FOUNDATIONAL MATHEMATICS

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Foundational Mathematics

Ashley McHale

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Glossary

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Licensing

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CHAPTER OVERVIEW

1: The Whole Numbers

Welcome to the study of prealgebra. In this first chapter of study, we will introduce the set of natural numbers, then follow with the set of whole numbers. We will then follow with a quick review of addition, subtraction, multiplication, and division skills involving whole numbers that are prerequisite for success in the study of prealgebra. Along the way we will introduce a number of properties of the whole numbers and show how that can be used to evaluate expressions involving whole number operations. We will also define what is meant by *prime* and *composite* numbers, discuss a number of divisibility tests, then show how any composite number can be written uniquely as a product of prime numbers. This will lay the foundation for requisite skills with fractional numbers in later chapters. Finally, we will introduce the concept of a *variable*, then introduce equations and technique required for their solution. We will use equations to model and solve a number of real-world applications along the way.

Let's begin the journey.

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1.1: An Introduction to the Whole Numbers

A *set* is a collection of objects. If the set is finite, we can describe the set completely by simply listing all the objects in the set and enclosing the list in curly braces. For example, the set

$$S = \{dog,\ \text{cat, parakeet}\ \}$$

is the set whose members are “dog”, “cat”, and “parakeet.” If the set is infinite, then we need to be more clever with our description. For example, the set of *natural numbers* (or *counting numbers*) is the set

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}.$$

Because this set is infinite (there are an infinite number of natural numbers), we can’t list all of them. Instead, we list the first few then follow with “three dots,” which essentially mean “etcetera.” The implication is that the reader sees the intended pattern and can then intuit the remaining numbers in the set. Can you see that the next few numbers are 6, 7, 8, 9, etc.?

If we add the number zero to the set of natural numbers, then we have a set of numbers that are called the *whole numbers*.

The Whole Numbers

The set

$$\mathbb{W} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

is called the set of *whole numbers*.

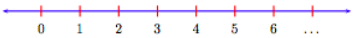
The whole numbers will be our focus in the remainder of this chapter.

Graphing Numbers on the Number Line

It is a simple matter to set up a correspondence between the whole numbers and points on a **number line**. First, draw a number line, then set a tick mark at zero.

The next step is to declare a unit length.

The remainder of the whole numbers now fall easily in place on the number line.



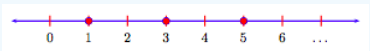
When asked to graph a whole number on a number line, shade in a solid dot at the position on the number line that corresponds to the given whole number.

Example 1.

Graph the whole numbers 1, 3, and 5 on the number line.

Solution

Shade the numbers 1, 3, and 5 on the number line as solid dots.



Exercise

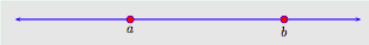
Graph the whole numbers 3, 4, and 6 on the number line.

Ordering the whole numbers

Now that we have a correspondence between the whole numbers and points on the number line, we can order the whole numbers in a natural way. Note that as you move to the left along the number line, the numbers get smaller; as you move to the right, the numbers get bigger. This inspires the following definition.

Ordering the Whole Numbers

Suppose that a and b are whole numbers located on the number line so that the point representing the whole number a lies to the left of the point representing the whole number b.



Then the whole number a is "less than" the whole number b and write

$$a < b.$$

Alternatively, we can also say that the whole number b is "greater than" the whole number a and write

$$b > a.$$

Comparison Property

When comparing two whole numbers a and b, only one of three possibilities is true:

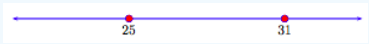
$$a < b \text{ or } a = b \text{ or } a > b.$$

Example 2.

Compare the whole numbers 25 and 31.

Solution

On the number line, 25 is located to the left of 31.



Therefore, 25 is less than 31 and write $25 < 31$. Alternatively, we could also note that 31 is located to the right of 25. Therefore, 31 is greater than 25 and write $31 > 25$.

Exercise

Compare the whole numbers 18 and 12.

Answer

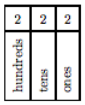
$18 > 12$.

Expanded Notation

The whole numbers

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

are called *digits* and are used to construct larger whole numbers. For example, consider the whole number 222 (pronounced "two hundred twenty two"). It is made up of three twos, but the position of each two describes a different meaning or value.



- The first two is in the "hundreds" position and represents two hundreds or 200.
- The second two is in the "tens" position and represents two tens or 20.
- The third two is in the "ones" position and represents two ones or 2.

Consider the larger number 123,456,789. The following table shows the place value of each digit.

1	2	3	4	5	6	7	8	9
hundred millions	ten millions	millions	hundred thousands	ten thousands	thousands	hundreds	tens	ones
millions			thousands			ones		

In “expanded notation,” we would write

1 hundred million + 2 ten millions + 3 millions + 4 hundred thousands + 5 ten thousands + 6 thousands + 7 hundreds + 8 tens + 9 ones.

We read the numeral 123,456,789 as “one hundred twenty three million, four hundred fifty six thousand, seven hundred eighty nine.”

Let’s look at another example.

Example 3.

Write the number 23,712 in expanded notation, then pronounce the result.

Solution

In expanded notation, 23,712 becomes

2 ten thousands + 3 thousands + 7 hundreds + 1 ten + 2 ones.

This is pronounced "twenty three thousand, seven hundred twelve."

Exercise

Write the number 54,615 in expanded notation. Pronounce the result.

Example 4.

Write the number 203,405 in expanded notation, then pronounce the result.

Solution

In expanded notation, 203,405 becomes

2 hundred thousands + 0 ten thousands + 3 thousands + 4 hundreds + 0 tens + 5 ones.

Since 0 ten thousands is zero and 0 tens is also zero, this can also be written

2 hundred thousands + 3 thousands + 4 hundreds + 5 ones.

This is pronounced “two hundred three thousand, four hundred five.”

Exercise

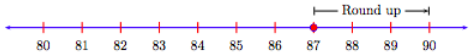
Write the number 430,705 in expanded notation. Pronounce the result.

Answer

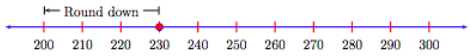
9,440

Rounding whole numbers

When less precision is needed, we round numbers to a particular place. For example, suppose a store owner needs approximately 87 boxes of ten-penny nails, but they can only be ordered in cartons containing ten boxes.



Note that 87 is located closer to 9 tens (or 90) than it is to 8 tens (or 80). Thus, rounded to the nearest ten, 87 ≈ 90 (87 approximately equals 90). The store owner decides that 90 boxes is probably a better fit for his needs. On the other hand, the same store owner estimates that he will need 230 bags of peatmoss for his garden section.



Note that 230 is closer to 2 hundreds (or 200) than it is to 3 hundreds (or 300). The store owner worries that might have overestimated his need, so he rounds down to the nearest hundred, 230 ≈ 200 (230 approximately equals 200). There is a simple set of rules to follow when rounding.

Rules for Rounding

To round a number to a particular place, follow these steps:

1. Mark the place you wish to round to. This is called the rounding digit.

2. Check the next digit to the right of your digit marked in step 1. This is called the test digit.

    a) If the test digit is greater than or equal to 5, add 1 to the rounding digit and replace all digits to the right of the rounding digit with zeros.

    b) If the test digit is less than 5, keep the rounding digit the same and replace all digits to the right of rounding digit with zero

Let's try these rules with an example or two.

Example 5

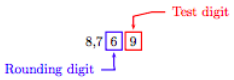
Round the number 8,769 to the nearest ten.

Solution

Mark the rounding and test digits.

Exercise

Round the number 9,443 to the nearest ten.



The test digit is greater than 5. The “Rules for Rounding” require that we add 1 to the rounding digit, then make all digits to the right of the rounding digit zeros. Thus, rounded to the nearest ten,

$$8,769 \approx 8,770.$$

That is, 8,769 is *approximately* equal to 8,770.

Mathematical Notation.

The symbol

$$\approx$$

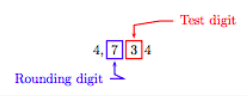
means *approximately equal*.

Example 6.

Round the number 4,734 to the nearest hundred.

Solution

Mark the rounding and test digits.



The test digit is less than 5. The “Rules for Rounding” require that we keep the rounding digit the same, then make all digits to the right of the rounding digit zeros. Thus, rounded to the nearest hundred,

$$4,734 \approx 4,700.$$

Exercise

Round the number 6,656 to the nearest hundred.

Answer

6,700

Table 1.1: Atmospheric CO_2 values (ppmv) derived from in situ air samples collected at Mauna Loa, Hawaii, USA.

Year	1965	1975	1985	1995	2005
Atmospheric CO_2	319	330	344	359	378

Tables and Graphs

Reading data in graphical form is an important skill. The data in Table 1.1 provides measures of the carbon dioxide content (CO_2) in the atmosphere, gathered in the month of January at the observatory atop Mauna Loa in Hawaii.

In Figure 1.1(a), a *bar graph* is used to display the carbon dioxide measurements. The year the measurement was taken is placed on the horizontal axis, and the height of each bar equals the amount of carbon dioxide in the atmosphere during that year.

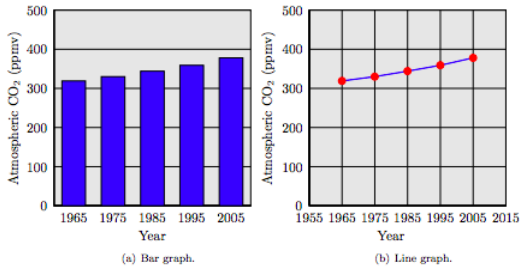


Figure 1.1: Using graphs to examine carbon dioxide data.

In Figure 1.1(b), a *line graph* is used to display the carbon dioxide measurements. Again, the dates of measurement are placed on the horizontal axis, and the amount of carbon dioxide in the atmosphere is placed on the vertical axis. Instead of using the height of a bar to represent the carbon dioxide measurement, we place a dot at a height that represents the carbon monoxide content. Once each data point is plotted, we connect consecutive data points with line segments.

Exercises

In Exercises 1-12, sketch the given whole numbers on a number line, then arrange them in order, from smallest to largest.

1. 2, 8, and 4
2. 2, 7, and 4
3. 1, 8, and 2
4. 0, 4, and 3
5. 0, 4, and 1
6. 3, 6, and 5
7. 4, 9, and 6

8. 2, 4, and 3
9. 0, 7, and 4
10. 2, 8, and 6
11. 1, 6, and 5
12. 0, 9, and 5

In Exercises 13-24, create a number line diagram to determine which of the two given statements is true.

13. $3 < 8$ or $3 > 8$
14. $44 < 80$ or $44 > 80$
15. $59 < 24$ or $59 > 24$
16. $15 < 11$ or $15 > 11$
17. $0 < 74$ or $0 > 74$
18. $11 < 18$ or $11 > 18$
19. $1 < 81$ or $1 > 81$
20. $65 < 83$ or $65 > 83$
21. $43 < 1$ or $43 > 1$
22. $62 < 2$ or $62 > 2$
23. $43 < 28$ or $43 > 28$
24. $73 < 21$ or $73 > 21$

---

25. Which digit is in the thousands column of the number 2,054,867,372?
26. Which digit is in the hundreds column of the number 2,318,999,087?
27. Which digit is in the hundred thousands column of the number 8,311,900,272?
28. Which digit is in the tens column of the number 1,143,676,212?
29. Which digit is in the hundred millions column of the number 9,482,616,000?
30. Which digit is in the hundreds column of the number 375,518,067?
31. Which digit is in the ten millions column of the number 5,840,596,473?
32. Which digit is in the hundred thousands column of the number 6,125,412,255?
33. Which digit is in the hundred millions column of the number 5,577,422,501?
34. Which digit is in the thousands column of the number 8,884,966,835?
35. Which digit is in the tens column of the number 2,461,717,362?
36. Which digit is in the ten millions column of the number 9,672,482,548?

---

37. Round the number 93, 857 to the nearest thousand.
38. Round the number 56, 872 to the nearest thousand.
39. Round the number 9, 725 to the nearest ten.
40. Round the number 6, 815 to the nearest ten.
41. Round the number 58, 739 to the nearest hundred.
42. Round the number 93, 146 to the nearest hundred.
43. Round the number 2, 358 to the nearest ten.

44. Round the number 8, 957 to the nearest ten.

45. Round the number 39, 756 to the nearest thousand.

46. Round the number 24, 965 to the nearest thousand.

47. Round the number 5, 894 to the nearest ten.

48. Round the number 3, 281 to the nearest ten.

49. Round the number 56, 123 to the nearest hundred.

50. Round the number 49, 635 to the nearest hundred.

51. Round the number 5, 483 to the nearest ten.

52. Round the number 9, 862 to the nearest ten.

---

53. According to the U.S. Census Bureau, the estimated population of the US is 304,059,724 as of July 2008. Round to the nearest hundred thousand.

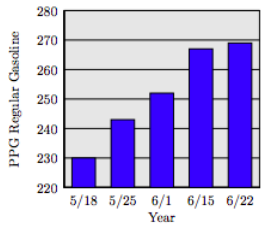
54. According to the U.S. Census Bureau, the estimated population of California is 36,756,666 as of July 2008. Round to the nearest hundred thousand.

55. According to the U.S. Census Bureau, the estimated population of Humboldt County is 129,000 as of July 2008. Round to the nearest ten thousand.

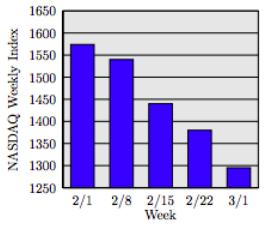
56. According to the U.S. Census Bureau, the estimated population of the state of Alaska was 686,293 as of July 2008. Round to the nearest ten thousand.

---

57. The following bar chart shows the average price (in cents) of one gallon of regular gasoline in the United States over five consecutive weeks in 2009, running from May 18 (5/18) through June 22 (6/22). What was the price (in cents) of one gallon of regular gasoline on June 1, 2009?



58. The following bar chart shows the average weekly NASDAQ index for five consecutive weeks in 2009, beginning with week starting February 1 (2/1) and ending with the week starting March 1 (3/1). What was the average NASDAQ index for the week starting February 8, 2009?



59. The population of Humboldt County is broken into age brackets in the following table. *Source: WolframAlpha*.

Age in years	Number
under 5	7,322
5 – 18	26,672
18 – 65	78,142
over 65	16,194

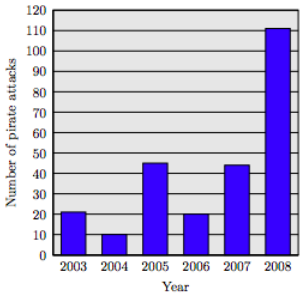
Create a bar chart for this data set with one bar for each age category.

60. The five cities with the largest number of reported violent crimes in the year 2007 are reported in the following table. *Source: Wikipedia.*

City	Violent Crimes
Setroit	2,289
St. Louis	2,196
Memphis	1,951
Oakland	1,918
Baltimore	1,631

Create a bar chart for this data set with one bar for each city.

61. The following bar chart tracks pirate attacks off the coast of Somalia.

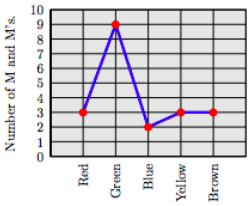


Source: ICC International Maritime Bureau, AP Times-Standard, 4/15/2009

a) How many pirate attacks were there in 2003?

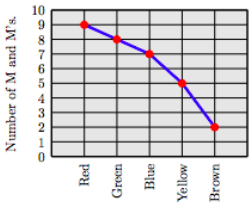
b) How many pirate attacks were there in 2008?

62. A team of students separated a small bowl of M and M's into five piles by color. The following line plot indicates the number of M and M's of each color.



How many red M and M's were in the bowl?

63. A team of students separated a small bowl of M and M's into five piles by color. The following line plot indicates the number of M and M's of each color.



How many red M and M's were in the bowl?

---

64. A team of students separated a small bowl of M and M's into five piles by color. The following table indicates the number of M and M's of each color.

Color	Number
Red	5
Green	9
Blue	7
Yellow	2
Brown	3

Create a lineplot for the M and M data. On the horizontal axis, arrange the colors in the same order as presented in the table above.

65. A team of students separated a small bowl of M and M's into five piles by color. The following table indicates the number of M and M's of every color.

Color	Number
Red	3
Green	7
Blue	2
Yellow	4
Brown	9

Create a lineplot for the M and M data. On the horizontal axis, arrange the colors in the same order as presented in the table above.

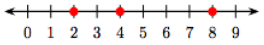
---

66. *Salmon count*. The table shows the number of adult coho salmon returning to the Shasta River over the past four years. Round the salmon count for each year to the nearest ten. *Times-Standard Shasta River coho rescue underway.*

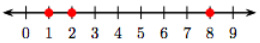
Year	Salmon count
2007	300
2008	31
2009	9
2010	4

Answers

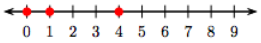
1. Smallest to largest: 2, 4, and 8.



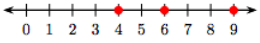
3. Smallest to largest: 1, 2, and 8.



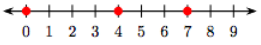
5. Smallest to largest: 0, 1, and 4.



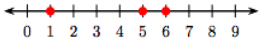
7. Smallest to largest: 4, 6, and 9.



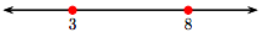
9. Smallest to largest: 0, 4, and 7.



11. Smallest to largest: 1, 5, and 6.



13.



Therefore, $3 < 8$.

15.

Therefore, $59 > 24$.

17.

Therefore, $0 < 74$.

19.

Therefore, $1 < 81$.

21.

Therefore, $43 > 1$.

23.

Therefore, $43 > 28$.

25. 7

27. 9

29. 4

31. 4

33. 5

35. 6

37. 94000

39. 9730

41. 58700

43. 2360

45. 40000

47. 5890

49. 56100

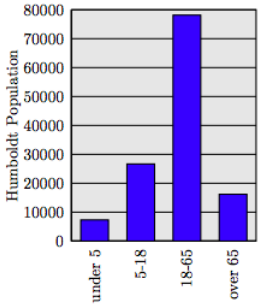
51. 5480

53. 304,100,000

55. 130,000

57. Approximately 252 cents

59.



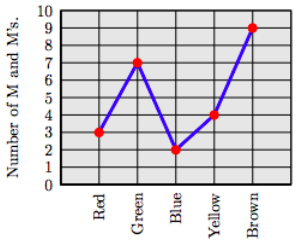
61.

a) Approximately 21

b) Approximately 111

63. 9

65.



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1.2: Adding and Subtracting Whole Numbers

In the expression 3 + 4, which shows the sum of two whole numbers, the whole numbers 3 and 4 are called *addends* or *terms*. We can use a visual approach to find the sum of 3 and 4. First, construct a number line as shown in Figure 1.2.

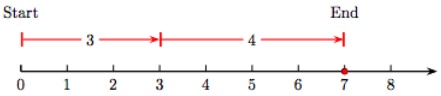


Figure 1.2: Adding whole numbers on the number line.

To add 3 and 4, proceed as follows.

1. Start at the number 0, then draw an arrow 3 units to the right, as shown in Figure 1.2. This arrow has magnitude (length) three and represents the whole number 3.
2. Draw a second arrow of length four, starting at the end of the first arrow representing the number 3. This arrow has magnitude (length) four and represents the whole number 4.
3. The sum of 3 and 4 could be represented by an arrow that starts at the number 0 and ends at the number 7. However, we prefer to mark this sum on the number line as a solid dot at the whole number 7. This number represents the sum of the whole numbers 3 and 4.

The Commutative Property of Addition

Let's change the order in which we add the whole numbers 3 and 4. That is, let's find the sum 4 + 3 instead.

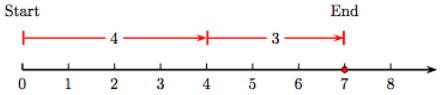


Figure 1.3: Addition is commutative; i.e., order doesn't matter.

As you can see in Figure 1.3, we start at zero then draw an arrow of length four, followed by an arrow of length three. However, the result is the same; i.e., 4 + 3 = 7.

Thus, the order in which we add three and four does not matter; that is,

$$3 + 4 = 4 + 3.$$

This property of addition of whole numbers is known as the *commutative property* of addition.

The Commutative Property of Addition

Let a and b represent two whole numbers. Then,

$$a + b = b + a.$$

Grouping Symbols

In mathematics, we use *grouping symbols* to affect the order in which an expression is evaluated. Whether we use parentheses, brackets, or curly braces, the expression inside any pair of grouping symbols must be evaluated first. For example, note how we first evaluate the sum in the parentheses in the following calculation.

$$(3 + 4) + 5 = 7 + 5$$

$$= 12$$

The rule is simple: Whatever is inside the parentheses is evaluated first.

Writing Mathematics

When writing mathematical statements, follow the mantra:

One equal sign per line.

We can use brackets instead of parentheses.

$$5 + [7 + 9] = 5 + 16$$
$$= 21$$

Again, note how the expression inside the brackets is evaluated first.

We can also use curly braces instead of parentheses or brackets.

$$\{2+3\} + 4 = 5 + 4$$
$$= 9$$

Again, note how the expression inside the curly braces is evaluated first.

If grouping symbols are nested, we evaluate the innermost parentheses first.

For example,

$$2 + [3 + (4 + 5)] = 2 + [3 + 9]$$
$$= 2 + 12$$
$$= 14.$$

Grouping Symbols

Use parentheses, brackets, or curly braces to delimit the part of an expression you want evaluated first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

The Associative Property of Addition

Consider the evaluation of the expression (2+3)+4. We evaluate the expression in parentheses first.

$$(2 + 3) + 4 = 5 + 4$$
$$= 9$$

Now, suppose we change the order of addition to 2 + (3 + 4). Then,

$$2 + (3 + 4) = 2 + 7$$
$$= 9.$$

Although the grouping has changed, the result is the same. That is,

$$(2 + 3) + 4 = 2 + (3 + 4).$$

This property of addition of whole numbers is called the *associate property* of addition.

Associate Property of Addition

Let a, b, and c represent whole numbers. Then,

$$(a + b) + c = a + (b + c).$$

Because of the associate property of addition, when presented with a sum of three numbers, whether you start by adding the first two numbers or the last two numbers, the resulting sum is the same.

The Additive Identity

Imagine a number line visualization of the sum of four and zero; i.e., 4 + 0.

In Figure 1.4, we start at zero, then draw an arrow of magnitude (length) four pointing to the right. Now, at the end of this arrow, attach a second arrow of length zero. Of course, that means that we remain right where we are, at 4. Hence the shaded dot at 4 is the sum. That is, 4 + 0 = 4.

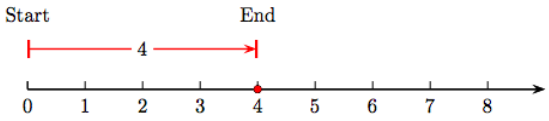


Figure 1.4: Adding zero to four.

The Additive Identity Property

The whole number zero is called the additive identity. If a is any whole number, the

$$a + 0 = a.$$

The number zero is called the additive identity because if you add zero to any number, you get the identical number back.

Adding Larger Whole Numbers

For completeness, we include two examples of adding larger whole numbers. Hopefully, the algorithm is familiar from previous coursework.

Example 1

Simplify: 1, 234 + 498.

Solution

Align the numbers vertically, then add, starting at the furthest column to the right. Add the digits in the ones column, 4 + 8 = 12. Write the 2, then carry a 1 to the tens column. Next, add the digits in the tens column, 3 + 9 = 12, add the carry to get 13, then write the 3 and carry a 1 to the hundreds column. Continue in this manner, working from right to left

$$\begin{array}{r} {\scriptstyle 11} \\ 1234 \\ +\ \ 498 \\ \hline 1732 \end{array}$$

Therefore, 1, 234 + 498 = 1, 732

Exercise

Simplify: 1,286 + 349.

Answer

1635

Add three or more numbers in the same manner.

Example 2

Simplify: 256 + 322 + 418.

Solution

Align the numbers vertically, then add, starting at the furthest column to the right. Add the digits in the ones column, 6 + 2 + 8 = 16. Write the 6, then carry a 1 to the tens column. Continue in this manner, working from right to left.

$$\begin{array}{r} 256 \\ 322 \\ +418 \\ \hline 996 \end{array}$$

Therefore, 256 + 322 + 418 = 996.

Exercise

Simplify: 256 + 342 + 283

Answer

881

Subtraction of Whole Numbers

The key idea is this: *Subtraction is the opposite of addition*. For example, consider the difference 7 - 4 depicted on the number line in Figure 1.5.

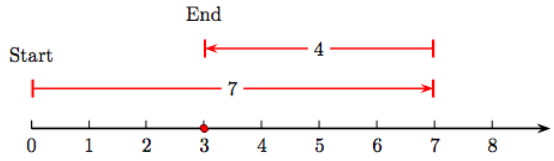


Figure 1.5: Subtraction means *add the opposite*.

If we were adding 7 and 4, we first draw an arrow starting at zero pointing to the right with magnitude (length) seven. Then, to add 4, we would draw a second arrow of magnitude (length) 4, attached to the end of the first arrow and pointing to the right.

However, because subtraction is the *opposite* of addition, in Figure 1.5 we attach an arrow of magnitude (length) four to the end of the first arrow, but *pointing in the opposite direction* (to the left). Note that this last arrow ends at the answer, which is a shaded dot on the number line at 3. That is, $7 - 4 = 3$.

Note that subtraction is **not commutative**; that is, it make no sense to say that $7 - 5$ is the same as $5 - 7$.

Subtraction is **not associative**. It is not the case that $(9 - 5) - 2$ is the same as $9 - (5 - 2)$. On the one hand,

$$(9 - 5) - 2 = 4 - 2$$
$$= 2,$$

but

$$9 - (5 - 2) = 9 - 3$$
$$= 6.$$

Subtracting Larger Whole Numbers

Much as we did with adding larger whole numbers, to subtract two large whole numbers, align them vertically then subtract, working from right to left. You may have to “borrow” to complete the subtraction at any step.

Example 3

Simplify: $1,755 - 328$.

Solution

Align the numbers vertically, then subtract, starting at the ones column, then working right to left. At the ones column, we cannot subtract 8 from 5, so we borrow from the previous column. Now, 8 from 15 is 7. Continue in this manner, working from right to left.

$$\begin{array}{rcccc} & & & {\scriptstyle 4} & \\ & 1 & 7 & \cancel{5} & {}^{1}5 \\ - & & 3 & 2 & 8 \\ \hline & 1 & 4 & 2 & 7 \end{array}$$

Therefore, 1, 755 − 328 = 1, 427.

Exercise

Simplify: 5,635 - 288.

Answer

5,347

Order of Operations

In the absence of grouping symbols, it is important to understand that addition holds no precedence over subtraction, and vice-versa.

> Perform all additions and subtractions in the order presented, moving left to right.

Let's look at an example.

Example 4

Simplify the expression $15-8+4$.

Solution

This example can be trickier than it seems. However, if we follow the rule (perform all additions and subtractions in the order presented, moving left to right), we should have no trouble. First comes fifteen minus eight, which is seven. Then seven plus four is eleven.

$$\begin{aligned} 15-8+4 &= 7+4 \\ &= 11. \end{aligned}$$

Exercise

Simplify: $25-10+8$.

Answer

23

Caution! Incorrect answer ahead!

Note that it is possible to arrive at a different (but incorrect) answer if we favor addition over subtraction in Example 4. If we first add eight and four, then 15 − 8 + 4 becomes 15 − 12, which is 3. However, note that **this is incorrect**, because it violates the rule "perform all additions and subtractions in the order presented, moving left to right."

Applications — Geometry

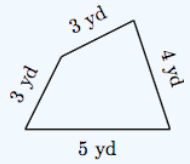
There are any number of applications that require a sum or difference of whole numbers. Let's examine a few from the world of geometry.

Perimeter of a Polygon

In geometry a polygon is a plane figure made up of a closed path of a finite sequence of segments. The segments are called the *edges* or *sides* of the polygon and the points where two edges meet are called the *vertices* of the polygon. The *perimeter* of any polygon is the sum of the lengths of its sides.

Example 5

A quadrilateral is a polygon with four sides. Find the perimeter of the quadrilateral shown below, where the sides are measured below.



Solution

To find the perimeter of the quadrilateral, find the sum of the lengths of the sides.

$$\text{Perimeter} = 3 + 3 + 4 + 5 = 15$$

Hence, the perimeter of the quadrilateral is 15 yards.

Exercise

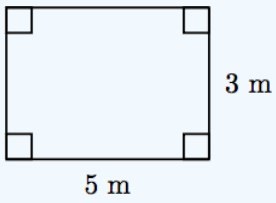
A quadrilateral has sides that measure in 4 in., 3 in., 5 in., and 5 in. Find the perimeter.

Answer

17 inches

Example 6

A quadrilateral (four sides) is a rectangle if all four of its angles are right angles. It can be shown that the opposite sides of a rectangle must be equal. Find the perimeter of the rectangle shown below, where the sides of the rectangle are measured in meters.



Solution

To find the perimeter of the rectangle, find the sum of the four sides. Because opposite sides have the same length, we have two sides of length 5 meters and two sides of length 3 meters. Hence,

$$\text{Perimeter} = 5 + 3 + 5 + 3 = 16.$$

Thus, the perimeter of the rectangle is 16 meters.

Exercise

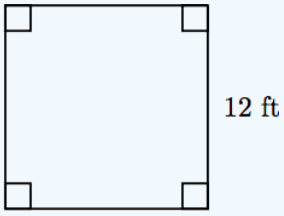
A rectangle has length 12 meters and width 8 meters. Find its perimeter.

Answer

40 meters

Example 7

A quadrilateral (four sides) is a *square* if all four of its sides measures 18 centimeters. Find its perimeter. are equal and all four of its angles are right angles. Pictured below is a square having a side of length 12 feet. Find the perimeter of the square.



Solution

Because the quadrilateral is a square, all four sides have the same length, namely 12 feet. To find the perimeter of the square, find the sum of the four sides. Perimeter = 12 + 12 + 12 + 12 = 48

Hence, the perimeter of the square is 48 feet.

Exercise

A square has a side that measures 18 centimeters. Find its perimeter.

Answer

72 centimeters

Application — Alternative Fuels

Automobiles that run on alternative fuels (other than gasoline) have increased in the United States over the years.

Example 8

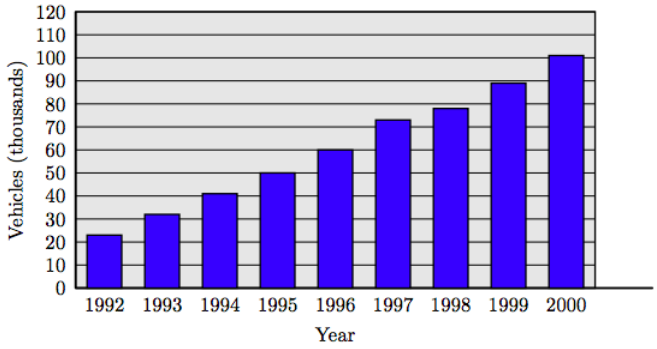
Table 1.2 show the number of cars (in thousands) running on compressed natural gas versus the year. Create a bar chart showing the number of cars running on compressed natural gas versus the year.

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000
Number	23	32	4`	50	60	73	78	89	101

Table 1.2: Number of vehicles (in thousands) running on compressed natural gas.

Solution

Place the years on the horizontal axis. At each year, sketch a bar having height equal to the number of cars in that year that are running on compressed natural gas. Scale the vertical axis in thousands.



Exercise

The following table shows the number of hybrid cars (in thousands) by country.

Country	Number
U.S.	279
Japan	77
Canada	17
U.K.	14
Netherlands	11

Create a bar chart showing the number of cars versus the country of use.

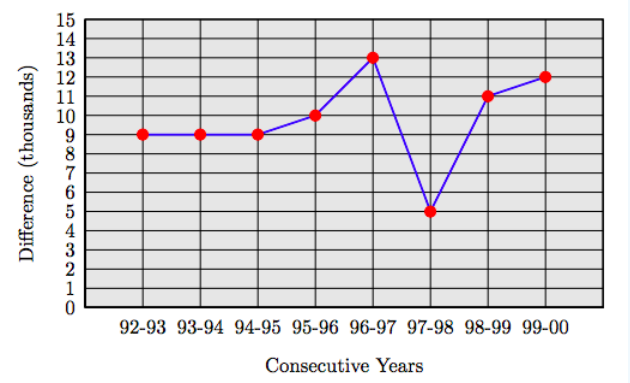
Example 9

Using the data in Table 1.2, crease a table that shows the differences in consecutive years, then create a line plot of the result. In what consecutive years did the United States see the greatest increase in cars powered by compressed natural gas?

Solution

Table 1.3 shows the differences in consecutive years.

Next, craft a line graph. Place consecutive years on the horizontal axis. At each consecutive year pair, plot a point at a height equal to the difference in alternative fuel vehicles. Connect the points with straight line segments.



Note how the line graph makes it completely clear that the greatest increase in vehicles powered by compressed natural gas occurred in the consecutive years 1996-1997, an increase of 13,000 vehicles.

Exercise

The following table show Alphonso's percentage scores on his examinations in mathematics.

Exam	Percentage
Exam #1	52
Exam #2	45
Exam #3	72
Exam #4	889
Exam #5	76

Construct a line graph of Alphonso's exam scores versus exam number.

Exercises

1. Sketch a number line diagram depicting the sum 3 + 2, as shown in Figure 1.2 in the narrative of this section.
2. Sketch a number line diagram depicting the sum 3 + 5, as shown in Figure 1.2 in the narrative of this section.
3. Sketch a number line diagram depicting the sum 3 + 4, as shown in Figure 1.2 in the narrative of this section.
4. Sketch a number line diagram depicting the sum 2 + 4, as shown in Figure 1.2 in the narrative of this section.
5. Sketch a number line diagram depicting the sum 4 + 2, as shown in Figure 1.2 in the narrative of this section.
6. Sketch a number line diagram depicting the sum 4 + 3, as shown in Figure 1.2 in the narrative of this section.
7. Sketch a number line diagram depicting the sum 2 + 5, as shown in Figure 1.2 in the narrative of this section.
8. Sketch a number line diagram depicting the sum 4 + 5, as shown in Figure 1.2 in the narrative of this section.
9. Sketch a number line diagram depicting the sum 4 + 4, as shown in Figure 1.2 in the narrative of this section.
10. Sketch a number line diagram depicting the sum 3 + 3, as shown in Figure 1.2 in the narrative of this section.

In Exercises 11-28, determine which property of addition is depicted by the given identity

11. 28 + 0 = 28
12. 53 + 0 = 53
13. 24 + 0 = 24
14. 93 + 0 = 93
15. (51 + 66) + 88 = 51 + (66 + 88)
16. (90 + 96) + 4 = 90 + (96 + 4)
17. 64 + 39 = 39 + 64
18. 68 + 73 = 73 + 68
19. (70 + 27) + 52 = 70 + (27 + 52)
20. (8 + 53) + 81 = 8 + (53 + 81)
21. 79 + 0 = 79
22. 42 + 0 = 42
23. 10 + 94 = 94 + 10
24. 55 + 86 = 86 + 55
25. 47 + 26 = 26 + 47

26. 62 + 26 = 26 + 62

27. (61 + 53) + 29 = 61 + (53 + 29)

28. (29 + 96) + 61 = 29 + (96 + 61)

---

29. Sketch a number line diagram depicting the difference 8−2, as shown in Figure 1.5 in the narrative of this section.

30. Sketch a number line diagram depicting the difference 8−4, as shown in Figure 1.5 in the narrative of this section.

31. Sketch a number line diagram depicting the difference 7−2, as shown in Figure 1.5 in the narrative of this section.

32. Sketch a number line diagram depicting the difference 9−5, as shown in Figure 1.5 in the narrative of this section.

33. Sketch a number line diagram depicting the difference 7−4, as shown in Figure 1.5 in the narrative of this section.

34. Sketch a number line diagram depicting the difference 6−4, as shown in Figure 1.5 in the narrative of this section.

35. Sketch a number line diagram depicting the difference 9−4, as shown in Figure 1.5 in the narrative of this section.

36. Sketch a number line diagram depicting the difference 6−5, as shown in Figure 1.5 in the narrative of this section.

37. Sketch a number line diagram depicting the difference 8−5, as shown in Figure 1.5 in the narrative of this section.

38. Sketch a number line diagram depicting the difference 9−3, as shown in Figure 1.5 in the narrative of this section.

In Exercises 39-50, simplify the given expression.

39. 16 − 8+2

40. 17 − 3+5

41. 20 − 5 + 14

42. 14 − 5+6

43. 15 − 2+5

44. 13 − 4+2

45. 12 − 5+4

46. 19 − 4 + 13

47. 12 − 6+4

48. 13 − 4 + 18

49. 15 − 5+8

50. 13 − 3 + 11

In Exercises 51-58, the width W and length L of a rectangle are given. Find the perimeter P of the rectangle.

51. W = 7 in, L = 9 in

52. W = 4 in, L = 6 in

53. W = 8 in, L = 9 in

54. W = 5 in, L = 9 in

55. W = 4 cm, L = 6 cm

56. W = 5 in, L = 8 in

57. W = 4 cm, L = 7 cm

58. W = 4 in, L = 9 in

In Exercises 59-66, the length s of a side of a square is given. Find the perimeter P of the square.

59. s = 25 cm

60. s = 21 in

61. s = 16 cm

62. s = 10 in

63. s = 18 in

64. s = 7 in

65. s = 3 in

66. s = 20 in

In Exercises 67-86, find the sum.

67. 3005 + 5217

68. 1870 + 5021

69. 575 + 354 + 759

70. 140 + 962 + 817

71. 472 + (520 + 575)

72. 318 + (397 + 437)

73. 274 + (764 + 690)

74. 638 + (310 + 447)

75. 8583 + 592

76. 5357 + 9936

77. 899 + 528 + 116

78. 841 + 368 + 919

79. (466 + 744) + 517

80. (899 + 996) + 295

81. 563 + 298 + 611 + 828

82. 789 + 328 + 887 + 729

83. 607 + 29 + 270 + 245

84. 738 + 471 + 876 + 469

85. (86 + 557) + 80

86. (435 + 124) + 132

In Exercises 87-104, find the difference.

87. 3493 − 2034 − 227

88. 3950 − 1530 − 2363

89. 8338 − 7366

90. 2157 − 1224

91. 2974 − 2374

92. 881 − 606

93. 3838 − (777 − 241)

94. 8695 − (6290 − 4233)

95. 5846 − 541 − 4577

96. 5738 − 280 − 4280

97. 3084 − (2882 − 614)

98. 1841 − (217 − 28)

99. 2103 − (1265 − 251)

100. 1471 − (640 − 50)

101. 9764 − 4837 − 150

102. 9626 − 8363 − 1052

103. 7095 − 226

104. 4826 − 1199

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105. **Water Subsidies**. Since the drought began in 2007, California farms have received $79 million in water subsidies. California cotton and rice farmers received an additional $439 million. How much total water subsidies have farmers received? *Associated Press Times-Standard 4/15/09*

106. **War Budget**. The 2010 Federal budget allocates $534 billion for the Department of Defense base programs and an additional $130 billion for the nation's two wars. How much will the Department of Defense receive altogether? *Associated Press Times-Standard 5/8/09*

107. **Sun Frost**. Arcata, CA is home to Sun Frost, a manufacturer of highly efficient refrigerators and freezers. The AC model RF12 refrigerator/freezer costs $2,279 while an R16 model refrigerator/freezer costs $3,017. How much more does the R16 model cost? Source: www.sunfrost.com/retail pricelist.html

108. **Shuttle Orbit**. The space shuttle usually orbits at 250 miles above the surface of the earth. To service the Hubble Space Telescope, the shuttle had to go to 350 miles above the surface. How much higher did the shuttle have to orbit?

109. **Earth's Orbit**. Earth orbits the sun in an ellipse. When earth is at its closest to the sun, called *perihelion*, earth is about 147 million kilometers. When earth is at its furthest point from the sun, called *aphelion*, earth is about 152 million kilometers from the sun. What's the difference in millions of kilometers between aphelion and perihelion?

110. **Pluto's Orbit**. Pluto's orbit is highly eccentric. Find the difference between Pluto's closest approach to the sun and Pluto's furthest distance from the sun if Pluto's perihelion (closest point on its orbit about the sun) is about 7 billion kilometers and its aphelion (furthest point on its orbit about the sun) is about 30 billion kilometers.

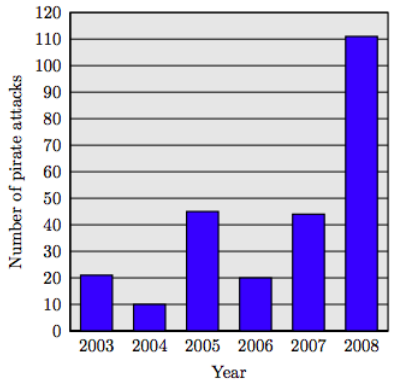
111. **Sunspot Temperature**. The surface of the sun is about 10,000 degrees Fahrenheit. Sunspots are darker regions on the surface of the sun that have a relatively cooler temperature of 6,300 degrees Fahrenheit. How many degrees cooler are sunspots?

112. **Jobs**. The Times-Standard reports that over the next year, the credit- and debitcard processing business Humboldt Merchant Services expects to cut 36 of its 80 jobs, but then turn around and hire another 21. How many people will be working for the company then? *Times-Standard 5/6/09*

113. **Wild tigers**. The chart shows the estimated wild tiger population, by region. According to this chart, what is the total wild tiger population worldwide? *Associated Press-Times-Standard 01/24/10 Pressure mounts to save the tiger*.

Region	Tiger population
India, Nepal and Bhutan	1650
China and Russia	450
Bangladesh	250
Sumatra (Indonesia)	400
Malaysia	500
other SE Asia	350

114. **Pirate Attacks**. The following bar chart tracks pirate attacks off the coast of Somalia.



Source: ICC International Maritime Bureau, AP Times-Standard, 4/15/2009

a) How many pirate attacks were there in 2003, 2004, and 2005 combined?

b) How many pirate attacks were there in 2006, 2007, and 2008 combined?

c) How many more pirate attacks were there in 2008 than in 2007?

115. Emily shows improvement on each successive examination throughout the term. Her exam scores are recorded in the following table.

Exam	**Score**
Exam #1	48
Exam #2	51
Exam #3	54
Exam #4	59
Exam #5	67
Exam #6	70

a) Create a bar plot for Emily's examination scores. Place the examination numbers on the horizontal axis in the same order shown in the table above.

b) Create a table that shows successive differences in examination scores. Make a line plot of these differences. Between which two exams did Emily show the greatest improvement?

116. Jason shows improvement on each successive examination throughout the term. His exam scores are recorded in the following table.

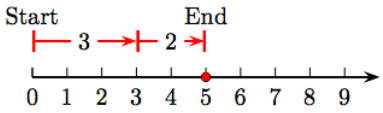
Exam	**Score**
Exam #1	34
Exam #2	42
Exam #3	45
Exam #4	50
Exam #5	57
Exam #6	62

a) Create a bar plot for Jason's examination scores. Place the examination numbers on the horizontal axis in the same order shown in the table above.

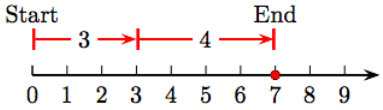
b) Create a table that shows successive differences in examination scores. Make a line plot of these differences. Between which two exams did Jason show the greatest improvement?

Answers

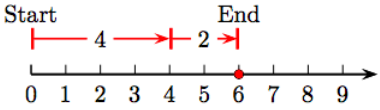
1. 3 + 2 = 5.



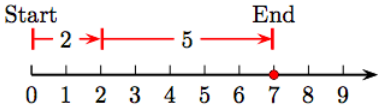
3. 3 + 4 = 7.



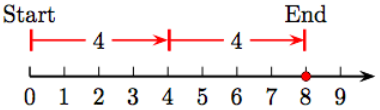
5. 4 + 2 = 6.



7. 2 + 5 = 7.



9. 4 + 4 = 8.



11. Additive identity property of addition.

13. Additive identity property of addition.

15. Associative property of addition

17. Commutative property of addition

19. Associative property of addition

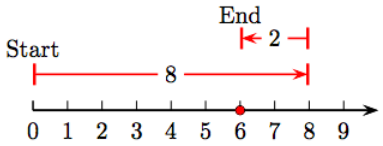
21. Additive identity property of addition.

23. Commutative property of addition

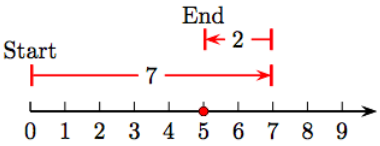
25. Commutative property of addition

27. Associative property of addition

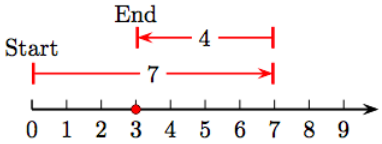
29. 8 − 2 = 6.



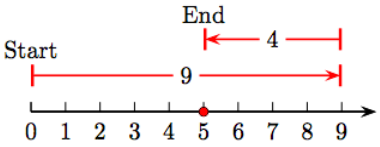
31. $7 - 2 = 5$.



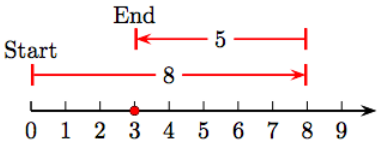
33. $7 - 4 = 3$.



35. $9 - 4 = 5$.



37. $8 - 5 = 3$.



39. 10

41. 29

43. 18

45. 11

47. 10

49. 18

51. P = 32 in

53. P = 34 in

55. P = 20 cm

57. P = 22 cm

59. P = 100 cm

61. P = 64 cm

63. P = 72 in

65. P = 12 in

67. 8222

69. 1688

https://math.libretexts.org/@go/page/46733

71. 1567

73. 1728

75. 9175

77. 1543

79. 1727

81. 2300

83. 1151

85. 723

87. 1232

89. 972

91. 600

93. 3302

95. 728

97. 816

99. 1089

101. 4777

103. 6869

105. $518 million

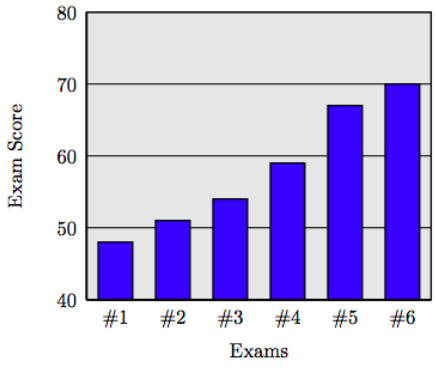
107. $738

109. 5 million kilometers

111. 3,700 degrees Fahrenheit

113. 3600

115. a) Bar chart.



b) Line plot of consecutive differences. The line plot of consecutive examination score differences.

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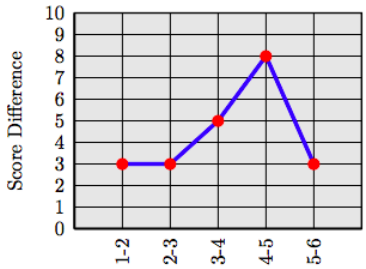

The largest improvement was between Exam #4 and Exam #5, where Emily improved by 8 points.

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1.3: Multiplication and Division of Whole Numbers

We begin this section by discussing multiplication of whole numbers. The first order of business is to introduce the various symbols used to indicate multiplication of two whole numbers.

Mathematical symbols that indicate multiplication

Symbol		Example
×	times symbol	3 × 4
·	dot	3 · 4
( )	parentheses	(3)(4) or 3(4) of (3)4

Products and Factors

In the expression $3 \cdot 4$, the whole numbers 3 and 4 are called the **factors** and $3 \cdot 4$ is called the **product**.

The key to understanding multiplication is held in the following statement.

> Multiplication is equivalent to repeated addition.

Suppose, for example, that we would like to evaluate the product $3 \cdot 4$. Because multiplication is equivalent to repeated addition, $3 \cdot 4$ is equivalent to adding three fours. That is,

$$3 \cdot 4 = \underbrace{4+4+4}_{\text{three fours}}$$

Thus, $3 \cdot 4 = 12$. You can visualize the product $3 \cdot 4$ as the sum of three fours on a number line, as shown in Figure 1.6.

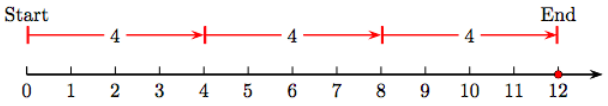


Figure 1.6: Note that 3 · 4 = 4 + 4 + 4. That is, 3 · 4 = 12.

Like addition, the order of the factors does not matter.

$$4 \cdot 3 = \underbrace{3+3+3+3}_{\text{four threes}}$$

Thus, $4 \cdot 3 = 12$. Consider the visualization of $4 \cdot 3$ in Figure 1.7.

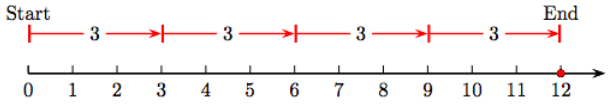


Figure 1.7: Note that 4 · 3 = 3 + 3 + 3 + 3. That is, 4 · 3 = 12.

The evidence in Figure 1.6 and Figure 1.7 show us that multiplication is commutative. That is,

$$3 \cdot 4 = 4 \cdot 3$$

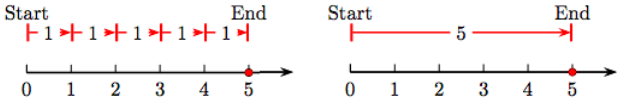
Commutative Property of Multiplication

If *a* and *b* are any whole numbers, then

$$a \cdot b = b \cdot a.$$

The Multiplicative Identity

In Figure 1.8(a), note that five ones equals 5; that is, $5 \cdot 1 = 5$. On the other hand, in Figure 1.8(b), we see that one five equals five; that is, $1 \cdot 5 = 5$.



(a) Note that $5 \cdot 1 = 1 + 1 + 1 + 1 + 1$. (b) Note that $1 \cdot 5 = 5$.

Figure 1.8: Note that 5 · 1 = 5 and 1 · 5 = 5.

Because multiplying a whole number by 1 equals that identical number, the whole number 1 is called the multiplicative identity.

The Multiplicative Identity Property

If a is any whole number, then

$$a \cdot 1 = a \text{ and } 1 \cdot a = a.$$

Multiplication by Zero

Because $3 \cdot 4 = 4 + 4 + 4$, we can say that the product $3 \cdot 4$ represents "3 sets of 4," as depicted in Figure 1.9, where three groups of four boxes are each enveloped in an oval.

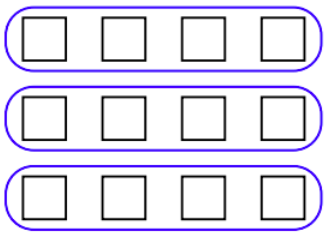


Figure 1.9: Three sets of four: 3 · 4 = 12.

Therefore, $0 \cdot 4$ would mean zero sets of four. Of course, zero sets of four is zero.

Multiplication by Zero.

If a represents any whole number, then

$$a \cdot 0 = 0 \text{ and } 0 \cdot a = 0.$$

The Associative Property of Multiplication

Like addition, multiplication of whole numbers is associative. Indeed,

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= 2 \cdot 12 \\ &= 24 \end{aligned}$$

and

$$\begin{aligned} (2 \cdot 3) \cdot 4 &= 6 \cdot 4 \\ &= 24. \end{aligned}$$

The Associative Property of Multiplication.

If a, b, and c are any whole numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Multiplying Larger Whole Numbers

Much like addition and subtraction of large whole numbers, we will also need to multiply large whole numbers. Again, we hope the algorithm is familiar from previous coursework.

Example 1

Simplify: $35 \cdot 127$.

Solution

Align the numbers vertically. The order of multiplication does not matter, but we'll put the larger of the two numbers on top of the smaller number. The first step is to multiply 5 times 127. Again, we proceed from right to left. So, 5 times 7 is 35. We write the 5, then carry the 3 to the tens column. Next, 5 times 2 is 10. Add the carry digit 3 to get 13. Write the 3 and carry the 1 to the hundreds column. Finally, 5 times 1 is 5. Add the carry digit to get 6.

$$\begin{array}{r} {\scriptstyle 1\ 3\ \ } \\ 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \end{array}$$

The next step is to multiply 3 times 127. However, because 3 is in the tens place, its value is 30, so we actually multiply 30 times 127. This is the same as multiplying 127 by 3 and placing a 0 at the end of the result.

$$\begin{array}{r} {\scriptstyle 2\ \ } \\ 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \\ 3\,8\,1\,0 \\ \hline \end{array}$$

After adding the 0, 3 times 7 is 21. We write the 1 and carry the 2 above the 2 in the tens column. Then, 3 times 2 is 6. Add the carry digit 2 to get 8. Finally, 3 times 1 is 1.

All that is left to do is to add the results.

$$\begin{array}{r} 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \\ 3\,8\,1\,0 \\ \hline 4\,4\,4\,5 \end{array}$$

Thus, $35 \cdot 127 = 4,445$.

Alternate Format

It does not hurt to omit the trailing zero in the second step of the multiplication, where we multiply 3 times 127. The result would look like this:

$$\begin{array}{r} 1\,2\,7 \\ \times\ 3\,5 \\ \hline 6\,3\,5 \\ 3\,8\,1\,0 \\ \hline 4\,4\,4\,5 \end{array}$$

In this format, the zero is understood, so it is not necessary to have it physically present. The idea is that with each multiplication by a new digit, we indent the product one space from the right.

Exercise

Simplify: $56 \cdot 335$

Answer

18,760

Division of Whole Numbers

We now turn to the topic of division of whole numbers. We first introduce the various symbols used to indicate division of whole numbers.

Mathematical Symbols that Indicate Division

Symbol		Example
$\div$	division symbol	$12 \div 4$
-	fraction bar	$\frac{12}{4}$
$\longdiv{}$	division bar	$4\longdiv{12}$

Note that each of the following say the same thing; that is, "12 divided by 4 is 3."

