task we will compute a number that measures how well a particular model fits the data we are working with.

Handling stray data points. It sometimes happens that a small subset of the points in a data set deviate substantially from the overall pattern. These "outlier" points sometimes convey important information, but other times simply reflect bad measurements. In either case, it is usually best to exclude them from the model-fitting process and report them separately. We will discuss how to do this without losing the efficiency of the automated-fitting approach.

Additional types of models. The model-fitting process that we have already applied to linear, quadratic, and exponential formulas can be used with any type of modeling formula. All that is needed for a new type of model formula is to enter the in C3 in spreadsheet format, with the parts that you wish to fit as parameters expressed as absolute references to column G cells. In this topic we will see how to quickly adapt a modeling spreadsheet to fit many different formulas or to provide model enhancements such as a non-zero baseline for an exponential model. In later topics we will extend our list of basic model formulas and discuss how these combinations of these basic models can be used to make predictive formulas for a variety of realistic situations.

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CHAPTER OVERVIEW

19: Topic M- Power Functions and Using Logarithmic Graphs

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19.1: M1.01- Introduction

Learning Objectives

1. Be able to evaluate “power function” modeling formulas that give output values proportional to a constant power of the input value.
2. Be able to find the best-fit power and scale parameters implied by a dataset.
3. Be able to find the best-fit inverse function of data with a power-function relationship.
4. Be able to use common logarithms to compress the range of one or both variables in a relationship so that a more informative graph can be produced of data with a large dynamic range.
5. Be able to create semi-log and log-log graphs of data, to determine when such graphs are appropriate, and to read data that is presented in such graphs.
6. Be able to determine whether data variables have an exponential relationship by examining a semi-logarithmic graph of the data points.
7. Be able to determine whether data variables have a power-function relationship by examining a log-log graph of the data points.

Overview

Some situations exist in which all the input or output values are positive, but the ratio between the largest and smallest values (the *dynamic range*) is very large. This usually makes it impossible to for a regular graph to show the details of the shape of the relationship. *Logarithms* are a standard mathematical tool that has been developed to address this issue.

Graphing software usually supports modes in which one or both of the axes are graphed with logarithmic spacing rather than the usual uniform spacing. Such graphs can display a much wider range of values, and are often used in application areas that produce data with a large dynamic range.

Because of the way exponential and power functions make use of exponents in their formulas, appropriate logarithmic graphs of data with such relationships form straight lines, which are very easy to recognize and use for estimation. This also makes it easy to detect outliers in such relationships.

In addition to being a function that can be directly used in modeling, Logarithms have special connections to two other modeling functions: the exponential function we have already discussed and the power function that often arises as a result of dimensional relationships. This is be because the logarithm of an exponential function is a straight line, and a power function graphs as a straight line if the logarithms of both the x and y values are used.

Graphs are often provided with semi-logarithmic and logarithmic scales to take advantage of these relationships, which make it easy to recognize when data has an exponential or power-function relationship. Spreadsheet can produce such graphs automatically.

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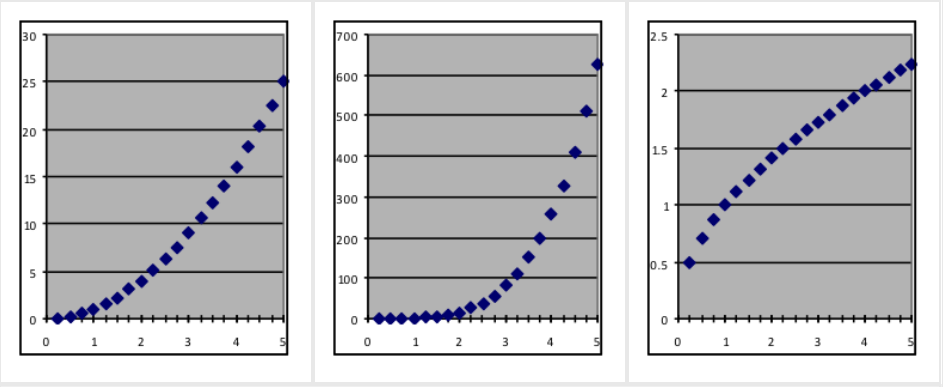
19.2: M1.02- Power Models Part I

Section 1: Dimensional relationships—the Power model $y=\text{scale}\cdot{x}^{\text{power}}$

Processes in which the relationship of input to output depends on volume, area, or distance often can be modeled by formulas in which output is computed by raising the input variable to a particular exponent power, then multiplying the result by a scaling factor. This differs from the earlier exponential formula because here the exponent is a parameter rather than being the input variable *x*, while *x* is used as the base rather than the exponent.

In addition to obvious relationships such as those between size, area, and volume for geometric shapes, power-function models are useful for many situations where the geometry involved is indirect. The rate at which animals use food, for example, depends on size based on interactions between weight (which grows rapidly with size) and breathing rate (which slows down), resulting in power-function relationships with fractional powers. The concentration of a fixed amount of a chemical dissolved in water has the reciprocal power-function relationship (where the power equals –1) to the amount of water. If the power equals exactly 1, the power function is a straight line through the origin; if the power is exactly 2, the power function is a parabola whose vertex is at the origin.

The *power* parameter may have any value, including fractional and negative numbers. However, for fractional powers (such as the *x0.5* for square roots) power functions are well-defined only for values of *x* that are greater than or equal to zero. The *scale* parameter can be any value, although it is usually positive (a zero value will make all output values zero and a negative value will flip all values around the *x* axis); the effect of the scale parameter is to uniformly stretch or shrink the graph vertically.



Power models can take many shapes, depending on the value of the power.

Examples of power-formula models:

y* = *122.3 x0.667 predicts the surface area of a steel ball in mm2 based on its mass in grams.

y* = *0.018 x3 predicts the weight of a cantaloupe in pounds based on its diameter in inches.

y* = *2×108 x-1 predicts daily visitors to web sites based on the order of popularity of the sites.

y* = *1.80 x0.667 predicts a planet's orbit radius in miles based on the length of its year in days.

The process of fitting a power model to a dataset is the same as for the other models you have studied—put the data into a worksheet in Models.xls, then use Solver to find the best-fit parameters. But in this case you will not have a preset worksheet template in which C3 already contains the right kind of formula. Instead, you will need to make a modeling worksheet yourself, or modify a copy of one of the ones you used earlier in Models.xls. The description below assumes that you will use the same row and column numbers as in Model.xls for the same kind of purposes, although you may vary them as long as you do so consistently.

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19.3: M1.03- Power Models Part II

To make a worksheet to fit the $y=\text{scale}\cdot{x}^{\text{power}}$ formula to data, we will make these changes (also put appropriate labels above or beside the active cells to help remember how they are being used):

- Decide on cells to use for each parameter. We will use G3 for *scale* and G4 for *power*
- Put the following formula into cell C3 as the model formula: **=$G$3*A3^$G$4**
- (This is the only step that is different between different types of model.)
- Put the formula **=B3-C3** into cell D3, to compute the Data-Model deviation.
- Put the formula **=D3^2** into cell E3, to compute the squared deviation.
- Put the formula **=SUM(E3:E100)** into H8 (or some other unused cell).
- To promote prompt convergence in the search process, set the initial values for the *scale* and *power* parameters so that the graph of the model roughly similar to the graph of the data (see note below).

Now this modified worksheet is a power-function template that can be used to fit a power model in the same way as the Linear, Quadratic, or Exponential templates—add the data, spread the formulas in C3, D3, and E3 down to match the data, make a graph and adjust the parameters to make the model similar to the data, then use Solver to find the best-fit parameters by minimizing H8, the sum of squared deviations.

***Optional: a* systematic way to set approximate initial values for power-function model parameters**

i. Set the *power* parameter equal to zero; this will make the model into a horizontal line at $y = scale$.
ii. Set the *scale* parameter equal to about the middle of the range of output values; the graph of the model will now pass through the data, with some data points above it and some below it.
iii. If the data is increasing as x increases, set the *power* parameter to 1; otherwise set it to -1.
iv. Repeat the sequence below until the graph of the model is roughly similar to the graph of the data:
   a. If the data is more curved than the model, double the *power* parameter; if less curved, halve it.
   b. If the data is all substantially smaller than the model (i.e., closer to the x-axis), double the *scale* parameter; if the data is all substantially larger than the model, halve the *scale* parameter.
v. Once the model is roughly similar to the data, use Solver starting with these parameter settings.

Example 1:

[a] Use the dataset on the right to find a model formula for the length of a planet's year in days based on its average distance from the sun in miles.

[b] Use that model to find year length for the dwarf planet Ceres, whose average distance from the sun is 256.1 million miles.

Solution approach:

[a] Make a power-law spreadsheet as described above, then put the distance data into column A and the year-length data into column B (don't copy the names). Then use Solver to get the best-fit model.

Planet	*Distance (millions of miles)*	*Length of Year (days)*
Mercury	36.1	88
Venus	66.7	226
Earth	92.6	365
Mars	140.8	687
Jupiter	481.8	4,332

[b] Once the model has been found, type the Ceres distance of 257.7 million miles into column A in the next row below the data (i.e., A11). The model's year-length prediction for Ceres will be in C11.

Answers: [a] The best-fit model is $y=0.410\cdot{x}^{1.50}$ [b] The predicted year for Ceres is **1,680 days**.

Both the square-root function $y = x^{0.5}$ and the square function $y = x^2$ are power functions. For positive values of x they are also inverse functions, since the square root of the square of a number reproduces the original number (and conversely). This occurs

because the numbers 0.5 and 2 are reciprocals of each other. This is true in general—the inverse of a power function is a different power function in which the new power is the reciprocal of the original one.

Example 2:

[a] Use the data from Example 8 to find a good model for using the length of a planets year to predict its average distance from the sun.

[b] Use that model to find the average distance from the sun of the asteroid Eros, whose year is 643 days long.

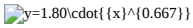
Solution approach:

[a] Make a power-law spreadsheet as described above, but this time put the distance data into column B and the year-length data into column A. Then use Solver to get the best-fit model.

[b] Once the model has been found, add the Eros year length of 643 days into column A in the next row (i.e., A11). The model's prediction for Eros's average distance to the sun will be in C11.

[reveal-answer q="680787"]Show Answer[/reveal-answer]

[hidden-answer a="680787"]

[a] The best-fit model is 

[b] The predicted average distance for Eros is **134.4 million miles**.

[/hidden-answer]

Planet	*Length of Year (days)*	*Distance (millions of miles)*
Mercury	88	36.1
Venus	226	66.7
Earth	365	92.6
Mars	687	140.8
Jupiter	4,332	481.8

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19.4: M1.04- Why We Use Logarithms

Section 2: Compressing the range of values by using logarithms

Many types of measurements cover an extremely wide range of values. An example is the energy released in earthquakes — there is a 100-billion-times difference between the energy of the smallest earthquake that a seismograph can measure and the largest ones that occur. In chemistry, concentrations of hydroxyl ion can vary by more than 100 trillion times between strong alkalis and strong acids.

Thus the same dataset might contain the relatively big value 4,200,000,000, the intermediate one 3,100, and the relatively small one 0.00025. In a regular graph of such a dataset, there is no scale that will show the data well—either the biggest number will be far off the scale at the top, or the middle number (which is less than a millionth of the big number) will be at the bottom and indistinguishable from the smallest number even though it is more than ten million times bigger.

The numbers above can be expressed as 4.2×109, 3.1×103, and 2.5×10-4, a style called scientific notation because it is used by scientists who often need to deal with very large or very small numbers. The idea of **base-10 logarithms** (also called "common" logarithms) carries this idea further by using decimal fractions in the exponents so that the initial number is not needed. Since 4,200,000,000109.623, we say that 9.623 is the logarithm of 4,200,000,000. Similarly, the logarithm of 3,100 is about 3.491 and that of 0.00025 is about –3.602 (all numbers between 1 and 0 have negative logarithms). So when using logarithms, the original range from 4,200,000,000 to 0.00025 becomes a compressed range from 9.623 to –3.602. On this scale, the intermediate value of 3.491 (the logarithm of 3,100) can be easily distinguished from both of the other values.

This is how our sense of hearing works. A whisper is a billion times less intense (in total energy into our ears) than a rock concert. In order to handle this range of input, our senses have evolved so that our perceived response to a stimulus is approximately proportional to the logarithm of its intensity. The use of logarithmic scales in mathematics and technology is a way of using this same tactic to deal with any range of numerical values where the ratio of the largest to the smallest (the *dynamic range*) is large.

The base of a logarithm does not have to be 10 (although only base-10, or "common", logarithms are used in this course and in most application areas). The non-10 bases for traditional logarithmic scales are typically those that make the results into convenient numbers (e.g., between 2 and 10 for earthquakes, between 1 and 6 for star brightness). A base of 2 is used in music because notes with a frequency ratio of 2 are harmonious.

Examples of logarithmic measurement scales

- Richter scale for earthquakes—each increase of one level means 32 times more energy
- pH for acidity/alkalinity—each increase of 1 in pH means 10 times more hydroxyl ions.
- Brightness of stars—each increase of 1 in stellar magnitude means a star is 2.51 times dimmer
- Sound—on the decibel scale, each bel (= 10 decibels) means the sound is 10 times more intense
- Music octaves—each one-octave increase in musical pitch means that vibration frequency doubles

[**Optional additional information about logarithm bases:** Any positive number except 1 can be used as a logarithmic base, resulting in logarithms that differ from "common" base-10 logarithms by a ratio equal to the common logarithm of the base chosen. Base-2 "binary" logarithms (3.322 times larger than common logarithms) are used mainly in computer-related fields. "Natural" logarithms (2.303 times larger than common logarithms) are used in calculus and based on the special number called *e* (approximately 2.71828). Natural logarithms are symbolized as "LN" on calculators, and calculated with the LN function in spreadsheets. On calculators, the LOG key computes base-10 logarithms, but spreadsheets use the LOG10 function for that purpose and use the LOG function only when the user is specifying which base to use. Thus the spreadsheet formula "=LOG(16,2)" evaluates to 4. This specify-the-base spreadsheet LOG function is useful for answering questions such as "How many years would it take for an investment at a 5% growth rate to double?" Since mathematically this is the same question as "What value of *x* will make (1.05)*x* equal to 2?", the answer 14.2 years is computed by the formula "=LOG(2,1.05)".]

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19.5: M1.05- Finding Logarithms

Logarithms are useful when the measurements in a dataset have a wide range of values, all greater than zero. Positive values are needed because **only positive numbers have logarithms**, since no exponent of 10 exists that gives a negative or zero result. Use of a negative value in the LOG10 spreadsheet function, or with the LOG key on a calculator, will give an error message.

The logarithm values themselves can be zero, in which case the original value is exactly 1, or negative, in which case the original value is less than 1 (e.g., -2 is the logarithm of 0.01 = 10-2).

Example 1: *For each of the values listed below, use reasoning to answer these two questions: [i] Does the value have a logarithm greater than 0? [ii] Does the value have a logarithm that is a whole number?*

[a] 582 [b] 10,000 [c] 0.23 [d] -48 [e] 6.2×1026 [f] 493.57285 [g] 0.001

Solution approach:

The whole-number powers of 10 are obvious: 101 = 10, 102 = 100, 103 = 1000, etc., as are the whole-number negative powers: 10-1 = 0.1, 10-2 = 0.01, 10-3 = 0.001, etc. The logarithm of any of these numbers is simply the corresponding exponent of 10. Also, 100 = 1 (any value to a zero power equals one), so numbers greater than 1 have positive logarithms and numbers less than one have negative logarithms.

[reveal-answer q="852352"]Show Answer[/reveal-answer]
[hidden-answer a="852352"]

[a] The logarithm of 582 is positive and is not an integer.

[b] The logarithm of 10,000 is positive and is an integer.

[c] The logarithm of 0.23 is negative and is not an integer.

[d] This number -48 is not positive, and therefore it does not have a logarithm.

[e] The logarithm of 6.2×1026 is positive and is not an integer.

[f] The logarithm of 493.57285 is positive and is not an integer.

[g] The logarithm of 0.001 is negative and is an integer.

[/hidden-answer]

Example 2: *For each of these logarithms, which two integer powers of 10 is the original value between?*

[a] 1.634 [b] 4.195 [c] -2.593 [d] -0.345 [e] 0.683

Answers:

[a] 101 = **10** and 102 = **100** (since 1.634 is between 1 and 2).

[b] 104 = **10,000** and 105 = **100,000** (since 4.195 is between 4 and 5).

[c] 10-2 = **0.01** and 10-3 = **0.001** (since -2.593 is between -2 and -3).

[d] 100 = **1** and 10-1 = **0.1** (since -0.345 is between 0 and -1).

[e] 100 = **1** and 101 = **10** (since 0.683 is between 0 and 1).

Finding logarithms of values, or values from their logarithms

In spreadsheets, we compute the base-10 logarithm of a number with the LOG10 function. Thus a cell containing the formula "=LOG10(3100)" will display the result 3.491361694. Note that logarithms, like trigonometric functions, almost always give values that are non-repeating decimals, so the logarithm values used are approximate rather than exact. The exception is for whole-number powers of the base, so that the base-10 logarithm of 1,000,000 is exactly 6, and that of 0.001 is exactly –3.

Since a logarithm is an exponent, you can always get back the original value by using the logarithm value as an exponent for the base. Thus 103.491 (the formula "=10^3.491" in a spreadsheet) will evaluate to nearly 3100, although there will a small difference due to rounding-error propagation because 3.491 is a rounded-off version of the logarithm. Most calculators have a LOG key that has the same effect as the LOG10 spreadsheet function. The inverse function for a calculator's LOG key is 10*x*, which coverts a base-10 logarithm back to the original value.

Example 3: *Find the common logarithms of these numbers, to three decimal places:*

[a] 48,300 [b] 2 [c] 0.055 [d] 7.2 [e] 2.6×1013 [f] 1.9×10-5

Solution approaches (you can use either one)**:**

[i] With a calculator, enter the value, then press the LOG key to see the logarithm. Use the EE or EXP key to enter the exponent of the numbers stated in scientific notation.

[ii] In a worksheet, enter a LOG10 formula with the value in parentheses, such as "=LOG10(7.2)". The numbers in scientific notation can be entered in "E format" – in the case as "2.5E13" and "1.9E-5"; you may wish to use the Format > Cells option to covert the format of result to Number or General.

[reveal-answer q="477554″]Show Answer[/reveal-answer]
[hidden-answer a="477554″]

[a] **4.684** [b] **0.301** [c] **–1.260** [d] **0.857** [e] **13.415** [f] **–4.721**

[/hidden-answer]

Example 4: *Find the values, to three significant digits, which have these numbers as logarithms:*

[a] 2.321 [b] –2.763 [c] 0.632 [d] –12.485 [e] 5.364 [f]26.931

Solution approaches (also remember to round to three significant digits)**:**

[i] With a calculator, enter the logarithm, then press the 10x key to see the value.

[ii] In a worksheet, enter a formula with the logarithm as the exponent of 10, such as "=10^2.321".

[reveal-answer q="944281″]Show Answer[/reveal-answer]
[hidden-answer a="944281″]

Answers: [a] **209** [b] **0.00173** [c] **4.29** [d] **3.27×10-13** [e] **23,100** [f] **8.53×1026**

[/hidden-answer]

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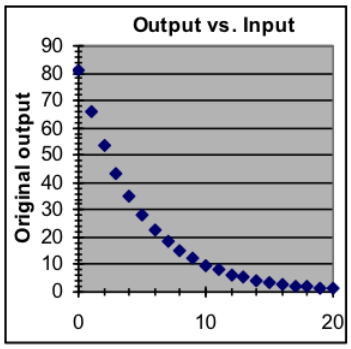
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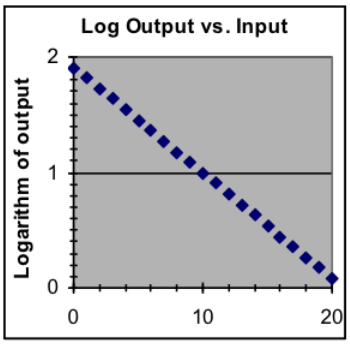
19.6: M1.06- Logarithmic Graphs Part I

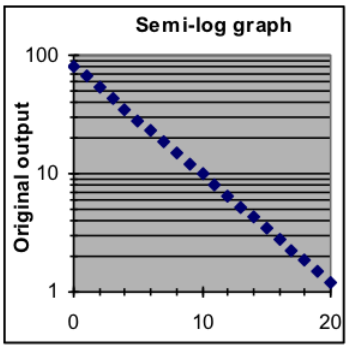
Section 3: Logarithmic graphs

Data that follows an exponential model is particularly well suited to logarithmic compression, because in that case the graph of the logarithms of the output values will form a straight line. Also, exponential data that covers several half-lives or doubling times will be more easily examined in the logarithmic form.

The display advantages of the logarithmic graph can be combined with the convenience in the original graph of having the scale on the left in the same form as the data. This is done with **semi-log** graphs, in which the logarithmic graph is displayed but the numbers shown in the *y* scale on the left are original, pre-logarithm values. The horizontal lines and numbers labeling them are placed at the vertical position that matches the corresponding logarithm.



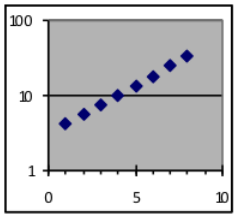


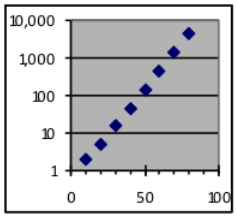


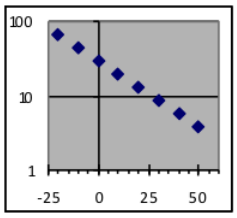
To make a semi-log graph, first make a regular graph, then format the y axis:

i. click on the vertical scale to select it,
ii. use Format > Selected Axis to display the Format Axis dialog box,
iii. click on the Scale tab to show the axis settings,
iv. put a check in the Logarithmic Scale box near the bottom.

Example 5:*Make semi-log graphs of these three datasets.*







x	y
1	4.1
2	5.5
3	7.4
4	10.0
5	13.5
6	18.2
7	24.5
8	33.1

x	y
10	2
20	5
30	15
40	47
50	145
60	449
70	1,394
80	4,329

x	y
-20	67.9
-10	45.1
0	30.0
10	19.9
20	13.3
30	8.8
40	5.9
50	3.9

Example 6: *For this exponential dataset, make a graph of the logarithms of the y data values and use that graph to estimate what y value is to be expected for an x value of 65.*

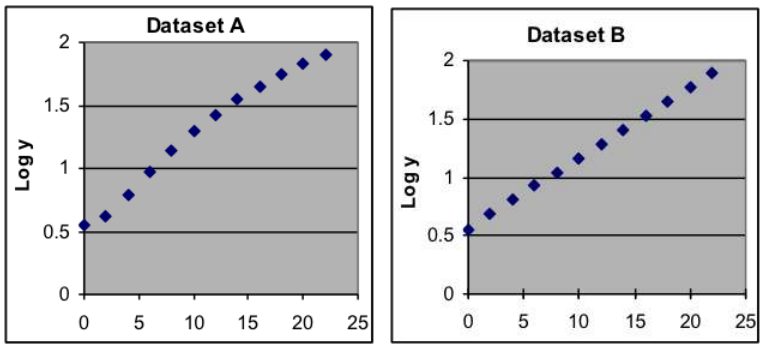
[reveal-answer q="167574"]Show Answer[/reveal-answer]
[hidden-answer a="167574"]

i. Copy the data to a blank worksheet, putting *x* values in column A and starting the numbers in row 3. Then add a third column of logarithm values by setting cell C3 to "=LOG10(B3)" and spreading that formula down beside all the data rows.
ii. Now make a scatter plot of columns A and C (but not B). This will plot *x* on the horizontal axis against the logarithm of *y* on the vertical axis, forming a straight line.
iii. By finding what place on that line is above the *x=65* position, we can estimate that the logarithm of *y* for that case is about 2.8. By evaluating 102.8 (with either a calculator 10x key or the "=10^2.8" spreadsheet formula), we find that the answer is about **630**.

[/hidden-answer]

x	*y*
0	26
10	69
20	181
30	482
40	1,279
50	3,393
60	9,002
70	23,885

Example 7: *One of these datasets has an exponential-growth pattern, while the other has a quadratic pattern (i.e., part of a parabola) whose shape is different but close enough that it is difficult to tell just by looking at the graphs. Use graphs of the logarithms of the y values to identify which dataset is exponential.*
[reveal-answer q="308585"]Show Answer[/reveal-answer]
[hidden-answer a="308585"]For each dataset, copy the data to a worksheet, add a third column that shows the logarithms of the *y* values, and make a scatter plot of the *x* values and those logarithms. Examine the two graphs (copies shown below) to see which one is a straight line, indicating that its original data was exponential. The graphs show that **dataset B is exponential** and dataset A is not exponential.



[/hidden-answer]

Dataset A	
x	*y*
0	3.5
2	4.1
4	6.1
6	9.3
8	13.7
10	19.5
12	26.5
14	34.9
16	44.5
18	55.3
20	67.5
22	80.9

Dataset B	
x	*y*
0	3.6
2	4.8
4	6.4
6	8.4
8	11.1
10	14.7
12	19.5
14	25.8
16	34.1
18	45.0
20	59.6
22	78.8

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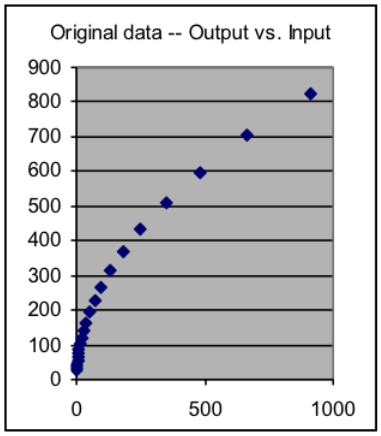
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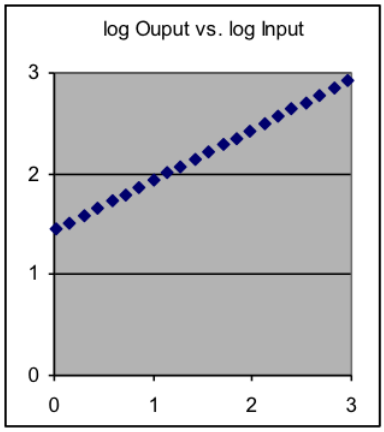
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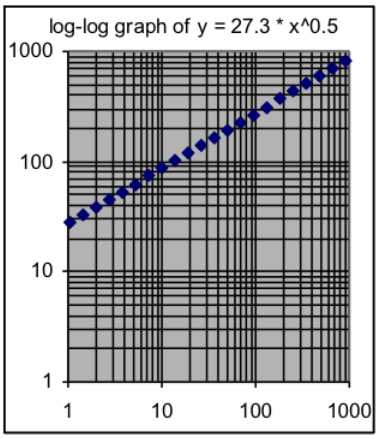
19.7: M1.07- Logarithmic Graphs Part II

Model-fitting using logarithms: It is possible, and sometimes useful, to fit models to the logarithms of the output values in a dataset, rather than to the output values themselves. For example, a linear model might be fit to the logarithm of data that has an exponential pattern. When this is done using minimization tools such as Solver, however, note that what is minimized is the standard deviation of the logarithms, not of the original values. In the case of logarithms, this has the same effect as minimizing relative standard deviation.

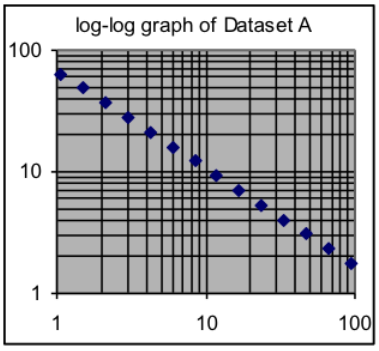
Log-log graphs: In situations where all *x* and *y* values are positive but there is a large dynamic range for both variables, it can help to use the logarithms of *x* values as well as for *y* values. A graph where this is done is called a "log-log" graph. On log-log graphs, straight lines indicate power-function relationships (in contrast to the exponential relationship indicated by straight lines on semi-log graphs). The slope of the line (0.5 on the graph of logarithms, below center) matches the *power* parameter in the model, and the intercept of the line (1.44 below) is the logarithm of the *scale* parameter 27.3.

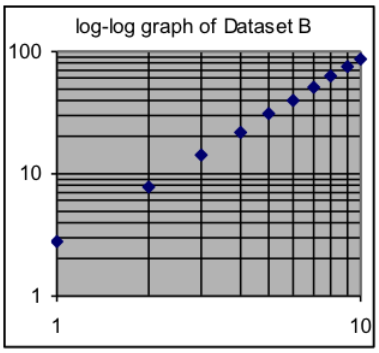






Example 8:*Use log-log graphs of each of these two datasets to determine whether their data follows a power-function pattern.*





Dataset A	
x	***y***
1.06	63.95
1.50	48.52
2.12	36.80
2.99	27.91
4.22	21.17
5.96	16.06
8.42	12.18
11.90	9.24
16.80	7.01
23.74	5.32
33.53	4.03
47.36	3.06
66.90	2.32
94.50	1.76

Dataset B	
x	***y***
1	2.75
2	7.78
3	14.29
4	22.00
5	30.75
6	40.42
7	50.93
8	62.23
9	74.25
10	86.96

[reveal-answer q="883494"]Show Answer[/reveal-answer]
[hidden-answer a="883494"]
The straight-line log-log graphs show that both datasets come from power-functions (a negative power for A, a positive power for B).
[/hidden-answer]

Using other functions of the output values: There are some situations in which it is convenient to apply other functions to the *y* data so that the result is close to a straight line. For example, the power function $y = x^2$ can be made linear by taking the square root of the *y* values. This tactic was used more often before computers were widely available to fit nonlinear formulas, but such "linerarizations" are still used in some areas because people are very good at judging the straightness of lines and at using linear graphs for interpolation and extrapolation. You can tell when an axis has been transformed in this way because the gridlines of the graph will be non-uniform in spacing, like those of semi-log graphs.

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19.8: M1.08- Exercises

Part I

Repeat the Examples 1-8

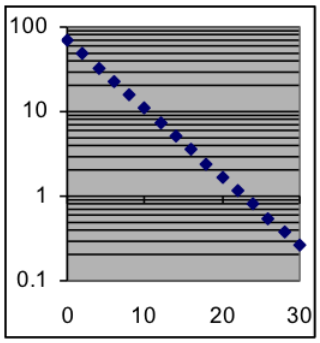
Part II

9. State the logarithms of these numbers, to three decimal places:
    a. 452.6
    b. 0.2
    c. 1,000
    d. 4.5×1015
    e. 0.00724
    f. 6.4×10-8
    g. 7.66
10. State the logarithms of these numbers, to three decimal places:
    a. 15,250
    b. 0.0001
    c. 1.4×10-5
    d. 1.11
    e. 200
    f. 7.5×1012
    g. 0.000215
11. What values have the following logarithms (round answers to three significant digits):
    a. 3.526
    b. 0.01
    c. -2.769
    d. 10
    e. -0.168
    f. 5.728
    g. 0
12. What values have the following logarithms (round answers to three significant digits):
    a. -2
    b. 1.592
    c. 5.923
    d. -4.511
    e. -1.735
    f. 0.301
    g. 23.301
13. Which of these values has a logarithm greater than 1?
    a. 45.8
    b. 8.3
    c. 195,680
    d. -15.2
    e. 0.783
    f. 5.4×1012
    g. 10
14. Which of these values has a logarithm less than 1?
    a. 6.8
    b. 55.5

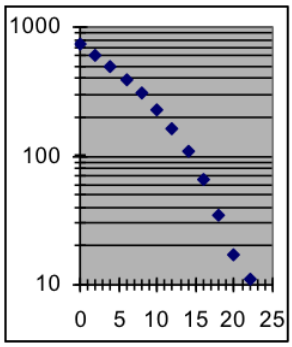
c. 4.2×109
d. 0.00002
e. 75
f. 9.27×10-3
g. 10

15. Which of the graphs below indicate that the corresponding dataset follows an exponential pattern?

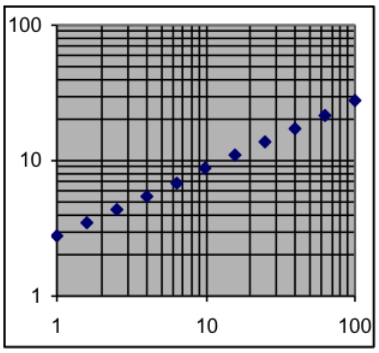
Graph A



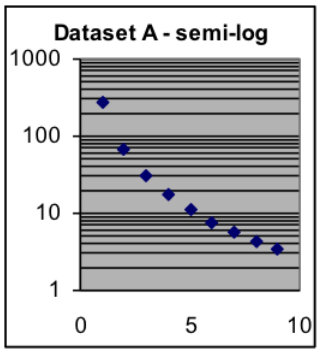
Graph B

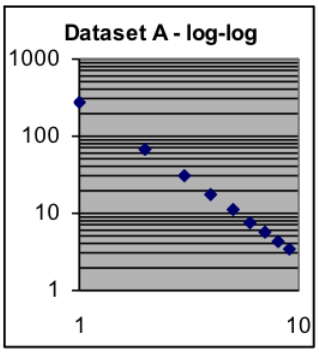


Graph C



16. Which of the graphs above indicate that the corresponding dataset follows a power-function pattern?
17. Use a semi-log graph to show whether dataset A is exponential.The point halfway between the x=1 and x=2 data points has a y value that is about halfway between y=100 and y=200.





Dataset A	
x	Y
1	275.3
2	68.8
3	30.6
4	17.2
5	11.0
6	7.6
7	5.6
8	4.3
9	3.4

Dataset B	
x	y
20	4
40	16
60	36
80	64
100	100
120	144
140	196
160	256
180	324

18. Use a semi-log graph to show whether dataset B is exponential.
19. Use the Exercise 17 graph to interpolate an estimated value for x = 1.5.
20. Use the Exercise 18 graph to interpolate an estimated value for x = 170.
21. Use a log-log graph to tell if dataset A approximates a power function.
22. Use a log-log graph to tell if dataset B approximates a power function.
23. Scientists have found that the total energy requirements of animals increase somewhat more slowly than body size. For example, a 1.2-pound mongoose requires 47 kilocalories per day, a 10-pound fox requires 240, a 22-pound bobcat requires 440, a 100-pound wolf requires 1350, a 300-pound lion requires 3100, a 400-pound tiger requires 3800, and a 700-pound polar bear requires 5900.
    a. What model is appropriate for predicting energy requirement from weight? (You will need to decide on the type of model, then find the best-fit parameters to this data for that type.)
    b. What daily energy requirement can be expected for a 45-pound lynx?

Infant data averages for Exercises 24–28		
Age (months)	Weight (pounds)	Length (inches)
3	13.0	24.0
6	17.2	26.7
9	20.3	28.6
12	22.2	30.0
15	24.0	31.4
18	25.3	32.5
21	26.6	33.6
24	27.8	34.5
(US Natl Cen Health Stat)		

24. Fit a power-function model to the infant data shown above, using age as the input variable and average weight as the output variable.
    a. State the best-fit scale and power parameters in an appropriate model formula.
    b. Is this model a good fit to the data?
    c. Compute the predicted average infant weight for an age of 20 months.
25. Fit a power-function model to the infant data shown above, using average weight as the input variable and age as the output variable.
    a. State the best-fit scale and power parameters in an appropriate model formula.
    b. What is the relationship of the power parameter in this model with the power parameter of the model found in the previous problem? Why?
26. Fit a power-function model to the infant data shown above, using age as the input variable and average length as the output variable.
    a. State the best-fit scale and power parameters in an appropriate model formula.
    b. Is this model a good fit to the data?
    c. Compute the predicted average infant length for an age of 20 months.
27. Fit a power-function model to the infant data shown above, using average length as the input variable and average weight as the output variable.
    a. State the best-fit scale and power parameters in an appropriate model formula.
    b. Use the predicted average infant length for an age of 20 months (computed in section c of the previous exercise) as input to the model found in this exercise, producing as output a predicted infant weight for an age of 20 months. What earlier exercise also predicted this quantity? How well do the predictions match?
28. Using the results of the previous exercise (without doing any more fitting), use a single computation to estimate what the best-fit power parameter would be for a model based on this data that predicts average length from average weight.

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CHAPTER OVERVIEW

20: Topic H- Linear Formulas – Word Problems

20.1: H1.01- Overview
20.2: H1.02- Example 1
20.3: H1.03- Example 2
20.4: H1.04- Example 3
20.5: H1.05- Example 3 Alternative Method
20.6: H1.06- Example 4
20.7: H1.07- Interpreting the Slope and Intercept
20.8: H1.08- Exercises

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20.1: H1.01- Overview

Topic H. Linear Formulas—Word Problems

Objectives:

1. Solve applications problems using linear models and interpret the results.
2. Re-define the input variable if needed to make the model easier to interpret in a useful manner.

Terminology:

We use the following instructions interchangeably:

Write a formula to express the linear relationship.

Write an equation to express the linear relationship.

Write an algebraic linear model to express the relationship.

Overview:

To solve the applications problems in this topic, follow these steps.

1. **Variables**:
   a. What are the names of the two variables?
   b. What are the units of each variable?
2. **Prediction**:
   a. Which variable will we predict in the main question? (That's the output variable, i.e. *y*-variable.)
   b. What letters will we use for each of the variables in our linear model?
   c. Are there any limits on the values of either variable? If so, what? Write them in mathematical notation.
   d. What are some points, that is, values of both variables for more than one point? Write these in appropriate ordered pair notation, with the *x*-variable first and the *y*-variable second.
3. **Is a linear model appropriate?** How do you know? (Use one of these three methods: (1) the problem says that a linear model should be used, (2) graph the data given and see that it forms a straight line, (3) the problem says that a certain increase in the input variable will always give the same increase in the output variable.)
4. **Slope**:
   a. Find the slope.
   b. Write a sentence to **interpret the slope**, using the units of each variable in the sentence.
5. Find the **formula** for the line.
6. Write a sentence to **interpret the *y*-intercept**, using the units of each variable in the sentence. If this value is out of the range of the acceptable values for the *y*-variable, comment on that.
7. Use the formula for the line to **answer the each prediction question**. Do the algebra and write a sentence, naming the variable and the units, interpreting your answer.
8. **Graphing and checking your algebraic work.**
   a. Make a graph appropriate to answer the prediction questions.
   (If you make the graph from only the information in the original problem, then you can check your work in developing the model by determining whether the results of the model are consistent with the graph. If you graph the formula you obtained, then you should check to be sure that the graph is consistent with the information given in the original problem.)
   b. Use the graph to answer the prediction questions. Did you find the same answers as when you used the formula?
   c. Do your answers to the prediction questions make sense? Explain something you looked at to see if they are reasonable.

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20.2: H1.02- Example 1

Example 1: During the first and second quarters of a year, a business had sales of $42,000 and $58,000, respectively. If the growth of sales follows a linear pattern for the next four years, what will sales be in the fourth quarter? In the 9th quarter? Use an algebraic method of solution.

Solution:

1. What are the two variables and their units?

Ans. Quarter and sales. Quarter is in numbers and sales are in dollars.

2. What will we predict? (Make it *y*.) What are the limits? What are some points?

Ans. We will predict sales. So *y* = sales. Then *x* = quarter and *x* will go from 1 to 16, in increments of 1.

Quarter 1 has sales of 42,000. So x = 1 and y = 42000. Use the point (1, 42000)

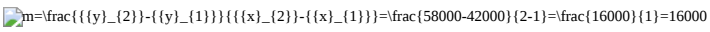
Quarter 2 has sales of 58,000. So x = 2 and y = 58000. Use the point (2, 58000)

3. Is a linear model appropriate?

We are told to use a linear model here to see what it would predict.

4. Slope: Find the slope: m = rise / run

Let (1,42000) be the first point and (2, 58000) be the second point.



Interpret the slope, using the units of the numbers in the problem:

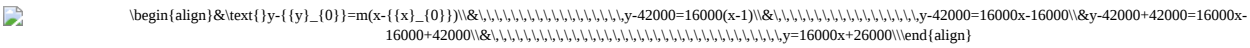
(As x increases by 1, y increases by m.)

For each quarter that goes by, sales will increase by $16,000.

5. Write the formula of the linear relationship:

(Choose either point and use the point-slope form of the line. Then simplify it to the slope-intercept form of the line.)

Choose (1,42000) for the point and use the slope of 16000 that was just computed.



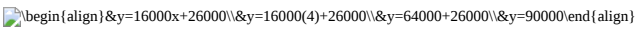
6. Interpret the y-intercept.

(The value of y is b when .)

For quarter 0, the sales would be $26,000. This isn't meaningful because the quarters start with the first quarter in this problem, not the zero-th quarter.

7. Use the formula to make the requested prediction. Write the result in a sentence, with units.

For the fourth quarter,  .



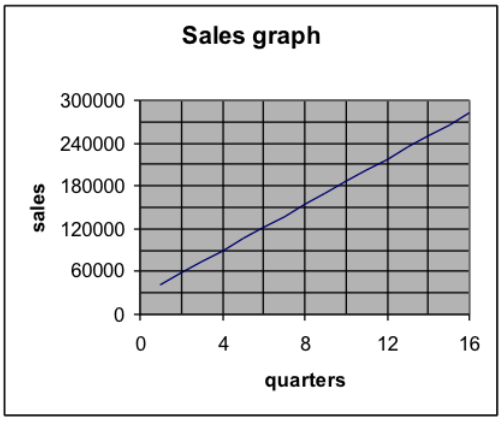
So, in the fourth quarter, the linear model predicts sales of $90,000.

In a similar manner, for the 9th quarter, the linear model predicts sales of $170,000, because y = 16000(9)+26000 = 170000

8. Sketch a graph from the original problem information, use it to estimate the answers to the prediction questions, and determine whether your answers from the algebraic formula are reasonable.

quarter	sales
1	42000
2	58000

By hand, we plot these points, draw the line, and then extend it far enough to use it to estimate the answers to the questions.



On this graph, we see that, when quarters = 4, the y-value is 90,000.

We also see that when quarters = 9, the y-value is $170,000.

So the answers from the graph are consistent with the answers we by doing the algebra and arithmetic using the formula.

These two answers for the predictions are reasonable.

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20.3: H1.03- Example 2

Example 2: Ajax Manufacturing bought a machine for $48,000. It is expected to last 15 years and, at the end of that time, have a salvage value of $7,000. Set up a linear depreciation model for this machine and find the worth at the end of 10 years.

Solution:

1. What are the two variables and their units?

Ans. Time and worth. Time is measured in years and worth in dollars.

2. What will we predict? (Make it y.) What are some points?

Ans. We will predict worth. So y = worth, which must always be above $7000. Then x = years and x will go from 0 to 15.

At zero years, the worth is 48000. So x = 0 and y = 48000. Use (0, 48000)

At 15 years, the worth is 7000. So x = 15 and y = 7000. Use (15, 7000)

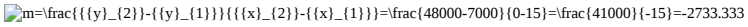
3. Is a linear model appropriate?

Yes, because the problem asks for linear depreciation.

4. Slope

Find the slope: m = rise / run

Let (0,48000) be the first point and (15,7000) be the second point.



Interpret the slope, using the units of the numbers in the problem:

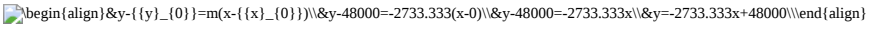
(As x increases by 1, y increases by m.)

For each year, the worth increases by -2733.33. But, of course, increasing by a negative amount means decreasing. So, for each year, the worth decreases by $2733.33.

5. Write the formula of the linear relationship:

(Choose either point and use the point-slope form of the line. Then simplify it to the slope-intercept form of the line.)

Choose (0, 48000) for the point and use the slope of -2733.333.



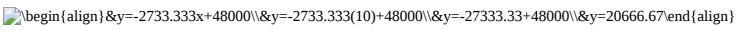
6. Interpret the y-intercept.

(The value of y is b when .)

At the beginning of the process, when time=0, the worth is $48,000.

7. Use the equation to make the requested prediction. Write the result in a sentence, with units.

At 10 years, x = 10:

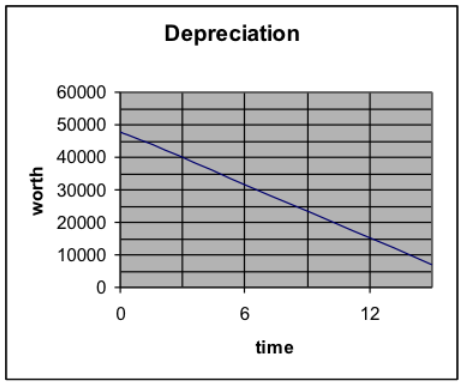


So the worth of the machine at 10 years is $20,666.67.

8. Make a graph and use it to check the prediction.

time	worth
0	48000
15	7000

By hand, we plot these points, draw the line, and use it to estimate the answer to the question.



Notice that the graph slants down, which means the slope is negative. And notice that you did get a negative slope using the formulas. That's good!

On the graph, look for time=10, and see that the worth is about $20,000.

That is consistent with the value obtained using the formula.

The prediction is reasonable.

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20.4: H1.04- Example 3

Example 3: For a certain type of letter sent by Federal Express, the charge is $8.50 for the first 8 ounces and $0.90 for each additional ounce (up to 16 ounces.) How much will it cost to send a 12-ounce letter?

Solution:

1. What are the two variables and their units?

Ans. Weight and cost. Weight is measured in ounces and cost is measured in dollars.

2. What will we predict? (Make it y.) What are some points?

Ans. We will predict cost. So y = cost. Then x = ounces and x must be greater than or equal to 8. Cost must be at least $8.50.

Eight ounces cost 8.50. So x = 8 and y = 8.50. Use (8, 8.50)

Nine ounces cost 9.40. So x = 9 and y = 9.40. Use (9, 9.40)

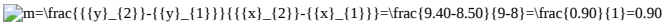
3. Is a linear model appropriate?

Yes, because the cost increases by the same amount for each additional ounce.

4. Slope

Find the slope: m = rise / run

Let (8, 8.50) be the first point and (9, 9.40) be the second point.



Interpret the slope, using the units of the numbers in the problem:

(As x increases by 1, y increases by m.)

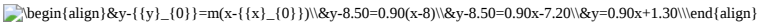
For each additional ounce, the cost increases by $0.90.

(Notice that was given in the problem. So really we didn't need to compute the slope. If we had recognized the interpretation of it, we could have just picked it out of the statement of the problem and skipped step 8.)

5. Write the formula for the linear relationship:

(Choose either point and use the point-slope form of the line. Then simplify it to the slope-intercept form of the line.)

Choose (8, 8.50) for the point and use the slope of 0.90.



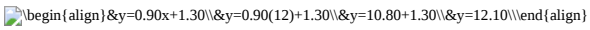
6. Interpret the y-intercept.

(The value of y is b when  *.)*

When the envelope weighs 0 ounces, the model says that the cost is $1.30. Since the model is stated to only be good for weights of at least 8 ounces, then this is not a useful observation.

7. Use the equation to make the requested prediction. Write the result in a sentence, with units.

For 12 ounces, x = 12:

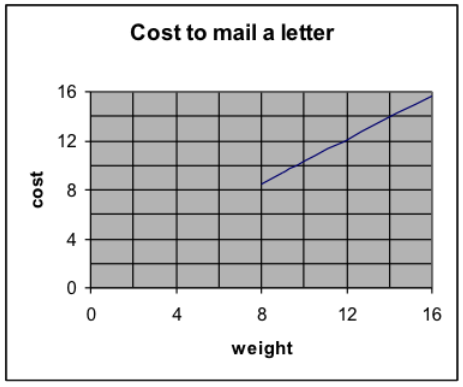


So a 12-ounce letter will cost $12.10.

8. Sketch a graph from the original problem information, use it to estimate the answers to the prediction questions, and determine whether your answers from the algebraic formula are reasonable.

weight	cost
8	8.50
9	9.40

By hand, plot these points, draw the line, extend it to the limits for the variables, and use it to estimate the answer to the question.



Using the graph, the cost for a letter of 12 ounces is a little more than $12.

That is consistent with the result from using the linear formula.

The prediction is reasonable.

Notice that the graph here only includes a line for the parts where the linear model is appropriate. In other words, it doesn't extend below a weight of 8 ounces. This helps illustrate why the y-intercept here is not very useful.

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20.5: H1.05- Example 3 Alternative Method

Alternative method of solution for Example 3:

Example 3 question: For a certain type of letter sent by Federal Express, the charge is $8.50 for the first 8 ounces and $0.90 for each additional ounce (up to 16 ounces.) How much will it cost to send a 12-ounce letter?

We saw here that it is awkward for the value to be so far from any useful values for the variables. This suggests that we re-define the variables to be more useful.

1-2. Define the variables and their units and possible values. List the points.

Let x = number of ounces above 8. So the possible values for x are 0 to 8 (which corresponds to the letter weighing 8 ounces to 16 ounces.)

Let y = cost. The possible values for y are $8.50 and larger.

A letter weighing 8 ounces costs $8.50. So x = 0. The point is (0, 8.50)

A letter weighing 9 ounces costs $8.50+0.90 = $9.40. So x = 1. The point is (1, 9.40)

3. Is a linear model appropriate?

Yes, because the cost increases by the same amount for each additional ounce.

4-5. Find the slope and the formula.

Using the points above, we have

6. Interpret the *y*-intercept.

When x = 0, the value for y is 8.50. Since x = weight – 8 ounces, that means that when the letter weighs 8 ounces, the cost is $8.50.

Notice that defining the x-variable in this manner has made the y-intercept more meaningful in the problem.

7. Make the prediction.

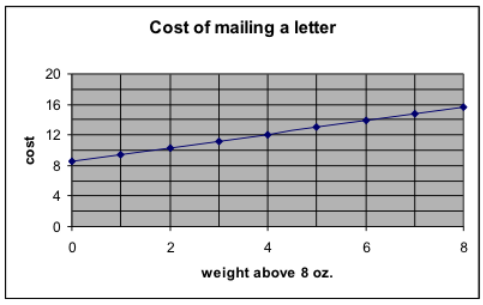
If the letter weighs 12 ounces, then we use x = 12 – 8 = 4. So $y=8.50+0.90\cdot4=8.50+3.60=12.10$.

This agrees with the answer we found by the first method.

8. Graph and check whether you obtain the same results.

weight	x = weight – 8	y=cost
8	0	8.5
9	1	9.4

By hand, plot these points, draw the line, extend it to the limits for the variables, and use it to estimate the answer to the question.



On the graph, look up a weight of 12 ounces, which is
x = 12 – 8 = 4.

The cost for x = 4 is a little larger than $12, which is consistent with the value we obtained from the formula.

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20.6: H1.06- Example 4

Example 4: Write a mathematical model for the population of this city over the given period of time and use that to predict the population in 2010.

Year	1970	1980	1990	2000
Pop'n (thousands)	234	289.5	345	400.5

Solution:

1. What are the two variables? And what are their units?

Ans. Year and population. Year is in years and population is in thousands of people.

2. What will we predict? (Make it y.) What are some points?

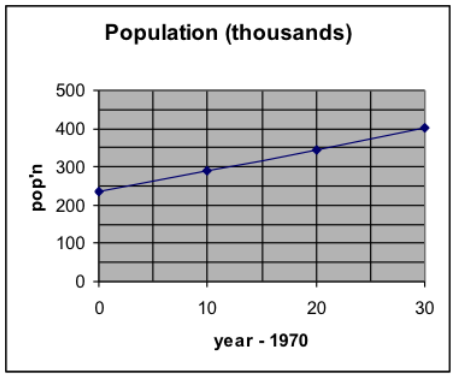
Ans. We will predict population. So y = population. Year is the input variable. However, as we saw Example 3, it will be more convenient to have the input variable x = years since 1970. Then x is greater than or equal to zero. Obviously y must also be greater than or equal to zero.

1970 has pop'n of 234. So x = 0 and y = 234. Use the point (0, 234)

1980 has pop'n of 289.5. So x = 10 and y = 289.5. Use the point (10,289.5)

3. Is a linear model appropriate?

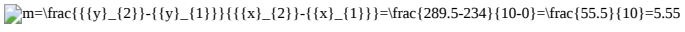
year	year – 1970	popn
1970	0	234
1980	10	289.5
1990	20	345
2000	30	400.5



The graph clearly indicates that a linear model is appropriate for this population growth over this time period.

4. Slope: Find the slope: m = rise / run

Let (0,234) be the first point and (10,289.5) be the second point.



Interpret the slope, using the units of the numbers in the problem:

(As x increases by 1, y increases by m.)

For each year that goes by, population will increase by 5.55 thousand people.

5. Write the formula for the linear relationship:

(Choose either point and use the point-slope form of the line. Then simplify it to the slope-intercept form of the line.)

Choose (0,234) for the point and use the slope of 5.55 that I just computed.

$$\begin{align}&y-{{y}_{0}}=m(x-{{x}_{0}})\\&y-234=5.55(x-0)\\&y-234=5.55x\\&y=5.55x+234\\\end{align}$$

6. Interpret the y-intercept.

(The value of y is b when $x=0$*.)*

When $x=0$, that is, in 1970, the population is 234 thousand people.

7. Use the formula to make the requested prediction. Write the result in a sentence, with units.

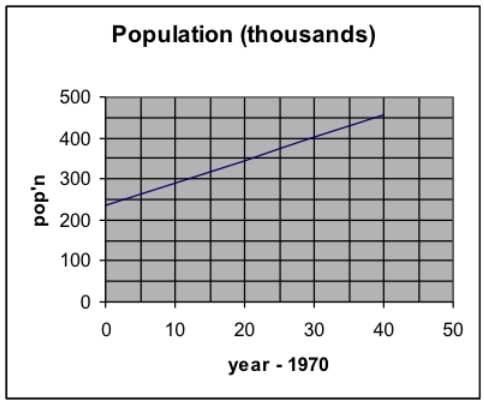
For 2010, $x=2010-1970=40$.

$$\begin{align}&y=5.55x+234\\&y=5.55(40)+234\\&y=222+234\\&y=456\\\end{align}$$

So, in 2010, the model predicts that the population will be 456 thousand people.

8. Sketch a graph from the original problem information, use it to estimate the answers to the prediction questions, and determine whether your answers from the algebraic formula are reasonable.

We take the same graph as before and extend it to the right so that we can estimate a population value for x = 40.



The value we see from the graph is approximately 450 thousand people. That is consistent with the value we obtained from the formula.

The prediction is reasonable.

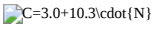
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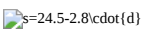
20.7: H1.07- Interpreting the Slope and Intercept

Students frequently have difficulty interpreting the slope and intercept in terms of the variables and units in the problem. Following are additional examples of interpretations and comments.

Example 5. When cigarettes are burned, one by-product in the smoke is carbon monoxide. Data is collected to determine whether the carbon monoxide emission can be predicted by the nicotine level of the cigarette. It is determined that the relationship is approximately linear when we predict carbon monoxide, *C*, from the nicotine level, *N*. Both variables are measured in milligrams. The formula for the model is 

Interpret the slope: If the amount of nicotine goes up by 1 mg, then we predict the amount of carbon monoxide in the smoke will increase by 10.3 mg.

Interpret the intercept: If the amount of nicotine is zero, then we predict that the amount of carbon monoxide in the smoke will be about 3.0 mg.

Example 6. Reinforced concrete buildings have steel frames. One of the main factors affecting the durability of these buildings is carbonation of the concrete (caused by a chemical reaction that changes the pH of the concrete) which then corrodes the steel reinforcing the building. Data is collected on specimens of the core taken from such buildings, where the depth, *d*, of the carbonation, in mm, and the strength, *s*, of the concrete, in mega-Pascals (MPa,) are measured. It is found that the model is 

Interpretation of the slope: If the depth of the carbonation increases by 1 mm, then the model predicts that the strength of the concrete will decrease by approximately 2.8 MPa.

Interpretation of the intercept: If the depth of the carbonation is 0 mm, then the model predicts that the strength of the concrete is approximately 24.5 MPa.

Comments: In this model, notice that the strength decreases as the carbonation increases, which is shown by the negative slope coefficient. When you interpret a negative slope, notice that you must say that, as the input variable increases, then the output variable decreases.

Notice that it isn't necessary to fully understand the units in which the variables are measured in order to correctly interpret these coefficients. While it is good to understand data thoroughly, it is also important to understand the structure of linear models and to be able to practice this on applied problems, even if they are not problems in your field.

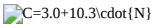
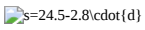
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20.8: H1.08- Exercises

Part I.

1. During the first and second quarters of a year, a business had sales of $42,000 and $58,000, respectively. If the growth of sales follows a linear pattern for the next four years, what will sales be in the fourth quarter? In the 9th quarter? Use an algebraic method of solution.
2. Ajax Manufacturing bought a machine for $48,000. It is expected to last 15 years and, at the end of that time, have a salvage value of $7,000. Set up a linear depreciation model for this machine and find the worth at the end of 10 years.
3. For a certain type of letter sent by Federal Express, the charge is $8.50 for the first 8 ounces and $0.90 for each additional ounce (up to 16 ounces.) How much will it cost to send a 12-ounce letter?
4. Write a mathematical model for the population of this city over the given period of time and use that to predict the population in 2010.

Year	1970	1980	1990	2000
Pop'n (thousands)	234	289.5	345	400.5

5. When cigarettes are burned, one by-product in the smoke is carbon monoxide. Data is collected to determine whether the carbon monoxide emission can be predicted by the nicotine level of the cigarette. It is determined that the relationship is approximately linear when we predict carbon monoxide, *C*, from the nicotine level, *N*. Both variables are measured in milligrams. The formula for the model is  .
   a. Interpret the slope.
   b. Interpret the intercept.
6. Reinforced concrete buildings have steel frames. One of the main factors affecting the durability of these buildings is carbonation of the concrete (caused by a chemical reaction that changes the pH of the concrete) which then corrodes the steel reinforcing the building. Data is collected on specimens of the core taken from such buildings, where the depth, *d*, of the carbonation, in mm, and the strength, *s*, of the concrete, in mega-Pascals (MPa,) are measured. It is found that the model is 
   a. Interpret the slope.
   b. Interpret the intercept.

Part II.

In addition to learning to work applied problems, one point of this lesson is to increase your ease and flexibility in reading and working problems. For that reason, some of the problems are stated somewhat differently than problems in the examples. Ask questions as needed.

Notice that many of the following problems have parts labeled with letters. In your solution, you are required to label each of the parts of your solution with the appropriate letter that shows which question you are answering.

Sometimes students prefer to work problems using exactly the same steps as the examples, numbering the steps in the same way. If you would prefer to work the problems in that way, then first do that, numbering the steps just as in the examples. AFTER you have completed that, then re-read the problem and go back and put additional labels onto your solution, to indicate where each part of the stated problem is solved.

7. A college had linear growth in enrollment over the period from 1993 – 2003. In 1997 they had 6754 students enrolled and in 2000 they had 8117 students enrolled. If the same pattern in growth continues, how many students do you expect they will have in 2010?
   a. Let *t* = number of years since 1993. Let *E* = enrollment. Make a table by hand or in a spreadsheet that shows the relationship of E to t over the given period of time.
   b. Graph the relationship.
   c. Write a linear model for the relationship of *E* to *t*.
   d. Interpret the slope and *y*-intercept in the formula, using the units in the problem.
   e. Use the formula to predict the enrollment in 2010.

f. Use your graph to predict the enrollment in 2010.
g. New question—not stated in the original problem: Approximately when do we expect the enrollment to be 11,380 students? Use either your graph or formula to do this. Explain which you used and how you did it.

8. A person deposits a certain amount of money in an account that pays simple interest.Thus the amount of money in the account at any time is a linear function of time. After 2 months, the amount in the account is $759.After 3 months, the amount in the account is $763.50.
   a. Do all the steps necessary to find a linear model to relate the amount in the account, *y*, to the number of months, *x*. Be sure to interpret the slope and intercept.
   b. Use your formula for the linear model to find the amount that will be in the account after 36 months.
   c. Use a spreadsheet to make a table of the amount in the account after each month up to 36 and check the answer you obtained using algebra.
9. One can measure temperature in degrees Celsius or degrees Fahrenheit. The two measurements are linearly related. The temperature at which water freezes is 0 degrees Celsius and 32 degrees Fahrenheit. The temperature at which water boils is 100 degrees Celsius and 212 degrees Fahrenheit. We want to predict the temperature Celsius.
   a. Let C = degrees Celsius and F = degrees Fahrenheit. Do all the steps necessary to find a linear model to describe this relationship.
   b. Interpret the slope and y-intercept of equation, using the units in the problem.
   c. When the temperature is 72 degrees Fahrenheit, what is the Celsius temperature?
   d. Using your formula, make a table that shows the temperature Fahrenheit from –20 degrees to 120 degrees and the predicted Celsius temperature for each of these. If you can use a spreadsheet for this, produce a table that goes in increments of 1 degree. If you are doing it by hand, produce a table that goes in increments of 10 degrees.
   e. Graph this relationship.
10. In 1991, the number of outlet shopping centers in the US was 142. By 1993, the number had increased to 249. (Hint: Let *t* = years since 1991 and then solve the problem using *t*.)
   a. If the number of outlet shopping centers continued to increase in a linear pattern, what would have been the number in 1994? Find a linear model and use it to make the prediction.
   b. In fact, the actual number of outlet shopping centers in 1994 was 300. Do you think that the increase in the number of outlet shopping centers was approximately linear?
   c. Make a graph of the data given in the initial set-up of the problem. Draw the line. Extend the line to 1994. What does your line predict the number of outlet shopping centers will be in 1994? Does that agree with the prediction your formula gave in part a?

For problems 11–14, write an algebraic linear model, interpret the slope and *y*-intercept, use the linear model to answer the question, and make a graph to check your work.

11. A cellular phone company has equipment that can service 80 thousand customers. In 2000 they had 57 thousand customers and, over the last few years, they have been adding about 3,000 customers per year. How many customers will they have in 2006? If this rate of increase continues, when will they need additional equipment?
12. The manager of a supermarket finds that she can sell 1130 gallons of milk per week at $3.99 per gallon and 1470 gallons of milk per week at $3.79. Assume that the sales, s, is a linear formula of the price, p. How many gallons would she expect to sell at $3.92 per gallon? (Hint: When you interpret the slope, it may seem strange to you. You might want to re-work the problem using cents instead of dollars for the price. That will make it easier to understand the interpretation of the slope.)
13. At 680 Fahrenheit, a certain species of cricket chirps 124 times per minute. At 400 Fahrenheit, the same cricket chirps 86 times per minute. Assume the chirps per minute, *C*, is linearly related to the temperature Fahrenheit, *T*. How many times per minute will the cricket chirp at 700 Fahrenheit? If you count the cricket chirps for a minute and find that it is 110 chirps, what is the temperature, to the nearest whole degree?
14. A bicycle manufacturer has daily fixed costs of $1500 and each bicycle costs $80 to manufacture. What is the cost of manufacturing 16 bicycles in a day? How many bicycles could be manufactured in one day for $3220?
15. Recall the example from Topic B. Formulas on “break-even” analysis. A company sells a catnip cat toy for $3 each and sells all that are produced. The fixed cost of production is $6000 and the variable cost is $1.20 each.
   a. Write a formula for the cost of producing *x* toys. [*ANSWER:Cost* = 6000 + 1.2 *x*]
   b. Write another formula for the revenue produced by selling *x* toys. [*ANSWER:Revenue* = 3 *x*]

c. Use a spreadsheet to graph both formulas.
d. Look at your graph to find the value of x for which the cost is equal to the revenue? (The point on the graph is called the "break-even" point.) [*ANSWER:* Breakeven is where cost and revenue lines cross.]
e. Use algebra to find the x-value of the break-even point and then use one of the formulas to find the y-value of the break-even point. Write your conclusion in a sentence.

16. A contractor purchases a piece of equipment for $65,000. The operating cost is $4.63 per hour (electricity, maintenance, etc.) and this year the operator is paid $15.37 per hour. That means that the total operating cost is $(4.63 + 15.37) per hour.
    a. Find a formula for the cost of the running the machine, C, where the input variable is t = number of hours run. (Be sure to include the purchase cost as the amount it will cost to run it zero hours. Do you see why that makes sense?)
    b. If the product generated by the machine in per hour is sold for $50, write a formula for the revenue, R, where the input variable is the number of hours run.
    c. If the machine is run for 1500 hours, what will the cost of running it be?
    d. If the machine is run for 1500 hours, what will the revenue from running it be?
    e. Using a spreadsheet, graph both the cost and the revenue formulas on the same axes. (You will need to decide on an appropriate set of t-values to use for your graph. You may need to do some and then extend it to more t-values.)
    f. For what t-value do the graphs of the formulas intersect? (Find it on the graph. Then use algebra to confirm that you have found it correctly.) Write a sentence to answer the question "How long will they have to run the machine to 'break even,' meaning that the revenue will equal the total cost?"

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CHAPTER OVERVIEW

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21.1: I1.01- Overview

Topic I—Linear and Quadratic Models

Objectives:

1. Recognize when a dataset shows a relationship between the variables that is approximately linear.
2. Use a spreadsheet to adjust the intercept and slope parameters of a linear formula so that the graph of corresponding points on the resulting line are close to the points graphed from a data set.
3. Use linear formula that best fits the data as a model for the data, predicting the output *y* value for any specified input *x* value.
4. Recognize when a dataset shows a relationship between the variables that is approximately quadratic.
5. Use a spreadsheet to adjust the location and scale parameters of a quadratic formula so that the graph of corresponding points on the resulting parabola are close to the points graphed from a data set.
6. Use the quadratic formula that best fits the data as a model for the data, predicting the output *y* value for any specified input *x* value.
7. Distinguish between appropriate and inappropriate extrapolation of a model.

Overview

In previous topics we have dealt with numbers produced from formulas, and separately with datasets showing the relationships between two variables. Now we are going to combine these perspectives and find formulas that approximately match the relationship between variables. Such formulas are *models* of the measurement data, and their graph will pass close to the data points.

The model formula is used to predict output values. In this topic we will examine models that are *linear* (that is, their graphs are straight lines), as well as one kind of non-linear model.

The models will not match the data exactly. There will always be some *noise* due to unavoidable random errors in the data-measurement process. Also, sometimes the actual pattern underlying the data will not match the model's formula (e.g., if the data has a curved graph and the model is a straight line). In that case even the best linear model will have to go above the data in some areas and below it in others.

Just as we computed deviations from the average when we analyzed the noise in repeated measurements, we will compute deviations from the model when we are trying to decide how well a particular model fits a dataset. A standard deviation based on these deviation values will be a numerical measure of how good the model is. We can also at the deviations to see if the model is too simple, since in an over-simple model most adjacent deviation values will have the same sign, positive or negative (in the correct model, the data will be randomly above or below the model values).

The data variable you want your model to predict should be used for the output *y* values in the dataset. Thus the other variable should be used for the input *x* values. Occasionally it is reasonable to also make use of the *inverse* model, where the role of the data variables is reversed and the second variable is used to predict the first one. If a model is linear, the inverse model for that data is also linear.

Note that which data variable is modeled as output can be different for people with different goals. One person might want to use temperature measurements to predict how long a metal bar will be, while someone else might to use the measured length of the bar to estimate what the temperature is. Both people could use the same set of calibration data, but would assign different *x* and *y* roles to the data variables when they make their predictive models.

In this topic we will focus on two simple models (linear and quadratic formulas), but the techniques shown will work in almost exactly the same way for fitting any kind of mathematical model to data. Some other useful models will be discussed in later topics.

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21.2: I1.02- Section 1

Section 1: Graphing data and model together

In order to find a good model for a dataset, we need to be able to compare the actual data values with the predictions of the model. This can be done by applying the model formula to each of the *x* values in Column A to compute "model *y*" values that are placed in Column C next to the corresponding "data *y*" value in Column B. Then a scatter plot that is made with all three columns selected will show both the data and the model predictions, in different colors. For a good model, the two kinds of points will be close to each other, although the data points will usually also include some random noise.

Example 1—Adding a model to a dataset and setting up a comparison graph

Input	*Output*
x	*data y*
0	6.6
1	9.3
2	9.2
3	11.5
4	12.9
5	15.2
6	14.4
7	17.5
8	19.3
9	19.8

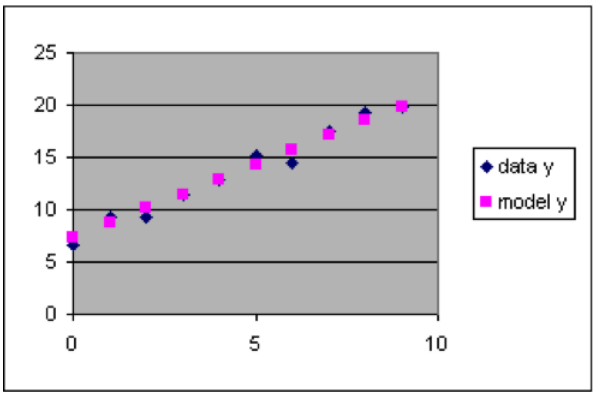
The dataset to the left has a relationship between *x* and *y* that is approximated by the linear formula $y = 1.4x + 7.3$. This formula can be used to compute *model y* values for each of the rows of the dataset, which we will put into column C next to the corresponding output *data y* value so that we can easily compare them.

1. Insert a new worksheet into a spreadsheet.
2. Copy the dataset so that the *x* and *y* values go into columns A and B, with the numbers starting in row 3. (That is, cell A3 will be 0 and B3 will be 7.41)
3. Put the labels "Model" into cell C1 and "model y" into cell C2.
4. Put the formula "=1.4*A3+7.3" into cell C3. (The result for C3 should be 7.3)
5. Spread the formula in C3 down column C to C12, next to all the data values in column B. (The results should be 8.7 for C4, 10.1 for C5, 11.5 for C6, etc.)
6. Select the rectangle from A2 to C12, and make a scatter plot.

When you have followed the steps listed above, you should have results that look about like this:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1	Input	Output	Prediction						
2	x	data y	model y						
3	0	6.6	7.3						
4	1	9.3	8.7						
5	2	9.2	10.1						
6	3	11.5	11.5						
7	4	12.9	12.9						
8	5	15.2	14.3						
9	6	14.4	15.7						
10	7	17.5	17.1						
11	8	19.3	18.5						

12	9	19.8	19.9						
13									



The graph above shows that *y = 1.4 x + 7.3* is a good model for this dataset, since the data points are close to the model points over the entire range of data, and the differences are randomly above and below the model.

But how did we know that the right model was y = 1.4 x + 7.3?

Good question. The example above shows how to *recognize* when a formula makes a good model, but does not show how to *find* a good model formula. What we really want is a method that will permit us to take any dataset that seems linear and quickly find what particular values for the slope and intercept parameters will make a linear formula that is a good model for that dataset (or a similar process for an appropriate nonlinear formula if the data pattern is not close to a straight line). The next section introduces a tool that provides an easy way to find the right parameter settings in a model formula.

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21.3: I1.03- Section 2

Section 2: Using a spreadsheet model to predict values

The benefit for having a good model formula for a relationship is that you can predict what output value should be expected for any input value, even if you have no data with that input value. This can be done simply by computing what *y* value the model evaluates to when the input value is used as the *x* value. For example, the model from the previous section predicts an output 21.3 for an input value of 10, because when 10 is substituted for *x*, the formula $y = 1.4\,x + 7.3$ becomes $y = 1.4 \times 10 + 7.3 = 21.3$.

Predictions are particularly easy to produce from a model if you already have the model formula entered into a worksheet similar to the one you made in the previous section. Whenever you enter a new *x* value into the cell in column A that is immediately below the last data value (such as A13), the spreadsheet automatically extends the formula in column C to that same row, showing the model's prediction for the new input value. (This process can be repeated for additional input values as needed. If your spreadsheet program does not automatically extend column C, you can spread the formula down by hand to get the same effect.)

Example 2: Using the worksheet from Section 1, predict the output (rounded to one decimal place) for these input values:
[a] x = 10 [b] x = 7.5 [c] x = -3 [d] x = 2.83 [e] x = 539 [f] x = -205[reveal-answer q="356906″]Show Answer[/reveal-answer]
[hidden-answer a="356906″][a] $y = 21.3$ [b] $y = 17.8$ [c] $y = 3.1$ [d] $y = 11.3$ [e] $y = 761.9$ [f] $y = -279.7$[/hidden-answer]

When adding values to columns A and C, what should be done to column B?

This is a natural question, since the empty cells (such as B13 and below) seem to leave a gap in the pattern of the worksheet. But **don't add to column B when predicting with the model**. You can make as many predictions with the model as you wish, *but you mustn't make up data*. Leave column B blank except for the data values that you are given to start with.

How reliable will the predictions of a model that closely matches the data be?

Notice that a model formula can be used both for *interpolation* for *x* values within the range of the data (such as in [b] and [d] above) and for *extrapolation*, in which the predicted point is for an *x* value that is outside the data range (such as in [a], [c], [e], and [f] above). Interpolation is generally dependable, although the noise in the relationship will usually keep it from exactly predicting future measurements. Extrapolation is less reliable, especially when the prediction being made is for a point that is very far away from the data on which the model is based. This is discussed in more detail in a later section.

Example of erroneous extrapolation *: For a given child, one can record the child's height and age in years. During the elementary-school years, that is a fairly linear relationship, with a growth rate of about 3 to 4 inches per year. But you would get very erroneous results if you use that model to estimate the typical height of 30-year-olds (who would be about 10 feet tall if the youthful pattern continued).*

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21.4: I1.04- Section 3 Part 1

Section 3: Using the *Models.xls* spreadsheet to find a linear model

The Models.xls spreadsheet available on the class web site is a template into which you can put any data values, then fit a mathematical model to closely match that data. It has all the needed formulas preset, including a formula in C3 that is based on the kind of model that is wanted (e.g., linear or quadratic). **You will use this or similar spreadsheets as your main tool in this topic and later modeling topics**, so it will save you a lot of time if you become skilled at using it.

Models.xls has multiple sheets, each of them preset to fit a particular type of model. In this section we will use the "Linear Model" sheet. Worksheets for Quadratic and Exponential are also included in Models.xls, and will be discussed in later sections and topics. You will find that there is very little difference in the process used for finding different models, so when you learn to use the Linear Model worksheet you will be able to quickly make use of any other model that turns out to be appropriate for the dataset you are fitting.

Steps for using the *Models.xls* spreadsheet to find a good linear model for a dataset:

1. Insert a new worksheet with the **Insert > Worksheet** menu choice, and label the tab at the bottom of this new sheet with an appropriate name (e.g., "Linear Model for Sediment Model").
2. Copy the contents of the worksheet labeled Linear Model (select it and then use a Ctrl-A, Ctrl-C, Ctrl-V sequence to paste a copy of its contents into the worksheet you inserted in step 1). (It will not work to use the Linear Model worksheet directly or to use the Edit > Move or Copy Sheet menu option, because the model-template worksheets are protected from change so that they are always available.)
3. Look at your data and **decide which column you want your model to predict**. That column of the data will be labeled as *y* values and compared to the output of the model. The other column will be labeled as *x* values, and will be used in the model formula to compute the model output. If you need to rearrange the columns to make the inverse graph or otherwise modify the dataset, use the Data Scratch Pad worksheet in Models.xls to get the data ready for step 4.
4. **Place the data into columns A and B**, starting the numbers at row 3 (you can put column labels in row 1 if you wish). Use column A for the input *x* data values and column B for the output *y* data values. Use as many rows as needed for the data; this may be different for different data sets.
5. **Spread the formula in cell C3 down** to as many rows as the data. (In the Linear Model sheet, C3 has been preset to "=$G$4*A3+$G$3", a linear formula that uses the value in cell G4 as slope and the value in cell G3 as intercept.)
6. **Spread the formulas in cell D3 and E3 down** beside the data and model rows. The formula "=B3-C3" in D3 computes the difference between the data and the model (called the *residualdeviation*), and the formula in E3 computes the square of that deviation. These column E values are used to compute (in G13) a numerical average of how well the model fits the data, which is called the *standard deviation*.
7. **Make a graph of the data and model together**. Select the data rows in columns A, B, and C, then select the Scatter Plot option from the Chart menu. The data and model points will be different colors, with one of each kind at each horizontal position.
8. **Adjust the "parameter" values at the top of column G to move the model close to the data**. Each type of model has different parameters (e.g., intercept and slope for linear models), but a similar process of adjustment can used in all cases.
9. **Decide if the model used is appropriate for the dataset**. A good model will come close over the whole range of input values, with deviations due mainly to noise in the data. If the best parameter settings cannot produce a good model (e.g., a line model fitted to parabolic data), then that model should not be used for predictions. A different model (e.g., a quadratic one) should be tried.
10. **Write the mathematical formula for this set of model parameters** (e.g., $y = 1.4\,x + 7.3$).

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Example 3: Using *Models.xls* to fit a linear model to a dataset

The table to the right gives data on the measured rate at which sediment built up in a factory holding tank during routine operation, after a cleaning process that is repeated a couple of times per year. The factory operators want to use this information to make a formula to predict sediment depth at any chosen time after a cleaning.

A preliminary graph of the data shows that the pattern of the points is reasonably close to a straight line. Therefore, the "Linear Model" worksheet in Models.xls is the appropriate one to use.

Days since cleaning	Depth (mm)
10	29.9
20	48.0
30	60.5
40	88.6
50	102.9
60	114.1
70	141.1
80	149.5

Solution:

In an earlier topic, you used a spreadsheet to adjust the intercept and slope of a linear equation and saw the resulting changes in the position of the straight-line graph. We will now use that same technique to make a good linear model for this data with Models.xls.

1. Insert a new worksheet into Models.xls, labeling its tab "Linear Sediment Model". Then copy into the new worksheet the contents of the read-only worksheet labeled "Linear Model Template".
2. In this case we want to predict sediment depth for any given number of days since the last cleaning. This means that we want to use day as the input variable x and depth as the output variable y.
3. Copy the data to the spreadsheet (columns A and B, rows 3 to 10 for the numbers), then label the top of the data columns with "Days" in A2 and "Depth" in B2.
4. Select C3 (which contains a preset linear formula based on the values in G3 and G4) and spread the formula down to row 10, matching the data. At first, these model values will be zeros.
5. Also select and spread D3 and E3 down to row 10. The values in columns D and E will not be very meaningful until you adjust the model to be a good fit.
6. Make a scatter plot of the data and model columns together (that is, the rectangle A1:C10). At first, the model points will lie on a horizontal line along the x-axis.
7. Adjust the parameters in G3 and G4 so that the model points are as close as you can get them to the data points.

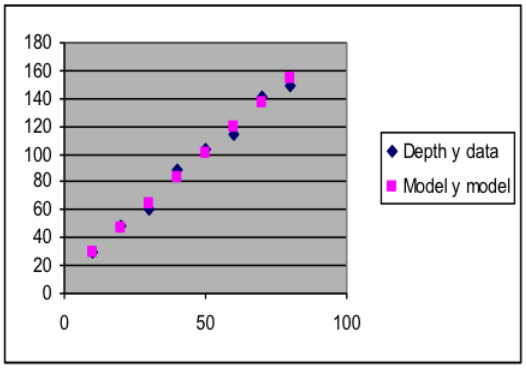
For a linear model, here is a good parameter-adjustment strategy:

1. **Set the intercept to approximately where the data trend crosses the y axis** (about 10 in this case, although you do not need to be exactly right since you will adjust the intercept again in [c] below),
2. **Adjust the slope to make the model line parallel to the data trend** (in this case, 1 is too low a value for the slope, and 2 is too high; 1.8 seems about right).
3. **Now adjust the intercept to its best value**, moving the model line without changing its slope until the model goes right through the data (in this case, a value of 11 for the intercept works well).

8. Check to see if the model is good. In this case, the model points are close to the data points over the whole data range, showing that a linear model is the correct type to use for this data.
9. Write the mathematical formula for the model you have found: ***y = 1.8 x + 11***

The Linear Sediment Model worksheet should look about like this at the end of this solution process:

	A	*B*	*C*	*D*	*E*	*F*	*G*	
1	x	y data	y model	Data-Model			***Linear model: y = m * x + b***	
2	Days	Depth	Prediction	deviation			***y = 1.8x+11***	
3	10	29.9	29	0.9			11	***b:* Intercept**
4	20	48.0	47	1			1.8	***m:* Slope**
5	30	60.5	65	-4.5				
6	40	88.6	83	5.6				
7	50	102.9	101	1.9				
8	60	114.1	119	-4.9				
9	70	141.1	137	4.1				
10	80	149.5	155	-5.5				
11								
12								
13								



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21.6: I1.06- Section 4

Section 4: Adjusting the input variables to simplify the intercept parameter of linear models

For the sediment data, both parameters for the model have natural meanings: the slope is the daily rate of sediment build-up, and the intercept is the sediment level immediately after cleaning. This is because the zero point of the input parameter, days since cleaning, has a meaning that is naturally related to the situation—zero corresponds to the date the cleaning took place.

However, sometimes the zero point for an input parameter is artificial, and has no natural relationship to the situation. When this is true, the intercept of a linear model for that data will not have a useful meaning. If the input parameter for a dataset of a company's annual sales is the calendar year, for example, then the intercept of a linear model fit to that data will be the model's "prediction" for the year 0, over 2000 years ago. This will probably be a very high or very low number that would be almost impossible to find just by making guesses to adjust the intercept parameter of the model. Even if found, the resulting model formula would be difficult to use because of the very large value it contains.

There are a couple of ways to avoid this problem. The one shown below, changing the input variable from "Year" to "Years since 1990", is the same technique you used in an earlier topic on graphing. When the input is redefined in this way, it becomes as simple to find the model as it was for the sediment data. In a later topic we will show how to adjust the model itself to give the same effect.

Example 4: *Find a good linear model for this data, redefining the input parameter as needed, and use the model to predict sales in 2010.*

Solution:

1. Redefine the input parameter to "Years since 1990", since that will cause the beginning of the data to have an input parameter of zero. To do this, copy the data from the table to a scratch-pad worksheet, make a column containing the Year data with 1990 subtracted, and copy the modified table to a copy of the linear model template.
2. Since the first row now has a zero input variable, use the Sales figure for that row, 453, as the initial setting for the model intercept parameter in cell G3.
3. Adjust the slope parameter in cell G4 until the model points are parallel to the trend of the data. A slope value of 26 is about right, but other nearby values would also be okay if the line looks correct.
4. Readjust the intercept value to ensure that the model goes through the middle of the data. Increasing it to 460 improves the fit (but again, a nearby value is okay if the graph shows a good fit).
5. The implied linear model is $\boldsymbol{y = 26\,x + 460}$.
6. Thus the prediction of the model for sales in 2010 (20 years after 1990) is **980**.

Year	House Sales
1990	426
1991	517
1992	500
1993	558
1994	611
1995	558
1996	601
1997	596
1998	683
1999	693
2000	708
2001	761
2002	771
2003	831
2004	897
2005	822
2006	889

To see what trouble was avoided by redefining the input variable, use this model to "predict" sales in the year 0, which of course is 1990 years before 1990. Since $26\cdot(-1990) + 460 = -51{,}280$, the corresponding sales model for the unmodified input values would be $y = 26\,x - 51{,}280$, whose intercept value would be very difficult to guess.

An alternate way of dealing with this same problem is to modify the model formula itself, substituting $(x - 1900)$ for x. If this were done, then the same model would be $y = 26 \cdot (x - 1990) + 460$. This approach has the advantage of not requiring a change in the data, which means that the horizontal scale on the graph would show the year number. On the other hand, the model would have a more complicated formula. Both techniques are used. We will examine this and other ways of modifying the model formula in a later topic.

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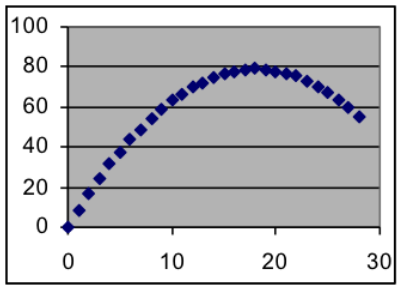
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21.7: I1.07- Section 5 Part 1

Section 5: Using *Models.xls* to find a quadratic model

time	height
x	y
0	0.0
1	8.6
2	16.8
3	24.4
4	31.5
5	37.4
6	43.8
7	48.9
8	54.4
9	58.9
10	63.3
11	66.5
12	69.7
13	72.2
14	74.5
15	76.2
16	77.8
17	78.1
18	79.0
19	78.5
20	78.0
21	76.9
22	75.6
23	73.0
24	70.3
25	67.3
26	63.7
27	60.0
28	55.0

If we wish to find a model for the data on the right, this quick scatter plot of the data shows that a straight line will not be sufficient. Such parabolic shapes (similar to the path of a thrown ball) is instead represented mathematically by a *quadratic* formula, in which the term that contains the input variable x is squared.



There are several ways to write a quadratic formula, all of which can make the same curves. For use in fitting data, the best kind of quadratic formula is $y = a\ (x - h)\ 2 + v$, where the parameters h and v are the x and y coordinates of the *vertex* of the parabola (that is, its highest or lowest point), and a is a "shape" parameter that determines how sharply (and in which direction) the parabola bends.

The quadratic formula pattern that is most convenient for fitting models: $y=a\cdot{{(x-h)}^{2}}+v$

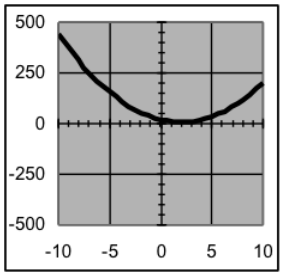
a is a "shape" parameter controlling how much (and in which direction) the parabola bends

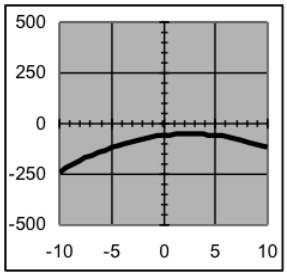
h is the x coordinate of the vertex (its horizontal distance from the origin)

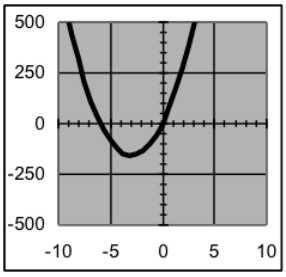
v is the y coordinate of the vertex (its vertical distance from the origin)

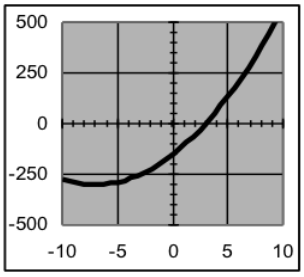
Examples: $y=3\cdot{{(x-2)}^{2}}+8$ $y=2.5\cdot{{(x-7)}^{2}}-33.4$ $y=-2\cdot{{(x+3.5)}^{2}}+37$

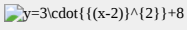
Examples of graphs of various quadratic formulas

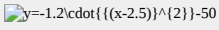


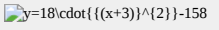


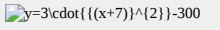












Example 5: *For each of the formulas above, state the location of the vertex of the parabola formed.*

Solution: Since the vertex is at (h,v) when a formula is expressed in the form, the coordinates for the vertices are: **(2, 8) (2.5, −50) (−3, −158) (−7, −300)**

Note that the sign of the x vertex coordinate is the opposite of the sign that the same number has in the formula, since the h value is subtracted when forming the formula.

Quadratic models are somewhat more complicated than linear ones, as is indicated by the fact that a quadratic model has three parameters instead of two. But there is really very little difference in the fitting process from what is done for straight lines: [1] put the data in the appropriate worksheet, [2] spread the C3:E3 formulas down beside the data, [3] make a graph and adjust the vertex (instead of the intercept) and the shape (instead of the slope) until the model and the data match, and [5] write down the formula or use it to predict any values you have been asked for.

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21.8: I1.08- Section 5 Part 2

Example 6: Fitting a quadratic model

The height of a basketball above the floor is measured at 0.1-second intervals from when it leaves the player's hand until it reaches the basket, with the results shown to the right.

1. *Find a quadratic model whose height predictions closely match this data.*
2. *How long after the shot begins does the ball reach the highest point in its path?*
3. *How high is the basketball above the floor at the highest point in its path?*
4. *How high was the basketball at 0.25 seconds into the shot?*

Time (secs)	Height (feet)
0.0	7.2
0.1	10.6
0.2	12.8
0.3	13.9
0.4	13.8
0.5	12.5
0.6	10.0

Answer:

We will put this data into a copy of the Quadratic Model worksheet of Models.xls, then adjust the model parameters until the model fits the data well.