$12 \div 4 = 3$ or $\frac{12}{4} = 3$ or $4\overline{)12}$

Quotients, Dividends, and Divisors

In the statement

$$\frac{3}{4)12}$$

the whole number 12 is called the *dividend*, the whole number 4 is called the *divisor*, and the whole number 3 is called the *quotient*. Note that this division bar notation is equivalent to

$$12 \div 4 = 3 \quad \text{and} \quad \frac{12}{4} = 3.$$

The expression a/b means "a divided by b," but this construct is also called a *fraction*.

Definition: Fraction

The expression

$$\frac{a}{b}$$

is called a **fraction**. The number a on top is called the **numerator** of the fraction; the number b on the bottom is called the **denominator** of the fraction.

The key to understanding division of whole numbers is contained in the following statement.

> Division is equivalent to repeated subtraction.

Suppose for example, that we would like to divide the whole number 12 by the whole number 4. This is equivalent to asking the question "how many fours can we subtract from 12?" This can be visualized in a number line diagram, such as the one in Figure 1.10.

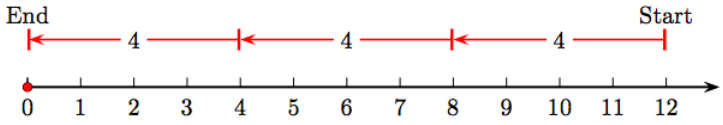


Figure 1.10: Division is repeated subtraction.

In Figure 1.10, note that we if we subtract three fours from twelve, the result is zero. In symbols,

$$12 - \underbrace{4 - 4 - 4}_{\text{three fours}} = 0.$$

Equivalently, we can also ask "How many groups of four are there in 12," and arrange our work as shown in Figure 1.11, where we can see that in an array of twelve objects, we can circle three groups of four ; i.e., 12 ÷ 4 = 3.

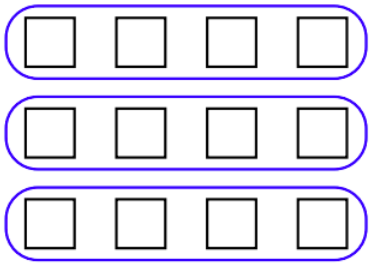


Figure 1.11: There are three groups of four in twelve.

In Figure 1.10 and Figure 1.11, note that the division (repeated subtraction) leaves no remainder. This is not always the case.

Example 2.

Divide 7 by 3.

Solution

In Figure 1.12, we can see that we can subtract two threes from seven, leaving a remainder of one.

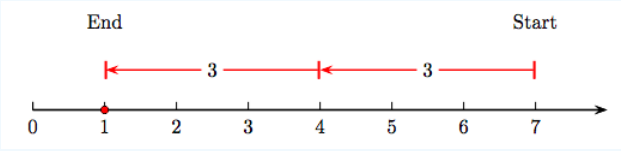


Figure 1.12: Division with a remainder.

Alternatively, in an array of seven objects, we can circle two groups of three, leaving a remainder of one.

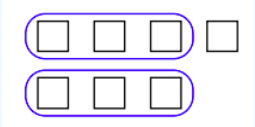


Figure 1.13: Dividing seven by three leaves a remainder of one.

Both Figure 1.12 and Figure1.13 show that there are two groups of three in seven, with one left over. We say "Seven divided by three is two, with a remainder of one.

Exercise

Use both the number line approach and the array of boxes approach to divide 12 by 5.

Division is not Commutative

When dividing whole numbers, the order matters. For example

$$12 \div 4 = 3,$$

but 4 ÷ 12 is not even a whole number. Thus, if a and b are whole numbers, then $a \div b$ does **not** have to be the same as $b \div a$.

Division is not Associative

When you divide three numbers, the order in which they are grouped will usually affect the answer. For example,

$$(48 \div 8) \div 2 = 6 \div 2$$
$$= 3,$$

but

$$48 \div (8 \div 2) = 48 \div 4$$
$$= 12.$$

Thus, if a, b, and c are whole numbers, $(a \div b) \div c$ does not have to be the same as $a \div (b \div c)$.

Division by Zero is Undefined

Suppose that we are asked to divide six by zero; that is, we are asked to calculate 6 ÷ 0. In Figure 1.14, we have an array of six objects.

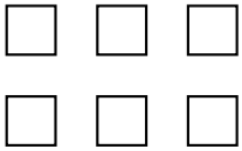


Figure 1.14: How many groups of zero do you see?

Now, to divide six by zero, we must answer the question "How many groups of zero can we circle in Figure 1.14?" Some thought will provide the answer: This is a meaningless request! It makes absolutely no sense to ask how many groups of zero can be circled in the array of six objects in Figure 1.14.

Division by Zero

Division by zero is **undefined**. Each of the expressions

$$6 \div 0 \quad \text{and} \quad \frac{6}{0} \quad \text{and} \quad 0\overline{)6}$$

is **undefined.**

On the other hand, it make sense to ask "What is zero divided by six?" If we create an array of zero objects, then ask how many groups of six we can circle, the answer is "zero groups of six." That is, zero divided by six is zero.

$$0 \div 6 = 0 \quad \text{and} \quad \frac{0}{6} = 0 \quad \text{and} \quad 6\frac{0}{0}$$

Dividing Larger Whole Numbers

We'll now provide a quick review of division of larger whole numbers, using an algorithm that is commonly called *long division*. This is not meant to be a thorough discussion, but a cursory one. We're counting on the fact that our readers have encountered this algorithm in previous courses and are familiar with the process.

Example 3

Simplify: 575/23.

Solution

We begin by estimating how many times 23 will divide into 57, guessing 1. We put the 1 in the quotient above the 7, multiply 1 times 23, place the answer underneath 57, then subtract.

$$23\overline{)\frac{1}{575}}$$
$$\frac{23}{34}$$

Because the remainder is larger than the divisor, our estimate is too small. We try again with an estimate of 2.

$$\begin{array}{r} 2 \\ 23\overline{)575} \\ \underline{46} \\ 11 \end{array}$$

That's the algorithm. Divide, multiply, then subtract. You may continue only when the remainder is smaller than the divisor.

To continue, bring down the 5, estimate that 115 divided by 23 is 5, then multiply 5 times the divisor and subtract.

$$\begin{array}{r} 25 \\ 23\overline{)575} \\ \underline{46} \\ 115 \\ \underline{115} \\ 0 \end{array}$$

Because the remainder is zero, 575/23 = 25.

? Exercise

Divide 980/35

Answer

28

Application — Counting Rectangular Arrays

Consider the rectangular array of stars in Figure 1.15. To count the number of stars in the array, we could use brute force, counting each star in the array one at a time, for a total of 20 stars. However, as we have four rows of five stars each, it is much faster to multiply: $4 \cdot 5 = 20$ stars.

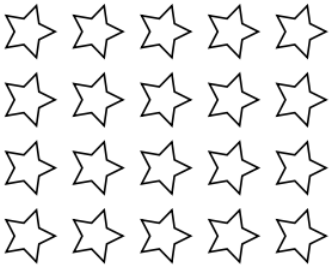


Figure 1.15: Four rows and five columns.

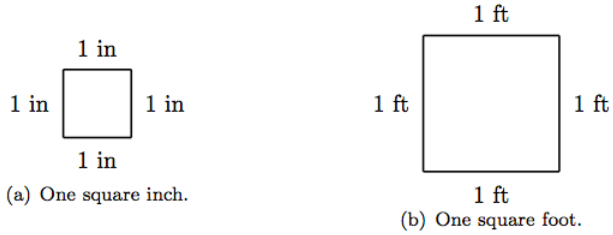


Figure 1.16: Measures of area are in square units.

Application — Area

In Figure 1.16(a), pictured is one square inch (1 in^2), a square with one inch on each side. In Figure 1.16(b), pictured is one square foot (1 ft^2), a square with one foot on each side. Both of these squares are measures of area. Now, consider the rectangle shown in Figure 1.17. The length of this rectangle is four inches (4 in) and the width is three inches (3 in).

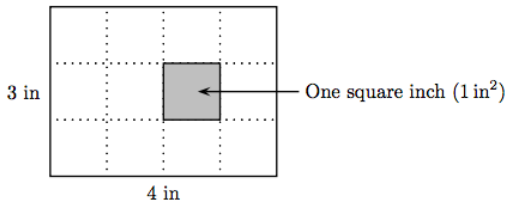


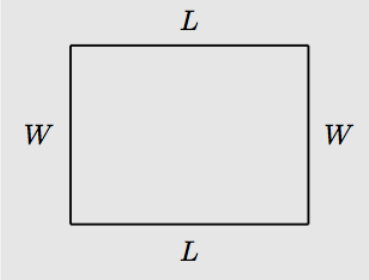
Figure 1.17: A rectangle with length 4 inches and width 3 inches.

To find the area of the figure, we can count the individual units of area that make up the area of the rectangle, twelve square inches (12 in^2) in all. However, as we did in counting the stars in the array in Figure 1.15, it is much faster to note that we have three rows of four square inches. Hence, it is much faster to multiply the number of squares in each row by the number of squares in each column: 4·3 = 12 square inches.

The argument presented above leads to the following rule for finding the area of a rectangle.

Area of a Rectangle

Let L and W represent the length and width of a rectangle, respectively.



To find the area of the rectangle, calculate the product of the length and width. That is, if A represents the area of the rectangle, then the area of the rectangle is given by the formula

$$A = LW.$$

Example 4

A rectangle has width 5 feet and length 12 feet. Find the area of the rectangle.

Solution

Substitute L = 12 ft and W = 5 ft into the area formula.

$$A = LW$$
$$= (12 \text{ ft})(5 \text{ ft})$$
$$= 60 \text{ ft}^2$$

Hence, the area of the rectangle is 60 square feet.

Exercise

A rectangle has width 17 inches and length 33 inches. Find the area of the rectangle.

Answer

561 square inches

Exercises

In Exercises 1-4 use number line diagrams as shown in Figure 1.6 to depict the multiplication.

1. $2 \cdot 4$.
2. $3 \cdot 4$.
3. $4 \cdot 2$.
4. $4 \cdot 3$.

In Exercises 5-16, state the property of multiplication depicted by the given identity.

5. $9 \cdot 8 = 8 \cdot 9$
6. $5 \cdot 8 = 8 \cdot 5$
7. $8 \cdot (5 \cdot 6) = (8 \cdot 5) \cdot 6$
8. $4 \cdot (6 \cdot 5) = (4 \cdot 6) \cdot 5$
9. $6 \cdot 2 = 2 \cdot 6$
10. $8 \cdot 7 = 7 \cdot 8$
11. $3 \cdot (5 \cdot 9) = (3 \cdot 5) \cdot 9$
12. $8 \cdot (6 \cdot 4) = (8 \cdot 6) \cdot 4$
13. $21 \cdot 1 = 21$
14. $39 \cdot 1 = 39$
15. $13 \cdot 1 = 13$
16. $44 \cdot 1 = 44$

In Exercises 17-28, multiply the given numbers.

17. $78 \cdot 3$
18. $58 \cdot 7$
19. $907 \cdot 6$
20. $434 \cdot 80$
21. $128 \cdot 30$
22. $454 \cdot 90$
23. $799 \cdot 60$
24. $907 \cdot 20$
25. $14 \cdot 70$
26. $94 \cdot 90$
27. $34 \cdot 90$
28. $87 \cdot 20$

In Exercises 29-40, multiply the given numbers.

29. $237 \cdot 54$
30. $893 \cdot 94$
31. $691 \cdot 12$
32. $823 \cdot 77$

33. 955 · 89

34. 714 · 41

35. 266 · 61

36. 366 · 31

37. 365 · 73

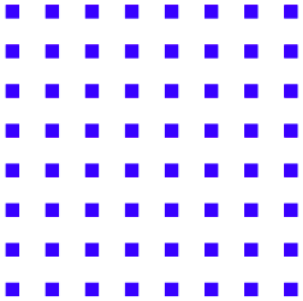
38. 291 · 47

39. 955 · 57

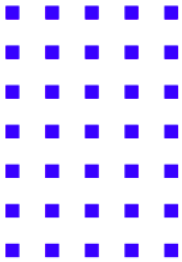
40. 199 · 33

---

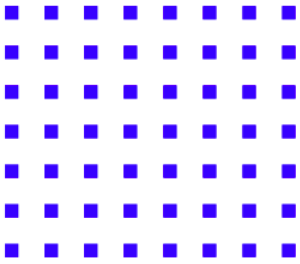
41. Count the number of objects in the array.



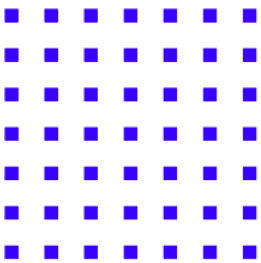
42. Count the number of objects in the array.



43. Count the number of objects in the array.



44. Count the number of objects in the array.



In Exercises 45-48, find the area of the rectangle having the given length and width.

45. L = 50 in, W = 25 in

46. L = 48 in, W = 24 in

47. L = 47 in, W = 13 in

48. L = 19 in, W = 10 in

In Exercises 49-52, find the perimeter of the rectangle having the given length and width.

49. L = 25 in, W = 16 in

50. L = 34 in, W = 18 in

51. L = 30 in, W = 28 in

52. L = 41 in, W = 25 in

---

53. A set of beads costs 50 cents per dozen. What is the cost (in dollars) of 19 dozen sets of beads?

54. A set of beads costs 60 cents per dozen. What is the cost (in dollars) of 7 dozen sets of beads?

55. If a math tutor worked for 47 hours and was paid $15 each hour, how much money would she have made?

56. If a math tutor worked for 46 hours and was paid $11 each hour, how much money would he have made?

57. There are 12 eggs in one dozen, and 12 dozen in one gross. How many eggs are in a shipment of 24 gross?

58. There are 12 eggs in one dozen, and 12 dozen in one gross. How many eggs are in a shipment of 11 gross?

59. If bricks weigh 4 kilograms each, what is the weight (in kilograms) of 5000 bricks?

60. If bricks weigh 4 pounds each, what is the weight (in pounds) of 2000 bricks?

In Exercises 61-68, which of the following four expressions differs from the remaining three?

61. $\frac{30}{5}$, $30 \div 5$, $5\longdiv{30}$, $5 \div 30$

62. $\frac{12}{2}$, $12 \div 2$, $2\longdiv{12}$, $2 \div 12$

63. $\frac{8}{2}$, $8 \div 2$, $2\longdiv{8}$, $8\longdiv{2}$

64. $\frac{8}{4}$, $8 \div 4$, $4\longdiv{8}$, $8\longdiv{4}$

65. $2\longdiv{14}$, $14\longdiv{2}$, $\frac{14}{2}$, $14 \div 2$

66. $9\longdiv{54}$, $54\longdiv{9}$, $\frac{54}{9}$, $54 \div 9$

67. $3\longdiv{24}$, $3 \div 24$, $\frac{24}{3}$, $24 \div 3$

68. $3\longdiv{15}$, $3 \div 15$, $\frac{15}{3}$, $15 \div 3$

In Exercises 69-82, simplify the given expression. If the answer doesn't exist or is undefined, write "undefined".

69. $0 \div 11$

70. $0 \div 5$

71. $17 \div 0$

72. $24 \div 0$

73. $10 \cdot 0$

74. $20 \cdot 0$

75. $\frac{7}{0}$

76. $\frac{23}{0}$

77. $16\longdiv{0}$

78. $25\longdiv{0}$

79. $\frac{0}{24}$

80. $\frac{0}{22}$

81. $0\longdiv{0}$

82. $0 \div 0$

In Exercises 83-94, divide the given numbers.

83. $\frac{2816}{44}$

84. $\frac{1998}{37}$

85. $\frac{2241}{83}$

86. $\frac{2716}{97}$

87. $\frac{3212}{73}$

88. $\frac{1326}{17}$

89. $\frac{8722}{98}$

90. $\frac{1547}{91}$

91. $\frac{1440}{96}$

92. $\frac{2079}{27}$

93. $\frac{8075}{85}$

94. $\frac{1587}{23}$

In Exercises 95-106, divide the given numbers.

95. $\frac{17756}{92}$

96. $\frac{46904}{82}$

97. $\frac{11951}{19}$

98. $\frac{22304}{41}$

99. $\frac{18048}{32}$

100. $\frac{59986}{89}$

101. $\frac{29047}{31}$

102. $\frac{33264}{86}$

103. $\frac{22578}{53}$

104. $\frac{18952}{46}$

105. $\frac{12894}{14}$

106. $\frac{18830}{35}$

---

107. A concrete sidewalk is laid in square blocks that measure 6 feet on each side. How many blocks will there be in a walk that is 132 feet long?

108. A concrete sidewalk is laid in square blocks that measure 5 feet on each side. How many blocks will there be in a walk that is 180 feet long?

109. One boat to the island can take 5 people. How many trips will the boat have to take in order to ferry 38 people to the island? (Hint: Round up your answer.)

110. One boat to the island can take 4 people. How many trips will the boat have to take in order to ferry 46 people to the island? (Hint: Round up your answer.)

111. If street lights are placed at most 145 feet apart, how many street lights will be needed for a street that is 4 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

112. If street lights are placed at most 70 feet apart, how many street lights will be needed for a street that is 3 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

113. A concrete sidewalk is laid in square blocks that measure 4 feet on each side. How many blocks will there be in a walk that is 292 feet long?

114. A concrete sidewalk is laid in square blocks that measure 5 feet on each side. How many blocks will there be in a walk that is 445 feet long?

115. One boat to the island can take 3 people. How many trips will the boat have to take in order to ferry 32 people to the island? (Hint: Round up your answer.)

116. One boat to the island can take 4 people. How many trips will the boat have to take in order to ferry 37 people to the island? (Hint: Round up your answer.)

117. If street lights are placed at most 105 feet apart, how many street lights will be needed for a street that is 2 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

118. If street lights are placed at most 105 feet apart, how many street lights will be needed for a street that is 3 miles long, assuming that there are lights at each end of the street? (Note: 1 mile = 5280 feet.)

---

119. **Writing articles**. Eli writes an average of 4 articles a day, five days a week, to support product sales. How many articles does Eli write in one week?

120. **Machine gun**. A 0.50-caliber antiaircraft machine gun can fire 800 rounds each minute. How many rounds could fire in three minutes? *Associated Press Times-Standard 4/15/09*

121. **Laps**. The swimming pool at CalCourts is 25 yards long. If one lap is up and back again, how many yards has Wendell swam doing 27 laps?

122. **Refrigerator wattage**. A conventional refrigerator will run about 12 hours each day can use 150 Watts of power each hour. How many Watts of power will a refrigerator use over the day?

123. **Horse hay**. A full-grown horse should eat a minimum of 12 pounds of hay each day and may eat much more depending on their weight. How many pounds minimum would a horse eat over a year?

124. **College costs**. After a $662 hike in fees, California residents who want to attend the University of California as an undergraduate should expect to pay $8,700 in for the upcoming academic year 2009- 2010. If the cost were to remain the same for the next several years, how much should a student expect to pay for a four-year degree program at a UC school?

125. **Non-resident costs**. Nonresident undergraduates who want to attend a University of California college should expect to pay about $22,000 for the upcoming academic year. Assuming costs remain the same, what can a four-year degree cost?

126. **Student tax**. The mayer of Providence, Rhode Island wants to tax its 25,000 Brown University students $150 each to contribute to tax receipts saying students should pay for the resources they use just like the town residents. How many dollars would the mayer generate?

127. **New iceberg**. A new iceberg, shaved off a glacier after a collision with another iceberg, measures about 48 miles long and 28 miles wide. What's the approximate area of the new iceberg? Associated PressTimes-Standard 02/27/10 2 Huge icebergs set loose off Antarctica's coast.

128. **Solar panels**. One of the solar panels on the International Space Station is 34 meters long and 11 meters wide. If there are eight of these, what's the total area for solar collection?

129. **Sidewalk**. A concrete sidewalk is to be 80 foot long and 4 foot wide. How much will it cost to lay the sidewalk at $8 per square foot?

130. **Hay bales**. An average bale of hay weighs about 60 pounds. If a horse eats 12 pounds of hay a day, how many days will one bale feed a horse?

131. **Sunspots**. Sunspots, where the sun's magnetic field is much higher, usually occur in pairs. If the total count of sunspots is 72, how many pairs of sunspots are there?

Answers

1. $2 \cdot 4 = \underbrace{4+4}_{2 \text{ times}} = 8$

3. $4 \cdot 2 = \underbrace{2+2+2+2}_{4 \text{ times}} = 8$

5. Commutative property of multiplication

7. Associative property of multiplication

9. Commutative property of multiplication

11. Associative property of multiplication

13. Multiplicative identity property

15. Multiplicative identity property

17. 234

19. 5442

21. 3840

23. 47940

25. 980

27. 3060

29. 12798

31. 8292

33. 84995

35. 16226

37. 26645

39. 54435

41. 64

43. 56

45. 1250 in^2

47. 611 in^2

49. 82 in

51. 116 in

53. 9.50

55. 705

57. 3456

59. 20000

61. $5 \div 30$

63. $8\sqrt{2}$

65. $14\sqrt{2}$

67. $3 \div 24$

69. 0

71. Undefined

73. 0

75. Undefined

77. 0

79. 0

81. Undefined

83. 64

85. 27

87. 44

89. 89

91. 15

93. 95

95. 193

97. 629

99. 564

101. 937

103. 426

105. 921

107. 22

109. 8

111. 147

113. 73

115. 11

117. 102

119. 20 articles

121. 1350 yards

123. 4380 pounds of hay

125. $88,000

127. 1344 mi^2

129. $2,560

131. 36

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1.4: Prime Factorization

In the statement $3 \cdot 4 = 12$, the number 12 is called the *product*, while 3 and 4 are called *factors*.

Example 1

Find all whole number factors of 18.

Solution

We need to find all whole number pairs whose product equals 18. The following pairs come to mind.

$$1 \cdot 18 = 18 \text{ and } 2 \cdot 9 = 18 \text{ and } 3 \cdot 6 = 18.$$

Hence, the factors of 18 are (in order) 1, 2, 3, 6, 9, and 18.

Exercise

Find all whole number factors of 21.

Answer

1, 3, 7, and 21

Divisibility

In Example 1, we saw $3 \cdot 6 = 18$, making 3 and 6 factors of 18. Because division is the inverse of multiplication, that is, division by a number undoes the multiplication of that number, this immediately provides

$$18 \div 6 = 3 \text{ and } 18 \div 3 = 6.$$

That is, 18 is divisible by 3 and 18 is divisible by 6. When we say that 18 is divisible by 3, we mean that when 18 is divided by 3, there is a zero remainder.

Divisible

Let a and b be whole numbers. Then a is divisible by b if and only if the remainder is zero when a is divided by b. In this case, we say that "b is a divisor of a."

Example 2

Find all whole number divisors of 18.

Solution

In Example 1, we saw that $3 \cdot 6 = 18$. Therefore, 18 is divisible by both 3 and 6 ($18 \div 3 = 6$ and $18 \div 6 = 3$). Hence, when 18 is divided by 3 or 6, the remainder is zero. Therefore, 3 and 6 are divisors of 18. Noting the other products in Example 1, the complete list of divisors of 18 is 1, 2, 3, 6, 9, and 18.

Exercise

Find all whole number divisors of 21.

Answer

1, 3, 7, and 21.

Example 1 and Example 2 show that when working with whole numbers, the words *factor* and *divisor* are interchangeable.

Factors and Divisors

If $c = a \cdot b$, then a and b are called *factors* of c. Both a and b are also called *divisors* of c.

Divisibility Tests

There are a number of very useful divisibility tests.

Divisible by 2. If a whole number ends in 0, 2, 4, 6, or 8, then the number is called an even number and is divisible by 2. Examples of even numbers are 238 and 1,246 (238 ÷ 2 = 119 and 1, 246 ÷ 2 = 623). A number that is not even is called an odd number. Examples of odd numbers are 113 and 2,339.

Divisible by 3. If the sum of the digits of a whole number is divisible by 3, then the number itself is divisible by 3. An example is 141. The sum of the digits is 1 + 4 + 1 = 6, which is divisible by 3. Therefore, 141 is also divisible by 3 (141 ÷ 3 = 47).

Divisible by 4. If the number represented by the last two digits of a whole number is divisible by 4, then the number itself is divisible by 4. An example is 11,524. The last two digits represent 24, which is divisible by 4 (24 ÷ 4 = 6). Therefore, 11,524 is divisible by 4 (11, 524 ÷ 4=2, 881).

Divisible by 5. If a whole number ends in a zero or a 5, then the number is divisible by 5. Examples are 715 and 120 (715÷5 = 143 and 120÷5 = 24).

Divisible by 6. If a whole number is divisible by 2 and by 3, then it is divisible by 6. An example is 738. First, 738 is even and divisible by 2. Second, 7+3+8=18, which is divisible by 3. Hence, 738 is divisible by 3. Because 738 is divisible by both 2 and 3, it is divisible by 6 (738 ÷ 6 = 123).

Divisible by 8. If the number represented by the last three digits of a whole number is divisible by 8, then the number itself is divisible by 8. An example is 73,024. The last three digits represent the number 24, which is divisible by 8 (24÷8 = 3). Thus, 73,024 is also divisible by 8 (73, 024÷8 = 9, 128).

Divisible by 9. If the sum of the digits of a whole number is divisible by 9, then the number itself is divisible by 9. An example is 117. The sum of the digits is 1 + 1 + 7 = 9, which is divisible by 9. Hence, 117 is divisible by 9 (117 ÷ 9 = 13).

Prime Numbers

We begin with the definition of a prime number.

Prime Number

A whole number (other than 1) is a prime number if its only factors (divisors) are 1 and itself. Equivalently, a number is prime if and only if it has exactly two factors (divisors).

Example 3

Which of the whole numbers 12, 13, 21, and 37 are prime numbers?

Solution

- The factors (divisors) of 12 are 1, 2, 3, 4, 6, and 12. Hence, 12 is not a prime number.
- The factors (divisors) of 13 are 1 and 13. Because its only divisors are 1 and itself, 13 is a prime number.
- The factors (divisors) of 21 are 1, 3, 7, and 21. Hence, 21 is not a prime number.
- The factors (divisors) of 37 are 1 and 37. Because its only divisors are 1 and itself, 37 is a prime number.

Exercise

Which of the whole numbers 15, 23, 51, and 59 are prime numbers?

Answer

23 and 59

Example 4

List all the prime numbers less than 20.

Solution

The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19.

You try it!

List all the prime numbers less than 100.

Composite Numbers

If a whole number is not a prime number, then it is called a *composite number*.

Example 5

Is the whole number 1,179 prime or composite?

Solution

Note that 1 + 1 + 7 + 9 = 18, which is divisible by both 3 and 9. Hence, 3 and 9 are both divisors of 1,179. Therefore, 1,179 is a composite number.

Exercise

Is the whole number 2,571 prime or composite?

Answer

Composite

Factor Trees

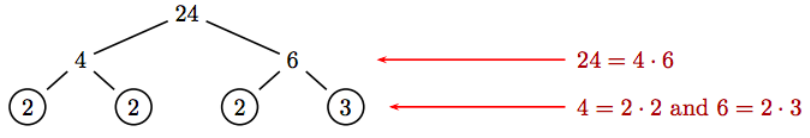
We will now learn how to express a composite number as a unique product of prime numbers. The most popular device for accomplishing this goal is the *factor tree*.

Example 6

Express 24 as a product of prime factors.

Solution

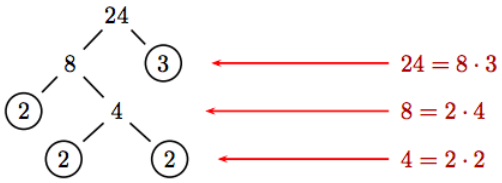
We use a factor tree to break 24 down into a product of primes.



At each level of the tree, break the current number into a product of two factors. The process is complete when all of the "circled leaves" at the bottom of the tree are prime numbers. Arranging the factors in the "circled leaves" in order,

$$24 = 2 \cdot 2 \cdot 2 \cdot 3.$$

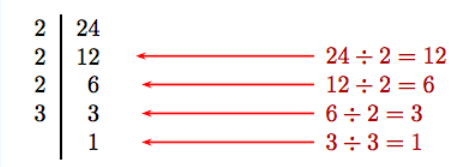
The final answer does not depend on product choices made at each level of the tree. Here is another approach.



The final answer is found by including all of the factors from the "circled leaves" at the end of each branch of the tree, which yields the same result, namely 24 = 2 · 2 · 2 · 3.

Alternate Approach

Some favor repeatedly dividing by 2 until the result is no longer divisible by 2. Then try repeatedly dividing by the next prime until the result is no longer divisible by that prime. The process terminates when the last resulting quotient is equal to the number 1.



The first column reveals the prime factorization; i.e., 24 = 2 · 2 · 2 · 3.

Exercise

Express 36 as a product of prime factors.

Answer

2 · 2 · 3 · 3.

The fact that the alternate approach in Example 6 yielded the same result is significant.

Unique Factorization Theorem

Every whole number can be **uniquely** factored as a product of primes.

This result guarantees that if the prime factors are ordered from smallest to largest, everyone will get the same result when breaking a number into a product of prime factors.

Exponents

We begin with the definition of an exponential expression.

Exponents

The expression a^m is defined to mean

$$a^m = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}$$

The number a is called the base of the exponential expression and the number m is called the exponent. The exponent m tells us to repeat the base a as a factor m times.

Example 7

Evaluate 2^5, 2^3, and 5^2.

Solution

- In the case of 2^5, we have

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$
$$= 32.$$

- In the case of 3^3, we have

$$3^3 = 3 \cdot 3 \cdot 3$$
$$= 27.$$

- In the case of 5^2, we have

$$5^2 = 5 \cdot 5$$
$$= 25.$$

Exercise

Evaluate: 3^5.

Answer

243.

Example 8

Express the solution to Example 6 in compact form using exponents.

Solution

In Example 6, we determined the prime factorization of 24.

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$

Because $2 \cdot 2 \cdot 2 = 23$, we can write this more compactly.

$$24 = 2^3 \cdot 3$$

Exercise

Prime factor 54.

Answer

$2 \cdot 3 \cdot 3 \cdot 3$

Example 9

Evaluate the expression $2^3 \cdot 3^2 \cdot 5^2$.

Solution

First raise each factor to the given exponent, then perform the multiplication in order (left to right).

$$2^3 \cdot 3^2 \cdot 5^2 = 8 \cdot 9 \cdot 25$$
$$= 72 \cdot 25$$
$$= 1800$$

Exercise

Evaluate: $3^3 \cdot 5^2$.

Answer

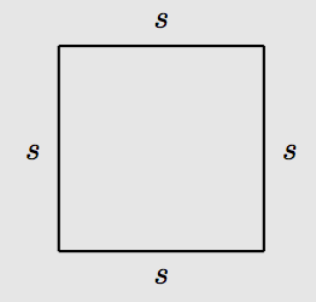
675

Application

A square is a rectangle with four equal sides.

Area of a Square

Let s represent the length of each side of a square.



Because a square is also a rectangle, we can find the area of the square by multiplying its length and width. However, in this case, the length and width both equal s, so $A = (s)(s) = s^2$. Hence, the formula for the area of a square is

$$A = s^2.$$

Example 10

The edge of a square is 13 centimeters. Find the area of a square.

Solution

Substitute $s = 13$ cm into the area formula.

$$A = s^2$$
$$= (13 \text{ cm})^2$$
$$= (13 \text{ cm})(13 \text{ cm})$$
$$= 169 \text{ cm}^2$$

Hence, the area of the square is 169 cm^2; i.e., 169 square centimeters.

Exercise

The edge of a square is 15 meters. Find the area of a square.

Answer

225 square meters

Exercises

In Exercises 1-12, find all divisors of the given number.

1. 30

2. 19

3. 83

4. 51

5. 91

6. 49

7. 75

8. 67

9. 64

10. 87

11. 14

12. 89

In Exercises 13-20, which of the following numbers is not divisible by 2?

13. 117, 120, 342, 230

14. 310, 157, 462, 160

15. 30, 22, 16, 13

16. 382, 570, 193, 196

17. 105, 206, 108, 306

18. 60, 26, 23, 42

19. 84, 34, 31, 58

20. 66, 122, 180, 63

In Exercises 21-28, which of the following numbers is not divisible by 3?

21. 561, 364, 846, 564

22. 711, 850, 633, 717

23. 186, 804, 315, 550

24. 783, 909, 504, 895

25. 789, 820, 414, 663

26. 325, 501, 945, 381

27. 600, 150, 330, 493

28. 396, 181, 351, 606

In Exercises 29-36, which of the following numbers is not divisible by 4?

29. 3797, 7648, 9944, 4048

30. 1012, 9928, 7177, 1592

31. 9336, 9701, 4184, 2460

32. 2716, 1685, 2260, 9788

33. 9816, 7517, 8332, 7408

34. 1788, 8157, 7368, 4900

35. 1916, 1244, 7312, 7033

36. 7740, 5844, 2545, 9368

In Exercises 37-44, which of the following numbers is not divisible by 5?

37. 8920, 4120, 5285, 9896
38. 3525, 7040, 2185, 2442
39. 8758, 3005, 8915, 3695
40. 3340, 1540, 2485, 2543
41. 2363, 5235, 4145, 4240
42. 9030, 8000, 5445, 1238
43. 1269, 5550, 4065, 5165
44. 7871, 9595, 3745, 4480

In Exercises 45-52, which of the following numbers is not divisible by 6?

45. 328, 372, 990, 528
46. 720, 288, 148, 966
47. 744, 174, 924, 538
48. 858, 964, 930, 330
49. 586, 234, 636, 474
50. 618, 372, 262, 558
51. 702, 168, 678, 658
52. 780, 336, 742, 312

In Exercises 53-60, which of the following numbers is not divisible by 8?

53. 1792, 8216, 2640, 5418
54. 2168, 2826, 1104, 2816
55. 8506, 3208, 9016, 2208
56. 2626, 5016, 1392, 1736
57. 4712, 3192, 2594, 7640
58. 9050, 9808, 8408, 7280
59. 9808, 1232, 7850, 7912
60. 3312, 1736, 9338, 3912

In Exercises 61-68, which of the following numbers is not divisible by 9?

61. 477, 297, 216, 991
62. 153, 981, 909, 919
63. 153, 234, 937, 675
64. 343, 756, 927, 891
65. 216, 783, 594, 928
66. 504, 279, 307, 432
67. 423, 801, 676, 936
68. 396, 684, 567, 388

In Exercises 69-80, identify the given number as prime, composite, or neither.

69. 19

70. 95

71. 41

72. 88

73. 27

74. 61

75. 91

76. 72

77. 21

78. 65

79. 23

80. 36

In Exercises 81-98, find the prime factorization of the natural number.

81. 224

82. 320

83. 108

84. 96

85. 243

86. 324

87. 160

88. 252

89. 32

90. 128

91. 360

92. 72

93. 144

94. 64

95. 48

96. 200

97. 216

98. 392

In Exercises 99-110, compute the exact value of the given exponential expression.

99. $5^2 \cdot 4^1$

100. $2^3 \cdot 4^1$

101. 0^1

102. 1^3

103. $3^3 \cdot 0^2$

104. $3^3 \cdot 2^2$

105. 4^1

106. 5^2

107. 4^3

108. 4^2

109. $3^3 \cdot 1^2$

110. $5^2 \cdot 2^3$

In Exercises 111-114, find the area of the square with the given side.

111. 28 inches

112. 31 inches

113. 22 inches

114. 13 inches

Create factor trees for each number in Exercises 115-122. Write the prime factorization for each number in compact form, using exponents.

115. 12

116. 18

117. 105

118. 70

119. 56

120. 56

121. 72

122. 270

---

123. **Sieve of Eratosthenes**. This exercise introduces the *Sieve of Eratosthenes*, an ancient algorithm for finding the primes less than a certain number n, first created by the Greek mathematician Eratosthenes. Consider the grid of integers from 2 through 100.

2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21
22	23	24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39	40	41
42	43	44	45	46	47	48	49	50	51
52	53	54	55	56	57	58	59	60	61
62	63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90	91
92	93	94	95	96	97	98	99	100	

To find the primes less than 100, proceed as follows.

i) Strike out all multiples of 2 (4, 6, 8, etc.)

ii) The list's next number that has not been struck out is a prime number.

iii) Strike out from the list all multiples of the number you identified in step (ii).

iv) Repeat steps (ii) and (iii) until you can no longer strike any more multiples.

v) All unstruck numbers in the list are primes.

Answers

1. 1, 2, 3, 5, 6, 10, 15, 30

3. 1, 83

5. 1, 7, 13, 91

7. 1, 3, 5, 15, 25, 75

9. 1, 2, 4, 8, 16, 32, 64

11. 1, 2, 7, 14

13. 117

15. 13

17. 105

19. 31

21. 364

23. 550

25. 820

27. 493

29. 3797

31. 9701

33. 7517

35. 7033

37. 9896

39. 8758

41. 2363

43. 1269

45. 328

47. 538

49. 586

51. 658

53. 5418

55. 8506

57. 2594

59. 7850

61. 991

63. 937

65. 928

67. 676

69. prime

71. prime

73. composite

75. composite

77. composite

79. prime

81. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7$

83. $2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

85. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

87. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

89. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

91. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$

93. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

95. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

97. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

99. 100

101. 0

103. 0

105. 4

107. 64

109. 27

111. 784 in^2

113. 484 in^2

115. $12 = 22 \cdot 3$

117. $105 = 3 \cdot 5 \cdot 7$

119. $56 = 23 \cdot 7$

121. $72 = 23 \cdot 32$

123. Unstruck numbers are primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

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1.5: Order of Operations

The order in which we evaluate expressions can be ambiguous. Take for example, the expression $4 + 3 \cdot 2$. If we do the addition first, then

$$4+3 \cdot 2=7 \cdot 2$$
$$= 14.$$

On the other hand, if we do the multiplication first, then

$$4+3 \cdot 2=4+6$$
$$= 10.$$

So, what are we to do? Of course, grouping symbols can remove the ambiguity

Grouping Symbols

Parentheses, brackets, or curly braces can be used to group parts of an expression. Each of the following are equivalent:

$$(4 + 3) \cdot 2 \text{ or } [4 + 3] \cdot 2 \text{ or } \{4+3\} \cdot 2$$

In each case, the rule is "evaluate the expression inside the grouping symbols first." If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

Thus, for example,

$$(4 + 3) \cdot 2=7 \cdot 2$$
$$= 14.$$

Note how the expression contained in the parentheses was evaluated first. Another way to avoid ambiguities in evaluating expressions is to establish an order in which operations should be performed. The following guidelines should always be strictly enforced when evaluating expressions.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in

Example 1

Evaluate $4 + 3 \cdot 2$.

Solution

Because of the established *Rules Guiding Order of Operations*, this expression is no longer ambiguous. There are no grouping symbols or exponents, so we immediately go to rule three, evaluate all multiplications and divisions in the order that they appear, moving left to right. After that we invoke rule four, performing all additions and subtractions in the order that they appear, moving left to right.

$$4+3\dot{2}=4+6$$
$$=10$$

Thus, $4 + 3 \cdot 2 = 10$.

Exercise

Simplify: 8 + 2 · 5.

Answer

18

Example 2

Evaluate 18 − 2 + 3.

Solution

Follow the *Rules Guiding Order of Operations*. Addition has no precedence over subtraction, nor does subtraction have precedence over addition. We are to perform additions and subtractions as they occur, moving left to right.

$$\begin{aligned} 18-2+3 &= 16+3 \quad \text{Subtract: } 18-2=16. \\ &= 19 \quad \text{Add: } 16+3=19. \end{aligned}$$

Thus, 18 − 2 + 3 = 19.

Exercise

Simplify: 17 − 8 + 2.

Answer

11

Example 3

Evaluate 54 ÷ 9 · 2.

Solution

Follow the *Rules Guiding Order of Operations*. Division has no precedence over multiplication, nor does multiplication have precedence over division. We are to perform divisions and multiplications as they occur, moving left to right.

$$\begin{aligned} 54 \div 9 \cdot 2 &= 6\dot{2} \quad \text{Divide: } 54 \div 9 = 6. \\ &= 12 \quad \text{Multiply: } 6 \cdot 2 = 12. \end{aligned}$$

Thus, 54 ÷ 9 · 2 = 12.

Exercise

Simplify: 72 ÷ 9 · 2.

Answer

16

Example 4

Evaluate $2 \cdot 3^2 - 12$.

Solution

Follow the *Rules Guiding Order of Operations*, exponents first, then multiplication, then subtraction.

$$\begin{aligned} 2 \cdot 3^2 - 12 &= 2\dot{9} - 12 && \text{Evaluate the exponent: } 3\hat{}2 = 9. \\ &= 18 - 12 && \text{Perform the multiplication: } 2 \cdot 9 = 18. \\ &= 6 && \text{Perform the subtraction: } 18 - 12 = 6. \end{aligned}$$

Thus, $2 \cdot 3^2 - 12 = 6$.

? Exercise

Simplify: $14 + 3 \cdot 4^2$

Answer

62

✓ Example 5

Evaluate $12 + 2(3 + 2 \cdot 5)^2$.

Solution

Follow the Rules Guiding Order of Operations, evaluate the expression inside the parentheses first, then exponents, then multiplication, then addition.

$$\begin{aligned} 12+2(3+5\cdot 5)^2 &= 12+2(3+10)^2 && \text{Multiply inside parentheses: } 2\ \cdot 5 = 10. \\ &= 12+2(13)^2 && \text{Add inside parentheses: } 3+10 = 13. \\ &= 12+2(169) && \text{Exponents are next: } (13)^2 = 169. \\ &= 12+338 && \text{Multiplication is next: } 2(169) = 338. \\ &= 350 && \text{Time to add: } 12+338 = 350. \end{aligned}$$

Thus, $12 + 2(3 + 2 \cdot 5)\ 2 = 350$.

? Exercise

Simplify: $3(2 + 3 \cdot 4)^2 - 11$.

Answer

577

✓ Example 6

Evaluate $2\{2 + 2[2 + 2]\}$.

Solution

When grouping symbols are nested, evaluate the expression between the pair of innermost grouping symbols first.

$$\begin{aligned} 2(2+2[2+2]) &= 2(2+2[4]) && \text{Innermost grouping first: } 2+2 = 4. \\ &= 2(2+8) && \text{Multiply next: } 2[4] = 8. \\ &= 2(10) && \text{Add inside braces: } 2+8 = 10. \\ &= 20 && \text{Multiply: } 2(10) = 20 \end{aligned}$$

Thus, $2(2 + 2[2 + 2]) = 20$.

? Exercise

Simplify: $2\{3 + 2[3 + 2]\}$.

Answer

26

Fraction Bars

Consider the expression

$$\frac{6^2+8^2}{(2+3)^2}$$

Because a fraction bar means division, the above expression is equivalent to

$$\left(6^2+8^2\right) \div (2+3)^2$$

The position of the grouping symbols signals how we should proceed. We should simplify the numerator, then the denominator, then divide.

Fractional Expressions

If a fractional expression is present, evaluate the numerator and denominator first, then divide.

Example 7

Evaluate the expression

$$\frac{6^2+8^2}{(2+3)^2}.$$

Solution

Simplify the numerator and denominator first, then divide.

$$\begin{aligned}
\frac{6^2+8^2}{(2+3)^2} &= \frac{6^2+8^2}{(5)^2} && \text{Parentheses in denominator first: } 2+3=5 \\
&= \frac{36+64}{25} && \text{Exponents are next: } 6^2=36,\ 8^2=64,\ 5^2=25. \\
&= \frac{100}{25} && \text{Add in numerator: } 36+64=100 \\
&= 4 && \text{Divide: } 100 \div 25 = 4.
\end{aligned}$$

Thus, $\frac{6^2+8^2}{(2+3)^2}=4$.

Exercise

Simplify: $\frac{12+3\cdot 2}{6}$ **Answer**

3

The Distributive Property

Consider the expression 2 · (3 + 4). If we follow the “Rules Guiding Order of Operations,” we would evaluate the expression inside the parentheses first. 2 · (3 + 4) = 2 · 7 Parentheses first: 3 + 4 = 7. = 14 Multiply: 2 · 7 = 14.

However, we could also choose to “distribute” the 2, first multiplying 2 times each addend in the parentheses.

$$\begin{aligned}
2\cdot(3+4) &= 2\cdot 3+2\cdot 4 && \text{Multiply 2 times both 3 and 4.} \\
&= 6+8 && \text{Multiply: } 2\cdot 3=6 \text{ and } 2\cdot 4=8. \\
&= 14 && \text{Add: } 6+8=14.
\end{aligned}$$

The fact that we get the same answer in the second approach is an illustration of an important property of whole numbers.[1]

The Distributive Property

Let a, b, and c be any whole numbers. Then,

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

We say that "multiplication is distributive with respect to addition."

Multiplication is distributive with respect to addition. If you are not computing the product of a number and a sum of numbers, the distributive property does not apply.

Caution! Wrong Answer Ahead!

If you are calculating the product of a number and the product of two numbers, the distributive property must not be used. For example, here is a common misapplication of the distributive property.

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= (2 \cdot 3) \cdot (2 \cdot 4) \\ &= 6 \cdot 8 \\ &= 48 \end{aligned}$$

This result is quite distant from the correct answer, which is found by computing the product within the parentheses first.

$$\begin{aligned} 2 \cdot (3 \cdot 4) &= 2 \cdot 12 \\ &= 24. \end{aligned}$$

In order to apply the distributive property, you must be multiplying times a sum.

Example 8

Use the distributive property to calculate $4 \cdot (5 + 11)$.

Solution

This is the product of a number and a sum, so the distributive property may be applied.

$$\begin{aligned} 4 \cdot (5+11) &= 4 \cdot 5 + 4 \cdot 11 \quad \text{Distribute the 4 times addend in the sum.} \\ &= 20 + 44 \quad \text{Multiply: } 4 \cdot 5 = 20 \text{ and } 4 \cdot 11 = 44. \\ &= 64 \quad \text{Add: } 20 + 44 = 64. \end{aligned}$$

Readers should check that the same answer is found by computing the sum within the parentheses first.

Exercise

Distribute: $5 \cdot (11 + 8)$.

Answer

95

The distributive property is the underpinning of the multiplication algorithm learned in our childhood years.

Example 9

Multiply: $6 \cdot 43$.

Solution

We'll express 43 as sum, then use the distributive property.

$$\begin{aligned} 6 \cdot 43 &= 6 \cdot (40+3) && \text{Express 43 as a sum: } 43 = 40+3 \\ &= 6 \cdot 40 + 6 \cdot 3 && \text{Distribute the 6.} \\ &= 240 + 18 && \text{Multiply: } 6 \cdot 40 = 240 \text{ and } 6 \cdot 3 = 18. \\ &= 258 && \text{Add: } 240 + 18 = 258. \end{aligned}$$

Readers should be able to see this application of the distributive property in the more familiar algorithmic form:

$$\begin{array}{r} 43 \\ \times 6 \\ \hline 18 \\ 240 \\ \hline 258 \end{array}$$

Or in the even more condensed form with "carrying:"

$$\begin{array}{r} {}^{1}43 \\ \times 6 \\ \hline 258 \end{array}$$

Exercise

Use the distributive property to evaluate $8 \cdot 92$.

Answer

736

Multiplication is also distributive with respect to subtraction.

The Distributive Property (Subtraction)

Let a, b, and c be any whole numbers. Then,

$$a \cdot (b - c) = a \cdot b - a \cdot c.$$

We say the multiplication is "distributive with respect to subtraction."

Example 10

Use the distributive property to simplify: $3 \cdot (12 - 8)$.

Solution

This is the product of a number and a difference, so the distributive property may be applied.

$$\begin{aligned} 3 \cdot (12-8) &= 3 \cdot 12 - 3 \cdot 8 && \text{Distribute the 3 times each term in the difference.} \\ &= 36 - 24 && \text{Multiply: } 3 \cdot 12 = 36 \text{ and } 3 \cdot 8 = 24. \\ &= 12 && \text{Subtract: } 36 - 24 = 12. \end{aligned}$$

Alternate solution

Note what happens if we use the usual "order of operations" to evaluate the expression.

$$\begin{aligned} 3 \cdot (12-8) &= 3 \cdot 4 && \text{Parentheses first: } 12 - 8 = 4. \\ &= 12 && \text{Multiply: } 3 \cdot 4 = 12. \end{aligned}$$

Same answer.

Exercise

Distribute: $8 \cdot (9 - 2)$.

Answer

56

Exercises

In Exercises 1-12, simplify the given expression.

1. $5+2 \cdot 2$
2. $5+2 \cdot 8$
3. $23-7 \cdot 2$
4. $37-3 \cdot 7$
5. $4 \cdot 3+2 \cdot 5$
6. $2 \cdot 5+9 \cdot 7$
7. $6 \cdot 5+4 \cdot 3$
8. $5 \cdot 2+9 \cdot 8$
9. $9+2 \cdot 3$
10. $3+6 \cdot 6$
11. $32-8 \cdot 2$
12. $24-2 \cdot 5$

In Exercises 13-28, simplify the given expression.

13. $45 \div 3 \cdot 5$
14. $20 \div 1 \cdot 4$
15. $2 \cdot 9 \div 3 \cdot 18$
16. $19 \cdot 20 \div 4 \cdot 16$
17. $30 \div 2 \cdot 3$
18. $27 \div 3 \cdot 3$
19. $8-6+1$
20. $15-5+10$
21. $14 \cdot 16 \div 16 \cdot 19$
22. $20 \cdot 17 \div 17 \cdot 14$
23. $15 \cdot 17+10 \div 10-12 \cdot 4$
24. $14 \cdot 18+9 \div 3-7 \cdot 13$
25. $22-10+7$
26. $29-11+1$
27. $20 \cdot 10+15 \div 5-7 \cdot 6$
28. $18 \cdot 19+18 \div 18-6 \cdot 7$

In Exercises 29-40, simplify the given expression.

29. $9+8 \div \{4+4\}$
30. $10+20 \div \{2+2\}$
31. $7 \cdot [8-5]-10$
32. $11 \cdot [12-4]-10$

33. $(18 + 10) \div (2 + 2)$

34. $(14 + 7) \div (2 + 5)$

35. $9 \cdot (10 + 7) - 3 \cdot (4 + 10)$

36. $9 \cdot (7 + 7) - 8 \cdot (3 + 8)$

37. $2 \cdot \{8 + 12\} \div 4$

38. $4 \cdot \{8+7\} \div 3$

39. $9+6 \cdot (12 + 3)$

40. $3+5 \cdot (10 + 12)$

In Exercises 41-56, simplify the given expression.

41. $2+9 \cdot [7 + 3 \cdot (9 + 5)]$

42. $6+3 \cdot [4 + 4 \cdot (5 + 8)]$

43. $7+3 \cdot [8 + 8 \cdot (5 + 9)]$

44. $4+9 \cdot [7 + 6 \cdot (3 + 3)]$

45. $6 - 5[11 - (2 + 8)]$

46. $15 - 1[19 - (7 + 3)]$

47. $11 - 1[19 - (2 + 15)]$

48. $9 - 8[6 - (2 + 3)]$

49. $4\{7[9 + 3] - 2[3 + 2]\}$

50. $4\{8[3 + 9] - 4[6 + 2]\}$

51. $9 \cdot [3 + 4 \cdot (5 + 2)]$

52. $3 \cdot [4 + 9 \cdot (8 + 5)]$

53. $3\{8[6 + 5] - 8[7 + 3]\}$

54. $2\{4[6 + 9] - 2[3 + 4]\}$

55. $3 \cdot [2 + 4 \cdot (9 + 6)]$

56. $8 \cdot [3 + 9 \cdot (5 + 2)]$

In Exercises 57-68, simplfiy the given expression.

57. $(5 - 2)^2$

58. $(5 - 3)^4$

59. $(4 + 2)^2$

60. $(3 + 5)^2$

61. $2^3 + 3^3$

62. $5^4 + 2^4$

63. $2^3 - 1^3$

64. $3^2 - 1^2$

65. $12 \cdot 5^2 + 8 \cdot 9+4$

66. $6 \cdot 3^2 + 7 \cdot 5 + 12$

67. $9 - 3 \cdot 2 + 12 \cdot 10^2$

68. $11 - 2 \cdot 3 + 12 \cdot 4^2$

In Exercises 69-80, simplify the given expression.

69. $4^2 - (13 + 2)$

70. $3^3 - (7 + 6)$

71. $3^3 - (7 + 12)$

72. $4^3 - (6 + 5)$

73. $19 + 3[12 - (2^3 + 1)]$

74. $13 + 12[14 - (2^2 + 1)]$

75. $17 + 7[13 - (2^2 + 6)]$

76. $10 + 1[16 - (2^2 + 9)]$

77. $4^3 - (12 + 1)$

78. $5^3 - (17 + 15)$

79. $5 + 7[11 - (2^2 + 1)]$

80. $10 + 11[20 - (2^2 + 1)]$

In Exercises 81-92, simplify the given expression.

81. $\frac{13+35}{3(4)}$

82. $\frac{35+28}{7(3)}$

83. $\frac{64-(8\cdot 6-3)}{4\cdot 7-9}$

84. $\frac{19-(4\cdot 3-2)}{6\cdot 3-9}$

85. $\frac{2+13}{4-1}$

86. $\frac{7+1}{8-4}$

87. $\frac{17+14}{9-8}$

88. $\frac{16+2}{13-11}$

89. $\frac{37+27}{8(2)}$

90. $\frac{16+38}{6(3)}$

91. $\frac{40-(3\cdot 7-9)}{8\cdot 2-2}$

92. $\frac{60-(8\cdot 6-3)}{5\cdot 4-5}$

In Exercises 93-100, use the distributive property to evaluate the given expression.