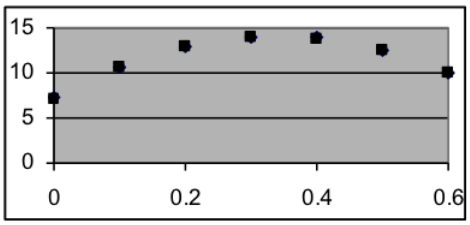
1. Insert a new worksheet into Models.xls, labeling its tab "Basketball height model". Then copy into this new worksheet the content of the read-only Quadratic Model Template worksheet.
2. Change the label in A2 from "Input" to "Time" for this data. Change B2 from "Output" to "Height".
3. Select the numbers from the dataset (not the labels), and paste them into A3. For this data, the numbers should fill columns A and B from row 3 to row 9.
4. Select cells C3, D3, and E3. Then spread them (and the formulas they already contain) down to row 9, matching the data. All three columns should now show numbers.
   Make a scatter plot of columns A, B, & C. This will show the data and the model on the same graph. At first, the model points will be on a horizontal line through the origin, but they will move as the quadratic model parameters (in G3, G4, and G5) are adjusted.
5. Adjust G3, G4, and G5 to make the model approximately match the data.
   1. Set the shape parameter G5 to a positive number (start with 1) if the data curves upward, and to a negative number (start with –1) if the data curves downward. Adjust G4 to make the bend of the model similar to the bend in the data (larger numbers cause sharper bends), but don't try to match the shape exactly until step [iii] below.
   2. Estimate x and y of the vertex of the data. Set G3 to the estimated x value and G4 to the y value.
   3. Now adjust G5 until the shape of the model is as close as possible to the shape of the data.
   4. Finally, adjust G3 and G4 to shift the model as needed so that it goes right through the data.
6. Now that you have a good model, you can add additional input values (e.g. 0.25 into A10), then read the result from column C (spread the C9 formula down if a value does not appear in C10. (Because there is no data value for 0.25, B10 should remain blank and D10 and E10 should be erased.)

When the instructions above are followed for this data, the result should be approximately the spreadsheet shown below. This spreadsheet provides the information to answer the questions asked in the problem:

1. A quadratic model matching this data is $y=-59\cdot{(x-0.34)}^{2}+14$
2. The vertex of the graph is at x=0.34, so the ball reached its peak at 0.34 seconds into the shot.
3. The vertex of the graph is at y=14, so the height of the ball at its peak was 14 feet.
4. The model predicts that the ball will have a height of 13.5 feet at 0.25 seconds.

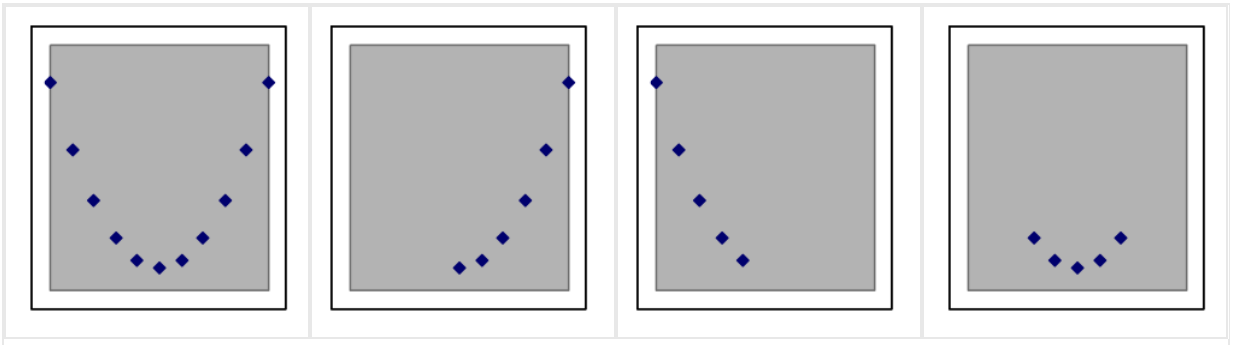
QUADRATIC-MODEL SPREADSHEET FILLED OUT WITH DATA FROM THIS EXAMPLE

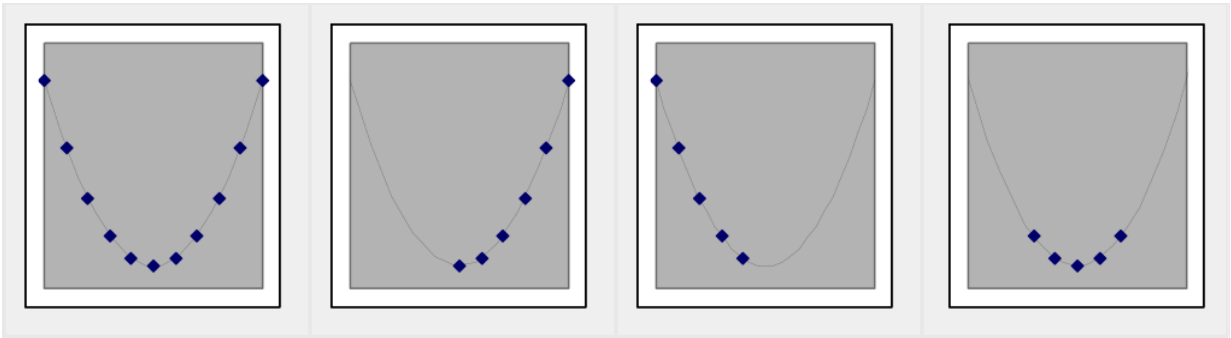
	A	B	C	D	E	F	G	H
1	x	y data	y model	Data-Model			Quadratic model: y = –$G$5*(x-$G$3)^2+$G$4	
2	Time	Height	Prediction	deviation			y = –59*(x-0.34)^2+14	
3	0.0	7.3	7.1796	0.0204			**0.34**	x coordinate of vertex
4	0.1	10.7	10.6016	-0.0016			**14**	y coordinate of vertex
5	0.2	12.9	12.8436	-0.0436			**-59**	shape coefficient
6	0.3	14.0	13.9056	-0.0056				
7	0.4	13.9	13.7876	0.0124				
8	0.5	12.6	12.4896	0.0104				
9	0.6	10.1	10.0116	-0.0116				
10	0.25		13.5221					
11								
12								
13								



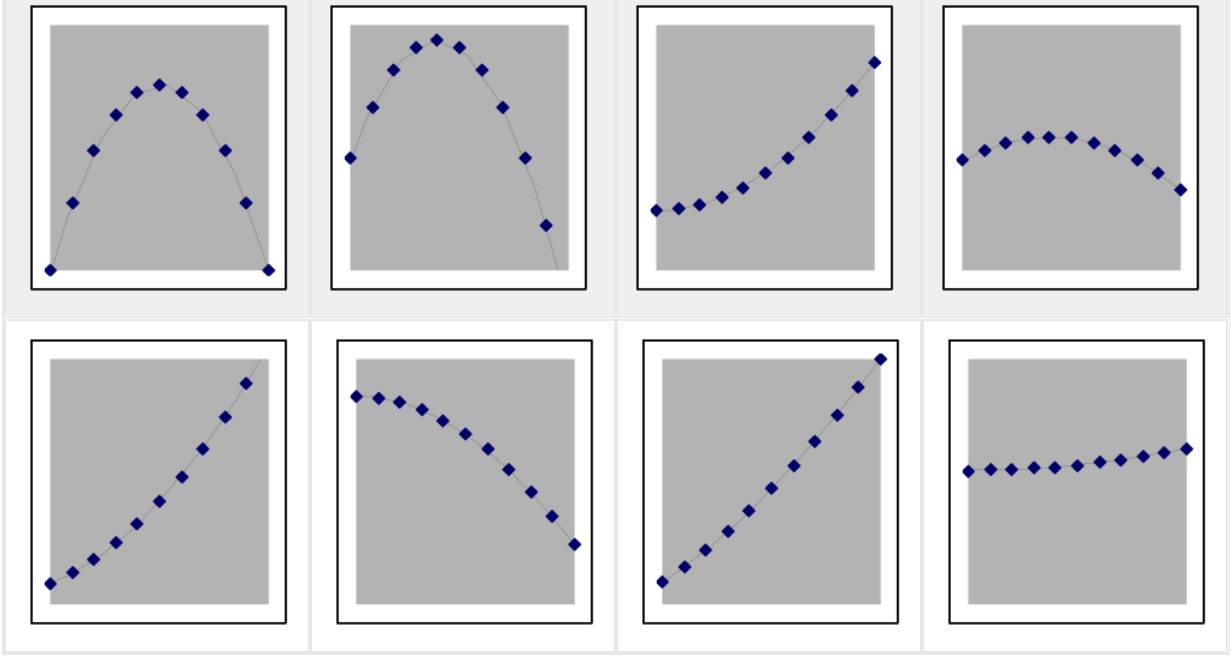
The shape of quadratic data can look different depending on where it is from on the parabola

A quadratic situation is easy to recognize when the data includes both sides of the peak or valley (as in the leftmost graph below), but if the data is all on one side you may need to look carefully to choose between a quadratic model and other nonlinear possibilities. Here are examples of how a quadratic model and data can match:





The above graphs are still quadratic if turned upside down, or uniformly stretched, compressed, or offset.
Each of the graphs below also shows data points from different quadratic models.



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21.9: I1.09- Section 6

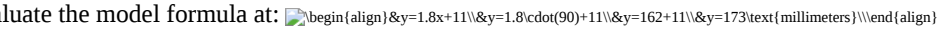
Section 6: Using models fitted with Models.xls to predict values

The main purpose of using data to make a model formula is that the formula can then be used to compute predictions of what the output y would be for any input x. This can be used to predict the future, to make inferences about the past (prior to the first data point), to find intermediate values between data points, and even to make a better estimate of what value you would get for the same measurement if you repeated it at one of the input values you already used.

Extrapolation

The $y = 1.8\,x + 11$ equation that fits the sediment-depth data well, for example, can be used to predict what sediment depth can be expected after day 80, the last day for which actual data was given. Using data to predict what measurements for a process will be outside the range of data input values is called *extrapolation*. ("extra" comes from a Latin word meaning "outside of").

Example 7: What sediment depth does the $y = 1.8\,x + 11$ model predict at 90 days after cleaning?

Answer: Evaluate the model formula at: 

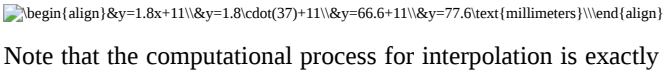
Warning: Extrapolation can be very useful, but is not always dependable since its accuracy depends on whether the process continues to change in the same way as it did during the time that the data for the model was taken. In general, people use extrapolation only for values that are within a limited distance from the last data point. It would be reasonable to extrapolate the sediment depth to 100 days, or perhaps even to 150 days, but not to 1000 days unless you have other information that indicates that the rate of increase is constant for that period.

Interpolation

The most dependable use of a model formula is to estimate what the output measurement would have been for some input value that is between two of the input values for the data. This process is called *interpolation*. Interpolation is dependably accurate if the model is a good model.

Example 8a: What sediment depth does the $y = 1.8\,x + 11$ model predict at 37 days after cleaning?

Answer: Evaluate the model formula at:

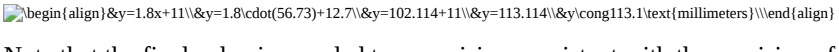


Note that the computational process for interpolation is exactly the same as for extrapolation. This is characteristic of the use of a model. You treat any input value the same way – just plug it into the formula and evaluate the result. You may decide you don't trust the answer (e.g., the 1000-day sediment-depth extrapolation), but the model gives answers the same way in all cases.

There is no need to limit interpolation or extrapolation to whole-number inputs when fractional inputs make sense. You could estimate the sediment depth at 22.5 days, or at 98.765 days. But use some judgment here – you would not want to estimate midnight traffic flow based on a history of measurements made at noon.

Example 8b: What sediment depth does the $y = 1.8\,x + 11$ model predict at 56.73 days after cleaning?

Answer: Evaluate the model formula at:



Note that the final value is rounded to a precision consistent with the precision of the data.

Backwards extrapolation

You could even compute the model's answer for an input value that comes before any of your data. That does not make sense for the sediment data (since we are told that the tank was changed abruptly by cleaning just before this data was taken), but in other situations it is often possible to make good estimates of what conditions were before data was taken.

Example 9: An accumulated coating of rust on the siding of a building is measured on June 1 for 15 successive years, and the these thickness measurements are found to fit a linear model, where y is the thickness in millimeters and x is the number of years since the first of these measurements in 1987. Estimate what the thickness of the coating was on June 1, 1980.

Answer: Evaluate the model formula at $x = -7$, since that corresponds to the year 1980 in the formula.

\begin{align}&y=0.085x+1.52\\&y=0.08\cdot(-7)+1.52\\&y=-0.56+1.52\\&y=0.96\text{millimeters}\\\end{align}

As with forward extrapolation, you have to use judgment about how far away from the data you can depend on backward extrapolation. Example 10's 7-years-before extrapolation is reasonable, but using this model to estimate the rust thickness in 1950 would not be reasonable, since that would give a negative thickness (which implies that the building may have been built after 1950).

What if the same data points are measured again?

For the sediment-depth data, the prediction of the model for sediment depth at 40 days after cleaning is 83 mm, which is 4.4 mm less than the actual data value of 88.6 mm. Which of these values would be best to use if we wanted to predict the depth at 40 days after some subsequent cleaning?

Deviations between data values and model predictions can come from two different kinds of sources:

- **Noise:** The deviations may just be random variations in the process, in which case we will do better to use the model, since next time the deviation is just as likely to be in the other direction. In a sense, the model is more accurate than the data in this case. The noise-suppressing smoothing effect that a model provides is an important benefit of the modeling approach.
- **Oversimplified models:** The relationship between the input and output variables for the process may not quite be a straight line, in which case even the best linear model will have errors that overestimate the data in some input ranges and underestimate it in others. If the linear model is oversimplified in this way, we will do better to use previous measurements. Better yet, we should use a non-linear model that has sufficient flexibility to follow the data more closely.

We can distinguish between these two cases by examination of the graph. If noise is the source of the deviations, the graph will show random placement of data points above and below the model. The deviations in the sediment-depth data were random, as shown by the graph from the fitting.

But when a straight-line model is used for data whose underlying relationship is curved, the model will pass above most data points in some part of the range, and below most points in other parts. In making predictions for such poor-model situations you should either use the previous measurements directly or, preferably, fit a more suitable model to the data and then use that model.

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21.10: I1.10- Section 7

Section 7: Looking at the data in the other direction—making an inverse model

Often it is obvious which of the variables from the data set you want to be predicted by the model you intend to fit to the data. But sometimes either choice makes sense, depending on what you are doing.

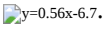
In the case of the sediment-depth data we started with the model that is appropriate if you want to predict the sediment depth at a specified time after cleaning. This model uses days as input and depth as output. But an inspector might decide to measure the sediment depth in order to estimate how long it has been since the last cleaning. He would need a model that uses depth as input and produces days as output. This can be produced in exactly the same way as before, but the two columns of data switch their roles.

Example 10: *Describe the model to estimate days since cleaning from measurements of sediment depth.*

Answer: Make a new copy of the spreadsheet with the data and the linear function settings, then:

1. Swap column A and column B, including the titles at the top, but still label A2 "*x*" and B2 "*y*".

2. Find good settings for the intercept and slope by the methods discussed earlier in this topic.

3. An example of a good model for this reversed data is  .

This fitted model is the *inverse* of the  model fitted earlier that used days as input and depth as output. This is clearer if names are used in the formulas rather than *x* and *y*:



In fact, we could have derived the inverse model from the original model by using algebra instead of model construction and have gotten almost the same answers (they might differ slightly because noise in the input data affects the fitting process somewhat differently than the same noise in the output data).

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21.11: I1.11- Section 8

Section 8: Systematically finding parameter values that fit a model to a dataset

The Models.xls workbook (or a similar workbook derived from it) provides a way for a **dataset** of *x* and *y* values (in columns A and B, respectively) to be matched to *y* predictions computed for that *x* value with a **model formula** (in column C). The model formula makes use of the values of a few **parameters** in column G. Changes to these parameters will thus change all the column C values. *For all types of model, the model-fitting process consists of finding values for the parameters which make the column C model values come as close as possible to the column B data values*.

The simplest way to find reasonable parameter settings is to make a scatter-plot graph of columns A, B, and C together, which will display the data and the model predictions on the same scale. When good parameter settings have been found, the two graphs will overlap as closely as possible. Once the graphs match well visually, the numerical differences between the data *y* value and the model *y* prediction (computed in column D) can sometimes be used to make fine adjustments to the parameter settings.

In general, even the best model will not fit the data exactly. If the deviations from the model are randomly above and below it, they represent noise in the data — in such a case, probably no other model would do better. On the other hand if most positive and negative deviations are grouped with several others of the same sign, this indicates the model does not fit the data well, perhaps because the kind of model formula being used cannot produce a shape similar to that of the data.

Summary of systematic curve-fitting techniques

- Make a graph showing the data and model points, so you can observe how good the fit is.
- First set the position parameters to approximate values, based on the kind of model:
  - Linear — estimate the intercept as the data *y* value for the *x* value closest to zero.
- (If no *x* value is close to zero, redefine the data or formula by subtracting the first *x* value)
  - Quadratic — estimate the vertex *x* and *y* parameters from the graph
- Estimate a beginning value for the other parameter from the graph or first two data points.
  - Linear slope—set to the difference of the *y* values divided by the difference in *x* values.
  - Quadratic shape—positive if the ends curve up, negative if the ends curve down
- Adjust the parameter values systematically to improve the initial estimates.
  - Change the first significant digit by 1, observing if this moves the model toward the data
  - Continue first-digit changes in the right direction until the model graph crosses the data.
  - Once the first digit is known, adjust the second digit.
  - Readjust the other parameter(s) to the same precision.

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21.12: I1.12- Exercises

Part I.

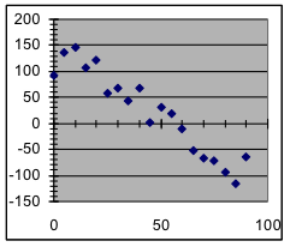
Reproduce the results in the Examples 1–10.

Part II.

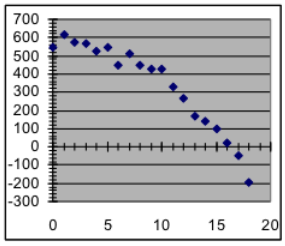
For each dataset in these problems, please do not type it in yourself, but find the text below on the course web page and “copy and paste” it into the spreadsheet. This will save you quite a lot of work.

11. Which of these graphs indicate data in which the variables have an approximately linear relationship?

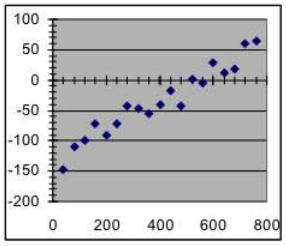
[a]



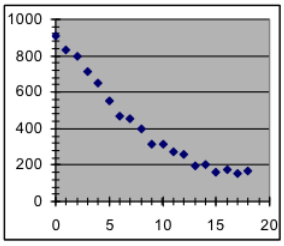
[b]



[c]



[d]



12. For which of the datasets graphed in the previous exercise will a linear model probably be a better predictor of future measurements than the corresponding measurements in the dataset?
13. A truck delivers loads of kerosene. The relationship between the gallons of kerosene it is carrying and the total weight in pounds of the truck follows a $y=6.23\text{}x+28540$ linear model, where x is the number of gallons and y is the weight. What is the expected weight of the truck when it is loaded with 5290 gallons of kerosene?
14. What linear model would be appropriate if you want to use the weight of the truck in the previous problem to predict how much kerosene it is loaded with?
15. Which of these datasets has a relationship between the variables whose trend is linear?

[a]

x	Y
30	-122.02
60	-112.01
90	-105.19
120	-92.76
150	-72.19
180	-62.13
210	-50.07
240	-37.11
270	-21.60
300	-12.85
330	-0.93
360	7.75
390	19.88
420	46.92
450	63.04
480	73.17
510	86.94
540	94.80
570	100.27

[b]

x	Y
-100	-182.39
-75	-154.87
-50	-141.15
-25	-122.83
0	-105.93
25	-87.78
50	-72.97
75	-60.00
100	-45.75
125	-32.09
150	-28.36
175	-18.14
200	-12.92
225	-0.25
250	-0.05
275	4.71
300	12.82
325	14.42
350	12.15

[c]

x	y
-400	461.43
-350	444.21
-300	401.07
-250	419.03
-200	348.53
-150	360.21
-100	275.67
-50	176.27
0	190.39
50	180.19
100	119.80
150	112.96
200	-19.65
250	41.88
300	-79.24
350	-72.22
400	-43.34
450	-125.61
500	-121.65

[d]

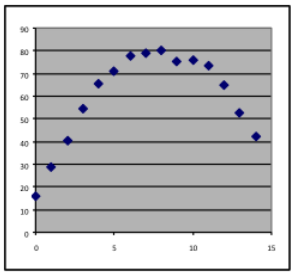
x	y
-45	-52.27
-40	-25.48
-35	-38.28
-30	-39.38
-25	-39.21
-20	-17.29
-15	-21.08
-10	-35.22
-5	-14.70
0	-15.89
5	1.54
10	-11.43
15	-24.28
20	-13.48
25	-20.92
30	-16.29
35	4.75
40	9.65
45	-7.65

[e]

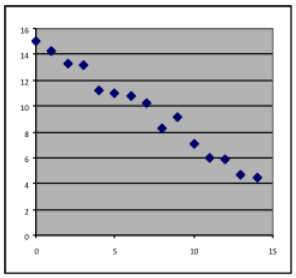
x	y
0	23.43
1	23.53
4	23.84
9	24.38
16	25.03
25	26.07
36	26.69
49	28.26
64	30.02
81	31.07
100	32.47
121	35.04
144	38.14
169	40.35
196	43.69
225	46.46
256	51.29
289	55.22
324	58.98

16. For each of the datasets in the previous problem that was identified as having a linear relationship, find and report a good linear model.
17. Use the model for dataset [a] above to predict the output y variable for input values of x at intervals of 100 from 0 to 600.
18. Use the model for dataset [e] above to predict the output y variable for input values of x at intervals of 50 from 0 to 350.
19. For each of the graphs below, identify where a linear or quadratic model would be appropriate, or whether neither of these. In each case, write a sentence stating what reason you have for the choice you make.

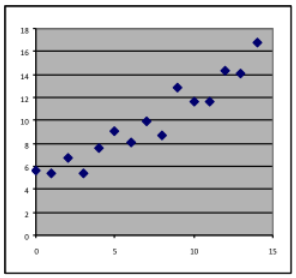
[19a]



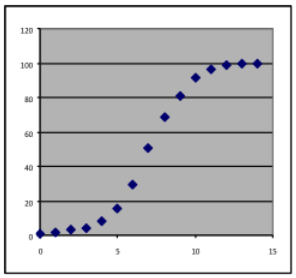
[19b]



[19c]



[19d]



year	Cable systems
1990	10,215
1991	10,704
1992	11,073
1993	11,108
1994	11,214
1995	11,215
1996	11,220
1997	10,943
1998	10,845
1999	10,700
2000	10,500

For each dataset in the following problems, please do not type it in yourself, but find the data text on the course web page and "copy and paste" it into the spreadsheet. This will save you quite a lot of work.

20. For the dataset to the right about cable systems:

    1. Identify the input and output variables for predicting cable system count.
    2. Restate the input variable in terms of years since the first year given.
    3. Use a spreadsheet to make a graph of the data and label the axes. (hand-labeling is okay)
    4. Use the quadratic-model spreadsheet to find a good quadratic model and state its formula.
    5. Using the model, predict the number of cable systems in 2003.
    6. What year does the model imply that the most cable systems existed?

MPG	MPH
17	10
25	20
30	30
32	40
31	50
28	60
22	70

21. For the dataset at the right about gas mileage (MPG) at different speeds (MPH):

    1. Identify the input and output variables appropriate for predicting mileage.
    2. Use a spreadsheet to make an appropriate graph and label the axes. (hand-labeling is okay)
    3. Use the quadratic-model spreadsheet to find a good quadratic model and state its formula.
    4. Using the model, predict the gas mileage for a speed of 25 mph.
    5. At what speed does the model imply that gas mileage is best?

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CHAPTER OVERVIEW

22: Topic J- Exponential Models and Model Comparison Techniques

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22.6: J1.06- Section 2 Part 1
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22.1: J1.01- Overview

Topic J—Exponential Models and Model Comparison Techniques

Objectives:

1. Recognize when a dataset shows an exponential relationship between the variables.
2. Use a spreadsheet to adjust the initial-value and slope parameters of an exponential formula so that the graph of corresponding points are close to the points graphed from a data set.
3. Use the exponential formula that best fits the data as a model for the data, predicting the output *y* value for any specified input *x* value.
4. Compare the extrapolation behavior of different types of model that fit the same data.
5. Choose among linear, quadratic, and exponential models based on a graph of the dataset.
6. Recognize the quality of a model based on whether positive and negative residual deviations are randomly distributed or grouped together.
7. Modify a model formula to simplify the fitting process without changing the dataset.

Overview: Additional tools for modeling

In an earlier topic, you learned how to find good model formulas for data whose pattern is a straight line or a parabola. In both cases, you used worksheets from Models.xls to find good parameters for the models. These worksheets were very similar, differing only in the formula placed in the C3 cell and in the names and meanings of the parameters in cell G3, G4, etc. Other models are just as easy.

In this topic we will add *exponential models*, useful when the output changes by the same percentage each step. We will also learn how to compare and choose among types of models. Finally, we will learn to modify the model formula when needed to make the graph or parameters more convenient.

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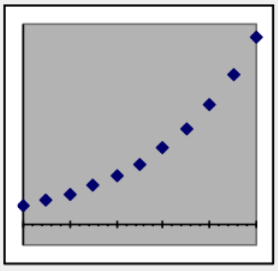
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22.2: J1.02- Section 1 Part 1

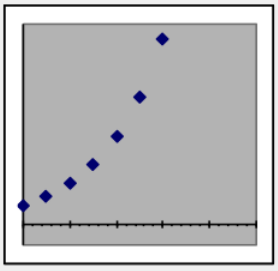
Section 1: Processes with constant-percentage growth/decay rates—*exponential* models

In situations where the cause of change in a quantity is the amount of that quantity that is currently present, the output variable changes by the same *percentage* for each equal step in the input variable. Such situations have an *exponential* model formula, which means that the input variable is used as an exponent in the formula. Accumulation of compound interest in a bank account is an example of exponential growth, and radioactive decay is an example of exponential decay. Both are modeled by exponential formulas, with the difference between growth and decay depending simply on whether the change rate is positive or negative.

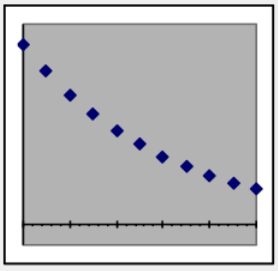
Relationship of different exponential models to the *x*-axis



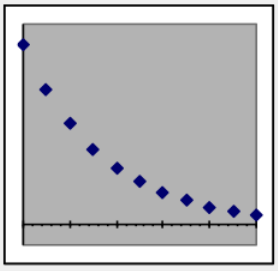
Exponential growth



Faster growth rate

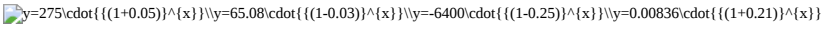


Exponential decay



Faster decay rate

Examples of exponential formulas:



Since any number raised to the power of zero equals exactly 1, the *y*-intercept of an exponential formula (i.e., the formula *y* value when *x* equals zero) will be just the multiplier term (such as 275, 65.08, 6400, and 0.00836 above). This intercept is one parameter of the exponential model, and reflects the starting value for processes that start at x=0.

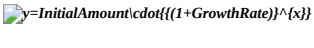
The other parameter reflects the rate at which the value of the formula will change. The simplest way of expressing it is to use the desired rate to make an appropriate base for the input parameter *x* as an exponent. This is the role of "1+0.05", "1-0.03", "1-0.25", and "1+0.21" in the examples above. The growth rates of 0.05, –0.03, –0.25, and 0.21 could instead be expressed as 5%, –3%, –25%, and 21%.

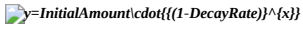
A negative growth rate is actually a decay rate, since it will result in values closer to zero as *x* becomes larger. When the growth rate is negative, the base of the exponent will be a number that is less than 1 but greater than 0. Thus the base for a –0.03 growth rate (that is, a 3% decay rate) would be 0.97, and the base for a –0.25 growth rate (i.e., a 25% decay rate) would be 0.75.

Warning: *An exponential decay rate can never be more than 100% (that is, have a growth rate of less than –1.0). Such a rate would not correspond to a realistic situation, and would make the base of the exponent a negative number. Spreadsheets will give error messages when negative numbers are used as a base in calculations with decimal exponents (unless the exponent is exactly a whole number).*

Exponential models include all 

forms such as:



or 

***InitialAmount* is the *y* value at *x* = 0**

Note that some data may have inputs starting at non-zero values (e.g., 1975), in which case it will usually make the model simpler if the inputs are redefined to start at zero (e.g., as "years since 1975") or if the model is modified to adjust for a non-zero starting value, as is illustrated in a later section.

***GrowthRate* is the relative increase in *y* when *x* increases by 1**

The growth rate is usually expressed in formulas as a decimal number but is also often described as a percentage (e.g., 0.05 is the decimal representing 5%). For exponential decay, the growth rate is negative but must be between zero and minus one (that is,

between 0% and –100%). A growth rate of zero would result in the output not changing from its initial value, since 1 raised to any power is still 1.

Often the formula will have the growth rate already combined with the 1, so that the growth term becomes (1.05)x or (105%)x. For negative rates, this term will be less than 1, such as (0.95)x or (95%)x.

Percentages must be changed to decimals when computing with them. Most spreadsheet programs will do this automatically if the percentage number is followed by a % character.

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22.3: J1.03- Section 1 Part 2

Example 1: Fitting an exponential model to find the growth rate

Use this data from the first 10 U.S. censuses to make an exponential model of U.S. population growth, then use that model to answer these questions: *[Use the Exponential Model worksheet in Models.xls.]*

a. Does an exponential model fit this data well? How do you know?
b. What was the average annual growth rate during this period?
c. What population does the model predict for 1900? For 2000?

Solution procedure:

[1] Redefine the input variable to "Years since 1780", adjusting the *x* values in column A to start at 0.

Year	Population (millions)
1780	2.8
1790	3.9
1800	5.3
1810	7.2
1820	9.6
1830	12.9
1840	17.1
1850	23.2
1860	31.4
1870	39.8

[2] Copy the redefined data values into a copy of the Exponential Model template. The numbers should fill rows 3 to 12 in columns A and B.

[3] Select cells C3, D3, and E3. Then spread them (and the formulas they already contain) down to row 12. All three columns should now show numbers. [The formula in C3 is "=$G$3*(1+$G$4)^A3".][2] Copy the redefined data values into a copy of the Exponential Model template. The numbers should fill rows 3 to 12 in columns A and B.

[4] Make a scatter plot of columns A, B, & C. This will show the data and the model on the same graph. At first, the model points will be on a horizontal line through the origin, but they will move as the exponential growth-rate model parameters (in G3 and G4) are adjusted.

[5] Adjust G3 (initial value) and G4 (growth rate) to make the model approximately match the data.

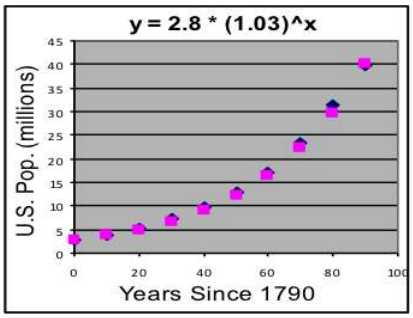
i. Set G3 to 2.8, the first value in the table. (We can later adjust this to reduce standard deviation).
ii. Set G4 to 0.01 (which is 1%), then adjust it until the shape of the model is close to that of the data at about 0.03 (which is 3%).

[6] Adjust the parameters using feedback from the graph until further adjustments do not significantly improve the fit.

EXPONENTIAL-GROWTH WORKSHEET FROM Models.xls FILLED OUT WITH EXAMPLE 1 DATA

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
1	x	y data	y model	Data-Model		Exponential model: y = a * (1+r)^x		
2	Year-1780	Height	Prediction	deviation			y = 2.8 * (1+0.03)^x	

3	0	2.8	2.80	0.0000			**2.8**	a: Initial value at x=0
4	10	3.9	3.76	0.1370			**0.03**	r: Growth rate
5	20	5.3	5.06	0.2429				
6	30	7.2	6.80	0.4037				
7	40	9.6	9.13	0.4663			*Model and data value counts*	
8	50	12.9	12.27	0.6251			2	Number of parameters
9	60	17.1	16.50	0.6035			10	Number of data points
10	70	23.2	22.17	1.0301				
11	80	31.4	29.79	1.6055			*Goodness of fit of this model*	
12	90	39.8	40.04	-0.2413			4.910062	Sum of squared deviat
13	120		97.19				**0.783427**	Standard deviation
14	220		1867.87					



Answers to questions asked:

[a] **Yes, this model fits the data well**, since the deviations are only a few percent of the typical population values.

[b] The model growth-rate parameter shows **the average annual growth rate of U.S. population from 1780 to 1870 was about 3.0%.**

[c] Evaluate the model at 120 (1900 is 120 years since 1780), and 220 (for 2000) to show that **a model based on the 1780-1870 census data predicts a U.S. population of about 97 million in 1900 and 1,868 million in 2000.** (Actual population was 76 million in 1900 and 281 million in 2000, showing that the rate of U.S. population growth slowed down substantially after 1870.)

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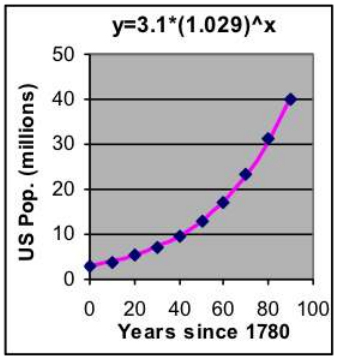
22.4: J1.04- Section 1 Part 3

Characteristics of exponential models

Doubling time and half-life: In an exponential model, equal steps in the input variable will always increase (or decrease) the value of the output variable by the same percentage. When the input variable is time, it is often useful to describe the process by how long it takes for the output to double (for growth) or decline to half (for decay) — these are called the "doubling time" or "half-life". These values can be estimated from the graph of the model by taking the y value in the model that is furthest from zero, drawing a horizontal line at half that height, then noting the difference between the x values of the original point and the point where the half-height line crosses the graph of the model.

Example 2: Estimate the doubling time of the model for the 1780-1870 U.S. census data.

Solution: The highest point on the graph of the model is (x=90, y=40.0), predicting a 1870 population (90 years after 1780) of 40.0 million people. Half that y value is 20.0, and we can see that the model graph crosses that value about x=65, halfway between the x=60 and x=70 data points. Subtracting 65 from 90 tells us that **the *y*values in themodel double in about 25 years**.



Comments on this solution: Notice that the model predicts a population of 10 million at about x=40, then 20 million at about x=65, then 40 million at about x=90 — this shows that the doubling time of 25 years is the same for different parts of the graph (this is true only for exponential models). Using the highest point graphed on the model makes it easier to estimate the coordinates (estimating the x position for y=5 would be harder, for example). For a decaying-exponential model, we still use the highest point but it is the first point on the left, so the time to the half-height point is a half-life rather than a doubling time.

Convergence to zero: Exponential decay models have a negative "growth" rate, so that at each step the output becomes smaller by a fixed percentage. This leads to output values that come closer and closer to zero but never quite reach it. But the difference from zero can quickly become small enough to be negligible for practical purposes—a process whose output value decreases by half every hour will be less than a ten-millionth of its original size a day later. Notice that if the initial value is negative, this convergence to zero is from below the x-axis, with increasing values of y that come closer and closer.

Example 3:*The intensity in Curies of a radioactive material that is used to make x-ray images of pipes is calibrated every 15 days, giving the dataset shown to the right. Fit an exponential model to the data to determine if this intensity exhibits exponential decay. If it does, find the decay rate of the material.*

Solution process:

1. Copy the data into the ExponentialModel.xls spreadsheet on the course website.
2. Make a scatter plot of the data and model (columns A, B, and C).
3. Set the InitialAmount parameter (in G2) equal to the first data *y* value.
4. Set the GrowthRate parameter to a small negative number (start with –0.02, which is a decay rate of 2% per day), then adjust this parameter so that the model matches the data well. We find that a growth rate of –0.0094 gives the best fit.

Answers:

Since the data fits the model well, the intensity exhibits exponential decay.

The growth rate of the model is –0.0094, a decay rate of 0.94% per day.

Example 4:*Estimate the half-life of the exponential decay process above.*

Solution process:

Examine the model graph (or the values in column C) to find when the output declines to half the initial amount of 160 Curies. This time is the half-life of the decay process.

Answer:

Since the intensity is 78.6 Curies at day 75, the half-life is about 75 days. (Note that the intensity at day 150 is about ¼ the original value, as it should be.)

Days	Curies
0	159.9
15	138.4
30	120.2
45	104.7
60	91.5
75	78.6
90	68.7
105	58.9
120	51.2
135	44.8
150	39.6
165	34.1
180	29.0
195	25.8
210	22.1
225	19.6
240	17.1
255	14.3
270	13.1

Unbounded growth: Exponential growth models become very large surprisingly quickly. This is because each increase causes later increases to be larger (since at each step the rate of increase depends on the current amount). An investment with a 7.2% annual return will double to 10 years, then redouble each decade to reach 1000 times the original value in a century. Under favorable conditions, bacteria can reproduce (and thus double their numbers) about every 30 minutes, leading to a million-fold increase in ten hours. The process by which scientists amplify the genetic material DNA so that it can be detected chemically doubles the number of DNA fragments every 2 minutes, leading in less than an hour to about 30 million copies of each original piece. Most explosions start with exponential growth, as each small reaction causes several others, which in turn each cause more, and so on. Notice that if the initial value is negative, the unbounded growth can be in a negative direction.

Such growth processes can not continue indefinitely. Although there are many natural processes that show exponential growth at certain stages, the steady build-up of the speed of exponential growth ensures that some limit (often exhaustion of some essential resource) is reached sooner or later. This should be kept in mind when modeling—data often show a pattern that will not continue.

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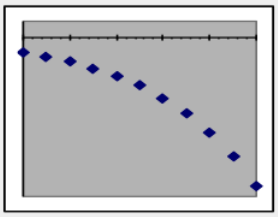
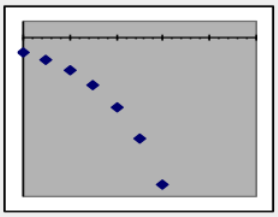
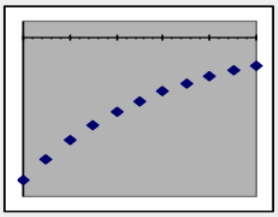
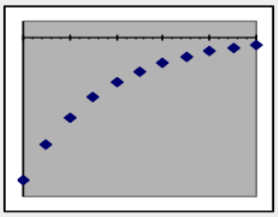
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22.5: J1.05- Section 1 Part 4

Negative output values in exponential models: When the initial amount is a negative number (this usually happens when the *y* value describes a difference from a reference value), the only difference is that all of the model's output values will be negative and the graphs will thus be reflected around the *x*-axis, "upside down" compared to models with positive initial values.

Relationship of exponential models with **negative initial amounts** to the *x*-axis

			
Exponential growth	*Faster growth rate*	*Exponential decay*	*Faster decay rate*

Example 5: *The table to the right shows the temperature of a soda can as it warms up in a room whose temperature is 75 ° F. Use an exponential model to: [a] estimate the decay rate per minute of the difference between the can temperature and the room temperature, [b] predict the can temperature at 20 minutes, and [c] estimate the half-life of the warming process.*

Solution:

Since the temperature of the can is not converging on zero, we can't directly use can temperature as a variable in an exponential-decay model. Instead, we will subtract the room temperature from the can temperature and use that *difference* as the output variable for the model.

Put this dataset into the exponential-model spreadsheet, then adjust the parameters to make the model fit the data. You will find that the data is well matched by an exponential model whose initial-value parameter is -40 and whose "growth" rate is -0.09 (which is really a decay rate since it is negative).

Original data

Minutes out of cooler	Temperature of can
0	35
5	50
10	59
15	65

Redefined data

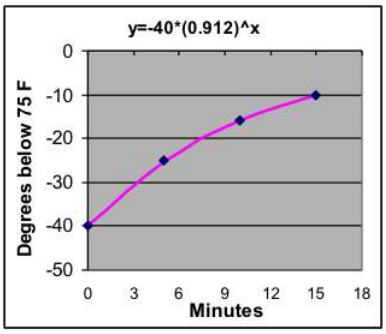
Minutes out of cooler	Degrees from 75° F
0	-40
5	-25
10	-16
15	-10

Evaluating the model for $x = 20$ gives a predicted difference at that time of 6.1 degrees below the room temperature of 75°F.

The data point whose *y* value is furthest from zero is the first one at (x=0, y=-40). The line halfway to the x-axis is at y=–20, which crosses the model graph at between 7 and 8 minutes.

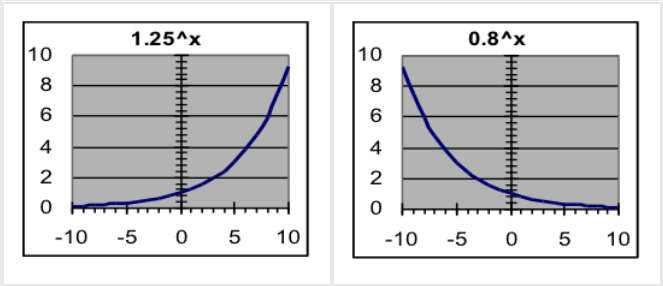
Answers:

a. The decay rate of the temperature difference is **8.8% per minute**.
b. The predicted temperature at 20 minutes is **68.9°F**.
c. The half-life of the warming process is **between 7 and 8 minutes**.



Symmetry of exponential growth and decay: Despite their different responses to large input valves (as described in the previous two paragraphs) exponential growth and decay are basically the same process, as shown by the growth and decay graphs shown to the right. These graphs are mirror images of each other, indicating that in this context growth is simply backwards decay, and vice versa.

This is why the same mathematical formula can be used for both growth and decay—the difference is just whether the base that is used is larger or smaller than 1.



Tip on fitting exponential models by hand: *If you get strange results in an exponential-modeling problem, check to see if you have made the rate too large (perhaps by entering a percentage without a percent character where a decimal was expected). This causes particularly dramatic errors when a negative number is entered, since attempting to raise a negative number to a fractional power will cause the spreadsheet to show an error message.*

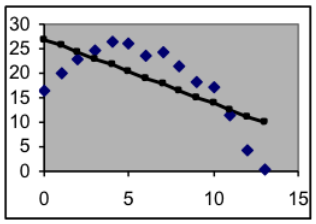
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22.6: J1.06- Section 2 Part 1

Section 2: Comparison of different models for the same data



This graph and its residual values below both show that even the best linear model fits this data poorly.

x	y	model	deviation
0	16.6	26.891	-10.322
1	20.2	25.586	-5.432
2	23.0	24.280	-1.279
3	24.8	22.975	1.837
4	26.3	21.669	4.620
5	26.0	20.364	5.603
6	23.5	19.059	4.484
7	24.2	17.753	6.482
8	21.3	16.448	4.868
9	18.3	15.142	3.191
10	17.3	13.837	3.459
11	11.4	12.531	-1.088
12	4.3	11.226	-6.925
13	0.4	9.920	-9.499

Sometimes different kinds of model can be fitted to the same data reasonably well. In such cases, any of the models may be used for interpolation. But different types of model give substantially different predictions when extrapolated, so it is important to have ways of choosing among them if prediction of future values is intended, as it often is. This is more of an issue when you are aware of the many different possibilities for model formulas, as you will be by the end of this course. Here are some of the ways that people use to decide among models.

- **Examine how well the different models fit the graph:** This is the basic test of a model, and you can eliminate any kind of model whose best-fit graph is clearly inferior to that of another possibility. If none of the kinds of models you know about can be made to fit the data well, you should avoid making predictions with those models for the process that produced that data.
- **Examine how positive and negative residual deviations are mixed:** For the best model, the sequence of deviations will be a random mix of positive and negative values, indicating that the data points are randomly scattered above and below the model points. Big adjacent groups of positive or negative residual values that cannot be avoided by changing the parameters indicate that the model does not match the data pattern.
- **Prior information:** Make use of information you have about the process being modeled other than the data values themselves. This may be your own knowledge about the kind of process that produced the data, or experts in the field may have already identified what kinds of models are best for the kind of data you have.