93. $5 \cdot (8 + 4)$

94. $8 \cdot (4 + 2)$

95. $7 \cdot (8 - 3)$

96. $8 \cdot (9 - 7)$

97. $6 \cdot (7 - 2)$

98. $4 \cdot (8 - 6)$

99. $4 \cdot (3 + 2)$

100. $4 \cdot (9 + 6)$

In Exercises 101-104, use the distributive property to evaluate the given expression using the technique shown in Example 9.

101. $9 \cdot 62$

102. $3 \cdot 76$

103. $3 \cdot 58$

104. $7 \cdot 57$

Answers

1. 9

3. 9

5. 22

7. 42

9. 15

11. 16

13. 75

15. 108

17. 45

19. 3

21. 266

23. 208

25. 19

27. 161

29. 10

31. 11

33. 7

35. 111

37. 10

39. 99

41. 443

43. 367

45. 1

47. 9

49. 296

51. 279

53. 24

55. 186

57. 9

59. 36

61. 35

63. 7

65. 376

67. 1203

69. 1

71. 8

73. 28

75. 38

77. 51

79. 47

81. 4

83. 1

85. 5

87. 31

89. 4

91. 2

93. 60

95. 35

97. 30

99. 20

101. 558

103. 174

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[1]Later, we'll see that this property applies to all numbers, not just whole numbers

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1.6: Solving Equations by Addition and Subtraction

Let's start with the definition of a variable.

Variable

A *variable* is a symbol (usually a letter) that stands for a value that may vary

Next we follow with the definition of an equation.

Equation

An *equation* is a mathematical statement that equates two mathematical expressions.

The key difference between a mathematical expression and an equation is the presence of an an equals sign. So, for example,

$$2 + 3[5 - 4 \cdot 2],\ x2 + 2x - 3, \text{ and } x + 2y + 3$$

are mathematical expressions (two of which contain variables), while

$$3 + 2(7 - 3) = 11,\ x + 3 = 4, \text{ and } 3x = 9$$

are equations. Note that each of the equations contain an equals sign, but the expressions do not.

Next we have the definition of a solution of an equation.

What it Means to be a Solution

A *solution* of an equation is a numerical value that satisfies the equation. That is, when the variable in the equation is replaced by the solution, a true statement results.

Example 1

Show that 3 is a solution of the equation $x + 8 = 11$.

Solution

Substitute 3 for x in the given equation and simplify.

$$\begin{aligned} x+8 &= 11 && \text{The given equation.} \\ 3+8 &= 11 && \text{Substitute 3 for } x. \\ 11 &= 11 && \text{Simplify both sides.} \end{aligned}$$

Since the left- and right-hand sides of the last line are equal, this shows that when 3 is substituted for x in the equation a true statement results. Therefore, 3 is a solution of the equation.

Exercise

Show that 27 is a solution of the equation $x - 12 = 15$

Example 2

Is 23 a solution of the equation $4 = y - 11$?

Solution

Substitute 23 for y in the given equation and simplify.

$$\begin{aligned} 4 &= y - 11 && \text{The given equation.} \\ 4 &= 23 - 11 && \text{Substitute 23 for } y \\ 4 &= 12 && \text{Simplify both sides.} \end{aligned}$$

Since the left- and right-hand sides of the last line are **not** equal, this shows that when 23 is substituted for y in the equation a false statement results. Therefore, 23 is **not** a solution of the equation.

Exercise

Is 8 a solution of $5 = 12 - y$?

Answer

No.

Equivalent Equations

We start with the definition of equivalent equations.

Equivalent Equations

Two equations are equivalent if they have the same solution set.

Example 3

Are the equations $x + 2 = 9$ and $x = 7$ equivalent?

Solution

The number 7 is the only solution of the equation $x + 2 = 9$. Similarly, 7 is the only solution of the equation $x = 7$. Therefore $x + 2 = 9$ and Answer: No. $x = 7$ have the same solution sets and are equivalent.

Exercise

Are the equations $x = 4$ and $x + 8 = 3$ equivalent?

Answer

No.

Example 4

Are the equations $x^2 = x$ and $x = 1$ equivalent?

Solution

By inspection, the equation $x^2 = x$ has two solutions, 0 and 1. On the other hand, the equation $x = 1$ has a single solution, namely 1. Hence, the equations $x^2 = x$ and $x = 1$ do not have the same solution sets and are **not** equivalent.

Exercise

Are the equations $x = 2$ and $x^2 = 2x$ equivalent?

Answer

No.

Operations that Produce Equivalent Equations

There are many operations that will produce equivalent operations. In this section we look at two: addition and subtraction.

Adding the Same Quantity to Both Sides of an Equation

Adding the same quantity to both sides of an equation does not change the solution set. That is, if

$$a = b,$$

then adding c to both sides of the equation produces the equivalent equation

$$a + c = b + c.$$

Let's see if this works as advertised. Consider the equation x − 4=3. By inspection, 7 is the only solution of the equation. Now, let's add 4 to both sides of the equation to see if the resulting equation is equivalent to x − 4 = 3.

$$\begin{aligned} x - 4 &= 3 && \text{The given equation.} \\ x - 4 + 4 &= 3 + 4 && \text{Add 4 to both sides of the equation.} \\ x &= 7 && \text{Simplify both sides of the equation.} \end{aligned}$$

The number 7 is the only solution of the equation x = 7. Thus, the equation x = 7 is equivalent to the original equation x − 4 = 3 (they have the same solutions).

Important Point

Adding the same amount to both sides of an equation does not change its solutions.

It is also a fact that subtracting the same quantity from both sides of an equation produces an equivalent equation.

Subtracting the Same Quantity from Both Sides of an Equation

Subtracting the same quantity from both sides of an equation does not change the solution set. That is, if

$$a = b,$$

then subtracting c from both sides of the equation produces the equivalent equation

$$a - c = b - c.$$

Let's also see if this works as advertised. Consider the equation

$$x + 4 = 9.$$

By inspection, 5 is the only solution of the equation. Now, let's subtract 4 from both sides of the equation to see if the resulting equation is equivalent to x + 4 = 9.

$$\begin{aligned} x + 4 &= 9 && \text{The given equation.} \\ x + 4 - 4 &= 9 - 4 && \text{Subtract 4 from both sides of the equation.} \\ x &= 5 && \text{Simplify both sides of the equation.} \end{aligned}$$

The number 5 is the only solution of the equation $x = 5$. Thus, the equation $x = 5$ is equivalent to the original equation $x + 4 = 9$ (they have the same solutions).

Important Point

Subtracting the same amount from both sides of an equation does not change its solutions.

Writing Mathematics

When solving equations, observe the following rules to neatly arrange your work:

1. **One equation per line**. This means that you should not arrange your work like this:

$$x+3=7 \quad x+3-3=7-3 \quad x=4$$

That's three equations on a line. Rather, arrange your work one equation per line like this:

$$\begin{aligned} x+3 &= 7 \\ x+3-3 &= 7-3 \\ x &= 4 \end{aligned}$$

2. **Add and subtract inline.** Don't do this:

$$\begin{array}{rcl} x-7 & = & 12 \\ +7 & & +7 \\ \hline x & = & 19 \end{array}$$

Instead, add 7 to both sides of the equation "inline."

$$\begin{aligned} x-7 &= 12 \\ x-7+7 &= 12+7 \\ x &= 19 \end{aligned}$$

Wrap and Unwrap

Suppose that you are wrapping a gift for your cousin. You perform the following steps in order.

1. Put the gift paper on.
2. Put the tape on.
3. Put the decorative bow on.

When we give the wrapped gift to our cousin, he politely unwraps the present, "undoing" each of our three steps in inverse order.

1. Take off the decorative bow.
2. Take off the tape.
3. Take off the gift paper.

This seemingly frivolous wrapping and unwrapping of a gift contains some deeply powerful mathematical ideas. Consider the mathematical expression $x+4$. To evaluate this expression at a particular value of *x*, we would start with the given value of *x*, then

1. Add 4.

Suppose we started with the number 7. If we add 4, we arrive at the following result: 11.

Now, how would we "unwrap" this result to return to our original number? We would start with our result, then

1. Subtract 4.

That is, we would take our result from above, 11, then subtract 4, which returns us to our original number, namely 7.

Addition and Subtraction as Inverse Operations

Two extremely important observations:

The inverse of addition is subtraction. If we start with a number *x* and add a number *a*, then subtracting *a* from the result will return us to the original number *x*. In symbols,

$$x+a-a=x.$$

The inverse of subtraction is addition. If we start with a number *x* and subtract a number *a*, then adding *a* to the result will return us to the original number *x*. In symbols,

$$x-a+a=x.$$

Example 5

Solve $x - 8 = 10$ for x.

Solution

To undo the effects of subtracting 8, we add 8 to both sides of the equation.

$$\begin{aligned} x - 8 &= 10 && \text{Original equation.} \\ x - 8 + 8 &= 10 + 8 && \text{Add 8 to both sides of the equation.} \\ x &= 18 && \text{On the left, adding "undoes" the effect of subtracting 8 and returns } x\text{. On the right, } 10 + 8 = 18. \end{aligned}$$

Therefore, the solution of the equation is 18.

Check

To check, substitute the solution 18 into the original equation.

$$\begin{aligned} x - 8 &= 10 && \text{Original equation.} \\ 18 - 8 &= 10 && \text{Substitute 18 for } x\text{.} \\ 10 &= 10 && \text{Simplify both sides.} \end{aligned}$$

The fact that the last line of our check is a true statement guarantees that 18 is a solution of x − 8 = 10.

Exercise

Solve $x + 5 = 12$ for x.

Answer

7.

Example 6

Solve $11 = y + 5$ for y.

Solution

To undo the effects of adding 5, we subtract 5 from both sides of the equation.

$$\begin{aligned} 11 &= y + 5 && \text{Original equation.} \\ 1 - 5 &= y + 5 - 5 && \text{Subtract 5 from both sides of the equation.} \\ 6 &= y && \text{On the right, subtracting "undoes" the effect of adding 5 and returns } y\text{. On the left, } 11 - 5 = 6. \end{aligned}$$

Therefore, the solution of the equation is 6.

Check

$$\begin{aligned} 11 &= y + 5 && \text{Original equation.} \\ 11 &= 6 + 5 && \text{Substitute 6 for } y\text{.} \\ 11 &= 11 && \text{Simplify both sides.} \end{aligned}$$

The fact that the last line of our check is a true statement guarantees that 6 is a solution of 11 = y + 5.

? Exercise

Solve $y - 8 = 11$ for y.

Answer

$y = 19$.

Word Problems

The solution of a word problem must incorporate each of the following steps.

Requirements for Word Problem Solutions

1. **Set up a Variable Dictionary**. You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
   1. Statements such as "Let P represent the perimeter of the rectangle."
   2. Labeling unknown values with variables in a table.
   3. Labeling unknown quantities in a sketch or diagram.
2. **Set up an Equation**. Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.
3. **Solve the Equation**. You must always solve the equation set up in the previous step.
4. **Answer the Question**. This step is easily overlooked. For example, the problem might ask for Jane's age, but your equation's solution gives the age of Jane's sister Liz. Make sure you answer the original question asked in the problem. Your solution should be written in a sentence with appropriate units.
5. **Look Back**. It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it's possible that your equation incorrectly models the problem's situation, so you could have a valid solution to an incorrect equation. The important question is: "Does your answer make sense based on the words in the original problem statement."

Let's give these requirements a test drive.

✓ Example 7

Four more than a certain number is 12. Find the number.

Solution

In our solution, we will carefully address each step of the Requirements for Word Problem Solutions.

1. *Set up a Variable Dictionary*. We can satisfy this requirement by simply stating "Let x represent a certain number."
2. *Set up an Equation*. "Four more than a certain number is 12" becomes

$$\begin{array}{lllll} 4 & \text{more than} & \text{a certain number} & \text{is} & 12 \\ 4+ & & x = & & 12 \end{array}$$

3. *Solve the Equation*. To "undo" the addition, subtract 4 from both sides of the equation.

$$\begin{aligned} 4 + x &= 12 && \text{Original equation.} \\ 4 + x - 4 &= 12 - 4 && \text{Subtract 4 from both sides of the equation.} \\ x &= 8 && \text{On the left, subtracting 4 "undoes" the effect of adding 4 and returns } x. \text{ On the right, } 12 - 4 = 8. \end{aligned}$$

4. *Answer the Question*. The number is 8.
5. *Look Back*. Does the solution 8 satisfy the words in the original problem? We were told that "four more than a certain number is 12." Well, four more than 8 is 12, so our solution is correct.

Exercise

12 more than a certain number is 19. Find the number.

Answer

7

Example 8

Amelie withdraws $125 from her savings account. Because of the withdrawal, the current balance in her account is now $1,200. What was the original balance in the account before the withdrawal?

Solution

In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary.* We can satisfy this requirement by simply stating "Let B represent the original balance in Amelie's account."

2. *Set up an Equation.* We can describe the situation in words and symbols.

Original Balance	minus	& is	Current Balance
B	$-$	$125 =$	1200

3. *Solve the Equation.* To "undo" the subtraction, add 125 to both sides of the equation.

$$\begin{aligned} B-125 &= 1200 && \text{Original equation.} \\ B-125+125 &= 1200+125 && \text{Add 125 to both sides of the equation.} \\ B &= 1325 && \text{On the left, adding 125 "undoes" the effect} \\ & && \text{of subtracting 125 and returns } B. \text{ On the right,} \\ & && 1200+125=1325. \end{aligned}$$

4. *Answer the Question.* The original balance was $1,325.

5. *Look Back.* Does the solution $1,325 satisfy the words in the original problem? Note that if Amelie withdraws $125 from this balance, the new balance will be $1,200. Hence, the solution is correct.

Exercise

Fred withdraws $230 from his account, lowering his balance to $3,500. What was his original balance?

Answer

$3730

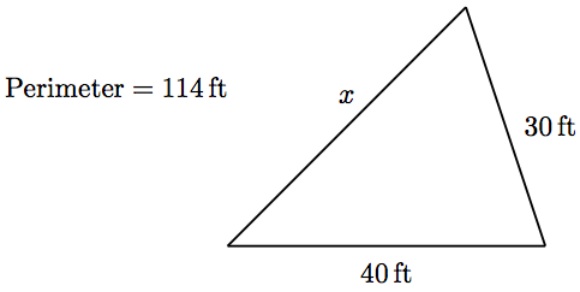
Example 9

The perimeter of a triangle is 114 feet. Two sides of the triangle measure 30 feet and 40 feet, respectively. Find the measure of the third side of the triangle.

Solution

In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions*.

1. Set up a Variable Dictionary. When geometry is involved, we can create our variable dictionary by labeling a carefully constructed diagram. With this thought in mind, we draw a triangle, then label its known and unknown sides and its perimeter.



The figure makes it clear that x represents the length of the unknown side of the triangle. The figure also summarizes information needed for the solution. 2. Set up an equation. We know that the perimeter of a triangle is found by finding the sum of its three sides; in words and symbols,

Perimeter	is	First Side	plus	Second Side	plus	Third Side
114	=	x	+	30	+	40

Simplify the right-hand side by adding 30 and 40; i.e., $30+40=70$.

$$114 = x + 70$$

3. Solve the Equation. To "undo" adding 70, subtract 70 from both sides of the equation.

$$\begin{aligned} 114 &= x + 70 && \text{Our equation.} \\ 114 - 70 &= x + 70 - 70 && \text{Subtract 70 from both sides.} \\ 44 &= x && \text{On the right, subtracting 70 "undoes" the effect of adding 70 and returns to } x. \text{ On the left, } 114 - 70 = 44. \end{aligned}$$

4. *Answer the Question*. The unknown side of the triangle is 44 feet.

5. *Look Back*. Does the solution 44 feet satisfy the words of the original problem? We were told that the perimeter is 114 feet and two of the sides have length 30 feet and 40 feet respectively. We found that the third side has length 44 feet. Now, adding the three sides, 30+ 40+ 44 = 114, which equals the given perimeter of 114 feet. The answer works!

? Exercise

The perimeter of a quadrilateral is 200 meters. If three of the sides measure 20, 40, and 60 meters, what is the length of the fourth side.

Answer

80 meters

Exercises

In Exercises 1-12, which of the numbers following the given equation are solutions of the given equation? Support your response with work similar to that shown in Examples 1 and 2.

1. x − 4 = 6; 10, 17, 13, 11
2. x − 9 = 7; 17, 23, 19, 16
3. x + 2 = 6; 5, 11, 7, 4
4. x + 3 = 9; 6, 9, 7, 13
5. x + 2 = 3; 8, 1, 4, 2
6. x + 2 = 5; 10, 3, 6, 4

7. x − 4 = 7; 12, 11, 18, 14

8. x − 6 = 7; 13, 16, 20, 14

9. x + 3 = 4; 8, 4, 2, 1

10. x + 5 = 9; 5, 11, 7, 4

11. x − 6 = 8; 17, 21, 14, 15

12. x − 2 = 9; 11, 14, 12, 18

In Exercises 13-52, solve the given equation for *x*.

13. x +5=6

14. x + 6 = 19

15. 5=4+ x

16. 10 = 8 + x

17. 13 + x = 17

18. 7+ x = 15

19. 9+ x = 10

20. 14 + x = 17

21. 19 = x − 3

22. 2= x − 11

23. x − 18 = 1

24. x − 20 = 8

25. x − 3 = 11

26. x − 17 = 18

27. 2+ x = 4

28. 1+ x = 16

29. x − 14 = 12

30. x − 1 = 17

31. x +2=8

32. x + 11 = 14

33. 11 + x = 17

34. 11 + x = 18

35. x + 13 = 17

36. x + 1 = 16

37. 20 = 3 + x

38. 9=3+ x

39. 20 = 8 + x

40. 10 = 3 + x

41. 3= x − 20

42. 13 = x − 15

43. x + 16 = 17

44. $x + 6 = 12$

45. $5 = x - 6$

46. $10 = x - 7$

47. $18 = x - 6$

48. $14 = x - 4$

49. $18 = 13 + x$

50. $17 = 5 + x$

51. $x - 9 = 15$

52. $x - 11 = 17$

---

53. 12 less than a certain number is 19. Find the number.

54. 19 less than a certain number is 1. Find the number.

55. A triangle has a perimeter of 65 feet. It also has two sides measuring 19 feet and 17 feet, respectively. Find the length of the third side of the triangle.

56. A triangle has a perimeter of 55 feet. It also has two sides measuring 14 feet and 13 feet, respectively. Find the length of the third side of the triangle.

57. Burt makes a deposit to an account having a balance of $1900. After the deposit, the new balance in the account is $8050. Find the amount of the deposit.

58. Dave makes a deposit to an account having a balance of $3500. After the deposit, the new balance in the account is $4600. Find the amount of the deposit.

59. 8 more than a certain number is 18. Find the number.

60. 3 more than a certain number is 19. Find the number.

61. Michelle withdraws a $120 from her bank account. As a result, the new account balance is $1000. Find the account balance before the withdrawal.

62. Mercy withdraws a $430 from her bank account. As a result, the new account balance is $1200. Find the account balance before the withdrawal.

63. *Foreclosures*. Between January and March last year, 650,000 homes received a foreclosure notice. Between the first three months of this year, there were 804,000 foreclosure notices. What was the increase in home foreclosure notices? *Associated Press Times-Standard 4/22/09*

64. *Home Price*. According to the Humboldt State University Economics Department's Humboldt Economic Index, the median home price in the US fell $1500 over the last month to $265,000. What was the median home price before the price drop?

65. *Unmanned Areal Vehicle*. Northrup Grumman's Global Hawk unmanned drone can fly at 65,000 feet, 40,000 feet higher than NASA's Ikhana unmanned aircraft. How high can the Ikhana fly?

66. *Tribal Land*. The Yurok Tribe has the option to purchase 47,000 acres in order to increase its ancestral territory. The first phase would include 22,500 acres in the Cappel and Pecman watersheds. The second phase plans for acreage in the Blue Creek area. How many acres could be purchased in the second phase? *Times-Standard 4/15/09*

Answers

1. 10

3. 4

5. 1

7. 11

9. 1

11. 14

13. 1

15. 1

17. 4

19. 1

21. 22

23. 19

25. 14

27. 2

29. 26

31. 6

33. 6

35. 4

37. 17

39. 12

41. 23

43. 1

45. 11

47. 24

49. 5

51. 24

53. 31

55. 29

57. $6150

59. 10

61. $1120

63. 154,000

65. 25,000 feet

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1.7: Solving Equations by Multiplication and Division

In Section 1.6, we stated that two equations that have the same solutions are *equivalent*. Furthermore, we saw that adding the same number to both sides of an equation produced an equivalent equation. Similarly, subtracting the same the number from both sides of an equation also produces an equivalent equation. We can make similar statements for multiplication and division.

Multiplying both Sides of an Equation by the Same Quantity

Multiplying both sides of an equation by the same quantity does not change the solution set. That is, if

$$a = b$$

then multiplying both sides of the equation by c produces the equivalent equation

$$a \cdot c = b \cdot c$$

provided $c \neq 0$.

A similar statement can be made about division.

Dividing both Sides of an Equation by the Same Quantity

Dividing both sides of an equation by the same quantity does not change the solution set. That is, if

$$a = b$$

then dividing both sides of the equation by c produces the equivalent equation

$$\frac{a}{c} = \frac{b}{c},$$

provided $c \neq 0$.

In Section 1.6, we saw that addition and subtraction were inverse operations. If you start with a number, add 4 and subtract 4, you are back to the original number. This concept also works for multiplication and division.

Multiplication and Division as Inverse Operations

Two extremely important observations:

The inverse of multiplication is division. If we start with a number x and multiply by a number a, then dividing the result by the number a returns us to the original number x. In symbols,

$$\frac{a \cdot x}{a} = x.$$

The inverse of division is multiplication. If we start with a number x and divide by a number a, then multiplying the result by the number a returns us to the original number x. In symbols,

$$a \cdot \frac{x}{a} = x.$$

Let's put these ideas to work.

Example 1

Solve the equation $3x = 24$ for x.

Solution

To undo the effects of multiplying by 3, we divide both sides of the equation by 3.

$$
\begin{aligned}
3x &= 24 && \text{Original equation.} \\
\frac{3x}{3} &= \frac{24}{3} && \text{Divide both sides of the equation by 3.} \\
x &= 8 && \text{On the left, dividing by 3 "undoes" the effect of multiplying by 3 and returns to } x\text{. On the right, } 24/3 = 8.
\end{aligned}
$$

To check, substitute the solution 8 into the original equation.

$$
\begin{aligned}
3x &= 24 && \text{Original equation.} \\
3(8) &= 24 && \text{Substitute 8 for } x\text{.} \\
24 &= 24 && \text{Simplify both sides.}
\end{aligned}
$$

That fact that the last line of our check is a true statement guarantees that 8 is a solution of 3x = 24.

? Exercise

Solve for *x*: 5*x* = 120.

Answer

24

✔ Example 2

Solve the following equation for *x*.

$$\frac{x}{7} = 12$$

Solution

To undo the effects of dividing by 7, we multiply both sides of the equation by 7.

$$
\begin{aligned}
\frac{x}{7} &= 12 && \text{Original equation.} \\
\frac{84}{7} &= 12 && \text{Multiply both sides of the equation by 7.} \\
x &= 84 && \text{On the left, multiplying by 7 "undoes" the effect of dividing by 7 and returns to } x\text{. On the right, } 7 \cdot 12 = 84.
\end{aligned}
$$

To check, substitute the solution 84 into the original equation.

$$
\begin{aligned}
\frac{x}{7} &= 12 && \text{Original equation.} \\
\frac{84}{7} &= 12 && \text{Substitute 84 for } x\text{.} \\
12 &= 12 && \text{Simplify both sides.}
\end{aligned}
$$

That fact that the last line of our check is a true statement guarantees that 84 is a solution of *x*/7 = 12.

? Exercise

Solve for *x*: *x*/2 = 19

Answer

38

Word Problems

In Section 1.6 we introduced *Requirements for Word Problem Solutions*. Those requirements will be strictly adhered to in this section.

Example 3

Fifteen times a certain number is 45. Find the unknown number.

Solution

In our solution, we will carefully address each step of the Requirements for Word Problem Solutions.

1. *Set up a Variable Dictionary*. We can satisfy this requirement by simply stating "Let x represent a certain number."

2. *Set up an equation*. "Fifteen times a certain number is 45" becomes

$$\begin{array}{ccccccc} \boxed{15} & \text{times} & \boxed{\text{a certain number}} & \text{is} & \boxed{45} \\ 15 & \cdot & x & = & 45 \end{array}$$

3. *Solve the Equation*. To "undo" the multiplication by 15, divide both sides of the equation by 15.

$$\begin{aligned} 15x &= 45 && \text{Original equation. Write } 15 \cdot x \text{ as } 15x \\ \frac{15x}{15} &= \frac{45}{15} && \text{Divide both sides of the equation by } 15. \\ x &= 3 && \text{On the left, dividing by 15 "undoes" the effect of multiplying by 15 and returns to } x. \text{ On the right, } 45/15 = 3. \end{aligned}$$

4. *Answer the Question*. The unknown number is 3.

5. *Look Back*. Does the solution 3 satisfy the words of the original problem? We were told that "15 times a certain number is 45." Well, 15 times 3 is 45, so our solution is correct.

Exercise

Seven times a certain number is one hundred five. Find the unknown number.

Answer

15

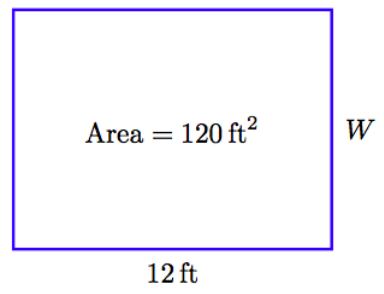
Example 4

The area of a rectangle is 120 square feet. If the length of the rectangle is 12 feet, find the width of the rectangle.

Solution

In our solution, we will carefully address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. When geometry is involved, we can create our variable dictionary by labeling a carefully constructed diagram. With this thought in mind, we draw a rectangle, then label its length, width, and area.



The figure makes it clear that W represents the width of the rectangle. The figure also summarizes information needed for the solution.

2. *Set up an equation*. We know that the area of a rectangle is found by multiplying its length and width; in symbols,

$$A = LW.$$

We're given the area is A = 120 ft^2 and the length is L = 12 ft. Substitute these numbers into the area formula (1.1) to get

$$120 = 12W.$$

3. *Solve the Equation*. To "undo" the multiplication by 12, divide both sides of the equation by 12.

$$\begin{aligned} 120 &= 12W && \text{Our equation.} \\ \frac{120}{12} &= \frac{12W}{12} && \text{Divide both sides of the equation by 12.} \\ 10 &= W && \text{On the right, dividing by 12 "undoes" the effect} \\ & && \text{of multiplying by 12 and returns to } W. \text{ On the left,} \\ & && 120/12 = 10. \end{aligned}$$

4. *Answer the Question*. The width is 10 feet.

5. *Look Back*. Does the found width satisfy the words of the original problem? We were told that the area is 120 square feet and the length is 12 feet. The area is found by multiplying the length and width, which gives us 12 feet times 10 feet, or 120 square feet. The answer works!

Exercise

The area of a rectangle is 3,500 square meters. If the width is 50 meters, find the length.

Answer

70 meters

Example 5

A class of 23 students averaged 76 points on an exam. How many total points were accumulated by the class as a whole?

Solution

In our solution, we will carefully address each step of the Requirements for Word Problem Solutions. 1. Set up a Variable Dictionary. We can set up our variable dictionary by simply stating "Let T represent the total points accumulated by the class." 2. Set up an equation. To find the average score on the exam, take the total points accumulated by the class, then divide by the number of students in the class. In words and symbols,

Total Points	divided by	Number of Students	equals	Average Score
T	$\div$	23	$=$	76

An equivalent representation is

$$\frac{T}{23} = 76.$$

3. *Solve the Equation*. To "undo" the division by 23, multiply both sides of the equation by 23.

$$\begin{aligned} \frac{T}{23} &= 76 && \text{Our equation.} \\ 23 \cdot \frac{T}{23} &= 76 \cdot 23 && \text{Multiply both sides of the equation by 23.} \\ T &= 1748 && \text{On the left, multiplying by 23 "undoes" the effect} \\ & && \text{of dividing by 23 and returns to } T. \text{ On the right,} \\ & && 76 \cdot 23 = 1748. \end{aligned}$$

4. *Answer the Question*. The total points accumulated by the class on the exam is 1,748.

5. *Look Back*. Does the solution 1,748 satisfy the words of the original problem? To find the average on the exam, divide the total points 1,748 by 23, the number of students in the class. Note that this gives an average score of 1748 ÷ 23 = 76. The answer works!

? Try it out!

A class of 30 students averaged 75 points on an exam. How many total points were accumulated by the class as a whole?

Answer

2,250

Exercises

In Exercises 1-12, which of the numbers following the given equation are solutions of the given equation?

1. $\frac{x}{6} = 4$; 24, 25, 27, 31
2. $\frac{x}{7} = 6$; 49, 42, 43, 45
3. $\frac{x}{2} = 3$; 6, 9, 13, 7
4. $\frac{x}{9} = 5$; 45, 46, 48, 52
5. $5x = 10$; 9, 2, 3, 5
6. $4x = 36$; 12, 16, 9, 10
7. $5x = 25$; 5, 6, 8, 12
8. $3x = 3$; 1, 8, 4, 2
9. $2x = 2$; 4, 8, 1, 2
10. $3x = 6$; 2, 9, 5, 3
11. $\frac{x}{8} = 7$; 57, 59, 63, 56
12. $\frac{x}{3} = 7$; 24, 21, 28, 22

In Exercises 13-36, solve the given equation for x.

13. $\frac{x}{6} = 7$
14. $\frac{x}{8} = 6$
15. $2x = 16$
16. $2x = 10$
17. $2x = 18$
18. $2x = 0$
19. $4x = 24$
20. $2x = 4$
21. $\frac{x}{4} = 9$
22. $\frac{x}{5} = 6$
23. $5x = 5$
24. $3x = 15$
25. $5x = 30$

26. $4x = 28$

27. $\frac{x}{3} = 4$

28. $\frac{x}{9} = 4$

29. $\frac{x}{8} = 9$

30. $\frac{x}{8} = 2$

31. $\frac{x}{7} = 8$

32. $\frac{x}{4} = 6$

33. $2x = 8$

34. $3x = 9$

35. $\frac{x}{8} = 5$

36. $\frac{x}{5} = 4$

---

37. The price of one bookcase is $370. A charitable organization purchases an unknown number of bookcases and the total price of the purchase is $4,810. Find the number of bookcases purchased.

38. The price of one computer is $330. A charitable organization purchases an unknown number of computers and the total price of the purchase is $3,300. Find the number of computers purchased.

39. When an unknown number is divided by 3, the result is 2. Find the unknown number.

40. When an unknown number is divided by 8, the result is 3. Find the unknown number.

41. A class of 29 students averaged 80 points on an exam. How many total points were accumulated by the class as a whole?

42. A class of 44 students averaged 87 points on an exam. How many total points were accumulated by the class as a whole?

43. When an unknown number is divided by 9, the result is 5. Find the unknown number.

44. When an unknown number is divided by 9, the result is 2. Find the unknown number.

45. The area of a rectangle is 16 square cm. If the length of the rectangle is 2 cm, find the width of the rectangle.

46. The area of a rectangle is 77 square ft. If the length of the rectangle is 7 ft, find the width of the rectangle.

47. The area of a rectangle is 56 square cm. If the length of the rectangle is 8 cm, find the width of the rectangle.

48. The area of a rectangle is 55 square cm. If the length of the rectangle is 5 cm, find the width of the rectangle.

49. The price of one stereo is $430. A charitable organization purchases an unknown number of stereos and the total price of the purchase is $6,020. Find the number of stereos purchased.

50. The price of one computer is $490. A charitable organization purchases an unknown number of computers and the total price of the purchase is $5,880. Find the number of computers purchased.

51. A class of 35 students averaged 74 points on an exam. How many total points were accumulated by the class as a whole?

52. A class of 44 students averaged 88 points on an exam. How many total points were accumulated by the class as a whole?

53. 5 times an unknown number is 20. Find the unknown number.

54. 5 times an unknown number is 35. Find the unknown number.

55. 3 times an unknown number is 21. Find the unknown number.

56. 2 times an unknown number is 10. Find the unknown number.

Answers

1. 24

3. 6

5. 2

7. 5

9. 1

11. 56

13. 42

15. 8

17. 9

19. 6

21. 36

23. 1

25. 6

27. 12

29. 72

31. 56

33. 4

35. 40

37. 13

39. 6

41. 2,320

43. 45

45. 8 cm

47. 7 cm

49. 14

51. 2,590

53. 4

55. 7

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1.8: Use the Language of Algebra (Part 1)

Learning Objectives

- Use variables and algebraic symbols
- Identify expressions and equations
- Simplify expressions with exponents
- Simplify expressions using the order of operations

Be Prepared!

Before you get started, take this readiness quiz.

a. Add: $43+69$. If you missed this problem, review Example 1.2.8.
b. Multiply: $(896)201$. If you missed this problem, review Example 1.4.11.
c. Divide: $7,263 \div 9$. If you missed this problem, review Example 1.5.8.

Use Variables and Algebraic Symbols

Greg and Alex have the same birthday, but they were born in different years. This year Greg is 20 years old and Alex is 23, so Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more, right?

In the language of algebra, we say that Greg's age and Alex's age are variable and the three is a constant. The ages change, or vary, so age is a variable. The 3 years between them always stays the same, so the age difference is the constant.

In algebra, letters of the alphabet are used to represent variables. Suppose we call Greg's age g. Then we could use $g+3$ to represent Alex's age. See Table 1.8.1.

Table 1.8.1

Greg's age	Alex's age
12	15
20	23
35	38
g	g + 3

Letters are used to represent variables. Letters often used for variables are $x, y, a, b,$ and c.

Definition: Variables and Constants

A variable is a letter that represents a number or quantity whose value may change.

A constant is a number whose value always stays the same.

To write algebraically, we need some symbols as well as numbers and variables. There are several types of symbols we will be using. In Whole Numbers, we introduced the symbols for the four basic arithmetic operations: addition, subtraction, multiplication, and division. We will summarize them here, along with words we use for the operations and the result.

Table 1.8.2

Operation	Notation	Say:	The result is...
Addition	a + b	a plus b	the sum of a and b
Subtraction	a − b	a minus b	the difference of a and b
Multiplication	a • b, (a)(b), (a)b, a(b)	a times b	the product of a and b

Operation	Notation	Say:	The result is...
Division	a ÷ b, a / b, $\frac{a}{b}$, $b\overline{)a}$	a divided by b	the quotient of a and b

In algebra, the cross symbol, $\times$, is not used to show multiplication because that symbol may cause confusion. Does $3xy$ mean $3 \times y$ (three times y) or $3 \bullet x \bullet y$ (three times x times y)? To make it clear, use $\bullet$ or parentheses for multiplication.

We perform these operations on two numbers. When translating from symbolic form to words, or from words to symbolic form, pay attention to the words *of* or *and* to help you find the numbers.

The *sum* ***of*** 5 ***and*** 3 means add 5 plus 3, which we write as $5+3$.

The *difference* ***of*** 9 ***and*** 2 means subtract 9 minus 2, which we write as $9-2$.

The *product* ***of*** 4 ***and*** 8 means multiply 4 times 8, which we can write as $4 \bullet 8$.

The *quotient* ***of*** 20 ***and*** 5 means divide 20 by 5, which we can write as $20 \div 5$.

Example 1.8.1: translate to words

Translate from algebra to words:

a. $12+14$
b. $(30)(5)$
c. $64 \div 8$
d. $x-y$

Solution

a.

12 + 14
12 plus 14
the sum of twelve and fourteen

b.

(30)(5)
30 times 5
the product of thirty and five

c.

64 ÷ 8
64 divided by 8
the quotient of sixty-four and eight

d.

x − y
x minus y
the difference of x and y

exercise 1.8.1

Translate from algebra to words.

a. $18+11$
b. $(27)(9)$
c. $84\div 7$
d. $p-q$

Answer a

18 plus 11; the sum of eighteen and eleven

Answer b

27 times 9; the product of twenty-seven and nine

Answer c

84 divided by 7; the quotient of eighty-four and seven

Answer d

p minus q; the difference of p and q

exercise 1.8.2

Translate from algebra to words.

a. $47-19$
b. $72\div 9$
c. $m+n$
d. $(13)(7)$

Answer a

47 minus 19; the difference of forty-seven and nineteen

Answer b

72 divided by 9; the quotient of seventy-two and nine

Answer c

m plus n; the sum of m and n

Answer d

13 times 7; the product of thirteen and seven

When two quantities have the same value, we say they are equal and connect them with an *equal sign*.

Definition: Equality Symbol

$a=b$ is read a is equal to b

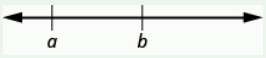
The symbol $=$ is called the equal sign.

An inequality is used in algebra to compare two quantities that may have different values. The number line can help you understand inequalities. Remember that on the number line the numbers get larger as they go from left to right. So if we know that b is greater than a, it means that b is to the right of a on the number line. We use the symbols "$<$" and "$>$" for inequalities.

Definition: Inequality

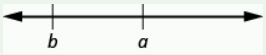
$a < b$ is read a is less than b

a is to the left of b on the number line



$a > b$ is read a is greater than b

a is to the right of b on the number line



The expressions $a < b$ and $a > b$ can be read from left-to-right or right-to-left, though in English we usually read from left-to-right. In general, $a < b$ is equivalent to $b > a$. For example, $7 < 11$ is equivalent to $11 > 7$. $a > b$ is equivalent to $b < a$. For example, $17 > 4$ is equivalent to $4 < 17$.

When we write an inequality symbol with a line under it, such as $a \leq b$, it means $a < b$ or $a = b$. We read this a is less than or equal to b. Also, if we put a slash through an equal sign, $\neq$, it means not equal.

We summarize the symbols of equality and inequality in Table 1.8.3.

Table 1.8.3

Algebraic Notation	Say
a = b	a is equal to b
a ≠ b	a is not equal to b
a < b	a is less than b
a > b	a is greater than b
a ≤ b	a is less than or equal to b
a ≥ b	a is greater than or equal to b

Definition: Symbols $<$ and $>$

The symbols $<$ and $>$ each have a smaller side and a larger side.

$$\text{smaller side} < \text{larger side}$$

$$\text{larger side} > \text{smaller side}$$

The smaller side of the symbol faces the smaller number and the larger faces the larger number.

Example 1.8.2: translate to words

Translate from algebra to words:

a. $20 \leq 35$
b. $11 \neq 15 - 3$
c. $9 > 10 \div 2$
d. $x + 2 < 10$

Solution

a.

$20 \leq 35$
20 is less than or equal to 35

b.

$11 \neq 15 - 3$
11 is not equal to 15 minus 3

c.

$9 > 10 \div 2$
9 is greater than 10 divided by 2

d.

$x + 2 < 10$
x plus 2 is less than 10

exercise 1.8.3

Translate from algebra to words.

a. $14 \leq 27$
b. $19 - 2 \neq 8$
c. $12 > 4 \div 2$
d. $x - 7 < 1$

Answer a

fourteen is less than or equal to twenty-seven

Answer b

nineteen minus two is not equal to eight

Answer c

twelve is greater than four divided by two

Answer d

x minus seven is less than one

exercise 1.8.4

Translate from algebra to words.

a. $19 \geq 15$
b. $7 = 12 - 5$
c. $15 \div 3 < 8$
d. $y - 3 > 6$

Answer a

nineteen is greater than or equal to fifteen

Answer b

seven is equal to twelve minus five

Answer c

fifteen divided by three is less than eight

Answer d

y minus three is greater than six

Example 1.8.3: translate

The information in Figure 1.8.1 compares the fuel economy in miles-per-gallon (mpg) of several cars. Write the appropriate symbol =, in each expression to compare the fuel economy of the cars.

Car	Prius	Mini Cooper	Toyota Corolla	Versa	Honda Fit
Fuel economy (mpg)	48	27	28	26	27

Figure 1.8.1: (credit: modification of work by Bernard Goldbach, Wikimedia Commons)

a. MPG of Prius _____ MPG of Mini Cooper
b. MPG of Versa _____ MPG of Fit
c. MPG of Mini Cooper _____ MPG of Fit
d. MPG of Corolla _____ MPG of Versa
e. MPG of Corolla_____ MPG of Prius

Solution

a.

	MPG of Prius____MPG of Mini Cooper
Find the values in the chart.	48____27
Compare.	48 > 27
	MPG of Prius > MPG of Mini Cooper

b.

	MPG of Versa____MPG of Fit
Find the values in the chart.	26____27
Compare.	26 < 27
	MPG of Versa < MPG of Fit

c.

	MPG of Mini Cooper____MPG of Fit
Find the values in the chart.	27____27
Compare.	27 = 27
	MPG of Mini Cooper = MPG of Fit

d.

	MPG of Corolla____MPG of Versa
Find the values in the chart.	28____26
Compare.	28 > 26

	MPG of Corolla > MPG of Versa

e.

	MPG of Corolla____MPG of Prius
Find the values in the chart.	28____48
Compare.	28 < 48
	MPG of Corolla < MPG of Prius

? exercise 1.8.5

Use Figure 1.8.1 to fill in the appropriate symbol, $=$, $<$, or $>$.

a. MPG of Prius_____MPG of Versa
b. MPG of Mini Cooper_____ MPG of Corolla

Answer a

$>$

Answer b

$<$

? exercise 1.8.6

Use Figure 1.8.1 to fill in the appropriate symbol, $=$, $<$, or $>$.

a. MPG of Fit_____ MPG of Prius
b. MPG of Corolla _____ MPG of Fit

Answer a

$<$

Answer b

$<$

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in written language. They indicate which expressions are to be kept together and separate from other expressions. Table 1.8.4 lists three of the most commonly used grouping symbols in algebra.

Table 1.8.4

Common Grouping Symbols	
parentheses	( )
brackets	[ ]
braces	{ }

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

$$8(14-8) \qquad 21-3[2+4(9-8)] \qquad 24\div 13-2[1(6-5)+4]$$

Identify Expressions and Equations

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. “Running very fast” is a phrase, but “The football player was running very fast” is a sentence. A sentence has a subject and a verb.

In algebra, we have *expressions* and *equations*. An expression is like a phrase. Here are some examples of expressions and how they relate to word phrases:

Table 1.8.5

Expression	Words	Phrase
3 + 5	3 plus 5	the sum of three and five
n - 1	n minus one	the difference of n and one
6 • 7	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Notice that the phrases do not form a complete sentence because the phrase does not have a verb. An equation is two expressions linked with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb. Here are some examples of equations:

Table 1.8.6

Equation	Sentence
3 + 5 = 8	The sum of three and five is equal to eight.
n − 1 = 14	n minus one equals fourteen.
6 • 7 = 42	The product of six and seven is equal to forty-two.
x = 53	x is equal to fifty-three.
y + 9 = 2y − 3	y plus nine is equal to two y minus three.

Definition: Expressions and Equations

An expression is a number, a variable, or a combination of numbers and variables and operation symbols.

An equation is made up of two expressions connected by an equal sign.

Example 1.8.4: expression or equation

Determine if each is an expression or an equation:

a. $16 - 6 = 10$
b. $4 \bullet 2 + 1$
c. $x \div 25$
d. $y + 8 = 40$

Solution

(a) 16 − 6 = 10	This is an equation—two expressions are connected with an equal sign.
(b) 4 • 2 + 1	This is an expression—no equal sign.
(c) x ÷ 25	This is an expression—no equal sign.

(d) y + 8 = 40	This is an equation—two expressions are connected with an equal sign.

? exercise 1.8.7

Determine if each is an expression or an equation:

a. $23+6=29$
b. $7 \bullet 3-7$

Answer a

equation

Answer b

expression

? exercise 1.8.8

Determine if each is an expression or an equation:

a. $y \div 14$
b. $x-6=21$

Answer a

expression

Answer b

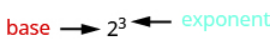
equation

Simplify Expressions with Exponents

To simplify a numerical expression means to do all the math possible. For example, to simplify $4 \bullet 2+1$ we'd first multiply $4 \bullet 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:

$$\begin{aligned} 4 \cdot 2+1 \\ 8+1 \\ 9 \end{aligned}$$

Suppose we have the expression $2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2$. We could write this more compactly using exponential notation. Exponential notation is used in algebra to represent a quantity multiplied by itself several times. We write $2 \bullet 2 \bullet 2$ as 2^3 and $2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2$ as 2^9. In expressions such as 2^3, the 2 is called the **base** and the 3 is called the **exponent**. The exponent tells us how many factors of the base we have to multiply.

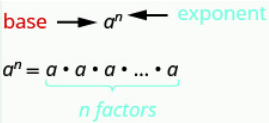


means multiply three factors of 2

We say 2^3 is in exponential notation and $2 \bullet 2 \bullet 2$ is in expanded notation.

Definition: Exponential Notation

For any expression a^n, a is a factor multiplied by itself n times if n is a positive integer.



The expression a^n is read a to the nth power.

For powers of $n = 2$ and $n = 3$, we have special names. a^2 is read as "a squared" a^3 is read as "a cubed" Table 1.8.7 lists some examples of expressions written in exponential notation.

Table 1.8.7

Exponential Notation	In Words
7^2	7 to the second power, or 7 squared
5^3	5 to the third power, or 5 cubed
9^4	9 to the fourth power
12^5	12 to the fifth power

Example 1.8.5: exponential form

Write each expression in exponential form:

a. $16 \bullet 16 \bullet 16 \bullet 16 \bullet 16 \bullet 16 \bullet 16$
b. $9 \bullet 9 \bullet 9 \bullet 9 \bullet 9$
c. $x \bullet x \bullet x \bullet x$
d. $a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a$

Solution

(a) The base 16 is a factor 7 times.	16^7
(b) The base 9 is a factor 5 times.	9^5
(c) The base x is a factor 4 times.	x^4
(d) The base a is a factor 8 times.	a^8

exercise 1.8.9

Write each expression in exponential form: $41 \bullet 41 \bullet 41 \bullet 41 \bullet 41$

Answer

41^5

exercise 1.8.10

Write each expression in exponential form: $7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7$

Answer

7^9

Example 1.8.6: expanded form

Write each exponential expression in expanded form:

a. 8^6
b. x^5

Solution

a. The base is 8 and the exponent is 6, so 8^6 means $8 \bullet 8 \bullet 8 \bullet 8 \bullet 8 \bullet 8$
b. The base is x and the exponent is 5, so x^5 means $x \bullet x \bullet x \bullet x \bullet x$

exercise 1.8.11

Write each exponential expression in expanded form:

a. 4^8
b. a^7

Answer a

$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

Answer b

$a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$

exercise 1.8.12

Write each exponential expression in expanded form:

a. 8^8
b. b^6

Answer a

$8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$

Answer b

$b \cdot b \cdot b \cdot b \cdot b \cdot b$

To simplify an exponential expression without using a calculator, we write it in expanded form and then multiply the factors.

Example 1.8.7: simplify

Simplify: 3^4.

Solution

Expand the expression.	$3^4 = 3 \cdot 3 \cdot 3 \cdot 3$
Multiply left to right.	$9 \cdot 3 \cdot 3 = 27 \cdot 3$
Multiply.	81

exercise 1.8.13

Simplify:

a. 5^3
b. 1^7

Answer a

125

Answer b

1

? exercise 1.8.14

Simplify:

a. 7^2
b. 0^5

Answer a

49

Answer b

0

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1.9: Use the Language of Algebra (Part 2)

Simplify Expressions Using the Order of Operations

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values. For example, consider the expression:

$$4+3\cdot 7$$

Some students say it simplifies to	49	Some students say it simplifies to	25
Since 4 + 3 gives 7.	4 + 3 • 7 = 7 • 7	Since 3 • 7 is 21.	4 + 3 • 7 = 4 + 21
And 7 • 7 is 49.	7 • 7 = 49	And 21 + 4 makes 25.	4 + 21 = 25

Imagine the confusion that could result if every problem had several different correct answers. The same expression should give the same result. So mathematicians established some guidelines called the order of operations, which outlines the order in which parts of an expression must be simplified.

Definition: Order of Operations

When simplifying mathematical expressions perform the operations in the following order:

1. **P**arentheses and other Grouping Symbols
- Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
2. **E**xponents
- Simplify all expressions with exponents.
3. **M**ultiplication and **D**ivision
- Perform all multiplication and division in order from left to right. These operations have equal priority.
4. **A**ddition and **S**ubtraction
- Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, "How will I remember the order?" Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase. **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally.

Table 1.9.8

Order of Operations	
Please	**P**arentheses
Excuse	**E**xponents
My **D**ear	**M**ultiplication and **D**ivision
Aunt **S**ally	**A**ddition and **S**ubtraction

It's good that '**M**y **D**ear' goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, '**A**unt **S**ally' goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

Example 1.9.8: simplify

Simplify the expressions:

a. $4+3 \bullet 7$
b. $(4+3) \bullet 7$

Solution

a.

Are there any parentheses? No. Are there any exponents? No. Is there any multiplication or division? Yes.	$4+3 \cdot 7$
Multiply first.	$4+3 \cdot 7$
Add.	$4+21$
	25

b.

Are there any parentheses? Yes.	$(4+3) \cdot 7$
Simplify inside the parentheses.	$(4+3) \cdot 7$
Are there any exponents? No. Is there any multiplication or division? Yes.	$(7)7$
Multiply.	49

exercise 1.9.15

Simplify the expressions:

a. $12-5 \bullet 2$
b. $(12-5) \bullet 2$

Answer a

2

Answer b

14

exercise 1.9.16

Simplify the expressions:

a. $8+3 \bullet 9$
b. $(8+3) \bullet 9$

Answer a

35

Answer b

99

Example 1.9.9: simplify

Simplify:

a. $18 \div 9 \bullet 2$
b. $18 \bullet 9 \div 2$

Solution

a.

Are there any parentheses? No. Are there any exponents? No. Is there any multiplication or division? Yes.	$18 \div 9 \cdot 2$
Multiply and divide from left to right. Divide.	$2 \cdot 2$
Multiply.	4

b.

Are there any parentheses? No. Are there any exponents? No. Is there any multiplication or division? Yes.	$18 \cdot 9 \div 2$
Multiply and divide from left to right. Multiply.	$162 \div 2$
Divide.	81

exercise 1.9.17

Simplify: $42 \div 7 \bullet 3$

Answer

18

exercise 1.9.18

Simplify: $12 \bullet 3 \div 4$

Answer

9

Example 1.9.10: simplify

Simplify: $18 \div 6 + 4(5 - 2)$.

Solution

Parentheses? Yes, subtract first.	$18 \div 6 + 4(5 - 2)$
Exponents? No. Multiplication or division? Yes.	$18 \div 6 + 4(3)$
Divide first because we multiply and divide left to right.	$3 + 4(3)$
Any other multiplication or division? Yes. Multiply.	$3 + 12$
Any other multiplication or division? No. Any addition or subtraction? Yes	15

exercise 1.9.19

Simplify: $30 \div 5 + 10(3-2)$

Answer

16

exercise 1.9.20

Simplify: $70 \div 10 + 4(6-2)$

Answer

23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

Example 1.9.11: simplify

Simplify: $5 + 2^3 + 3[6 - 3(4-2)]$.

Solution

Are there any parentheses (or other grouping symbol)? Yes.	$5 + 2^3 + 3[6 - 3(4-2)]$
Focus on the parentheses that are inside the brackets.	$5 + 2^3 + 3[6 - 3(4-2)]$
Subtract.	$5 + 2^3 + 3[6 - 3(2)]$
Continue inside the brackets and multiply.	$5 + 2^3 + 3[6 - 6]$
Continue inside the brackets and subtract.	$5 + 2^3 + 3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes.	$5 + 2^3 + 3[0]$
Simplify exponents.	$5 + 8 + 3[0]$
Is there any multiplication or division? Yes.	$5 + 8 + 3[0]$
Multiply.	$5 + 8 + 0$
Is there any addition or subtraction? Yes.	$textcolorred5 + 8 + 0$
Add.	$13 + 0$
Add.	13

exercise 1.9.21

Simplify: $9 + 5^3 - [4(9+3)]$

Answer

86

exercise 1.9.22

Simplify: $7^2 - 2[4(5+1)]$

Answer

1

Example 1.9.12: simplify

Simplify: $2^3 + 34 \div 3 - 5^2$.

Solution

If an expression has several exponents, they may be simplified in the same step.	$2^3 + 3^4 \div 3 - 52$
Simplify exponents.	$2^3 + 3^4 \div 3 - 5^2$
Divide.	$8 + 81 \div 3 - 25$
Add.	$8 + 27 - 25$
Subtract.	$35 - 25$
	10

exercise 1.9.23

Simplify: $3^2 + 2^4 \div 2 + 4^3$

Answer

81

exercise 1.9.24

Simplify: $6^2 - 5^3 \div 5 + 8^2$

Answer

75

Access Additional Online Resources

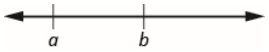
- Order of Operations
- Order of Operations – The Basics
- Ex: Evaluate an Expression Using the Order of Operations
- Example 3: Evaluate an Expression Using The Order of Operations

Key Concepts

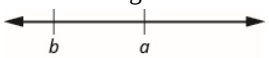
Operation	Notation	Say:	The result is…
Addition	a+b	a plus b	the sum of a and b
Multiplication	a·b,(a)(b),(a)b,a(b)	a times b	The product of a and b
Subtraction	a−b	a minus b	the difference of a and b

Operation	Notation	Say:	The result is…
Division	$a \div b, a/b, \frac{a}{b}, a\overline{)b}$	a divided by b	The quotient of a and b

- **Equality Symbol**
  - $a = b$ is read as a is equal to b
  - The symbol $=$ is called the equal sign.
- **Inequality**
  - $a < b$ is read a is less than b
  - a is to the left of b on the number line

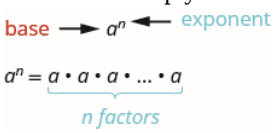


  - $a > b$ is read a is greater than b
  - a is to the right of b on the number line



Algebraic Notation	Say
a=b	a is equal to b
a≠b	a is not equal to b
a<b	a is less than b
a>b	a is greater than b
a≤b	a is less than or equal to b
a≥b	a is greater than or equal to b

- **Exponential Notation**
  - For any expression a^n is a factor multiplied by itself n times, if n is a positive integer.
  - a^n means multiply n factors of a



  - The expression of a^n is read a to the nth power.

Order of Operations When simplifying mathematical expressions perform the operations in the following order:

- Parentheses and other Grouping Symbols: Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
- Exponents: Simplify all expressions with exponents.
- Multiplication and Division: Perform all multiplication and division in order from left to right. These operations have equal priority.
- Addition and Subtraction: Perform all addition and subtraction in order from left to right. These operations have equal priority.

Glossary

expressions

An expression is a number, a variable, or a combination of numbers and variables and operation symbols.

equation

An equation is made up of two expressions connected by an equal sign.

Practice Makes Perfect

Use Variables and Algebraic Symbols

In the following exercises, translate from algebraic notation to words.

1. $16 - 9$
2. $25 - 7$
3. $5 \cdot 6$
4. $3 \cdot 9$
5. $28 \div 4$
6. $45 \div 5$
7. $x + 8$
8. $x + 11$
9. $(2)(7)$
10. $(4)(8)$
11. $14 < 21$
12. $17 < 35$
13. $36 \geq 19$
14. $42 \geq 27$
15. $3n = 24$
16. $6n = 36$
17. $y - 1 > 6$
18. $y - 4 > 8$
19. $2 \leq 18 \div 6$
20. $3 \leq 20 \div 4$
21. $a \neq 7 \cdot 4$
22. $a \neq 1 \cdot 12$

Identify Expressions and Equations

In the following exercises, determine if each is an expression or an equation.

23. $9 \cdot 6 = 54$
24. $7 \cdot 9 = 63$
25. $5 \cdot 4 + 3$
26. $6 \cdot 3 + 5$
27. $x + 7$
28. $x + 9$
29. $y - 5 = 25$
30. $y - 8 = 32$

Simplify Expressions with Exponents

In the following exercises, write in exponential form.

31. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
32. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
33. $x \cdot x \cdot x \cdot x \cdot x$
34. $y \cdot y \cdot y \cdot y \cdot y \cdot y$

In the following exercises, write in expanded form.

35. 5^3
36. 8^3
37. 2^8
38. 10^5

Simplify Expressions Using the Order of Operations

In the following exercises, simplify.

39. (a) 3 + 8 • 5 (b) (3+8) • 5
40. (a) 2 + 6 • 3 (b) (2+6) • 3
41. $2^3 - 12 \div (9 - 5)$
42. $3^2 - 18 \div (11 - 5)$
43. 3 • 8 + 5 • 2
44. 4 • 7 + 3 • 5
45. 2 + 8(6 + 1)
46. 4 + 6(3 + 6)
47. 4 • 12 / 8
48. 2 • 36 / 6
49. 6 + 10 / 2 + 2
50. 9 + 12 / 3 + 4
51. (6 + 10) ÷ (2 + 2)
52. (9 + 12) ÷ (3 + 4)
53. 20 ÷ 4 + 6 • 5
54. 33 ÷ 3 + 8 • 2
55. 20 ÷ (4 + 6) • 5
56. 33 ÷ (3 + 8) • 2
57. $4^2 + 5^2$
58. $3^2 + 7^2$
59. $(4 + 5)^2$
60. $(3 + 7)^2$
61. $3(1 + 9 \cdot 6) - 4^2$
62. $5(2 + 8 \cdot 4) - 7^2$
63. 2[1 + 3(10 − 2)]
64. 5[2 + 4(3 − 2)]

Everyday Math

65. **Basketball** In the 2014 NBA playoffs, the San Antonio Spurs beat the Miami Heat. The table below shows the heights of the starters on each team. Use this table to fill in the appropriate symbol (=, <, >).

Spurs	Height	Heat	Height
Tim Duncan	83″	Rashard Lewis	82"
Boris Diaw	80"	LeBron James	80"
Kawhi Leonard	79"	Chris Bosh	83"
Tony Parker	74"	Dwyane Wade	76"
Danny Green	78"	Ray Allen	77"

a. Height of Tim Duncan____Height of Rashard Lewis
b. Height of Boris Diaw____Height of LeBron James
c. Height of Kawhi Leonard____Height of Chris Bosh
d. Height of Tony Parker____Height of Dwyane Wade
e. Height of Danny Green____Height of Ray Allen

66. **Elevation** In Colorado there are more than 50 mountains with an elevation of over 14,000 feet. The table shows the ten tallest. Use this table to fill in the appropriate inequality symbol.

Mountain	Elevation

Mountain	Elevation
Mt. Elbert	14,433′
Mt. Massive	14,421′
Mt. Harvard	14,420′
Blanca Peak	14,345′
La Plata Peak	14,336′
Uncompahgre Peak	14,309′
Crestone Peak	14,294′
Mt. Lincoln	14,286′
Grays Peak	14,270′
Mt. Antero	14,269′

a. Elevation of La Plata Peak____Elevation of Mt. Antero
b. Elevation of Blanca Peak____Elevation of Mt. Elbert
c. Elevation of Gray's Peak____Elevation of Mt. Lincoln
d. Elevation of Mt. Massive____Elevation of Crestone Peak
e. Elevation of Mt. Harvard____Elevation of Uncompahgre Peak

Writing Exercises

67. Explain the difference between an expression and an equation.
68. Why is it important to use the order of operations to simplify an expression?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use variables and algebraic symbols.			
identify expressions and equations.			
simplify expressions with exponents.			
simplify expressions using the order of operations.			

(b) If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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1.10: Evaluate, Simplify, and Translate Expressions (Part 1)

Learning Objectives

- Evaluate algebraic expressions
- Identify terms, coefficients, and like terms
- Simplify expressions by combining like terms
- Translate word phrases to algebraic expressions

Be prepared!

Before you get started, take this readiness quiz.

1. Is $n \div 5$ an expression or an equation? If you missed this problem, review Example 2.1.4.
2. Simplify 4^5. If you missed this problem, review Example 2.1.6.
3. Simplify $1 + 8 \bullet 9$. If you missed this problem, review Example 2.1.8.

Evaluate Algebraic Expressions

In the last section, we simplified expressions using the order of operations. In this section, we'll evaluate expressions—again following the order of operations.

To **evaluate** an algebraic expression means to find the value of the expression when the variable is replaced by a given number. To evaluate an expression, we substitute the given number for the variable in the expression and then simplify the expression using the order of operations.

Example 1.10.1: evaluate

Evaluate $x + 7$ when

a. $x = 3$
b. $x = 12$

Solution

a. To evaluate, substitute 3 for x in the expression, and then simplify.

	$x + 7$
Substitute.	$3 + 7$
Add.	10

When $x = 3$, the expression $x + 7$ has a value of 10.

b. To evaluate, substitute 12 for x in the expression, and then simplify.

	$x + 7$
Substitute.	$12 + 7$
Add.	19

When $x = 12$, the expression $x + 7$ has a value of 19. Notice that we got different results for parts (a) and (b) even though we started with the same expression. This is because the values used for x were different. When we evaluate an expression, the value varies depending on the value used for the variable.

exercise 1.10.1

Evaluate: $y+4$ when

a. $y=6$
b. $y=15$

Answer a

10

Answer b

19

exercise 1.10.2

Evaluate: $a-5$ when

a. $a=9$
b. $a=17$

Answer a

4

Answer b

12

Example 1.10.2

Evaluate $9x-2$, when

a. $x=5$
b. $x=1$

Solution

Remember ab means a times b, so $9x$ means 9 times x.

a. To evaluate the expression when $x=5$, we substitute 5 for x, and then simplify.

	$9x-2$
Substitute 5 for x.	$9\cdot 5-2$
Multiply.	$45-2$
Subtract.	43

b. To evaluate the expression when $x=1$, we substitute 1 for x, and then simplify.

	$9x-2$
Substitute 1 for x.	$9\cdot 1-2$
Multiply.	$9-2$
Subtract.	7

Notice that in part (a) that we wrote $9\bullet 5$ and in part (b) we wrote $9(1)$. Both the dot and the parentheses tell us to multiply.

exercise 1.10.3

Evaluate: $8x - 3$, when

a. $x = 2$
b. $x = 1$

Answer a

13

Answer b

5

exercise 1.10.4

Evaluate: $4y - 4$, when

a. $y = 3$
b. $y = 5$

Answer a

8

Answer b

16

Example 1.10.3: evaluate

Evaluate x^2 when $x = 10$.

Solution

We substitute 10 for x, and then simplify the expression.

	x^2
Substitute 10 for x.	10^2
Use the definition of exponent.	$10 \cdot 10$
Multiply	100

When $x = 10$, the expression x^2 has a value of 100.

exercise 1.10.5

Evaluate: x^2 when $x = 8$.

Answer

64

exercise 1.10.6

Evaluate: x^3 when $x = 6$.

Answer

216

Example 1.10.4: evaluate

Evaluate 2^x when $x = 5$.

Solution

In this expression, the variable is an exponent.

	2^x
Substitute 5 for x.	2^5
Use the definition of exponent.	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
Multiply	32

When $x = 5$, the expression 2^x has a value of 32.

exercise 1.10.7

Evaluate: 2^x when $x = 6$.

Answer

64

exercise 1.10.8

Evaluate: 3^x when $x = 4$.

Answer

81

Example 1.10.5: evaluate

Evaluate $3x + 4y - 6$ when $x = 10$ and $y = 2$.

Solution

This expression contains two variables, so we must make two substitutions.

	$3x + 4y - 6$
Substitute 10 for x and 2 for y.	$3(10) + 4(2) - 6$
Multiply.	$30 + 8 - 6$
Add and subtract left to right.	32

When $x = 10$ and $y = 2$, the expression $3x + 4y - 6$ has a value of 32.

exercise 1.10.9

Evaluate: $2x + 5y - 4$ when $x = 11$ and $y = 3$

Answer

33

exercise 1.10.10

Evaluate: $5x - 2y - 9$ when $x = 7$ and $y = 8$

Answer

10

Example 1.10.6: evaluate

Evaluate $2x^2 + 3x + 8$ when $x = 4$.

Solution

We need to be careful when an expression has a variable with an exponent. In this expression, $2x^2$ means $2 \bullet x \bullet x$ and is different from the expression $(2x)^2$, which means $2x \bullet 2x$.

	$2x^2 + 3x + 8$
Substitute 4 for each x.	$2(4)^2 + 3(4) + 8$
Simplify 4^2.	$2(16) + 3(4) + 8$
Multiply.	$32 + 12 + 8$
Add.	52

exercise 1.10.11

Evaluate: $3x^2 + 4x + 1$ when $x = 3$.

Answer

40

exercise 1.10.12

Evaluate: $6x^2 - 4x - 7$ when $x = 2$.

Answer

9

Identify Terms, Coefficients, and Like Terms

Algebraic expressions are made up of *terms*. A **term** is a constant or the product of a constant and one or more variables. Some examples of terms are 7, y, $5x^2$, $9a$, and $13xy$.

The constant that multiplies the variable(s) in a term is called the **coefficient**. We can think of the coefficient as the number *in front of* the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1 \bullet x$. Table 1.10.1 gives the coefficients for each of the terms in the left column.

Table 1.10.1

Term	Coefficient
7	7
9a	9
y	1

Term	Coefficient
$5x^2$	5

An algebraic expression may consist of one or more terms added or subtracted. In this chapter, we will only work with terms that are added together. Table $1.10.2$ gives some examples of algebraic expressions with various numbers of terms. Notice that we include the operation before a term with it.

Table 1.10.2

Expression	Terms
7	7
y	y
x + 7	x, 7
2x + 7y + 4	2x, 7y, 4
$3x^2 + 4x^2 + 5y + 3$	$3x^2$, $4x^2$, 5y, 3

Example 1.10.7:

Identify each term in the expression $9b+15x^2+a+6$. Then identify the coefficient of each term.

Solution

The expression has four terms. They are $9b$, $15x^2$, a, and 6.

The coefficient of $9b$ is 9.

The coefficient of $15x^2$ is 15.

Remember that if no number is written before a variable, the coefficient is 1. So the coefficient of a is 1.

The coefficient of a constant is the constant, so the coefficient of 6 is 6.

exercise 1.10.13

Identify all terms in the given expression, and their coefficients: $4x+3b+2$

Answer

The terms are $4x, 3b,$ and 2. The coefficients are $4, 3,$ and 2.

exercise 1.10.14

Identify all terms in the given expression, and their coefficients: $9a+13a^2+a^3$

Answer

The terms are $9a, 13a^2,$ and a^3, The coefficients are $9, 13,$ and 1.

Some terms share common traits. Look at the following terms. Which ones seem to have traits in common?

$5x, 7, n^2, 4, 3x, 9n^2$

Which of these terms are like terms?

- The terms 7 and 4 are both constant terms.
- The terms $5x$ and $3x$ are both terms with x.
- The terms n^2 and $9n^2$ both have n^2.

Terms are called like terms if they have the same variables and exponents. All constant terms are also like terms. So among the terms $5x, 7, n^2, 4, 3x, 9n^2, 7$and 4 are like terms, $5x$ and $3x$ are like terms, and n^2 and $9n^2$ are like terms.

Definition: Like terms

Terms that are either constants or have the same variables with the same exponents are like terms.

Example 1.10.8: identify

Identify the like terms:

a. $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$
b. $4x^2 + 2x + 5x^2 + 6x + 40x + 8xy$

Solution

a. $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$

Look at the variables and exponents. The expression contains y^3, x^2, x, and constants. The terms y^3 and $4y^3$ are like terms because they both have y^3. The terms $7x^2$ and $5x^2$ are like terms because they both have x^2. The terms 14 and 23 are like terms because they are both constants. The term $9x$ does not have any like terms in this list since no other terms have the variable x raised to the power of 1.

b. $4x^2 + 2x + 5x^2 + 6x + 40x + 8xy$

Look at the variables and exponents. The expression contains the terms $4x^2, 2x, 5x^2, 6x, 40x$and $8xy$The terms $4x^2$ and $5x^2$ are like terms because they both have x^2. The terms $2x, 6x$, and $40x$ are like terms because they all have x. The term $8xy$ has no like terms in the given expression because no other terms contain the two variables xy.

exercise 1.10.15

Identify the like terms in the list or the expression: $9, 2x^3, y^2, 8x^3, 15, 9y, 11y^2$

Answer

$9, 15$; $2x^3$ and $8x^3$, y^2, and $11y^2$

exercise 1.10.16

Identify the like terms in the list or the expression: $4x^3 + 8x^2 + 19 + 3x^2 + 24 + 6x^3$

Answer

$4x^3$ and $6x^3$; $8x^2$ and $3x^2$; 19 and 24

Simplify Expressions by Combining Like Terms

We can simplify an expression by combining the like terms. What do you think $3x + 6x$ would simplify to? If you thought $9x$, you would be right!

We can see why this works by writing both terms as addition problems.

$$\overbrace{x+x+x}^{3x} + \overbrace{x+x+x+x+x+x}^{6x}$$
$$+$$
$$9x$$

Add the coefficients and keep the same variable. It doesn't matter what x is. If you have 3 of something and add 6 more of the same thing, the result is 9 of them. For example, 3 oranges plus 6 oranges is 9 oranges. We will discuss the mathematical properties behind this later.

The expression $3x + 6x$ has only two terms. When an expression contains more terms, it may be helpful to rearrange the terms so that like terms are together. The Commutative Property of Addition says that we can change the order of addends without changing the sum. So we could rearrange the following expression before combining like terms.

$$3x + 4y - 2x + 6y$$
$$3x - 2x + 4y + 6y$$

Now it is easier to see the like terms to be combined.

HOW TO: COMBINE LIKE TERMS

Step 1. Identify like terms.

Step 2. Rearrange the expression so like terms are together.

Step 3. Add the coefficients of the like terms.

Example 1.10.9: simplify

Simplify the expression: $3x + 7 + 4x + 5$.

Solution

	$3x + 7 + 4x + 5$
Identify the like terms	$3x + 7 + 4x + 5$
Rearrange the expression, so the like terms are together.	$3x + 4x + 7 + 5$
Add the coefficients of the like terms.	$7x + 12$
The original expression is simplified to...	$7x + 12$

exercise 1.10.17

Simplify: $7x + 9 + 9x + 8$

Answer

$16x + 17$

exercise 1.10.18

Simplify: $5y + 2 + 8y + 4y + 5$

Answer

$17y + 7$

Example 1.10.10: simplify

Simplify the expression: $7x^2 + 8x + x^2 + 4x$.

Solution

	$7x^2 + 8x + x^2 + 4x$
Identify the like terms.	$7x^2 + 8x + x^2 + 4x$
Rearrange the expression so like terms are together.	$7x^2 + x^2 + 8x + 4x$

Add the coefficients of the like terms.	$8x^2 + 12x$

These are not like terms and cannot be combined. So $8x^2 + 12x$ is in simplest form.

? exercise 1.10.19

Simplify: $3x^2 + 9x + x^2 + 5x$

Answer

$4x^2 + 14x$

? exercise 1.10.20

Simplify: $11y^2 + 8y + y^2 + 7y$

Answer

$12y^2 + 15y$

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1.11: Evaluate, Simplify, and Translate Expressions (Part 2)

Translate Words to Algebraic Expressions

In the previous section, we listed many operation symbols that are used in algebra, and then we translated expressions and equations into word phrases and sentences. Now we'll reverse the process and translate word phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. They are summarized in Table 1.11.3.

Table 1.11.3

Operation	Phrase	Expression
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	a + b
Subtraction	a minus b the difference of a and b b subtracted from a a decreased by b b less than a	a - b
Multiplication	a times b the product of a and b	a • b, ab, a(b), (a)(b)
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	a ÷ b, a / b, $\frac{a}{b}$, $b\overline{)a}$

Look closely at these phrases using the four operations:

- the sum of a and b
- the difference of a and b
- the product of a and b
- the quotient of a and b

Each phrase tells you to operate on two numbers. Look for the words ***of*** and ***and*** to find the numbers.

Example 1.11.11: translate

Translate each word phrase into an algebraic expression:

a. the difference of 20 and 4
b. the quotient of $10x$ and 3

Solution

a. The key word is *difference*, which tells us the operation is subtraction. Look for the words *of* and *and* to find the numbers to subtract.

$$\text{the difference of } 20 \text{ and } 4$$

$$20 \text{ minus } 4$$

$$20 - 4$$

b. The key word is *quotient*, which tells us the operation is division.

the quotient of $10x$ and 3

divide $10x$ by 3

$$10x \div 3$$

This can also be written as $10x/3$ or $\frac{10x}{3}$

? exercise 1.11.21

Translate the given word phrase into an algebraic expression:

a. the difference of 47 and 41
b. the quotient of $5x$ and 2

Answer a

$47 - 41$

Answer b

$5x \div 2$

? exercise 1.11.22

Translate the given word phrase into an algebraic expression:

a. the sum of 17 and 19
b. the product of 7 and x

Answer a

$17 + 19$

Answer b

$7x$

How old will you be in eight years? What age is eight more years than your age now? Did you add 8 to your present age? Eight *more than* means eight added to your present age.

How old were you seven years ago? This is seven years less than your age now. You subtract 7 from your present age. Seven *less than* means seven subtracted from your present age.

✓ Example 1.11.12: translate

Translate each word phrase into an algebraic expression:

a. Eight more than y
b. Seven less than $9z$

Solution

a. The key words are more than. They tell us the operation is addition. More than means "added to".

Eight more than y

Eight added to y

$$y + 8$$

b. The key words are less than. They tell us the operation is subtraction. Less than means "subtracted from".

Seven less than $9z$

Seven subtracted from $9z$

$$9z-7$$

exercise 1.11.23

Translate each word phrase into an algebraic expression:

a. Eleven more than x
b. Fourteen less than $11a$

Answer a

$x+11$

Answer b

$11a-14$

exercise 1.11.24

Translate each word phrase into an algebraic expression:

a. 19 more than j
b. 21 less than $2x$

Answer a

$j+19$

Answer b

$2x-21$

Example 1.11.13: translate

Translate each word phrase into an algebraic expression:

a. five times the sum of m and n
b. the sum of five times m and n

Solution

a. There are two operation words: times tells us to multiply and sum tells us to add. Because we are multiplying 5 times the sum, we need parentheses around the sum of m and n.

$$\text{five times the sum of } m \text{ and } n$$
$$5(m+n)$$

b. To take a sum, we look for the words of and and to see what is being added. Here we are taking the sum of five times m and n.

$$\text{the sum of five times } m \text{ and } n$$
$$5m+n$$

Notice how the use of parentheses changes the result. In part (a), we add first and in part (b), we multiply first.

exercise 1.11.25

Translate the word phrase into an algebraic expression:

a. four times the sum of p and q
b. the sum of four times p and q

Answer a

$4(p+q)$

Answer b

$4p+q$

exercise 1.11.26

Translate the word phrase into an algebraic expression:

a. the difference of two times x and 8
b. two times the difference of x and 8

Answer a

$2x-8$

Answer b

$2(x-8)$

Later in this course, we'll apply our skills in algebra to solving equations. We'll usually start by translating a word phrase to an algebraic expression. We'll need to be clear about what the expression will represent. We'll see how to do this in the next two examples.

Example 1.11.14: write an expression

The height of a rectangular window is 6 inches less than the width. Let w represent the width of the window. Write an expression for the height of the window.

Solution

Write a phrase about the height.	6 less than the width
Substitute w for the width.	6 less than w
Rewrite 'less than' as 'subtracted from'.	6 subtracted from w
Translate the phrase into algebra.	w - 6

exercise 1.11.27

The length of a rectangle is 5 inches less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Answer

$w-5$

exercise 1.11.28

The width of a rectangle is 2 meters greater than the length. Let l represent the length of the rectangle. Write an expression for the width of the rectangle.

Answer

$l+2$

Example 1.11.15: write an expression

Blanca has dimes and quarters in her purse. The number of dimes is 2 less than 5 times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution

Write a phrase about the number of dimes.	two less than five times the number of quarters
Substitute q for the number of quarters.	2 less than five times q
Translate 5 *times* q.	2 less than 5q
Translate the phrase into algebra.	5q - 2

exercise 1.11.29

Geoffrey has dimes and quarters in his pocket. The number of dimes is seven less than six times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Answer

$6q-7$

exercise 1.11.30

Lauren has dimes and nickels in her purse. The number of dimes is eight more than four times the number of nickels. Let n represent the number of nickels. Write an expression for the number of dimes.

Answer

$4n+8$

Access Additional Online Resources

- Algebraic Expression Vocabulary

Key Concepts

- Combine like terms.
  - Identify like terms.
  - Rearrange the expression so like terms are together.
  - Add the coefficients of the like terms

Glossary

term

A term is a constant or the product of a constant and one or more variables.

coefficient

The constant that multiplies the variable(s) in a term is called the coefficient.

like terms

Terms that are either constants or have the same variables with the same exponents are like terms.

evaluate

To evaluate an algebraic expression means to find the value of the expression when the variable is replaced by a given number.

Practice Makes Perfect

Evaluate Algebraic Expressions

In the following exercises, evaluate the expression for the given value.

69. $7x + 8$ when $x = 2$
70. $9x + 7$ when $x = 3$
71. $5x - 4$ when $x = 6$
72. $8x - 6$ when $x = 7$
73. x^2 when $x = 12$
74. x^3 when $x = 5$
75. x^5 when $x = 2$
76. x^4 when $x = 3$
77. 3^x when $x = 3$
78. 4^x when $x = 2$
79. $x^2 + 3x - 7$ when $x = 4$
80. $x^2 + 5x - 8$ when $x = 6$
81. $2x + 4y - 5$ when $x = 7, y = 8$
82. $6x + 3y - 9$ when $x = 6, y = 9$
83. $(x - y)^2$ when $x = 10, y = 7$
84. $(x + y)^2$ when $x = 6, y = 9$
85. $a^2 + b^2$ when $a = 3, b = 8$
86. $r^2 - s^2$ when $r = 12, s = 5$
87. $2l + 2w$ when $l = 15, w = 12$
88. $2l + 2w$ when $l = 18, w = 14$

Identify Terms, Coefficients, and Like Terms

In the following exercises, list the terms in the given expression.

89. $15x^2 + 6x + 2$
90. $11x^2 + 8x + 5$
91. $10y^3 + y + 2$
92. $9y^3 + y + 5$

In the following exercises, identify the coefficient of the given term.

93. $8a$
94. $13m$
95. $5r^2$
96. $6x^3$

In the following exercises, identify all sets of like terms.

97. x^3, $8x$, 14, $8y$, 5, $8x^3$
98. $6z$, $3w^2$, 1, $6z^2$, $4z$, w^2
99. $9a$, a^2, $16ab$, $16b^2$, $4ab$, $9b^2$
100. 3, $25r^2$, $10s$, $10r$, $4r^2$, $3s$

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the given expression by combining like terms.

101. $10x + 3x$
102. $15x + 4x$
103. $17a + 9a$
104. $18z + 9z$
105. $4c + 2c + c$
106. $6y + 4y + y$

107. $9x + 3x + 8$
108. $8a + 5a + 9$
109. $7u + 2 + 3u + 1$
110. $8d + 6 + 2d + 5$
111. $7p + 6 + 5p + 4$
112. $8x + 7 + 4x - 5$
113. $10a + 7 + 5a - 2 + 7a - 4$
114. $7c + 4 + 6c - 3 + 9c - 1$
115. $3x^2 + 12x + 11 + 14x^2 + 8x + 5$
116. $5b^2 + 9b + 10 + 2b^2 + 3b - 4$

Translate English Phrases into Algebraic Expressions

In the following exercises, translate the given word phrase into an algebraic expression.

117. The sum of 8 and 12
118. The sum of 9 and 1
119. The difference of 14 and 9
120. 8 less than 19
121. The product of 9 and 7
122. The product of 8 and 7
123. The quotient of 36 and 9
124. The quotient of 42 and 7
125. The difference of x and 4
126. 3 less than x
127. The product of 6 and y
128. The product of 9 and y
129. The sum of 8x and 3x
130. The sum of 13x and 3x
131. The quotient of y and 3
132. The quotient of y and 8
133. Eight times the difference of y and nine
134. Seven times the difference of y and one
135. Five times the sum of x and y
136. Nine times five less than twice x

In the following exercises, write an algebraic expression.

137. Adele bought a skirt and a blouse. The skirt cost $15 more than the blouse. Let b represent the cost of the blouse. Write an expression for the cost of the skirt.
138. Eric has rock and classical CDs in his car. The number of rock CDs is 3 more than the number of classical CDs. Let c represent the number of classical CDs. Write an expression for the number of rock CDs.
139. The number of girls in a second-grade class is 4 less than the number of boys. Let b represent the number of boys. Write an expression for the number of girls.
140. Marcella has 6 fewer male cousins than female cousins. Let f represent the number of female cousins. Write an expression for the number of boy cousins.
141. Greg has nickels and pennies in his pocket. The number of pennies is seven less than twice the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.
142. Jeannette has $5 and $10 bills in her wallet. The number of fives is three more than six times the number of tens. Let t represent the number of tens. Write an expression for the number of fives.

Everyday Math

In the following exercises, use algebraic expressions to solve the problem.

143. **Car insurance** Justin's car insurance has a $750 deductible per incident. This means that he pays $750 and his insurance company will pay all costs beyond $750. If Justin files a claim for $2,100, how much will he pay, and how much will his

insurance company pay?

144. **Home insurance** Pam and Armando's home insurance has a $2,500 deductible per incident. This means that they pay $2,500 and their insurance company will pay all costs beyond $2,500. If Pam and Armando file a claim for $19,400, how much will they pay, and how much will their insurance company pay?

Writing Exercises

145. Explain why "the sum of x and y" is the same as "the sum of y and x," but "the difference of x and y" is not the same as "the difference of y and x." Try substituting two random numbers for x and y to help you explain. 146. Explain the difference between "4 times the sum of x and y" and "the sum of 4 times x and y."

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate algebraic expressions.			
identify terms, coefficients, and like terms.			
simplify expressions by combining like terms.			
translate word phrases to algebraic expressions.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

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1.12: Solving Equations Using the Subtraction and Addition Properties of Equality (Part 1)

Learning Objectives

- Determine whether a number is a solution of an equation
- Model the Subtraction Property of Equality
- Solve equations using the Subtraction Property of Equality
- Solve equations using the Addition Property of Equality
- Translate word phrases to algebraic equations
- Translate to an equation and solve

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $x+8$ when $x=11$. If you missed this problem, review Example 2..2.1.
2. Evaluate $5x-3$ when $x=9$. If you missed this problem, review Example 2.2.2.
3. Translate into algebra: the difference of x and 8. If you missed this problem, review Example 2.2.11.

When some people hear the word *algebra*, they think of solving equations. The applications of solving equations are limitless and extend to all careers and fields. In this section, we will begin solving equations. We will start by solving basic equations, and then as we proceed through the course we will build up our skills to cover many different forms of equations.

Determine Whether a Number is a Solution of an Equation

Solving an equation is like discovering the answer to a puzzle. An algebraic equation states that two algebraic expressions are equal. To solve an equation is to determine the values of the variable that make the equation a true statement. Any number that makes the equation true is called a **solution** of the equation. It is the answer to the puzzle!

Definition: Solution of an Equation

A solution to an equation is a value of a variable that makes a true statement when substituted into the equation. The process of finding the solution to an equation is called solving the equation.

To find the solution to an equation means to find the value of the variable that makes the equation true. Can you recognize the solution of $x+2=7$? If you said 5, you're right! We say 5 is a solution to the equation $x+2=7$ because when we substitute 5 for x the resulting statement is true.

$$\begin{aligned} x+2 &= 7 \\ 5+2 &\stackrel{?}{=} 7 \\ 7 &= 7 \checkmark \end{aligned}$$

Since $5+2=7$ is a true statement, we know that 5 is indeed a solution to the equation. The symbol $\stackrel{?}{=}$ asks whether the left side of the equation is equal to the right side. Once we know, we can change to an equal sign ($=$) or not-equal sign ($\neq$).

HOW TO: DETERMINE WHETHER A NUMBER IS A Solution TO AN EQUATION

Step 1. Substitute the number for the variable in the equation.

Step 2. Simplify the expressions on both sides of the equation.

Step 3. Determine whether the resulting equation is true.

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

Example 1.12.1

Determine whether $x = 5$ is a solution of $6x - 17 = 16$.

Solution

	$6x - 17 = 16$
Substitute 5 for x.	$6 \cdot 5 - 17 \stackrel{?}{=} 16$
Multiply.	$30 - 17 \stackrel{?}{=} 16$
Subtract	$13 \neq 16$

So $x = 5$ is not a solution to the equation $6x - 17 = 16$.

exercise 1.12.1

Is $x = 3$ a solution of $4x - 7 = 16$?

Answer

no

exercise 1.12.2

Is $x = 2$ a solution of $6x - 2 = 10$?

Answer

yes

Example 1.12.2

Determine whether $y = 2$ is a solution of $6y - 4 = 5y - 2$.

Solution

Here, the variable appears on both sides of the equation. We must substitute 2 for each y.

	$6y - 4 = 5y - 2$
Substitute 2 for y.	$6(2) - 4 \stackrel{?}{=} 5(2) - 2$
Multiply.	$12 - 4 \stackrel{?}{=} 10 - 2$
Subtract	$8 = 8$ ✓

Since $y = 2$ results in a true equation, we know that 2 is a solution to the equation $6y - 4 = 5y - 2$.

exercise 1.12.3

Is $y = 3$ a solution of $9y - 2 = 8y + 1$?

Answer

yes

? exercise 1.12.4

Is $y = 4$ a solution of $5y - 3 = 3y + 5$?

Answer

yes

Model the Subtraction Property of Equality

We will use a model to help you understand how the process of solving an equation is like solving a puzzle. An envelope represents the variable – since its contents are unknown – and each counter represents one.

Suppose a desk has an imaginary line dividing it in half. We place three counters and an envelope on the left side of desk, and eight counters on the right side of the desk as in Figure 1.12.1. Both sides of the desk have the same number of counters, but some counters are hidden in the envelope. Can you tell how many counters are in the envelope?

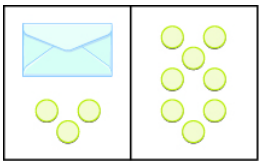

Figure 1.12.1

What steps are you taking in your mind to figure out how many counters are in the envelope? Perhaps you are thinking "I need to remove the 3 counters from the left side to get the envelope by itself. Those 3 counters on the left match with 3 on the right, so I can take them away from both sides. That leaves five counters on the right, so there must be 5 counters in the envelope." Figure 1.12.2 shows this process.

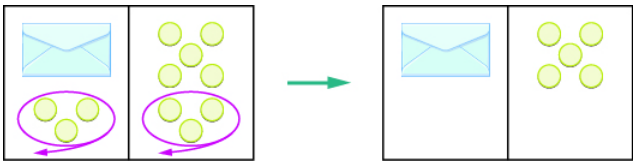

Figure 1.12.2

What algebraic equation is modeled by this situation? Each side of the desk represents an expression and the center line takes the place of the equal sign. We will call the contents of the envelope x, so the number of counters on the left side of the desk is $x + 3$. On the right side of the desk are 8 counters. We are told that $x + 3$ is equal to 8 so our equation is $x + 3 = 8$.

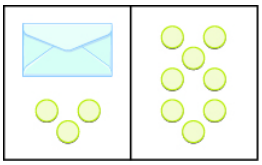

Figure 1.12.3

$$x + 3 = 8$$

Let's write algebraically the steps we took to discover how many counters were in the envelope.

	$x + 3 = 8$
First, we took away three from each side.	$x + 3 - 3 = 8 - 3$
Then we were left with five.	$x = 5$

Now let's check our solution. We substitute 5 for x in the original equation and see if we get a true statement.

$$x+3=8$$
$$5+3\stackrel{?}{=}8$$
$$8=8\ \checkmark$$

Our solution is correct. Five counters in the envelope plus three more equals eight

Example 1.12.3

Write an equation modeled by the envelopes and counters, and then solve the equation:

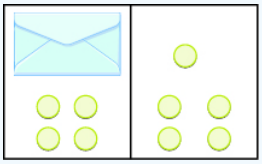

Figure 1.12.4

Solution

On the left, write x for the contents of the envelope, add the 4 counters, so we have x + 4.	x + 4
On the right, there are 5 counters.	5
The two sides are equal.	x + 4 = 5
Solve the equation by subtracting 4 counters from each side.	

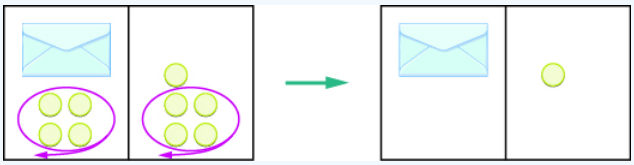

Figure 1.12.5

We can see that there is one counter in the envelope. This can be shown algebraically as:

$$x+4=5$$
$$x+4-4=5-4$$
$$x=1$$

Substitute 1 for x in the equation to check.

$$x+4=5$$
$$1+4\stackrel{?}{=}5$$
$$5=5\ \checkmark$$

Since $x=1$ makes the statement true, we know that 1 is indeed a solution.

Exercise 1.12.5

Write the equation modeled by the envelopes and counters, and then solve the equation:

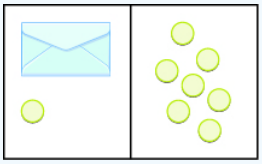

Figure 1.12.6

Answer

$x+1=7; x=6$

Exercise 1.12.6

Write the equation modeled by the envelopes and counters, and then solve the equation:

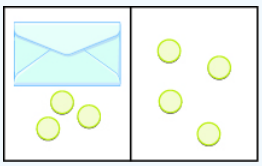

Figure 1.12.7

Answer

$x+3=4; x=1$

Solve Equations Using the Subtraction Property of Equality

Our puzzle has given us an idea of what we need to do to solve an equation. The goal is to isolate the variable by itself on one side of the equations. In the previous examples, we used the Subtraction Property of Equality, which states that when we subtract the same quantity from both sides of an equation, we still have equality.

Definition: Subtraction Property of Equality

For any numbers a, b, and c, if $a=b$ then $a-c=b-c$

Think about twin brothers Andy and Bobby. They are 17 years old. How old was Andy 3 years ago? He was 3 years less than 17, so his age was $17-3$, or 14. What about Bobby's age 3 years ago? Of course, he was 14 also. Their ages are equal now, and subtracting the same quantity from both of them resulted in equal ages 3 years ago.

$$\begin{aligned} a &= b \\ a-3 &= b-3 \end{aligned}$$

HOW TO: SOLVE AN EQUATION USING THE SUBTRACTION PROPERTY OF EQUALITY

Step 1. Use the Subtraction Property of Equality to isolate the variable.

Step 2. Simplify the expressions on both sides of the equation.

Step 3. Check the solution.

Example 1.12.4: solve

Solve: $x+8=17$.

Solution

We will use the Subtraction Property of Equality to isolate x.

	$x+8=17$
Subtract 8 from both sides.	$x+8-8=17-8$
Simplify.	$x=9$

Check the solution.	$9+8=17$
	$17=17$ ✓

Since $x=9$ makes $x+8=17$ a true statement, we know 9 is the solution to the equation.

Exercise 1.12.7

Solve: $x+6=19$

Answer

$x=13$

Exercise 1.12.8

Solve: $x+9=14$

Answer

$x=5$

Example 1.12.5: solve

Solve: $100=y+74$.

Solution

To solve an equation, we must always isolate the variable—it doesn't matter which side it is on. To isolate y, we will subtract 74 from both sides.

	$100=y+74$
Subtract 74 from both sides.	$100-74=y+74-74$
Simplify.	$26=y$
Substitute 26 for y to check	$100\stackrel{?}{=}26+74$
	$100=100$ ✓

Since $y=26$ makes $100=y+74$ a true statement, we have found the solution to this equation.

Exercise 1.12.9

Solve: $95=y+67$

Answer

$y=28$

Exercise 1.12.10

Solve: $91=y+45$

Answer

$y=46$

Solve Equations Using the Addition Property of Equality

In all the equations we have solved so far, a number was added to the variable on one side of the equation. We used subtraction to "undo" the addition in order to isolate the variable.

But suppose we have an equation with a number subtracted from the variable, such as $x-5=8$. We want to isolate the variable, so to "undo" the subtraction we will add the number to both sides.

We use the Addition Property of Equality, which says we can add the same number to both sides of the equation without changing the equality. Notice how it mirrors the Subtraction Property of Equality.

Definition: Addition Property of Equality

For any numbers a, b, and c, if $a=b$ then $a+c=b+c$

Remember the 17-year-old twins, Andy and Bobby? In ten years, Andy's age will still equal Bobby's age. They will both be 27.

$$\begin{aligned} a &= b \\ a+10 &= b+10 \end{aligned}$$

We can add the same number to both sides and still keep the equality.

HOW TO: SOLVE AN EQUATION USING THE ADDITION PROPERTY OF EQUALITY

Step 1. Use the Addition Property of Equality to isolate the variable.

Step 2. Simplify the expressions on both sides of the equation.

Step 3. Check the solution.

Example 1.12.6: solve

Solve: $x-5=8$.

Solution

We will use the Addition Property of Equality to isolate the variable.

	$x-5=8$
Add 5 to both sides.	$x-5+5=8+5$
Simplify	$x=13$
Now we can check. Let x = 13.	$13-5\stackrel{?}{=}8$
	$8=8$ ✓

Exercise 1.12.11

Solve: $x-9=13$

Answer

$x=22$

Exercise 1.12.12

Solve: $y-1=3$

Answer

$y = 4$

Example 1.12.7: solve

Solve: $27 = a - 16$.

Solution

We will add 16 to each side to isolate the variable.

	$27 = a - 16$
Add 16 to each side.	$27 + 16 = a - 16 + 16$
Simplify.	$43 = a$
Now we can check. Let a = 43.	$27 \stackrel{?}{=} 43 - 16$
	$27 = 27$ ✓

The solution to $27 = a - 16$ is $a = 43$.

Exercise 1.12.13

Solve: $19 = a - 18$

Answer

$a = 37$

Exercise 1.12.14

Solve: $27 = n - 14$

Answer

$n = 41$

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1.13: Solving Equations Using the Subtraction and Addition Properties of Equality (Part 2)

Translate Word Phrases to Algebraic Equations

Remember, an equation has an equal sign (=) between two algebraic expressions. So if we have a sentence that tells us that two phrases are equal, we can translate it into an equation. We look for clue words that mean equals. Some words that translate to the equal sign are:

- is equal to
- is the same as
- is
- gives
- was
- will be

It may be helpful to put a box around the equals word(s) in the sentence to help you focus separately on each phrase. Then translate each phrase into an expression, and write them on each side of the equal sign.

We will practice translating word sentences into algebraic equations. Some of the sentences will be basic number facts with no variables to solve for. Some sentences will translate into equations with variables. The focus right now is just to translate the words into algebra.

Example 1.13.8: translate

Translate the sentence into an algebraic equation: The sum of 6 and 9 is 15.

Solution

The word is tells us the equal sign goes between 9 and 15.

Locate the “equals” word(s).	The sum of 6 and 9 is 15
Write the = sign.	The sum of 6 and 9 = 15
Translate the words to the left of the *equals* word into an algebraic expression.	6 + 9 = _____
Translate the words to the right of the *equals* word into an algebraic expression.	6 + 9 = 15

Exercise 1.13.15

Translate the sentence into an algebraic equation: The sum of 7 and 6 gives 13.

Answer

$7+6=13$

Exercise 1.13.16

Translate the sentence into an algebraic equation: The sum of 8 and 6 is 14.

Answer

$8+6=14$

Example 1.13.9: translate

Translate the sentence into an algebraic equation: The product of 8 and 7 is 56.

Solution

The location of the word is tells us that the equal sign goes between 7 and 56.

Locate the “equals” word(s).	The product of 8 and 7 is 56
Write the = sign.	The product of 8 and 7 = 56
Translate the words to the left of the *equals* word into an algebraic expression.	8 • 7 = _____
Translate the words to the right of the *equals* word into an algebraic expression.	8 • 7 = 56

Exercise 1.13.17

Translate the sentence into an algebraic equation: The product of 6 and 9 is 54.

Answer

$6 \cdot 9 = 54$

Exercise 1.13.18

Translate the sentence into an algebraic equation: The product of 21 and 3 gives 63.

Answer

$21 \cdot 3 = 63$

Example 1.13.10: translate

Translate the sentence into an algebraic equation: Twice the difference of x and 3 gives 18.

Solution

Locate the “equals” word(s).	Twice the difference of x and 3 gives 18.
Recognize the key words: *twice*; *difference of … and …*	*Twice* means two times.
Translate.	Twice (2) the difference of x and 3 ($x - 3$) gives (=) 18.

Exercise 1.13.19

Translate the given sentence into an algebraic equation: Twice the difference of x and 5 gives 30.

Answer

$2(x - 5) = 30$

Exercise 1.13.20

Translate the given sentence into an algebraic equation: Twice the difference of y and 4 gives 16.

Answer

$2(y-4)=16$

Translate to an Equation and Solve

Now let's practice translating sentences into algebraic equations and then solving them. We will solve the equations by using the Subtraction and Addition Properties of Equality.

Example 1.13.11: translate and solve

Translate and solve: Three more than x is equal to 47.

Solution

	Three more than x is equal to 47.
Translate.	$x+3=47$
Subtract 3 from both sides of the equation.	$x+3-3=47-3$
Simplify	$x=44$
We can check. Let x = 44.	$44+3\stackrel{?}{=}47$
	$47=47$ ✓

So $x=44$ is the solution.

Exercise 1.13.21

Translate and solve: Seven more than x is equal to 37.

Answer

$x+7=37; x=30$

Exercise 1.13.22

Translate and solve: Eleven more than y is equal to 28.

Answer

$y+11=28; y=17$

Example 1.13.12: translate and solve

Translate and solve: The difference of y and 14 is 18.

Solution

	The difference of y and 14 is 18.
Translate.	$y-14=18$
Add 14 to both sides.	$y-14+14=18+14$

Simplify.	$y = 32$
We can check. Let y = 32.	$32 - 14 \stackrel{?}{=} 18$
	$18 = 18$ ✓

So $y = 32$ is the solution.

? Exercise 1.13.23

Translate and solve: The difference of z and 17 is equal to 37.

Answer

$z - 17 = 37; z = 54$

? Exercise 1.13.24

Translate and solve: The difference of x and 19 is equal to 45.

Answer

$x - 19 = 45; x = 64$

Access Additional Online Resources

- Solving One Step Equations By Addition and Subtraction

Key Concepts

- Determine whether a number is a solution to an equation.
  1. Substitute the number for the variable in the equation.
  2. Simplify the expressions on both sides of the equation.
  3. Determine whether the resulting equation is true. If it is true, the number is a solution.

  If it is not true, the number is not a solution.
- Subtraction Property of Equality
  - For any numbers a, b, and c,

if	a=b
then	a−c=b−c

- Solve an equation using the Subtraction Property of Equality.
  1. Use the Subtraction Property of Equality to isolate the variable.
  2. Simplify the expressions on both sides of the equation.
  3. Check the solution.
- Addition Property of Equality
  - For any numbers a, b, and c,

if	a=b
then	a+c=b+c

- Solve an equation using the Addition Property of Equality.
  1. Use the Addition Property of Equality to isolate the variable.
  2. Simplify the expressions on both sides of the equation.

3. Check the solution.

Glossary

solution of an equation

A solution to an equation is a value of a variable that makes a true statement when substituted into the equation. The process of finding the solution to an equation is called solving the equation.

Practice Makes Perfect

Determine Whether a Number is a Solution of an Equation

In the following exercises, determine whether each given value is a solution to the equation.

147. $x + 13 = 21$
   a. $x = 8$
   b. $x = 34$

148. $y + 18 = 25$
   a. $y = 7$
   b. $y = 43$

149. $m - 4 = 13$
   a. $m = 9$
   b. $m = 17$

150. $n - 9 = 6$
   a. $n = 3$
   b. $n = 15$

151. $3p + 6 = 15$
   a. $p = 3$
   b. $p = 7$

152. $8q + 4 = 20$
   a. $q = 2$
   b. $q = 3$

153. $18d - 9 = 27$
   a. $d = 1$
   b. $d = 2$

154. $24\,f - 12 = 60$
   a. $f = 2$
   b. $f = 3$

155. $8u - 4 = 4u + 40$
   a. $u = 3$
   b. $u = 11$

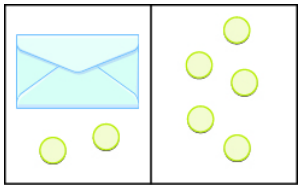
156. $7v - 3 = 4v + 36$
   a. $v = 3$
   b. $v = 11$

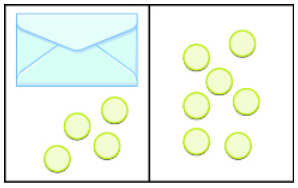
157. $20h - 5 = 15h + 35$
   a. $h = 6$
   b. $h = 8$

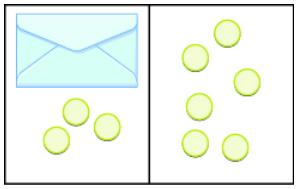
158. $18k - 3 = 12k + 33$
   a. $k = 1$
   b. $k = 6$

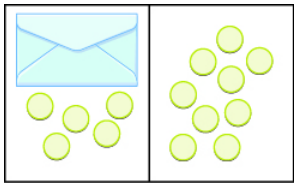
Model the Subtraction Property of Equality

In the following exercises, write the equation modeled by the envelopes and counters and then solve using the subtraction property of equality.

159. 

160. 

161. 

162. 

Solve Equations using the Subtraction Property of Equality

In the following exercises, solve each equation using the subtraction property of equality.

163. $a + 2 = 18$
164. $b + 5 = 13$
165. $p + 18 = 23$
166. $q + 14 = 31$
167. $r + 76 = 100$
168. $s + 62 = 95$
169. $16 = x + 9$
170. $17 = y + 6$
171. $93 = p + 24$
172. $116 = q + 79$
173. $465 = d + 398$
174. $932 = c + 641$

Solve Equations using the Addition Property of Equality

In the following exercises, solve each equation using the addition property of equality.

175. $y - 3 = 19$
176. $x - 4 = 12$
177. $u - 6 = 24$
178. $v - 7 = 35$
179. $f - 55 = 123$
180. $g - 39 = 117$
181. $19 = n - 13$
182. $18 = m - 15$

183. $10 = p - 38$
184. $18 = q - 72$
185. $268 = y - 199$
186. $204 = z - 149$

Translate Word Phrase to Algebraic Equations

In the following exercises, translate the given sentence into an algebraic equation.

187. The sum of 8 and 9 is equal to 17.
188. The sum of 7 and 9 is equal to 16.
189. The difference of 23 and 19 is equal to 4.
190. The difference of 29 and 12 is equal to 17.
191. The product of 3 and 9 is equal to 27.
192. The product of 6 and 8 is equal to 48.
193. The quotient of 54 and 6 is equal to 9.
194. The quotient of 42 and 7 is equal to 6.
195. Twice the difference of n and 10 gives 52.
196. Twice the difference of m and 14 gives 64.
197. The sum of three times y and 10 is 100.
198. The sum of eight times x and 4 is 68.

Translate to an Equation and Solve

In the following exercises, translate the given sentence into an algebraic equation and then solve it.

199. Five more than p is equal to 21.
200. Nine more than q is equal to 40.
201. The sum of r and 18 is 73.
202. The sum of s and 13 is 68.
203. The difference of d and 30 is equal to 52.
204. The difference of c and 25 is equal to 75.
205. 12 less than u is 89.
206. 19 less than w is 56.
207. 325 less than c gives 799.
208. 299 less than d gives 850.

Everyday Math

209. **Insurance** Vince's car insurance has a $500 deductible. Find the amount the insurance company will pay, p, for an $1800 claim by solving the equation $500 + p = 1800$.
210. **Insurance** Marta's homeowner's insurance policy has a $750 deductible. The insurance company paid $5800 to repair damages caused by a storm. Find the total cost of the storm damage, d, by solving the equation $d - 750 = 5800$.
211. **Sale purchase** Arthur bought a suit that was on sale for $120 off. He paid $340 for the suit. Find the original price, p, of the suit by solving the equation $p - 120 = 340$.
212. **Sale purchase** Rita bought a sofa that was on sale for $1299. She paid a total of $1409, including sales tax. Find the amount of the sales tax, t, by solving the equation $1299 + t = 1409$.

Writing Exercises

213. Is $x = 1$ a solution to the equation $8x - 2 = 16 - 6x$? How do you know?
214. Write the equation $y - 5 = 21$ in words. Then make up a word problem for this equation.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether a number is a solution of an equation.			
model the subtraction property of equality.			
solve equations using the subtraction property of equality.			
solve equations using the addition property of equality.			
translate word phrases to algebraic equations.			
translate to an equation and solve.			

(b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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1.E: Introduction to the Language of Algebra (Exercises)

2.1 - Use the Language of Algebra

Use Variables and Algebraic Symbols

In the following exercises, translate from algebra to English.

317. $3 \cdot 8$
318. $12 - x$
319. $24 \div 6$
320. $9 + 2a$
321. $50 \geq 47$
322. $3y < 15$
323. $n + 4 = 13$
324. $32 - k = 7$

Identify Expressions and Equations

In the following exercises, determine if each is an expression or equation.

325. $5 + u = 84$
326. $36 - 6s$
327. $4y - 11$
328. $10x = 120$

Simplify Expressions with Exponents

In the following exercises, write in exponential form.

329. $2 \cdot 2 \cdot 2$
330. $a \cdot a \cdot a \cdot a \cdot a$
331. $x \cdot x \cdot x \cdot x \cdot x \cdot x$
332. $10 \cdot 10 \cdot 10$

In the following exercises, write in expanded form.

333. 8^4
334. 3^6
335. y^5
336. n^4

In the following exercises, simplify each expression.

337. 3^4
338. 10^6
339. 2^7
340. 4^3

Simplify Expressions Using the Order of Operations

In the following exercises, simplify.

341. $10 + 2 \cdot 5$
342. $(10 + 2) \cdot 5$
343. $(30 + 6) \div 2$
344. $30 + 6 \div 2$
345. $7^2 + 5^2$
346. $(7 + 5)^2$
347. $4 + 3(10 - 1)$
348. $(4 + 3)(10 - 1)$

2.2 - Evaluate, Simplify, and Translate Expressions

Evaluate an Expression

In the following exercises, evaluate the following expressions.

349. $9x - 5$ when $x = 7$
350. y^3 when $y = 5$
351. $3a - 4b$ when $a = 10, b = 1$
352. bh when $b = 7, h = 8$

Identify Terms, Coefficients and Like Terms

In the following exercises, identify the terms in each expression.

353. $12n^2 + 3n + 1$
354. $4x^3 + 11x + 3$

In the following exercises, identify the coefficient of each term.

355. $6y$
356. $13x^2$

In the following exercises, identify the like terms.

357. $5x^2$, 3, $5y^2$, $3x$, x, 4
358. 8, $8r^2$, $8r$, $3r$, r^2, $3s$

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the following expressions by combining like terms.

359. $15a + 9a$
360. $12y + 3y + y$
361. $4x + 7x + 3x$
362. $6 + 5c + 3$
363. $8n + 2 + 4n + 9$
364. $19p + 5 + 4p - 1 + 3p$
365. $7y^2 + 2y + 11 + 3y^2 - 8$
366. $13x^2 - x + 6 + 5x^2 + 9x$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate the following phrases into algebraic expressions.