- **Extrapolation behavior of the model:** If extending the model a moderate distance forward or back from the data values gives unreasonable predictions (such as negative values for population), that model should be avoided, especially if one of the other possibilities has better extrapolation behavior.
- **Numerical measures:** In a later topic you will learn how to use and interpret the *standard deviation*, a numerical measure of how good the fit is between a dataset and a particular model. This value is based on the differences (in column D) between the data (in column B) and the model prediction (in column C). The best model will usually have the smallest standard deviation.

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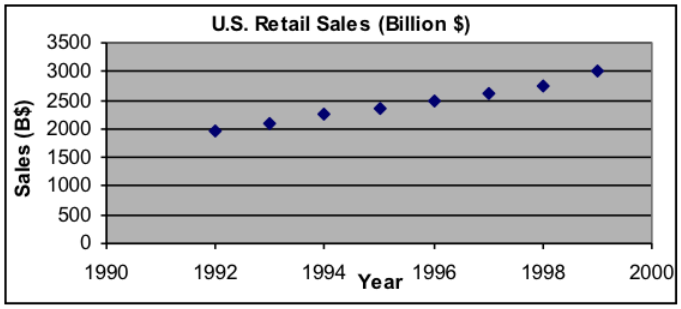
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22.7: J1.07- Section 2 Example 6

Example 6: Comparing different models of sales-data history (when both fit the data)

The dataset below of total retail sales in the United States from 1992 to 1999 looks as if it could be fit pretty closely by a linear model, in which sales increase by the same amount each year. However, most economic trends increase by about the same percentage each year on the average, so an exponential model might be more appropriate and would also be a good fit. To choose between these alternatives, we will find both best-fit models and examine what happens to them when extrapolated back 20 years.

Year	Sales (B$)
1992	1,952
1993	2,082
1994	2,248
1995	2,359
1996	2,502
1997	2,611
1998	2,746
1999	2,995



Question: For this data, which model has more reasonable extrapolation behavior—linear or exponential?

Solution approach:

[1] Copy the dataset into the Data Scratch Pad worksheet in *Models.xls* and modify it so that the input variable is years since 1992. This will make the model parameters easier to find. (The redefined dataset is shown to the right.)

[2] Use a copy of the Linear Model template to find the best linear model, which is about $y=141x+1943$

[3] Use the Exponential Model template to find the best exponential model, which is about $y=1971\cdot{(1.06)}^{x}$

[5] Note that the graphs of the two models both fit the data well.

Since 1992	Sales (B$)
0	1,952
1	2,082
2	2,248
3	2,359
4	2,502
5	2,611
6	2,746
7	2,995

In both the Linear Model and Exponential Model worksheets, follow these steps:

[a] Insert the values from -1 to -20 into column A below the existing data. These new input values correspond to the years from 1991 back to 1972. [6] In both the Linear Model and Exponential Model worksheets, follow these steps:

[b] Spread the formula in column C down as far as the new input values. This will give the sales prediction of that model for the previous 20 years. The results for the two models should look like the worksheet extracts shown below.

Extracts from Models.xls worksheets, with extrapolations back for 20 years

Linear Model y=141x+1943

	A	B	C
1	x	*y data*	*y model*
2	Year-1992	Sales (B$)	Prediction
3	0	1,952	1943
4	1	2,082	2084
5	2	2,248	2225
6	3	2,359	2366
7	4	2,502	2507
8	5	2,611	2648
9	6	2,746	2789
10	7	2,995	2930
11	-1		1802
12	-2		1661
13	-3		1520
14	-4		1379
15	-5		1238
16	-6		1097
17	-7		956
18	-8		815
19	-9		674
20	-10		533
21	-11		392
22	-12		251
23	-13		110
24	-14		-31
25	-15		-172
26	-16		-313
27	-17		-454
28	-18		-595
29	-19		-736
30	-20		-877

Exponential Model y=1971\cdot{(1+0.06)}^{x}

	A	B	C
1	x	*y data*	*y model*
2	Year-1992	Sales (B$)	Prediction
3	0	1,952	1971.00
4	1	2,082	2089.26
5	2	2,248	2214.62
6	3	2,359	2347.49
7	4	2,502	2488.34
8	5	2,611	2637.64
9	6	2,746	2795.90
10	7	2,995	2963.66
11	-1		1859.43
12	-2		1754.18
13	-3		1654.89
14	-4		1561.22
15	-5		1472.85
16	-6		1389.48
17	-7		1310.83
18	-8		1236.63
19	-9		1166.63
20	-10		1100.60
21	-11		1038.30
22	-12		979.53
23	-13		924.08
24	-14		871.78
25	-15		822.43
26	-16		775.88
27	-17		731.96
28	-18		690.53
29	-19		651.44
30	-20		614.57

[c] Make a new graph of the data and model (the A1:C30 rectangle of cells). In the graphs below showing the extrapolations, the scales have been set to equal and gridlines have been omitted, to make comparisons easier. But default-setting graphs will have the same information.

[6] Compare the extrapolated models. In this case, the linear extrapolation back 20 years predicts negative sales totals in 1972, which is obviously unrealistic. The exponential model remains positive regardless of how far back it is extrapolated, which shows that the exponential model is a much better choice for a sales model.

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22.7: J1.07- Section 2 Example 6 is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

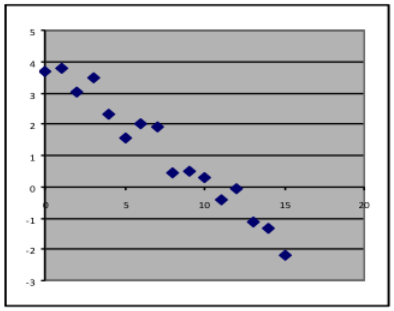
22.8: J1.08- Section 3

Section 3: Choosing a trial model based on the pattern of the data graph

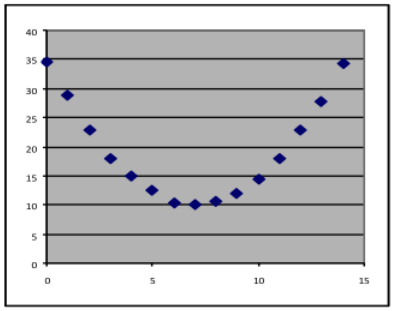
In most cases, it will not be difficult to decide which type of model will best fit the trendline of the data. Each type of model has its own characteristics, which usually means that only one of them, if any, will be capable of fitting the data well. If you identify a good candidate for a model, you can test it by using the appropriate worksheet to find its best-fit parameters and see how good the match is. If no model you know of fits the data, explain what aspect of the data pattern lets you know that model won't work.

Example 7—for each graph below, is a linear, exponential, quadratic, or some other model appropriate?

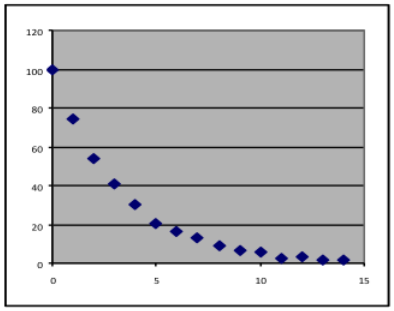
[a]



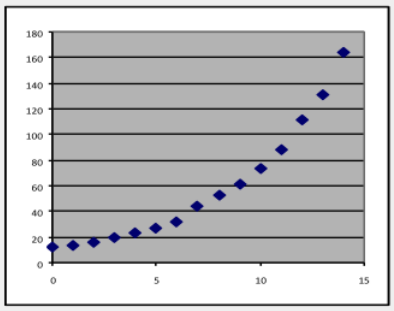
[b]



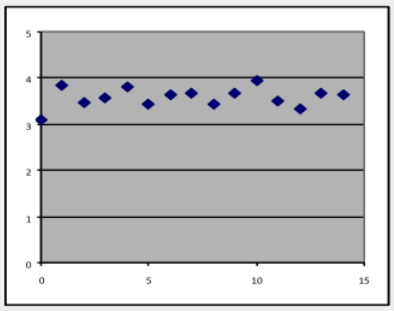
[c]



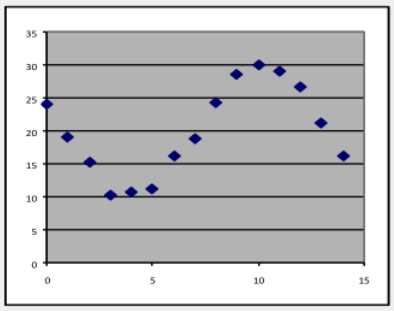
[d]



[e]



[f]



Answers:

a. Linear. While the graph is noisy, the trendline cannot be quadratic because it shows no sign of bending. It cannot be exponential since it isn't showing any sign of bending to approach the *x*-axis. The best-fit straight line would leave only random noise, so it is the best model in this case.

b. Quadratic. Since the graph changes direction from downward to upward as it goes from left to right, it cannot be linear or exponential. The graph is symmetric enough that the parabola that a quadratic model produces a trendline would be a good fit.

c. A decaying exponential. The trendline is clearly not linear, and it approaches but does not cross the x-axis. A quadratic would also fit reasonably well (in that case the curve would go up again if extrapolated to the right), but an exponential would fit better and also is a simpler formula.

d. A growth exponential. The data trend starts with a positive intercept and about doubles for each five steps. This is a faster increase than the best-fit quadratic, which would not quite follow the data

e. Linear, although with significant noise. The best-fit line is about horizontal, so its slope is close to zero. A dataset that consists of repeated measurements of the same thing (as is often done to test the stability of the measurement process) would look similar to this.

f. None of the listed models would work. This graph changes between upward and downward twice, while a quadratic can only change once and the other listed models cannot make that much of a direction change at all. As we will see in a later topic, there are other models to use in this case.

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22.9: J1.09- Section 4 Part 1

Section 4: Building redefinition of variables into model formulas

Redefinition of input parameters

In several different modeling problems so far (including those using population and sales data in this topic), we have redefined the input variable (e.g., to "Years since 1992" or "Years since 1790") so that the parameter values for the model did not refer to projected values at 0 AD. We did this by replacing the "Year" column with a "Years since …" column, changing the numerical values appropriately by subtracting the first year in the dataset. This approach produces correct answers with reasonably-sized parameter values for the model, but having to change back and forth between the "year" and the "years since" version of the parameter is awkward, and graphs made with the "years since" version do not directly display the year, as would be convenient.

These problems can be avoided by some simple changes to the formula of the model. The basic idea is that for a problem where we previously would have used "Years since 1992" and subtracted 1992 from each year listed in the data table, we will now leave the data table alone but instead replace each "*x*" in the model formula by the expression "(*x*–1992)". The examples below use the retail sales data from an earlier section to show what effect this model-modification strategy would have for linear and exponential models of that dataset. Remember that the formula in cell C3 is expressed in spreadsheet format, so that what we do there is replace "A3" with "(A3-1992)". [*Note: Always include the parentheses shown, so that the subtraction is done before any other operation.*]

US Retail Sales

Year	Sales (B$)
1992	1,952
1993	2,082
1994	2,248
1995	2,359
1996	2,502
1997	2,611
1998	2,746
1999	2,995

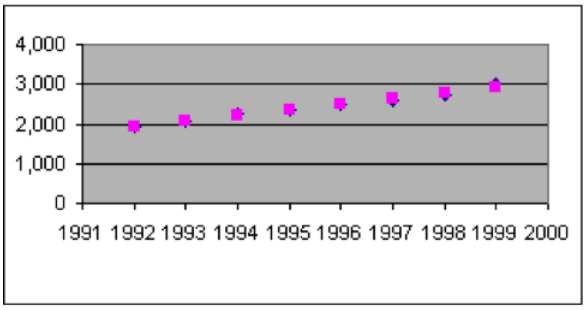
Example 8: *Modify the linear model for US Retail Sales so that the parameters are easy to find, without subtracting 1992 from each of the input x values in column A.*

1. Insert a new Linear Model worksheet.
2. Copy the data into columns A and B, leaving the year values as they are.
3. Set cell H3 to the phrase "Model output in 1992" (replacing the label "Intercept").
4. Modify the content of cell G1 by replacing the "x" in the expression by "(x-1992)".
5. Modify the content of cell C3 by replacing "A3" by "(A3-1992)".
6. Spread the modified formula in cell C3 down to row 10 to match the data.
7. Spread the formulas in D3 and E3 down to row 10.
8. Make a scatter plot of columns A, B, and C (select the rectangle A1:C10) to show the model and the data together so that you can watch how well they match while you adjust the parameters.
9. Adjust the model parameters, using the following steps:
    a. Set G3 to 1,952, which is the dataset value for 1992 (this may be adjusted further in step [c]).
    b. Adjust the G3 parameter until the model graph closely follows the shape of the data.
    c. If needed for the best fit, make small adjustments to G3 so that the model moves up or down.
10. Use the model to predict sales for any specified year (e.g., 2000) by entering the year number for which you want a prediction at the bottom of column A.

Resulting worksheet:

	A	B	C	D	E	F	G	H
1	x	y data	y model				***Linear model: y = m * (x-1992) + b***	
2	Year	Sales (B$)	prediction				**y = 140 * (x – 1992) + 1952**	
3	1992	1,952	1,952				**1,952**	**b: Model output in 1992**
4	1993	2,082	2,092				**140**	**m: Slope**
5	1994	2,248	2,232					
6	1995	2,359	2,372					
7	1996	2,502	2,512					
8	1997	2,611	2,652					
9	1998	2,746	2,792					
10	1999	2,995	2,932					

The formula in C3 is now "=$G$4*(A3-1992)+$G$3"



Things to notice about this process:

i. The values on the *x*-axis of the graph are the actual year numbers, not years since 1992.
ii. The final parameter values are the same values that were found by the subtract-1992 method.
iii. When making predictions with the model, there is no need to subtract 1992 from the *x* value.
iv. There is nothing in this process that makes use of the fact that the formula was linear—this means that **exactly the same instructions can be followed for any model** whose G3 parameter gives the prediction of the model when $x = 0$.

Example 9: Modify the exponential model for US Retail Sales so that the parameters are easy to find, without subtracting 1992 from each of the input x values in column A.

Follow the instructions for Example 8, except use a copy of the Exponential Model worksheet

Top of the resulting spreadsheet:

	A	B	C	D	E	F	G	H
1	x	y data	y model				*Exponential model: y = a * (1+r)^(x-1992)*	
2	Year	Sales (B$)	prediction				**y = 1952 * (1 + 0.06) ^ (x – 1992)**	
3	1992	1,952	1,952				**1,952**	**a: Model output in 1992**
4	1993	2,082	2,069				**0.06**	**r: Growth rate**

The formula in C3 is now "=$G$3*(1+$G$4)^(A3-1992)

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22.10: J1.10- Section 4 Part 2

Redefinition of output parameters

In the example discussed earlier where a soda can warms up in a 75º room, subtraction of the room temperature from the column B temperature-data values was needed in order to use an exponential-decay model to predict the process. This was necessary because the standard exponential-decay model can only converge on zero as its asymptotic value, so when using it we need to model the temperature *difference*, not the temperature itself.
But we can avoid the need for changing the data if we modify the model formula to add on the room temperature to the prediction. The exponential part of the formula will still converge on zero, but the formula as a whole will now converge on the room temperature, as is needed for correct predictions.

Soda-can warm-up in a 75º room	
Minutes out of cooler	Temp-erature of can
0	35
5	50
10	59
15	65

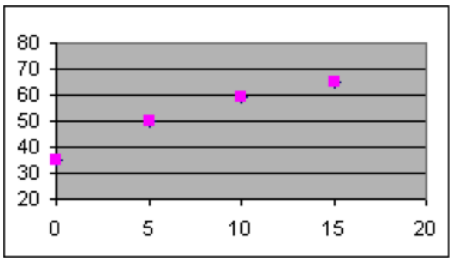
Example 10: *Modify the exponential model formula for the warming of a soda can in a room whose air temperature is 75* ºF, *without subtracting 75 from each of the data y values in column B.*

1. Insert a new worksheet and label it “Adjusted soda warm-up model”, then copy into this new worksheet the content of the Exponential Model Template worksheet.
2. Copy the data into columns A and B, leaving the temperature values as they are.
3. Set H3 to the phrase “Temperature at 0 minutes” as a reminder of what is being predicted.
4. Modify the content of cell G1 by adding “+75” at the end just before the last quotation mark.
5. Modify the content of cell C3 by adding “+75” at the end.
6. Spread the modified formula in cell C3 down to row 6 to match the data.
7. Spread the formula in D3 down to row 6.
8. Make a scatter plot of columns A, B, and C (select the rectangle A1:C6) to show the model and the data together so that you can watch how well they match while you adjust the parameters.
9. Adjust the model parameters, using the following steps:
   a. Set G3 to 35, which is the dataset value for 0 minutes.
   b. Adjust the G4 growth-rate parameter until the model graph closely follows the shape of the data. The decay rate is the same 8.8% that was found by the subtract-room-temperature method.
   c. If needed for the best fit, make small adjustments to G3 so that the model moves up or down.
10. Use the model to predict temperature for any specified time (e.g., 20 minutes) by entering the minute number for which you want a prediction at the bottom of column A.

Resulting worksheet:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
1	x	y data	y model				*Exponential model: y = a * (1+r)^x + 75*	
2	Minutes	Temperature	Prediction				y = –40 * (1 – 0.088)^x + 75	
3	0	35	35.00				–40	**a: Initial temperature difference**

4	5	50	49.76				–0.088	**r: Growth rate**
5	10	59	59.08					
6	15	65	64.95					
7								
8								



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22.11: J1.11- Exercises

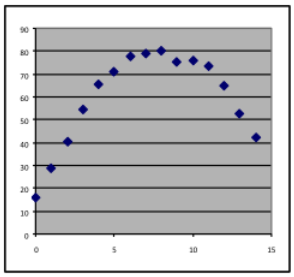
Part I.

Reproduce the results in Examples 1–10.

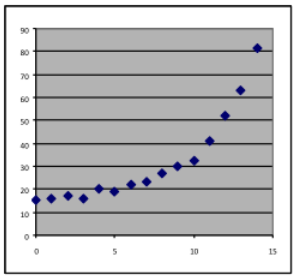
Part II.

11 & 12. For each of the graphs below, identify where a linear, quadratic, or exponential model would be appropriate, or where none of these three choices would work. In each case, write a sentence stating what reason you have for the choice you make.

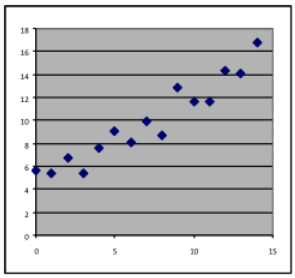
[11a]



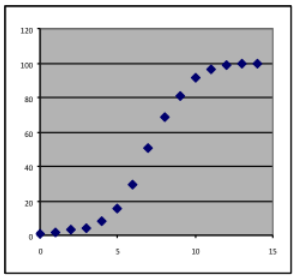
[11b]



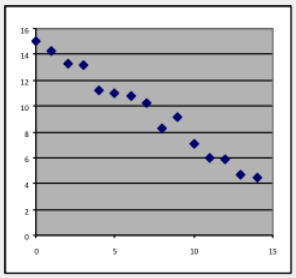
[11c]



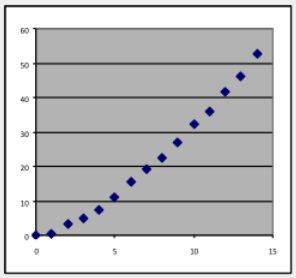
[11d]



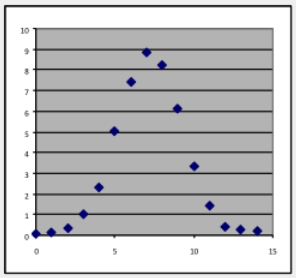
[12a]



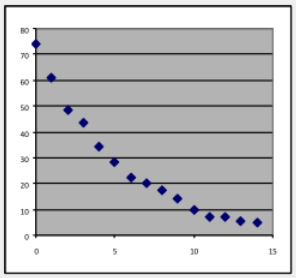
[12b]



[12c]



[12d]



For each dataset in the following problems, please do not type it in yourself, but find the data table shown below on the course web page and "copy and paste" it into the spreadsheet. This will save you quite a lot of work.

Instructions on using modeling spreadsheets to build best-fit models (for use on exercises 13-20):

1. **Copy the data into the Data Scratch Pad worksheet** in *Models.xls* and make a scatter plot to look at to decide what kind of formula (linear, quadratic, or exponential) to use for the model.
2. **Copy the Models.xls template** designed to fit the chosen kind of model into a new worksheet, labeling the new worksheet with a reference to the problem.
3. **Copy the data** into the worksheet, putting the input data values in column A and the output data values in column B, with the numbers starting at line 3.
4. Select C3:E3, then **spread the formulas** in these cells down to all the rows that have data values.
5. Make a scatter-plot of columns A to C, which should give you **a graph of the data** in one color combined with a **graph of the model values** in a different color.
6. **Adjust the parameters** until the model graph is as close to the data as possible, indicating that the current parameter settings for the chosen model fit the data about as well as a model of that kind can.
7. **Examine the graph** to see whether this is a good model (the data points are randomly above and below the model) or whether a better model should be looked for.

Drug concentrations

Elapsed hours	Concentration (*mg/ml*)
1	17.4
2	11.2
3	7.5
4	4.7
5	3.2
6	2.1

Coffee cool-off in 70º room

Minutes since pouring	Coffee temp. °F
0	112.7
2	103.6
4	96.8
6	91.2
8	86.7
10	83.3

Number of web sites

Month/Year	Sites
June 1993	130
Dec 1993	623
June 1994	2,738
Dec 1994	10,022
June 1995	23,500
Dec 1995	92,000

World Population Growth

Year	Population
1900	1,550,000
1910	1,750,000
1920	1,860,000
1930	2,070,000
1940	2,300,000
1950	2,400,000

Stopping Distances

Speed (mph)	Distance (feet)
20	41
30	71
40	118
50	176
60	240
70	316

UT Austin Tuition

Year	Tuition
1989	410
1990	440
1991	510
1992	590
1993	715
1994	770
1995	930
1996	1,085
1997	1,200

13. The concentration of a drug in the bloodstream is measured hourly after injection, giving the results shown in the “Drug Concentrations” dataset above:
    a. Identify the input and output variables.
    b. Use a spreadsheet to make an appropriate graph and label the axes. (hand-labeling is okay)
    c. Use the exponential-model spreadsheet to find a good exponential model and state its equation.
    d. Using the model, predict the concentration at 9 hours.
    e. How long after the injection does the model predict that the concentration will have fallen to 10.0 mg/ml?
    f. What decay rate does the model imply for the concentration of the drug?
14. The temperature of a cup of coffee is measured every 2 minutes, giving the “Coffee cool-off” dataset shown above. The room temperature is 70° F.
    a. Identify the input and output variables.
    b. Restate the temperatures into a form suitable for an exponential model.
    c. Use a spreadsheet to make an appropriate graph and label the axes. (hand-labeling is okay)
    d. Use the exponential-model spreadsheet to find a good exponential model for the temperature difference and state its equation.

e. What decay rate does the model imply for the temperature difference?
f. Using the model, predict the temperature 15 minutes after pouring.
g. About how long after pouring will the temperature be 80° F?

15. Solve the previous exercise by modifying the model formula rather than restating the temperatures.
16. The number of web sites grew rapidly once the world wide web was invented in 1993. Use the "Number of web sites" dataset above to:
    a. Identify the input and output variables.
    b. Restate the date variable in months since January 1993.
    c. Use a spreadsheet to make an appropriate graph and label the axes. (hand-labeling is okay)
    d. Use the exponential-model spreadsheet to find an exponential model that fits the data well and state its equation.
    e. State the growth rate per month implied by this model.
    f. How many web sites does this model predict for Dec 1997?
17. The population of the world for the first six decades of the twentieth century is given in the "World Population Growth" dataset above.
    a. Identify the input and output variables.
    b. Modify the data to restate the date variable in years since 1900.
    c. Use a spreadsheet to make an appropriate graph and label the axes. (hand-labeling is okay)
    d. Use the exponential-model spreadsheet to find an exponential model that fits the data well and state its equation.
    e. State the growth rate per year implied by this model.
    f. What world population does this model predict for 2000?
18. Solve the previous exercise by modifying the model formula rather than restating the date variable.
19. For the "Stopping Distances" dataset above giving the stopping distance of a car at various speeds,
    a. Identify the input and output variables.
    b. Fit an exponential model to the data and state the model formula.
    c. Fit a quadratic model to the data and state the model formula.
    d. Which model better fits the data? How can you tell?
    e. Use the better model to predict the stopping distance at 80 mph.
    f. What stopping distance does the model predict for zero mph? Does this prediction make sense?
20. For the "UT Austin Tuition" dataset shown above,
    a. Identify the input and output variables.
    b. Fit a linear model to the data and state the model formula.
    c. Use the linear model to predict 2006 tuition.
    d. Fit an exponential model to the data and state the model formula.
    e. Use the exponential model to predict 2006 tuition.
    f. State which model you think is better for the tuition dataset, and what arguments you could make to convince someone else that your conclusion is correct.
21. What percentage growth or decay rate is implied by these exponential models?
    a. $y = 443.2 \times (1.07)x$
    b. $y = 0.134 \times (1.025)x$
    c. $y = 5.3 \times (0.97)x$
22. What percentage growth or decay rate is implied by these exponential models?
    a. $y = 3.52 \times (0.94)x$
    b. $y = 69 \times (1.11)x$
    c. $y = 0.825 \times (1.075)x$
23. What prediction does the model $y = 3.52 \times (0.94)x$ make for:
    a. $x = 10$
    b. $x = 3.178$
    c. $x = 0$
24. What prediction does the model $443.2 \times (1.07)x$ make for:

a. $x = 10$
b. $x = 3.178$
c. $x = 0$

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CHAPTER OVERVIEW

23: Topic C- Communicating Precision of Approximate Numbers

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23.1: C1.01 Overview

Topic C. Communicating the Precision of Measured Numbers

This topic includes

Section 1. Reporting measurements

Section 2. Rounding

Section 3. Precision of a number, including the concept of significant digits

Section 4. Identifying the interval of actual values for a rounded number.

Objectives:

1. Practice the usual rules for rounding.
2. Round a number to a given precision.
3. Determine the implied precision of a rounded number, giving it in words, or by a number, or by underlining the significant digits.
4. Understand why we don't do two sequential rounding operations on a number, but do all rounding in one step.
5. Understand that rounding numbers ending in exactly half is sometimes done in a different way.
6. On a number line, graph the interval of actual values which are consistent with a rounded value.
7. Use interval notation to give the interval of actual values which are consistent with a rounded value.
8. Use inequality notation to give the interval of actual values which are consistent with a rounded value.
9. For a given rounded number, find the maximum error due to rounding and express it as a percentage of the rounded number.

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23.2: C1.02- Reporting

Section 1. Reporting Measurements

The following examples use the metric system for measuring lengths. The ideas are applicable to measurement in other contexts, such as volume or area, and using the English system as well as the metric system. These examples were chosen because it is easier to convey precision with decimals than fractions, and the metric system makes it easy to use decimals. Look at your ruler and notice that the centimeter is divided into tenths (each of those is called a millimeter) and the inch is divided into either eighths or sixteenths. So when we want to make measurements with a ruler smaller than one centimeter, we can easily use decimals, but when we want to make measurements smaller than one inch with a ruler, it is natural to use fractions rather than decimals.

Suppose we are using a ruler to measure the lengths of some pieces of cardboard and we give each measurement, along with a phrase describing how precisely we measured it.

Measurement		
12 cm, correct to the nearest cm.		
18.2 cm, correct to the nearest tenth of a cm.		
33 cm, correct to the nearest tenth of a cm.		
30 cm, correct to the nearest cm.		
Estimate of 20 cm, correct to the nearest ten cm.		
Estimate of 180 cm, correct to the nearest ten cm.		
Estimate of 300 cm, correct to the nearest ten cm.		
Estimate of 300 cm, correct to the nearest hundred cm.		

It would be more convenient to give each number in a way that conveys the precision instead of having to write the phrase afterwards. The generally accepted method for doing that is to report exactly the same number of digits as were observed. That is straightforward in many situations. It is somewhat less straightforward when the most precise digit observed is a “trailing zero.” But that is also easily taken care of for the third number in our list, 33 cm, rounded to the nearest cm. Here we use a trailing zero after the decimal place. In arithmetic courses, we learned that when we compute with the numbers 33 and 33.0, we will obtain the same results. So if someone writes 33.0 instead of 33, they must be intending to convey something more than just the exact value of 33. They are conveying that the number is approximately 33 and also conveying the precision of that approximation.

The fourth number in our list, 30 cm, correct to the nearest cm, can be written as 30. cm. When we explicitly include a decimal we indicate that each of the digits before the decimal was measured precisely. I think of this as only a fairly clear report rather than a clear report, because a decimal point at the end of the number is so easily overlooked or forgotten in copying the number.

Measurement	**Completely clear correct report**	**Fairly clear correct report**
12 cm, correct to the nearest cm.	12 cm	
18.2 cm, correct to the nearest tenth of a cm.	18.2 cm	
33 cm, correct to the nearest tenth of a cm.	33.0 cm	
30 cm, correct to the nearest cm.		30. cm
Estimate of 20 cm, correct to the nearest ten cm.		
Estimate of 180 cm, correct to the nearest ten cm.		
Estimate of 300 cm, correct to the nearest ten cm.		

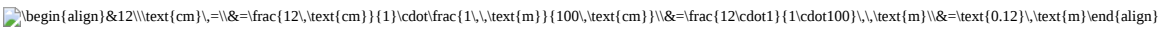
Estimate of 300 cm, correct to the nearest hundred cm.		

The next four numbers in our list are somewhat more difficult to report clearly without words. For a fairly clear correct report, we could say that we will assume that all trailing zeros in the number that would be needed to indicate the size are NOT to be interpreted as conveying precision. If we do that, then three of these four would be interpreted correctly.

Measurement	**Completely clear correct report**	**Fairly clear correct report**
12 cm, correct to the nearest cm.	12 cm	
2.7 cm, correct to the nearest tenth of a cm.	2.7 cm	
18.2 cm, correct to the nearest tenth of a cm.	18.2 cm	
33 cm, correct to the nearest tenth of a cm.	33.0 cm	
30 cm, correct to the nearest cm.		30. cm
Estimate of 20 cm, correct to the nearest ten cm.		20 cm
Estimate of 180 cm, correct to the nearest ten cm.		180 cm
Estimate of 300 cm, correct to the nearest ten cm.		
Estimate of 300 cm, correct to the nearest hundred cm.		300 cm

We see that there is an advantage in having the most-precise digit in the number after the decimal point when we want to report the number concisely and have the precision be clear. We could do that by changing our scale so that the relevant measurements are all decimals. In this case, since 100 centimeters is 1 meter, we could simply change our measurements to meters from centimeters and then use trailing zeros as needed to convey the precision completely clearly.

The conversion calculation needed for the first measurement in our table is



Review. It is important to be able to convert measurements between different types of units using the proportion method, as illustrated above. Notice that the proportion method enables you to keep track of the units algebraically. This is an important skill. See the course web pages for additional explanations and examples of this method of measurement conversion.

Measurement	**Completely clear correct report in centimeters**	**Completely clear correct report in meters**
12 cm, correct to the nearest cm.	12 cm	0.12 m
2.7 cm, correct to the nearest tenth of a cm.	2.7 cm	0.027 m
18.2 cm, correct to the nearest tenth of a cm.	18.2 cm	0.182 m
33 cm, correct to the nearest tenth of a cm.	33.0 cm	0.330 m
30 cm, correct to the nearest cm.		0.30 m
Estimate of 20 cm, correct to the nearest ten cm.		0.2 m
Estimate of 180 cm, correct to the nearest ten cm.		1.8 m
Estimate of 300 cm, correct to the nearest ten cm.		3.0 m

Estimate of 300 cm, correct to the nearest hundred cm.		3 m

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23.3: C1.03- Rounding

Section 2. Rounding.

Suppose we want to make a graph to summarize the heights of a class of 50 people. And we have their heights measured to the nearest tenth of an inch. The purpose of making this graph is to get a feeling for the variability of their heights. So the numbers as measured are more accurate than we really need – heights to the nearest inch would be adequate and easier to handle. Here is a portion of the dataset.

Original	61.3	68.5	71.4	65.8	64.3	63.4	67.2	72.3	69.5	70.1	62.8	63.7	65.2
Rounded	61	69	71	66	64	63	67	72	70	70	63	64	65

The usual rule for rounding is that, when the part you will drop is less than half, you "go down" and when the part you will drop is equal to or more than half, you "go up."

In a later section in this Topic, we will also learn to "think backwards" to see the interval of actual values that are consistent with a particular rounded value.

Example 1. Round to the nearest hundredth.

a. (a) 3.14738 goes to 3.15
b. (b) 0.73372 goes to 0.73
c. (c) 0.0032 goes to 0.00

Example 2. Round to the nearest ten.

a. (a) 817 goes to 820
b. (b) –1123 goes to –1120
c. (c) 74.567 goes to 70

Example 3. **Important idea: Sequential rounding does not always give correct results.** You must do all the necessary rounding in one step in order to obtain correct results.

Problem: Round 64.7 to the nearest ten.

Correct Solution: The nearest ten means that the answer must either be 60 or 70. Since 4.7 is less than half of ten, then when we round to the nearest ten, we have 60.

Incorrect solution: If we round 64.7 to the nearest one first, it is between 64 and 65, and 0.7 is more than half, so it rounds to 65. Then if we round that result to the nearest ten, it is between 60 and 70 and 5 is half of ten, so we round this to 70.

Notice that we did not obtain the same answer by these two solution methods. Yet, clearly, 64.7 is closer to 60 than to 70, so the first solution method must be correct. This illustrates the incorrect answer that sometimes arises if we do our rounding in sequential steps rather than all in one step.

Discussion. What about one-half?

Bookkeepers have noticed that, if you systematically round all numbers to the nearest dollar, rounding half-dollars up, and then take sums of those rounded numbers to estimate the sums of the original values, those estimates are a bit too high to be accurate. The problem is that there is some non-symmetry in the rounding rule. All those less than half "go down" and all those more than half "go up." So far that's symmetric. The problem is that the ones that are exactly half all go the same direction, which is "up." So that's not symmetric. So the rounded values are, on the average, overall, just a little bit higher than the original numbers.

Going deeper. In situations where dealing with one-half in a non-symmetric manner might be a problem, a more sophisticated rounding rule is adopted. See the course web pages for additional discussion of more sophisticated rounding rules.

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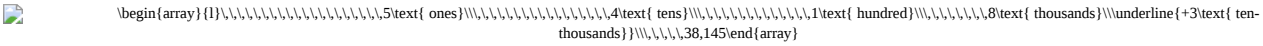
23.4: C1.04- Precision

Section 3. Precision of a number

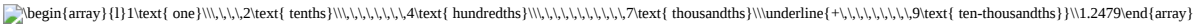
In elementary school we learned that doing arithmetic with the number 18 and the number 18.0 gives the same results. But when we are thinking of approximate numbers, those two ways of reporting a number do not imply the same thing. In particular, they imply a different rounding precision for the number, so they imply a different set of actual values that could have led to this rounded number. These are important distinctions when we work with measured numbers.

Review. Names of the places in a number.

Example 1. Consider the number 38,145. The places, from the right, are the ones, tens, hundreds, thousands, ten thousands. So this number is



Example 2. Consider the number 1.2479. The left-most place, before the decimal point, is the ones place. Immediately after the decimal is the tenths place, then the hundredths place, then the thousandths place, then the ten-thousandths place.



Precision: We can define the precision of a number in three different ways.

1. State it in words.
2. State it with a number.
3. Imply it by how the number is written.

Example 3: The amount $5200 is measured to the nearest hundred dollars. That can be stated as

1. The amount $5200 is measured to the nearest hundred dollars.
2. The amount $5200 is measured to the nearest 100 dollars.
3. The value $5200 has the obvious implied precision.

Example 4: The number 73.123 is rounded to the nearest one-thousandth. That can be stated as

1. The number 73.123 is rounded to the nearest one-thousandth.
2. The number 73.123 is rounded to the nearest 0.001.
3. The number 73.123 has the obvious implied precision.

Example 5: Round 18.038 to the nearest tenth.

Solution: Since we need to cut off everything past the tenths place, we must cut off the marked-out part here 18.038. Since the 3 at the beginning of the 38 is less than 5, we round down and the answer is 18.0. We could communicate that as “The answer is 18, rounded to the nearest tenth.” But no one would do that because it is confusing. It is much less confusing if we always include the tenths digit when we report the result of rounding to the nearest tenth. So here we could say “the answer is 18.0, rounded to the nearest tenth.” In fact, technical people would not give the words here because everyone would understand that the number 18.0 implies that it is rounded to the nearest tenth.

Example 6. Round 2.1397354 to the nearest 0.001.

Solution. Since we need to cut off everything past the thousandths place, we cut off the marked-out part here 2.1397354 and notice that the first digit of the 7354 that we need to cut off is greater than half, so we must round up. But the previous digit is 9, so to round up we must go to 10. That means the answer is 2.140 since the 3 in front of the 9 must go up 1. The answer is 2.140.

Example 7: We measured the length of pipe as 6.30 meters. What is the implied precision of that number?

Solution: Since this number is given in hundredths of a meter, and there is no reason to write the zero on the end except to give the hundredths place, this implies that the number is measured to the nearest hundredth, that is, the nearest 0.01.

Example 8: In this summary of a financial report of a small business, the amount spent on utilities last year is given as $13,000. What is the implied rounding precision of that number?

Solution: The implied rounding precision is that the number is rounded to the nearest thousand dollars.

Caution: (Extending the previous example.) If the actual value was $12,973.37 and we wanted to round to the nearest hundred dollars, the answer would also be $13,000. In our previous examples, where we were doing all the rounding somewhere after the decimal point, we could easily communicate the rounding precision by extra zeros as needed, so the rounding precision could always be given unambiguously. But **when we have large numbers, where the extra zeros are not after the decimal, then the rounding precision cannot always be given unambiguously by just writing the number**. In those cases, if we want to be completely clear in reporting a rounded number, we must state the rounding precision in words or numbers and not rely on our readers just using the implied rounding precision. When we are reading rounded numbers reported by others, we should look at other information given besides the actual number to see whether the implied rounding precision is consistent with that.

When scientists and others working in technical fields report approximate numbers, they sometimes use the concept of identifying significant digits. **The significant digits in a number are those which give an actual value that was measured or recorded, as opposed to the digits in a number which merely to indicate the size of the number**. (Think of the usual meaning of the word "significant." These digits are significant because they are actually measured.) Identifying the significant digits is simply another way of identifying the implied precision of the number.

Number	Implied precision in words	Implied precision in numbers	Significant digits underlined
17.3	Tenths	0.1	17.3
18	Ones	1	18
18.0	Tenths	0.1	18.0
100.6	Tenths	0.1	100.6
83.20	hundredths	0.01	83.20
97.1080	Ten-thousandths	0.001	97.1080
13000	Thousands	1000	13000
20800	Hundreds	100	20800

In identifying significant digits, we have focused here mainly on identifying the right-most significant digit to be consistent with the implied rounding precision. However, when we change the units of a measurement after the measurement has been made, we don't want that to change the number of significant digits. Specifically notice the second number in our previous table.

Measurement and its precision	Completely clear correct report in centimeters	Fairly clear correct report	Not a clear report	Completely clear correct report in meters	# of sig digits
12 cm, 1	12 cm			0.12 m	2
2.7 cm, 0.1	2.7 cm			0.027	2
18.2 cm, 0.1	18.2 cm			0.182 m	3
33 cm, 0.01	33.0 cm			0.330 m	3
30 cm, 1		30. cm		0.30 m	2
Estimate of 20 cm, 10		20 cm		0.2 m	1
Estimate of 180 cm, 10		180 cm		1.8 m	2
Estimate of 300 cm, 10			300 cm	3.0 m	2

Estimate of 300 cm, 100		300 cm		3 m	1

Some books give “rules for identifying significant digits.” These include

1. leading zeros in a decimal number less than 1 are not significant
2. trailing zeros in a number greater than 1 are not significant

The reason behind these rules is that, unless you are specifically told otherwise, these zeros are in the number just to indicate the size of the number and not to indicate an actual measured value. The second and last rows in the table are examples of these situations.

Going Deeper:

In scientific and technical work, the measurement units will usually be chosen so that the rounding precision is either to the nearest whole number or is in the decimal portion so that it can be given unambiguously. Sometimes they will use scientific notation to convey the rounding precision unambiguously. See the course web pages for supplemental information for this Topic. This includes a discussion of how scientists and other technical workers usually choose measurement units so that the implied precision is almost always at the level of 1, 0.1, or 0.01, so that the numbers’ implied precision is easy to read and unambiguous. Some discussion of the use of scientific notation is also included.

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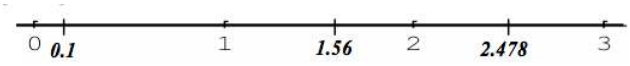
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23.5: C1.05- Interval Part 1

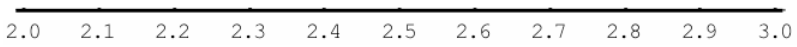
Section 4. Write an interval of actual values that are consistent with a rounded value.

We can represent all possible real numbers on a number line. We can only label a few of the points because if we tried to label more, the picture would become confusing. But you should think of being able to graph on a number line any particular number you are given. (When doing these problems, estimate distances rather than trying to measure them exactly.)

Example 1. Draw a number line that includes the values from 0 to 3. On that number line, label the points corresponding to the three numbers 0.1, 1.56, and 2.478.



Example 2. Draw a number line that includes the values from 2.0 to 3.0 and label the points corresponding to these numbers: 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, and 3.0.

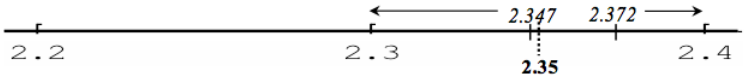


Example 3. Draw a number line that includes the values from 2.2 to 2.4, and label 2.2, 2.3, and 2.4.
(We will now take some numbers in that range with more decimal places, graph them on the number line, and then consider how to round each of those numbers to the nearest tenth.)

Consider the two numbers 2.347 and 2.372. Place each of them reasonably correctly on the number line.

Round each of these numbers to the nearest tenth and use an arrow to show which number it is rounded to.

What is the cut-off value that separates the numbers that are rounded to 2.3 and those that are rounded to 2.4? Indicate that value on the number line and label it.

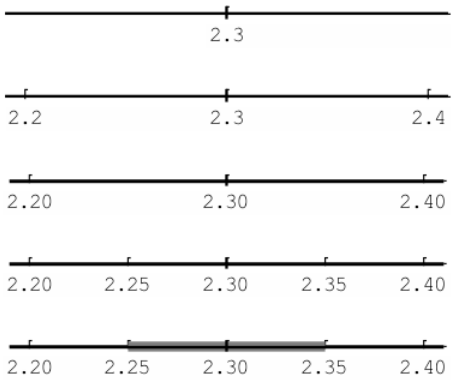


Notice that the cut-off value separating the numbers that are rounded to 2.3 and those rounded to 2.4 is 2.35, which is halfway between 2.3 and 2.4.

Example 4. We will draw the interval of actual values that are consistent with the rounded value of 2.3.

Procedure:

- Think of numbers near 2.3 that have the same precision. Since the precision of 2.3 is one-tenth, then other nearby numbers are 2.1, 2.2, 2.3, 2.4, 2.5, etc.
- Choose numbers with the same precision—the given 2.3 and the ones on either side, which are 2.2 and 2.4. Put them on a number line.
- What is the cut-off value that separates the number that are rounded to 2.2 and the numbers that are rounded to 2.3? Indicate that value on the number line.
- To label that value, it is easiest to go back and put an extra zero on all the other values, so you have 2.20, 2.30, and 2.40. That helps clarify that the value half-way between 2.20 and 2.30 is 2.25. Label it.
- What is the cut-off value that separates the numbers that are rounded to 2.3 and the numbers that are rounded to 2.4? Indicate that value on the number line and label it.
- Indicate the interval of values that are rounded to 2.3.



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23.6: C1.06- Interval Part 2

Discussion: What about the end points of this interval?

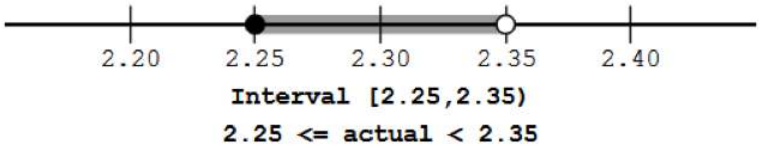
By drawing these intervals and end points, we see that numbers between 2.25 and 2.35, when rounded to the nearest tenth, are rounded to 2.3.

But, of course, we remember that the exact value 2.35, when rounded to the nearest tenth, goes to 2.4. So we really shouldn't include 2.35 in the interval.

So how far back should we go? Well, what about rounding 2.349999 to the nearest tenth? That goes to 2.3.

So we need all numbers less than 2.35 to be rounded to 2.3. We need some illustration or notation to indicate all the numbers up to, but not including 2.35.

These are several different ways of indicating this:



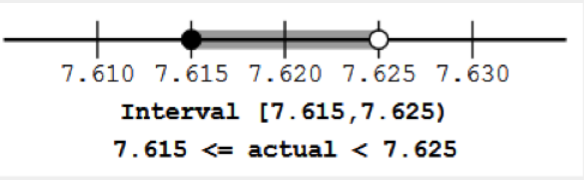
Each of these indicates that the left endpoint is included and the right endpoint is not included.

In this course, when we discuss the interval of values of the actual data that are consistent with a rounded value, it will be adequate for you to merely give the endpoints and not specifically indicate that the left end point is included and the right end point is not.

Example 5. Consider numbers rounded to the nearest hundredth.

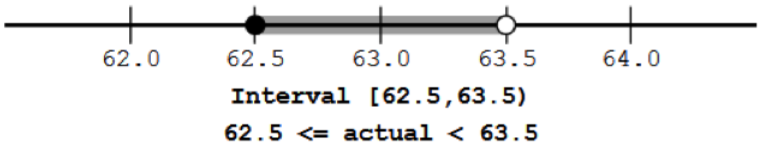
a. Write several numbers between 7.6000 and 7.6500 and round each of them to the nearest hundredth.
b. Draw a number line and use it to illustrate which numbers are to be rounded to 7.63.
c. Suppose we report a number as 7.63, rounded to the nearest hundredth. Write an interval that gives the actual values that are consistent with a rounded value of 7.63

Solution:

a.		b. and c.
7.6278 goes to 7.63	7.6251 goes to 7.63	
7.6421 goes to 7.64	7.6250 goes to 7.63	
7.6189 goes to 7.62	7.6249 goes to 7.62	
7.6317 goes to 7.63		
7.6048 goes to 7.60		

Example 6: If a measured number is reported as 63 feet, rounded to the nearest foot, what interval of possible actual values are consistent with that?

Solution: (All numbers here are in feet.)



Method: To determine what range of actual values are consistent with a given rounded number,

a. Write the given rounded number and several numbers near it with the same precision as the given number.

b. Graph a number line with the next-smaller and the next-larger number with the same precision on either side of the given rounded number.
c. If the numbers end in decimals, then add a zero at the end of each number in order to make it easier to label the numbers halfway in between.
d. On that number line, cut the intervals between the two rounded numbers in half.
e. Label those two cut-off points.
f. Mark the interval between those two cut-off points as the range of actual values that are consistent with the rounded number.
g. To be completely accurate in writing the interval, indicate that the smaller of the two cut-off values is included in the interval and the larger of the two cut-off values is not included in the interval.

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23.7: C1.07- Interval Part 3

Precision of end points: The numbers at the ends of the interval of actual values will **always** have exactly one more decimal place of precision than the given rounded number.

Discussion: Notation.

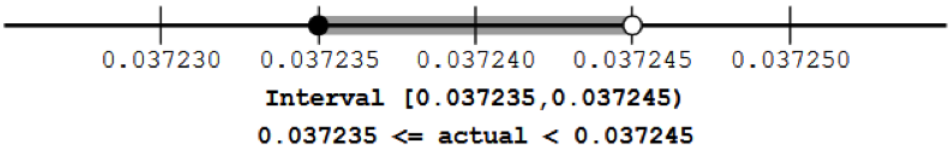
In the examples below, the same answer is given with several different forms of notation. You do not have to write all of those different ways for every answer. You must understand all the ways, but pick the one you prefer and write your answers with just that one form of notation.

Example 7: A measured number is reported as 0.03724 kilometers

a. state the implied precision as a number
b. state the implied precision in words
c. underline the significant digits
d. find the interval of possible actual values consistent with this rounded number.

Solution:

a. The precision here is implied to be 0.00001
b. The precision is one hundred-thousandth.
c. The significant digits are underlined: 0.0<u>3724</u> kilometers
d. The next-smaller number with the same precision is 0.03723 and the next-larger number with the same precision is 0.03725. Write those as 0.037230 and 0.037240 and 0.037250. Then find and label the half-way points.



The actual number is between 0.037235 and 0.037245 kilometers.

Example 8: A measured number is reported as <u>1.27</u> liters.

a. state the implied precision as a number
b. state the implied precision in words
c. underline the significant digits
d. what interval of possible actual values are consistent with that and what are several ways this might be reported?

Solution:

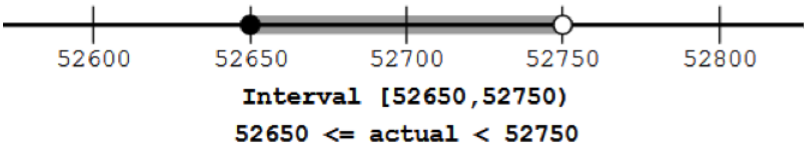
a. The rounding precision here is implied to be 0.01.
b. The precision is one-hundredth.
c. The significant digits are underlined: 1.27 liters.
d. The next-smaller number with the same precision is 1.26 and the next-larger number with the same precision is 1.28. Write those as 1.260, 1.270, and 1.280. Then find and label the half-way points.

Or it might be reported as 1.27 ± 0.005 liters

Another method of reporting it would be $1.27_{+0.005}^{-0.005}$ liters, which is usually only used if the distances on the two sides are not equal.

Example 9: If a measured number is reported as 52700 feet, rounded to the nearest hundred feet, underline the significant digits and identify the interval of possible actual values.

Solution: (All numbers here are in feet.) The significant digits are underlined: <u>527</u>00 feet



This might also be reported as 52700 ± 50 feet.

Another method of reporting it would be 52700_{+50}^{-50} feet, which is usually only used if the distances on the two sides are not equal.

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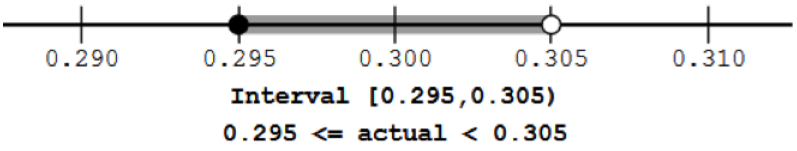
23.7: C1.07- Interval Part 3 is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

23.8: C1.08- Interval Part 4

Another method of reporting it would be 52700_{+50}^{-50} feet, which is usually only used if the distances on the two sides are not equal.

Example 10: If a measured number is reported as 0.30 meters, what is the rounding precision, underline the significant digits, and find the interval of possible actual values.

Solution: The rounding precision is 0.01, which is one-hundredth. The significant digits are underlined: 0.30 The next-smaller rounded number is 0.29 and the next-larger rounded number is 0.31. We'll write those as 0.290 and 0.300 and 0.310. (All numbers here are in meters.)



This might also be reported as 0.30 ± 0.005 meters

Another method of reporting it would be $0.30_{+0.005}^{-0.005}$ meters, which is usually only used if the distances on the two sides are not equal.

Example 11: If a measured number is reported as 12 meters,

a. what is the interval of actual values consistent with that,
b. what is the maximum amount that the actual value could differ from the reported value
c. what is the maximum amount of error as a percentage of the reported number?

Solution:

a. The interval of actual values is between 11.5 and 12.5 meters.
b. The maximum amount that the actual value could differ from the reported value is 0.5 meters.
c. To find the percentage of the maximum error as a percentage of the reported value, we must write it as a fraction, and then do the division and get a decimal fraction. Then convert that decimal to a percentage. $\frac{0.5\text{ m}}{12\text{ m}}=0.041666667$, which is 4.16666667%.

So **the maximum error is about 4% of the reported value**. This is called the *relative error*.

Example 12: If a measured number is reported as 12.0 meters,

a. what is the interval of actual values consistent with that,
b. what is the maximum amount that the actual value could differ from the reported value
c. what is the maximum amount of error as a percentage of the reported number?

Solution:

a. The interval of actual values is between 11.95 and 12.05 meters.
b. The maximum amount that the actual value could differ from the reported value is 0.05 meters.
c. To find the percentage of the maximum error as a percentage of the reported value, we must write it as a fraction, and then do the division and get a decimal fraction. Then convert that decimal to a percentage. $\frac{0.05\text{ m}}{12\text{ m}}=0.0041666667$, which is 0.416666667%.

So **the maximum error is about 0.4% of the reported value**. This relative error is one-tenth of the relative error in Example 11, reflecting the increased precision of the reported number.

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23.9: C1.09- Exercises

In the examples 7-12 above, the same answer was given with several different forms of notation. You do not have to write all of those different ways for every answer, unless you are explicitly asked to give several different forms of the answer. You must understand how to write all the forms, of course.

Part I.

1. Round to the nearest hundredth. (a) 3.14738 (b) 0.73372 (c) 0.0032
2. Round to the nearest ten. (a) 817 (b) –1123 (c) 74.567
3. Round 64.7 to the nearest ten.
   a. Do it in one step, correctly.
   b. Illustrate the error that occurs if you try to do it in two steps – first rounding to the nearest one and then rounding that result to the nearest ten.
4. For each of the following numbers, fill in the blanks in the table.

Number	Implied precision in words	Implied precision in numbers	Significant digits underlined
17.3			
18			
18.0			
100.6			
83.20			
97.1080			
13000			
20800			

5. Draw a number line that includes the values from 0 to 3. On that number line, label the points corresponding to the three numbers 0.1, 1.56, and 2.478.
6. Draw a number line that includes the values from 2.0 to 3.0 and label the points corresponding to these numbers: 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, and 3.0.
7. Draw a number line that includes the values from 2.2 to 2.4, and label 2.2, 2.3, and 2.4. Then find the cut-off values for the numbers that would round to 2.3.
8. Consider numbers rounded to the nearest hundredth.
   a. Write several numbers between 7.6000 and 7.6500 and round each of them to the nearest hundredth.
   b. Draw a number line and use it to illustrate which numbers are to be rounded to 7.63.
   c. Suppose we report a number as 7.63, rounded to the nearest hundredth. Write an interval that gives the actual values that are consistent with a rounded value of 7.63
9. If a measured number is reported as 63 feet, rounded to the nearest foot, what interval of possible actual values are consistent with that?
10. A measured number is reported as 0.03724 kilometers
    a. state the implied precision as a number
    b. state the implied precision in words
    c. underline the significant digits
    d. find the interval of possible actual values consistent with this number.
11. A measured number is reported as 1.27 liters
    a. state the implied precision as a number

b. state the implied precision in words
c. underline the significant digits
d. find the interval of possible actual values consistent with this number and what are several ways this might be reported?

12. Pay careful attention to when trailing zeros are conveying precision.
    a. If a measured number is reported as 52700 feet, rounded to the nearest hundred feet, underline the significant digits and find the interval of possible actual values that are consistent with that.
    b. If a measured number is reported as 0.30 meters, rounded to the nearest hundredth of a meter, underline the significant digits and find the interval of possible actual values that are consistent with that.
13. If a measured number is reported as 12 meters,
    a. what is the interval of actual values consistent with that,
    b. what is the maximum amount that the actual value could differ from the reported value
    c. what is the maximum amount of error as a percentage of the reported number?
14. If a measured number is reported as 12.0 meters,
    a. what is the interval of actual values consistent with that,
    b. what is the maximum amount that the actual value could differ from the reported value
    c. what is the maximum amount of error as a percentage of the reported number?

Part II.

15. Consider 12.1464.
    a. Round 12.1464 to the nearest hundredth.
    b. Round 12.1464 to the nearest tenth.
    c. Round 12.1464 to the nearest hundredth and then round that result to the nearest tenth. Is this a good method of getting a rounded number for 12.1464 to the nearest tenth? Explain.
16. Consider 8.5478
    a. Round 8.5478 to the nearest hundredth.
    b. Round 8.5478 to the nearest tenth.
    c. Round 8.5478 to the nearest hundredth and then round that result to the nearest tenth. Is this a good method of getting a rounded number for 8.5478 to the nearest tenth? Explain.

For each of the given numbers in problems 17-24,

a. state the implied rounding precision as a number
b. state the implied rounding precision in words
c. underline the significant digits
d. find the interval of possible actual values consistent with this rounded number.
e. what is the maximum amount that the actual value could differ from the reported value?
f. what is the maximum amount of error as a percentage of the reported number?

17. The rounded number 0.71
18. The rounded number 8.93
19. The measured number $54,000
20. The measured number $157,000
21. The measured number 2.347 liters
22. The measured number 18.978
23. The measured number 0.0072
24. The measured number 0.0378

For each of the given numbers in problems 25-28,

a. underline the significant digits
b. find the interval of possible actual values consistent with this rounded number.

25. The measured number 0.00406
26. The measured number 0.0003802
27. The measured number 12.000406

28. The measured number 9.0382
29. Consider these three rounded numbers
    a. For the measured number 7 meters, underline the significant digits and find the interval of actual values.
    b. For the measured number 7.0 meters, underline the significant digits and find the interval of actual values.
    c. For the measured number 7.00 meters, underline the significant digits and find the interval of actual values.
    d. Do these three reported rounded numbers convey identical information?
30. Consider these three rounded numbers.
    a. For the measured number 82 meters, underline the significant digits and find the interval of actual values.
    b. For the measured number 82.0 meters, underline the significant digits and find the interval of actual values.
    c. For the measured number 82.00 meters, underline the significant digits and find the interval of actual values.
    d. Do these three reported rounded numbers convey identical information?

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CHAPTER OVERVIEW

24: Topic E- Using a Spreadsheet

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24.1: E1.11- Exercises

Exercises: For each of the following problems, after you have completed it on the spreadsheet, print the spreadsheet to turn it in with your homework.

a. Organize your work so that everything you want to print appears in Columns A-I. Material that appears to the right of that will not print on the same piece of 8.5 by 11 paper.
b. If you have the graph selected when you choose “Print” it will only print the graph. Since you need to show the numerical work as well as the graph for these, before you print, make sure that nothing is selected.
c. Never print long columns of numbers. These are only useful to view or compute with—not to read.

Part I.

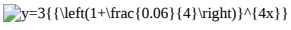
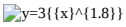
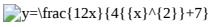
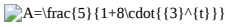
1. Graph $y=4+2{{(x-3)}^{2}}$ on the values $0\le{x}\le12$ and, on the same axes, graph $y=10x$.
   a. Determine what range of x-values has the first formula value smaller than the second formula value. (Where is the curved graph below the straight line?)
   b. Use the graph to estimate the *y*-value of $y=4+2{{(x-3)}^{2}}$ when $x=14$.
2. Use the graph to estimate which *x* gives the lowest value for *y* when $y=4+2{{(x-3)}^{2}}$ on the values $0\le{x}\le12$.
3. Use a spreadsheet to graph $y=\sqrt{12x+3}$ for input values between 0 and 6.
4. In the spreadsheet, put =PI() into a cell. See that it gives 3.14… with high precision.
5. Use a spreadsheet to compute the average of these six test grades: 72, 85, 69, 79, 92, 71. Then change the first score to 102 and see how the average updates to give a new value.
6. Use a spreadsheet to find the maximum of all the data values from the set of data below. Then replace the first value in the “Men” column, 67, with the number 85. Notice that the computed maximum changes to 85 as soon as you do this.

Men	Women
67	62
57	55
63	59
49	68
52	42

7. Use copy and paste to get the data from column A in Example 17 to put into column A of a spreadsheet. Use formulas to find the average, standard deviation, maximum, and minimum of the set of values, putting the appropriate formulas into column C and labels in column D describing what the formula cell to the left is computing.
8. Use the STDEV function to compute the standard deviation of these grades: 72, 85, 69, 79, 92, 71 (the same values listed in Exercise 5 above).
9. Use the MAX and MIN formulas to compute the largest and smallest grades in Exercise 5.
10. Consider the formula $y=2{{(x-3)}^{2}}+4$. Make a spreadsheet to graph that formula in a more general way, as $y=a{{(x-h)}^{2}}+k$.
    a. Then use it to investigate the effects of changing *h* on the shape and position of the graph.
    b. Then use it to investigate the effects of changing *k* on the shape and position of the graph.
    c. Then use it to investigate the effects of changing *a* on the shape and position of the graph.

Part II.

In 11-14, graph each of the following formulas using a spreadsheet. In 11 & 12, use input values from 0 to 20, with an increment of 1. In 13 & 14, use input values between 0 and 5, with an increment of 0.25.

11. 	12. 	13. 	14. 

15. Use the AVERAGE function to find the average of these 8 quiz scores: 7, 9, 10, 5, 8, 9, 6, 7

16. Set up a spreadsheet to calculate the standard deviation for the set of quiz grads shown to the right. Once you have the spreadsheet set up, explore the effects of the numbers on the standard deviation by making at least three changes in grades.
Report what changes you make, and what standard deviation results after each change.

Grades
74
72
90
84
86
95
97
72

17. Using the set of measurement values listed to the right:
[a] Calculate the average, standard deviation, and maximum for the set of measurement values listed to the right.
[b] Does the maximum differ from the average by more or less than the standard deviation?

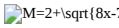
Measurements
433
442
439
436
436
447
439
440
427

18. Calculate the average, standard deviation, and maximum for the set of measurement values listed to the right.

Measurements
1,330
1,344
1,365
1,337
1,322
1,327
1,347
1,342
1,325

19. Calculate the average, standard deviation, maximum, and minimum for the set of measurement values listed to the right.

Measurements
0.398
0.376
0.423
0.416
0.355
0.422
0.360
0.416
0.374
0.379
0.446

20. Use the SQRT function to graph 

for values of x between 1 and 12.
21. For Topic B. Formulas and Graphing problem in Part II about the drug dosage, use a spreadsheet to make a more comprehensive graph to estimate more accurately the time for the concentration to be reduced to 2 ml/l. Give your estimate in hours with decimals, not in hours and minutes.
22. For Topic B. Formulas and Graphing problem in Part II about the landing speed, use a spreadsheet to make a more comprehensive graph in the appropriate part of the graph to estimate the weight which will have a landing speed of 100 ft/sec. to the nearest hundred pounds.
23. For Topic B. Formulas and Graphing problem in Part II about the logistic formula, use a spreadsheet to make a more comprehensive graph in the appropriate part of the graph to estimate the time when the value of P will be 900.

24. For Topic B. Formulas and Graphing problem in Part II about the break-even point, use a spreadsheet to make a more comprehensive graph so that you can estimate the break-even point to the nearest hundred thermometers.
25. Consider the formula $y=2{{(1.5)}^{x}}$. Make a spreadsheet to graph that exponential formula in a more general way, as $y=a{{(b)}^{x}}$ for values of *x* between -6 and 6. Use that spreadsheet to investigate the effect of changing the value of the base, *b*, where $b>0$. Choose some values for the base less than 1 and some greater than 1. In your solution, write a summary of what you found and give at least three graphs to support your summary.
26. Consider the formula $y={{x}^{2}}$. Make a spreadsheet to graph that power formula in a more general way, as $y={{x}^{p}}$ for values of *x* between 0 and 6. Use that spreadsheet to investigate the effect of changing the value of *p*, where $0.3<p<4$. Choose some values for the power, *p*, less than 1 and some greater than 1. In your solution, write a summary of what you found and give at least three graphs to support your summary.

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24.2: E1.01- Overview

Topic E. Using a Spreadsheet

The illustrations in this Topic are from Excel, but all spreadsheets work similarly on the basic topics. Practice on whatever spreadsheet is available to you. Expect to have questions. Please allow time for that and make a place in your notebook to write those questions and the answers you find. And NEVER SPEND MORE THAN 5 MINUTES STUCK on something involving computer software. Instead, write down your question and ask it during the next class or office hours.

Objectives:

1. Learn the details of the particular computer you're using. Turn it on, find and open the program, open it.
2. Basics:
   a. Enter data by typing it in.
   b. Enter patterned data efficiently (usually for the input values, or the x-values)
   c. Enter a formula (usually for the output values, or the y-values)
   d. Copy a formula into other cell.
   e. Print and/or save the worksheet.
   f. Copy a set of data from a table on the web into the spreadsheet.
   g. Relative cell locations versus absolute cell locations.
3. Graphs
   a. Make a graph of a formula onto a rectangular coordinate system.
   b. Make a graph of two different formulas on the same axes, so that they can easily be compared.
   c. For a given input value, use the graph to estimate the output value.
   d. For a given output value, use the graph to estimate the input value that would give that output.
   e. Use the spreadsheet to look "more closely" at some particular portion of the graph.
   f. Optional: Changing the titles, labels, and scale along the axes.
4. Formulas and functions.
   a. Know when and how to insert extra parentheses into formulas to make Excel compute correctly.
   b. Use these built-in functions: SQRT, AVERAGE, MAX, MIN, STDEV.
5. Explore how the graphs of formulas change as the parameters change.

Section 1. Starting the program.

You'll use the computer in the classroom and then a computer at home or in a computer lab to do your homework. Fill in this chart.

	Classroom	**Where I do homework**
Location		
Login (Username / password)		
Name of the spreadsheet program		

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24.3: E1.02- Section 2 Part 1

Section 2: Basics

Example 1. Entering data.

When you put data into the worksheet, it is important to include labels and explanations.

Often we will want to graph data. In that case the columns of numbers should be next to each other. If there are more than two columns of numbers, it is must useful if they are all together. That is, if there are other things to include, such as explanations, or other constant values, etc., those should be completely separated from the columns of data by at least one blank column.

	A	*B*	*C*	*D*	*E*	*F*	*G*
1		G	L				
2	Goat	151	12				
3	Lion	108	10				
4	Pig	115	10				
5	Cat	63	11		G = Gestation period (days)		
6	Dog	63	11		L = Life Expectancy (years)		
7	Squirrel	44	9				
8	Rabbit	31	7				
9	Duck	28	10				
10	Chicken	22	7.5				
11	Parakeet	18	8				

When you enter data into the spreadsheet, if you will use it later, it is a good idea to include some explanations, as indicated here. Notice that, when you are typing the names of the variables, they are typed into one cell each, but since they are longer, they show spread over several cells.

Example 2. Enter a patterned set of numbers efficiently.

In graphing, we often need a long sequence of x-values and it is tedious to type them in one by one. They can be entered more quickly.

Label the first column as the x variable, and then we will enter the numbers 0, 1, 2, 3, …, 20 quickly.

Start by typing in the *x* and then entering the first two numbers in the next two cells. (This sets up the pattern.)

	A	*B*
1	x	
2	0	
3	1	
4		

Next, select those TWO CELLS at the same time by clicking on the first cell, holding down the left mouse button and then moving to the second cell, and then letting up the left mouse button. There will be a small rectangle at the bottom left corner of the bottom

cell. Move your mouse on top of that and see a + symbol appear. Left-click on that small rectangle and drag it down several cells. Then let up on the left mouse button and the cells will fill in with the next numbers in the sequence.

Continue by selecting the bottom two cells and extending it further, until you have completed the task.

Spend about 2 minutes practicing this. **If you can't make it work, ask for help.**

Example 3. Notice the cell locations.

Notice that, in the worksheet, all the cells are labeled. Along the top is a letter and along the side is a number. Can you find the cell A2? Can you find the cell D15? In other spreadsheet programs, they might be labeled differently, but they will still have labels. This is a very important aspect of spreadsheets.

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24.4: E1.03- Section 2 Part 2

Example 4. Entering a formula into a spreadsheet.

We will set up the data that will allow us to graph the formula $y=4+2{{(x-3)}^{2}}$ on the values $0\lex\le9$.

- Set up the *x*-values from 0 to 20 as in the previous example, with the label *x* at the top of that column.
- Label the second column as *y* and then enter the following formula into the second cell in that column. The spaces aren't needed. I show this with spaces only to make it easier to read.
  =4+2*(A2–3)^2 (Enter) (It isn't necessary to type capital letters for the cell locations. Excel will change them to capital letters automatically.)
- Notice that after you punched Enter, the formula disappeared in the cell and was replaced by the value of the formula when $x=0$. That's because, in place of the *x* in the formula, you entered a cell location, so the spreadsheet used the value in that cell location and evaluated the formula. Did it evaluate the formula correctly? You should check it. Since $y=4+2{{(x-3)}^{2}}=4+2{{(0-3)}^{2}}=4+18=22$, the value for cell B2 should be 22. Is it?
- When you put your cursor on the cell where you typed the formula, which has a number in it now, notice that the formula itself appears in a line above the spreadsheet, so that you can still see it.
- Put the formula into the next cell. Copy the formula by highlighting the cell (left-click on the cell, then **Edit>Copy**.) Move the cursor to the next cell and then choose **Edit>Paste**. That should put the formula into the second cell. Notice that the second cell has a number, but the updated formula is in the line above the spreadsheet. How is the formula updated? Do you see that now it has the cell location A3 in it? So it's using the next value for x, which is exactly what you want.
- Continue copying and pasting the formula until you have completed the spreadsheet. You can do that one cell at a time or you can copy into several cells at once by selecting several cells before you Paste. Experiment with this. Write notes about how to do it here.

	A	*B*
1	x	y
2	0	22
3	1	12
4	2	6
5	3	4
6	4	6
7	5	12
8	6	22
9	7	36
10	8	54
11	9	76

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24.5: E1.04- Section 2 Part 3

Summary. Saving the worksheet and printing the worksheet.

If you have spent much time preparing a worksheet, it is a good idea to save it. I recommend that you keep a folder on your computer and in a "cloud" location or on a flash drive for these files. Name the folder for this class and the files by the name of the topic and then by the page number or problem number. Examples: Dpage3, Dproblem9.

When you are doing classwork, there is no need to print. I prefer that you not print then, because all the computers in the room share the same printer and it will cause problems for us if everyone tries to print at the same time. Also, the printers in the classrooms are often not working well. When you are doing homework, you will need to print some of your output. If you don't have a printer at home, you can bring your file to computer lab to print it. Be conservative about how much you print. Generally speaking, don't print until you have finished the problem and are confident it is correct. If you need to ask questions about your work, an electronic file is much more useful than a printout.

1. Organize your work so that everything you want to print appears in Columns A-I. Material that appears to the right of that will not print on the same piece of 8.5 by 11 paper.
2. If you have the graph selected when you choose "Print" it will only print the graph. To print your numerical values as well as your graph, make sure that nothing is selected.
3. Never print long columns of numbers. These are only useful to view or compute with – not to read. When you have used long columns of numbers, then copy and paste the material you actually want to print to another page before you print.
4. To save a file choose **File > Save As**
5. To print a file choose **File > Print**

Example 5. Copying data from the web to a worksheet.

If you find data in an electronic file that is already organized into a table, you can just "Copy and Paste" it into a worksheet. This is very convenient!

Practice this by going to this course web page, open this document (in the .rtf file) and go to the animal data from Example 1 in this section. Copy and paste it into your worksheet.

Example 6. Relative cell locations versus absolute cell locations

In Example 4, when you copied a formula and pasted it into another cell, you noticed that it updated the reference to another cell to keep it in the same "relative location." This is a very important aspect of spreadsheets, because it enables you to easily input a formula into a whole series of cells. Occasionally, however, in a formula you will want to use a cell reference that doesn't get updated. Do that by using dollar signs. So A3 is a relative cell location, but $A$3 is an absolute cell location and does not get updated when you copy and paste a formula including it to another place.

Make a worksheet that will enable me to look at any of $y={x}^{2}$ or $y={x}^{3}$ or $y={x}^{6}$ very easily.

The formula in cell B3 is =A3^$D$1. Below on the left is the worksheet as I originally entered it. On the right is the same worksheet after I changed the entry in cell D1 to the number 6.

	A	B	C	D
1				2
2	x	y = x^exponent		
3	0	0		
4	1	1		
5	2	4		
6	3	9		
7	4	16		
8	5	25		
9	6	36		

	A	B	C	D
1				6
2	x	y = x^exponent		
3	0	0		
4	1	1		
5	2	64		
6	3	729		
7	4	4096		
8	5	15625		
9	6	46656		

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24.6: E1.05- Graphs Part 1

Section 3. Graphs

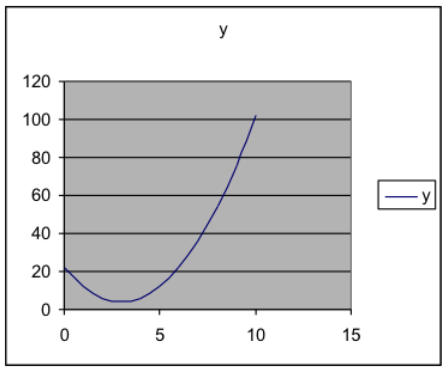
First we'll investigate how to use a spreadsheet to graph one formula onto a rectangular coordinate system, just like the graphs you have done by hand in algebra classes. **You MUST put the values for the input variable and the output variable in two columns, with the input variable on the left.** Some of the details of exactly what menus to use after that may vary from one spreadsheet program to another. The description here is for Excel. Ask for help about the exact sequence to use if you have a different spreadsheet program, such as Open Office. (The Microsoft Works spreadsheet program will not graph.)

Then, in Excel, choose the **Insert > Chart**

Example 7. Graph $y=4+2{(x-3)}^{2}$ on the values using a rectangular coordinate system and connecting the points with a line/curve.

Solution:

- First enter the *x*-values in the first column, including the label at the top of the column. There is no clear answer about exactly how many values to enter here, but be sure they include some at each end of the range of *x*-values.
- Second, enter the formula as described in Section 2, Example 4, including the label at the top of the column. (Look back at that example to see the values.)
- Select the cells that contain the labels and the data to graph.
- In the menus at the top of the page, choose **Insert>Chart** to begin a graph.
- Choose **X-Y Scatter** for the type. (Gives the rectangular coordinate system that we want.)
- You'll have a choice of connected or unconnected lines. In this case, **choose the connected lines**. Sometimes in this course we'll choose unconnected lines.
- Go through the rest of the choices quickly, not changing anything, and click Finish.



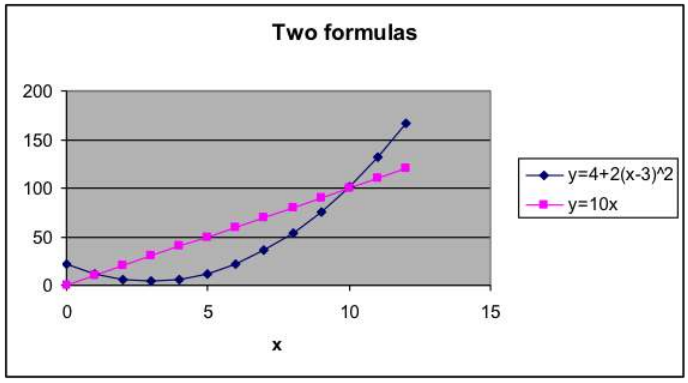
Check: In this course, you will graph many formulas that you haven't seen before, so you won't know what they are supposed to look like. The best way to check this is to evaluate at least one of the values by hand (using a calculator) as we did when entering the formula, just to be sure you have done what you intended when you put the formula in. But, if you have entered the formula correctly, then those values will be correct and this will be the correct graph.

Example 8. Graph $y=4+2{(x-3)}^{2}$ on the values $0\le{x}\le12$ and, on the same axes, graph $y=10x$. Determine what range of x-values has the first formula value smaller than the second formula value. (Where is the curved graph below the straight line?)

Solution:

- Repeat the first few steps in the previous example to obtain the first formula values. This time extend the x-values to 12.
- In the third column of the spreadsheet, put the values for the formula $y=10x$.
- Select all three columns of numbers, as far down as you have values.
- Use **Insert>Chart** and choose X-Y Scatter. Choose the connected lines. I encourage you to experiment a bit.

	A	B	C
1	x	y=4+2(x-3)^2	y=10x
2	0	22	0
3	1	12	10
4	2	6	20
5	3	4	30
6	4	6	40
7	5	12	50
8	6	22	60
9	7	36	70
10	8	54	80
11	9	76	90
12	10	102	100
13	11	132	110
14	12	166	120



The range of values is about $x=1$ to $x=10$ for which the values of the curved graph are below the values of the straight-line graph.

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Example 9. Just by sketching some more of the graph, estimate the *y*-value of $y=4+2{{(x-3)}^{2}}$ when $x=14$.

Solution: Extend the graph a bit and find that it appears to give $y=250$.

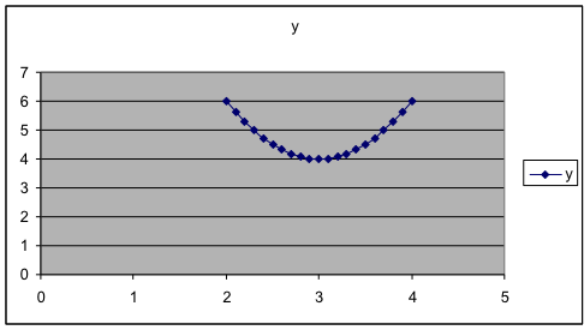
Check: Check this by plugging $x=14$ into the formula, $y=4+2{{(x-3)}^{2}}=4+2{{(14-3)}^{2}}=4+2{{(11)}^{2}}=246$

Example 10. Use the graph to estimate which *x* gives the lowest value for *y* when $y=4+2{{(x-3)}^{2}}$ on the values $0\le{x}\le12$.

Solution: That x-value is clearly between 0 and 5. It appears to be a bit above halfway. So we estimate that it is about $x=3$.

Example 11. Let's "magnify" the portion of the graph near $x=3$ in order to see very precisely where the minimum value is. Actually, we leave the old dataset and graph alone and produce a new one. This time, we'll just use *x* values near 3. So we'll graph $y=4+2{{(x-3)}^{2}}$ on the values $2\le{x}\le4$ where we increase the *x*-values in increments of 0.1. Use the same technique as before. Here is the middle part of the data table and the graph. This makes clear that $x=3$ gives the minimum value for *y*.

	A	*B*
1	x	y
2	2.5	4.5
3	2.6	4.32
4	2.7	4.18
5	2.8	4.08
6	2.9	4.02
7	3	4
8	3.1	4.02
9	3.2	4.08
10	3.3	4.18



Check: To see that this is right, plug in and find the y-value. Then plug in a couple of other x-values close by and see if their y-values are lower or higher.

So $x=3$ and then $y=4+2{{(x-3)}^{2}}=4+2{{(3-3)}^{2}}=4+2{{(0)}^{2}}=4$

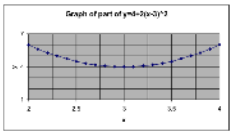
Now $x=3.02$ and then $y=4+2{{(x-3)}^{2}}=4+2{{(3.02-3)}^{2}}=4+2{{(0.02)}^{2}}=4.0008$

Now $x=2.99$ and then $y=4+2{{(x-3)}^{2}}=4+2{{(2.99-3)}^{2}}=4+2{{(0.01)}^{2}}=4.0002$

These make it clear that the value of y at is smaller than the value of y at values for x near 3. Putting that together with what we saw on the graph, it is very clear now that $x=3$ is the correct answer.

Example 12. **Optional:** Changing the titles, labels, and scale along the axes.

In Example 4, we might prefer to have a graph with more labels, more grid lines, and different scales for the axes. Here is the result we want and how to obtain it in Excel. This may be done differently or may not be possible in other spreadsheet programs.

	While making the graph, in Step 3 of 4, click on some of the tabs at the top and change something. **Titles**: Put the labels you want. **Gridlines**: Check all four boxes for major and minor gridlines for both variables. **Legend**: Uncheck “Show Legend.” **After the graph is made**, move your cursor on top of some value on the *x*-axis and double click. In the resulting box, choose the Scale tab and then fill in the four boxes. I used 2 to 4, with major gridlines at 0.5 and minor gridlines at 0.25. Then do the same thing for the y-axis, but from 1 to 7, with major gridlines at 3 and minor gridlines at 1.

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24.8: E1.07- Section 4

Section 4. Formulas and functions

We have similar issues about order of operations when we enter formulas into a spreadsheet as when we enter them into a calculator. In these two types of problems below, it is necessary to insert parentheses. When we write these by hand, the order of operations is implied by the placement of the expressions, so that when we type them in, we must use parentheses to compensate for that.

Evaluate each expression		In a spreadsheet, if the x-value is in cell A2
, where $x=3$		=3^(2*A2)
$\frac{{x}^{2}-6x}{4x+2}$, where $x=3$		=(A2^2-6*A2)/(4*A2+2)
$\sqrt{4x+9}$		=SQRT(4*A2+9) **or** =(4*A2+9)^0.5

In addition to the usual mathematical operations, there are various named functions you may use in spreadsheet formulas.

Named function	**Example (usual notation)**	**Example (spreadsheet notation)**
Square Root	$\sqrt{4x-5}$	=SQRT(4*A2-5)
Average (of a set of five numbers, in cells A2 through A6)	$\frac{{{x}_{1}}+{{x}_{2}}+{{x}_{3}}+{{x}_{4}}+{{x}_{5}}}{5}$	=AVERAGE(A2:A6)
Maximum (of a set of seven numbers, in B2 through B8)	$\max({{x}_{1}},{{x}_{2}},{{x}_{3}},{{x}_{4}},{{x}_{5}},{{x}_{6}},{{x}_{7}})$	=MAX(B2:B8)
Pi (the value 3.14…)	π	=PI()

To find additional functions, in Excel, choose **Insert > Function** and explore the dialog box. Many of the functions will be things you don't know about, but some will have familiar names. Explore them.

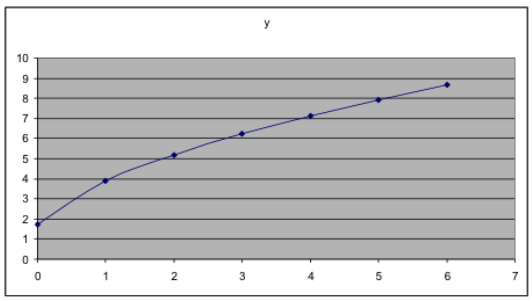
Example 13. Use a spreadsheet to graph $y=\sqrt{12x+3}$ for input values between 0 and 6.

Solution: Put in the values for x, then insert the formula. Then graph.

	A	*B*
1	x	y
2	0	1.732051
3	1	3.872983
4	2	5.196152
5	3	6.244998
6	4	7.141428
7	5	7.937254
8	6	8.660254

The formula in cell B2 is
=SQRT(12*A2+3)

Then that formula was copied to cells B3 through B8.



Example 14. Put the formula for the value into the spreadsheet and see what number it gives.

Solution. In cell A1, put =PI(). Enter that. See that it gives the value 3.14 … with high precision, as does the key on a calculator.

Example 15. Use a spreadsheet to compute the average of these six test grades: 72, 85, 69, 79, 92, 71

Solution:

	A
1	72
2	85
3	69
4	79
5	92
6	71

Cell C4:

78	average

First, type the scores into a blank spreadsheet. (In this case, copying and pasting doesn't work because the scores above are not in separate cells of a table in the original document.) If you put them in the upper left corner, they will be in cells A1 through A6.

Next choose another cell to put the formula in. If you choose cell C4, then in C4, put this formula =AVERAGE(A1:A6)

As soon as you enter that formula, the average of the six numbers, which is 78, will appear in the cell.

To "play" with this, go back to the cells with the data and replace the first value of 72 with 102. Notice that the average in cell C4 changes to 83.

Example 16. Use a spreadsheet to find the maximum of all the output data values in the dataset below.

Solution:

	A	*B*	*C*
1	Year	Men	Women
2	2000	67	62
3	2001	57	55
4	2002	63	59
5	2003	49	68
6	2004	52	42

Cell F3:

68	maximum

First, copy and paste these data values into a blank spreadsheet. If you paste the whole set into the upper left corner, the output data will be in cells B2 through C6

Next choose another cell to put the formula in. If you choose cell F3 then, in cell F3, put this formula =MAX(B2:C6)

As soon as you enter this formula, that cell should show 68, which is the maximum of these numbers.

Now, to "play" with this, go back to the cells with the data and replace the first value in the "Men" column, 67, with the number 85. Notice that the maximum, over in cell F3, changes to the new maximum 85.

Note: well-organized data will have all the data in a single column (or sometimes a single row), except when there are subcategories like those shown here.

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24.9: E1.08- Section 5

Section 5. Using spreadsheets to compute values describing sets of measurement data.

Several of the functions provided in spreadsheets are designed specifically for providing summary information about sets of measurement data (such summaries are called *statistics*). In particular, the AVERAGE function provides information on the typical value for a set of measurements, and the STDEV function provides information about the typical amount that individual measurements differ from the average. The MAX and MIN functions can also be used to find the measurements that are furthest from the average in each direction (i.e., the maximum and minimum measurement values).

All of these functions are computed in the spreadsheet in the same way. A cell is set to a formula where the function name is followed by a range of cells in parentheses, as was shown in the examples above for "=AVERAGE(A1:A6)" and "=MAX(A1:C3)". The range contains the values used in the computation, and can be as large as desired, meaning that once they are in the spreadsheet it is just as easy to compute the average of 1000 values as the average of 10 values.

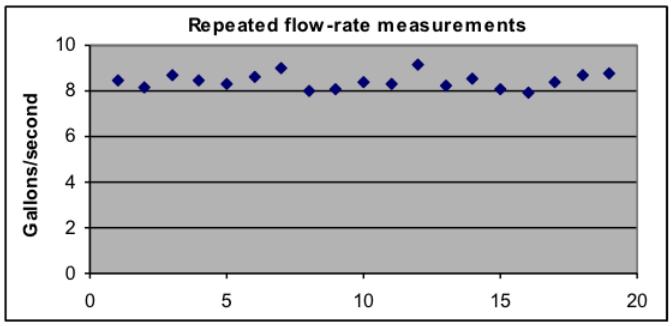
Average, maximum, and minimum are familiar concepts that are often used with numbers. The STDEV function will be new to many students. It computes the *standard deviation* of the values in the range. Standard deviation is a function that was specifically designed for use with measurements, and is the main way used in mathematics and science to describe how much a set of numerical values differ among themselves. For situations where the same object is being measured multiple times, the value of STDEV reflects the amount of noise in the measurement process.

The mathematical process used to calculate standard deviation is rather complicated, and is described in a later topic. But you will never need to calculate standard deviation by hand, since all spreadsheets have a function like STDEV. We will discuss standard deviation more in several later topics, but here is an example of a spreadsheet using it and the other statistical functions.

Example 17: Describe the average, standard deviation, maximum, and minimum for a measurement set.

	A	*B*	*C*	*D*	*E*
1	Flow rate				
2	8.475		8.432128	Average	— C2 contains the formula =AVERAGE(A2:A20)
3	8.179				
4	8.660		0.332176	Std. dev.	— C4 contains the formula =STDEV(A2:A20)
5	8.483				
6	8.304		9.148	Maximum	— C6 contains the formula =MAX(A2:A20)
7	8.599				
8	8.981		7.896	Minimum	— C8 contains the formula =MIN(A2:A20)
9	8.004				
10	8.080				
11	8.368				
12	8.287				
13	9.148				
14	8.218				
15	8.549				
16	8.071				
17	7.896				

18	8.419				
19	8.708				
20	8.780				



Note that the average and the standard deviation show many decimal places, even though the data used only has three decimal places. This is typical of computer calculations. In the later topics we will discuss how to decide how many digits should be reported for such results.

If you copy the data and make a similar spreadsheet, you can "play" with it by varying the data values.

Example 18: Compute the standard deviation of the six text grades in Example 15: *72, 85, 69, 79, 92, 71*

Solution:

	A
1	72
2	85
3	69
4	79
5	92
6	71

Cell C6:

9.077445	Standard deviation

First, type the scores into a blank spreadsheet (or start with the spreadsheet you already made for Example 15, making sure the values are reset when you played with changing them). If you put the scores in the upper left corner, they will be in cells A1 through A6, as shown to the left.

Next choose another cell (such as C6) to enter the formula "=STDEV(A1:A6)" into. As soon as you enter the formula, 9.077445, which is the standard deviation of the six grades, should appear in that cell. This indicates that about 9 points is the typical amount of variation of this student's test scores from his or her average.

To "play" with this, go back to the cells with the data and replace the first value of 72 with 42. Notice that the standard deviation in cell C6 almost doubles, changing to 17.44706. The standard deviation is particularly sensitive to large deviations from the average value.

Example 19: Use a spreadsheet formula to compute the maximum and minimum grades in Example 18.

Solution: You already have put the data into a spreadsheet (although you need to restore any data values you changed while "playing". The all you need to do is put "=MAX(A1:A6)" into one cell (such as C8) and "=MIN(A1:A6)" into another cell (such as C10). This should give the correct values of 92 and 71, respectively.