367. the difference of x and 6
368. the sum of 10 and twice a
369. the product of $3n$ and 9
370. the quotient of s and 4
371. 5 times the sum of y and 1
372. 10 less than the product of 5 and z
373. Jack bought a sandwich and a coffee. The cost of the sandwich was $3 more than the cost of the coffee. Call the cost of the coffee c. Write an expression for the cost of the sandwich.
374. The number of poetry books on Brianna's bookshelf is 5 less than twice the number of novels. Call the number of novels n. Write an expression for the number of poetry books.

2.3 - Solve Equations Using the Subtraction and Addition Properties of Equality

Determine Whether a Number is a Solution of an Equation

In the following exercises, determine whether each number is a solution to the equation.

375. $y + 16 = 40$

    a. 24

b. 56

376. $d - 6 = 21$

a. 15
b. 27

377. $4n + 12 = 36$

a. 6
b. 12

378. $20q - 10 = 70$

a. 3
b. 4

379. $15x - 5 = 10x + 45$

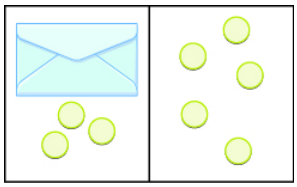
a. 2
b. 10

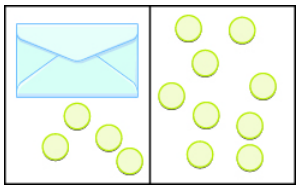
380. $22p - 6 = 18p + 86$

a. 4
b. 23

Model the Subtraction Property of Equality

In the following exercises, write the equation modeled by the envelopes and counters and then solve the equation using the subtraction property of equality

381. 

382. 

Solve Equations using the Subtraction Property of Equality

In the following exercises, solve each equation using the subtraction property of equality.

383. $c + 8 = 14$
384. $v + 8 = 150$
385. $23 = x + 12$
386. $376 = n + 265$

Solve Equations using the Addition Property of Equality

In the following exercises, solve each equation using the addition property of equality.

387. $y - 7 = 16$
388. $k - 42 = 113$
389. $19 = p - 15$
390. $501 = u - 399$

Translate English Sentences to Algebraic Equations

In the following exercises, translate each English sentence into an algebraic equation.

391. The sum of 7 and 33 is equal to 40.

392. The difference of 15 and 3 is equal to 12.
393. The product of 4 and 8 is equal to 32.
394. The quotient of 63 and 9 is equal to 7.
395. Twice the difference of n and 3 gives 76.
396. The sum of five times y and 4 is 89.

Translate to an Equation and Solve

In the following exercises, translate each English sentence into an algebraic equation and then solve it.

397. Eight more than x is equal to 35.
398. 21 less than a is 11.
399. The difference of q and 18 is 57.
400. The sum of m and 125 is 240.

Mixed Practice

In the following exercises, solve each equation.

401. $h - 15 = 27$
402. $k - 11 = 34$
403. $z + 52 = 85$
404. $x + 93 = 114$
405. $27 = q + 19$
406. $38 = p + 19$
407. $31 = v - 25$
408. $38 = u - 16$

2.4 - Find Multiples and Factors

Identify Multiples of Numbers

In the following exercises, list all the multiples less than 50 for each of the following.

409. 3
410. 2
411. 8
412. 10

Use Common Divisibility Tests

In the following exercises, using the divisibility tests, determine whether each number is divisible by 2, by 3, by 5, by 6, and by 10.

413. 96
414. 250
415. 420
416. 625

Find All the Factors of a Number

In the following exercises, find all the factors of each number.

417. 30
418. 70
419. 180
420. 378

Identify Prime and Composite Numbers

In the following exercises, identify each number as prime or composite.

421. 19
422. 51

423. 121

2.5 - Prime Factorization and the Least Common Multiple

Find the Prime Factorization of a Composite Number

In the following exercises, find the prime factorization of each number.

425. 84
426. 165
427. 350
428. 572

Find the Least Common Multiple of Two Numbers

In the following exercises, find the least common multiple of each pair of numbers.

429. 9, 15
430. 12, 20
431. 25, 35
432. 18, 40

Everyday Math

433. Describe how you have used two topics from The Language of Algebra chapter in your life outside of your math class during the past month.

PRACTICE TEST

In the following exercises, translate from an algebraic equation to English phrases.

434. $6 \cdot 4$
435. $15 - x$

In the following exercises, identify each as an expression or equation.

436. $5 \cdot 8 + 10$
437. $x + 6 = 9$
438. $3 \cdot 11 = 33$
439. (a) Write $n \cdot n \cdot n \cdot n \cdot n \cdot n$ in exponential form. (b) Write 3^5 in expanded form and then simplify.

In the following exercises, simplify, using the order of operations.

440. $4 + 3 \cdot 5$
441. $(8 + 1) \cdot 4$
442. $1 + 6(3 - 1)$
443. $(8 + 4) \div 3 + 1$
444. $(1 + 4)^2$
445. $5[2 + 7(9 - 8)]$

In the following exercises, evaluate each expression.

446. $8x - 3$ when $x = 4$
447. y^3 when $y = 5$
448. $6a - 2b$ when $a = 5, b = 7$
449. hw when $h = 12, w = 3$
450. Simplify by combining like terms.
    a. $6x + 8x$
    b. $9m + 10 + m + 3$

In the following exercises, translate each phrase into an algebraic expression.

451. 5 more than x

452. the quotient of 12 and y
453. three times the difference of a and b
454. Caroline has 3 fewer earrings on her left ear than on her right ear. Call the number of earrings on her right ear, r. Write an expression for the number of earrings on her left ear.

In the following exercises, solve each equation.

455. n − 6 = 25
456. x + 58 = 71

In the following exercises, translate each English sentence into an algebraic equation and then solve it.

457. 15 less than y is 32.
458. the sum of a and 129 is 164.
459. List all the multiples of 4, that are less than 50.
460. Find all the factors of 90.
461. Find the prime factorization of 1080.
462. Find the LCM (Least Common Multiple) of 24 and 40.

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1.E: Whole Numbers (Exercises)

1.1 - Introduction to Whole Numbers

Identify Counting Numbers and Whole Numbers

In the following exercises, determine which of the following numbers are (a) counting numbers (b) whole numbers.

453. 0, 2, 99
454. 0, 3, 25
455. 0, 4, 90
456. 0, 1, 75

Model Whole Numbers

In the following exercises, model each number using base-10 blocks and then show its value using place value notation.

457. 258
458. 104

Identify the Place Value of a Digit

In the following exercises, find the place value of the given digits.

459. 472,981

(a) 8 (b) 4 (c) 1 (d) 7 (e) 2

459. 12,403,295

(a) 4 (b) 0 (c) 1 (d) 9 (e) 3

Use Place Value to Name Whole Numbers

In the following exercises, name each number in words.

461. 5,280
462. 204,614
463. 5,012,582
464. 31,640,976

Use Place Value to Write Whole Numbers

In the following exercises, write each number as a whole number using digits.

465. six hundred two
466. fifteen thousand, two hundred fifty-three
467. three hundred forty million, nine hundred twelve thousand, sixty-one
468. two billion, four hundred ninety-two million, seven hundred eleven thousand, two

Round Whole Numbers

In the following exercises, round to the nearest ten.

469. 412
470. 648
471. 3,556
472. 2,734

In the following exercises, round to the nearest hundred.

473. 38,975
474. 26,849
475. 81,486
476. 75,992

1.2 - Add Whole Numbers

Use Addition Notation

In the following exercises, translate the following from math notation to words.

477. 4 + 3
478. 25 + 18
479. 571 + 629
480. 10,085 + 3,492

Model Addition of Whole Numbers

In the following exercises, model the addition.

481. 6 + 7
482. 38 + 14

Add Whole Numbers

In the following exercises, fill in the missing values in each chart.

483.

+	0	1	2	3	4	5	6	7	8	9
0	0	1		3	4		6	7		9
1	1	2	3	4			7	8	9	
2		3	4	5	6	7	8		10	11
3	3		5		7	8		10		12
4	4	5			8	9			12	
5	5		7	8			11		13	
6	6	7	8		10			13		15
7			9			12	13		15	16
8	8	9		11			14		16	
9	9	10	11		13	14			17	

484.

+	3	4	5	6	7	8	9
6							
7							
8							
9							

In the following exercises, add.

485. (a) 0 + 19 (b) 19 + 0
486. (a) 0 + 480 (b) 480 + 0
487. (a) 7 + 6 (b) 6 + 7
488. (a) 23 + 18 (b) 18 + 23
489. 44 + 35
490. 63 + 29
491. 96 + 58
492. 375 + 591
493. 7,281 + 12,546
494. 5,280 + 16,324 + 9,731

Translate Word Phrases to Math Notation

In the following exercises, translate each phrase into math notation and then simplify.

495. the sum of 30 and 12
496. 11 increased by 8
497. 25 more than 39
498. total of 15 and 50

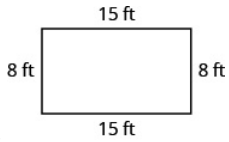
Add Whole Numbers in Applications

In the following exercises, solve.

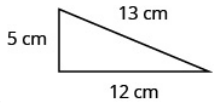
499. **Shopping for an interview** Nathan bought a new shirt, tie, and slacks to wear to a job interview. The shirt cost $24, the tie cost $14, and the slacks cost $38. What was Nathan's total cost?
500. **Running** Jackson ran 4 miles on Monday, 12 miles on Tuesday, 1 mile on Wednesday, 8 miles on Thursday, and 5 miles on Friday. What was the total number of miles Jackson ran?

In the following exercises, find the perimeter of each figure.

501.



502.



1.3 - Subtract Whole Numbers

Use Subtraction Notation

In the following exercises, translate the following from math notation to words.

503. $14 - 5$
504. $40 - 15$
505. $351 - 249$
506. $5{,}724 - 2{,}918$

Model Subtraction of Whole Numbers

In the following exercises, model the subtraction.

507. $18 - 4$
508. $41 - 29$

Subtract Whole Numbers

In the following exercises, subtract and then check by adding.

509. $8 - 5$
510. $12 - 7$
511. $23 - 9$
512. . $46 - 21$
513. $82 - 59$
514. $110 - 87$
515. $539 - 217$
516. $415 - 296$
517. $1{,}020 - 640$
518. $8{,}355 - 3{,}947$
519. $10{,}000 - 15$
520. $54{,}925 - 35{,}647$

Translate Word Phrases to Math Notation

In the following exercises, translate and simplify.

521. the difference of nineteen and thirteen
522. subtract sixty-five from one hundred
523. seventy-four decreased by eight
524. twenty-three less than forty-one

Subtract Whole Numbers in Applications

In the following exercises, solve.

525. **Temperature** The high temperature in Peoria one day was 86 degrees Fahrenheit and the low temperature was 28 degrees Fahrenheit. What was the difference between the high and low temperatures?
526. **Savings** Lynn wants to go on a cruise that costs $2,485. She has $948 in her vacation savings account. How much more does she need to save in order to pay for the cruise?

1.4 - Multiply Whole Numbers

Use Multiplication Notation

In the following exercises, translate from math notation to words.

527. 8 × 5
528. 6 • 14
529. (10)(95)
530. 54(72)

Model Multiplication of Whole Numbers

In the following exercises, model the multiplication.

531. 2 × 4
532. 3 × 8

Multiply Whole Numbers

In the following exercises, fill in the missing values in each chart.

533.

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0		0		0	0
1	0	1	2		4	5	6	7		9
2	0		4		8	10		14	16	
3		3		9			18		24	
4	0	4		12			24			36
5	0	5	10		20		30	35	40	45
6			12	18			36	42		54
7	0	7		21		35			56	63
8	0	8	16		32		48		64	
9			18	27	36			63	72	

534.

×	3	4	5	6	7	8	9
6							
7							
8							
9							

In the following exercises, multiply.

535. 0 • 14
536. (256)0
537. 1 • 99
538. (4,789)1
539. (a) 7 • 4 (b) 4 • 7
540. (25)(6)
541. 9,261 × 3
542. 48 • 76
543. 64 • 10
544. 1,000(22)
545. 162 × 493
546. (601)(943)

547. 3,624 × 517
548. 10,538 • 22

Translate Word Phrases to Math Notation

In the following exercises, translate and simplify.

549. the product of 15 and 28
550. ninety-four times thirty-three
551. twice 575
552. ten times two hundred sixty-four

Multiply Whole Numbers in Applications

In the following exercises, solve.

553. **Gardening** Geniece bought 8 packs of marigolds to plant in her yard. Each pack has 6 flowers. How many marigolds did Geniece buy?
554. **Cooking** Ratika is making rice for a dinner party. The number of cups of water is twice the number of cups of rice. If Ratika plans to use 4 cups of rice, how many cups of water does she need?
555. **Multiplex** There are twelve theaters at the multiplex and each theater has 150 seats. What is the total number of seats at the multiplex?
556. **Roofing** Lewis needs to put new shingles on his roof. The roof is a rectangle, 30 feet by 24 feet. What is the area of the roof?

1.5 - Divide Whole Numbers

Use Division Notation

Translate from math notation to words.

557. $54 \div 9$
558. $42 / 7$
559. $\frac{72}{8}$
560. $6\overline{)48}$

Model Division of Whole Numbers

In the following exercises, model.

561. $8 \div 2$
562. $3\overline{)12}$

Divide Whole Numbers

In the following exercises, divide. Then check by multiplying.

563. $14 \div 2$
564. $\frac{32}{8}$
565. $52 \div 4$
566. $26\overline{)26}$
567. $\frac{97}{1}$
568. $0 \div 52$
569. $100 \div 0$
570. $\frac{355}{5}$
571. $3828 \div 6$
572. $31\overline{)1,519}$
573. $\frac{7505}{25}$

574. 5,166 ÷ 42

Translate Word Phrases to Math Notation

In the following exercises, translate and simplify.

575. the quotient of 64 and 16
576. the quotient of 572 and 52

Divide Whole Numbers in Applications

In the following exercises, solve.

577. **Ribbon** One spool of ribbon is 27 feet. Lizbeth uses 3 feet of ribbon for each gift basket that she wraps. How many gift baskets can Lizbeth wrap from one spool of ribbon?
578. **Juice** One carton of fruit juice is 128 ounces. How many 4 ounce cups can Shayla fill from one carton of juice?

PRACTICE TEST

579. Determine which of the following numbers are (a) counting numbers (b) whole numbers.

$$0, 4, 87 \qquad (1.E.1)$$

580. Find the place value of the given digits in the number 549,362.

(a) 9 (b) 6 (c) 2 (d) 5

581. Write each number as a whole number using digits.

(a) six hundred thirteen (b) fifty-five thousand two hundred eight

582. Round 25,849 to the nearest hundred.

Simplify.

583. 45 + 23
584. 65 − 42
585. 85 ÷ 5
586. 1,000 × 8
587. 90 − 58
588. 73 + 89
589. (0)(12,675)
590. 634 + 255
591. $\frac{0}{9}$
592. $8\overline{)128}$
593. 145 − 79
594. 299 + 836
595. 7 • 475
596. 8,528 + 704
597. 35(14)
598. $\frac{26}{0}$
599. 733 − 291
600. 4,916 − 1,538
601. 495 ÷ 45
602. 52 × 983

Translate each phrase to math notation and then simplify.

603. The sum of 16 and 58
604. The product of 9 and 15
605. The difference of 32 and 18

606. The quotient of 63 and 21
607. Twice 524
608. 29 more than 32
609. 50 less than 300

In the following exercises, solve.

610. LaVelle buys a jumbo bag of 84 candies to make favor bags for her son's party. If she wants to make 12 bags, how many candies should she put in each bag?
611. Last month, Stan's take-home pay was $3,816 and his expenses were $3,472. How much of his take-home pay did Stan have left after he paid his expenses?
612. Each class at Greenville School has 22 children enrolled. The school has 24 classes. How many children are enrolled at Greenville School?
613. Clayton walked 12 blocks to his mother's house, 6 blocks to the gym, and 9 blocks to the grocery store before walking the last 3 blocks home. What was the total number of blocks that Clayton walked?

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1.S: Introduction to the Language of Algebra (Summary)

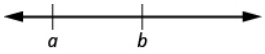
Key Terms

coefficient	The constant that multiplies the variable(s) in a term.
composite number	A composite number is a counting number that is not prime.
divisibility	If a number m is a multiple of n, then we say that m is divisible by n.
equation	An equation is made up of two expressions connected by an equal sign.
evaluate	To evaluate an algebraic expression means to find the value of the expression when the variable is replaced by a given number.
expression	An expression is a number, a variable, or a combination of numbers and variables and operation symbols.
least common multiple (LCM)	The smallest number that is a multiple of two numbers.
like terms	Terms that are either constants or have the same variables with the same exponents.
multiple of a number	A number is a multiple of n if it is the product of a counting number and n.
prime factorization	The product of prime numbers that equals the number.
prime number	A counting number greater than 1 whose only factors are 1 and itself.
solution of an equation	A value of a variable that makes a true statement when substituted into the equation. The process of finding the solution to an equation is called solving the equation.
term	A constant or the product of a constant and one or more variables.

Key Concepts

2.1 - Use the Language of Algebra

Operation	Notation	Say:	The result is…
Addition	a + b	a plus b	The sum of a and b
Multiplication	a • b, (a)(b), (a)b, a(b)	a times b	The product of a and b
Subtraction	a - b	a minus b	The difference of a and b
Division	a ÷ b, a / b, $\frac{a}{b}$, $b\overline{)a}$	a divided by b	The quotient of a and b

- **Equality Symbol**
  - a = b is read as a is equal to b
  - The symbol = is called the equal sign.
- **Inequality**
  - a < b is read a is less than b
  - a is to the left of b on the number line: 

  - a > b is read a is greater than b

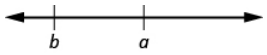
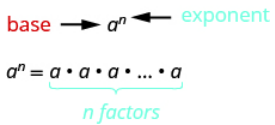
- a is to the right of b on the number line: 

Table 2.77

Algebraic Notation	Say
a = b	a is equal to b
a ≠ b	a is not equal to b
a < b	a is less than b
a > b	a is greater than b
a ≤ b	a is less than or equal to b
a ≥ b	a is greater than or equal to b

- **Exponential Notation**
  - For any expression a n is a factor multiplied by itself n times, if n is a positive integer.
  - a^n means multiply n factors of a
  - The expression of a^n is read a to the nth power



- **Order of Operations**: When simplifying mathematical expressions perform the operations in the following order:

1. Parentheses and other Grouping Symbols: Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
2. Exponents: Simplify all expressions with exponents.
3. Multiplication and Division: Perform all multiplication and division in order from left to right. These operations have equal priority.
4. Addition and Subtraction: Perform all addition and subtraction in order from left to right. These operations have equal priority.

2.2 - Evaluate, Simplify, and Translate Expressions

- **Combine like terms**.

1. Identify like terms.
2. Rearrange the expression so like terms are together.
3. Add the coefficients of the like terms

2.3 - Solving Equations Using the Subtraction and Addition Properties of Equality

- **Determine whether a number is a solution to an equation.**

1. Substitute the number for the variable in the equation.
2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true. If it is true, the number is a solution. If it is not true, the number is not a solution.

- **Subtraction Property of Equality**
  - For any numbers a, b, and c, if a = b, then a - c = b - c.
- Solve an equation using the Subtraction Property of Equality.

1. Use the Subtraction Property of Equality to isolate the variable.
2. Simplify the expressions on both sides of the equation.
3. Check the solution.

- **Addition Property of Equality**

- For any numbers a, b, and c, if a = b, then a + c = b + c.

- Solve an equation using the Addition Property of Equality.

1. Use the Addition Property of Equality to isolate the variable.
2. Simplify the expressions on both sides of the equation.
3. Check the solution.

2.4 - Find Multiples and Factors

Divisibility Tests	
A number is divisible by	
2	if the last digit is 0, 2, 4, 6, or 8
3	if the sum of the digits is divisible by 3
5	if the last digit is 5 or 0
6	if divisible by both 2 and 3
10	if the last digit is 0

- **Factors:** If a • b = m, then a and b are factors of m, and m is the product of a and b.
- **Find all the factors of a counting number.**

1. Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.
   a. If the quotient is a counting number, the divisor and quotient are a pair of factors.
   b. If the quotient is not a counting number, the divisor is not a factor.
2. List all the factor pairs.
3. Write all the factors in order from smallest to largest.

- **Determine if a number is prime**.

1. Test each of the primes, in order, to see if it is a factor of the number.
2. Start with 2 and stop when the quotient is smaller than the divisor or when a prime factor is found.
3. If the number has a prime factor, then it is a composite number. If it has no prime factors, then the number is prime.

2.5 - Prime Factorization and the Least Common Multiple

- **Find the prime factorization of a composite number using the tree method**.

1. Find any factor pair of the given number, and use these numbers to create two branches.
2. If a factor is prime, that branch is complete. Circle the prime.
3. If a factor is not prime, write it as the product of a factor pair and continue the process.
4. Write the composite number as the product of all the circled primes.

- **Find the prime factorization of a composite number using the ladder method**.

1. Divide the number by the smallest prime.
2. Continue dividing by that prime until it no longer divides evenly.
3. Divide by the next prime until it no longer divides evenly.
4. Continue until the quotient is a prime.
5. Write the composite number as the product of all the primes on the sides and top of the ladder.

- **Find the LCM by listing multiples**

1. List the first several multiples of each number.
2. Look for multiples common to both lists. If there are no common multiples in the lists, write out additional multiples for each number.
3. Look for the smallest number that is common to both lists.
4. This number is the LCM.

- **Find the LCM using the prime factors method**.

1. Find the prime factorization of each number.
2. Write each number as a product of primes, matching primes vertically when possible.
3. Bring down the primes in each column.
4. Multiply the factors to get the LCM.

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1.S: Whole Numbers (Summary)

Key Terms

coordinate	A number paired with a point on a number line.
counting numbers	The counting numbers are the numbers 1, 2, 3, ….
difference	The result of subtracting two or more numbers.
dividend	The number being divided when dividing two numbers.
divisor	The number dividing the dividend when dividing two numbers.
number line	A number line is used to visualize numbers. The numbers on the number line get larger as they go from left to right, and smaller as they go from right to left.
origin	The point labeled 0 on a number line.
place value system	Our number system is called a place value system because the value of a digit depends on its position, or place, in a number.
product	The result of multiplying two or more numbers.
quotient	The result of dividing two numbers.
rounding	The process of approximating a number.
sum	The result of adding two or more numbers.
whole numbers	The whole numbers are the numbers 0, 1, 2, 3, ….

Key Concepts

1.1 - Introduction to Whole Numbers

Place Value														
Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
								5	2	7	8	1	9	4

- **Name a whole number in words.**
  1. Starting at the digit on the left, name the number in each period, followed by the period name. Do not include the period name for the ones.
  2. Use commas in the number to separate the periods.
- **Use place value to write a whole number.**
  1. Identify the words that indicate periods. (Remember the ones period is never named.)
  2. Draw three blanks to indicate the number of places needed in each period.
  3. Name the number in each period and place the digits in the correct place value position.
- **Round a whole number to a specific place value.**
  1. Locate the given place value. All digits to the left of that place value do not change.

2. Underline the digit to the right of the given place value.
3. Determine if this digit is greater than or equal to 5. If yes—add 1 to the digit in the given place value. If no—do not change the digit in the given place value.
4. Replace all digits to the right of the given place value with zeros.

1.2 - Add Whole Numbers

• **Addition Notation:** To describe addition, we can use symbols and words.

Operation	Notation	Expression	Read as	Result
Addition	+	3 + 4	three plus four	the sum of 3 and 4

- **Identity Property of Addition:** The sum of any number a and 0 is the number.

$$a+0=a$$
$$0+a=a$$

- **Commutative Property of Addition:** Changing the order of the addends a and b does not change their sum.

$$a+b=b+a. \tag{1.S.1}$$

- **Add whole numbers**.

1. Write the numbers so each place value lines up vertically.
2. Add the digits in each place value. Work from right to left starting with the ones place. If a sum in a place value is more than 9, carry to the next place value.
3. Continue adding each place value from right to left, adding each place value and carrying if needed.

1.3 - Subtract Whole Numbers

Operation	Notation	Expression	Read as	Result
Subtraction	-	7 - 3	seven minus three	the difference of 7 and 3

- **Subtract whole numbers**.

1. Write the numbers so each place value lines up vertically.
2. Subtract the digits in each place value. Work from right to left starting with the ones place. If the digit on top is less than the digit below, borrow as needed.
3. Continue subtracting each place value from right to left, borrowing if needed.
4. Check by adding.

1.4 - Multiply Whole Numbers

Operation	Notation	Expression	Read as	Result
Multiplication	×	3 × 8	three times eight	the product of 3 and 8
	•	3 • 8		
	()	3(8)		

- **Multiplication Property of Zero:** The product of any number and 0 is 0.

$$a\cdot 0=0$$
$$0\cdot a=0$$

- **Identity Property of Multiplication:** The product of any number and 1 is the number.

$$1\cdot a=a$$
$$a\cdot 1=a$$

- **Commutative Property of Multiplication:** Changing the order of the factors does not change their product.

$$a \cdot b = b \cdot a. \tag{1.S.2}$$

- **Multiply two whole numbers to find the product**.

1. Write the numbers so each place value lines up vertically.
2. Multiply the digits in each place value.
3. Work from right to left, starting with the ones place in the bottom number.
4. Multiply the bottom number by the ones digit in the top number, then by the tens digit, and so on.
5. If a product in a place value is more than 9, carry to the next place value.
6. Write the partial products, lining up the digits in the place values with the numbers above. Repeat for the tens place in the bottom number, the hundreds place, and so on.
7. Insert a zero as a placeholder with each additional partial product.
8. Add the partial products.

1.5 - Divide Whole Numbers

Operation	Notation	Expression	Read as	Result
Division	$\div$	$12 \div 4$	Twelve divided by four	the quotient of 12 and 4
	$\frac{a}{b}$	$\frac{12}{4}$		
	$b\overline{)a}$	$4\overline{)12}$		
	a / b	12 / 4		

- **Division Properties of One:**
  - Any number (except 0) divided by itself is one.

$$a \div a = 1 \tag{1.S.3}$$

  - Any number divided by one is the same number.

$$a \div 1 = a \tag{1.S.4}$$

- **Division Properties of Zero:**
  - Zero divided by any number is 0.

$$0 \div a = 0 \tag{1.S.5}$$

  - Dividing a number by zero is undefined.

$$a \div 0 = undefined \tag{1.S.6}$$

- **Divide whole numbers**.

1. Divide the first digit of the dividend by the divisor. If the divisor is larger than the first digit of the dividend, divide the first two digits of the dividend by the divisor, and so on.
2. Write the quotient above the dividend.
3. Multiply the quotient by the divisor and write the product under the dividend.
4. Subtract that product from the dividend.
5. Bring down the next digit of the dividend.
6. Repeat from Step 1 until there are no more digits in the dividend to bring down.
7. Check by multiplying the quotient times the divisor.

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CHAPTER OVERVIEW

2: Integers

At over 29,000 feet, Mount Everest stands as the tallest peak on land. Located along the border of Nepal and China, Mount Everest is also known for its extreme climate. Near the summit, temperatures never rise above freezing. Every year, climbers from around the world brave the extreme conditions in an effort to scale the tremendous height. Only some are successful. Describing the drastic change in elevation the climbers experience and the change in temperatures requires using numbers that extend both above and below zero. In this chapter, we will describe these kinds of numbers and operations using them.

2.1: Introduction to Integers (Part 1)
2.2: Introduction to Integers (Part 2)
2.3: Add Integers (Part 1)
2.4: Add Integers (Part 2)
2.5: Subtract Integers (Part 1)
2.6: Subtract Integers (Part 2)
2.7: Multiply and Divide Integers (Part 1)
2.8: Multiply and Divide Integers (Part 2)
2.9: Square Roots
2.10: What to Combine? Order of Operations with Integers
2.11: Mathematical Expressions (NOT Equations)
2.12: Evaluating (NOT Solving) Algebraic Expressions
2.13: Simplifying (Still Not Solving) Algebraic Expressions)
  2.13.1: Simplifying Algebraic Expressions
  2.13.2: Combining Like Terms
2.14: Mean
2.15: Solve Equations Using Integers; The Division Property of Equality (Part 1)
2.16: Solve Equations Using Integers; The Division Property of Equality (Part 2)
2.E: Integers (Exercises)
2.S: Integers (Summary)

Figure 3.1 - The peak of Mount Everest. (credit: Gunther Hagleitner, Flickr)

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2.1: Introduction to Integers (Part 1)

Learning Objectives

- Locate positive and negative numbers on the number line
- Order positive and negative numbers
- Find opposites
- Simplify expressions with absolute value
- Translate word phrases to expressions with integers

be prepared!

Before you get started, take this readiness quiz.

1. Plot 0, 1, and 3 on a number line. If you missed this problem, review Example 1.1.1.
2. Fill in the appropriate symbol: ($=$, $<$, or $>$): $2\_\_\_4$ If you missed this problem, review Example 2.1.2.

Locate Positive and Negative Numbers on the Number Line

Do you live in a place that has very cold winters? Have you ever experienced a temperature below zero? If so, you are already familiar with negative numbers. A **negative number** is a number that is less than 0. Very cold temperatures are measured in degrees below zero and can be described by negative numbers. For example, -1° F (read as "negative one degree Fahrenheit") is 1 degree below 0. A minus sign is shown before a number to indicate that it is negative. Figure $2.1.1$ shows -20° F, which is 20 degrees below 0.

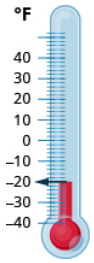


Figure $2.1.1$: Temperatures below zero are described by negative numbers.

Temperatures are not the only negative numbers. A bank overdraft is another example of a negative number. If a person writes a check for more than he has in his account, his balance will be negative.

Elevations can also be represented by negative numbers. The elevation at sea level is 0 feet. Elevations above sea level are positive and elevations below sea level are negative. The elevation of the Dead Sea, which borders Israel and Jordan, is about $1,302$ feet below sea level, so the elevation of the Dead Sea can be represented as $-1,302$ feet. See Figure $2.1.2$.

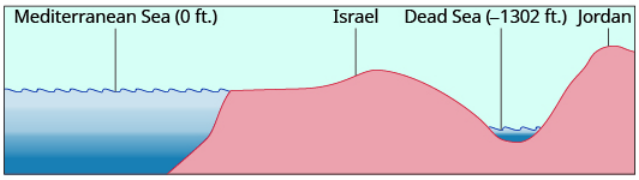


Figure $2.1.2$: The surface of the Mediterranean Sea has an elevation of 0 ft. The diagram shows that nearby mountains have higher (positive) elevations whereas the Dead Sea has a lower (negative) elevation.

Depths below the ocean surface are also described by negative numbers. A submarine, for example, might descend to a depth of 500 feet. Its position would then be -500 feet as labeled in Figure $2.1.3$.

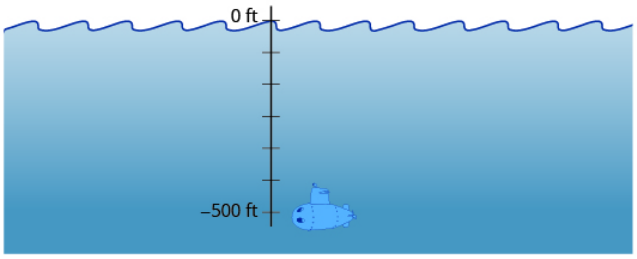


Figure 2.1.3: Depths below sea level are described by negative numbers. A submarine 500 ft below sea level is at −500 ft.

Both positive and negative numbers can be represented on a number line. Recall that the number line created in Add Whole Numbers started at 0 and showed the counting numbers increasing to the right as shown in Figure 2.1.4. The counting numbers ($1, 2, 3, \ldots$) on the number line are all positive. We could write a plus sign, $+$, before a positive number such as $+2$ or $+3$, but it is customary to omit the plus sign and write only the number. If there is no sign, the number is assumed to be positive.

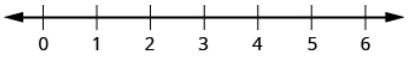


Figure 2.1.4

Now we need to extend the number line to include negative numbers. We mark several units to the left of zero, keeping the intervals the same width as those on the positive side. We label the marks with negative numbers, starting with -1 at the first mark to the left of 0, -2 at the next mark, and so on. See Figure 2.1.5.

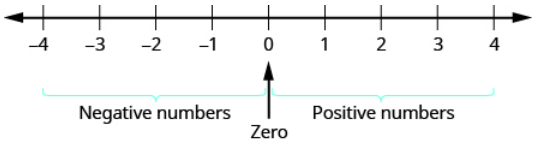


Figure 2.1.5: On a number line, positive numbers are to the right of zero. Negative numbers are to the left of zero. What about zero? Zero is neither positive nor negative.

The arrows at either end of the line indicate that the number line extends forever in each direction. There is no greatest positive number and there is no smallest negative number.

Example 2.1.1: plot on the number line

Plot the numbers on a number line:

a. 3
b. -3
c. -2

Solution

Draw a number line. Mark 0 in the center and label several units to the left and right.

a. To plot 3, start at 0 and count three units to the right. Place a point as shown in Figure 2.1.6.

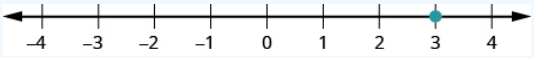


Figure 2.1.6

b. To plot -3, start at 0 and count three units to the left. Place a point as shown in Figure 2.1.7.

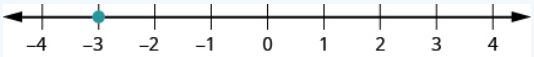


Figure 2.1.7

c. To plot -2, start at 0 and count two units to the left. Place a point as shown in Figure 2.1.8.

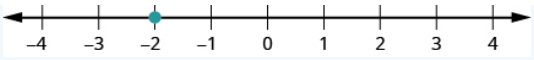


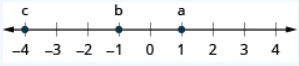
Figure 2.1.8

? Exercise 2.1.1

Plot the numbers on a number line.

$1, -1, -4$

Answer

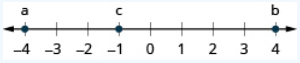


? Exercise 2.1.2

Plot the numbers on a number line.

$-4, 4, -1$

Answer



Order Positive and Negative Numbers

We can use the number line to compare and order positive and negative numbers. Going from left to right, numbers increase in value. Going from right to left, numbers decrease in value. See Figure 2.1.9.

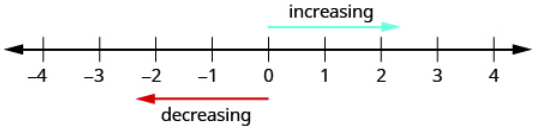


Figure 2.1.9

Just as we did with positive numbers, we can use inequality symbols to show the ordering of positive and negative numbers. Remember that we use the notation $a < b$ (read a is less than b) when a is to the left of b on the number line. We write $a > b$ (read a is greater than b) when a is to the right of b on the number line. This is shown for the numbers 3 and 5 in Figure 2.1.10.

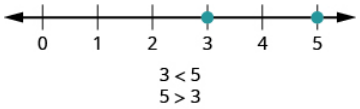
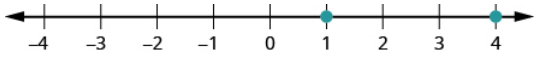


Figure 2.1.10: The number 3 is to the left of 5 on the number line. So 3 is less than 5, and 5 is greater than 3.

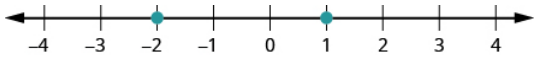
The numbers lines to follow show a few more examples.

a.



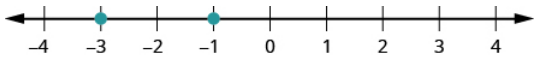
4 is to the right of 1 on the number line, so $4 > 1$. 1 is to the left of 4 on the number line, so $1 < 4$.

b.



-2 is to the left of 1 on the number line, so $-2 < 1$. 1 is to the right of -2 on the number line, so $1 > -2$.

c.



-1 is to the right of -3 on the number line, so $-1 > -3$. -3 is to the left of -1 on the number line, so $-3 < -1$.

Example 2.1.2: order the pairs

Order each of the following pairs of numbers using $<$ or $>$:

a. $14 \_\_\_ 6$
b. $-1 \_\_\_ 9$
c. $-1 \_\_\_ -4$
d. $2 \_\_\_ -20$

Solution

Begin by plotting the numbers on a number line as shown in Figure 2.1.11.

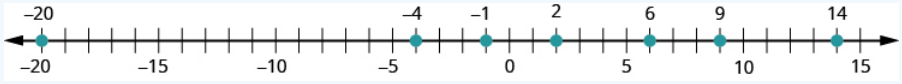


Figure 2.1.11

a.

Compare 14 and 6.	$14 \_\_\_ 6$
14 is to the right of 6 on the number line.	$14 > 6$

b.

Compare -1 and 9.	$-1 \_\_\_ 9$
-1 is to the left of 9 on the number line.	$-1 < 9$

c.

Compare -1 and -4.	$-1 \_\_\_ -4$
-1 is to the right of -4 on the number line.	$-1 > -4$

d.

Compare 2 and -20.	$2 \_\_\_ -20$
2 is to the right of -20 on the number line.	$2 > -20$

Exercise 2.1.3

Order each of the following pairs of numbers using $<$ or $>$.

a. $15 \_\_\_ 7$
b. $-2 \_\_\_ 5$
c. $-3 \_\_\_ -7$
d. $5 \_\_\_ -17$

Answer a

$>$

Answer b

$<$

Answer c

$>$

Answer d

$>$

Exercise 2.1.4

Order each of the following pairs of numbers using $<$ or $>$.

a. $8 \_\_\_ 13$
b. $3 \_\_\_ -4$
c. $-5 \_\_\_ -2$
d. $9 \_\_\_ -21$

Answer a

$<$

Answer b

$>$

Answer c

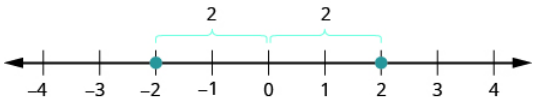
$<$

Answer d

$>$

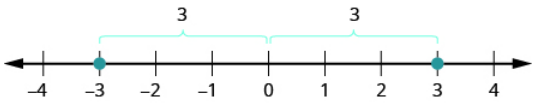
Find Opposites

On the number line, the negative numbers are a mirror image of the positive numbers with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, they are called **opposites**. The opposite of 2 is -2, and the opposite of -2 is 2 as shown in Figure $2.1.12a$. Similarly, 3 and -3 are opposites as shown in Figure $2.1.12b$



The numbers –2 and 2 are opposites.

(a)



The numbers –3 and 3 are opposites.

(b)

Figure 2.1.12

Definition: opposite

The opposite of a number is the number that is the same distance from zero on the number line, but on the opposite side of zero.

Example 2.1.3:

Find the opposite of each number:

a. 7

b. -10

Solution

a. The number -7 is the same distance from 0 as 7, but on the opposite side of 0. So -7 is the opposite of 7 as shown in Figure 2.1.13.

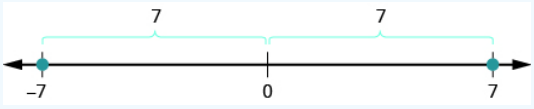


Figure 2.1.13

b. The number 10 is the same distance from 0 as -10, but on the opposite side of 0. So 10 is the opposite of -10 as shown in Figure 2.1.14.

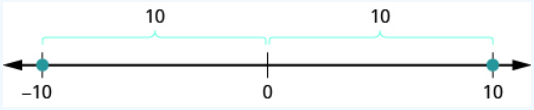


Figure 2.1.14

? Exercise 2.1.5

Find the opposite of each number:

a. 4
b. -3

Answer a

-4

Answer b

3

? Exercise 2.1.6

Find the opposite of each number:

a. 8
b. -5

Answer a

-8

Answer b

5

Opposite Notation

Just as the same word in English can have different meanings, the same symbol in algebra can have different meanings.

The specific meaning becomes clear by looking at how it is used. You have seen the symbol "$-$", in three different ways.

10 - 4	Between two numbers, the symbol indicates the operation of subtraction. We read $10 - 4$ as 10 minus 4.
-8	In front of a number, the symbol indicates a negative number. We read -8 as negative eight.

-x	In front of a variable or a number, it indicates the opposite. We read −x as the opposite of x .
− (−2)	Here we have two signs. The sign in the parentheses indicates that the number is negative 2. The sign outside the parentheses indicates the opposite. We read − (−2) as the opposite of −2.

Definition: Opposite Notation

$-a$ means the opposite of the number a.

The notation $-a$ is read the opposite of a.

Example 2.1.4: simplify

Simplify: $-(-6)$.

Solution

	$-(-6)$
The opposite of −6 is 6.	6

Exercise 2.1.7

Simplify: $-(-1)$

Answer

1

Exercise 2.1.8

Simplify: $-(-5)$

Answer

5

Integers

The set of counting numbers, their opposites, and 0 is the set of integers.

Definition: integers

Integers are counting numbers, their opposites, and zero.

$$\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$$

We must be very careful with the signs when evaluating the opposite of a variable.

Example 2.1.5: evaluate

Evaluate $-x$:

a. when $x = 8$
b. when $x = -8$

Solution

a.

To evaluate −x when x = 8, substitute 8 for x.	$-x$
Substitute 8 for x.	$-(8)$
Simplify.	-8

b.

To evaluate −x when x = −8, substitute 8 for x.	$-x$
Substitute -8 for x.	$-(-8)$
Simplify.	-8

Exercise 2.1.9

Evaluate $-n$:

a. when $n = 4$
b. when $n = -4$

Answer a

-4

Answer b

4

Exercise 2.1.10

Evaluate $-m$:

a. when $m = 11$
b. when $m = -11$

Answer a

-11

Answer b

11

Simplify Expressions with Absolute Value

We saw that numbers such as 5 and -5 are opposites because they are the same distance from 0 on the number line. They are both five units from 0. The distance between 0 and any number on the number line is called the absolute value of that number. Because distance is never negative, the absolute value of any number is never negative. The symbol for absolute value is two vertical lines on either side of a number. So the absolute value of 5 is written as $|5|$, and the absolute value of -5 is written as $|-5|$ as shown in Figure 2.1.15.

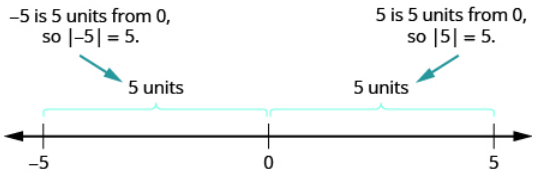


Figure 2.1.15

Definition: Absolute Value

The absolute value of a number is its distance from 0 on the number line.

The absolute value of a number n is written as $|n|$.

$$|n| \geq 0 \ for\ all\ numbers$$

Example 3.6:

Simplify:

a. $|3|$
b. $|-44|$
c. $|0|$

Solution

a.

		3
3 is 3 units from zero.	3

b.

	|-44
-44 is 44 units from zero.	44

c.

		0
0 is already at zero.	0

Exercise 2.1.11

Simplify:

a. $|12|$
b. $-|-28|$

Answer a

12

Answer b

-28

Exercise 3.12:

Simplify:

a. $|9|$
b. $-|37|$

Answer a

9

Answer b

-37

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2.2: Introduction to Integers (Part 2)

We treat absolute value bars just like we treat parentheses in the order of operations. We simplify the expression inside first.

Example 2.2.7: evaluate

Evaluate:

a. $|x|$ when $x = -35$
b. $|-y|$ when $y = -20$
c. $-|u|$ when $u = 12$
d. $-|p|$ when $p = -14$

Solution

a.

To find |x| when x = −35:	$\|x\|$
Substitute -35 for x.	$\|-35\|$
Take the absolute value.	35

b.

To find |− y| when y = −20:	$\|-y\|$
Substitute -20 for y.	$\|-(-20)\|$
Simplify.	$\|20\|$
Take the absolute value.	20

c.

To find -|u| when u = 12:	$-\|u\|$
Substitute 12 for u.	$-\|12\|$
Take the absolute value.	-12

d.

To find −|p| when p = −14:	$-\|p\|$
Substitute -14 for p.	$-\|-14\|$
Take the absolute value.	-14

Notice that the result is negative only when there is a negative sign outside the absolute value symbol

Exercise 2.2.13

Evaluate:

a. $|x|$ when $x = -17$
b. $|-y|$ when $y = -39$
c. $-|m|$ when $m = 22$
d. $-|p|$ when $p = -11$

Answer a

17

Answer b

39

Answer c

-22

Answer d

11

Exercise 2.2.14

Evaluate:

a. $|y|$ when $y = -23$
b. $|-y|$ when $y = -21$
c. $-|n|$ when $n = 37$
d. $-|q|$ when $q = -49$

Answer a

23

Answer b

21

Answer c

-37

Answer d

-49

Example 2.2.8: compare expressions

Fill in $<$, $>$ or $=$ for each of the following:

a. $|-5|\_\_\_-|-5|$
b. $8\_\_\_-|-8|$
c. $-9\_\_\_-|-9|$
d. $-|-7|\_\_\_-7$

Solution

To compare two expressions, simplify each one first. Then compare.

a.

	|−5|___− |−5|
Simplify.	5___−5
Order.	5 > −5

b.

	8___− |−8|
Simplify.	8___−8

Order.	$8 > -8$

c.

	$-9 \_\_\_ -\lvert -9 \rvert$
Simplify.	$-9 \_\_\_ -9$
Order.	$-9 = -9$

d.

	$-\lvert -7 \rvert \_\_\_ -7$
Simplify.	$-7 \_\_\_ -7$
Order.	$-7 = -7$

Exercise 2.2.15

Fill in $<$, $>$, or $=$ for each of the following:

a. $|-9| \_\_\_ -|-9|$
b. $2 \_\_\_ -|-2|$
c. $-8 \_\_\_ |-8|$
d. $-|-5| \_\_\_ -5$

Answer a

$>$

Answer b

$>$

Answer c

$<$

Answer d

$=$

Exercise 2.2.16

Fill in $<$, $>$, or $=$ for each of the following:

a. $7 \_\_\_ -|-7|$
b. $-|-11| \_\_\_ -11$
c. $|-4| \_\_\_ -|-4|$
d. $-1 \_\_\_ |-1|$

Answer a

$>$

Answer b

$=$

Answer c

$>$

Answer d

$<$

Absolute value bars act like grouping symbols. First simplify inside the absolute value bars as much as possible. Then take the absolute value of the resulting number, and continue with any operations outside the absolute value symbols.

Example 2.2.9: simplify

Simplify:

a. $|9-3|$
b. $4|-2|$

Solution

For each expression, follow the order of operations. Begin inside the absolute value symbols just as with parentheses.

a.

Simplify inside the absolute value sign.	$\|9-3\| = \|6\|$
Take the absolute value.	6

b.

Take the absolute value.	$4\|-2\| = 4 \cdot 2$
Multiply.	8

Exercise 2.2.17

Simplify:

a. $|12-9|$
b. $3|-6|$

Answer a

3

Answer b

18

Exercise 2.2.18

Simplify:

a. $|27-16|$
b. $9|-7|$

Answer a

11

Answer b

63

Example 2.2.10: simplify

Simplify: $|8+7|-|5+6|$.

Solution

For each expression, follow the order of operations. Begin inside the absolute value symbols just as with parentheses.

Simplify inside each absolute value sign.	|8+7|−|5+6| = |15|−|11|
Subtract.	4

Exercise 2.2.19

Simplify: $|1+8|-|2+5|$

Answer

2

Exercise 2.2.20

Simplify: $|9-5|-|7-6|$

Answer

3

Example 2.2.11: simplify

Simplify: $24-|19-3(6-2)|$.

Solution

We use the order of operations. Remember to simplify grouping symbols first, so parentheses inside absolute value symbols would be first.

Simplify in the parentheses first.	24 − |19 − 3(6 − 2)| = 24 − |19 − 3(4)|
Multiply 3(4).	24 − |19 − 12|
Subtract inside the absolute value sign.	24 − |7|
Take the absolute value.	24 - 7
Subtract.	17

Exercise 2.2.21

Simplify: $19-|11-4(3-1)|$

Answer

16

Exercise 2.2.22

Simplify: $9-|8-4(7-5)|$

Answer

9

Translate Word Phrases into Expressions with Integers

Now we can translate word phrases into expressions with integers. Look for words that indicate a negative sign. For example, the word *negative* in "negative twenty" indicates -20. So does the word *opposite* in "the opposite of 20."

Example 2.2.12: translate

Translate each phrase into an expression with integers:

a. the opposite of positive fourteen
b. the opposite of -11
c. negative sixteen
d. two minus negative seven

Solution

a. the opposite of fourteen -14
b. the opposite of $-11-(-11)$
c. negative sixteen -16
d. two minus negative seven $2-(-7)$

Exercise 2.2.23

Translate each phrase into an expression with integers:

a. the opposite of positive nine
b. the opposite of -15
c. negative twenty
d. eleven minus negative four

Answer a

-9

Answer b

15

Answer c

-20

Answer d

$11-(-4)$

Exercise 2.2.24

Translate each phrase into an expression with integers:

a. the opposite of negative nineteen
b. the opposite of twenty-two
c. negative nine
d. negative eight minus negative five

Answer a

19

Answer b

-22

Answer c

-9

Answer d

$-8-(-5)$

As we saw at the start of this section, negative numbers are needed to describe many real-world situations. We'll look at some more applications of negative numbers in the next example.

Translate into an expression with integers:

a. The temperature is 12 degrees Fahrenheit below zero.
b. The football team had a gain of 3 yards.
c. The elevation of the Dead Sea is $1,302$ feet below sea level.
d. A checking account is overdrawn by $\$40$.

Solution

Look for key phrases in each sentence. Then look for words that indicate negative signs. Don't forget to include units of measurement described in the sentence.

a.

	The temperature is 12 degrees Fahrenheit below zero.
Below zero tells us that 12 is a negative number.	−12 ºF

b.

	The football team had a gain of 3 yards.
A *gain* tells us that 3 is a positive number.	3 yards

c.

	The elevation of the Dead Sea is 1,302 feet below sea level.
Below sea level tells us that 1,302 is a negative number.	−1,302 feet

d.

	A checking account is overdrawn by $40.
Overdrawn tells us that 40 is a negative number.	−$40

? Exercise 2.2.25

Translate into an expression with integers: The football team had a gain of 5 yards.

Answer

5 yards

Exercise 2.2.26

Translate into an expression with integers: The scuba diver was 30 feet below the surface of the water.

Answer

-30 feet

Access Additional Online Resources

- Introduction to Integers
- Simplifying the Opposites of Negative Integers
- Comparing Absolute Value of Integers
- Comparing Integers Using Inequalities

Key Concepts

- Opposite Notation
  - $-a$ means the opposite of the number a
  - The notation $-a$ is read the opposite of a.
- Absolute Value Notation
  - The absolute value of a number n is written as $|n|$.
  - $|n| \geq 0$ for all numbers.

Glossary

absolute value

The absolute value of a number is its distance from 0 on the number line.

integers

Integers are counting numbers, their opposites, and zero ... $-3, -2, -1, 0, 1, 2, 3 \ldots$

negative number

A negative number is less than zero.

opposites

The opposite of a number is the number that is the same distance from zero on the number line, but on the opposite side of zero.

Practice Makes Perfect

Locate Positive and Negative Numbers on the Number Line

In the following exercises, locate and label the given points on a number line.

1. (a) 2 (b) −2 (c) −5
2. (a) 5 (b) −5 (c) −2
3. (a) −8 (b) 8 (c) −6
4. (a) −7 (b) 7 (c) −1

Order Positive and Negative Numbers on the Number Line

In the following exercises, order each of the following pairs of numbers, using < or >.

5. (a) 9__4 (b) −3__6 (c) −8__−2 (d) 1__−10
6. (a) 6__2; (b) −7__4; (c) −9__−1; (d) 9__−3
7. (a) −5__1; (b) −4__−9; (c) 6__10; (d) 3__−8
8. (a) −7__3; (b) −10__−5; (c) 2__−6; (d) 8__9

Find Opposites

In the following exercises, find the opposite of each number.

9. (a) 2 (b) −6
10. (a) 9 (b) −4
11. (a) −8 (b) 1
12. (a) −2 (b) 6

In the following exercises, simplify.

13. −(−4)
14. −(−8)
15. −(−15)
16. −(−11)

In the following exercises, evaluate.

17. −m when (a) m = 3 (b) m = −3
18. −p when (a) p = 6 (b) p = −6
19. −c when (a) c = 12 (b) c = −12
20. −d when (a) d = 21 (b) d = −21

Simplify Expressions with Absolute Value

In the following exercises, simplify each absolute value expression.

21. (a) |7| (b) |−25| (c) |0|
22. (a) |5| (b) |20| (c) |−19|
23. (a) |−32| (b) |−18| (c) |16|
24. (a) |−41| (b) |−40| (c) |22|

In the following exercises, evaluate each absolute value expression.

25. (a) |x| when x = −28 (b) | − u| when u = −15
26. (a) |y| when y = −37 (b) | − z| when z = −24
27. (a) − |p| when p = 19 (b) − |q| when q = −33
28. (a) − |a| when a = 60 (b) − |b| when b = −12

In the following exercises, fill in , or = to compare each expression.

29. (a) −6__|−6| (b) − |−3|__−3
30. (a) −8__|−8| (b) − |−2|__−2
31. (a) |−3|__− |−3| (b) 4__− |−4|
32. (a) |−5|__− |−5| (b) 9__− |−9|

In the following exercises, simplify each expression.

33. |8 − 4|
34. |9 − 6|
35. 8|−7|
36. 5|−5|
37. |15 − 7| − |14 − 6|
38. |17 − 8| − |13 − 4|
39. 18 − |2(8 − 3)|
40. 15 − |3(8 − 5)|
41. 8(14 − 2|−2|)
42. 6(13 − 4|−2|)

Translate Word Phrases into Expressions with Integers

Translate each phrase into an expression with integers. Do not simplify.

43. (a) the opposite of 8 (b) the opposite of −6 (c) negative three (d) 4 minus negative 3
44. (a) the opposite of 11 (b) the opposite of −4 (c) negative nine (d) 8 minus negative 2
45. (a) the opposite of 20 (b) the opposite of −5 (c) negative twelve (d) 18 minus negative 7
46. (a) the opposite of 15 (b) the opposite of −9 (c) negative sixty (d) 12 minus 5
47. a temperature of 6 degrees below zero
48. a temperature of 14 degrees below zero
49. an elevation of 40 feet below sea level
50. an elevation of 65 feet below sea level
51. a football play loss of 12 yards
52. a football play gain of 4 yards
53. a stock gain of $3
54. a stock loss of $5
55. a golf score one above par
56. a golf score of 3 below par

Everyday Math

57. **Elevation** The highest elevation in the United States is Mount McKinley, Alaska, at 20,320 feet above sea level. The lowest elevation is Death Valley, California, at 282 feet below sea level. Use integers to write the elevation of: (a) Mount McKinley (b) Death Valley
58. **Extreme temperatures** The highest recorded temperature on Earth is 58° Celsius, recorded in the Sahara Desert in 1922. The lowest recorded temperature is 90° below 0° Celsius, recorded in Antarctica in 1983. Use integers to write the: (a) highest recorded temperature (b) lowest recorded temperature
59. **State budgets** In June, 2011, the state of Pennsylvania estimated it would have a budget surplus of $540 million. That same month, Texas estimated it would have a budget deficit of $27 billion. Use integers to write the budget: (a) surplus (b) deficit
60. **College enrollments** Across the United States, community college enrollment grew by 1,400,000 students from 2007 to 2010. In California, community college enrollment declined by 110,171 students from 2009 to 2010. Use integers to write the change in enrollment: (a) growth (b) decline

Writing Exercises

61. Give an example of a negative number from your life experience.
62. What are the three uses of the "−" sign in algebra? Explain how they differ.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
locate positive and negative numbers on the number line.			
order positive and negative numbers.			
find opposites.			
simplify expressions with absolute value.			
translate word phrases to expressions with integers.			

(b) If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don’t get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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2.3: Add Integers (Part 1)

Learning Objectives

- Model addition of integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate word phrases to algebraic expressions
- Add integers in applications

be prepared!

Before you get started, take this readiness quiz.

1. Evaluate $x+8$ when $x=6$. If you missed this problem, review Example 2.2.1.
2. Simplify: $8+2(5+1)$. If you missed this problem, review Example 2.1.8.
3. Translate the sum of 3 and negative 7 into an algebraic expression. If you missed this problem, review Table 2.2.3.

Model Addition of Integers

Now that we have located positive and negative numbers on the number line, it is time to discuss arithmetic operations with integers.

Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more difficult. This difficulty relates to the way the brain learns.

The brain learns best by working with objects in the real world and then generalizing to abstract concepts. Toddlers learn quickly that if they have two cookies and their older brother steals one, they have only one left. This is a concrete example of $2-1$. Children learn their basic addition and subtraction facts from experiences in their everyday lives. Eventually, they know the number facts without relying on cookies.

Addition and subtraction of negative numbers have fewer real world examples that are meaningful to us. Math teachers have several different approaches, such as number lines, banking, temperatures, and so on, to make these concepts real.

We will model addition and subtraction of negatives with two color counters. We let a blue counter represent a positive and a red counter will represent a negative.

If we have one positive and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero as summarized in Figure 2.3.1.

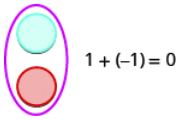


Figure 2.3.1: A blue counter represents +1. A red counter represents −1. Together they add to zero.

We will model four addition facts using the numbers 5, -5 and 3, -3.

$$5+3 \qquad -5+(-3) \qquad -5+3 \qquad 5+(-3)$$

Example 2.3.1: model the expression

Model: $5+3$.

Solution

Interpret the expression.	5 + 3 means the sum of 5 and 3.

Model the first number. Start with 5 positives.	5
Model the second number. Add 3 positives.	5 3
Count the total number of counters.	8 positives
The sum of 5 and 3 is 8.	5 + 3 = 8

Exercise 2.3.1

Model the expression. $2+4$

Answer

6

Exercise 2.3.2

Model the expression. $2+5$

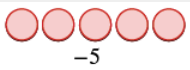
Answer

7

Example 2.3.2: model the expression

Model: $-5+(-3)$.

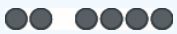
Solution

Interpret the expression.	−5 + (−3) means the sum of −5 and −3.
Model the first number. Start with 5 negatives.	 −5
Model the second number. Add 3 negatives.	−5 −3
Count the total number of counters.	8 negatives
The sum of −5 and −3 is −8.	−5 + −3 = −8

Exercise 2.3.3

Model the expression. $-2+(-4)$

Answer



-6

Exercise 2.3.4

Model the expression. $-2+(-5)$

Answer

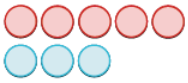
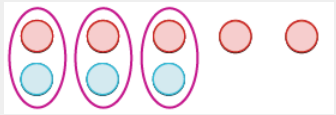
-7

Example 2.3.1 and Example 2.3.2 are very similar. The first example adds 5 positives and 3 positives—both positives. The second example adds 5 negatives and 3 negatives—both negatives. In each case, we got a result of 8 -- either 8 positives or 8 negatives. When the signs are the same, the counters are all the same color. Now let's see what happens when the signs are different.

Example 2.3.3: model the expression

Model: $-5+3$.

Solution

Interpret the expression.	−5 + 3 means the sum of −5 and 3.
Model the first number. Start with 5 negatives.	
Model the second number. Add 3 positives.	
Remove any neutral pairs.	
Count the result.	2 negatives
The sum of −5 and 3 is −2.	−5 + 3 = −2

Notice that there were more negatives than positives, so the result is negative.

Exercise 2.3.5

Model the expression, and then simplify: $2+(-4)$

Answer

-2

Exercise 2.3.6

Model the expression, and then simplify: $2+(-5)$

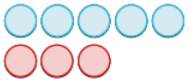
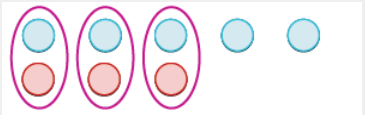
Answer

-3

Example 2.3.4: model the expression

Model: $5+(-3)$.

Solution

Interpret the expression.	5 + (−3) means the sum of 5 and −3.
Model the first number. Start with 5 positives.	
Model the second number. Add 3 negatives.	
Remove any neutral pairs.	
Count the result.	2 positives
The sum of 5 and −3 is 2.	5 + (−3) = 2

Exercise 2.3.7

Model the expression, and then simplify: $(-2)+4$

Answer

2

Exercise 2.3.8

Model the expression: $(-2)+5$

Answer

3

Example 2.3.5: model the addition

Model each addition.

a. $4+2$
b. $-3+6$
c. $4+(-5)$

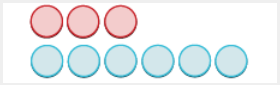
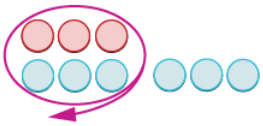
d. $-2+(-3)$

Solution

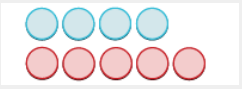
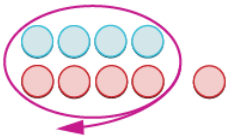
a. $4+2$

Start with 4 positives.	
Add two positives.	
How many do you have? 6.	4 + 2 = 6

b. $-3+6$

Start with 3 negatives.	
Add 6 positives.	
Remove neutral pairs.	
How many are left?	
3.	−3 + 6 = 3

c. $4+(-5)$

Start with 4 positives.	
Add 5 negatives.	
Remove neutral pairs.	
How many are left?	
−1.	4 + (−5) = −1

d. $-2+(-3)$

Start with 2 negatives.	
Add 3 negatives.	
How many do you have? −5.	−2 + (−3) = −5

Exercise 2.3.9

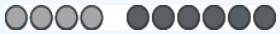
Model each addition.

a. $3+4$
b. $-1+4$
c. $4+(-6)$
d. $-2+(-2)$

Answer a

Answer b

Answer c



Answer d

Exercise2.3.10

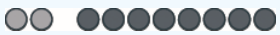
Model each addition.

a. $5+1$
b. $-3+7$
c. $2+(-8)$
d. $-3+(-4)$

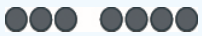
Answer a

Answer b

Answer c



Answer d



Simplify Expressions with Integers

Now that you have modeled adding small positive and negative integers, you can visualize the model in your mind to simplify expressions with any integers.

For example, if you want to add $37+(-53)$, you don't have to count out 37 blue counters and 53 red counters.

Picture 37 blue counters with 53 red counters lined up underneath. Since there would be more negative counters than positive counters, the sum would be negative. Because $53-37=16$, there are 16 more negative counters.

$$37+(-53)=-16$$

Let's try another one. We'll add $-74+(-27)$. Imagine 74 red counters and 27 more red counters, so we have 101 red counters all together. This means the sum is -101.

$$-74+(-27)=-101$$

Look again at the results of Example 2.3.1 - Example 2.3.2.

Table 2.3.1: Addition of Positive and Negative Integers

5 + 3	−5 + (−3)
both positive, sum positive	both negative, sum negative
When the signs are the same, the counters would be all the same color, so add them.	
−5 + 3	5 + (−3)
different signs, more negatives	different signs, more positives
sum negative	sum positive
When the signs are different, some counters would make neutral pairs; subtract to see how many are left.	

Example 2.3.6: simplify

Simplify:

a. $19+(-47)$
b. $-32+40$

Solution

a. Since the signs are different, we subtract 19 from 47. The answer will be negative because there are more negatives than positives. $19+(-47)=-28$
b. The signs are different so we subtract 32 from 40. The answer will be positive because there are more positives than negatives $-32+40=8$

Exercise 2.3.11

Simplify each expression:

a. $15+(-32)$
b. $-19+76$

Answer a

-17

Answer b

57

Exercise 2.3.12

Simplify each expression:

a. $-55+9$
b. $43+(-17)$

Answer a

-46

Answer b

26

Example 2.3.7: simplify

Simplify: $-14+(-36)$.

Solution

Since the signs are the same, we add. The answer will be negative because there are only negatives.

$$-14+(-36)=-50$$

Exercise 2.3.13

Simplify the expression: $-31+(-19)$

Answer

-50

Exercise 2.3.14

Simplify the expression: $-42+(-28)$

Answer

-70

The techniques we have used up to now extend to more complicated expressions. Remember to follow the order of operations.

Example 2.3.8: simplify

Simplify: $-5+3(-2+7)$.

Solution

Simplify inside the parentheses.	−5 + 3(5)
Multiply.	−5 + 15
Add left to right.	10

Exercise 2.3.15

Simplify the expression: $-2+5(-4+7)$

Answer

13

Exercise 2.3.16

Simplify the expression: $-4+2(-3+5)$

Answer

0

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2.4: Add Integers (Part 2)

Evaluate Variable Expressions with Integers

Remember that to evaluate an expression means to substitute a number for the variable in the expression. Now we can use negative numbers as well as positive numbers when evaluating expressions.

Example 2.4.9: evaluate

Evaluate $x+7$ when

a. $x=-2$
b. $x=-11$

Solution

a. Evaluate $x+7$ when $x=-2$

Substitute -2 for x.	$-2+7$	$(2.4.1)$
Simplify.	5	$(2.4.2)$

b. Evaluate $x+7$ when $x=-11$

Substitute -11 for x.	$-11+7$	$(2.4.3)$
Simplify.	-4	$(2.4.4)$

Exercise 2.4.17

Evaluate each expression for the given values: $x+5$ when

a. $x=-3$
b. $x=-17$

Answer a

2

Answer b

-12

Exercise 2.4.18

Evaluate each expression for the given values: $y+7$ when

a. $y=-5$
b. $y=-8$

Answer a

2

Answer b

-1

Example 2.4.10: evaluate

When $n = -5$, evaluate

a. $n + 1$
b. $-n + 1$

Solution

a.

Evaluate n + 1 when n = −5	$n + 1$	(2.4.5)
Substitute -5 for n.	$-5 + 1$	(2.4.6)
Simplify.	-4	(2.4.7)

b.

Evaluate −n + 1 when n = −5	$-n + 1$	(2.4.8)
Substitute -5 for n.	$-(-5) + 1$	(2.4.9)
Simplify.	$5 + 1$	(2.4.10)
Add.	6	(2.4.11)

Exercise 2.4.19

When $n = -8$, evaluate

a. $n + 2$
b. $-n + 2$

Answer a

-6

Answer b

10

Exercise 2.4.20

When $y = -9$, evaluate

a. $y + 8$
b. $-y + 8$

Answer a

-1

Answer b

17

Next we'll evaluate an expression with two variables.

Example 2.4.11: evaluate

Evaluate $3a+b$ when $a=12$ and $b=-30$.

Solution

Substitute 12 for a and -30 for b.	$3(12)+(-30)$
Multiply.	$36+(-30)$
Add.	6

Exercise 2.4.21

Evaluate the expression: $a+2b$ when $a=-19$ and $b=14$.

Answer

9

Exercise 2.4.22

Evaluate the expression: $5p+q$ when $p=4$ and $q=-7$.

Answer

13

Example 2.4.12: evaluate

Evaluate \((x + y)^2\) when $x=-18$ and $y=24$.

Solution

This expression has two variables.

Substitute -18 for x and 24 for y.	$(-18+24)^2$	(2.4.12)
Add inside the parentheses.	$(6)^2$	(2.4.13)
Simplify.	36	(2.4.14)

Exercise 2.4.23

Evaluate: $(x+y)^2$ when $x=-15$ and $y=29$.

Answer

196

Exercise 2.4.24

Evaluate: $(x+y)^3$ when $x=-8$ and $y=10$.

Answer

8

Translate Word Phrases to Algebraic Expressions

All our earlier work translating word phrases to algebra also applies to expressions that include both positive and negative numbers. Remember that the phrase *the sum* indicates addition.

Example 2.4.13: translate

Translate and simplify: the sum of -9 and 5.

Solution

The sum of −9 and 5 indicates addition.	the sum of −9 and 5
Translate.	−9 + 5
Simplify.	−4

Exercise 2.4.25

Translate and simplify the expression: the sum of -7 and 4

Answer

$-7+4=-3$

Exercise 2.4.26

Translate and simplify the expression: the sum of -8 and -6

Answer

$-8+(-6)=-14$

Example 2.4.14: translate

Translate and simplify: the sum of 8 and -12, increased by 3.

Solution

The phrase *increased by* indicates addition.

Translate.	[8 + (−12)] + 3
Simplify.	−4 + 3
Add.	−1

Exercise 2.4.27

Translate and simplify: the sum of 9 and -16, increased by 4.

Answer

$[9+(-16)]+4=-3$

Exercise 2.4.28

Translate and simplify: the sum of -8 and -12, increased by 7.

Answer

$[-8+(-12)]+7=-13$

Add Integers in Applications

Recall that we were introduced to some situations in everyday life that use positive and negative numbers, such as temperatures, banking, and sports. For example, a debt of $\$5$ could be represented as $-\$5$. Let's practice translating and solving a few applications.

Solving applications is easy if we have a plan. First, we determine what we are looking for. Then we write a phrase that gives the information to find it. We translate the phrase into math notation and then simplify to get the answer. Finally, we write a sentence to answer the question.

Example 2.4.15: evaluate

The temperature in Buffalo, NY, one morning started at 7 degrees below zero Fahrenheit. By noon, it had warmed up 12 degrees. What was the temperature at noon?

Solution

We are asked to find the temperature at noon.

Write a phrase for the temperature.	The temperature warmed up 12 degrees from 7 degrees below zero.
Translate to math notation.	−7 + 12
Simplify.	5
Write a sentence to answer the question.	The temperature at noon was 5 degrees Fahrenheit.

Exercise 2.4.29

The temperature in Chicago at 5 A.M. was 10 degrees below zero Celsius. Six hours later, it had warmed up 14 degrees Celsius. What is the temperature at 11 A.M.?

Answer

4 degrees Celsius

Exercise 2.4.30

A scuba diver was swimming 16 feet below the surface and then dove down another 17 feet. What is her new depth?

Answer

-33 feet

Example 2.4.16: evaluate

A football team took possession of the football on their 42-yard line. In the next three plays, they lost 6 yards, gained 4 yards, and then lost 8 yards. On what yard line was the ball at the end of those three plays?

Solution

We are asked to find the yard line the ball was on at the end of three plays.

Write a word phrase for the position of the ball.	Start at 42, then lose 6, gain 4, lose 8.
Translate to math notation	$42 - 6 + 4 - 8$
Simplify.	32
Write a sentence to answer the question.	At the end of the three plays, the ball is on the 32-yard line.

Exercise 2.4.31

The Bears took possession of the football on their 20-yard line. In the next three plays, they lost 9 yards, gained 7 yards, then lost 4 yards. On what yard line was the ball at the end of those three plays?

Answer

14 yard line

Exercise 2.4.32

The Chargers began with the football on their 25-yard line. They gained 5 yards, lost 8 yards and then gained 15 yards on the next three plays. Where was the ball at the end of these plays?

Answer

37 yard line

Access Additional Online Resources

- Adding Integers with Same Sign Using Color Counters
- Adding Integers with Different Signs Using Counters
- Ex1: Adding Integers
- Ex2: Adding Integers

Key Concepts

Addition of Positive and Negative Integers

$5+3$	$-5+(-3)$
both positive, sum positive	both negative, sum negative
When the signs are the same, the counters would be all the same color, so add them.	
$-5+3$	$5+(-3)$
different signs, more negatives	different signs, more positives
Sum negative	sum positive
When the signs are different, some counters would make neutral pairs; subtract to see how many are left.	

Practice Makes Perfect

Model Addition of Integers

In the following exercises, model the expression to simplify.

63. 7 + 4
64. 8 + 5
65. −6 + (−3)
66. −5 + (−5)
67. −7 + 5
68. −9 + 6
69. 8 + (−7)
70. 9 + (−4)

Simplify Expressions with Integers

In the following exercises, simplify each expression.

71. −21 + (−59)
72. −35 + (−47)
73. 48 + (−16)
74. 34 + (−19)
75. −200 + 65
76. −150 + 45
77. 2 + (−8) + 6
78. 4 + (−9) + 7
79. −14 + (−12) + 4
80. −17 + (−18) + 6
81. 135 + (−110) + 83
82. 140 + (−75) + 67
83. −32 + 24 + (−6) + 10
84. −38 + 27 + (−8) + 12
85. 19 + 2(−3 + 8)
86. 24 + 3(−5 + 9)

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

87. 87. x + 8 when
    a. x = −26
    b. x = −95
88. y + 9 when
    a. y = −29
    b. y = −84
89. y + (−14) when
    a. y = −33
    b. y = 30
90. x + (−21) when
    a. x = −27
    b. x = 44
91. When a = −7, evaluate:
    a. a + 3
    b. −a + 3
92. When b = −11, evaluate:

a. $b + 6$
b. $-b + 6$

93. When $c = -9$, evaluate:
    a. $c + (-4)$
    b. $-c + (-4)$
94. When $d = -8$, evaluate:
    a. $d + (-9)$
    b. $-d + (-9)$
95. $m + n$ when, $m = -15$, $n = 7$
96. $p + q$ when, $p = -9$, $q = 17$
97. $r-3s$ when, $r = 16$, $s = 2$
98. $2t + u$ when, $t = -6$, $u = -5$
99. $(a + b)^2$ when, $a = -7$, $b = 15$
100. $(c + d)^2$ when, $c = -5$, $d = 14$
101. $(x + y)^2$ when, $x = -3$, $y = 14$
102. $(y + z)^2$ when, $y = -3$, $z = 15$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate each phrase into an algebraic expression and then simplify.

103. The sum of -14 and 5
104. The sum of -22 and 9
105. 8 more than -2
106. 5 more than -1
107. -10 added to -15
108. -6 added to -20
109. 6 more than the sum of -1 and -12
110. 3 more than the sum of -2 and -8
111. the sum of 10 and -19, increased by 4
112. the sum of 12 and -15, increased by 1

Add Integers in Applications

In the following exercises, solve.

113. **Temperature** The temperature in St. Paul, Minnesota was -19 °F at sunrise. By noon the temperature had risen 26 °F. What was the temperature at noon?
114. **Temperature** The temperature in Chicago was -15 °F at 6 am. By afternoon the temperature had risen 28 °F. What was the afternoon temperature?
115. **Credit Cards** Lupe owes $73 on her credit card. Then she charges $45 more. What is the new balance?
116. **Credit Cards** Frank owes $212 on his credit card. Then he charges $105 more. What is the new balance?
117. **Weight Loss** Angie lost 3 pounds the first week of her diet. Over the next three weeks, she lost 2 pounds, gained 1 pound, and then lost 4 pounds. What was the change in her weight over the four weeks?
118. **Weight Loss** April lost 5 pounds the first week of her diet. Over the next three weeks, she lost 3 pounds, gained 2 pounds, and then lost 1 pound. What was the change in her weight over the four weeks?
119. **Football** The Rams took possession of the football on their own 35-yard line. In the next three plays, they lost 12 yards, gained 8 yards, then lost 6 yards. On what yard line was the ball at the end of those three plays?
120. **Football** The Cowboys began with the ball on their own 20-yard line. They gained 15 yards, lost 3 yards and then gained 6 yards on the next three plays. Where was the ball at the end of these plays?
121. **Calories** Lisbeth walked from her house to get a frozen yogurt, and then she walked home. By walking for a total of 20 minutes, she burned 90 calories. The frozen yogurt she ate was 110 calories. What was her total calorie gain or loss?
122. **Calories** Ozzie rode his bike for 30 minutes, burning 168 calories. Then he had a 140-calorie iced blended mocha. Represent the change in calories as an integer.

Everyday Math

123. **Stock Market** The week of September 15, 2008, was one of the most volatile weeks ever for the U.S. stock market. The change in the Dow Jones Industrial Average each day was: Monday −504, Tuesday +142, Wednesday −449, Thursday +410, Friday +369. What was the overall change for the week?
124. **Stock Market** During the week of June 22, 2009, the change in the Dow Jones Industrial Average each day was: Monday −201, Tuesday −16, Wednesday −23, Thursday +172, Friday −34. What was the overall change for the week?

Writing Exercises

125. Explain why the sum of −8 and 2 is negative, but the sum of 8 and −2 and is positive.
126. Give an example from your life experience of adding two negative numbers.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
model addition of integers.			
simplify expressions with integers.			
evaluate variable expressions with integers.			
translate word phrases to algebraic expressions.			
add integers in applications.			

(b) After reviewing this checklist, what will you do to become confident or all objectives?

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2.5: Subtract Integers (Part 1)

Learning Objectives

- Model subtraction of integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate words phrases to algebraic expressions
- Subtract integers in applications

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: $12-(8-1)$. If you missed this problem, review Example 2.1.8.
2. Translate the *difference of* \(20\) *and* -15 into an algebraic expression. If you missed this problem, review Example 1.3.11.
3. Add: $-18+7$. If you missed this problem, review Example 3.2.6.

Model Subtraction of Integers

Remember the story in the last section about the toddler and the cookies? Children learn how to subtract numbers through their everyday experiences. Real-life experiences serve as models for subtracting positive numbers, and in some cases, such as temperature, for adding negative as well as positive numbers. But it is difficult to relate subtracting negative numbers to common life experiences. Most people do not have an intuitive understanding of subtraction when negative numbers are involved. Math teachers use several different models to explain subtracting negative numbers.

We will continue to use counters to model subtraction. Remember, the blue counters represent positive numbers and the red counters represent negative numbers.

Perhaps when you were younger, you read $5-3$ as *five take away three*. When we use counters, we can think of subtraction the same way.

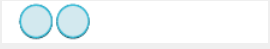
We will model four subtraction facts using the numbers 5 and 3.

$$5-3 \qquad -5-(-3) \qquad -5-3 \qquad 5-(-3)$$

Example 2.5.1: model

Model: $5-3$.

Solution

Interpret the expression.	5 − 3 means 5 take away 3.
Model the first number. Start with 5 positives.	5
Take away the second number. So take away 3 positives.	2 positives
Find the counters that are left.	

The difference between 5 and 3 is 2.

Exercise 2.5.1

Model the expression: $6 - 4$

Answer

2

Exercise 2.5.2

Model the expression: $7 - 4$

Answer

3

Example 2.5.2: model

Model: $-5 - (-3)$.

Solution

Interpret the expression.	−5 − (−3) means −5 take away −3.
Model the first number. Start with 5 negatives.	
Take away the second number. So take away 3 negatives.	
Find the number of counters that are left.	

The difference between -5 and -3 is -2.

Exercise 2.5.3

Model the expression: $-6 - (-4)$

Answer

-2

Exercise 2.5.4

Model the expression: $-7 - (-4)$

Answer

-3

Notice that Example 2.5.1 and Example 2.5.2 are very much alike.

- First, we subtracted 3 positives from 5 positives to get 2 positives.
- Then we subtracted 3 negatives from 5 negatives to get 2 negatives.

Each example used counters of only one color, and the "take away" model of subtraction was easy to apply.

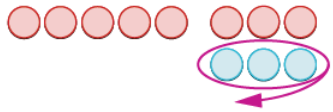
Figure 2.5.1

Now let's see what happens when we subtract one positive and one negative number. We will need to use both positive and negative counters and sometimes some neutral pairs, too. Adding a neutral pair does not change the value.

Example 2.5.3: model

Model: $-5-3$.

Solution

Interpret the expression.	−5 − 3 means −5 take away 3.
Model the first number. Start with 5 negatives.	
Take away the second number. So we need to take away 3 positives.	
But there are no positives to take away. Add neutral pairs until you have 3 positives.	
Now take away 3 positives.	
Count the number of counters that are left.	

The difference of -5 and 3 is -8.

Exercise 2.5.5

Model the expression: $-6-4$

Answer

-10

Exercise 2.5.6

Model the expression: $-7-4$

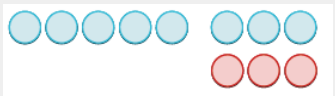
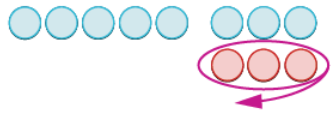
Answer

-11

Example 2.5.4: model

Model: $5-(-3)$.

Solution

Interpret the expression.	5 − (−3) means 5 take away −3.
Model the first number. Start with 5 positives.	
Take away the second number, so take away 3 negatives.	
But there are no negatives to take away. Add neutral pairs until you have 3 negatives.	
Then take away 3 negatives.	
Count the number of counters that are left.	8 positives

The difference of 5 and -3 is 8.

Exercise 2.5.7

Model the expression: $6-(-4)$

Answer

10

Exercise 2.5.8

Model the expression: $7-(-4)$

Answer

11

Example 2.5.5: model

Model each subtraction.

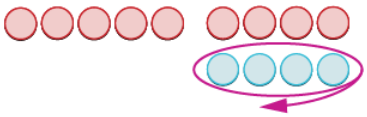
a. $8 - 2$
b. $-5 - 4$
c. $6 - (-6)$
d. $-8 - (-3)$

Solution

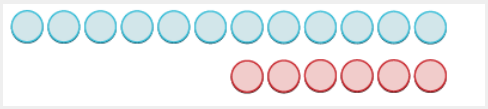
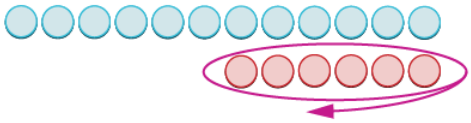
a. $8 - 2$: This means 8 take away 2.

Start with 8 positives.	
Take away 2 positives.	
How many are left?	8 − 2 = 6

b. $-5 - 4$: This means -5 take away 4.

Start with 5 negatives.	
You need to take away 4 positives. Add 4 neutral pairs to get 4 positives.	
Take away 4 positives.	
How many are left?	−5 − 4 = −9

c. $6 - (-6)$: This means 6 take away -6.

Start with 6 positives.	
Add 6 neutrals to get 6 negatives to take away.	
Remove 6 negatives.	
How many are left?	6 − (−6) = 12

d. $-8 - (-3)$: This means -8 take away -3.

Start with 8 negatives.	

Take away 3 negatives.	
How many are left?	$-8-(-3)=-5$

Exercise 2.5.9

Model each subtraction.

a. $7-(-8)$
b. $-7-(-2)$
c. $4-1$
d. $-6-8$

Answer a

Answer b

Answer c

Answer d

Exercise 2.5.10

Model each subtraction.

a. $4-(-6)$
b. $-8-(-1)$
c. $7-3$
d. $-4-2$

Answer a

Answer b

Answer c

Answer d

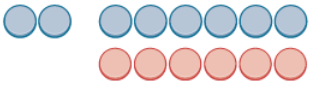
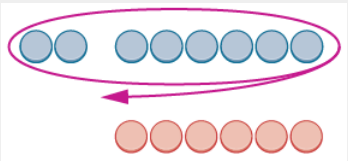
Example 2.5.6: model

Model each subtraction expression:

a. $2-8$
b. $-3-(-8)$

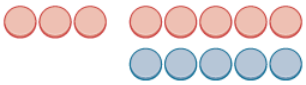
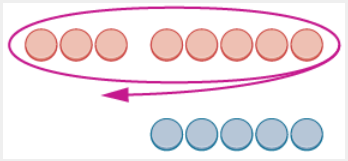
Solution

a.

We start with 2 positives.	2
We need to take away 8 positives, but we have only 2.	
Add neutral pairs until there are 8 positives to take away.	
Then take away eight positives.	
Find the number of counters that are left. There are 6 negatives.	6 negatives

$2-8=-6$

b.

We start with 3 negatives.	3
We need to take away 8 negatives, but we have only 3.	
Add neutral pairs until there are 8 negatives to take away	
Then take away the 8 negatives.	
Find the number of counters that are left. There are 5 positives.	5 positives

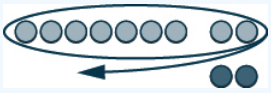
$-3-(-8)=5$

Exercise 2.5.11

Model each subtraction expression.

a. $7 - 9$
b. $-5 - (-9)$

Answer a



-2

Answer b

4

Exercise 2.5.12

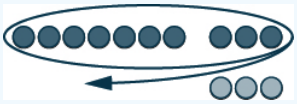
Model each subtraction expression.

a. $4 - 7$
b. $-7 - (-10)$

Answer a

-3

Answer b



3

Simplify Expressions with Integers

Do you see a pattern? Are you ready to subtract integers without counters? Let's do two more subtractions. We'll think about how we would model these with counters, but we won't actually use the counters.

- Subtract $-23 - 7$. Think: We start with 23 negative counters. We have to subtract 7 positives, but there are no positives to take away. So we add 7 neutral pairs to get the 7 positives. Now we take away the 7 positives. So what's left? We have the original 23 negatives plus 7 more negatives from the neutral pair. The result is 30 negatives. $-23 - 7 = -30$ Notice, that to subtract 7, we added 7 negatives.
- Subtract $30 - (-12)$. Think: We start with 30 positives. We have to subtract 12 negatives, but there are no negatives to take away. So we add 12 neutral pairs to the 30 positives. Now we take away the 12 negatives. What's left? We have the original 30 positives plus 12 more positives from the neutral pairs. The result is 42 positives. $30 - (-12) = 42$ Notice that to subtract -12, we added 12.

While we may not always use the counters, especially when we work with large numbers, practicing with them first gave us a concrete way to apply the concept, so that we can visualize and remember how to do the subtraction without the counters.

Have you noticed that subtraction of signed numbers can be done by adding the opposite? You will often see the idea, the Subtraction Property, written as follows:

Definition: Subtraction Property

$$a-b=a+(-b) \tag{2.5.1}$$

Look at these two examples.

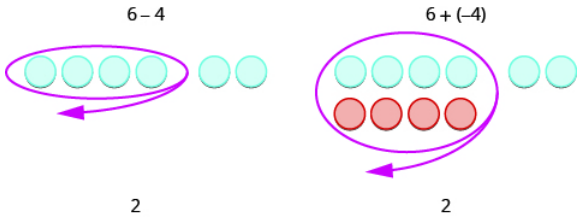


Figure 2.5.2

We see that $6-4$ gives the same answer as $6+(-4)$.

Of course, when we have a subtraction problem that has only positive numbers, like the first example, we just do the subtraction. We already knew how to subtract $6-4$ long ago. But knowing that $6-4$ gives the same answer as $6+(-4)$ helps when we are subtracting negative numbers.

Example 2.5.7: simplify

Simplify:

a. $13-8$ and $13+(-8)$
b. $-17-9$ and $-17+(-9)$

Solution

a.

Subtract to simplify.	13 − 8 = 5
Add to simplify.	13 + (−8) = 5
Subtracting 8 from 13 is the same as adding −8 to 13.	

b.

Subtract to simplify.	−17 − 9 = −26
Add to simplify.	−17 + (−9) = −26
Subtracting 9 from −17 is the same as adding −9 to −17.	

Exercise 2.5.13

Simplify each expression:

a. $21-13$ and $21+(-13)$
b. $-11-7$ and $-11+(-7)$

Answer a

8, 8

Answer b

$-18, -18$

Exercise 2.5.14

Simplify each expression:

a. $15-7$ and $15+(-7)$
b. $-14-8$ and $-14+(-8)$

Answer a

$8, 8$

Answer b

$-22, -22$

Now look what happens when we subtract a negative.

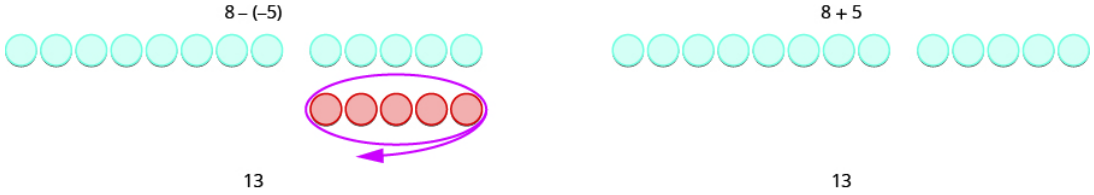


Figure 2.5.3

We see that $8-(-5)$ gives the same result as $8+5$. Subtracting a negative number is like adding a positive.

Example 2.5.8: simplify

Simplify:

a. $9-(-15)$ and $9+15$
b. $-7-(-4)$ and $-7+4$

Solution

a. $9-(-15)$ and $9+15$

Subtract to simplify.	9 − (−15) = 24
Add to simplify.	9 + 15 = 24

Subtracting -15 from 9 is the same as adding 15 to 9.

b. $-7-(-4)$ and $-7+4$

Subtract to simplify.	−7 − (−4) = −3
Add to simplify.	−7 + 4 = −3

Subtracting -4 from -7 is the same as adding 4 to -7.

Exercise 2.5.15

Simplify each expression:

a. $6-(-13)$ and $6+13$
b. $-5-(-1)$ and $-5+1$

Answer a

$19, 19$

Answer b

$-4, -4$

? Exercise 2.5.16

Simplify each expression:

a. $4-(-19)$ and $4+19$
b. $-4-(-7)$ and $-4+7$

Answer a

$23, 23$

Answer b

$3, 3$

Look again at the results of Example 2.5.1 - Example 2.5.4.

Table 2.5.1: Subtraction of Integers

5 – 3	–5 – (–3)
2	–2
2 positives	2 negatives
When there would be enough counters of the color to take away, subtract.	
–5 – 3	5 – (–3)
–8	8
5 negatives, want to subtract 3 positives	5 positives, want to subtract 3 negatives
need neutral pairs	need neutral pairs
When there would not be enough of the counters to take away, add neutral pairs.	

Contributors and Attributions

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2.6: Subtract Integers (Part 2)

Example 2.6.9: simplify

Simplify: $-74-(-58)$.

Solution

We are taking 58 negatives away from 74 negatives.	−74 − (−58)
Subtract.	−16

Exercise 2.6.17

Simplify the expression: $-67-(-38)$

Answer

-29

Exercise 2.6.18

Simplify the expression: $-83-(-57)$

Answer

-26

Example 2.6.10

Simplify: $7-(-4-3)-9$.

Solution

We use the order of operations to simplify this expression, performing operations inside the parentheses first. Then we subtract from left to right.

Simplify inside the parentheses first.	7 − (−7) − 9
Subtract from left to right.	14 − 9
Subtract.	5

Exercise 2.6.19

Simplify the expression: $8-(-3-1)-9$

Answer

3

Exercise 2.6.20

Simplify the expression: $12-(-9-6)-14$

Answer

13

Example 2.6.11: simplify

Simplify: $3 \bullet 7 - 4 \bullet 7 - 5 \bullet 8$.

Solution

We use the order of operations to simplify this expression. First we multiply, and then subtract from left to right.

Multiply first.	21 - 28 - 40
Subtract from left to right.	-7 - 40
Subtract.	-47

Exercise 2.6.21

Simplify the expression: $6 \bullet 2 - 9 \bullet 1 - 8 \bullet 9$

Answer

-69

Exercise 2.6.22

Simplify the expression: $2 \bullet 5 - 3 \bullet 7 - 4 \bullet 9$

Answer

-47

Evaluate Variable Expressions with Integers

Now we'll practice evaluating expressions that involve subtracting negative numbers as well as positive numbers.

Example 2.6.12: evaluate

Evaluate $x - 4$ when

a. $x = 3$
b. $x = -6$

Solution

a. To evaluate $x - 4$ when $x = 3$, substitute 3 for x in the expression.

Substitute 3 for x.	$3 - 4$
Subtract.	-1

b. To evaluate $x - 4$ when $x = -6$, substitute -6 for x in the expression.

Substitute -6 for x.	$-6 - 4$
Subtract.	-10

Exercise 2.6.23

Evaluate each expression: $y-7$ when

a. $y=5$
b. $y=-8$

Answer a

-2

Answer b

-15

Exercise 2.6.24

Evaluate each expression: $m-3$ when

a. $m=1$
b. $m=-4$

Answer a

-2

Answer b

-7

Example 2.6.13: evaluate

Evaluate $20-z$ when

a. $z=12$
b. $z=-12$

Solution

a. To evaluate $20-z$ when $z=12$, substitute 12 for z in the expression.

Substitute 12 for z.	$20-12$
Subtract.	8

b. To evaluate $20-z$ when $z=-12$, substitute -12 for z in the expression.

Substitute -12 for z.	$20-(-12)$
Subtract.	32

Exercise 2.6.25

Evaluate each expression: $17-k$ when

a. $k=19$
b. $k=-19$

Answer a

-2

Answer b

36

Exercise 2.6.26

Evaluate each expression: $-5-b$ when

a. $b=14$
b. $b=-14$

Answer a

-19

Answer b

9

Translate Word Phrases to Algebraic Expressions

When we first introduced the operation symbols, we saw that the expression $a-b$ may be read in several ways as shown below.

$a-b$
a minus b
the difference of a and b
subtract b from a
b subtracted from a
b less than a

Figure 2.6.4

Be careful to get a and b in the right order!

Example 2.6.14: translate and simplify

Translate and then simplify:

a. the difference of 13 and -21
b. subtract 24 from -19

Solution

a. A *difference* means subtraction. Subtract the numbers in the order they are given.

Translate.	13 - (-21)
Simplify	34

b. *Subtract* means to take 24 away from -19.

Translate.	-19 - 24
Simplify	-43

Exercise 2.6.27

Translate and simplify:

a. the difference of 14 and -23
b. subtract 21 from -17

Answer a

$-14-(-23)=37$

Answer b

$-17-21=-38$

Exercise 2.6.28

Translate and simplify:

a. the difference of 11 and -19
b. subtract 18 from -11

Answer a

$11-(-19)=30$

Answer b

$-11-18=-29$

Subtract Integers in Applications

It's hard to find something if we don't know what we're looking for or what to call it. So when we solve an application problem, we first need to determine what we are asked to find. Then we can write a phrase that gives the information to find it. We'll translate the phrase into an expression and then simplify the expression to get the answer. Finally, we summarize the answer in a sentence to make sure it makes sense.

HOW TO: SOLVE APPLICATION PROBLEMS.

Step 1. Identify what you are asked to find.

Step 2. Write a phrase that gives the information to find it.

Step 3. Translate the phrase to an expression.

Step 4. Simplify the expression.

Step 5. Answer the question with a complete sentence.

Example 2.6.15: simplify

The temperature in Urbana, Illinois one morning was 11 degrees Fahrenheit. By mid-afternoon, the temperature had dropped to -9 degrees Fahrenheit. What was the difference between the morning and afternoon temperatures?

Solution

Step 1. Identify what we are asked to find.	the difference between the morning and afternoon temperatures
Step 2. Write a phrase that gives the information to find it.	the difference of 11 and −9
Step 3. Translate the phrase to an expression. The word *difference* indicates subtraction.	11 − (−9)
Step 4. Simplify the expression.	20
Step 5. Write a complete sentence that answers the question.	The difference in temperature was 20 degrees Fahrenheit.

Exercise 2.6.29

The temperature in Anchorage, Alaska one morning was 15 degrees Fahrenheit. By mid-afternoon the temperature had dropped to 30 degrees below zero. What was the difference between the morning and afternoon temperatures?

Answer

45 degrees Fahrenheit

Exercise 2.6.30

The temperature in Denver was -6 degrees Fahrenheit at lunchtime. By sunset the temperature had dropped to -15 degree Fahrenheit. What was the difference between the lunchtime and sunset temperatures?

Answer

9 degrees Fahrenheit

Geography provides another application of negative numbers with the elevations of places below sea level.

Example 2.6.16: simplify

Dinesh hiked from Mt. Whitney, the highest point in California, to Death Valley, the lowest point. The elevation of Mt. Whitney is $14,497$ feet above sea level and the elevation of Death Valley is 282 feet below sea level. What is the difference in elevation between Mt. Whitney and Death Valley?

Solution

Step 1. What are we asked to find?	The difference in elevation between Mt. Whitney and Death Valley
Step 2. Write a phrase.	elevation of Mt. Whitney−elevation of Death Valley
Step 3. Translate.	14,497 − (−282)
Step 4. Simplify.	14,779
Step 5. Write a complete sentence that answers the question.	The difference in elevation is 14,779 feet.

Exercise 2.6.31

One day, John hiked to the $10,023$ foot summit of Haleakala volcano in Hawaii. The next day, while scuba diving, he dove to a cave 80 feet below sea level. What is the difference between the elevation of the summit of Haleakala and the depth of the cave?

Answer

$10,103$ feet

Exercise 2.6.32

The submarine Nautilus is at 340 feet below the surface of the water and the submarine Explorer is 573 feet below the surface of the water. What is the difference in the position of the Nautilus and the Explorer?

Answer

233 feet

Managing your money can involve both positive and negative numbers. You might have overdraft protection on your checking account. This means the bank lets you write checks for more money than you have in your account (as long as they know they can get it back from you!)

Example 2.6.17: simplify

Leslie has $25 in her checking account and she writes a check for $8.

a. What is the balance after she writes the check?
b. She writes a second check for $20. What is the new balance after this check?
c. Leslie's friend told her that she had lost a check for $10 that Leslie had given her with her birthday card. What is the balance in Leslie's checking account now?

Solution

a.

What are we asked to find?	The balance of the account
Write a phrase.	$25 minus $8
Translate.	$25 - $8
Simplify.	$17
Write a sentence answer.	The balance is $17.

b.

What are we asked to find?	The new balance
Write a phrase.	$17 minus $20
Translate.	$17 - $20
Simplify.	-$3
Write a sentence answer.	She is overdrawn by $3.

c.

What are we asked to find?	The new balance
Write a phrase.	$10 more than −$3
Translate.	-$3 + $10
Simplify.	$7
Write a sentence answer.	The balance is now $7.

Exercise 2.6.33

Araceli has $75 in her checking account and writes a check for $27.

a. What is the balance after she writes the check?
b. She writes a second check for $50. What is the new balance?
c. The check for $20 that she sent a charity was never cashed. What is the balance in Araceli's checking account now?

Answer a

$48

Answer b

$-\$2$

Answer c

$\$18$

? Exercise 2.6.34

Genevieve's bank account was overdrawn and the balance is $-\$78$.

a. She deposits a check for $\$24$ that she earned babysitting. What is the new balance?
b. She deposits another check for $\$49$. Is she out of debt yet? What is her new balance?

Answer a

$-\$54$

Answer b

No, $-\$5$

Access Additional Online Resources

- Adding and Subtracting Integers
- Subtracting Integers with Color Counters
- Subtracting Integers Basics
- Subtracting Integers
- Integer Application

Key Concepts

- Subtraction of Integers

5–3	–5–(–3)
2	–2
2 positives	2 negatives
When there would be enough counters of the color to take away, subtract.	
–5–3	5–(–3)
–8	8
5 negatives, want to subtract 3 positives	5 positives, want to subtract 3 negatives
need neutral pairs	need neutral pairs
When there would not be enough of the counters to take away, add neutral pairs.	

- Subtraction Property
  - $a-b=a+(-b)$
  - $a-(-b)=a+b$
- Solve Application Problems
  - Step 1. Identify what you are asked to find.
  - Step 2. Write a phrase that gives the information to find it.
  - Step 3. Translate the phrase to an expression.
  - Step 4. Simplify the expression.

- Step 5. Answer the question with a complete sentence.

Practice Makes Perfect

Model Subtraction of Integers

In the following exercises, model each expression and simplify.

127. 8 − 2
128. 9 − 3
129. −5 − (−1)
130. −6 − (−4)
131. −5 − 4
132. −7 − 2
133. 8 − (−4)
134. 7 − (−3)

Simplify Expressions with Integers

In the following exercises, simplify each expression.

135. (a) 15 − 6 (b) 15 + (−6)
136. (a) 12 − 9 (b) 12 + (−9)
137. (a) 44 − 28 (b) 44 + (−28)
138. (a) 35 − 16 (b) 35 + (−16)
139. (a) 8 − (−9) (b) 8 + 9
140. (a) 4 − (−4) (b) 4 + 4
141. (a) 27 − (−18) (b) 27 + 18
142. (a) 46 − (−37) (b) 46 + 37

In the following exercises, simplify each expression.

143. 15 − (−12)
144. 14 − (−11)
145. 10 − (−19)
146. 11 − (−18)
147. 48 − 87
148. 45 − 69
149. 31 − 79
150. 39 − 81
151. −31 − 11
152. −32 − 18
153. −17 − 42
154. −19 − 46
155. −103 − (−52)
156. −105 − (−68)
157. −45 − (−54)
158. −58 − (−67)
159. 8 − 3 − 7
160. 9 − 6 − 5
161. −5 − 4 + 7
162. −3 − 8 + 4
163. −14 − (−27) + 9
164. −15 − (−28) + 5
165. 71 + (−10) − 8
166. 64 + (−17) − 9
167. −16 − (−4 + 1) − 7
168. −15 − (−6 + 4) − 3

169. $(2 - 7) - (3 - 8)$
170. $(1 - 8) - (2 - 9)$
171. $-(6 - 8) - (2 - 4)$
172. $-(4 - 5) - (7 - 8)$
173. $25 - [10 - (3 - 12)]$
174. $32 - [5 - (15 - 20)]$
175. $6.3 - 4.3 - 7.2$
176. $5.7 - 8.2 - 4.9$
177. $5^2 - 6^2$
178. $6^2 - 7^2$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression for the given values.

179. $x - 6$ when (a) $x = 3$ (b) $x = -3$
180. $x - 4$ when (a) $x = 5$ (b) $x = -5$
181. $5 - y$ when (a) $y = 2$ (b) $y = -2$
182. $8 - y$ when (a) $y = 3$ (b) $y = -3$
183. $4x^2 - 15x + 1$ when $x = 3$
184. $5x^2 - 14x + 7$ when $x = 2$
185. $-12 - 5x^2$ when $x = 6$
186. $-19 - 4x^2$ when $x = 5$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate each phrase into an algebraic expression and then simplify.

187. (a) The difference of 3 and −10 (b) Subtract −20 from 45
188. (a) The difference of 8 and −12 (b) Subtract −13 from 50
189. (a) The difference of −6 and 9 (b) Subtract −12 from −16
190. (a) The difference of −8 and 9 (b) Subtract −15 from −19
191. (a) 8 less than −17 (b) −24 minus 37
192. (a) 5 less than −14 (b) −13 minus 42
193. (a) 21 less than 6 (b) 31 subtracted from −19
194. (a) 34 less than 7 (b) 29 subtracted from −50

Subtract Integers in Applications

In the following exercises, solve the following applications.

195. **Temperature** One morning, the temperature in Urbana, Illinois, was 28° Fahrenheit. By evening, the temperature had dropped 38° Fahrenheit. What was the temperature that evening?
196. **Temperature** On Thursday, the temperature in Spincich Lake, Michigan, was 22° Fahrenheit. By Friday, the temperature had dropped 35° Fahrenheit. What was the temperature on Friday?
197. **Temperature** On January 15, the high temperature in Anaheim, California, was 84° Fahrenheit. That same day, the high temperature in Embarrass, Minnesota was −12° Fahrenheit. What was the difference between the temperature in Anaheim and the temperature in Embarrass?
198. **Temperature** On January 21, the high temperature in Palm Springs, California, was 89°, and the high temperature in Whitefield, New Hampshire was −31°. What was the difference between the temperature in Palm Springs and the temperature in Whitefield?
199. **Football** At the first down, the Warriors football team had the ball on their 30-yard line. On the next three downs, they gained 2 yards, lost 7 yards, and lost 4 yards. What was the yard line at the end of the third down?
200. **Football** At the first down, the Barons football team had the ball on their 20-yard line. On the next three downs, they lost 8 yards, gained 5 yards, and lost 6 yards. What was the yard line at the end of the third down?
201. **Checking Account** John has $148 in his checking account. He writes a check for $83. What is the new balance in his checking account?

202. **Checking Account** Ellie has $426 in her checking account. She writes a check for $152. What is the new balance in her checking account?
203. **Checking Account** Gina has $210 in her checking account. She writes a check for $250. What is the new balance in her checking account?
204. **Checking Account** Frank has $94 in his checking account. He writes a check for $110. What is the new balance in his checking account?
205. **Checking Account** Bill has a balance of −$14 in his checking account. He deposits $40 to the account. What is the new balance?
206. **Checking Account** Patty has a balance of −$23 in her checking account. She deposits $80 to the account. What is the new balance?

Everyday Math

207. **Camping** Rene is on an Alpine hike. The temperature is −7°. Rene's sleeping bag is rated "comfortable to −20°". How much can the temperature change before it is too cold for Rene's sleeping bag?
208. **Scuba Diving** Shelly's scuba watch is guaranteed to be watertight to −100 feet. She is diving at −45 feet on the face of an underwater canyon. By how many feet can she change her depth before her watch is no longer guaranteed?

Writing Exercises

209. Explain why the difference of 9 and −6 is 15.
210. Why is the result of subtracting 3 − (−4) the same as the result of adding 3 + 4?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section

I can...	Confidently	With some help	No-I don't get it!
model subtraction of integers.			
simplify expressions with integers.			
evaluate variable expressions with integers.			
translate word phrases to algebraic expressions.			
subtract integers in applications.			

(b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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2.7: Multiply and Divide Integers (Part 1)

Learning Objectives

- Multiply integers
- Divide integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate word phrases to algebraic expressions

be prepared!

Before you get started, take this readiness quiz.

1. Translate the quotient of 20 and 13 into an algebraic expression. If you missed this problem, review Example 1.5.12.
2. Add: $-5+(-5)+(-5)$. If you missed this problem, review Example 3.2.8.
3. Evaluate $n+4$ when $n=-7$. If you missed this problem, review Example 3.2.10.

Multiply Integers

Since multiplication is mathematical shorthand for repeated addition, our counter model can easily be applied to show multiplication of integers. Let's look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction.

We remember that $a \bullet b$ means add a, b times. Here, we are using the model shown in Figure $2.7.1$ just to help us discover the pattern.

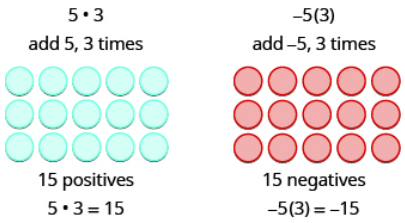

Figure 2.7.1

Now consider what it means to multiply 5 by -3. It means subtract 5, 3 times. Looking at subtraction as taking away, it means to take away 5, 3 times. But there is nothing to take away, so we start by adding neutral pairs as shown in Figure $2.7.2$.

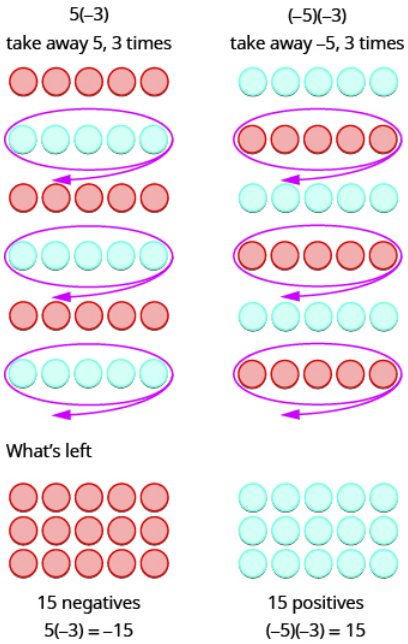

Figure 2.7.2

In both cases, we started with 15 neutral pairs. In the case on the left, we took away 5, 3 times and the result was -15. To multiply $(-5)(-3)$, we took away -5, 3 times and the result was 15. So we found that

5(3) = 15	-5(3) = -15
5(-3) = -15	(-5)(-3) = 15

Notice that for multiplication of two signed numbers, when the signs are the same, the product is positive, and when the signs are different, the product is negative.

Definition: Multiplication of Signed Numbers

The sign of the product of two numbers depends on their signs.