Of course for a set of six grades you can easily find the maximum and minimum grades just by looking at the list. But what if you needed to find the maximum of a list of hundreds of grades? An advantage of spreadsheets is that they work just as easily for large amounts of data as for small amounts.

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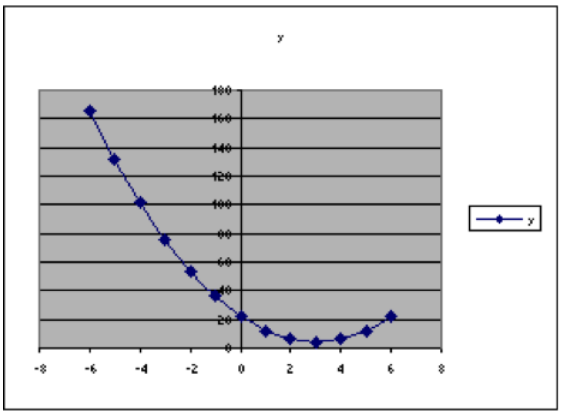
Section 6. Explore how the graphs of formulas change as the parameters change.

Example 20. Consider this formula $y=a{{(x-h)}^{2}}+k$, which is a generalization of the formula we have graphed in several examples in this topic. That example is $y=4+2{{(x-3)}^{2}}=2{{(x-3)}^{2}}+4$, so there $a=2$, $h=3$, and $k=4$. In this section, we will use the spreadsheet to graph this in a way that will make it easy to explore what happens when we change one or more of (which we call parameters.)

Setting up the spreadsheet:

1. We will graph this for values of *x* from -6 to +6. So put the label x in cell A1 and then start in A2 with these values and continue through A14.
2. Instead of putting the formula directly into column B, we will enter the three initial values $a=2$, $h=3$, and $k=4$ in cells over to the side. Please put the values in column G, cells G2, G3, and G4, respectively and the labels in column H.
3. Label column B as *y*. That is, enter y into cell B1. Then enter the formula in cell B2. Notice that the cell references for *a*, *h*, and *k* must be entered with absolute references, so the formula that would have been =2*(A2-3)^2+4 is, instead =$G$2*(A2-$G$3)^2+$G$4
4. When this is entered correctly, the numerical value will be 166.
5. After the formula is entered correctly, then copy and paste it into the remainder of the cells in column B, which is through B14.
6. Then select the data, with the labels. Highlight it. Then choose **Insert > Chart** and XY Scatter. Choose one of the options that connects the data with curves. Then keep clicking on “Next” until you click on “Finish.” This will produce a graph of the formula with the given values in it.

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
1	x	y						
2	-6	166					2	a
3	-5	132					3	h
4	-4	102					4	k
5	-3	76						
6	-2	54						
7	-1	36						
8	0	22						
9	1	12						
10	2	6						
11	3	4						
12	4	6						
13	5	12						
14	6	22						
15								



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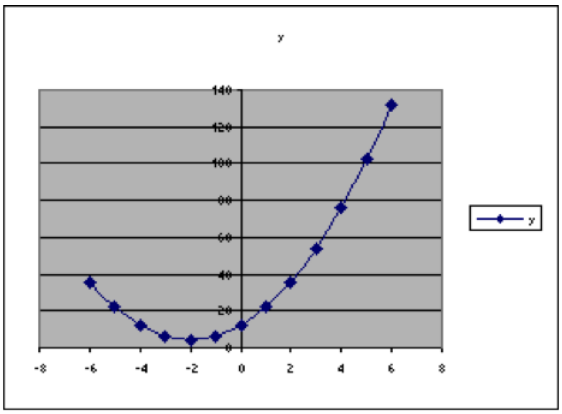
24.11: E1.10- Section 6 Part 2

Start exploring. How does changing *h* change the graph?

Start by changing *h* to -2.

That makes the spreadsheet look like the illustration below.

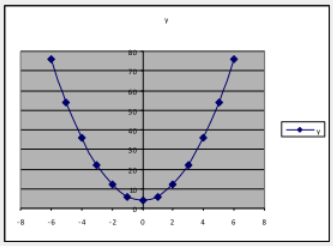
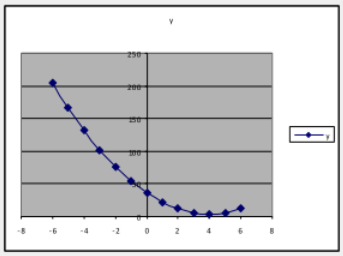
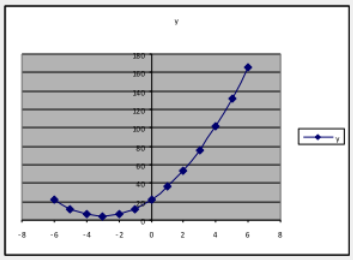
	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
1	x	y						
2	-6	36					2	a
3	-5	22					-2	h
4	-4	12					4	k
5	-3	6						
6	-2	4						
7	-1	6						
8	0	12						
9	1	22						
10	2	36						
11	3	54						
12	4	76						
13	5	102						
14	6	132						
15								
16								



Now we can notice that, when $h=3$, the lowest point on the graph is at $x=3$, and when $h=-2$, then the lowest point on the graph is at $x=-2$.

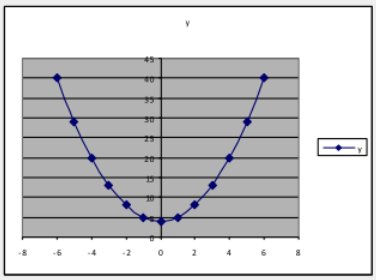
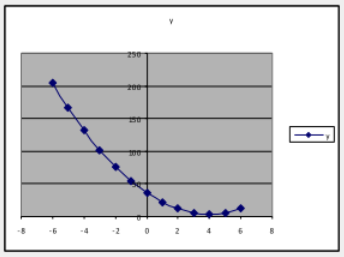
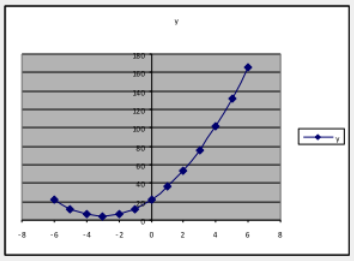
This suggests that maybe the value that is subtracted from *x* in the original formula is the one that determines where the lowest *y*-value is – that is, where the lowest point on the graph is.

Try $h=0$, $h=4$, and $h=-3$.

$h=0$ (leaving $a=2$ and $k=4$)	$h=4$ (leaving $a=2$ and $k=4$)	$h=-3$ (leaving $a=2$ and $k=4$)
		

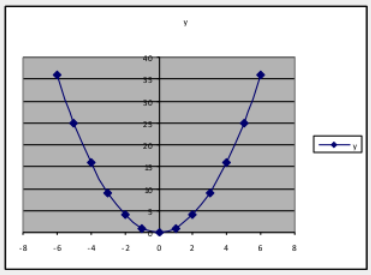
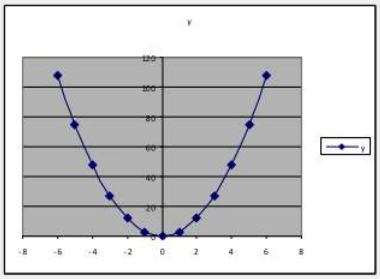
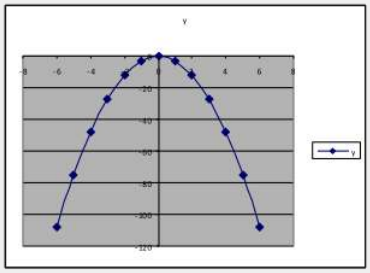
Do these results support the conjecture we made in the previous sentence? Answer: Yes.

Example 21. Using the same formula and spreadsheet as in Example 18, use the values $a=1$, $h=0$, and explore the effect of changing *k*.

$k=4$ (leaving $a=1$ and $h=0$)	$k=0$ (leaving $a=1$ and $h=0$)	$k=-7$ (leaving $a=1$ and $h=0$)
		

We find that changing k alone changes how far up or down the lowest point on the graph is. It appears that the *y*-value of that lowest point is *k*.

Example 22. Using the same formula and spreadsheet as in Example 17, use $h=0$ and $k=0$, and explore the effect of changing *a*.

$a=1$ (with $h=0$ and $k=0$)	$a=3$ (with $h=0$ and $k=0$)	$a=-3$ (with $h=0$ and $k=0$)
		

We find that changing a from a positive to a negative number makes the graph change from opening upward to opening downward. Making a larger (from 1 to 3) changes how large the y-values are, so that the y-values for $a=3$ are three times as large as those when $a=1$.

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CHAPTER OVERVIEW

25: Topic O- Combining Modeling Formulas

25.1: O.01- Overview
25.2: O.02- Section 1 Part 1
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25.1: O.01- Overview

Modeling, Part VI. Combining Modeling Formulas

Objectives:

1. Be able to make and use compound models, in which two basic models are added together.
2. Detect and avoid confounding of parameters in compound models.
3. Construct and use composite models by combining formulas with IF, MAX, or MIN functions.
4. Use compound or composite models to separate different effects reflected in a dataset.

Overview

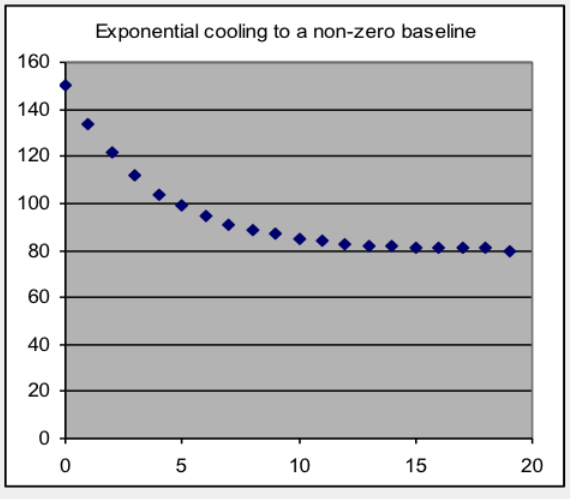
Combining models by addition:

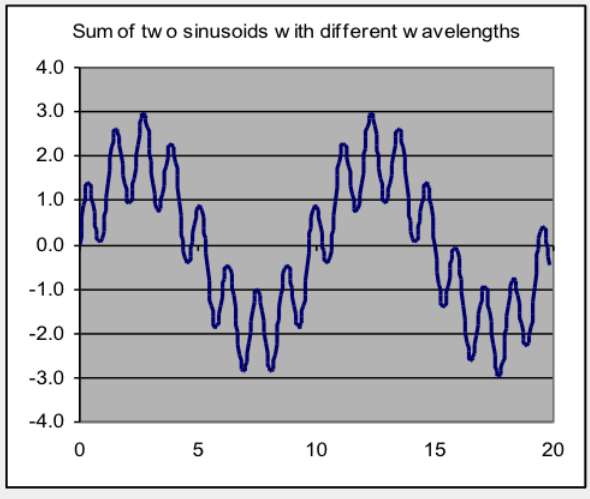
Nine different kinds of basic models have been discussed in this course so far: constant average, linear, quadratic, exponential, logistic, normal, sinusoidal, power and logarithmic functions. Each reflects a particular type of relationship that frequently occurs between two variables. Thus often the best-fit settings of one of these basic models will fully match a measurement set, except for random noise.

But sometimes more than one type of process significantly influences the relationship between two variables. In such cases, combining two or more basic models is needed to provide a formula that fits the data well. This increases the number of model parameters, but the automated fitting that Solver supplies makes it almost as easy to fit a six-parameter combined model as a two-parameter basic model.

Most often the combination is just the sum of two basic model formulas (such as an exponential model plus a baseline value, as would be appropriate to model the temperature of an object cooling off to an unknown room temperature). It may even be useful to combine two models of the same type but with different parameter settings – a model of outdoor temperature might consist of one sinusoid with a one-day wavelength (matched to day/night effects) added to another sinusoid with a one-year wavelength (reflecting seasonal changes). Occasionally what is needed for a good fit is more complicated, such as when the output of one basic model is used as the input to another type of model.

Examples of compound models, formed by addition of two basic models





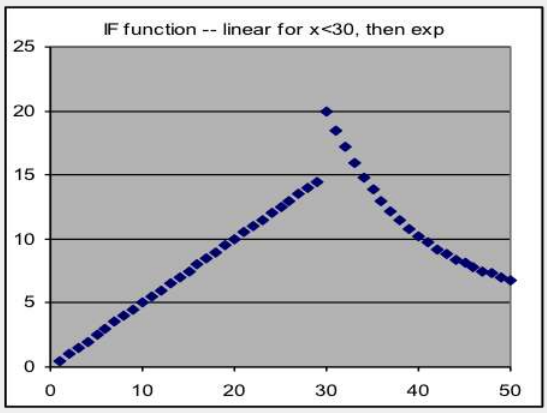
The main potential difficulty that can arise from combining models is that sometimes particular parameters in the two models control the same thing (e.g., vertical offset), a situation called *confounding*. When Solver searches for a fit in such a case, it is unpredictable which of these parameters it will change, or what final values they will reach. The solution to this difficulty is to drop one of the parameters from the compound model – which one is usually obvious when the meaning of the parameters is considered.

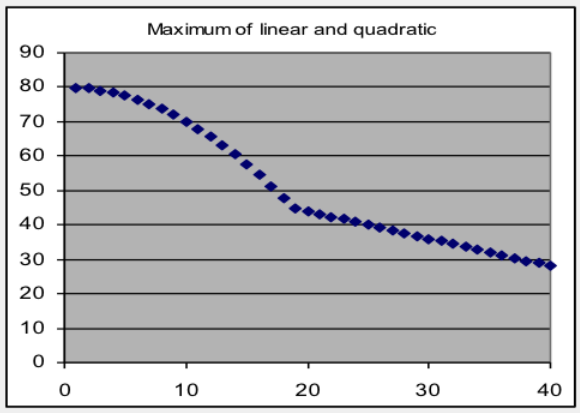
Combining models by composition (using different models for different ranges of input values):

Sometimes a relationship between two variables matches different formulas for different parts of the range of input values. One way this may be modeled in a spreadsheet is by using the different formulas with the IF function – for example, if C3 contains "=IF(A3>100,400*A3,4*A3^2)", the model function will change from linear to quadratic for values of x greater than 100. In a real modeling formula, the two formulas used in the IF function would also include parameter references such as $G$3 and $G$4. If the transition point is not known from the problem, it can be made a parameter, and thus deduced from the data (such a formula might begin "=IF(A3>$G$3,…").

Another way that different formulas can be used in different regions is to use the MAX (or MIN) functions so that the larger (or smaller) result of two formulas is used. This is often appropriate in situations where two separate processes are acting to limit the relationship. An example is the speed of a car as it brakes to a stop as fast as it can, where the rate of deceleration is limited by whichever of two effects is smaller: the friction of the tires with the road, and how much heat the brakes can dissipate.

Examples of composite models, formed by using different model formulas for different input ranges





An abrupt change in slope or value in graphs of models produced by these composition methods generally makes it obvious where the model switches from one formula to the other.

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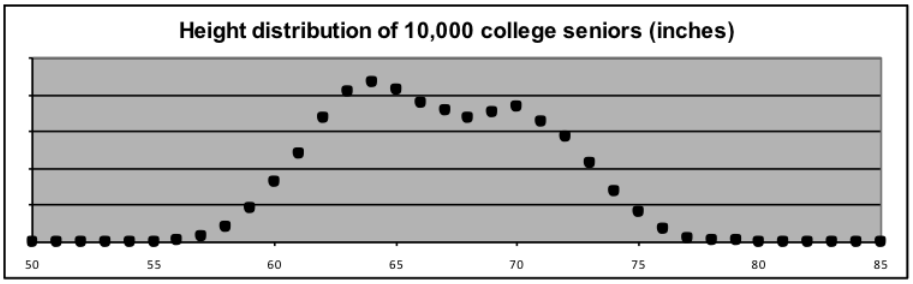
Section 1: Compound models—adding two model formulas to model combined effects

Compound-model application 1: Modeling mixed populations

In an earlier topic we discussed normal-distribution models, whose graph is a "bell-shaped" curve with a single peak. This model is appropriate in many situations where some characteristic of a group of generally-similar objects is distributed around an average value. But when two groups with different averages are mixed, the resulting distribution may have two partially-overlapping peaks, as illustrated below by a dataset on the height of a combined population of male and female students. A single normal distribution will not do a good job of modeling this data.

Example 1:

Height data from a co-educational population



The data that produced the above graph is shown in a table to the right. A plausible model for data of this kind is the sum of two normal distributions. This will entail six parameters: total, average, and width for each normal. The formula to be placed into C3 will thus be "=$G$3*NORMDIST(A3,$G$4,$G$5,FALSE)+ $G$6*NORMDIST(A3,$G$7,$G$8,FALSE)"; as always, it should be spread down column C beside all the data rows.As usual, we will set the initial values for the parameters to reasonable approximations before using Solver. In this case, we can do that by setting each "total" parameter to 5,000 (half the overall total), setting the "average" parameters to the approximately the *x* positions of the two peaks (64 and 70 are close enough), and setting both "width" parameters to a value, such as 2 or 3, that gives a model that is reasonably similar to the data. Then use Solver to minimize the sum of squared deviations, answering "G3:G8" in the "By changing cells" entry field so that all six parameters will be used. As shown below, a good fit is found.

Parameters (two-normal sum)	
5529.128	*Total 1*
63.72487	*Average 1*
2.685645	*Width 1*
4569.312	*Total 2*
70.38119	*Average 2*
2.728443	*Width 2*

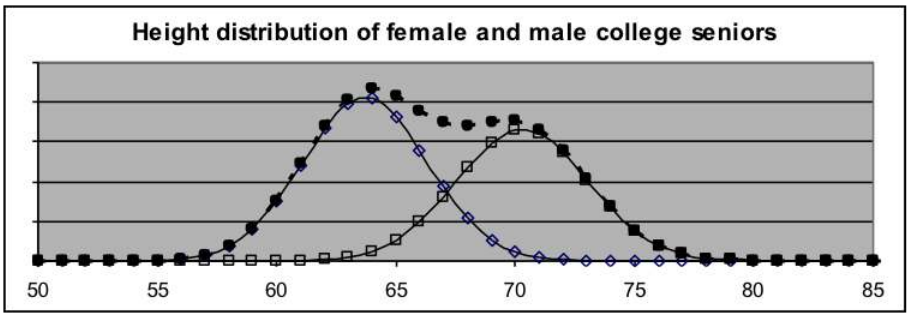
Goodness of fit of this model	
1710.994	Sum of squared deviations
7.429222	**Standard deviation**

Height	Students
48	0
49	0
50	0
51	0
52	0
53	0
54	0
55	2
56	12
57	29
58	79
59	180
60	326
61	481
62	673
63	813
64	864

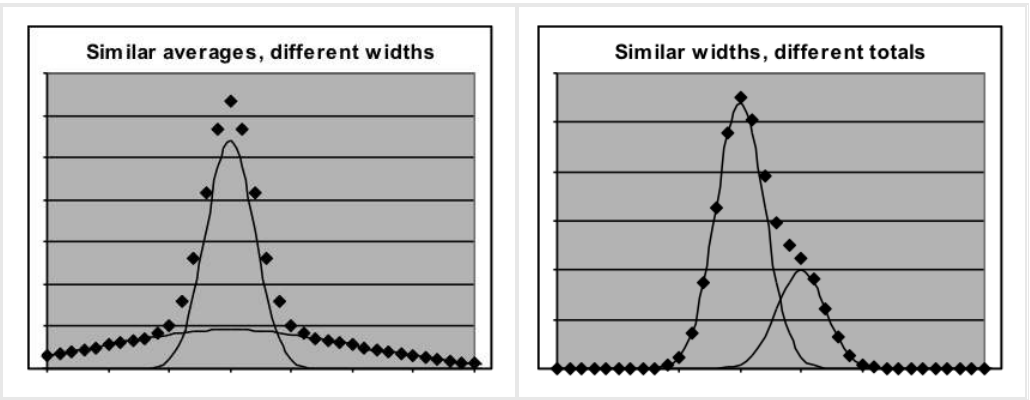
This best-fit model does more than give a formula for describing the height distribution of the combined male-female population. Because it fits the data so well, we can confidently deduce the characteristics of the male and female distributions from the parameters for the two normal distributions, even though we do not have any data that identifies gender, and even though both males and females contribute to each height total. Clearly the first normal describes the height distribution of females and the second normal describes the height distribution of males. Thus we can use these results to see that 45.7% is the answer to the question "*How much of this data came from males?*", and that 63.7 inches is the answer to the question "*What was the average height of the female part of this sample?*".

65	825
66	751
67	717
68	677
69	700
70	735
71	658
72	568
73	424
74	280
75	167
76	75
77	25
78	13
79	7
80	0
81	1
82	1
83	0
84	0

If we individually graph the two normal distributions found here along with their sum we can see the components that combined to form the data distribution:



This same sum-of-two-normal-distributions approach can be used in a variety of situations. Sometimes the component distributions have about the same average but greatly different widths, so that the resultant distribution has "fat tails". Other times the distributions have substantially different totals but nearby averages, so that the result looks like the larger distribution with a bump added on one side. It is often the case that an investigator is interested in only one of the component distributions, with the other values being ignored after the fitting process. An example of each of these kinds of two-normal situations are given below, with the components shown (thin lines) as well as the sum (dots) that reflects the data that would be observed.



Examples of the sum of two differing-parameter normal-distribution functions

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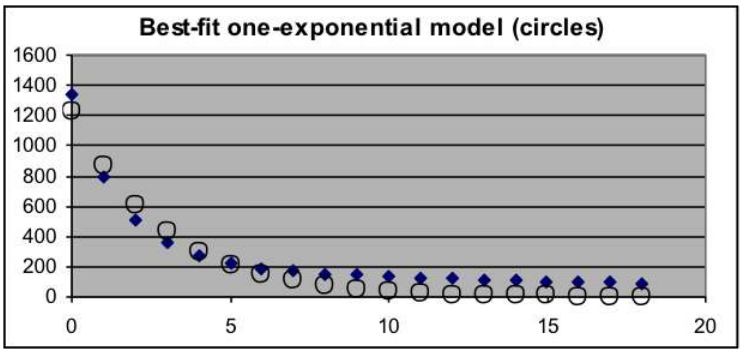
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25.3: O.03- Section 1 Part 2

Compound-model application 2: Determining the contents of a mixture of radioisotopes

Materials with small amounts of radioactivity can be used in medical diagnosis and treatment. The way these are made (irradiation in a nuclear reactor) often results in a mixture of different radioisotopes. Each type of radioisotope has an exponential decay pattern with a specific decay rate. Thus the best model for the overall radioactivity of the material at each time is the sum of two or more basic exponential models. Fitting the data with such a compound model will then show the decay rates and the relative amounts of each radioisotope formed. Since scientists have identified the decay rates of all the different radioisotopes, the fitting results usually are enough to identify them.

Example 2:*The data to the right show radioactivity measurements taken at one-hour intervals. Even though the data graph (the solid dots in graph below) has a shape similar to that of a decaying exponential, a single exponential-decay model (the circles in the graph below) does not fit the data well. Test whether the data could be fit well by the sum of two exponential models. If so, report the two decay rates and the relative activity of the two components.*

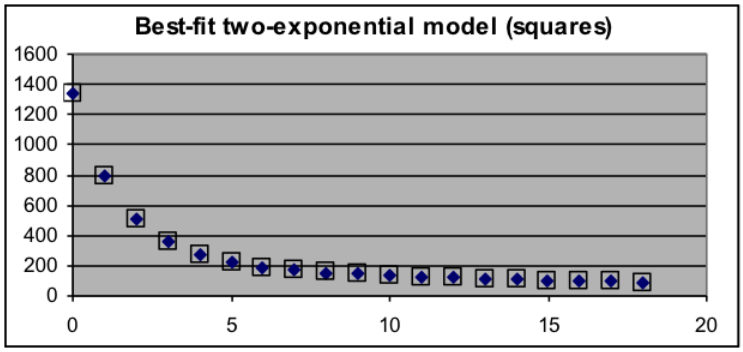


Hours	Activity
0	1333
1	799
2	513
3	359
4	270
5	220
6	191
7	173
8	154
9	145
10	137
11	134
12	122
13	113
14	109
15	104
16	95
17	94
18	88

Solution:

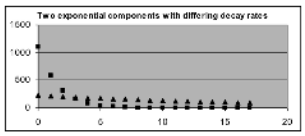
To fit the data to a model based on the sum of two basic exponential-decay models, four parameters will be needed (the initial value and growth/decay rate for each of the basic models), so the formula placed in C3 will be "=$G$3*(1+$G$4)^A3+$G$5*(1+$G$6)^A3", which will be spread down column C beside all the data values. Set the initial-value parameters G3 and G5 to about 600 (half the first data value). Set the growth-rate parameters G4 and G6 to different negative values (such as –10% and –30%) that make the model roughly match the data. Then use Solver to minimize the sum of squared deviations in H12 by changing the G3:G6 range of parameters. This produces the results below:

1108.925	*Initial value 1*
-47.40%	*Growth rate 1*
227.0572	*Initial value 2*
-5.11%	*Growth rate 2*
30.6017	Sum of squared dev.
1.428325	Standard deviation



Clearly, the sum-of-two-exponentials model fits this data much better than the one-exponential model. The best-fit parameters show that the measured radioactivity is produced by a fast-decaying component (a decay rate of 47.4% per hour) that starts out producing 83% of the activity, as well as by a slower-decaying component (a decay rate of 5.1% per hour) that starts out producing 17% of the activity.

Now that the parameters of the two components are known, the exponential model for each component can be graphed individually, showing how the slower-decay component becomes the main source of activity after about three hours.



If the model had not fit the data well, this would imply that there are more than two radioactive components, either in the initial mixture or because one or both of the components are decaying to a additional radioisotope. Such a situation would require a more sophisticated analysis to fully understand, but the analysis process above would still be enough to determine whether a two-radioisotope explanation is adequate.

If the oversimplified one-exponential model had been used, it could lead to some dangerous decisions, even though the values of that model were moderately close to the data over much of its range. The danger is because one of the most important issues about radioisotopes is how long a radioactive sample needs to be shielded before it is safe. The best-fit single-exponential model for this data has a decay rate of 30% per hour, which will predict about the true level of activity at 5 hours, but at 24 hours will predict an activity level that is over 300 times smaller than the actual one, since by that time the activity is almost completely due to the component whose decay rate is only 5.1% (99.99999% of the other component has already decayed by that time). Even at 18 hours (the last data point), the actual activity is more than 50 times the level predicted by the single-exponential model.

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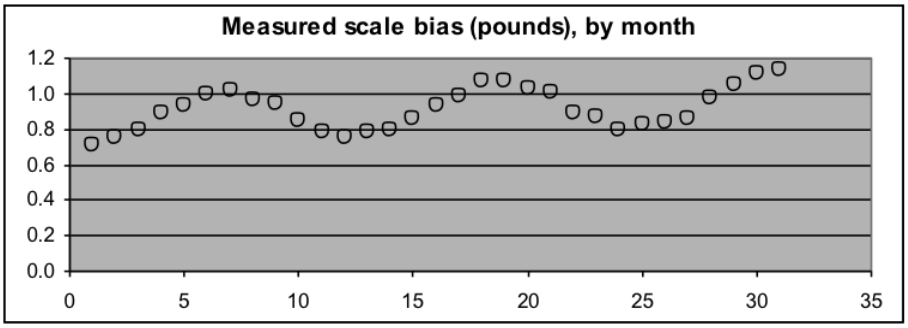
25.4: O.04- Section 2

Section 2: Avoiding confounded parameters—don't use two parameters to control the same thing

Models formed by adding two basic models together are very useful, but in some cases a problem arises because both models have parameters that control the same thing (e.g., vertical offset). When this is true, there is not any "best-fit" solution for these parameters, since any combination of vertical-offset values that gives a good fit could be replaced by other values which add up to the same thing. In such a situation, what values Solver will find for these "confounded" parameters depends unpredictably on their initial settings.

This problem can be avoided by eliminating one of the confounded parameters. If the compound model is the sum of a linear model and a sinusoidal model, for example, the linear *intercept* parameter and the sinusoidal *baseline* parameter both control the vertical offset. In this case, it would be best to leave out the sinusoidal baseline parameter (use only wavelength, amplitude, and phase), because the natural way to think about data of this kind is as a straight line with sinusoidal deviations.

Example 3:*A series of monthly calibration measurements of the bias in pounds of an outdoor scale produces the data shown to the right. An examination of the graph of the data (shown below) indicates that its pattern is a combination of a gradual multi-year trend (probably due to wear of some part) and a repeating seasonal variation (probably due to temperature variation). Use a compound model combining a linear model with sinusoidal variation to [i] determine the annual rate of change shown by the multi-year trend and [ii] to predict the bias 8 months after the last data point shown.*



Month	Bias
1	0.712
2	0.749
3	0.800
4	0.895
5	0.937
6	0.998
7	1.015
8	0.966
9	0.950
10	0.847
11	0.788
12	0.752
13	0.783
14	0.794
15	0.857
16	0.933
17	0.992
18	1.071
19	1.072
20	1.031
21	1.006
22	0.895
23	0.871

Solution:

The compound model should have the two linear parameters of *intercept* and *slope*, but it should use only three of the four sinusoidal parameters, since the *average* parameter is added on to the result in the same way as the *intercept*. So the formula of the model is *y = intercept + x * slope + amplitude**sin(2π*(*x+phase)/wavelength*), so C3 needs to be "=$G$3+A3*$G$4+$G$6*SIN(2*PI()*(A3+$G$7)/$G$5)", which is to be spread down beside the data as usual.

Once the spreadsheet is set up, we can use Solver to minimize the sum of squared deviations by changing the five parameters G3:G7.

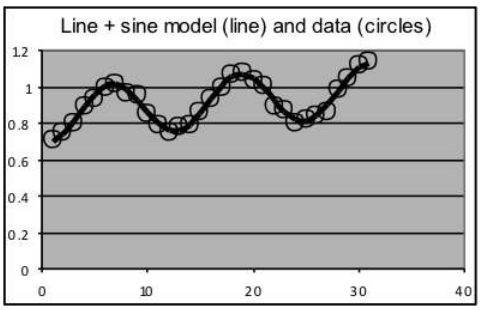
When you have a lot of parameters, it becomes more important to start Solver with initial values that ensure that are reasonably correct, so that Solver does not get lost in its search for the best-fit values. This is not difficult if you use the data+model graph for feedback, and make some simple estimates from the data. There is no need to start with a close match – Solver will do that work – but you want to avoid situations where Solver tries a very incorrect value for a parameter like wavelength, for example.

In this case, it will work to use values such as 1.0 for the intercept, 12 for the wavelength (since the temperature variations should have a 12-month cycle), and about 0.1 for the amplitude (since the valley-to-peak variation is about 0.2. An initial slope of zero is okay, since the true value is a small positive number, and an initial phase value of about 8 looks about right on the graph.

24	0.797
25	0.825
26	0.841
27	0.861
28	0.982
29	1.050
30	1.116
31	1.136

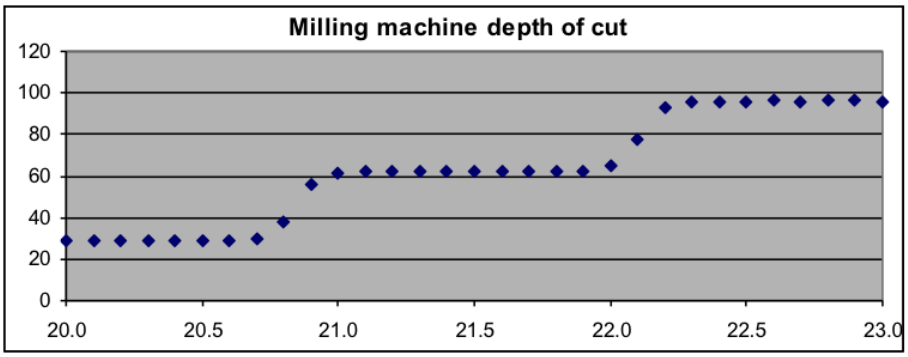
Solver produces these results:

Line+Sinusoidal Parameters	
0.837518	*Intercept*
0.004593	*Slope*
12.14912	*Wavelength*
0.143756	*Amplitude*
8.580468	*Phase offset*
Goodness of fit of this model	
0.006831	Sum of squared dev.
0.016209	Standard deviation



Answers: The fitting results show that the multi-year trend in the bias is 0.004593 pounds per month (the linear slope parameter), which is about **0.055 pounds per year**. Evaluating the model at 39 months (8 months after the last data value) gives a prediction for bias at that time of about **0.945 pounds**.

Note that the predicted bias value at 39 months is lower than the bias shown in the last data value, indicating that between these times the seasonal variation is larger than the long-term upward trend. Another interesting aspect of this problem is that simply fitting a linear model to the data would not have given good results; since the 2½-year pattern includes three upward-sloping segments and only two downward-sloping segments, a plain linear model would give a result more than 20% too large for the long-term trend.



Confounded-parameter problems can usually be avoided by thinking about the meaning of each parameter in the context of the data. If the information it conveys is already being supplied by an earlier parameter, leave it out. In the example below, a three-stage process (with two transitions) can be modeled by adding two logistic models. Since a logistic model has four parameters (rate, center, height, and floor), one might expect eight parameters in the compound model. But are any of these parameters redundant?

Example 4:*The data to the right record the depth of the cut of a milling machine during a portion of a production run. Fit an appropriate model to the data to make estimates of the time, to the nearest millisecond, of the midpoint of each transition.*

Solution:

Step transitions can be modeled by logistic functions; for this data, the sum of two logistic functions would be suitable.

The horizontal asymptotes in a logistic graph are controlled by the *floor* and *height* parameters. But in this sum-of-logistics model, *floor2* (the floor of the second logistic) will equal *floor1+height1*, and does not need a separate parameter in the model.

In this case, it appears that the model can be further simplified by using the same *height* and *rate* parameters for both transitions, leaving the model with five parameters: *floor*, *height*, *rate*, *center1* ,and *center2*, (use cells G3–G7 for these).

Secs	Depth
20.0	28.46
20.1	28.47
20.2	28.50
20.3	28.58
20.4	28.61
20.5	28.69
20.6	28.63
20.7	29.89
20.8	38.27
20.9	56.00
21.0	61.44
21.1	62.05
21.2	62.34
21.3	62.35
21.4	62.37
21.5	62.22
21.6	62.19
21.7	62.37
21.8	62.43
21.9	62.49
22.0	64.81
22.1	78.21
22.2	92.96
22.3	95.77
22.4	95.95
22.5	95.93
22.6	95.99
22.7	96.08
22.8	96.06
22.9	96.20
23.0	96.25

The model to be used is thus
y = *floor* + *height* / (1+0.018316^(*rate**(*x*–*center1*))) + *height* / (1+0.018316^(*rate**(*x*–*center2*)))

This means that the formula to be placed into C3 (and spread down beside the data) should be =$G$3+$G$4/(1+0.018316^($G$5*(A3-$G$6)))+$G$4/(1+0.018316^($G$5*(A3-$G$7)))

Set initial values for the parameters from the data, checking them by looking at the data+model graph. In this case, the *floor* is about 30, the step *height* is about 30, and the transition *centers* are at roughly 21 and 22. The *rate* is positive, so start with 1 (which the graph shows is too slow) and adjust it to 10.

Answers:
The best-fit results produced by Solver are shown to the right. The parameters relevant to the question asked are *center1* and *center2* , so the times of the middle of the two transitions are **20.837 seconds** and **22.105 seconds**.

28.4643	Floor
33.8087	Height
5.8546	Rate
20.8371	Center 1
22.1047	Center 2

An extreme case of confounding parameters can occur when the effect of all of the parameters in one of the models can be replicated by adjustments in the parameters of the other model. In such a case, there is no need for a compound model. Of the basic modeling functions we have discussed in this course, this situation arises when linear models and/or quadratic models are combined.

The sum of two or more linear models can always be expressed as a single new linear model that will give exactly the same output. The sum of two or more quadratic models, or the sum of linear and quadratic models, can be expressed as a single new quadratic model. Similar combinations occur for other polynomial models, with the result matching the highest-order polynomial used in the sum.

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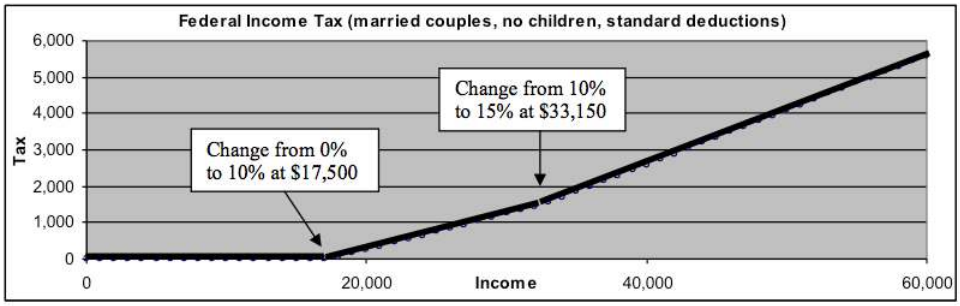
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25.5: O.05- Section 3

Section 3: Composite models—Using different formulas for different parts of the graph

Compound models add two or more modeling functions together. The other main way in which models are combined is to "compose" a new model by using different modeling formulas depending on what the input values are. The graphs of such composite models will usually have a sharp break, in value or slope or both, at each point in the input range at which there is a switch from one function to another.



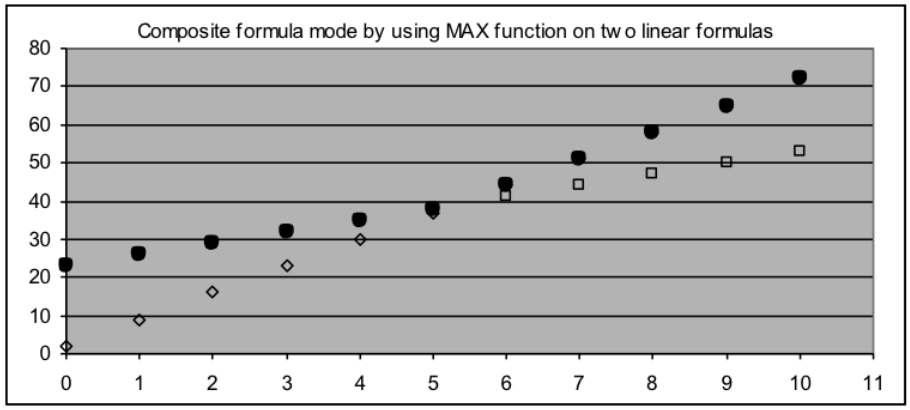
Income taxes due are described by a composite model with a change in slope for each tax bracket. In this case each piece of the model is a straight line segment starting at the end of the previous one.

Each segment of a composite model can be built of any kind of modeling formula. The model output can have abrupt jumps in value, although it is more usual for changes at the transition values to be in the slope, since few processes have big output changes for small input changes. Sometimes the position of the transition between formulas is a parameter of the model, so that the fitting process lets the data tell you where the transition is. In the common situation where the transition is at the intersection of two fitted formulas, the transition can usually be deduced algebraically by setting the two formulas equal to each other (using the fitted parameters as coefficients) and solving for the x and y of the crossing point.

Two main approaches are used to "composing" a composite model. The simplest approach is to use each input value to evaluate two or more formulas, then pick the largest of the results to use as the output value. The effect is to make a model whose graph follows the top of the graphs of each of the component formulas. This can be implemented in a spreadsheet with the MAX function, whose value is equal to the largest of its arguments (e.g., the value of "=MAX(14,22,-30,5)" is 22).

The graph and table below show the output of a model formula that uses two linear formulas ($y=3x+23$ and $y=7x+2$) as arguments to the MAX function. The solid circles are the output of the composite model that results from the MAX function, while the empty triangles and squares show the portions of the two linear formulas that are not used because the other formula is larger at that x value.

x	*y*
0	23
1	26
2	29
3	32
4	35
5	38
6	44
7	51
8	58
9	65
10	72



Example 5:
A cylindrical water tank drains through two outlets, one near the bottom of the tank and the other near the middle. Each outlet pumps at a steady (but different) rate until the water drops below the opening to its outlet. The dataset to the right shows the water depth measured in feet at one-minute intervals during drainage. Based on this data, find the answer to these questions:
[a] What percentage of the total pumping capacity is provided by the outlet at the bottom of the tank?
[b] What is the depth of the tank at its upper outlet?

Minutes	Depth
0	53.3
1	45.0
2	36.8
3	28.6
4	24.0
5	21.6
6	19.2
7	16.9
8	14.5

Solution:

Steady drainage of a cylindrical tank is a linear process, so it should work to model this situation with two straight lines – one to match the part of the process where the tank is being emptied through both outputs (the left part of the data graph), and the other line for the part of the process where the tank is emptying only through the outlet at the bottom (the right part of the graph). As the model to fit, we will use the MAX function with two linear formulas.

This model will have four parameters: *intercept1*, *slope1*, *intercept2*, and *slope2*; if cells G3:G6 are used, the formula in C3 should be “=MAX($G$3+$G$4*A3,$G$5+$G$6*A3)” (this should be spread down beside the data as usual).

Use a graph of the data to choose and check initial values for the parameters. The intercept for the first segment is near 50, and its slope is roughly –10; the second segment has an intercept of roughly 30 (extend the points after the bend back to the y-axis), and the second slope is less steep, roughly –3 or so.

[a] Since pumping capacity is proportional to the slope of the lines, the percentage requested in the problem can be computed from the ratio of *slope2*, which reflects the activity of the bottom pump, to *slope1*, which reflects the activity of both pumps together. This ratio is 0.28797, so **the bottom pump is 28.8% of total capacity**.

[b] To determine the height of the higher outlet, we need to know the location of the point at which the two lines cross. This can be done with algebra by solving the equation that results from equating the two linear formulas to each other, using the best-fit numerical coefficients that were found with Solver.



$$\begin{align}&\text{At the crossing point,}{{y}_{cross}}=53.27-8.23{{x}_{cross}}\text{ and }{{y}_{cross}}=33.46-2.37{{x}_{cross}}\\&\text{Therefore: }53.27-8.23{{x}_{cross}}=33.46-2.37{{x}_{cross}}\\&\text{}-8.23{{x}_{cross}}+2.37{{x}_{cross}}=33.46-53.27\\&\text{}-5.86{{x}_{cross}}=-19.81\\&\text{}{{x}_{cross}}=3.3805\text{ minutes}\\&\text{So}{{y}_{cross}}=53.27-8.23{{x}_{cross}}=53.27-8.23\cdot3.3805=53.27-27.82=25.45\text{ feet}\\\end{align}$$

Best-fit Solver results

53.27	*Intercept 1*
-8.23	*Slope 1*
33.46	*Intercept 2*
-2.37	*Slope 2*

The same approach can be used with the MIN function, which evaluates to the *lowest* value among its arguments (e.g., the value of "=MIN(14,22,-30,5)" is –30), but in this case the graph of the resulting model follows the bottom of the graphs of the component formulas, rather than the top.

The other main approach to creating a composite function is to explicitly test the input *x* value and select between two different formulas based on whether the test is true or false. This is done with the IF function, which uses the format IF(*test_expression*, *formula_if_true*, *formula_if_false*). For example, the formula "=IF(A3<6,4*A3,A3^2)" will use the 4*A3 formula (evaluating to 20) if A3=5, but will use the A3^2 formula (evaluating to 49) if A3=7. This approach is most useful when the test expression uses a model parameter that is also used in one or both of the formulas, since otherwise it is simpler to simply divide the data into several sections based on *x* values and to separately model each section.

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25.6: O.06- Section 4

Section 4: Miscellaneous advanced modeling issues and topics for further study

Unnecessary parameters: The ability to add more parameters by adding basic models together raises the question of where to stop this process. The additional parameters almost always enable the model to come closer to the data points, thus reducing the sum-of-squared-deviations figure—if the number of parameters is increased until there are as many parameters as data points, it is likely that the model can go exactly through every point. However, such a model is usually not better than some simpler one at predicting other output values.

The way to assess this is to find the model that produces the minimum standard deviation. Since the standard deviation is computed by an averaging process that increases the result when more parameters are used, using more parameters than are needed to follow the non-random trend of the data will result in a larger standard deviation.

A model that has more parameters than are needed is said to be *over-fitted*. This can be especially misleading for some kinds of modeling formulas whose extrapolation behavior is poor, such as high-order polynomials, because it results in the model replicating some of the noise in a particular data set rather than extracting the repeatable aspects of the process being measured. An example of an over-fitted model is show to the right.

Multiple and local solutions: Solver finds solutions by making small changes in each of the indicated parameters and keeping any changes that result in a smaller value for the goodness-of-fit indicator chosen (usually the sum-of-squared-deviations or the standard deviation). This process is continued until none of the changes improves the quality of the fit, even for very small changes. The search thus stops in a “valley” (actually more like the bottom of a bowl), where changing any parameter in any direction will make the fit worse.

Usually this process gives good model parameters even if the initial parameter values are quite different from the best-fit values (the search keeps moving downhill until it reaches the bottom of the bowl). But for some functions the fitting process has more than one solution; in such cases, the solution that is found depends on the initial conditions (and to a lesser extent on the details of the search process). If all the solutions are of equal quality, the search is said to have multiple solutions. But it is sometimes possible to have a “local minimum” solution, in which an answer is found that is better than any other nearby parameters settings, but still is significantly larger than some other solution.

Splines: A *spline* is a complex type of composite function that is often used during mechanical design processes to create smoothly-varying curves that do not match a simple mathematical formula. Splines break up the points along the desired path into overlapping groups, then fit modeling formulas to each group, with added provision to ensure that the slope of the curve changes gradually in the overlapping parts. Although splines are now generated by complex software using advanced versions of techniques similar to those introduced in this course, they were originally made by inserting thin flexible pieces of wood between pegs placed along the desired path, with the path of the bent wood measured or traced to produce a natural-looking curve.

Stability of solutions: Since best-fit parameter values are often used to explain aspects of the situation that was measured, it is of interest to know how dependable they are. How much will the best-fit parameters change if a new set of measurement (with new noise, as always) is taken? One way to find out is to add a small amount of random noise to the data and then re-fit the model. Doing this several times and observing the effects will provide information about the sensitivity of each parameter to noise.

It is possible to have parameters that are quite sensitive to noise even though the predictions of the model are much less sensitive. This can occur when small changes in the data lead the fitting process to change two different parameters in ways that almost cancel out in the result of the prediction formula. This is a milder version of the “confounding” that was discussed in Section 2 of this topic.

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CHAPTER OVERVIEW

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26.1: N1.01- Overview

Topic N. Additional Useful Modeling Formulas

Objectives:

1. Be able to use and understand the parameters of normal, logistic, and sinusoidal models.
2. Be able to use logarithmic models to make inverse models for exponential data.
3. Be able to select appropriate models based on the nature of the process producing measurements.

Overview

Additional useful modeling formulas:

While there are a great many types of mathematical functions that could be used for modeling in addition to the types discussed previously (constant, linear, quadratic, exponential, and power-function), three more types are particularly useful in practical applications: normal, logistic, and sinusoidal.

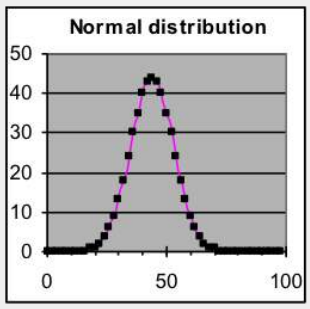
Normal models produce the “bell-shaped curve” graph that usually is a good fit to the distribution of some characteristic around its average value for a population (e.g., the distribution of adult heights). Normal models are particularly relevant to measurement because the distribution of noise values usually follows such a model. The parameters we will use for normal curves are the population size, the average value, and the width of the peak.

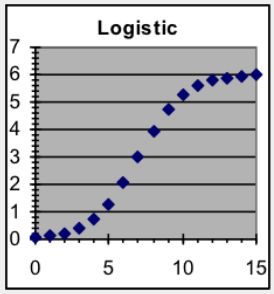
Logistic models describe a smooth transition from one steady value to a different steady value (e.g., the temperature of water coming from the hot water faucet when the heater is far from the faucet). A process that shows exponential growth at some times must eventually level off—a logistic model can often be used to describe the entire process in such a case. The parameters of a logistic model are the starting and ending output values, the center (the input value at which the output is halfway between the two limits), and the rate of change at the center.

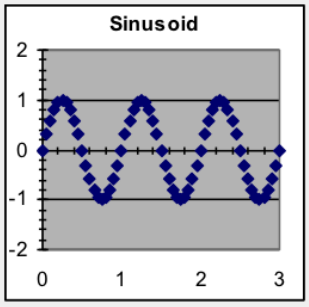
Sinusoidal models (based on the *sine* function discussed in the trigonometric topics) can be used to describe a great many processes that involve waves, vibrations, or rotation. This is the only model we will discuss that has a repeating pattern, and most repeating pattern in nature match some sinusoidal model. The parameters for a sinusoid are its *wavelength* (how often the pattern repeats), its *baseline* (the average output value over a full wavelength), its *amplitude* (how far output varies above and below the average), and its *phase* (the relative position of the peaks and valleys).

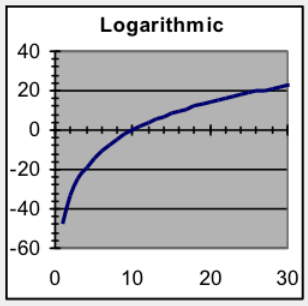
We will examine one additional type of model, even though it is used less often. Logarithmic models are the inverse of exponential models, and are thus appropriate in situations where equal-percentage changes of the input variable result in equal-amount changes of the output variable. The parameters used for a logarithmic model are closely related to those for the corresponding exponential model that would apply if x and y columns were exchanged: *growth rate* (or decay rate, if negative) and an *x-intercept* value (which would be the y-intercept value in an exponential model).

Illustrative graphs of the additional models discussed in this topic









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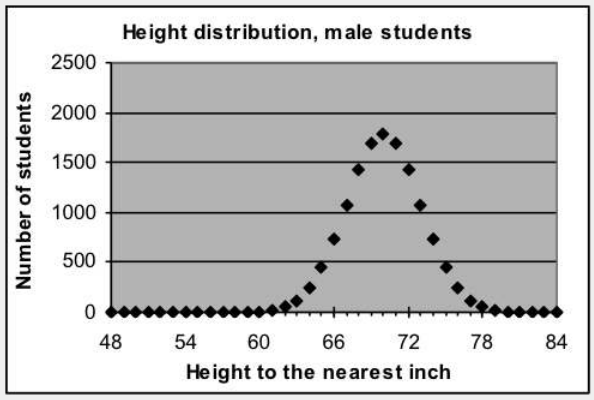
26.2: N1.02- Section 1

Section 1: Bell-shaped curves—Normal distribution models

It is often useful to model how some property is distributed among a collection of objects that are similar but not exactly alike. Such models are the mathematical answers to questions such as *How tall are college students, to the nearest inch?* or *If you weigh soda cans to the nearest gram, how many have each weight?* One good way to answer these questions is to fit a smooth model to data telling how many items have each value. Very often such a model will be close to the bell-shaped curve that mathematicians call a "normal" distribution because it usually results when many independent effects are combined.

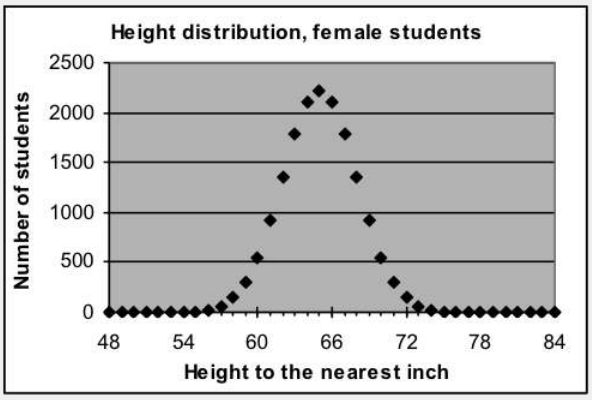
A typical application of normal models is the distribution of demographic information

Normal-curve model for heights to the nearest inch
parameters: total =13,379, average = 5'10", width = ±3″



Formula used: =13379*NORMDIST(A3,70,3,FALSE)

Normal-curve model for heights to the nearest inch
parameters: total =16,708, average = 5'5", width = ±3″



Formula used: =16708*NORMDIST(A3,65,3,FALSE)

Normal-distribution models are particularly interesting for measurements because noise usually results in repeated measurements of the same object having a normal distribution around the average value. The standard deviation of the noise corresponds to the width of the normal distribution.

Spreadsheets provide a predefined NORMDIST function that can be used in making normal-distribution models. The arguments supplied to the NORMDIST function are the average value for all items (this is the *x* value where the peak of the graph will be) and the width value that indicates how close a typical item is to the average. (NORMDIST also requires a logical constant as a third argument; this will always be FALSE for the way we are using the function.)

Math note: It is possible to compute the normal distribution with a standard mathematical formula, but it is more convenient to use the predefined NORMDIST function. Judge for yourself — the corresponding spreadsheet formula would be =0.3989*$G$3/$G$5*(0.6065^(((A3-$G$4)/$G$5)^2))

***Example 1:*Distribution of weights in a manufactured product**

The exact weight of standard manufactured products often varies slightly due to small random changes in the machinery that makes them. Measurements of how the weights of a large test sample of individual items are distributed provides information about how dependable the declared weights are. The dataset to the right shows the percentage breakdown of measurements rounded to the nearest gram for a product whose intended weight is 915 grams (about 2 pounds).

[a] Fit a normal-curve model to this data and report the parameters found.

[b] Do you think that it is accurate for the manufacturers to put a statement on the package of this item that says "Weight 915 grams"? Why or why not?

[c] Can you think of way to describe this weight distribution using two numbers that is more informative that just stating the average?

Weight Distribution	
Weight (grams)	Percent of total
911	1
912	3
913	8
914	17
915	25
916	23
917	15
918	6
919	2

Solution approach:

[i] Make a normal-curve template from the General Model template in *Models.xls*, with appropriate parameters: G3 for *Total* (label in H3), G4 for *Average* (label in H4), and G5 for *Width* (label in H5).

[ii] Put this formula in C3: **=$G$3*NORMDIST(A3,$G$4,$G$5,FALSE)**

[iii] Place the weight-distribution data in the template, then as usual spread the formulas in columns C, D, and E down to match the data (to row 11).

[iv] Make a graph of the data and model so that you can see how well the model and data match.

[v] Set the *Average* and *Width* parameters in G4 and G5 to values that are approximately correct, such as 915 and 1 in this case. (A close match for the graphs is not needed – just get close enough that you can see the peak in both the model and the data. Rough parameter settings are a good idea in all fitting processes, but they are essential for normal-curve models because a *Width* value of zero gives a computation error in NORMDIST, and if *Average* is far away from the correct value all the modeled points will be zero, giving Solver nothing to work with.)

[vi] Use Solver to find the best-fit parameter values for this model.

Answers:

[a] The best-fit normal curve has parameters Total=99.8, Average=915.4 gm, and Width=1.58 gm.

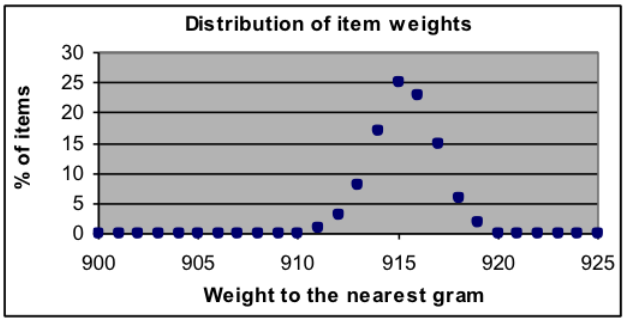
[b] Since 915 grams is less than 0.1% away from the average and most values are within a fraction of a percent of it, 915 seems to be the best way to describe the weight using a single number. But since different people might use the weight information in different ways, it would be better to make it clear that the number is an average, so that they are warned that some items will be heavier and others lighter.

[c] The best description of a characteristic that has a normal distribution is to report both the average and the width of the distribution. In this case we might say: "These items weigh 915.4 ±1.6 grams."

The worksheet for this example should be similar to this

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
1	Weight	Percent	y model	Residual	Squared			

2	(grams)	of total	prediction	deviation	Deviation		***Normal-curve parameters***	
3	911	1	0.542948	0.457052	0.208897		915.3682	*Average*
4	912	3	2.575238	0.424762	0.180423		1.576364	*Width*
5	913	8	8.167791	-0.167791	0.028154		99.76079	*Total*
6	914	17	17.32288	-0.322883	0.104253			
7	915	25	24.56767	0.432333	0.186911		**Model and data value counts**	
8	916	23	23.29893	-0.298926	0.089356		3	Number of parameters
9	917	15	14.77529	0.224708	0.050494		9	Number of dev. averaged
10	918	6	6.265626	-0.265626	0.070557			
11	919	2	1.776729	0.223271	0.04985		**Goodness of fit of this model**	
12							0.968896	Sum of squared dev.
13							**0.401849**	**Standard deviation**



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26.3: N1.03- Section 2 Part 1

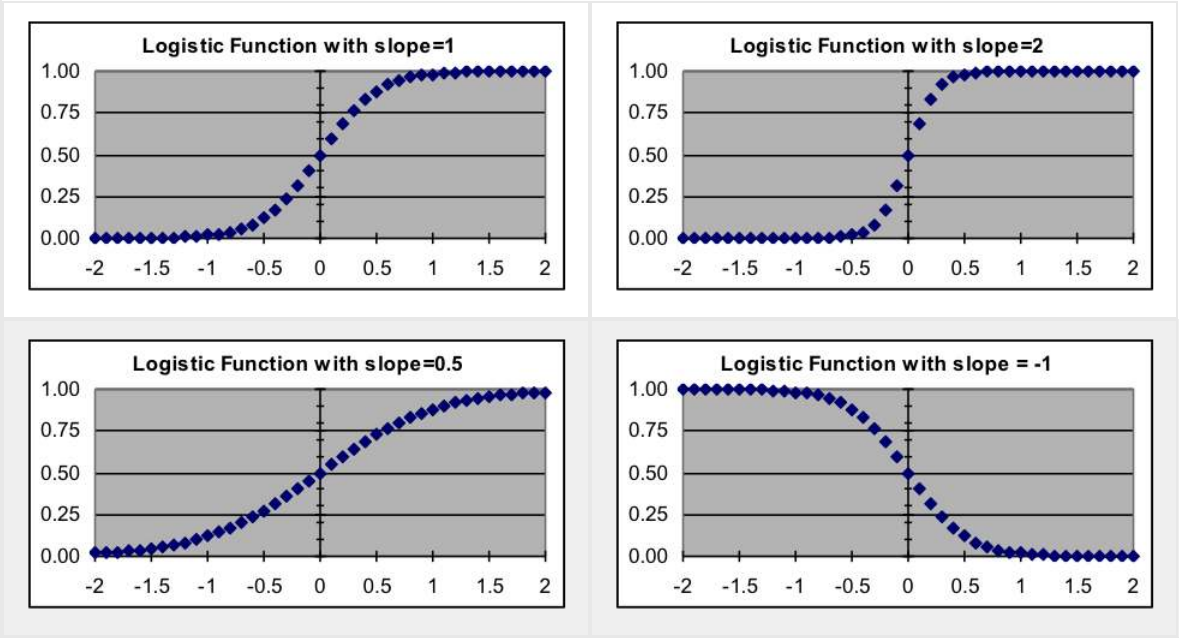
Section 2: Gradual transition between two steady values—Logistic models

Basic logistic formula: $y=\frac{1}{1+{{0.018316}^{{rate\cdot{x}}}}}$

Even though the formula above looks somewhat strange, its graph is actually rather simple. The graph starts by being almost horizontal barely above zero at the far left, then smoothly transitions through a midpoint to again become almost horizontal barely below one on the far right. The basic logistic function has its largest rate of change at the midpoint, where its slope equals the *rate* parameter.

[*Optional math fact:* Where does the 0.018316 in the formula come from? Actually, any positive number except 1.0 could be used as the base of the exponent and you would still get a logistic graph – it would just have a different slope at the center. The value 0.018316 is used because it makes the slope at the center equal the rate parameter.]

In the basic logistic formula, the only parameter is the rate of change at the center. Here are some examples of the effect of different values for that slope. In all cases, the *x*-axis scale is from –2 to +2.



Spreadsheet formula for basic logistic: =1/(1+0.018316^($G$3*A3)

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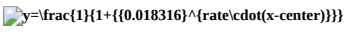
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26.4: N1.04- Section 2 Part 2

Making the logistic formula more versatile, stage one—shifting right or left

While it is useful to be able to control the rate of transition by using the rate parameter, more adjustability is needed to make this formula a good match to typical transition situations whose graphs have shapes similar to the basic logistic. We can handle this by adding more parameters to control the size and position of the curve.

So in addition to the rate/slope parameter, we will add a *center* parameter that tells the *x* value where the midpoint occurs, enabling us to shift the curve to the right (with positive *center* values) or to the left (with negative *center* values). Since the logistic formula equals ½ when the exponent is equal to zero, the way to put the *center* parameter into the formula is to subtract *center* from the *x* value before multiplying by the rate. This leads to the following version of the logistic formula, in which both slope and left-right position can be controlled:

Simple logistic formula (with center adjustment): 

The advantage of having the *center* parameter is that we can use it for data where we can see that there is a logistic transition from zero to one, but we don't know exactly when that transition occurred. By fitting a simple logistic model to the data, we can have the data tell us both when the transition occurred (the best-fit value for the *center* parameter) and what its maximum rate of change was (the best-fit value for the *rate* parameter).

Example 2: Wine fermentation time

As wine matures, a small amount of yeast grows and converts the sugar in the grapes to alcohol. At first growth is rapid because there is plenty of sugar and no alcohol; as more alcohol accumulates, the rate slows down as the increasing amount of alcohol and decreasing amount of sugar interferes with yeast metabolism, finally stopping when the maximum alcohol level for that type of wine is reached..

For the data shown below, [a] When was half the alcohol produced? [b] How fast was the percentage of alcohol compared to the saturation alcohol level increasing at that time?

Solution approach:

The data shows a transition from zero to one (since 100% = 1), so it can be modeled with the simple logistic model constructed from the General Model template. The appropriate spreadsheet formula to put in cell C3 is **=1/(1+0.018316^($G$3*(A3-$G$4)))** if we use cell G3 for the *rate* parameter and cell G4 for the *center* parameter.

Answers:

[a] Half the alcohol was produced by **8.68 days** of fermentation (from the best-fit value of the *center* parameter).

[b] At the midpoint of fermentation, the alcohol level was growing at the rate of **16.5% of the saturation level per day** (from the best-fit value of the *rate* parameter).

Days of fermentation	*Saturation level %*
0	0%
3	2%
6	15%
9	55%
12	90%
15	99%
18	100%
21	100%

The worksheet for this example should be similar to this:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
	X	y data	y model	Residual	Squared		y = 1/(1+0.018316^($G$3*(x-$G$4)))	
1	Days	Saturation	prediction	deviation	deviation		***Simple-logistic parameters:***	
2	0	0%	0.003294	-0.003294	1.09E-05		0.164513	*Rate*
3	3	2%	0.023243	-0.003243	1.05E-05		8.680789	*Center*
4	6	15%	0.146279	0.003721	1.38E-05			
5	9	55%	0.552322	-0.002322	5.39E-06		**Model and data value counts**	
6	12	90%	0.898822	0.001178	1.39E-06		2	Number of parameters
7	15	99%	0.984607	0.005393	2.91E-05		8	Number of deviations averaged
8	18	100%	0.997834	0.002166	4.69E-06			
9	21	100%	0.999699	0.000301	9.09E-08		**Goodness of fit of this model**	
10							7.59E-05	Sum of squared deviations
11							**0.003556**	**Standard deviation**

The data and model match well, but it would be difficult to give an accurate value for the midpoint or the rate of change at the center by visually reading the graph. This is where the numerical aspect of the model is particularly useful.

Didn't we do something like this before? The technique used here of implementing a shift in *x* position by subtracting the amount of the shift from the *x* value used in the formula should be familiar. It is essentially the same approach that we used in an earlier topic when we subtracted the year 1992 from *x* in the model formula rather than subtracting it from the input values in the dataset. But the difference here is that the amount of the shift is a *parameter*, so we don't have to know its value in advance—we can instead get the model-fitting process to tell us the exact amount of shift that the data implies.

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26.5: N1.05- Section 2 Part 3

Making the logistic formula more versatile, stage two—different step sizes, up/down shifts

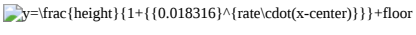
Transitions that follow a logistic pattern need not involve only steps from zero to one. To complete our generalization of the logistic formula, we will add parameters that let us adjust the curve to take other step sizes, and to have other baselines.

Both new parameters are straightforward to implement mathematically. To make the amount of vertical change something different than 1, we will multiply the simple formula by a scale factor. We will call this the *height* parameter since the curve will now have *y* values ranging from zero to *height*.

Moving the baseline up or down is even simpler mathematically. We will add a *floor* parameter to the scaled formula. Positive values for *floor* will move the curve up; negative values for *floor* will move the curve down. (Note that we add *floor* at the very end. It is not multiplied by *height*.)

So the full logistic model we have constructed has four parameters, and lets the data (via the model-fitting process) determine what their values are:

- *rate* (cell G3) — The controls the slope of the graph at the midpoint, which reflects how fast (and in which direction) the transition occurs. The slope at the midpoint equals *rate*×*height*.
- *center* (cell G4) — The midpoint of a transition is not usually at an input value of zero, even though the midpoint of the graph will be when the exponent expression is equal to zero. The formula can be made to reflect this by subtracting a *center* parameter from the input variable *x*.
- *height* (cell G5) — The amount of change in output will not always be 1.0. Multiplying the basic logistic model by a *height* scaling parameter will make its values go from zero to *height* instead of from zero to one.
- *floor* (cell G6) — Not all transitions have zero as the lower limit for the output value. We can allow for this with a *floor* parameter that is always added to the basic logistic formula.

The resulting full logistic model formula is 

Corresponding spreadsheet formula:=$G$5/(1+0.018316^($G$3*(A3-$G$4)))+$G$6

Example 3: *Fit a full logistic model to the dataset to the right showing water temperature during the first 10 seconds after a hot-water faucet is opened, then answer these questions about the model:*

[a] What real-world quantity corresponds to the *floor* parameter? What is its value for the best-fit model for this data?

[b] What real-world quantity corresponds to the *center* parameter? What is its value for the best-fit model for this data?

[c] How can you compute an estimate of the hot-water temperature from the best-fit parameters? What is its value in this case?

[d] How fast was the water warming up when the water temperature was halfway through the transition?

Water Temperature	
Seconds	Degrees F
1	68.8
2	68.7
3	70.9
4	86.9
5	115.5
6	124.5
7	125.4
8	125.5

9	125.7
10	125.7

[i] Use the General Model template to make a logistic worksheet, using the formula shown above.***Solution approach:***

[ii] Copy the data to the worksheet and make a graph of data and model.

[iii] Preset to approximate values to speed up Solver's search: *rate*=1, *floor*=70, *height*=55, *center*=4

[iv] Use Solver to find the best-fit parameters.

Answers:

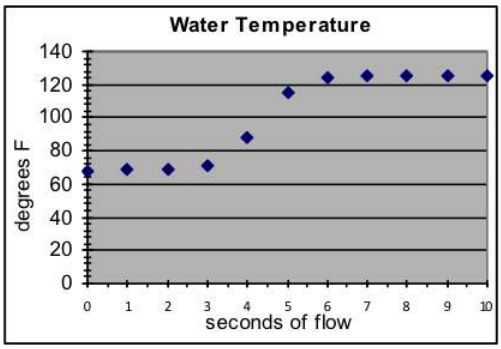
[a] The *floor* parameter corresponds to the temperature of the cooled-off water that has been standing in the hot-water pipes. In this case, that temperature was 68.5 °F.

[b] The *center* parameter corresponds to the time in seconds from when data-taking was started until the water temperature had risen halfway to the hot-water temperature. In this case, the midpoint was at 4.31 seconds.

[c] In this model, the hot-water temperature is the sum of the *floor* and *height* parameters, 125.6 °F.

[d] This is the product of the *rate* and *height* parameters, which is 31.9 °F/second in this case.

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
1	X	y data	y model	Residual	Squared			
2	Secs	Deg F	prediction	deviation	deviation		***Logistic parameters***	
3	1	68.5	68.56844	-0.068438	0.004684		0.559017	*Rate*
4	2	69.1	68.85719	0.242814	0.058959		4.310686	*Center*
5	3	71.2	71.42354	-0.223536	0.049968		57.05165	*Height*
6	4	87.6	87.53105	0.068945	0.004753		68.53369	*Floor*
7	5	115.5	115.5249	-0.024861	0.000618			
8	6	124.2	124.3091	-0.109083	0.011899			
9	7	125.5	125.4462	0.053849	0.0029			
10	8	125.4	125.5704	-0.170428	0.029046		**Model and data value counts**	
11	9	125.8	125.5837	0.216258	0.046767		4	Number of parameters
12	10	125.6	125.5852	0.014834	0.00022		10	Number of dev. averaged
13								
14							**Goodness of fit of this model**	
15							0.209814	Sum of squared deviations
16							**0.187**	**Standard deviation**



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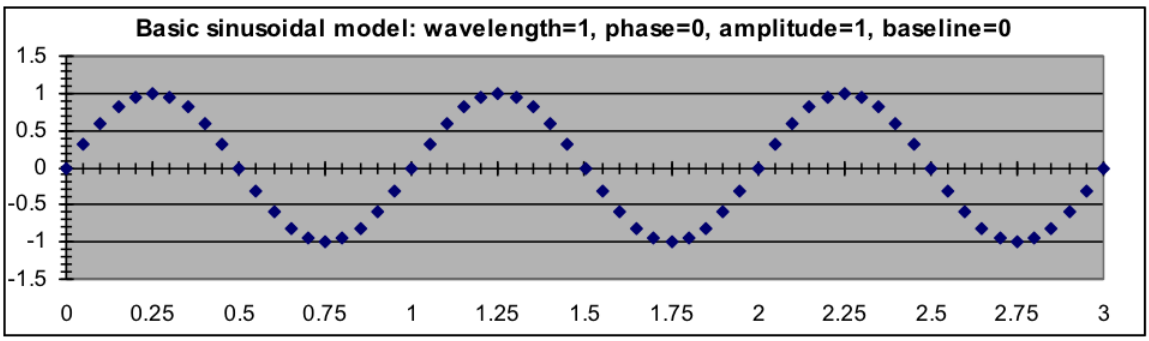
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26.6: N1.06- Section 3 Part 1

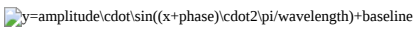
Section 3: Sinusoidal Models

Repeating behavior is very often encountered in the world, with examples such as ocean waves, heartbeats, vibrations of a spring, rotation of wheels, the phases of the moon, and the pattern of the seasons. Because none of the modeling functions we have discussed so far have repeating features, we need something more to be able to model repetitive phenomena.

For most repetitive situations, models based on the same sine function used in trigonometry will serve our needs. This may seem surprising—what do triangles and waves have to do with each other? One reason that mathematics is so powerful is that the same patterns show up in a great variety of different situations, which means that mathematical tools that were developed for one purpose often turn out to be useful in other contexts.



For modeling, we will want to be able to modify the shape of the standard sine function in several ways. First, we want to control how often the pattern repeats—the **wavelength**—so that we are not limited to the wavelength of the sine function (2π, or about 6.2832). Secondly, we need to be able to shift the pattern to the right or left so that the peaks and valleys of the model match the peaks and valleys of the data—this shift is called the **phase** of the model. Thirdly, we need to be able to match the amount of up-and-down variation in the model—the **amplitude**—to the amount of variation in the data. When the average value is not zero, we will also need to add a constant **baseline** to the model. The mathematical formula that serves these modeling needs will thus make use of four parameters:

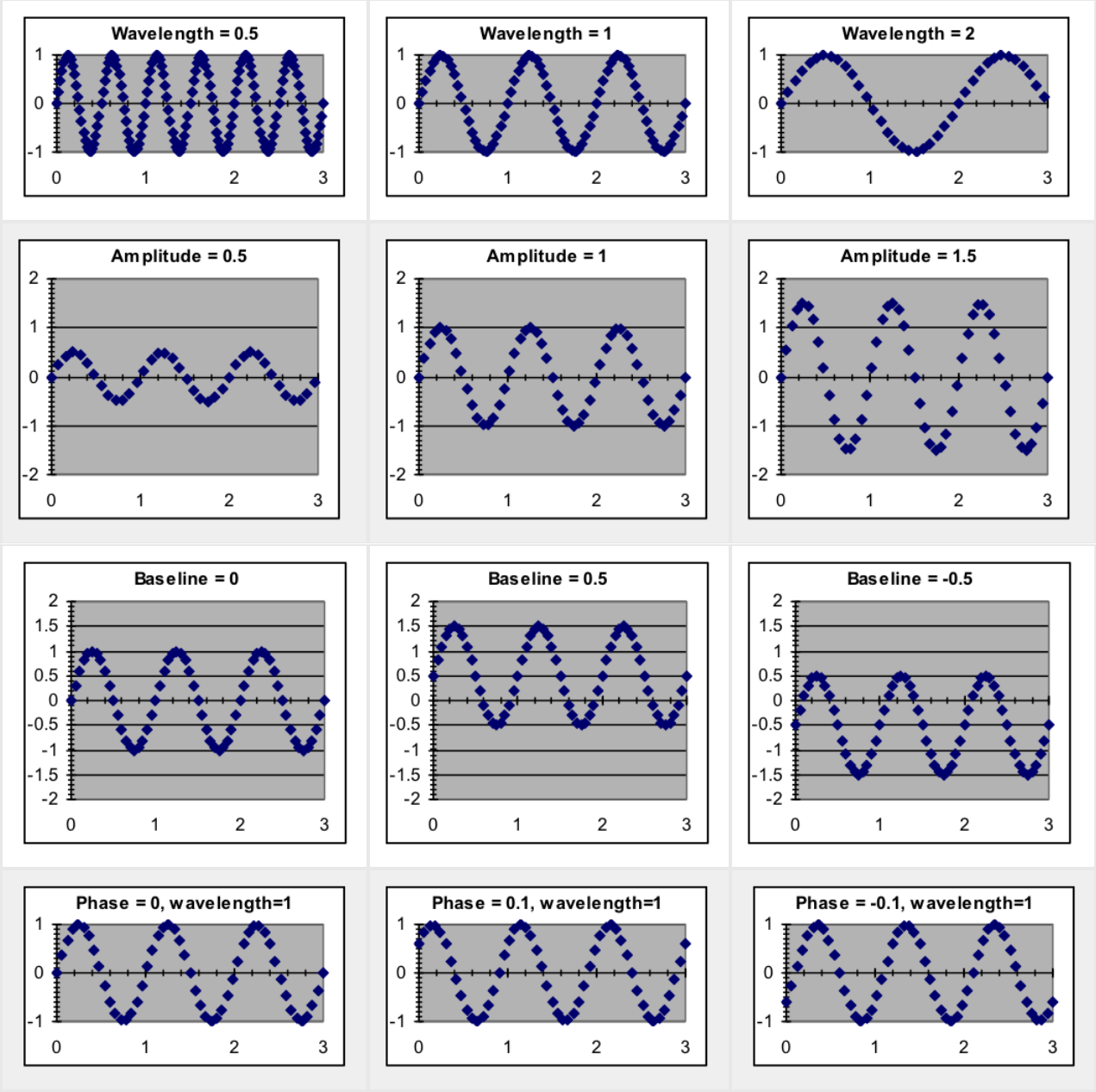


If the wavelength, phase, amplitude, and baseline parameters are in G3, G4, G5, and G6, respectively, then the corresponding spreadsheet formula to be put into cell C3 of the model template is:

=$G$5*SIN(6.2832*(A3+$G$4)/$G$3)+$G$6

Summary of sinusoidal parameters

- **Wavelength**—How often does the pattern repeat? Wavelength is in the same units as the *x* data, and the ratio of *x* to the model wavelength is multiplied by 2π because that is the natural wavelength of the SIN function. When the *x* values are based on time, the wavelength is sometimes referred to as the *period* or as a *cycle* of the sinusoid.
- **Phase** (or **phase shift**)—Where in the pattern is the *x* = 0 point? The basic pattern (which has a phase shift of zero) can be shifted to the left or the right. Note that because the pattern repeats, a shift to the left of ¼ wavelength would result in the same pattern as a shift to the right of ¾ wavelength.
- **Amplitude**—This is simply a scale factor that determines how far the maximum and minimum *y* are from the average. While it is mathematically possible to use a negative amplitude (the pattern turns upside down), this is not done since the same effect can be obtained by a half-wavelength phase shift.
- **Baseline**—The average around which the pattern varies. This can be modeled by addition to the result of the SIN function after multiplying by the amplitude.



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Month	Temp
1	47.8
2	45.3
3	54.0
4	72.6
5	83.4
6	91.9
7	92.9
8	84.8
9	79.5
10	69.1
11	60.1
12	46.4
13	50.8
14	47.4
15	62.8
16	71.9
17	83.1
18	91.2
19	93.4
20	88.4
21	75.5
22	67.0
23	59.6
24	44.3

Example 4: *Report and interpret the best-fit sinusoidal model for the dataset to the right, which records the amount that the average Fahrenheit temperature for a month is above or below the average temperature for the entire two-year period. The months are numbered starting with 1 for January 2003.*

Solution approach:

[i] Copy the dataset to the Data Scratch Pad worksheet and display a scatter plot.

[ii] From the graph, make rough estimates of the amplitude and wavelength for the data, so that you will have values to use as initial parameter settings.

[iii] Use the model formula explained in this section to create a sinusoidal model worksheet from the General Model template in Models.xls, with parameters for wavelength, phase, and amplitude. Set the wavelength and amplitude parameters to the values you estimated in step [ii] above (leaving these parameters blank or zero will cause errors or confuse Solver).

[iv] Copy the dataset to the worksheet, then spread the formulas in columns C, D, and E down to match the data, as usual.

[v] Use Solver to find the best-fit parameters, and report your results.

Answer:

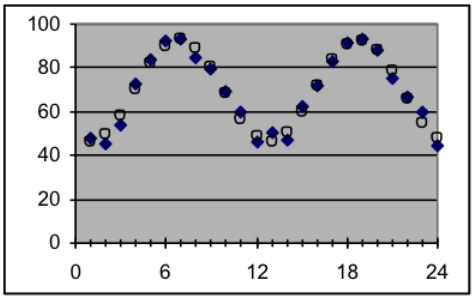
The best-fit sinusoidal model to this data has a **wavelength of 11.9 months**, a **phase shift of 4.0 months**, an **amplitude of 23.3 degrees F**, and a **baseline average of 69.1 degrees F**. The standard deviation for the fit is 2.8 degrees F.

Interpretation: The monthly temperature-difference averages at the location this data came from range by about 23 degrees above and below the yearly average of about 69 degrees. The pattern repeats about every 12 months, with the maximum-positive-slope point (i.e., spring) in April (month 4). Individual monthly averages can be expected to be within 2.8 degrees of this model about 2/3 of the time.

The final worksheet should look similar to this:

x	y data	y model	Residual	Squared		y = $G$5 * SIN((x-$G$4)*6.2832/$G$3)	
Input	Output	prediction	deviation	deviation		***Sinusoidal parameters***	
1	47.8	46.37	1.43	2.0577		11.85328	*Wavelength*

2	45.3	49.43	-4.13	17.0726	3.960422	*Phase shift*
3	54	58.07	-4.07	16.5328	23.28311	*Amplitude*
4	72.6	69.89	2.71	7.3542		
5	83.4	81.64	1.76	3.0990		
6	91.9	90.08	1.82	3.3073	**Model and data value counts**	
7	92.9	92.89	0.01	0.0002	3	Number of parameters
8	84.8	89.28	-4.48	20.0901	24	Number of dev. averaged
9	79.5	80.26	-0.76	0.5799		
10	69.1	68.31	0.79	0.6226	**Goodness of fit of this model**	
11	60.1	56.72	3.38	11.3938	169.8698	Sum of squared deviations
12	46.4	48.70	-2.30	5.2707	**2.844124**	**Standard deviation**
13	50.8	46.44	4.36	19.0293		
14	47.4	50.57	-3.17	10.0663		
15	62.8	59.96	2.84	8.0596		
16	71.9	72.01	-0.11	0.0132		
17	83.1	83.41	-0.31	0.0974		
18	91.2	91.01	0.19	0.0357		
19	93.4	92.72	0.68	0.4664		
20	88.4	88.06	0.34	0.1155		
21	75.5	78.32	-2.82	7.9745		
22	67	66.19	0.81	0.6531		
23	59.6	55.01	4.59	21.0864		
24	44.3	47.85	-3.55	12.6379		



Sinusoidal cycles and phases

Cycles: There is a close relationship between sinusoids and rotation. A spinning object repeats its sequence of positions every rotation, and the height of a point on a circle rotating at a constant speed varies over time according to a sinusoidal pattern. The wavelength of this sinusoid equals the rotation time of the circle, and the amplitude equals the radius of the circle. The phase is

how long it has been since the point last passed the midpoint in an upward direction. The baseline is the height of the center of the circle.

Left-right shifts of the sinusoidal pattern: The standard sinusoid crosses the *y* axis where the curve has the steepest upward slope, where it is increasing past the average value. But the curve can be shifted right or left so that it crosses the *y* axis at some other point of the cycle. The distance from the nearest standard steepest-slope point to the actual crossing point is called the **phase** or the **phase shift** of the sinusoid. The phase shift is added to the *x* value before it is used inside the SIN function.

Two sinusoidal processes with the same wavelength can be *in phase*, in which case they will add together if they apply to the same measurement, or be *out of phase*, in which case there may be partial cancellation. Sometimes phase shifts are talked about in terms of angular degrees (e.g., "The electric and magnetic waves are 90 degrees out of phase"); this is based on taking a wavelength as being 360 degrees due to the analogy with rotation), so a phase shift of 90° means a phase shift of ¼ of the wavelength.

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26.8: N1.08- Section 4

Section 4: Logarithmic models (the inverse model for exponential data)

The last basic models we will introduce in this course are *logarithmic* models. These are based on the LOG spreadsheet function discussed in an earlier topic. The main reason that logarithmic models are useful is that they are the inverse of exponential models —for a logarithmic model, equal-percentage changes in the input variable *x* result in equal-amount steps in the output variable *y*.

An example of a common use of a logarithmic model is in computing how old an archeological object is from how much radioactive Carbon-14 remains in it. The radioactivity of several objects of known age (such as tree rings) is measured, and then a model that predicts age from radioactivity level is fit to this data. Finally, measurements are made of the radioactivity of other similar objects (such as wooden tools) for which the age is unknown, and the model is used to estimate their age. (If you wanted to predict the level of radioactivity from the age you would use an exponential model, but here the roles of the variables are reversed.)

The two parameters of a logarithmic model are closely related to those for exponential models. The *growth rate* (or *decay rate*, if negative*)* parameter is the same as for exponential models, although in this case it is used as the second parameter (the logarithmic base) of the spreadsheet LOG function. Since the roles of *x* and *y* have been reversed, the initial-value (i.e., y-intercept) parameter from the exponential model become an *x-intercept* parameter in the logarithmic model. While there are several different ways that a logarithmic modeling formula could be written, the most convenient for our purposes will be , which implies a formula in C3 of "=LOG(A3/$G$3,1+$G$4)".

Example 5: *For a particular measurement apparatus and technique, radiation readings of 1274, 1085, 865, 697, and 276 were observed for objects with known ages of about 1100, 2400, 4300, 6100, and 13800 years, respectively. Use this data to create a model that predicts the age of similar objects, and use that model to estimate the age of such an object whose radiation measurement is 515.*

Solution:
[i] First, we must make an appropriate data table. Since we will be using the radiation readings to predict age, we will put the radiation readings into the column A and the respective age values to the right in column B.
[ii] Make a modeling spreadsheet, setting C3 to =LOG(A3/$G$3,1+$G$4) and spreading that model formula down to row 7 beside the data. Put the labels "x-intercept" and "rate" to the right of the parameter cells G3 and G4.
[iii] As usual, include deviations in column D and squared deviations in column E, and enter a sum-of-squared-deviations formula into a cell such as H8.
[iv] Adjust the parameters in G3 and G4 to initial values that make the model graph roughly match the data (example: G3 = 2000 & G4 = −0.0001)
[v] Use Solver to find the best-fit parameters: G3 = 1451 and G4 = −0.0001203; the implied model formula is *age* = LOG(*radiation*/1451, 1−0.0001203).
[vi] Enter the radiation level 515 into A8, giving an estimated age of 8,612 years.

Radiation reading	*Age in years*
1,274	1,100
1,085	2,400
865	4,300
697	6,100
276	13,800

NOTE: What is the real-world meaning of these model parameters? The growth/decay rate of −0.0001203 implies that 0.01203% of the Carbon-14 atoms decay each year. The "x-intercept" parameter shows what radiation-reading input would give an age output of zero; that is, what the radiation reading would be for a brand-new object.

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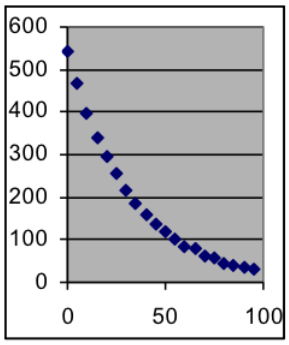
26.9: N1.09- Section 5

Section 5: Recognition of appropriate models

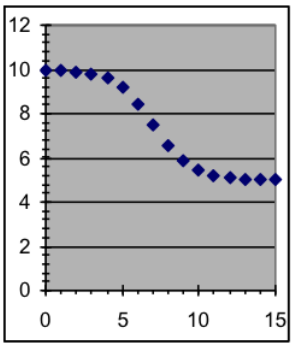
Many other model formulas exist (including many possible combinations of models, as will be discussed in a later topic), but the set of basic models that have been discussed in this course cover the most important patterns that arise in application settings. This will be useful to you even if you do not do more model-fitting yourself, since it gives you ways to talk about patterns you encounter and to know what is being referred to in written material.

Example 6: For each of these graphs, which of the basic models discussed in this course best matches it: linear, quadratic, exponential, power-function, normal, logistic, sinusoidal, or logarithmic?

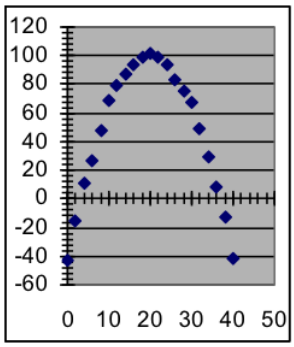
[a]



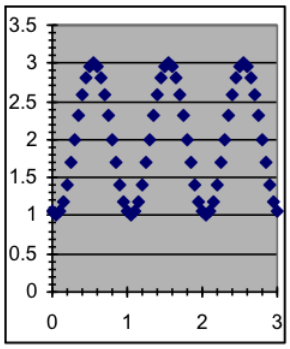
[b]



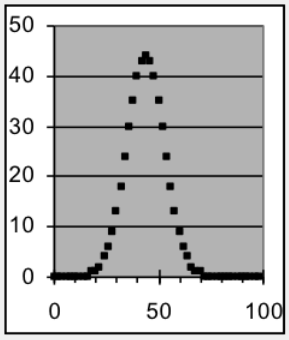
[c]



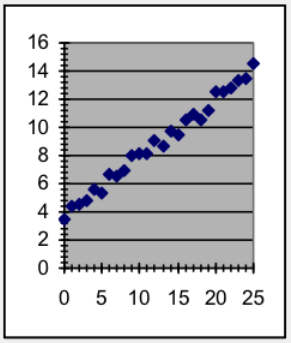
[d]



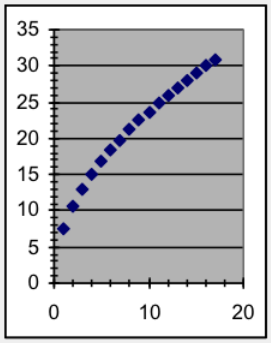
[e]



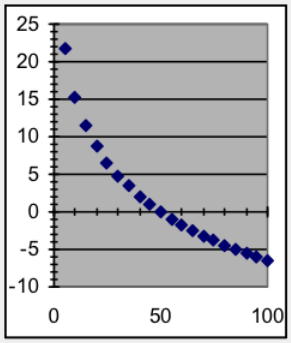
[f]



[g]



[h]



Try to identify the type of model in each case before looking at the answers.

[reveal-answer q="721610"]Answers to Example 6:[/reveal-answer]
[hidden-answer a="721610"]

[a] exponential. [b] logistic [c] quadratic [d] sinusoidal [e] normal [f] linear [g] power [h] logarithmic

[/hidden-answer]

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26.10: N1.10- Exercises

Part I

Reproduce the results in Examples 1 – 6.