Same signs	Product
Two positives	Positive
Two negatives	Positive

Different signs	Product
Positive • negative	Negative
Negative • positive	Negative

Example 2.7.1:multiply

Multiply each of the following:

a. $-9 \bullet 3$
b. $-2(-5)$
c. $4(-8)$
d. $7 \bullet 6$

Solution

a.

Multiply, noting that the signs are different and so the product is negative.	–9 • 3 = –27

b.

Multiply, noting that the signs are the same and so the product is positive.	–2(–5) = 10

c.

Multiply, noting that the signs are different and so the product is negative.	4(–8) = –32

d.

The signs are the same, so the product is positive.	7 • 6 = 42

Exercise 2.7.1

Multiply:

a. $-6 \bullet 8$
b. $-4(-7)$
c. $9(-7)$
d. $5 \bullet 12$

Answer a

-48

Answer b

28

Answer c

-63

Answer d

60

Exercise 2.7.2

Multiply:

a. $-8 \bullet 7$
b. $-6(-9)$
c. $7(-4)$
d. $3 \bullet 13$

Answer a

-56

Answer b

54

Answer c

-28

Answer d

39

When we multiply a number by 1, the result is the same number. What happens when we multiply a number by -1? Let's multiply a positive number and then a negative number by -1 to see what we get.

$-1 \cdot 4$	$-1(-3)$
-4	3
-4 is the opposite of 4	3 is the opposite of -3

Each time we multiply a number by -1, we get its opposite.

Definition: Multiplication by -1

Multiplying a number by -1 gives its opposite.

$$-1 \cdot a = -a \tag{2.7.1}$$

Example 2.7.2: multiply

Multiply each of the following:

a. $-1 \bullet 7$
b. $-1(-11)$

Solution

a.

The signs are different, so the product will be negative.	$-1 \cdot 7$
Notice that -7 is the opposite of 7.	-7

b.

The signs are the same, so the product will be positive.	$-1(-11)$
Notice that 11 is the opposite of -11.	11

Exercise 2.7.3

Multiply.

a. $-1 \bullet 9$
b. $-1 \bullet (-17)$

Answer a

-9

Answer b

17

Exercise 2.7.4

Multiply.

a. $-1 \bullet 8$
b. $-1 \bullet (-16)$

Answer a

-8

Answer b

16

Divide Integers

Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $5 \bullet 3 = 15$ In words, this expression says that 15 can be divided into 3 groups of 5 each because adding five three times gives 15. If we look at some examples of multiplying integers, we might figure out the rules for dividing integers.

5 • 3 = 15 so 15 ÷ 3 = 5	−5(3) = −15 so −15 ÷ 3 = −5
(−5)(−3) = 15 so 15 ÷ (−3) = −5	5(−3) = −15 so −15 ÷ −3 = 5

Division of signed numbers follows the same rules as multiplication. When the signs are the same, the quotient is positive, and when the signs are different, the quotient is negative.

Definition: Division of Signed Numbers

The sign of the quotient of two numbers depends on their signs.

Same signs	Quotient
Two positives	Positive
Two negatives	Positive

Different signs	Quotient
Positive & negative	Negative
Negative & positive	Negative

Remember, you can always check the answer to a division problem by multiplying.

Example 2.7.3: divide

Divide each of the following:

a. $-27 \div 3$
b. $-100 \div (-4)$

Solution

a.

Divide, noting that the signs are different and so the quotient is negative.	–27 ÷ 3 = –9

b.

Divide, noting that the signs are the same and so the quotient is positive.	–100 ÷ (–4) = 25

Exercise 2.7.5

Divide:

a. $-42 \div 6$
b. $-117 \div (-3)$

Answer a

-7

Answer b

39

Exercise 2.7.6

Divide:

a. $-63 \div 7$
b. $-115 \div (-5)$

Answer a

-9

Answer b

23

Just as we saw with multiplication, when we divide a number by 1, the result is the same number. What happens when we divide a number by -1? Let's divide a positive number and then a negative number by -1 to see what we get.

$8 \div (-1)$	$-9 \div (-1)$
-8	9
-8 is the opposite of 8	9 is the opposite of -9

When we divide a number by, -1 we get its opposite.

Definition: Division by -1

Dividing a number by -1 gives its opposite.

$$a \div (-1) = -a \tag{2.7.2}$$

Example 2.7.4: divide

Divide each of the following:

a. $16 \div (-1)$
b. $-20 \div (-1)$

Solution

a.

The dividend, 16, is being divided by –1.	$16 \div (-1)$
Dividing a number by –1 gives its opposite.	–16

Notice that the signs were different, so the result was negative.

b.

The dividend, –20, is being divided by –1.	$-20 \div (-1)$
Dividing a number by –1 gives its opposite.	20

Notice that the signs were the same, so the quotient was positive.

Exercise 2.7.7

Divide:

a. $6 \div (-1)$
b. $-36 \div (-1)$

Answer a

-6

Answer b

36

Exercise 2.7.8

Divide:

a. $28 \div (-1)$
b. $-52 \div (-1)$

Answer a

-28

Answer b

52

Simplify Expressions with Integers

Now we'll simplify expressions that use all four operations–addition, subtraction, multiplication, and division–with integers. Remember to follow the order of operations.

Example 2.7.5: simplify

Simplify: $7(-2)+4(-7)-6$.

Solution

We use the order of operations. Multiply first and then add and subtract from left to right.

Multiply first.	$-14 + (-28) - 6$
Add.	$-42 - 6$

Subtract.	−48

Exercise 2.7.9

Simplify: $8(-3)+5(-7)-4$

Answer

-63

Exercise 2.7.10

Simplify: $9(-3)+7(-8)-1$

Answer

-84

Example 2.7.6: simplify

Simplify:

a. $(-2)^4$
b. -2^4

Solution

The exponent tells how many times to multiply the base.

a. The exponent is 4 and the base is -2. We raise -2 to the fourth power.

Write in expanded form.	(−2)(−2)(−2)(−2)
Multiply.	4(−2)(−2)
Multiply.	−8(−2)
Multiply.	16

b. The exponent is 4 and the base is 2. We raise 2 to the fourth power and then take the opposite.

Write in expanded form.	−(2 • 2 • 2 • 2)
Multiply.	−(4 • 2 • 2)
Multiply.	−(8 • 2)
Multiply.	−16

Exercise 2.7.11

Simplify:

a. $(-3)^4$
b. -3^4

Answer a

81

Answer b

-81

Exercise 2.7.12

Simplify:

a. $(-7)^2$
b. -7^2

Answer a

49

Answer b

-49

Example 2.7.7: simplify

Simplify: $12-3(9-12)$.

Solution

According to the order of operations, we simplify inside parentheses first. Then we will multiply and finally we will subtract.

Subtract the parentheses first.	12 − 3(−3)
Multiply.	12 − (−9)
Subtract.	21

Exercise 2.7.13

Simplify: $17-4(8-11)$

Answer

29

Exercise 2.7.14

Simplify: $16-6(7-13)$

Answer

52

Example 2.7.8: simplify

Simplify: $8(-9)\div(-2)^3$.

Solution

We simplify the exponent first, then multiply and divide.

Simplify the exponent.	8(−9) ÷ (−8)
Multiply.	−72 ÷ (−8)

Divide.	9

Exercise 2.7.15

Simplify: $12(-9) \div (-3)^3$

Answer

4

Exercise 2.7.16

Simplify: $18(-4) \div (-2)^3$

Answer

9

Example 2.7.9: simplify

Simplify: $-30 \div 2 + (-3)(-7)$.

Solution

First we will multiply and divide from left to right. Then we will add.

Divide.	$-15 + (-3)(-7)$
Multiply.	$-15 + 21$
Add.	6

Exercise 2.7.17

Simplify: $-27 \div 3 + (-5)(-6)$

Answer

21

Exercise 2.7.18

Simplify: $-32 \div 4 + (-2)(-7)$

Answer

6

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2.8: Multiply and Divide Integers (Part 2)

Evaluate Variable Expressions with Integers

Now we can evaluate expressions that include multiplication and division with integers. Remember that to evaluate an expression, substitute the numbers in place of the variables, and then simplify.

Example 2.8.10: evaluate

Evaluate $2x^2 - 3x + 8$ when $x = -4$.

Solution

Substitute -4 for x.	$2(-4)^2 - 3(-4) + 8$
Simplify exponents.	$2(16) - 3(-4) + 8$
Multiply.	$32 - (-12) + 8$
Subtract.	$44 + 8$
Add.	52

Keep in mind that when we substitute -4 for x, we use parentheses to show the multiplication. Without parentheses, it would look like $2 \bullet -4^2 - 3 \bullet -4 + 8$.

Exercise 2.8.19

Evaluate: $3x^2 - 2x + 6$ when $x = -3$

Answer

39

Exercise 2.8.20

Evaluate: $4x^2 - x - 5$ when $x = -2$

Answer

13

Example 2.8.11: evaluate

Evaluate $3x + 4y - 6$ when $x = -1$ and $y = 2$.

Solution

Substitute x = -1 and y = 2.	$3(-1) + 4(2) - 6$
Multiply.	$-3 + 8 - 6$
Simplify.	-1

Exercise 2.8.21

Evaluate: $7x + 6y - 12$ when $x = -2$ and $y = 3$

Answer

-8

Exercise 2.8.22

Evaluate: $8x-6y+13$ when $x=-3$ and $y=-5$

Answer

19

Translate Word Phrases to Algebraic Expressions

Once again, all our prior work translating words to algebra transfers to phrases that include both multiplying and dividing integers. Remember that the key word for multiplication is *product* and for division is *quotient*.

Example 2.8.12: translate

Translate to an algebraic expression and simplify if possible: the product of -2 and 14.

Solution

The word *product* tells us to multiply.

Translate.	$(-2)(14)$
Simplify.	-28

Exercise 2.8.23

Translate to an algebraic expression and simplify if possible: the product of -5 and 12

Answer

$-5(12)=-60$

Exercise 2.8.24

Translate to an algebraic expression and simplify if possible: the product of 8 and -13

Answer

$8(-13)=-104$

Example 2.8.13

Translate to an algebraic expression and simplify if possible: the quotient of -56 and -7.

Solution

The word *quotient* tells us to divide.

Translate.	$-56 \div (-7)$
Simplify.	8

Exercise 2.8.25

Translate to an algebraic expression and simplify if possible: the quotient of -63 and -9

Answer

$-63 \div -9 = 7$

Exercise 2.8.26

Translate to an algebraic expression and simplify if possible: the quotient of -72 and -9

Answer

$-72 \div -9 = 8$

Access Additional Online Resources

- Multiplying Integers Using Color Counters
- Multiplying Integers Using Color Counters With Neutral Pairs
- Multiplying Integers Basics
- Dividing Integers Basics
- Ex. Dividing Integers
- Multiplying and Dividing Signed Numbers

Key Concepts

- Multiplication of Signed Numbers
  - To determine the sign of the product of two signed numbers:

Same Signs	Product
Two positives Two negatives	Positive Positive

Different Signs	Product
Positive • negative Negative • positive	Negative Negative

- Division of Signed Numbers
  - To determine the sign of the quotient of two signed numbers:

Same Signs	Quotient
Two positives Two negatives	Positive Positive

Different Signs	Quotient
Positive • negative Negative • Positive	Negative Negative

- Multiplication by -1
  - Multiplying a number by -1 gives its opposite: $-1a = -a$

- Division by -1
  - Dividing a number by -1 gives its opposite: $a \div (-1) = -a$

Practice Makes Perfect

Multiply Integers

In the following exercises, multiply each pair of integers.

211. $-4 \cdot 8$
212. $-3 \cdot 9$
213. $-5(7)$
214. $-8(6)$
215. $-18(-2)$
216. $-10(-6)$
217. $9(-7)$
218. $13(-5)$
219. $-1 \cdot 6$
220. $-1 \cdot 3$
221. $-1(-14)$
222. $-1(-19)$

Divide Integers

In the following exercises, divide.

223. $-24 \div 6$
224. $-28 \div 7$
225. $56 \div (-7)$
226. $35 \div (-7)$
227. $-52 \div (-4)$
228. $-84 \div (-6)$
229. $-180 \div 15$
230. $-192 \div 12$
231. $49 \div (-1)$
232. $62 \div (-1)$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

233. $5(-6) + 7(-2) - 3$
234. $8(-4) + 5(-4) - 6$
235. $-8(-2) - 3(-9)$
236. $-7(-4) - 5(-3)$
237. $(-5)^3$
238. $(-4)^3$
239. $(-2)^6$
240. $(-3)^5$
241. -4^2
242. -6^2
243. $-3(-5)(6)$
244. $-4(-6)(3)$
245. $-4 \cdot 2 \cdot 11$
246. $-5 \cdot 3 \cdot 10$
247. $(8 - 11)(9 - 12)$
248. $(6 - 11)(8 - 13)$
249. $26 - 3(2 - 7)$

250. $23 - 2(4 - 6)$
251. $-10(-4) \div (-8)$
252. $-8(-6) \div (-4)$
253. $65 \div (-5) + (-28) \div (-7)$
254. $52 \div (-4) + (-32) \div (-8)$
255. $9 - 2[3 - 8(-2)]$
256. $11 - 3[7 - 4(-2)]$
257. $(-3)^2 - 24 \div (8 - 2)$
258. $(-4)^2 - 32 \div (12 - 4)$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

259. $-2x + 17$ when (a) $x = 8$ (b) $x = -8$
260. $-5y + 14$ when (a) $y = 9$ (b) $y = -9$
261. $10 - 3m$ when (a) $m = 5$ (b) $m = -5$
262. $18 - 4n$ when (a) $n = 3$ (b) $n = -3$
263. $p^2 - 5p + 5$ when $p = -1$
264. $q^2 - 2q + 9$ when $q = -2$
265. $2w^2 - 3w + 7$ when $w = -2$
266. $3u^2 - 4u + 5$ when $u = -3$
267. $6x - 5y + 15$ when $x = 3$ and $y = -1$
268. $3p - 2q + 9$ when $p = 8$ and $q = -2$
269. $9a - 2b - 8$ when $a = -6$ and $b = -3$
270. $7m - 4n - 2$ when $m = -4$ and $n = -9$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

271. The product of -3 and 15
272. The product of -4 and 16
273. The quotient of -60 and -20
274. The quotient of -40 and -20
275. The quotient of -6 and the sum of a and b
276. The quotient of -7 and the sum of m and n
277. The product of -10 and the difference of p and q
278. The product of -13 and the difference of c and d

Everyday Math

279. **Stock market** Javier owns 300 shares of stock in one company. On Tuesday, the stock price dropped $12 per share. What was the total effect on Javier's portfolio?
280. **Weight loss** In the first week of a diet program, eight women lost an average of 3 pounds each. What was the total weight change for the eight women?

Writing Exercises

281. In your own words, state the rules for multiplying two integers.
282. In your own words, state the rules for dividing two integers.
283. Why is $-2^4 \neq (-2)^4$?
284. Why is $-4^2 \neq (-4)^2$?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
multiply integers.			
divide integers.			
simplify expressions with integers.			
evaluate variable expressions with integers.			
translate word phrases to algebraic expressions.			

(b) On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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2.9: Square Roots

Recall that

$$x^2 = x \cdot x.$$

The Square of a Number

The number x^2 is called the square of the number x.

Thus, for example:

- $9^2 = 9 \cdot 9 = 81$. Therefore, the number 81 is the square of the number 9.
- $(-4)^2 = (-4)(-4) = 16$. Therefore, the number 16 is the square of the number −4.

In the margin, we've placed a "List of Squares" of the whole numbers ranging from 0 through 25, inclusive.

x	x^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

Square Roots

Once you've mastered the process of squaring a whole number, then you are ready for the inverse of the squaring process, taking the *square root* of a whole number.

- Above, we saw that $9^2 = 81$. We called the number 81 the square of the number 9. Conversely, we call the number 9 a *square root* of the number 81.
- Above, we saw that $(-4)^2 = 16$. We called the number 16 the *square* of the number −4. Conversely, we call the number −4 a *square root* of the number 16.

x	$\sqrt{x}$
0	0
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
81	9
100	10
121	11
144	12
169	13
196	14
225	15
256	16
289	17
324	18
361	19
400	20
441	21
484	22
529	23
576	24
625	25

Square Root

If $a^2 = b$, then a is called a square root of the number b.

Example 1

Find the square roots of the number 49.

Solution

To find a square root of 49, we must think of a number a such that $a^2 = 49$. Two numbers come to mind.

- $(-7)^2 = 49$. Therefore, -7 is a square root of 49.
- $7^2 = 49$. Therefore, 7 is a square root of 49.

Note that 49 has two square roots, one of which is positive and the other one is negative.

Exercise

Find the square roots of 256.

Answer

$-16, 16$

Example 2

Find the square roots of the number 196. Solution. To find a square root of 196, we must think of a number a such that a^2 = 196. With help from the "List of Squares," two numbers come to mind.

- $(-14)^2$ = 196. Therefore, −14 is a square root of 196.
- 14^2 = 196. Therefore, 14 is a square root of 196.

Note that 196 has two square roots, one of which is positive and the other one is negative.

Exercise

Find the square roots of 625.

Answer

−25, 25

Example 3

Find the square roots of the number 0.

Solution

To find a square root of 0, we must think of a number *a* such that a^2 = 0. There is only one such number, namely zero. Hence, 0 is the square root of 0.

Exercise

Find the square roots of 9.

Answer

−3, 3

Example 4

Find the square roots of the number −25.

Solution

To find a square root of −25, we must think of a number a such that a^2 = −25. This is impossible because no square of a real number (whole number, integer, fraction, or decimal) can be negative. Positive times positive is positive and negative times negative is also positive. You cannot square and get a negative answer. Therefore, −25 has no square roots[2].

Exercise

Find the square roots of −81.

Answer

There are none.

[2]At least not in Prealgebra. In later courses, you will be introduced to the set of complex numbers, where −25 will have two square roots.

Radical Notation

Because $(-3)^2$ = 9 and 3^2 = 9, both −3 and 3 are square roots of 9. Special notation, called *radical notation*, is used to request these square roots.

- The radical notation $\sqrt{9}$, pronounced "the nonnegative square root of 9," calls for the nonnegative[3] square root of 9. Hence,

$$\sqrt{9} = 3.$$

- The radical notation $-\sqrt{9}$, pronounced "the negative square root of 9," calls for the negative square root of 9. Hence,

$$-\sqrt{9} = -3.$$

Radical Notation

In the expression $\sqrt{9}$, the symbol $\sqrt{\ }$ is called a *radical* and the number within the radical, in this case the number 9, is called the *radicand*.

For example,

- In the expression $\sqrt{529}$, the number 529 is the radicand.
- In the expression $\sqrt{a^2 + b^2}$, the expression $a2^+b^2$ is the radicand.

Radical Notation and Square Root

If *b* is a positive number, then

1. $\sqrt{b}$ calls for the nonnegative square root of *b*.
2. $-\sqrt{b}$ calls for the negative square root of *b*.

Note: Nonnegative is equivalent to saying "not negative;" i.e., positive or zero.

Example 5

Simplify: (a) $\sqrt{121}$, (b) $-\sqrt{625}$, and (c) $\sqrt{0}$.

Solution

(a) Referring to the list of squares, we note that 11^2 = 121 and $(-11)^2$ = 121. Therefore, both 11 and −11 are square roots of 121. However, $\sqrt{121}$ calls for the nonnegative square root of 121. Thus,

$$\sqrt{121} = 11.$$

(b) Referring to the list of squares, we note that 25^2 = 625 and $(-25)^2$ = 625. Therefore, both 25 and −25 are square roots of 625. However, $-\sqrt{625}$ calls for the negative square root of 625. Thus,

$$-\sqrt{625} = -25.$$

(c) There is only one square root of zero. Therefore,

$$\sqrt{0} = 0.$$

Exercise

Simplify: a) $\sqrt{144}$ b) $-\sqrt{324}$

Answer

(a) 12 (b) −18

Example 6

Simplify: (a) $-\sqrt{25}$, and (b) $-\sqrt{25}$

Solution

(a) Because 5^2 = 25 and $(-5)^2$ = 25, both 5 and −5 are square roots of 25. However, the notation $-\sqrt{25}$ calls for the negative square root of 25. Thus, $-\sqrt{25} = -5$.

(b) It is not possible to square a real number (whole number, integer, fraction, or decimal) and get −25. Therefore, there is no real square root of −25. That is, $\sqrt{-25}$ is not a real number. It is undefined.[4]

Exercise

Simplify: a) $-\sqrt{36}$ b) $\sqrt{-36}$

Answer

(a) −6 (b) undefined

[3]Nonnegative is equivalent to saying "not negative;" i.e., positive or zero.

[4]At least in Prealgebra. In later courses you will be introduced to the set of complex numbers, where $\sqrt{-25}$ will take on a new meaning.

Order of Operations

With the addition of radical notation, the *Rules Guiding Order of Operations* change slightly.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents and radicals that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

The only change in the rules is in item #2, which says: "Evaluate all exponents and radicals that appear in the expression," putting radicals on the same level as exponents.

Example 7

Simplify: $-3\sqrt{9}+12\sqrt{4}$.

Solution

According to the *Rules Guiding Order of Operations*, we must evaluate the radicals in this expression first.

$$\begin{aligned} -3\sqrt{9}+12\sqrt{4} &= -3(3)+12(2) && \text{Evaluate radicals first: } \sqrt{9}=3 \\ & && \text{and } \sqrt{4}=2. \\ &= -9+24 && \text{Multiply: } -3(3)=-9 \text{ and } 12(2)=24. \\ &= 15 && \text{Add: } -9+24=15. \end{aligned}$$

Exercise

Simplify: $2\sqrt{4}-3\sqrt{9}$

Answer

−5

Example 8

Simplify: $-2-3\sqrt{36}$.

Solution

According to the *Rules Guiding Order of Operations*, we must evaluate the radicals in this expression first, moving left to right.

$$\begin{aligned} -2-3\sqrt{36} &= -2-36 && \text{Evaluate radicals first: } \sqrt{36}=6 \\ &= -2-18 && \text{Multiply: } 3(6)=18. \\ &= -20 && \text{Subtract: } -2-18=-2+(-18)=-20. \end{aligned}$$

? Exercise

Simplify: $5-8\sqrt{169}$

Answer

−99

✔ Example 9

Simplify: (a) $\sqrt{9+16}$ and (b) $\sqrt{9}+\sqrt{16}$.

Solution

Apply the *Rules Guiding Order of Operations*.

a) In this case, the radical acts like grouping symbols, so we must evaluate what is inside the radical first.

$$\begin{aligned} \sqrt{9+16} &= \sqrt{25} && \text{Add: } 9+16=25. \\ &= 5 && \text{Take nonnegative square root: } \sqrt{25}=5. \end{aligned}$$

b) In this example, we must evaluate the square roots first.

$$\begin{aligned} \sqrt{9}+\sqrt{16} &= 3+4 && \text{Square root: } \sqrt{9}=3 \text{ and } \sqrt{16}=4. \\ &= 7 && \text{Add: } 3+4=7. \end{aligned}$$

? Exercise

Simplify: a) $\sqrt{25+144}$ b) $\sqrt{25}+\sqrt{144}$

Answer

(a) 13 (b) 17

Fractions and Decimals

We can also find square roots of fractions and decimals.

✔ Example 10

Simplify: (a) $\sqrt{49}$, and (b) $-\sqrt{0.49}$.

Solution

(a) Because $\left(\frac{2}{3}\right)^2=\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)=\frac{4}{9}$, then

$$\sqrt{\frac{4}{9}}=\frac{2}{3}.$$

(b) Because $(0.7)^2 = (0.7)(0.7) = 0.49$ and $(-0.7)^2 = (-0.7)(-0.7) = 0.49$, both 0.7 and −0.7 are square roots of 0.49. However, $-\sqrt{0.49}$ calls for the negative square root of 0.49. Hence,

$$-\sqrt{0.49}=-0.7$$

Exercise

Simplify: a) $\sqrt{\frac{25}{49}}$ b) $\sqrt{0.36}$

Answer

(a) 5/7 (b) 0.6

Estimating Square Roots

The squares in the "List of Squares" are called *perfect squares*. Each is the square of a whole number. Not all numbers are perfect squares. For example, in the case of $\sqrt{24}$, there is no whole number whose square is equal to 24. However, this does not prevent $\sqrt{24}$ from being a perfectly good number.

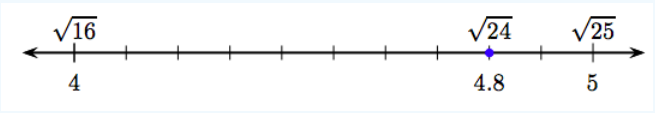
We can use the "List of Squares" to find decimal approximations when the radicand is not a perfect square.

Example 11

Estimate $\sqrt{24}$ by guessing. Use a calculator to find a more accurate result and compare this result with your guess.

Solution

From the "List of Squares," note that 24 lies betwen 16 and 25, so 24 will lie between 4 and 5, with √24 much closer to 5 than it is to 4.



Let's guess

$$\sqrt{24} \approx 4.8.$$

As a check, let's square 4.8.

$$(4.8)^2 = (4.8)(4.8) = 23.04$$

Not quite 24! Clearly, $\sqrt{24}$ must be a little bit bigger than 4.8.

Let's use a scientific calculator to get a better approximation. From our calculator, using the square root button, we find

$$\sqrt{24} \approx 4.89897948557.$$

Even though this is better than our estimate of 4.8, it is still only an approximation. Our calculator was only capable of providing 11 decimal places. However, the exact decimal representation of $\sqrt{24}$ is an infinite decimal that never terminates and never establishes a pattern of repetition.

Just for fun, here is a decimal approximation of $\sqrt{24}$ that is accurate to 1000 places, courtesy of http://www.wolframalpha.com/.

4.89897948556635619639456814941178278393189496131334025686538513450192075491463005307971886620928046963718920245322837824971773091967551468325156790247455710565782549505535314249526021054182354044696262135797338170726488670509120806761761787874917113569314944872260828854054043234840367660016317961567602617940145738798726167431618880160088747737509832902930787829002408945289626663258702188948362702657099008893234345326285099529663624900802313209072918018687172335863967331332533818263813071727532210516312358732472358220589344176709151025767105979664820111738041001283093224823470679882086211598579693467906510557472083659310343660782073560076724633259464660565809954782094852720141025275395093777354012819859118514346569290057761830288514926052059059264741510500684551198309085256259600612934415988485060457568524106813589572009319387995987119508123342717309306912496416512553772738561882612744867017729603144969267446489475909097628876958672740183948202955704655118212631969215662073401907064945 3

If you were to multiply this number by itself (square the number), you would get a number that is extremely close to 24, but it would not be exactly 24. There would still be a little discrepancy.

? Exercise

Estimate: $\sqrt{83}$.

Answer

9.1

Important Observation

A calculator can only produce a finite number of decimal places. If the decimal representation of your number does not terminate within this limited number of places, then the number in your calculator window is only an approximation.

- The decimal representation of 1/8 will terminate within three places, so most calculators will report the exact answer, 0.125.
- For contrast, 2/3 does not terminate. A calculator capable of reporting 11 places of accuracy produces the number 0.666666666667. However, the exact decimal representation of 2/3 is 0.6. Note that the calculator has rounded in the last place and only provides an approximation of 2/3. If your instructor asks for an exact answer on an exam or quiz then 0.666666666667, being an approximation, is not acceptable. You must give the exact answer 2/3.

Exercises

In Exercises 1-16, list all square roots of the given number. If the number has no square roots, write “none”.

1. 256
2. 361
3. −289
4. −400
5. 441
6. 36
7. 324
8. 0
9. 144
10. 100
11. −144
12. −100
13. 121
14. −196
15. 529
16. 400

In Exercises 17-32, compute the exact square root. If the square root is undefined, write “undefined”.

17. $\sqrt{-9}$
18. $-\sqrt{-196}$
19. $\sqrt{576}$
20. $\sqrt{289}$

21. $\sqrt{-529}$
22. $\sqrt{-256}$
23. $-\sqrt{25}$
24. $\sqrt{225}$
25. $-\sqrt{484}$
26. $-\sqrt{36}$
27. $-\sqrt{196}$
28. $-\sqrt{289}$
29. $\sqrt{441}$
30. $\sqrt{324}$
31. $-\sqrt{4}$
32. $\sqrt{100}$

In Exercises 33-52, compute the exact square root.

33. $\sqrt{0.81}$
34. $\sqrt{5.29}$
35. $\sqrt{3.61}$
36. $\sqrt{0.09}$
37. $\sqrt{\frac{225}{16}}$
38. $\sqrt{\frac{100}{81}}$
39. $\sqrt{3.24}$
40. $\sqrt{5.76}$
41. $\sqrt{\frac{121}{49}}$
42. $\sqrt{\frac{625}{324}}$
43. $\sqrt{\frac{529}{121}}$
44. $\sqrt{\frac{4}{121}}$
45. $\sqrt{2.89}$
46. $\sqrt{4.41}$
47. $\sqrt{\frac{144}{25}}$
48. $\sqrt{\frac{49}{36}}$
49. $\sqrt{\frac{256}{361}}$
50. $\sqrt{\frac{529}{16}}$
51. $\sqrt{0.49}$
52. $\sqrt{4.84}$

In Exercises 53-70, compute the exact value of the given expression.

53. $6-\sqrt{576}$
54. $-2-7\sqrt{576}$
55. $\sqrt{8^2+15^2}$
56. $\sqrt{7^2+24^2}$
57. $6\sqrt{16}-9\sqrt{49}$
58. $3\sqrt{441}+6\sqrt{484}$
59. $\sqrt{5^2+12^2}$
60. $\sqrt{15^2+20^2}$
61. $\sqrt{3^2+4^2}$
62. $\sqrt{6^2+8^2}$
63. $-2\sqrt{324}-6\sqrt{361}$
64. $-6\sqrt{576}-8\sqrt{121}$
65. $-4-3\sqrt{529}$
66. $-1+\sqrt{625}$
67. $-9\sqrt{484}+7\sqrt{81}$
68. $-\sqrt{625}-5\sqrt{576}$
69. $2-\sqrt{16}$
70. $8-6\sqrt{400}$

In Exercises 71-76, complete the following tasks to estimate the given square root.

a) Determine the two integers that the square root lies between.

b) Draw a number line, and locate the approximate location of the square root between the two integers found in part (a).

c) Without using a calculator, estimate the square root to the nearest tenth.

71. $\sqrt{58}$
72. $\sqrt{27}$
73. $\sqrt{79}$
74. $\sqrt{12}$
75. $\sqrt{44}$
76. $\sqrt{88}$

In Exercises 77-82, use a calculator to approximate the square root to the nearest tenth.

77. $\sqrt{469}$
78. $\sqrt{73}$
79. $\sqrt{615}$
80. $\sqrt{162}$
81. $\sqrt{444}$
82. $\sqrt{223}$

Answers

1. 16, −16

3. none

5. 21, −21

7. 18, −18

9. 12, −12

11. none

13. 11, −11

15. 23, −23

17. undefined

19. 24

21. undefined

23. −5

25. −22

27. −14

29. 21

31. −2

33. 0.9

35. 1.9

37. $\frac{15}{4}$

39. 1.8

41. 11 7

43. $\frac{23}{11}$

45. 1.7

47. $\frac{12}{5}$

49. $\frac{16}{19}$

51. 0.7

53. −18

55. 17

57. −39

59. 13

61. 5

63. −150

65. −73

67. −135

69. − 2

71. 7.6

73. 8.9

75. 6.6

77. 21.7

79. 24.8

81. 21.1

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2.10: What to Combine? Order of Operations with Integers

For convenience, we repeat the "Rules Guiding Order of Operations" first introduced in Section 1.5.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

Let's look at a number of examples that require the use of these rules.

Example 1

Simplify: (a) $(-3)^2$ and (b) -3^2

Solution

Recall that for any integer a, we have $(-1)a = -a$. Because negating is equivalent to multiplying by -1, the "Rules Guiding Order of Operations" require that we address grouping symbols and exponents before negation.

a) Because of the grouping symbols, we negate first, then square. That is,

$$\begin{aligned}(-3)^2 &= (-3)(-3)\\ &= 9.\end{aligned}$$

b) There are no grouping symbols in this example. Thus, we must square first, then negate. That is,

$$\begin{aligned}-3^2 &= -(3 \cdot 3)\\ &= -9.\end{aligned}$$

Exercise

Simplify: -2^2.

Answer

-4

Example 2

Simplify: $-2 - 3(5 - 7)$.

Solution

Grouping symbols first, then multiplication, then subtraction.

$$\begin{aligned}-2-3(5-7) &= -2-3(-2) && \text{Perform subtraction within parentheses.}\\ &= 2-2-(-6) && \text{Multiply: } 3(-2) = -6.\\ &= -2+6 && \text{Add the opposite.}\\ &= 4\end{aligned}$$

Exercise

Simplify: $-3-2(6-8)$.

Answer

1

Example 3

Simplify: $-2(2-4)^2-3(3-5)^3$.

Solution

Grouping symbols first, then multiplication, and subtraction, in that order.

\[\begin{aligned} -2(2-4)^2 -3(3-5)^3 = -2(-2)^2 -3(-2)^3 ~ & \textcolor{red}{ \text{ Perform subtraction within parentheses first.} } \\ =2 (4) -3(-8) ~ & \textcolor{red}{ \text{ Exponents are next.} } \\ =-8-(-24) ~ & \textcolor{red}{ \text{ Multiplications are next.} } \\ =-8+24 ~ & \textcolor{red}{ \text{ Add the opposite.} } \\ =16 \end{aligned}\nonumber \]

Exercise

Simplify: $-2(5-6)^3-3(5-7)^2$

Answer

-10

Example 4

Simplify: $-24 \div 8(-3)$.

Solution

Division has no preference over multiplication, or vice versa. Divisions and multiplications must be performed in the order that they occur, moving left to right.

$$\begin{aligned} -24 \div 8(-3) &= -3(-3) \quad \text{Division first: } -24 \div 8 = -3. \\ &= 9 \end{aligned}$$

Note that if you multiply first, which would be incorrect, you would get a completely different answer.

Exercise

Simplify: $-48 \div 6(-2)$.

Answer

16

Example 5

Simplify: $(-2)(-3)(-2)^3$.

Solution

Exponents first, then multiplication in the order that it occurs, moving left to right.

$$\begin{aligned} (-2)(-3)(-2)^3 &= (-2)(-3)(-8) \quad \text{Exponent first: } (-2)^3 = -8. \\ &= 6(-8) \quad \text{Multiply from left to right: } (-2)(-3) = 6. \\ &= -48 \end{aligned}$$

? You try it!

Simplify: $(-4)(-2)^2(-1)^3$.

Answer

16

Evaluating Fractions

If a fraction bar is present, evaluate the numerator and denominator separately according to the "Rules Guiding Order of Operations," then perform the division in the final step.

✔ Example 6

Simplify:

$$\frac{-5-5(2-4)^3}{-22-3(-5)}$$

Solution

Evaluate numerator and denominator separately, then divide.

$$\begin{aligned}
\frac{-5-5(2-4)^3}{-22-3(-5)} &= \frac{-5-5(-2)^3}{-22-(-15)} && \text{Numerator: parentheses first.}\\
&= \frac{-5-5(-8)}{-22+15} && \text{Denominator: multiply } 3(-5)=-15.\\
&= \frac{-5-(-40)}{-7} && \text{Numerator: exponent } (-2)^3=-8.\\
&= \frac{-5+40}{-7} && \text{Denominator: add the opposite.}\\
&= \frac{35}{-7} && \text{Numerator: multiply } 5(-8)=-40.\\
&= -5 && \text{Denominator: add } -22+15=-7.\\
& && \text{Numerator: add the opposite.}\\
& && \text{Numerator: } -5+40=35.\\
& && \text{Divide: } 35/-7=-5.
\end{aligned}$$

? Exercise

Simplify:

$$\frac{6-2(-6)}{-2-(-2)^2}$$

Answer

-3

Absolute Value

Absolute value calculates the magnitude of the vector associated with an integer, which is equal to the distance between the number and the origin (zero) on the number line. Thus, for example, |4| = 4 and | − 5| = 5.

But absolute value bars also act as grouping symbols, and according to the "Rules Guiding Order of Operations," you should evaluate the expression inside a pair of grouping symbols first.

Example 7

Simplify: (a) $-(-3)$ and (b) $-|-3|$.

Solution

There is a huge difference between simple grouping symbols and absolute value.

a) This is a case of $-(-a) = a$. Thus, $-(-3) = 3$.

b) This case is much different. The absolute value of -3 is 3, and then the negative of that is -3. In symbols,

$$-|-3| = -3$$

Exercise

Simplify: $-|-8|$.

Answer

-8

Example 8

Simplify: $-3 - 2|5 - 7|$.

Solution

Evaluate the expression inside the absolute value bars first. Then multiply, then subtract.

$$-3 - 2|5 - 7| = -3 - 2|-2|$$

Exercise

Simplify: $-2 - 4|6 - 8|$.

Answer

-10

Exercises

In Exercises 1-40, compute the exact value of the given expression.

1. $7 - \frac{-14}{2}$
2. $-2 - \frac{-16}{4}$
3. $-7 - \frac{-18}{9}$
4. $-6 - \frac{-7}{7}$
5. -54
6. -33
7. $9 - 1(-7)$
8. $85 - 8(9)$
9. -63
10. -35
11. $3 + 9(4)$

12. $6 + 7(-1)$
13. $10 - 72 \div 6 \cdot 3+8$
14. $8 - 120 \div 5 \cdot 6+7$
15. $6 + \frac{14}{2}$
16. $16 + \frac{8}{2}$
17. -34
18. -22
19. $3 - 24 \div 4 \cdot 3+4$
20. $4 - 40 \div 5 \cdot 4+9$
21. $64 \div 4 \cdot 4$
22. $18 \div 6 \cdot 1$
23. $-2 - 3(-5)$
24. $64 - 7(7)$
25. $15 \div 1 \cdot 3$
26. $30 \div 3 \cdot 5$
27. $8 + 12 \div 6 \cdot 1 - 5$
28. $9 + 16 \div 2 \cdot 4 - 9$
29. $32 \div 4 \cdot 4$
30. $72 \div 4 \cdot 6$
31. $-11 + \frac{16}{16}$
32. $4 + \frac{-20}{4}$
33. -52
34. -43
35. $10 + 12(-5)$
36. $4 + 12(4)$
37. $2+6 \div 1 \cdot 6 - 1$
38. $1 + 12 \div 2 \cdot 2 - 6$
39. $40 \div 5 \cdot 4$
40. $30 \div 6 \cdot 5$

In Exercises 41-80, simplify the given expression.

41. $-11 + |-1 - (-6)^2|$
42. $13 + |-21 - (-4)^2|$
43. $|0(-4)| - 4(-4)$
44. $|10(-3)| - 3(-1)$
45. $(2 + 3 \cdot 4) - 8$
46. $(11 + 5 \cdot 2) - 10$
47. $(8 - 1 \cdot 12) + 4$
48. $(9 - 6 \cdot 1) + 3$

49. $(6 + 10 \cdot 4) - 6$

50. $(8 + 7 \cdot 6) - 12$

51. $10 + (6 - 4)^3 - 3$

52. $5 + (12 - 7)^2 - 6$

53. $(6 - 8)^2 - (4 - 7)^3$

54. $(3 - 8)^2 - (4 - 9)^3$

55. $|0(-10)| + 4(-4)$

56. $|12(-5)| + 7(-5)$

57. $|8(-1)| - 8(-7)$

58. $|6(-11)| - 7(-1)$

59. $3 + (3 - 8)^2 - 7$

60. $9 + (8 - 3)^3 - 6$

61. $(4 - 2)^2 - (7 - 2)^3$

62. $(1 - 4)^2 - (3 - 6)^3$

63. $8 - |- 25 - (-4)^2|$

64. $20 - |- 22 - 4^2|$

65. $-4 - |30 - (-5)^2|$

66. $-8 - |- 11 - (-6)^2|$

67. $(8 - 7)^2 - (2 - 6)^3$

68. $(2 - 7)^2 - (4 - 7)^3$

69. $4 - (3 - 6)^3 + 4$

70. $6 - (7 - 8)^3 + 2$

71. $-3 + | - 22 - 5^2|$

72. $12 + |23 - (-6)^2|$

73. $(3 - 4 \cdot 1) + 6$

74. $(12 - 1 \cdot 6) + 4$

75. $1 - (1 - 5)^2 + 11$

76. $9 - (3 - 1)^3 + 10$

77. $(2 - 6)^2 - (8 - 6)^3$

78. $(2 - 7)^2 - (2 - 4)^3$

79. $|9(-3)| + 12(-2)$

80. $|0(-3)| + 9(-7)$

In Exercises 81-104, simplify the given expression.

81. $\frac{4(-10)-5}{-9}$

82. $\frac{-4 \cdot 6-(-8)}{-4}$

83. $\frac{10^2-4^2}{2 \cdot 6-10}$

84. $\frac{3^2-9^2}{2 \cdot 7-5}$

85. $\frac{3^2+6^2}{5-1\cdot 8}$

86. $\frac{10^2+4^2}{1-6\cdot 5}$

87. $\frac{-8-4}{7-13}$

88. $\frac{13-1}{8-4}$

89. $\frac{2^2+6^2}{11-4\cdot 4}$

90. $\frac{7^2+3^2}{10-8\cdot 1}$

91. $\frac{1^2-5^2}{9\cdot 1-5}$

92. $\frac{5^2-7^2}{2\cdot 2-12}$

93. $\frac{4^2-8^2}{6\cdot 3-2}$

94. $\frac{7^2-6^2}{6\cdot 3-5}$

95. $\frac{10^2+2^2}{10-2\cdot 7}$

96. $\frac{2^2+10^2}{10-2\cdot 7}$

97. $\frac{16-(-2)}{19-1}$

98. $\frac{-8-20}{-15-(-17}$

99. $\frac{15-(-15)}{13-(-17)}$

100. $\frac{7-(-9)}{-1-1}$

101. $\frac{4\cdot 5-(-19)}{3}$

102. $\frac{10\cdot 7-(-11)}{-3}$

103. $\frac{-6\cdot 9-(-4)}{2}$

104. $\frac{-6\cdot 2-10}{-11}$

Answers

1. 14

3. −5

5. −625

7. 16

9. −216

11. 39

13. −18

15. 13

17. −81

19. −11

21. 64

23. 13

25. 45

27. 5

29. 32

31. −10

33. −25

35. −50

37. 37

39. 32

41. 26

43. 16

45. 6

47. 0

49. 40

51. 15

53. 31

55. −16

57. 64

59. 21

61. −121

63. −33

65. −9

67. 65

69. 35

71. 44

73. 5

75. −4

77. 8

79. 3

81. 5

83. 42

85. −15

87. 2

89. −8

91. −6

93. −3

95. −8

97. 1

99. 1

101. 13

103. −25

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2.11: Mathematical Expressions (NOT Equations)

Recall the definition of a *variable* presented in Section 1.6.

Definition: Variable

A variable is a symbol (usually a letter) that stands for a value that may vary.

Let's add the definition of a *mathematical expression*.

Definition: Mathematical Expression

When we combine numbers and variables in a valid way, using operations such as addition, subtraction, multiplication, division, exponentiation, and other operations and functions as yet unlearned, the resulting combination of mathematical symbols is called a *mathematical expression*.

Thus,

$$2a,\ x + 5, \text{ and } y^2,$$

being formed by a combination of numbers, variables, and mathematical operators, are valid mathematical expressions. A mathematical expression must be *well-formed*. For example,

$$2 + \div 5x$$

is *not a valid expression* because there is no term following the plus sign (it is not valid to write +÷ with nothing between these operators). Similarly,

$$2 + 3(2$$

is not well-formed because parentheses are not balanced.

Translating Words into Mathematical Expressions

In this section we turn our attention to translating word phrases into mathematical expressions. We begin with phrases that translate into *sums*. There is a wide variety of word phrases that translate into sums. Some common examples are given in Table 2.11.1*a*, though the list is far from complete. In like manner, a number of phrases that translate into differences are shown in Table 2.11.1*b*

Table 2.11.1: Translating words into symbols.

Phrase	Translates to:	Phrase	Translates to:
sum of x and 12	$x + 12$	difference of x and 12	$x - 12$
4 greater than b	$b + 4$	4 less than b	$b - 4$
6 more than y	$y + 6$	7 subtracted from y	$y - 7$
44 plus r	$44 + r$	44 minus r	$44 - r$
3 larger than z	$z + 3$	3 smaller than z	$z - 3$
a) Phrases that are sums		b) Phrases that are differences	

Let's look at some examples, some of which translate into expressions involving sums, and some of which translate into expressions involving differences.

Example 1

Translate the following phrases into mathematical expressions:

a. "12 larger than x,"
b. "11 less than y," and

c. "r decreased by 9."

Solution

Here are the translations.

a. “12 larger than x” becomes $x + 12$.
b. “11 less than y” becomes $y - 11$.
c. “r decreased by 9” becomes $r - 9$.

Exercise

Translate the following phrases into mathematical expressions:

a. "13 more than x" and
b. "12 fewer than y".

Answer

(a) $x + 13$ and

(b) $y - 12$

Example 2

Let W represent the width of the rectangle. The length of a rectangle is 4 feet longer than its width. Express the length of the rectangle in terms of its width W.

Solution

We know that the width of the rectangle is W. Because the length of the rectangle is 4 feet longer that the width, we must add 4 to the width to find the length.

Length	is	4	more than	the width
Length	$=$	4	$+$	W

Thus, the length of the rectangle, in terms of its width W, is $4 + W$.

Exercise

The width of a rectangle is 5 inches shorter than its length L. Express the width of the rectangle in terms of its length L.

Answer

$L - 5$

Example 3

A string measures 15 inches is cut into two pieces. Let x represent the length of one of the resulting pieces. Express the length of the second piece in terms of the length x of the first piece.

Solution

The string has original length 15 inches. It is cut into two pieces and the first piece has length x. To find the length of the second piece, we must subtract the length of the first piece from the total length.

Length of the second piece	is	Total length	minus	the length of the first piece
Length of the second piece	$=$	15	$-$	x

Thus, the length of the second piece, in terms of the length x of the first piece, Answer: $12 + x$ is $15 - x$.

Exercise

A string is cut into two pieces, the first of which measures 12 inches. Express the total length of the string as a function of x, where x represents the length of the second piece of string.

Answer

$12 + x$

There is also a wide variety of phrases that translate into products. Some examples are shown in Table 3.2(a), though again the list is far from complete. In like manner, a number of phrases translate into quotients, as shown in Table 3.2(b).

Table 2.11.2: Translating words into symbols.

Phrase	Translates to:	Phrase	Translates to:
product of x and 12	$12x$	quotient of x and 12	$x/12$
4 times b	$4b$	4 divided by b	$4/b$
twice r	$2r$	the ratio of 44 to r	$44/r$
a) Phrases that are products.		b) Phrases that are differences.	

Let's look at some examples, some of which translate into expressions involving products, and some of which translate into expressions involving quotients.

Example 4

Translate the following phrases into mathematical expressions: (a) "11 times x," (b) "quotient of y and 4," and (c) "twice a."

Solution

Here are the translations. a) "11 times x" becomes $11x$. b) "quotient of y and 4" becomes $y/4$, or equivalently, $\frac{y}{4}$. c) "twice a" becomes $2a$.

Exercise

Translate into mathematical symbols: (a) "the product of 5 and x" and (b) "12 divided by y."

Answer

(a) $5x$ and (b) $12/y$.

Example 5

A plumber has a pipe of unknown length x. He cuts it into 4 equal pieces. Find the length of each piece in terms of the unknown length x.

Solution

The total length is unknown and equal to x. The plumber divides it into 4 equal pieces. To find the length of each pieces, we must divide the total length by 4.

$$\begin{array}{cccccc} \text{Length of each piece} & \text{is} & \text{Total length} & \text{divided by} & 4 \\ \text{Length of each piece} & = & x & \div & 4 \end{array}$$

Thus, the length of each piece, in terms of the unknown length x, is $x/4$, or equivalently, $\frac{x}{4}$.

Exercise

A carpenter cuts a board of unknown length L into three equal pieces. Express the length of each piece in terms of L.

Answer

L/3

Example 6

Mary invests A dollars in a savings account paying 2% interest per year. She invests five times this amount in a certificate of deposit paying 5% per year. How much does she invest in the certificate of deposit, in terms of the amount A in the savings account?

Solution

The amount in the savings account is A dollars. She invests five times this amount in a certificate of deposit.

$$\begin{array}{cccccc} \text{Amount in CD} & \text{is} & 5 & \text{times} & \text{Amount in savings} \\ \text{Amount in CD} & = & 5 & \cdot & A \end{array}$$

Thus, the amount invested in the certificate of deposit, in terms of the amount A in the savings account, is $5A$.

Exercise

David invests K dollars in a savings account paying 3% per year. He invests half this amount in a mutual fund paying 4% per year. Express the amount invested in the mutual fund in terms of K, the amount invested in the savings account.

Answer

$\frac{1}{2}K$

Combinations

Some phrases require combinations of the mathematical operations employed in previous examples.

Example 7

Let the first number equal x. The second number is 3 more than twice the first number. Express the second number in terms of the first number x.

Solution

The first number is x. The second number is 3 more than twice the first number.

$$\begin{array}{cccc} \text{Second number} & \text{is} & 3 \text{ more than} & \text{twice the first number} \\ \text{Second number} = & & 3+ & 2x \end{array}$$

Therefore, the second number, in terms of the first number x, is $3 + 2x$.

Exercise

A second number is 4 less than 3 times a first number. Express the second number in terms of the first number y.

Answer

$3y - 4$

Example 8

The length of a rectangle is L. The width is 15 feet less than 3 times the length. What is the width of the rectangle in terms of the length L?

Solution

The length of the rectangle is L. The width is 15 feet less than 3 times the length.

Width	is	3 times the length	less	15
Width $=$		$3L-$		15

Therefore, the width, in terms of the length L, is $3L - 15$.

Exercise

The width of a rectangle is W. The length is 7 inches longer than twice the width. Express the length of the rectangle in terms of its length L.

Answer

$2W + 7$

Exercises

In Exercises 1-20, translate the phrase into a mathematical expression involving the given variable.

1. “8 times the width n ”
2. “2 times the length z ”
3. “6 times the sum of the number n and 3”
4. “10 times the sum of the number n and 8”
5. “the demand b quadrupled”
6. “the supply y quadrupled”
7. “the speed y decreased by 33”
8. “the speed u decreased by 30”
9. “10 times the width n ”
10. “10 times the length z ”
11. “9 times the sum of the number z and 2”
12. “14 times the sum of the number n and 10”
13. “the supply y doubled”
14. “the demand n quadrupled”
15. “13 more than 15 times the number p ”
16. “14 less than 5 times the number y ”
17. “4 less than 11 times the number x ”
18. “13 less than 5 times the number p ”
19. “the speed u decreased by 10”
20. “the speed w increased by 32”

---

21. Representing Numbers. Suppose n represents a whole number.

i) What does n + 1 represent?

ii) What does n + 2 represent?

iii) What does n − 1 represent?

22. Suppose 2n represents an even whole number. How could we represent the next even number after 2n?

23. Suppose 2n + 1 represents an odd whole number. How could we represent the next odd number after 2n + 1?

24. There are b bags of mulch produced each month. How many bags of mulch are produced each year?

25. Steve sells twice as many products as Mike. Choose a variable and write an expression for each man's sales.

26. Find a mathematical expression to represent the values.

i) How many quarters are in d dollars?

ii) How many minutes are in h hours?

iii) How many hours are in d days?

iv) How many days are in y years?

v) How many months are in y years?

vi) How many inches are in f feet?

vii) How many feet are in y yards?

Answers

1. 8n

3. 6(n + 3)

5. 4b

7. y − 33

9. 10n

11. 9(z + 2)

13. 2y

15. 15p + 13 17.

11x − 4

19. u − 10

21.

i) n+1 represents the next whole number after n.

ii) n+2 represents the next whole number after n + 1, or, two whole numbers after n.

iii) n − 1 represents the whole number before n.

23. 2n + 3

25. Let Mike sell p products. Then Steve sells 2p products.

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2.12: Evaluating (NOT Solving) Algebraic Expressions

In this section we will evaluate *algebraic expressions* for given values of the variables contained in the expressions. Here are some simple tips to help you be successful.

Tips for Evaluating Algebraic Expressions

1. Replace all occurrences of variables in the expression with open parentheses. Leave room between the parentheses to substitute the given value of the variable.
2. Substitute the given values of variables in the open parentheses prepared in the first step.
3. Evaluate the resulting expression according to the Rules Guiding Order of Operations.

Let's begin with an example.

Example 1

Evaluate the expression $x^2-2xy+y^2$ at $x=-3$ and $y=2$.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $x^2 - 2xy + y^2$ with open parentheses.

$$x^2-2xy+y^2=(\,)^2-2(\,)(\,)+(\,)^2$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned} x^2-2xy+y^2 &\quad \text{Original expression.}\\ =(-3)^2-2(-3)(2)+(2)^2 &\quad \text{Substitute } -3 \text{ for } x \text{and } 2 \text{ for } y.\\ =9-2(-3)(2)+4 &\quad \text{Evaluate exponents first.}\\ =9-(-6)(2)+4 &\quad \text{Left to right, multiply } 2(-3)=-6.\\ =9-(-12)+4 &\quad \text{Left to right, multiply: } (-6)(2)=-12.\\ =9+12+4 &\quad \text{Add the opposite.}\\ =25 &\quad \text{Add.} \end{aligned}$$

Exercise

If $x = -2$ and $y = -1$, evaluate $x^3 - y^3$.