Part II

Work the assigned problems.

[7] The water temperature at a faucet was measured (on the Fahrenheit temperature scale) each second after the hot-water tap is turned on. The results were: 72° at 1 second, 72° at 2 seconds, 75° at 3 seconds, 82° at 4 seconds, 95° at 5 seconds, 103° at 6 seconds, 105° at 7 seconds, and 105° at 8 seconds.

a. What type of model is appropriate for this situation?
b. At what time is the temperature changing most rapidly?
c. About how fast is the temperature changing when it changes fastest?

[8] The weight of babies at birth to the nearest pound is tabulated from birth records, with these results: 1% weigh 2 pounds or less, 1% weigh 3 pounds, 2% weigh 4 pounds, 5% weigh 5 pounds, 19% weigh 6 pounds, 36% weigh 7 pounds, 26% weigh 8 pounds, 8% weigh 9 pounds, 1% weigh 10 pounds or more. If a normal-distribution model is fit to this data, what is the best-fit value for the width parameter?

[9] A bank balance earning a constant rate of compound interest has these values: $1550 after 5 years, $2002 after 10 years, $2585 after 15 years, $3339 after 20 years, and $4313 after 25 years. Find a model formula to compute how long it took for the balance to equal $2,938.

[10] Because the earth's orbit around the sun is an ellipse, the distance between them varies according to the time of year. The table to the right shows the distance in miles at 50-day intervals (the first data point is the 50th day of the year, February 19th). Fit a sinusoidal model to this data, and then use the parameters of the model to compute the closest distance that occurs.

Day	Distance
50	91,907,193
100	93,146,982
150	94,243,003
200	94,460,990
250	93,660,185
300	92,366,382
350	91,477,551

Problems 11–20 have the same instructions, applied to different datasets. Copy and paste the datasets from the course web site copy of this topic into a spreadsheet, rather than retyping them.

For each of the datasets listed below (copy and paste it from the course website and)

a. Display the dataset and determine which model type discussed in this course is most suitable.
b. Write the best-fit formula that shows how to compute the y values from the x values.

[11] Dataset A

x	y
1	26.52
2	26.41
3	26.12
4	25.43
5	23.91
6	21.05
7	17.03
8	13.16
9	10.60
10	9.29
11	8.70
12	8.46
13	8.37
14	8.33
15	8.31
16	8.30

[12] Dataset B

x	y
0	172
1	195
2	216
3	230
4	244
5	256
6	261
7	266
8	264
9	262
10	255
11	247

[13] Dataset C

x	y
0	66.8
2	65.3
4	64.2
6	63.6
8	63.6
10	64.2
12	65.4
14	66.9
16	68.4
18	69.7
20	70.6
22	70.9
24	70.5
26	69.6
28	68.2
30	66.7

[14] Dataset D

x	y
1	239.7
2	296.6
3	386.6
4	469.9
5	597.6
6	777.3
7	952.2
8	1180.0
9	1424.4
10	1682.6
11	1980.3
12	2309.7

[15] Dataset E

x	y
1992	45,619
1993	49,529
1994	53,405
1995	57,228
1996	60,877
1997	65,003
1998	68,849
1999	72,399
2000	76,529
2001	80,448
2002	84,030
2003	88,027

[16] Dataset F

x	y
1991	0.5%
1992	1.1%
1993	2.2%
1994	4.5%
1995	8.8%
1996	16.5%
1997	28.9%
1998	45.5%
1999	63.2%
2000	77.9%
2001	87.9%
2002	93.7%
2003	96.8%
2004	98.4%
2005	99.2%
2006	99.6%

[17] Dataset G

x	y
0	10.65
1	8.46
2	7.10
3	5.60
4	4.74
5	3.90
6	3.09
7	2.62
8	2.00
9	1.55
10	1.47
11	0.98
12	0.76
13	0.97
14	0.62
15	0.49
16	0.42
17	0.29
18	0.27
19	0.21
20	0.20

[18] Dataset H

x	y
0	0.0%
1	0.1%
2	2.2%
3	15.0%
4	36.8%
5	33.3%
6	11.1%
7	1.4%
8	0.1%
9	0.0%
10	0.0%
11	0.0%
12	0.0%

[19] Dataset I

x	y
0	0.00
10	52.18
20	73.57
30	90.41
40	104.30
50	116.60
60	127.82
70	138.09
80	147.46
90	156.65
100	164.93
110	172.89
120	180.86

[20] Dataset J

x	y
0	0
2	0
4	2
6	4
8	9
10	17
12	32
14	48
16	61
18	67
20	69
22	62
24	48
26	33
28	19
30	11
32	5
34	2
36	1
38	0
40	0

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CHAPTER OVERVIEW

27: Topic B- Solving Equations, Evaluating Expressions, and Checking Your Work

27.1: B1.01- Introduction
27.2: B1.02- Section 1
27.3: B1.03- Section 2
27.4: B1.04- Section 3
27.5: B1.05- Section 4
27.6: B1.06- Exercises

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27.1: B1.01- Introduction

Objectives

1. Use additional resources to practice algebra, as needed.
2. Learn to check answers in some way other than looking “in the back of the book.”
3. Use the distributive property and other properties to simplify algebraic expressions and check your work.
4. Solve and simplify linear equations and check solutions of equations by plugging them back in.
5. Solve equations with variables in the denominator and check solutions by plugging them in.
6. Solve equations of those two types with multiple variables for one variable, resulting in a formula for that one variable.
7. Evaluate variable expressions when given the value of each variable, using the correct order of operations.

Overview

The prerequisite for these materials includes mastery of basic math skills and some algebra, as needed to achieve “college-level” on Texas’ TSI test. Many students who have learned math at this level forget it easily, so this review section of some algebra is provided. Some students in the class will go through this material quickly and others will need to go much more slowly and do more review work. Additional optional practice problems are provided on the course web pages. The next few topics in the do not use this algebra, so it isn’t necessary to master everything in this topic immediately.

A major difference between this course and most other math courses involves “checking your answer by looking in the back of the book.” In standard algebra classes, students practice quite a lot of similar problems and are expected to check many of the answers by looking in the back of the book. When we use mathematics in applications in the real world, there is no “back of the book.” In this course, we will learn methods for independently checking our results to see if they are correct, or at least reasonable, without having to rely on answers that someone else gives us. Most students find this awkward at first, but when they persist in practicing the checking methods, develop more confidence in using mathematics and in their problem-solving skills. Even when a solution is provided in the “back of the book” we expect you to pay attention to other ways of checking your work, and to use those frequently (including on test questions.)

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27.2: B1.02- Section 1

Use the distributive property and other properties to simplify expressions and check your work.

Distributive Property:

For any real numbers *a*, *b*, and *c*: $a(b + c) = ab + ac$.

For any real numbers *a*, *b*, and *c*: $a(b – c) = ab – ac$

The fact that two expressions are equal means that they are equal for any values of the variable. So you can check by taking a few values for the variable and making sure that those do make the two sides equal. Usually we don't use the values 0 or 1 or 2 as the value for the variable and it is best to avoid numbers that already appear in the problem.

Example 1

Simplify $10(4x+5)$.

[reveal-answer q="894348″]Show Answer[/reveal-answer]
[hidden-answer a="894348″]

Solution:	Partial Check: Use
$10(4x+5)=40x+50$	$\begin{aligned}&\,10(4x+5)=40x+50\\&10(4\cdot3+5)\,?=?\,40\cdot3+50\\&\,10(12+5)\,?=?\,120+50\\&\,10(17)\,?=?\,170\\&\,170\,=\,170\\\end{aligned}$

[/hidden-answer]

Example 2

Simplify $-6(x+9)$.

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[hidden-answer a="113961″]

Solution:	Partial Check: Use $x=5$
$-6(x+9)=-6x-54$	$\begin{aligned}&\,-6(x+9)=-6x-54\\&-6(5+9)\,?=?\,-6*5-54\\&\,-6(14)\,?=?\,-30-54\\&\,-84\,=\,-84\\\end{aligned}$

[/hidden-answer]

Example 3

Simplify $2(3x-7)$.

[reveal-answer q="802251″]Show Answer[/reveal-answer]
[hidden-answer a="802251″]

Solution:	Partial Check: Use $x=5$
$\begin{aligned}&2(3x-7)=2\cdot3x-2\cdot7\\&\,=6x-14\\\end{aligned}$	$\begin{aligned}&2(3\cdot5-7)\,?=?\,6\cdot5-14\\&\,2(15-7)\,?=?\,30-14\\&\,2(8)\,=\,16\\\end{aligned}$

[/hidden-answer]

Example 4

Simplify $17-5(4x-12)$.

[reveal-answer q="785026″]Show Answer[/reveal-answer]
[hidden-answer a="785026″]

Solution:	Partial Check: Use $x=3$
$\begin{aligned}&17-5(4x-12)=17-20x-(-60)\\&\,=17-20x+60\\&\,=77-20x\\\end{aligned}$	$\begin{aligned}&17-5(4\cdot3-12)\,?=?\,77-20\cdot3\\&\,17-5(12-12)\,?=?\,77-60\\&\,17-0\,=\,17\\\end{aligned}$

[/hidden-answer]

Caution: Checking your work on problems like these is not completely satisfactory, because students who make a mistake in the original solution often make a corresponding mistake in the checking process. So the fact that it checks does not provide complete confidence that the solution is correct. But it's still a good idea to check because you'll catch most of your mistakes.

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27.3: B1.03- Section 2

To solve equations, write a simpler equivalent equation using the following rules. Keep writing simpler equivalent equations until you have simplified it enough that one side has the variable alone and the other has a number. That gives the solution.

1. Add or subtract the same number or expression to both sides of the equation.
2. Multiply or divide both sides of the equation by the same nonzero number or expression.

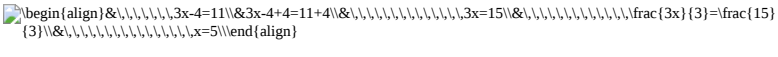
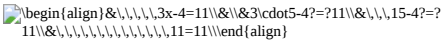
You can check the solution to an equation by plugging the number into the original equation and seeing if the resulting statement is true.

For additional explanation and examples, see the course web page for links. These include links to instructional materials and a link to an equation solver, where you can enter any equation and have it find the solution. You can use that to check your work on as many problems as you have the time and energy to work.

Example 1

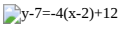
Solve  .

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[hidden-answer a="765770"]

Solution: x	Check:
 .	

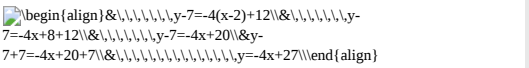
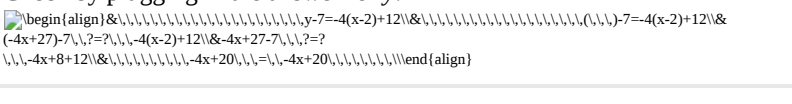
[/hidden-answer]

Example 2

Find a formula for *y* (that is, solve for *y* in terms of *x)*: 

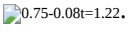
[reveal-answer q="731148"]Show Answer[/reveal-answer]
[hidden-answer a="731148"]

Discussion: This is more complicated than a problem like Example 1. *However, it uses exactly the same techniques*. Problems like this arise when we begin to investigate graphing lines. Probably you have done problems like these in a previous algebra course, but maybe you haven't done many of them. We will practice many problems like these in the first few weeks of the course. Don't practice these until you can do problems like Example 1 fairly easily.

Solution:	Check by plugging in the answer for *y*:
	

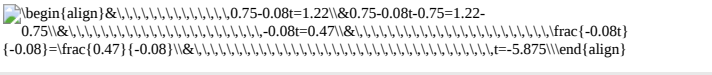
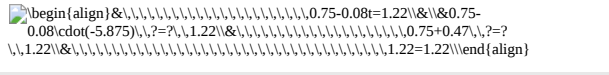
[/hidden-answer]

Example 3

Solve  .

[reveal-answer q="606202"]Show Answer[/reveal-answer]
[hidden-answer a="606202"]

Discussion: This is basically like Example 1, except that the coefficients are decimals and the variable is *t* rather than *x*. In your previous algebra classes, you may not have done much work with decimal numbers, but we will use them often in this class. We will practice many problems like these in the first few weeks of the course. Don't practice these until you can do problems like Example 1 fairly easily.

Solution:	Check:
	

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27.4: B1.04- Section 3

Example 1

Solve $\frac{14}{3}=\frac{8}{x}$.

Discussion: Sometimes students learn to solve problems like this one by “cross-multiplying.” That is correct. However, most math teachers prefer to think of solving this by multiplying both sides by the same thing—in this case the product of the two denominators. The result is the same. Both methods are shown below.

[reveal-answer q=”639200″]Show Answer[/reveal-answer]
[hidden-answer a=”639200″]

Solution Method: *Cross-multiply:*	Alternate Solution Method: *Multiply by common denominator*	Check:
$\begin{align}&\frac{14}{3}=\frac{8}{x}\\&14x=3\cdot8\\&14x=24\\&\frac{14x}{14}=\frac{24}{14}\\&x=1.7143\\\end{align}$	$\begin{align}&\frac{14}{3}=\frac{8}{x}\\&3\cdot{x}\cdot\frac{14}{3}=3\cdot{x}\cdot\frac{8}{x}\\&\frac{3}{3}\cdot{x}\cdot14=\frac{x}{x}\cdot3\cdot8\\&1\cdot{x}\cdot14=1\cdot3\cdot8\\\end{align}$ $\begin{align}&14x=24\\&\frac{14x}{14}=\frac{24}{14}\\&x=1.7143\\\end{align}$	$\begin{align}&\frac{14}{3}=\frac{8}{x}\\&\,\,\,?=\,\,\frac{8}{1.7143}\\&4.66667\,\,?=\,\,4.66663\\\end{align}$ Here, the two sides aren’t exactly equal. But that isn’t surprising, because we rounded the final answer to the original problem, so we don’t expect the two sides here to be exactly equal.

Notice that the quotient, which is the final answer, didn’t come out even, so I had to round off. I chose to round off to four decimal places. We’ll talk more specifically about how exactly answers should be given later in the course. For now, when you have to round in the final computation, always keep at least three decimal places. That will ensure that your checking will produce answers close enough that you can recognize them as being essentially the same, so your checking is useful.

[/hidden-answer]

Example 2

Solve $\frac{7}{33}=\frac{x}{5}$

Discussion: Here the variable isn’t in the denominator, but this illustrates that the basic principle of multiplying both sides by the same non-zero expression works here too.

[reveal-answer q=”478965″]Show Answer[/reveal-answer]
[hidden-answer a=”478965″]

Solution Method: *Cross-multiply:*	Alternate Solution Method: *Multiply by common denominator*	Check:
$\begin{align}&\frac{7}{33}=\frac{x}{5}\\&33x=7\cdot5\\&33x=35\\&\frac{33x}{33}=\frac{35}{33}\\&x=1.0606\\\end{align}$	$\begin{align}&\frac{7}{33}=\frac{x}{5}\\&33\cdot5\cdot\frac{7}{33}=33\cdot5\cdot\frac{x}{5}\\&\frac{33}{33}\cdot5\cdot7=\frac{5}{5}\cdot33\cdot{x}\\&1\cdot{x}\cdot35=1\cdot33\cdot{x}\\&33x=35\\&\frac{33x}{33}=\frac{35}{33}\\&x=1.0606\\\end{align}$	$\begin{align}&\frac{7}{33}=\frac{x}{5}\\&\,\,\,?=\,\,\frac{7}{33}\\&\,\,\,\frac{1.0606}{5}\\&\,\,\,0.21212\,\,\,=\,\,\,0.21212\\\end{align}$

[/hidden-answer]

Example 3

Solve $\frac{12}{x}=6$

[reveal-answer q=”654143″]Show Answer[/reveal-answer]
[hidden-answer a=”654143″]

Solution Method: *Cross-multiply:*	Solution Method: *Multiply by common denominator*	Check:
$\begin{align}&\frac{12}{x}=\frac{6}{1}\\&6x=12\\&x={}^{12}\!\!\diagup\!\!{}_{6}\;=2\\\end{align}$	$\begin{align}&\frac{12}{x}=6\\&x\cdot\frac{12}{x}=x\cdot6\\&\frac{x}{x}\cdot12=6x\\&12=6x\\&x={}^{12}\!\!\diagup\!\!{}_{6}\;=2\\\end{align}$	$\begin{align}&\frac{12}{x}=6\\&\,\,\,\frac{12}{2}\,\,=\,6\\\end{align}$

[/hidden-answer]

Example 4

Find a formula for *h* (that is, solve for *h*.) $\frac{h}{36}=\frac{m}{k}$.

[reveal-answer q="39989"]Show Answer[/reveal-answer]
[hidden-answer a="39989"]

Solution Method: *Cross-multiply:*	Alternate Solution Method: *Multiply by common denominator*	Check:
$\begin{aligned}&\frac{h}{36}=\frac{m}{k}\\&hk=36m\\&\frac{hk}{k}=\frac{36m}{k}\\&h=\frac{36m}{k}\end{aligned}$	$\begin{aligned}&\frac{h}{36}=\frac{m}{k}\\&36k\frac{h}{36}=36k\frac{m}{k}\\&hk=36m\\&\frac{hk}{k}=\frac{36m}{k}\\&h=\frac{36m}{k}\end{aligned}$	$\frac{h}{36}=\frac{m}{k}$ Substitute $\begin{aligned}&\frac{\frac{36m}{k}}{36}=\\&\frac{36\cdot m}{k}\div36=\\&\frac{36\cdot m}{k}\cdot\frac{1}{36}=\\&=\frac{m}{k}\end{aligned}$

[/hidden-answer]

Example 5

Find a formula for *d* (that is, solve for *d*.) $\frac{a}{0.37}=\frac{r}{d}$.

[reveal-answer q="353261"]Show Answer[/reveal-answer]
[hidden-answer a="353261"]

Solution Method: *Cross-multiply:*	Alternate Solution Method: *Multiply by common denominator*	Check:
$\begin{aligned}&\frac{a}{0.37}=\frac{r}{d}\\&0.37r=a\cdot d\\&d\frac{0.37r}{a}=\frac{a\cdot d}{a}\\&\frac{0.37r}{a}=d\\&d=\frac{0.37r}{a}\end{aligned}$	$\begin{aligned}&\frac{a}{0.37}=\frac{r}{d}\\&0.37d\cdot\frac{a}{0.37}=0.37d\cdot\frac{r}{d}\\&d\cdot a=0.37r\\&\frac{d\cdot a}{a}=\frac{0.37r}{a}\\&d=\frac{0.37r}{a}\end{aligned}$	$\frac{a}{0.37}=\frac{r}{d}$ Substitute for d on the right. $\begin{aligned}&\frac{r}{d}=\frac{r}{\frac{0.37r}{a}}\\&=r\div\frac{0.37r}{a}\\&=\frac{r}{1}\div\frac{0.37r}{a}\\&=\frac{r}{1}\cdot\frac{a}{0.37r}\\&=\frac{ra}{0.37r}\\&=\frac{a}{0.37}\end{aligned}$

[/hidden-answer]

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27.5: B1.05- Section 4

Evaluate variable expressions when given the value of each variable, using the correct order of operations.

Recall these rules about order of operations. Do them in this order.

1. All operations inside symbols of grouping (parentheses) from the inside out.
2. All operations of exponents or roots.
3. All multiplications and divisions, in order from left to right.
4. All additions and subtractions, in order from left to right.

The best method I know is to write the expression first with the variable, then with parentheses in place of the variable, and then with the values inserted into the parentheses.

Then you can remove any of those parentheses that aren't needed to keep the negative numbers clear and to keep the products of two numbers clear.

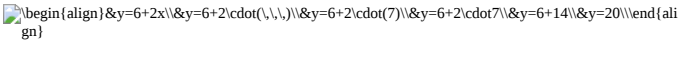
Then, begin to evaluate the expression according to the order of operations, doing one operation per step.

For problems in this course, the most important of these rules are to do exponents first and then to do multiplications/divisions before additions/subtractions.

Example 1

Evaluate  when  .

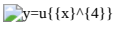
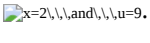
[reveal-answer q="992565"]Show Answer[/reveal-answer]
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Solution: 	Notice that the 7 here is in parentheses, but it is not an OPERATION in parentheses, so there is no need to think about that first rule in the order of operations.

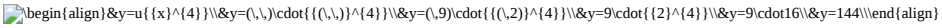
Discussion: Later in the course, some students frequently do problems like this incorrectly because they do the addition before the multiplication. It is important to learn this rule well.

[/hidden-answer]

Example 2

Evaluate  when  .

[reveal-answer q="657780"]Show Answer[/reveal-answer]
[hidden-answer a="657780"]



Discussion: Later in the course, some students have trouble remembering not to do the multiplication of 9 times 2 here first. Many of our formulas will include exponents such as this, so it is important to learn this rule well.

[/hidden-answer]

Checking your work

When evaluating expressions, the only clear method to check your work is to simply re-work it. That's not completely satisfactory because if you have a mistake in understanding, you're likely to make it both times.

However, in this course, most of the times when we will evaluate an expression, it is in the context of a larger problem for which there is a good method of checking available.

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27.6: B1.06- Exercises

After you work each of these problems, use some method of checking your answer and show that check right beside your solution. Don't forget to prepare and fill out the homework cover sheet as you do the problems.

Part I

1. Simplify $10(4x+5)$
2. Simplify $-6(x+9)$
3. Simplify $2(3x-7)$
4. Simplify $17-5(4x-12)$
5. Solve $3x-4=11$
6. Find a formula for y (that is, solve for *y* in terms of *x*): $y-7=-4(x-2)+12$
7. Solve $0.75-0.08t=1.22$
8. Solve $\frac{14}{3}=\frac{8}{x}$
9. Solve $\frac{7}{33}=\frac{x}{5}$
10. Solve $\frac{12}{x}=6$
11. Find a formula for *h* (that is, solve for *h*.) $\frac{h}{36}=\frac{m}{k}$.
12. Find a formula for *d* (that is, solve for *d*.) $\frac{a}{0.37}=\frac{r}{d}$.
13. Evaluate $y=6+2x$ when $x=7$
14. Evaluate $y=u{x}^{4}$ when $x=2$ and $u=9$

Part II

1. Solve $8x+7=31$
2. Solve $6x-14=40$
3. Simplify $13(x+2)$
4. Simplify $8(x-2)$
5. Solve $\frac{5}{2}=\frac{35}{x}$
6. Solve $\frac{9}{r}=\frac{54}{30}$
7. Solve $\frac{7}{5}=\frac{x}{35}$
8. Solve $17=-3x+2$
9. Find a formula for y (that is, solve for *y*): $y-7=4(x-3)$
10. Find a formula for y (that is, solve for *y*): $y-15=3(x-7)$
11. Find a formula for *k* (that is, solve for *k*.) $\frac{m}{2.1}=\frac{k}{t}$.
12. Find a formula for *m* (that is, solve for *m*.) $\frac{b}{m}=\frac{w}{17}$.
13. Evaluate $L=A\cdot{r}^{t}$ where $A=10,\,\,\,\,r=2,\,\,\,\,t=5$
14. Evaluate $y=7x-3$ where $x=5$

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CHAPTER OVERVIEW

28: Topic D- Formulas—Computing and Graphing

28.1: D1.01- Introduction
28.2: D1.02- Examples 2–5
28.3: D1.03- Examples 6–9
28.4: D1.04- Examples 10–11
28.5: D1.05- Exercises

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28.1: D1.01- Introduction

Objectives

1. Distinguish between input values and output values in formulas.
2. Use the order of operations correctly evaluating formulas.
3. Choose an appropriate scale for each axis and graph the formula by hand.
4. Use the graph to find what input value will give a particular output value.
5. Use numerical methods to check that answer and to refine it to get a somewhat more accurate answer. (After learning to graph using a spreadsheet.)
6. Use formulas with several different variables as input.
7. Understand the use of subscripts in formulas.

Many formulas are used in geometry and other applications. It is important to be able to correctly plug in values and compute the result. In this course we will also learn to graph some of these formulas so that we can investigate the patterns more fully.

Definition: Usually when people refer to an equation as a formula it has the output variable alone on one side of the equation. When the equation is written in that form, it is easy to evaluate the output variable at different input values and to easily graph the formula. **When we use the word "formula" in this course, we will always mean that the output variable is alone on one side of the equation**.

Example 1

Example 1. In Canada and Mexico, weather reports report temperature using the Centigrade (or Celsius) scale. In the US, temperature is reported using the Fahrenheit scale. To convert temperature C to temperature F, we use the formula $F=\frac{9}{5}C+32$. When the temperature C is 23°, what is the temperature F?

[reveal-answer q="752745"]Show Answer[/reveal-answer]
[hidden-answer a="752745"]

$$\begin{align}&F=\frac{9}{5}C+32\\&F=\frac{9}{5}(23)+32\\&F=41.4+32\\&F=73.4\\\end{align}$$

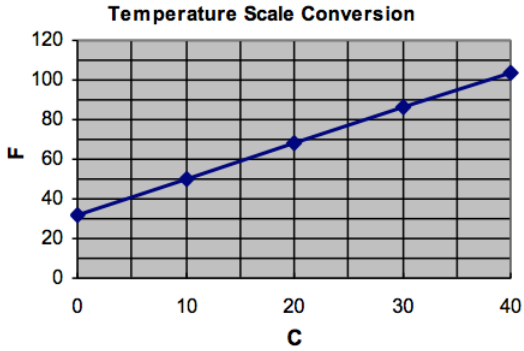
Thus the temperature F is 73.4°.

Notice that we used the correct order of operations here, performing the multiplication before the addition.

Here we call the 23° the input value and the 73.4° the output value. (In algebra class, we call the set of input values the "domain" and the set of output values the "range".)

When we graph a formula, we will put the input value on the horizontal axis, sometimes called the *x*-axis, and the output value on the vertical axis, sometimes called the *y*-axis. We can use different letters for the variables besides *x* and *y*, but when we think of the formula graphically, we must be clear about which variable plays the role of *x* and which plays the role of *y*. (You should be able to do these graphs by hand as well as with a spreadsheet.)

Formula input and output	
C	**F**
23	73.4
0	32
10	50
20	68
30	86
40	104



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28.2: D1.02- Examples 2–5

Example 2

If we invest $750 at 6% annual interest, compounded quarterly, the formula for the amount *A* that the investment is worth after *t* years is $A=750\cdot{\left(1+\frac{0.06}{4}\right)}^{4t}$. Find the amount the investment is worth after five years. Then evaluate this for several values of *t* between 0 and 30 years, and sketch a graph of this formula.

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First, recall how to get an exponent in your calculator. On most calculators, it's a ^ key or a

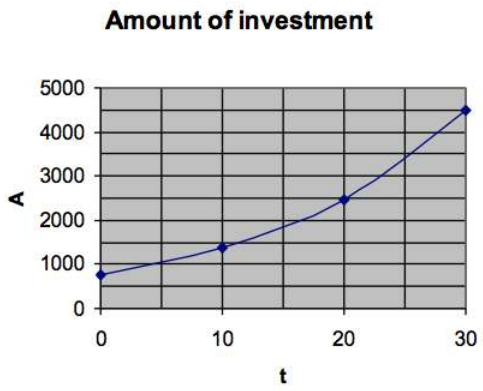
key. Find that key and practice using it to evaluate . When you can do that correctly, then evaluate the following expression, writing these intermediate values as indicated below.

$A=750\cdot{\left(1+\frac{0.06}{4}\right)}^{4t}=750\cdot{\left(1+\frac{0.06}{4}\right)}^{4\cdot5}}=750\cdot{\left(1.015\right)}^{20}}=750\cdot1.346855007=1010.141255$

So the investment is worth $1010.14 after 5 years.

It is more convenient to be able to put all of these values into the calculator at the same time rather than writing intermediate steps. We will learn to do that in this course, but that's not the point of this lesson. The point here is to practice making graphs. This will be useful to you in checking your work when you evaluate formulas. If one of the values you compute doesn't seem to fit the pattern of the others, that warns you to check your computation again.

Formula input and output	
t	**A**
5	1010.141
0	750.000
10	1360.514
20	2467.997
30	4476.992



We can use the graph to estimate at what time the amount of the investment will be about $4000. Look at the graph to find 4000 on the vertical axis and then notice that the corresponding value on the horizontal axis is about 27.5. So at approximately 27.5 years, the amount of the investment will be $4000.

Check that by plugging it into the formula. $A=750\cdot{\left(1+\frac{0.06}{4}\right)}^{4\cdot27.5}}=\,\,\,\text{etc.}\,\,=3857.68$

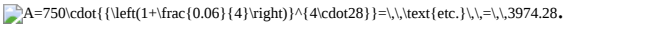
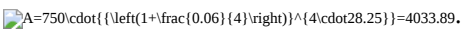
This is as close as we could reasonably expect from using a graph to approximate the input value.

[/hidden-answer]

Example 3

Consider our approximation from Example 2. We wanted to find the number of years to leave the money in so that the amount of the investment would be $4000. The graph suggested that  =27.5 years. But then we found that after 27.5 years, the amount was only $3857.68. Clearly we should leave the money in somewhat longer. How much longer?

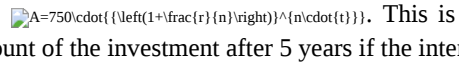
[reveal-answer q="137839"]Show Answer[/reveal-answer]
[hidden-answer a="137839"]

If we know how to do some complicated algebra (using logarithms) we can quickly obtain an answer to "how long should we leave the money to earn $4000?" We will not do that particular type of algebra in this course. Instead we will use numerical work to get refine our estimate from the graph. Here, when we look at the graph and the numbers, it is clear that 30 years is too long and 27.5 years is too short. So we try some value in between. Let's try 28 years.  . This is much closer to the $4000 which was our goal. If we wanted to try to get it even more accurately, we would use a value larger than 28 years, but only very slightly larger. Since the amount is compounded quarterly, the next time the amount will increase is at 28.25 years, so let's try that.  . That is further away from $4000 than the value for 28 years, so the best answer for this question is that the amount of the investment will be at about $4000 at 28 years.

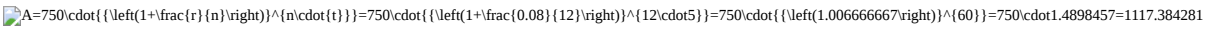
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Example 4

Following Example 2, we'd like to have a formula that allows us to vary the interest rate and number of times per year it is compounded as well as varying the number of years of the investment. So let *r* be the interest rate, converted to a decimal and *n* be the number of times per year it is compounded, and, as before *t* is the number of years and *A* is the amount of the investment.

Then the formula is  . This is an example of a formula with several input values. Use this formula to find the amount of the investment after 5 years if the interest rate is 8% and it is compounded monthly.

[reveal-answer q="261282"]Show Answer[/reveal-answer]
[hidden-answer a="261282"]



So after 5 years, this investment is worth $1117.38.

We can't graph this easily, because we would need more than a two-dimensional graph. We need one dimension for the output value and one dimension for every input value and investigating four-dimensional graphs is beyond the scope of this course! However, in most practical applications, technicians and scientists isolate one or two input variables that they are most interested in and then analyze the problem with a two-dimensional graph (for one input variable) or a three-dimensional graph (for two input variables.) We will look at some three-dimensional graphs later in the course.

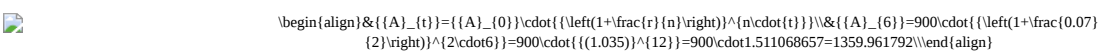
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Example 5

Following Example 3, suppose we want to allow the initial amount of the investment to change. So we need a variable for that. Since it is an amount, we'd like to call it *A*, but we already have an *A* in this formula that means something else. We could use a different letter, but in applications problems we often choose to call both values *A* and distinguish between them by a subscript. In this problem, we would usually call the original amount of money

and the final amount of money after *t* years . So the formula is . Use this formula to find the amount of an investment after 6 years if the initial amount is $900, the annual rate is 0.07 and it is compounded twice a year.

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[hidden-answer a="61734"]



So, after 6 years, this investment is worth $1359.96.

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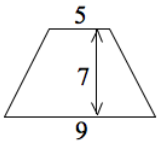
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28.3: D1.03- Examples 6–9

Example 6

Find the area of this trapezoid, where the numbers represent feet.



[reveal-answer q="593926"]Show Answer[/reveal-answer]
[hidden-answer a="593926"]

The formula is $A=\frac{1}{2}h\left({{b}_{1}}+{{b}_{2}}\right)$where h is the height,

and are the lengths of the two bases, and A is the area. Notice that the subscripts are used because both these sides are called bases, so b is a reasonable letter to use, but there are two different ones. We use subscripts on the b's to distinguish between the two different bases.

[/hidden-answer]

Example 7

Find the volume of a sphere with radius 3 inches. The formula is $V=\frac{4}{3}\pi{{r}^{3}}$.

[reveal-answer q="603839"]Show Answer[/reveal-answer]
[hidden-answer a="603839"]$V=\frac{4}{3}\pi{{r}^{3}}=\frac{4}{3}\pi{{\left(3\right)}^{3}}=\frac{4}{3}\pi\cdot27=113.097$.

So the volume of this sphere is 113.097 cubic inches.

[/hidden-answer]

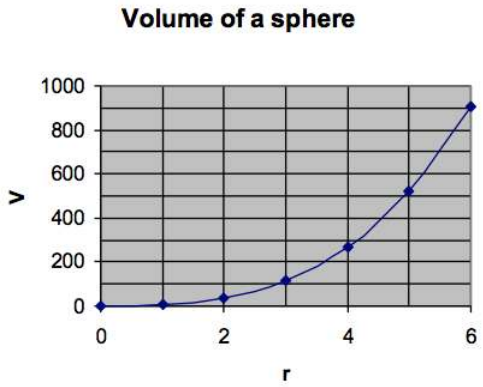
Example 8

Graph the formula $V=\frac{4}{3}\pi{{r}^{3}}$ for the values of r from 0 to 6 feet and use the graph to approximate what radius will give a volume of 400 cubic feet.

[reveal-answer q="417361"]Show Answer[/reveal-answer]
[hidden-answer a="417361"]

Notice that r is the input value and V is the output value.

r	V
0	0
1	4.188787
2	33.51029
3	113.0972
4	268.0823
5	523.5983
6	904.7779



Using the graph, we follow the line for 400 on the vertical axis across to the graph and then down to see the corresponding r value, which is approximately $r=4.5$ feet. We can check this by plugging in and finding that $V=\frac{4}{3}\pi{{(4.5)}^{3}}=381.7$

If we wanted to get a better estimate, we would try a somewhat higher value for *r*.

[/hidden-answer]

Example 9

Consider the problem of Example 8. Use graphical and numerical methods to find a value for the radius that will give a volume of 400 cubic feet, correct to within 10 cubic feet.

[reveal-answer q="257185"]Show Answer[/reveal-answer]
[hidden-answer a="257185"]

The graph tells us that a good estimate for the radius is 4.5 feet. Next we check that by plugging in and finding that $V=\frac{4}{3}\pi{{(4.5)}^{3}}=381.7$cubic feet. That's a bit too low, so let's try a slightly larger value for the radius. Let's try 4.6 feet. $V=\frac{4}{3}\pi{{(4.6)}^{3}}=407.72$ cubic feet. That's an adequate answer, according to the tolerance level stated in the problem. Of course, if we wanted a more accurate answer, we could continue to work numerically by taking a value for the radius just a bit smaller than 4.6 feet and finding the volume for that.

[/hidden-answer]

Perhaps you have noticed that you could use algebra (involving cube roots) instead of the graph on the problem of Examples 8 and 9 to obtain a quite exact answer for the radius needed to obtain a volume of 400 cubic feet. And that's a fine thing to do. But, in this course, we are practicing this technique that will allow you to solve problems of this kind even if the formula is so complicated that the algebra needed to "invert" the formula is harder than anything you've learned in your algebra courses. In fact, some formulas are so complicated that even mathematicians find these values in just the same way you're learning here. Using graphs in this way is a very powerful and important tool in working on technical problems!

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28.4: D1.04- Examples 10–11

Example 10

A certain river in Georgia is polluted with coliform bacteria. This has been blamed primarily on the dairy farmers whose farms are in the watershed of the river. This formula is used to model the cost in dollars, y, to remove p% of the bacterial pollution from the river. How much of the pollution can be removed if $1.0 million is available to spend on this? $y=\frac{380,000p}{100-p}$

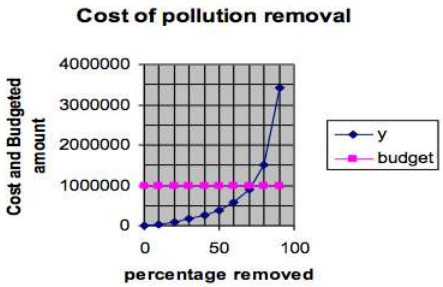
[reveal-answer q="270002"]Show Answer[/reveal-answer]
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We must graph this formula over the range of values $0\le{p}<100$ and then see for what values of p the y value is less than or equal to $1.0 million.

Use parentheses when putting this formula into your calculator. When we write this by hand it is obvious that the entire numerator and entire denominator should be evaluated separately before dividing. But when we enter it into a calculator, we have to use parentheses to indicate that.

For p = 20, use (380000*20)/(100 – 20) =

p	y
0	0
10	42222.22
20	95000.00
30	162857.1
40	253333.3
50	380000.0
60	570000.0
70	886666.7
80	1520000.0
90	3420000.0



We graphed the formula and then put a line across at $1 million. This crosses the cost graph at a bit more than. So we estimate that about 72% of the pollution can be removed for $1.0 million.

We could expand the scale or do more numerical computation to estimate this more precisely. That would be tedious to do by hand, but would be easy to do with a spreadsheet.

[/hidden-answer]

Example 11

A certain kind of appliance is sold for $10 each and the manufacturer can sell all she produces at that price. To produce these appliances requires a fixed cost of $14,220 per month (equipment depreciation, salaries, utilities, etc.) and the variable cost per appliance is $2.10.

1. Write a formula for the cost of producing *x* appliances.
2. Write another formula for the revenue produced by selling *x* appliances.
3. Graph both formulas for $0\le{x}\le2500$.
4. For what value of *x* is the cost equal to the revenue? (That point on the graph is called the “break-even” point.)

[reveal-answer q=”988402″]Show Answer[/reveal-answer]
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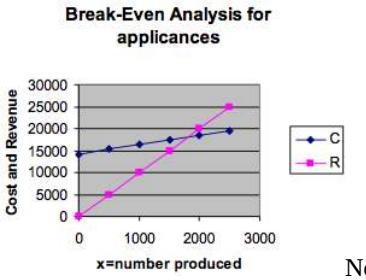
Let C = total cost of *x* appliances. C = fixed cost + variable cost times number of appliances.

Thus $C=14220+2.10\cdot{x}$

Let R = total revenue from *x* appliances. R = price per appliance times number of appliances

Thus $R=10.00\cdot{x}$

x	C	R
0	14220	0
500	15270	5000
1000	16320	10000
1500	17370	15000
2000	18420	20000
2500	19470	25000



Notice that these are equal for some x-value about halfway between 1500 and 2000. So we plot more values in this range and find that they are equal at $x=1800$.

We plot more values in this range and find that the cost and the revenue are equal at $x=1800$.

So, for 1800 appliances, both the cost and the revenue are $18,000. That means that, if we make 1800 appliances, we will “break even.”

[/hidden-answer]

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28.4: D1.04- Examples 10–11 is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

28.5: D1.05- Exercises

Use a calculator and make the graphs by hand to work the following problems. Don't calculate very many points—use those given. In this class we will usually use a spreadsheet to automate the calculations. The work in this lesson is mainly to help you recall how to graph formulas by hand so that you will fully understand the graphs you produce with a spreadsheet.

Part I

1. If we invest $750 at 6% annual interest, compounded quarterly, the formula for the amount A that the investment is worth after t years is $A=750\cdot{\left(1+\frac{0.06}{4}\right)}^{4t}$.
    1. Find the amount the investment is worth after five years. Then evaluate this for two more values of t between 0 and 30 years. Use the rest of the values that were given in the example, and sketch a graph of this formula.
    2. For the previous problem, how long should we leave the investment so that it will be worth $4000? Estimate the time, correct to the nearest year.
2. Then the formula is $A=750\cdot{\left(1+\frac{r}{n}\right)}^{n\cdot{t}}$. This is an example of a formula with several input values. Use this formula to find the amount of the investment after 5 years if the interest rate is 8% and it is compounded monthly.
3. Find the area of a trapezoid with bases 5 feet and 9 feet and height 7 feet. $A=\frac{1}{2}h\left({b}_{1}+{b}_{2}\right)$
4. Graph the formula for the volume of a sphere, $V=\frac{4}{3}\pi{r}^{3}$, for the values of r from 0 to 6 feet and use the graph to approximate what radius will give a volume of 400 cubic feet, correct to within 10 cubic feet. Evaluate about three values by hand/calculator and then use the rest from the example.
5. A certain river in Georgia is polluted with coliform bacteria. This has been blamed primarily on the dairy farmers whose farms are in the watershed of the river. This formula is used to model the cost in dollars, y, to remove p% of the bacterial pollution from the river. How much of the pollution can be removed if $1.0 million is available to spend on this? Evaluate about three values by hand/calculator and then use the rest form the example. $y=\frac{380,000p}{100-p}$
6. A certain kind of appliance is sold for $10 each and the manufacturer can sell all she produces at that price. To produce these appliances requires a fixed cost of $14,220 per month (equipment depreciation, salaries, utilities, etc.) and the variable cost per appliance is $2.10.
    1. Write a formula for the cost of producing x appliances.
    2. Write another formula for the revenue produced by selling x appliances.
    3. Graph both formulas for $0\le{x}\le2500$.
    4. For what value of x is the cost equal to the revenue? (That is called the "break-even" point.

Part II

7. The concentration of a particular drug in a patient's blood is given by the formula $C=\frac{7}{0.3{t}^{2}+1.1}$ where t is the number of hours after the patient takes the medicine and C is the concentration in milliliters per liter of blood. Of course, $t\ge0$. (Hint: When you are using your calculator to evaluate this, be sure to put parentheses around the entire denominator of the fraction.)

t	C
0	6.363636
1	5.000000
2	3.043478
3	
4	1.186441
5	0.813953

a. Use your calculator to confirm at least one of the values of C in the table of values and then fill in the missing value.
b. Which is the input variable? (When you graph this, put that along the horizontal axis.)
c. Using graph paper, graph this formula by hand.
d. It is recommended that the patient not take another dose until the concentration is less than 2.0 ml/l. According to your graph of the formula, approximately how long will that take? (Estimate to the nearest hour.)

8. The landing speed S , in feet per second, of a particular type of small plane can be modeled by the formula $S=\sqrt{1.496w}$ where w is the weight in pounds of the plane.

w	S

a. Use your calculator to confirm at least one of the values of C in the table of values and then fill in the

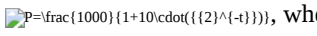
4000	77.35632
6000	94.74175
8000	
10000	122.3111

missing value..

b. Which is the input variable? (When you graph this, put that along the horizontal axis.)

c. Using graph paper, graph this formula by hand.

d. What weight would correspond to a landing speed of 100 ft/sec? Use your graph to estimate this to the nearest thousand pounds.

9. The logistic formula can be used to describe population growth in a situation with limited resources. Consider this logistic formula  , where P is the population at time t. (Hint: When you are using your calculator to evaluate this, be sure to put parentheses around the entire denominator of the fraction.)

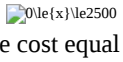
t	P(t)
0	90.9091
1	166.6667
2	285.7143
3	444.4444
4	
5	761.9048
6	864.8649
7	
8	962.4060
9	980.8429

a. Use your calculator to confirm at least one of the values of P in the table of values and then fill in the missing value.

b. Which is the input variable? (When you graph this, put that along the horizontal axis.)

c. Using graph paper, graph this formula by hand.

d. What time would we expect to have a value for P of 900? Use your graph to estimate this.

10. A certain kind of thermometer is sold for $5 each and the manufacturer can sell all she produces at that price. To produce these thermometers requires a fixed cost of $3500 per month (equipment depreciation, salaries, utilities, etc.) and the variable cost per thermometer is $1.50.
    1. Write a formula for the cost of producing x thermometers.
    2. Write a formula for the revenue produced by selling x thermometers.
    3. Graph both formulas for  in increments of 500.
    4. For what value of x is the cost equal to the revenue? (That point on the graph is called the “break-even” point.)
    5. What is the total cost of making that many thermometers?
    6. What is the total revenue earned from that many thermometers?

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