Answer

-7

Example 2

Evaluate the expression $(a - b)^2$ If $a = 3$ and $b = -5$, at $a = 3$ and $b = -5$.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $(a - b)^2$ with open parentheses.

$$(a-b)^2=((\,)-(\,))^2$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$
\begin{aligned}
(a-b)^2 &= ((3)-(-5))^2 && \text{Substitute 3 for } a \text{ and } -5 \text{ for } b.\\
&= (3+5)^2 && \text{Add the opposite: } (3)-(-5)=3+5\\
&= 8^2 && \text{Simplify inside parentheses: } 3+5=8\\
&= 64 && \text{Evaluate exponent: } 8^2=64
\end{aligned}
$$

? Exercise

If $a = 3$ and $b = -5$, evaluate $a^2 - b^2$.

Answer

−16

✓ Example 3

Evaluate the expression |a|−|b| at a = 5 and b = −7.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression |a|−|b| with open parentheses.

$$|a|-|b| = |(\,)|-|(\,)|$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$
\begin{aligned}
|a|-|b| = |(5)| &= |(-7)| && \text{Substitute } 5 \text{ for } a \text{ and } -7 \text{ for } b.\\
&= 5-7 && \text{Absolute values first: } |(5)|=5 \text{ and } |(-7)|=7|\\
&= 5+(-7) && \text{Add the opposites: } 5-7=5+(-7).\\
&= -2 && \text{Add: } 5+(-7)=-2.
\end{aligned}
$$

? Exercise

If $a = 5$ and $b = -7$, evaluate $2|a| - 3|b|$.

Answer

−11

✓ Example 4

Evaluate the expression $|a - b|$ at $a = 5$ and $b = -7$.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression $|a - b|$ with open parentheses.

$$|a-b| = |(\,)-(\,)|$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$
\begin{aligned}
|a-b| &= |(5)-(-7)| && \text{Substitute } 5 \text{ for } a \text{ and } -7 \text{ for } b.\\
&= |5+7| && \text{Add the opposite: } 5-(-7)=5+7.\\
&= |12| && \text{Add: } 5+7=12.\\
&= 12 && \text{Take the absolute value: } |12|=12.
\end{aligned}
$$

Exercise

If $a = 5$ and $b = -7$, evaluate $|2a - 3b|$.

Answer

31

Example 5

Evaluate the expression

$$\frac{ad-bc}{a+b}$$

at $a = 5$, $b = -3$, $c = 2$, and $d = -4$.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of variables in the expression with open parentheses.

$$\frac{ad-bc}{a+b} = \frac{(\,)(\,)-(\,)(\,)}{(\,)+(\,)}$$

Secondly, replace each variable with its given value, and thirdly, follow the "Rules Guiding Order of Operations" to evaluate the resulting expression.

$$\begin{aligned}
\frac{ad-bc}{a+b} &= \frac{(5)-(-3)(2)}{(5)+(-3)} && \text{Substitute: } 5 \text{ for } a,\ -3 \text{ for } b,\ 2 \text{ for } c,\ -4 \text{ for } d.\\
&= \frac{-20-(-6)}{2} && \text{Numerator: } (5)(=4) = -20,\ (-3)(2) = -6.\\
&&& \text{Denominator: } 5+(-3) = 2.\\
&= \frac{-20+6}{2} && \text{Numerator: Add the opposite.}\\
&= \frac{-14}{2} && \text{Numerator: } -20+6 = -14.\\
&= -7 && \text{Divide.}
\end{aligned}$$

Exercise

If a = −7, b = −3, c = −15, 15, and $d = -14$, evaluate:

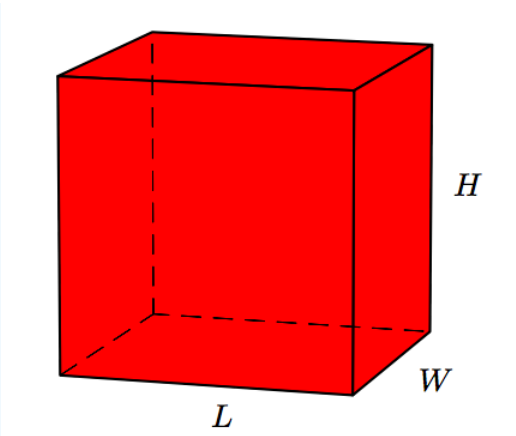
$$\frac{a^2+b^2}{c+d}$$

Answer

−2

Example 6

Pictured below is a rectangular prism.



The volume of the rectangular prism is given by the formula

$$V = LWH,$$

where L is the length, W is the width, and H is the height of the rectangular prism. Find the volume of a rectangular prism having length 12 feet, width 4 feet, and height 6 feet.

Solution

Following "Tips for Evaluating Algebraic Expressions," first replace all occurrences of of L, W, and H in the formula

$$V = LWH$$

with open parentheses.

$$V = (\,)(\,)(\,)$$

Next, substitute 12 ft for L, 4 ft for W, and 6 ft for H and simplify.

$$\begin{aligned} V &= (12\text{ft})(4\text{ft})(6\text{ft}) \\ &= 288\text{ft}^3 \end{aligned}$$

Hence, the volume of the rectangular prism is 288 cubic feet.

? Exercise

The surface area of the prism pictured in this example is given by the following formula:

$$S = 2(WH + LH + LW)$$

If L = 12, W = 4, and H = 6 feet, respectively, calculate the surface area.

Answer

288 square feet

Exercises

In Exercises 1-12, evaluate the expression at the given value of x.

1. $-3x^2 - 6x + 3$ at $x = 7$
2. $7x^2 - 7x + 1$ at $x = -8$
3. $-6x - 6$ at $x = 3$
4. $6x - 1$ at $x = -10$
5. $5x^2 + 2x + 4$ at $x = -1$

6. $4x^2 - 9x + 4$ at $x = -3$

7. $-9x - 5$ at $x = -2$

8. $-9x + 12$ at $x = 5$

9. $4x^2 + 2x + 6$ at $x = -6$

10. $-3x^2 + 7x + 4$ at $x = -7$

11. $12x + 10$ at $x = -12$

12. $-6x + 7$ at $x = 11$

In Exercises 13-28, evaluate the expression at the given values of x and y.

13. $|x|-|y|$ at $x = -5$ and $y = 4$

14. $|x|-|y|$ at $x = -1$ and $y = -2$

15. $-5x^2 + 2y^2$ at $x = 4$ and $y = 2$

16. $-5x^2 - 4y^2$ at $x = -2$ and $y = -5$

17. $|x|-|y|$ at $x = 0$ and $y = 2$

18. $|x|-|y|$ at $x = -2$ and $y = 0$

19. $|x - y|$ at $x = 4$ and $y = 5$

20. $|x - y|$ at $x = -1$ and $y = -4$

21. $5x^2 - 4xy + 3y^2$ at $x = 1$ and $y = -4$

22. $3x^2 + 5xy + 3y^2$ at $x = 2$ and $y = -1$

23. $|x - y|$ at $x = 4$ and $y = 4$

24. $|x - y|$ at $x = 3$ and $y = -5$

25. $-5x^2 - 3xy + 5y^2$ at $x = -1$ and $y = -2$

26. $3x^2 - 2xy - 5y^2$ at $x = 2$ and $y = 5$

27. $5x^2 + 4y^2$ at $x = -2$ and $y = -2$

28. $-4x^2 + 2y^2$ at $x = 4$ and $y = -5$

In Exercises 29-40, evaluate the expression at the given value of x.

29. $\frac{9+9x}{-x}$ at $x = -3$

30. $\frac{9-2x}{-x}$ at $x = -1$

31. $\frac{-8x+9}{-9+x}$ at $x = 10$

32. $\frac{2x+4}{1+x}$ at $x = 0$

33. $\frac{-4+9x}{7x}$ at $x = 2$

34. $\frac{-1-9x}{x}$ at $x = -1$

35. $\frac{-12-7x}{x}$ at $x = -1$

36. $\frac{12+11x}{3x}$ at $x = -6$

37. $\frac{6x-10}{5}$ + x at $x = -6$

38. $\frac{11x+11}{-4}$ + x at $x = 5$

39. $\frac{10x+11}{5}$ + x at $x = -4$

40. $\frac{6x+12}{-3}$ + x at x = 2

41. The formula

$$d = 16t^2$$

gives the distance (in feet) that an object falls from rest in terms of the time t that has elapsed since its release. Find the distance d (in feet) that an object falls in t = 4 seconds.

42. The formula

$$d = 16t^2$$

gives the distance (in feet) that an object falls from rest in terms of the time t that has elapsed since its release. Find the distance d (in feet) that an object falls in t = 24 seconds.

43. The formula

$$C = \frac{5(F-32)}{9}$$

gives the Celcius temperature C in terms of the Fahrenheit temperature F. Use the formula to find the Celsius temperature (◦ C) if the Fahrenheit temperature is F = 230◦ F.

44. The formula

$$C = \frac{5(F-32)}{9}$$

gives the Celcius temperature C in terms of the Fahrenheit temperature F. Use the formula to find the Celsius temperature (°C) if the Fahrenheit temperature is F = 95 °F.

45. The Kelvin scale of temperature is used in chemistry and physics. Absolute zero occurs at 0° K, the temperature at which molecules have zero kinetic energy. Water freezes at 273° K and boils at K = 373° K. To change Kelvin temperature to Fahrenheit temperature, we use the formula

$$F = \frac{9(K-273)}{5} + 32.$$

Use the formula to change 28° K to Fahrenheit.

46. The Kelvin scale of temperature is used in chemistry and physics. Absolute zero occurs at 0° K, the temperature at which molecules have zero kinetic energy. Water freezes at 273° K and boils at K = 373° K. To change Kelvin temperature to Fahrenheit temperature, we use the formula

$$F = \frac{9(K-273)}{5} + 32.$$

Use the formula to change 248° K to Fahrenheit.

47. A ball is thrown vertically upward. Its velocity t seconds after its release is given by the formula

$$v = v0 - gt,$$

where v_0 is its initial velocity, g is the acceleration due to gravity, and v is the velocity of the ball at time t. The acceleration due to gravity is g = 32 feet per second per second. If the initial velocity of the ball is v_0 = 272 feet per second, find the speed of the ball after t = 6 seconds.

48. A ball is thrown vertically upward. Its velocity t seconds after its release is given by the formula

$$v = v_0 - gt,$$

where v_0 is its initial velocity, g is the acceleration due to gravity, and v is the velocity of the ball at time t. The acceleration due to gravity is g = 32 feet per second per second. If the initial velocity of the ball is v_0 = 470 feet per second, find the speed of the ball after t = 4 seconds.

49. **Even numbers**. Evaluate the expression 2n for the following values:

i) n = 1

ii) n = 2

iii) n = 3

iv) n = −4

v) n = −5

vi) Is the result always an even number? Explain.

50. **Odd numbers**. Evaluate the expression 2n + 1 for the following values:

i) n = 1

ii) n = 2

iii) n = 3

iv) n = −4

v) n = −5

vi) Is the result always an odd number? Explain.

Answers

1. −186
3. −24
5. 7
7. 13
9. 138
11. −134
13. 1
15. −72
17. −2
19. 1
21. 69
23. 0
25. 9
27. 36
29. −6
31. −71
33. 1
35. 5
37. 46
39. −29
41. 256 feet
43. 110 degrees
45. −409° F

47. 80 feet per second

49.

i) 2

ii) 4

iii) 6

iv) −8

v) −10

vi) Yes, the result will always be an even number because 2 will always be a factor of the product 2n.

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2.13: Simplifying (Still Not Solving) Algebraic Expressions)

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2.13.1: Simplifying Algebraic Expressions

Recall the commutative and associative properties of multiplication.

The Commutative Property of Multiplication. If a and b are any integers, then

$$a \cdot b = b \cdot a, \text{ or equivalently, } ab = ba.$$

The Associative Property of Multiplication. If a, b, and c are any integers, then

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \text{ or equivalently, } (ab)c = a(bc).$$

The commutative property allows us to change the order of multiplication without affecting the product or answer. The associative property allows us to regroup without affecting the product or answer.

Example 1

Simplify: $2(3x)$.

Solution

Use the associative property to regroup, then simplify.

$$\begin{aligned} 2(3x) &= (2 \cdot 3)x \quad \text{Regrouping with the associative property.} \\ &= 6x \quad \text{Simplify: } 2 \cdot 3 = 6. \end{aligned}$$

Exercise

Simplify: $-5(7y)$

Answer

$-35y$

The statement $2(3x)=6x$ is an *identity*. That is, the left-hand side and right-hand side of $2(3x)=6x$ are the same for all values of x. Although the derivation in Example 1 should be the proof of this statement, it helps the intuition to check the validity of the statement for one or two values of x.

If $x = 4$, then

$$\begin{aligned} 2(3x) &= 2(3(4)) \\ &= 2(12) \\ &= 24 \end{aligned} \quad \text{and} \quad \begin{aligned} 6x &= 6(4) \\ &= 24 \end{aligned}$$

If $x = -5$, then

$$\begin{aligned} 2(3x) &= 2(3(-5)) \\ &= 2(-15) \\ &= -30 \end{aligned} \quad \text{and} \quad \begin{aligned} 6x &= 6(-5) \\ &= -30 \end{aligned}$$

The above calculations show that $2(3x)=6x$ for both $x = 4$ and $x = -5$. Indeed, the statement $2(3x)=6x$ is true, regardless of what is substituted for x.

Example 2

Simplify: $(-3t)(-5)$.

Solution

In essence, we are multiplying three numbers, −3, *t*, and −5, but the grouping symbols ask us to multiply the −3 and the t first. The associative and commutative properties allow us to change the order and regroup.

$$\begin{aligned}(-3t)(-5) &= ((-3)(-5))t \quad \text{Change the order and regroup.}\\ &= 15t \quad \text{Multiply: } (-3)(-5) = 15.\end{aligned}$$

? Exercise

Simplify: (−8*a*)(5)

Answer

−40*a*

✓ Example 3

Simplify: (−3*x*)(−2*y*)

Solution

In essence, we are multiplying four numbers, −3, x, −2, and y, but the grouping symbols specify a particular order. The associative and commutative properties allow us to change the order and regroup.

$$\begin{aligned}(-3x)(-2y) &= ((-3)(-2))(xy) \quad \text{Change the order and regroup.}\\ &= 6xy \quad \text{Multiply: } (-3)(-2) = 6.\end{aligned}$$

? Exercise

Simplify: (−4*a*)(5*b*)

Answer

−20*ab*

Speeding Things Up

The meaning of the expression $2 \cdot 3 \cdot 4$ is clear. Parentheses and order of operations are really not needed, as the commutative and associative properties explain that it doesn't matter which of the three numbers you multiply together first.

- You can multiply 2 and 3 first:

$$\begin{aligned}2 \cdot 3 \cdot 4 &= (2 \cdot 3) \cdot 4\\ &= 6 \cdot 4\\ &= 24.\end{aligned}$$

- Or you can multiply 3 and 4 first:

$$\begin{aligned}2 \cdot 3 \cdot 4 &= 2 \cdot (3 \cdot 4)\\ &= 2 \cdot 12\\ &= 24.\end{aligned}$$

- Or you can multiply 2 and 4 first:

$$\begin{aligned}2 \cdot 3 \cdot 4 &= (2 \cdot 4) \cdot 3\\ &= 8 \cdot 3\\ &= 24.\end{aligned}$$

So, it does not matter which two factors you multiply first.

Of course, this would not be the case if there were a mixture of multiplication and other operators (division, addition, subtraction). Then we would have to strictly follow the "Rules Guiding Order of Operations." But if the only operator is multiplication, the order

of multiplication is irrelevant.

Thus, when we see 2(3x), as in Example 1, we should think "It's all multiplication and it doesn't matter which two numbers I multiply together first, so I'll multiply the 2 and the 3 and get 2(3x)=6x."

Our comments apply equally well to a product of four or more factors. It simply doesn't matter how you group the multiplication. So, in the case of (−3x)(−2y), as in Example 3, find the product of −2 and −3 and multiply the result by the product of x and y. That is, (−3x)(−2y)=6xy.

Example 4

Simplify: (2a)(3b)(4c).

Solution

The only operator is multiplication, so we can order and group as we please. So, we'll take the product of 2, 3, and 4, and multiply the result by the product of a, b, and c. That is,

$$(2a)(3b)(4c) = 24abc$$

Exercise

Simplify: (−3x)(−2y)(−4z)

Answer

−24xyz.

The Distributive Property

Multiplication is distributive with respect to addition.

The Distributive Property

If a, b, and c are any integers, then

$a \cdot (b + c) = a \cdot b + a \cdot c$, or equivalently, $a(b + c) = ab + ac$.

For example, if we follow the "Rules Guiding Order of Operations" and first evaluate the expression inside the parentheses, then

$$\begin{aligned} 3(4+5) &= 3(9) && \text{Parentheses first: } 4+5=9. \\ &= 27. && \text{Multiply: } 3(9)=27. \end{aligned}$$

But if we "distribute" the 3, we get the same answer.

$$\begin{aligned} 3(4+5) &= 3(4+5) && \text{Each number in parentheses is multiplied by the number 3 outside the parentheses.} \\ &= 3(4)+3(5) \\ &= 12+15 && \text{Multiply first: } 3(4)=12,\ 3(5)=15. \\ &= 27 && \text{Add.} \end{aligned}$$

Example 5

Use the distributive property to simplify: 3(4x + 5).

Solution

Distribute the 3.

$$\begin{aligned} 3(4x+5) &= 3(4x)+3(5) && \text{Each number in parentheses is multiplied by the number 3 outside the parentheses.} \\ &= 12x+15 && \text{Multiply first: } 3(4x)=12x,\ 3(5)=15. \end{aligned}$$

Exercise

Use the distributive property to simplify: 2(5z+7).

Answer

10z + 14

Multiplication is also distributive with respect to subtraction.

The Distributive Property

If a, b, and c are any integers, then

$$a \cdot (b - c) = a \cdot b - a \cdot c, \text{ or equivalently, } a(b - c) = ab - ac.$$

The application of this form of the distributive property is identical to the first, the only difference being the subtraction symbol.

Example 6

Use the distributive property to simplify: 5(3x − 2).

Solution

Distribute the 5.

$$\begin{aligned} 5(3x-2) &= 5(3x)-5(2) && \text{Each number in parentheses is multiplied by the number 5 outside the parentheses.} \\ &= 15x-10 && \text{Multiply first: } 5(3x)=15x, 5(2)=10. \end{aligned}$$

Exercise

Use the distributive property to simplify: 7(4a − 5).

Answer

28a − 35

Example 7

Remove parentheses: (a) −9(2t + 7), and (b) −5(4 − 3y).

Solution

a) Use the distributive property.

$$\begin{aligned} -9(2t+7) &= -9(2t)+(-9)(7) && \text{Distribute multiplication by } -9. \\ &= -18t+(-63) && \text{Multiply: } -9(2t)=-18t \text{ and } -9(7)=-63. \\ &= -18t-63 && \text{Write the answer in simpler form.} \\ & && \text{Adding } -63 \text{ is the same as subtracting } 63. \end{aligned}$$

b) Use the distributive property.

$$\begin{aligned} -5(4-3y) &= -5(4)-(-5)(3y) && \text{Distribute multiplication by } -5. \\ &= -20-(-15y) && \text{Multiply: } -5(4)=-20 \text{ and } -5(3y)=-15y. \\ &= -18t-63 && \text{Write the answer in simpler form.} \\ & && \text{Subtracting } -15y \text{ is the same as adding } 15y. \end{aligned}$$

Exercise

Remove parentheses: $-3(4t - 11)$.

Answer

$-12t + 33$

Writing Mathematics

Example 7 stresses the importance of using as few symbols as possible to write your final answer. Hence, $-18t - 63$ is favored over $-18t + (-63)$ and $-20 + 15y$ is favored over $-20 - (-15y)$. You should always make these final simplifications.

Moving a Bit Quicker

Once you've applied the distributive property to a number of problems, showing all the work as in Example 7, you should try to eliminate some of the steps. For example, consider again Example 7(a). It's not difficult to apply the distributive property without writing down a single step, getting:

$$-9(2t+7) = -18t - 63.$$

Here's the thinking behind this technique:

1. First, multiply -9 times $2t$, getting $-18t$.
2. Second, multiply -9 times $+7$, getting -63.

Note that this provides exactly the same solution found in Example 7(a).

Let try this same technique on Example 7(b).

$$-5(4 - 3y) = -20 + 15y$$

Here's the thinking behind this technique.

1. First, multiply -5 times 4, getting -20.
2. Second, multiply -5 times $-3y$, getting $+15y$.

Note that this provides exactly the same solution found in Example 7(b).

Extending the Distributive Property

Suppose that we add an extra term inside the parentheses.

Distributive Property

If a, b, c, and d are any integers, then

$$a(b + c + d) = ab + ac + ad.$$

Note that we "distributed" the a times each term inside the parentheses. Indeed, if we added still another term inside the parentheses, we would "distribute" a times that term as well.

Example 8

Remove parentheses: $-5(2x - 3y + 8)$.

Solution

We will use the "quicker" technique, "distributing" -5 times each term in the parentheses mentally.

$$-5(2x - 3y + 8) = -10x + 15y - 40$$

Here is our thought process:

1. First, multiply −5 times 2x, getting −10x.
2. Second, multiply −5 times −3y, getting +15y.
3. Third, multiply −5 times +8, getting −40.

Exercise

Remove parentheses: −3(4a − 5b + 7)

Answer

−12a + 15b − 21

Example 9

Remove parentheses: −4(−3a + 4b − 5c + 12).

Solution

We will use the "quicker" technique, "distributing" −4 times each term in the parentheses mentally.

$$-4(-3a+4b-5c+12)=12a-16b+20c-48$$

Here is our thought process:

1. First, multiply −4 times −3a, getting 12a.
2. Second, multiply −4 times +4b, getting −16b.
3. Third, multiply −4 times −5c, getting +20c.
4. Fourth, multiply −4 times +12, getting −48.

Exercise

Remove parentheses: −2(−2x + 4y − 5z − 11).

Answer

4x − 8y + 10z + 22

Distributing a Negative

It is helpful to recall that negating is equivalent to multiplying by −1.

Multiplying by −1

Let a be any integer, then

$$(-1)a = -a \text{ and } -a = (-1)a.$$

We can use this fact, combined with the distributive property, to negate a sum.

Example 10

Remove parentheses: −(a + b).

Solution

Change the negative symbol into multiplying by −1, then distribute the −1.

$$\begin{aligned} -(a+b) &= (-1)(a+b) && \text{Negating is equivalent to multiplying by } -1. \\ &= -a-b && \text{Distribute the } -1. \end{aligned}$$

We chose to use the "quicker" technique of "distributing" the −1. Here is our thinking:

1. Multiply -1 times a, getting $-a$.
2. Multiply -1 times $+b$, getting $-b$.

Exercise

Remove parentheses: $-(4a - 3c)$

Answer

$-4a + 3c$

The results in Example 10 and Example 11 show us how to negate a sum: Simply negate each term of the sum. Positive terms change to negative, negative terms turn to positive.

Negating a Sum

To negate a sum, simply negate each term of the sum. For example, if a and b are integers, then

$$-(a + b) = -a - b \text{ and } -(a - b) = -a + b.$$

Example 12

Remove parentheses: $-(5 - 7u + 3t)$.

Solution

Simply negate each term in the parentheses.

$$-(5 - 7u + 3t) = -5+7u - 3t$$

Exercise

Remove parentheses: $-(5 - 2x + 4y - 5z)$

Answer

$-5+2x - 4y + 5z$

Exercises

In Exercises 1-20, use the associative and commutative properties of multiplication to simplify the expression.

1. $10(-4x)$
2. $7(-8x)$
3. $(-10x)(-3)$
4. $(-5x)(-8)$
5. $-5(3x)$
6. $9(6x)$
7. $(-4x)10$
8. $(-10x)(-6)$
9. $(5x)3$
10. $(3x)3$
11. $(5x)10$
12. $(-2x)(-10)$

13. −9(−7x)
14. −10(5x)
15. 6(2x)
16. 3(−10x)
17. −8(−9x)
18. 3(−3x)
19. (6x)7
20. (−8x)(−5)

In Exercises 21-44, simplify the expression.

21. 8(7x + 8)
22. −2(5x + 5)
23. 9(−2 + 10x)
24. −9(4 + 9x)
25. −(−2x + 10y − 6)
26. −(−6y + 9x − 7)
27. 2(10 + x)
28. 2(10 − 6x)
29. 3(3 + 4x)
30. 3(4 + 6x)
31. −(−5 − 7x + 2y)
32. −(4x − 8 − 7y)
33. 4(−6x + 7)
34. 6(4x + 9)
35. 4(8x − 9)
36. 10(−10x + 1)
37. −(4 − 2x − 10y)
38. −(−4x + 6 − 8y)
39. −(−5x +1+9y)
40. −(−10 − 5x − 4y)
41. −(6x + 2 − 10y)
42. −(6x + 4 − 10y)
43. −(−3y − 4+4x)
44. −(−7 − 10x + 7y)

Answers

1. −40x
3. 30x
5. −15x

7. −40x

9. 15x

11. 50x

13. 63x

15. 12x

17. 72x

19. 42x

21. 56x + 64

23. −18 + 90x

25. 2x − 10y + 6

27. 20 + 2x

29. 9 + 12x

31. 5+7x − 2y

33. −24x + 28

35. 32x − 36

37. −4+2x + 10y

39. 5x − 1 − 9y

41. −6x − 2 + 10y

43. 3y + 4 − 4x

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2.13.2: Combining Like Terms

We begin our discussion with the definition of a *term*.

Definition: Term

A **term** is a single number or variable, or it can be the product of a number (called its coefficient) and one or more variables (called its variable part). The terms in an algebraic expression are separated by addition symbols.

Example 1

Identify the terms in the algebraic expression

$$3x^2 + 5xy + 9y^2 + 12$$

For each term, identify its coefficient and variable part.

Solution

In tabular form, we list each term of the expression 3x^2 + 5xy + 9y^2 + 12, its coefficient, and its variable part.

Term	Coefficient	Variable Part
$3x^2$	3	x^2
$5xy$	5	xy
$9y^2$	9	y^2
12	12	None

Exercise

How many terms are in the algebraic expression $3x^2 + 2xy - 3y^2$?

Answer

3

Example 2

Identify the terms in the algebraic expression

$$a^3 - 3a^2b + 3ab^2 - b^3$$

For each term, identify its coefficient and variable part.

Solution

The first step is to write each difference as a sum, because the terms of an expression are defined above to be those items separated by addition symbols.

$$a^3 + (-3a^2b) + 3ab^2 + (-b^3)$$

In tabular form, we list each term of the expression $a^3 + (-3a^2b) + 3ab^2 + (-b^3)$, its coefficient, and its variable part.

Term	Coefficient	Variable Part
a^3	1	a^3
$-3a^2b$	-3	a^2b

Term	Coefficient	Variable Part
$3ab^2$	3	ab^2
$-b^3$	-1	b^3

Exercise

How many terms are in the algebraic expression $11 - a^2 - 2ab + 3b^2a$?

Answer

4

Like Terms

We define what is meant by "like terms" and "unlike terms."

Definition

Like and Unlike Terms. The variable parts of two terms determine whether the terms are like terms or unlike terms.

Like Terms. Two terms are called like terms if they have identical variable parts, which means that the terms must contain the same variables raised to the same exponential powers.

Unlike Terms. Two terms are called unlike terms if their variable parts are different.

Example 3

Classify each of the following pairs as either *like terms* or *unlike terms*: (a) $3x$ and $-7x$, (b) $2y$ and $3y^2$, (c) $-3t$ and $5u$, and (d) $-4a^3$ and $3a^3$.

Solution

Like terms must have *identical* variable parts.

a. $3x$ and $-7x$ have identical variable parts. They are "like terms."
b. $2y$ and $3y^2$ do not have identical variable parts (the exponents differ). They are "unlike terms."
c. $-3t$ and $5u$ do not have identical variable parts (different variables). They are "unlike terms."
d. $-4a^3$ and $3a^3$ have identical variable parts. They are "like terms."

Exercise

Are $-3xy$ and $11xy$ *like* or *unlike* terms?

Answer

Like terms

Combining Like Terms

When using the distributive property, it makes no difference whether the multiplication is on the left or the right, one still distributes the multiplication times each term in the parentheses.

Distributive Property

If a, b, and c are integers, then

$$a(b + c) = ab + ac \text{ and } (b + c)a = ba + ca.$$

In either case, you distribute a times each term of the sum.

"Like terms" can be combined and simplified. The tool used for combining like terms is the distributive property. For example, consider the expression $3y + 7y$, composed of two "like terms" with a common variable part. We can use the distributive property and write

$$3y + 7y = (3 + 7)y$$

Note that we are using the distributive property in reverse, "factoring out" the common variable part of each term. Checking our work, note that if we redistribute the variable part y times each term in the parentheses, we are returned to the original expression $3y + 7y$.

Example 4

Use the distributive property to combine like terms (if possible) in each of the following expressions: (a) $-5x^2 - 9x^2$, (b) $-5ab + 7ab$, (c) $4y^3 - 7y^2$, and (d) $3xy^2 - 7xy^2$.

Solution

If the terms are "like terms," you can use the distributive property to "factor out" the common variable part.

a) Factor out the common variable part x^2.

$$\begin{aligned} -5x^2 - 9x^2 &= (-5 - 9)x^2 \quad \text{Use the distributive property.} \\ &= -14x^2 \quad \text{Simplify: } -5 - 9 = -5 + (-9) = -14. \end{aligned}$$

b) Factor out the common variable part ab.

$$\begin{aligned} -5ab + 7ab &= (-5 + 7)ab \quad \text{Use the distributive property.} \\ &= 2ab \quad \text{Simplify: } -5 + 7 = 2. \end{aligned}$$

c) The terms in the expression $4y^3 - 7y^2$ have different variable parts (the exponents are different). These are "unlike terms" and cannot be combined.

d) Factor out the common variable part xy^2.

$$\begin{aligned} 3xy^2 - 7xy^2 &= (3 - 7)xy^2 \quad \text{Use the distributive property.} \\ &= -4xy^2 \quad \text{Simplify: } 3 - 7 = 3 + (-7) = -4. \end{aligned}$$

Exercise

Simplify: $-8z - 11z$

Answer

$-19z$

Speeding Things Up a Bit

Once you've written out all the steps for combining like terms, like those shown in Example 4, you can speed things up a bit by following this rule:

Combining Like Terms

To combine like terms, simply add their coefficients and keep the common variable part.

Thus for example, when presented with the sum of two like terms, such as in $5x + 8x$, simply add the coefficients and repeat the common variable part; that is, $5x + 8x = 13x$.

Example 5

Combine like terms:

a. $-9y - 8y$,
b. $-3y^5 + 4y^5$ and
c. $-3u^2 + 2u^2$.

Solution

a) Add the coefficients and repeat the common variable part. Therefore,

$$-9y - 8y = -17y.$$

b) Add the coefficients and repeat the common variable part. Therefore,

$$-3y^5 + 4y^5 = 1y^5.$$

However, note that $1y^5 = y^5$. Following the rule that the final answer should use as few symbols as possible, a better answer is $-3y^5 + 4y^5 = y^5$.

c) Add the coefficients and repeat the common variable part. Therefore,

$$-3u^2 + 2u^2 = (-1)u^2.$$

However, note that $(-1)u^2 = -u^2$. Following the rule that the final answer should use as few symbols as possible, a better answer is $-3u^2 + 2u^2 = -u^2$.

Exercise

Combine: $-3x$, and $-4x^2$

Answer

$-7x^2$

Simplify

A frequently occurring instruction asks the reader to *simplify* an expression.

Simplify

The instruction simplify is a generic term that means "try to write the expression in its most compact form, using the fewest symbols possible."

One way you can accomplish this goal is by combining like terms when they are present.

Example 6

Simplify: $2x + 3y - 5x + 8y$.

Solution

Use the commutative property to reorder terms and the associative and distributive properties to regroup and combine like terms.

$$\begin{aligned} 2x + 3y - 5x + 8y &= (2x - 5x) + (3y + 8y) && \text{Reorder and regroup.} \\ &= -3x + 11y && \text{Combine like terms:} \\ & && 2x - 5x = -3x \text{ and } 3y + 8y = 11y. \end{aligned}$$

Alternate solution

Of course, you do not need to show the regrouping step. If you are more comfortable combining like terms in your head, you are free to present your work as follows:

$$2x+3y-5x+8y=-3x+11y.$$

Exercise

Simplify: $-3a + 4b - 7a - 9b$

Answer

$-10a - 5b$

Example 7

Simplify: $-2x - 3 - (3x + 4)$.

Solution

First, distribute the negative sign.

$$-2x-3-(3x+4)=-2x-3-3x-4 \quad -(3x+4)=-3x-4.$$

Next, use the commutative property to reorder, then the associative property to regroup. Then combine like terms.

$$\begin{aligned} &=(-2x-3x)+(-3-4) && \text{Reorder and regroup.}\\ &=-5x+(-7) && \text{Combine like terms:}\\ & && -2x-3x=-5x.\\ &=-5x-7 && \text{Simplify:}\\ & && -5x+(-7)=-5x-7. \end{aligned}$$

Alternate solution

You may skip the second step if you wish, simply combining like terms mentally. That is, it is entirely possible to order your work as follows:

$$\begin{aligned} -2x-3-(3x+4)&=-2x-3-3x-4 && \text{Distribute negative sign.}\\ &=-5x-7 && \text{Combine like terms.} \end{aligned}$$

Exercise

Simplify: $-9a - 4 - (4a - 8)$

Answer

$-13a + 4$

Example 8

Simplify: $2(5 - 3x) - 4(x + 3)$.

Solution

Use the distributive property to expand, then use the commutative and associative properties to group the like terms and combine them.

\[\begin{aligned} 2(5-3x)-4(x+3) = 10-6x-4x-12 ~ & \textcolor{red}{ \text{ Use the distributive property.}} \\ =(-6x-4x)+(10-12) ~ & \textcolor{red}{ \text{ Group like terms.}} \\ =-10x-2 ~ & \textcolor{red}{ \text{ Combine like terms: }} \\ ~ & \textcolor{red}{-6x-4x=-10x \text{ and}} \\ ~ & \textcolor{red}{10-12=-2.} \end{aligned}\nonumber \]

Alternate solution

You may skip the second step if you wish, simply combining like terms mentally. That is, it is entirely possible to order your work as follows:

$$\begin{aligned} 2(5-3x)-4(x+3) &= 10-6x-4x-12 \quad \text{Distribute.} \\ &= -10x-2 \quad \text{Combine like terms.} \end{aligned}$$

Exercise

Simplify: $-2(3a-4)-2(5-a)$

Answer

$-4a-2$

Example 9

Simplify: $-8(3x^2y-9xy)-8(-7x^2-8xy)$

Solution

We will proceed a bit quicker with this solution, using the distributive property to expand, then combining like terms mentally.

$$\begin{aligned} -8(3x^2y-9xy)-8(-7x^2y-8xy) &= -24x^2y+72xy+56x^2y+64xy \\ &= 32x^2y+136xy \end{aligned}$$

Exercise

Simplify: $(a^2-2ab)-2(3ab+a^2)$

Answer

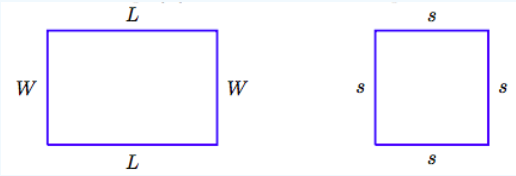
$-a^2-8ab$

Applications

We can simplify a number of useful formulas by combining like terms.

Example 10

Find the perimeter P of the (a) rectangle and (b) square pictured below. Simplify your answer as much as possible.



Solution

The perimeter of any polygonal figure is the sum of the lengths of its sides.

a) To find the perimeter P of the rectangle, sum its four sides.

$$P = L+W+L+W.$$

Combine like terms.

$$P = 2L+2W.$$

b) To find the perimeter P of the square, sum its four sides.

$$P = s + s + s + s.$$

Combine like terms.

$$P = 4s.$$

Exercise

A regular hexagon has six equal sides, each with length *x*. Find its perimeter in terms of *x*.

Answer

P = 6*x*

Sometimes it is useful to replace a variable with an expression containing another variable.

Example 11

The length of a rectangle is three feet longer than twice its width. Find the perimeter *P* of the rectangle in terms of its width alone.

Solution

From the previous problem, the perimeter of the rectangle is given by

$$P = 2L + 2W,$$

where *L* and *W* are the length and width of the rectangle, respectively. This equation gives the perimeter in terms of its length and width, but we're asked to get the perimeter in terms of the width alone. However, we're also given the fact that the length is three feet longer than twice the width.

$$\begin{array}{rcrcr} cyanLength & \text{is} & \text{Three Feet} & \text{longer than} & \text{Twice the Width} \\ L & = & 3 & + & 2W \end{array}$$

Because *L* = 3+2*W*, we can replace *L* with 3+2*W* in the perimeter equation 3.1.

$$P = 2L + 2W$$

$$P = 2(3 + 2W) + 2W$$

Use the distributive property, then combine like terms.

$$P = 6 + 4W + 2W$$

$$P = 6 + 6W.$$

This last equation gives the perimeter *P* in terms of the width *W* alone.

Exercise

The length *L* of a rectangle is 5 meters longer than twice its width *W*. Find the perimeter *P* of the rectangle in terms of its width *W*.

Answer

Answer: *P* = 6*W* + 10

Example 12

The width of a rectangle is two feet less than its length. Find the perimeter P of the rectangle in terms of its length alone.

Solution

Again, the perimeter of a rectangle is given by the equation

$$P = 2L + 2W,$$

where L and W are the length and width of the rectangle, respectively. This equation gives the perimeter in terms of its length and width, but we're asked to get the perimeter in terms of the length alone.

However, we're also given the fact that the width is two feet less than the length.

Width	is	Length	minus	Two feet
W	$=$	L	$-$	2

Because $W = L-2$, we can replace W with $L-2$ in the perimeter equation 3.2.

$$P = 2L + 2W$$

$$P = 2L + 2(L - 2)$$

Use the distributive property, then combine like terms.

$$P = 2L + 2L - 4$$

$$P = 4L - 4.$$

This last equation gives the perimeter P in terms of the length L alone.

Exercise

The width W of a rectangle is 5 feet less than twice its width L. Find the perimeter P of the rectangle in terms of its length L.

Answer

$P = 6L - 10$

Exercises

In Exercises 1-16, combine like terms by first using the distributive property to factor out the common variable part, and then simplifying.

1. $17xy^2 + 18xy^2 + 20xy^2$
2. $13xy - 3xy + xy$
3. $-8xy^2 - 3xy^2 - 10xy^2$
4. $-12xy - 2xy + 10xy$
5. $4xy - 20xy$
6. $-7y^3 + 15y^3$
7. $12r - 12r$
8. $16s - 5s$
9. $-11x - 13x + 8x$
10. $-9r - 10r + 3r$
11. $-5q + 7q$
12. $17n + 15n$

13. $r - 13r - 7r$

14. $19m + m + 15m$

15. $3x^3 - 18x^3$

16. $13x^2y + 2x^2y$

In Exercises 17-32, combine like terms by first rearranging the terms, then using the distributive property to factor out the common variable part, and then simplifying.

17. $-8 + 17n + 10 + 8n$

18. $11 + 16s - 14 - 6s$

19. $-2x^3 - 19x^2y - 15x^2y + 11x^3$

20. $-9x^2y - 10y^3 - 10y^3 + 17x^2y$

21. $-14xy - 2x^3 - 2x^3 - 4xy$

22. $-4x^3 + 12xy + 4xy - 12x^3$

23. $-13 + 16m + m + 16$

24. $9 - 11x - 8x + 15$

25. $-14x^2y - 2xy^2 + 8x^2y + 18xy^2$

26. $-19y^2 + 18y^3 - 5y^2 - 17y^3$

27. $-14x^3 + 16xy + 5x^3 + 8xy$

28. $-16xy + 16y^2 + 7xy + 17y^2$

29. $9n + 10 + 7 + 15n$

30. $-12r + 5 + 17 + 17r$

31. $3y + 1 + 6y + 3$

32. $19p + 6 + 8p + 13$

In Exercises 33-56, simplify the expression by first using the distributive property to expand the expression, and then rearranging and combining like terms mentally.

33. $-4(9x^2y + 8) + 6(10x^2y - 6)$

34. $-4(-4xy + 5y^3) + 6(-5xy - 9y^3)$

35. $3(-4x^2 + 10y^2) + 10(4y^2 - x^2)$

36. $-7(-7x^3 + 6x^2) - 7(-10x^2 - 7x^3)$

37. $-s + 7 - (-1 - 3s)$

38. $10y - 6 - (-10 - 10y)$

39. $-10q - 10 - (-3q + 5)$

40. $-2n + 10 - (7n - 1)$

41. $7(8y + 7) - 6(8 - 7y)$

42. $-6(-5n - 4) - 9(3 + 4n)$

43. $7(10x^2 - 8xy^2) - 7(9xy^2 + 9x^2)$

44. $10(8x^2y - 10xy^2) + 3(8xy^2 + 2x^2y)$

45. $-2(6 + 4n) + 4(-n - 7)$

46. $-6(-2 - 6m) + 5(-9m + 7)$

47. $8 - (4 + 8y)$

48. $-1 - (8 + s)$

49. $-8(-n + 4) - 10(-4n + 3)$

50. $3(8r - 7) - 3(2r - 2)$

51. $-5 - (10p + 5)$

52. $-1 - (2p - 8)$

53. $7(1 + 7r) + 2(4 - 5r)$

54. $(5 - s) + 10(9 + 5s)$

55. $-2(-5 - 8x^2) - 6(6)$

56. $8(10y^2 + 3x^3) - 5(-7y^2 - 7x^3)$

---

57. The length L of a rectangle is 2 feet longer than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

58. The length L of a rectangle is 7 feet longer than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

59. The width W of a rectangle is 8 feet shorter than its length L. Find the perimeter of the rectangle in terms of its length alone.

60. The width W of a rectangle is 9 feet shorter than its length L. Find the perimeter of the rectangle in terms of its length alone.

61. The length L of a rectangle is 9 feet shorter than 4 times its width W. Find the perimeter of the rectangle in terms of its width alone.

62. The length L of a rectangle is 2 feet shorter than 6 times its width W. Find the perimeter of the rectangle in terms of its width alone.

Answers

1. $55xy^2$

3. $-21xy^2$

5. $-16xy$

7. 0

9. $-16x$

11. $2q$

13. $-19r$

15. $-15x^3$

17. $2 + 25n$

19. $9x^3 - 34x^2y$

21. $-18xy - 4x^3$

23. $3 + 17m$

25. $-6x^2y + 16xy^2$

27. $-9x^3 + 24xy$

29. $24n + 17$

31. $9y + 4$

33. $24x^2y - 68$

35. $-22x^2 + 70y^2$

37. $2s + 8$

39. $-7q - 15$

41. $98y + 1$

43. $7x^2 - 119xy^2$

45. $-40 - 12n$

47. $4 - 8y$

49. $48n - 62$

51. $-10 - 10p$

53. $15 + 39r$

55. $-26 + 16x^2$

57. $4 + 14W$

59. $4L - 16$

61. $10W - 18$

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2.14: Mean

Lesson

Let's explore the mean of a data set and what it tells us.

Exercise 2.14.1: Close to Four

Use the digits 0–9 to write an expression with a value as close as possible to 4. Each digit can be used only one time in the expression.

$(\square + \square + \square + \square) \div 4$

Figure 2.14.1

Exercise 2.14.2: Spread Out and Share

1. The kittens in a room at an animal shelter are arranged in five crates, as shown.

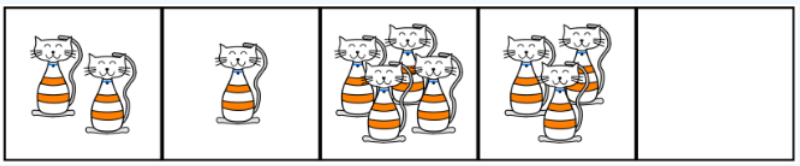

Figure 2.14.2: By Clker-Free-Vector-Images. Public Domain. Pixabay. Source.

   a. The manager of the shelter wants the kittens distributed equally among the crates. How might that be done? How many kittens will end up in each crate?
   b. The number of kittens in each crate after they are equally distributed is called the **mean** number of kittens per crate, or the **average** number of kittens per crate.
   Explain how the expression $10 \div 5$ is related to the average.
   c. Another room in the shelter has 6 crates. No two crates contain the same number of kittens, and there is an average of 3 kittens per crate.
   Draw or describe at least two different arrangements of kittens that match this description. You may choose to use the applet to help.

2. Five servers were scheduled to work the number of hours shown. They decided to share the workload, so each one would work equal hours.
   Server A: 3
   Server B: 6
   Server C: 11
   Server D: 7
   Server E: 4

   a. On the grid on the left, draw 5 bars whose heights represent the hours worked by servers A, B, C, D, and E.

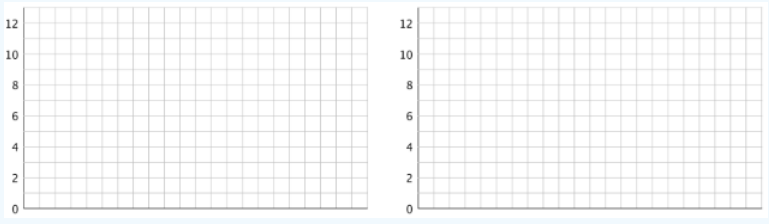

Figure 2.14.3

   b. Think about how you would rearrange the hours so that each server gets a fair share. Then, on the grid on the right, draw a new graph to represent the rearranged hours. Be prepared to explain your reasoning.
   c. Based on your second drawing, what is the average or mean number of hours that the servers will work?
   d. Explain why we can also find the mean by finding the value of $31 \div 5$.
   e. Which server will see the biggest change to work hours? Which server will see the least change?

Are you ready for more?

Server F, working 7 hours, offers to join the group of five servers, sharing their workload. If server F joins, will the mean number of hours worked increase or decrease? Explain how you know.

Exercise 2.14.3: Getting to School

For the past 12 school days, Mai has recorded how long her bus rides to school take in minutes. The times she recorded are shown in the table.

9	8	6	9	10	7	6	12	9	8	10	8

1. Find the mean for Mai's data. Show your reasoning.
2. In this situation, what does the mean tell us about Mai's trip to school?
3. For 5 days, Tyler has recorded how long his walks to school take in minutes. The mean for his data is 11 minutes. Without calculating, predict if each of the data sets shown could be Tyler's. Explain your reasoning.
   - data set A: 11, 8, 7, 9, 8
   - data set B: 12, 7, 13, 9, 14
   - data set C: 11, 20, 6, 9, 10
   - data set D: 8, 10, 9, 11, 11
4. Determine which data set is Tyler's. Explain how you know.

Summary

Sometimes a general description of a distribution does not give enough information, and a more precise way to talk about center or spread would be more useful. The **mean**, or **average**, is a number we can use to summarize a distribution.

We can think about the mean in terms of "fair share" or "leveling out." That is, a mean can be thought of as a number that each member of a group would have if all the data values were combined and distributed equally among the members.

For example, suppose there are 5 bottles which have the following amounts of water: 1 liter, 4 liters, 2 liters, 3 liters, and 0 liters.

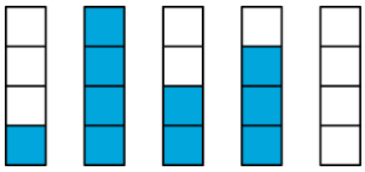

Figure 2.14.4: There are 5 identical tape diagrams that are each partitioned into 4 equal parts. The first diagram has 1 part shaded. The second diagram has 4 parts shaded. The third diagram has 2 parts shaded. The fourth diagram has 3 parts shaded. The fifth diagram has no parts shaded.

To find the mean, first we add up all of the values. We can think of as putting all of the water together: $1+4+2+3+0=10$.

Figure 2.14.5

To find the "fair share," we divide the 10 liters equally into the 5 containers: $10 \div 5 = 2$.

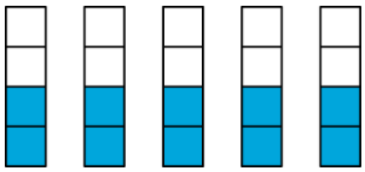

Figure 2.14.6

Suppose the quiz scores of a student are 70, 90, 86, and 94. We can find the mean (or average) score by finding the sum of the scores $(70+90+86+94=340)$ and dividing the sum by four $(340 \div 4 = 85)$. We can then say that the student scored, on average, 85 points on the quizzes.

In general, to find the mean of a data set with n values, we add all of the values and divide the sum by n.

Glossary Entries

Definition: Average

The average is another name for the mean of a data set.

For the data set 3, 5, 6, 8, 11, 12, the average is 7.5.

$3+5+6+8+11+12=45$

$45\div 6=7.5$

Definition: Mean

The mean is one way to measure the center of a data set. We can think of it as a balance point. For example, for the data set 7, 9, 12, 13, 14, the mean is 11.

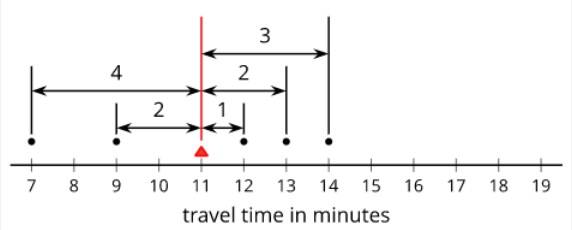

Figure 2.14.7

To find the mean, add up all the numbers in the data set. Then, divide by how many numbers there are. $7+9+12+13+14=55$ and $55\div 5=11$.

Practice

Exercise 2.14.4

A preschool teacher is rearranging four boxes of playing blocks so that each box contains an equal number of blocks. Currently Box 1 has 32 blocks, Box 2 has 18, Box 3 has 41, and Box 4 has 9.

Select **all** the ways he could make each box have the same number of blocks.

A. Remove all the blocks and make four equal piles of 25, then put each pile in one of the boxes.
B. Remove 7 blocks from Box 1 and place them in Box 2.
C. Remove 21 blocks from Box 3 and place them in Box 4.
D. Remove 7 blocks from Box 1 and place them in Box 2, and remove 21 blocks from Box 3 and place them in Box 4.
E. Remove 7 blocks from Box 1 and place them in Box 2, and remove 16 blocks from Box 3 and place them in Box 4.

Exercise 2.14.5

In a round of mini-golf, Clare records the number of strokes it takes to hit the ball into the hole of each green.

2 3 1 4 5 2 3 4 3

She said that, if she redistributed the strokes on different greens, she could tell that her average number of strokes per hole is 3. Explain how Clare is correct.

Exercise 2.14.6

Three sixth-grade classes raised $25.50, $49.75, and $37.25 for their classroom libraries. They agreed to share the money raised equally. What is each class's equal share? Explain or show your reasoning.

Exercise 2.14.7

In her English class, Mai's teacher gives 4 quizzes each worth 5 points. After 3 quizzes, she has the scores 4, 3, and 4. What does she need to get on the last quiz to have a mean score of 4? Explain or show your reasoning.

Exercise 2.14.8

An earthworm farmer examined two containers of a certain species of earthworms so that he could learn about their lengths. He measured 25 earthworms in each container and recorded their lengths in millimeters.

Here are histograms of the lengths for each container.

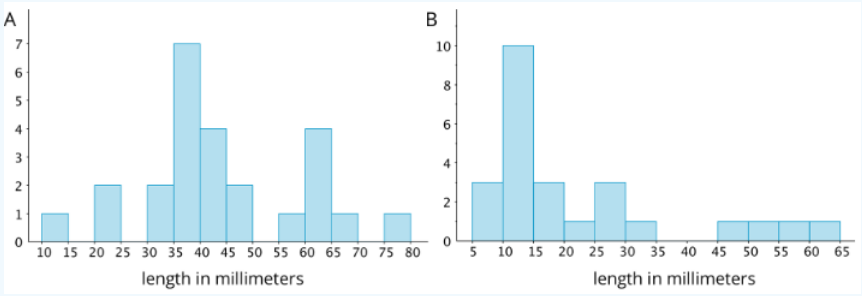

Figure 2.14.8: Two histograms, length in millimeters. Histogram A from 10 to 80 by fives, Beginning at 10 up to but not including 15, height of bar at each interval is 1, 0, 2, 0, 2, 7, 4, 2, 0, 1, 4, 1, 0, 1. Histogram B from 5 to 65 by fives. Beginning at 5 up to but not including 10, height of bar at each interval is 3, 10, 3, 1, 3, 1, 0, 0, 1, 1, 1, 1.

1. Which container tends to have longer worms than the other container?
2. For which container would 15 millimeters be a reasonable description of a typical length of the worms in the container?
3. If length is related to age, which container had the most young worms?

(From Unit 8.2.5)

Exercise 2.14.9

Diego thinks that $x = 3$ is a solution to the equation $x^2 = 16$. Do you agree? Explain or show your reasoning.

(From Unit 6.3.4)

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2.15: Solve Equations Using Integers; The Division Property of Equality (Part 1)

Learning Objectives

- Determine whether an integer is a solution of an equation
- Solve equations with integers using the Addition and Subtraction Properties of Equality
- Model the Division Property of Equality
- Solve equations using the Division Property of Equality
- Translate to an equation and solve

be prepared!

Before you get started, take this readiness quiz.

1. Evaluate $x+4$ when $x=-4$. If you missed this problem, review Example 3.2.9.
2. Solve: $y-6=10$. If you missed this problem, review Example 2.3.6.
3. Translate into an algebraic expression 5 less than x. If you missed this problem, review Table 1.3.1.

Determine Whether a Number is a Solution of an Equation

In Solve Equations with the Subtraction and Addition Properties of Equality, we saw that a solution of an equation is a value of a variable that makes a true statement when substituted into that equation. In that section, we found solutions that were whole numbers. Now that we've worked with integers, we'll find integer solutions to equations.

The steps we take to determine whether a number is a solution to an equation are the same whether the solution is a whole number or an integer.

HOW TO: DETERMINE WHETHER A NUMBER IS A Solution TO AN EQUATION.

Step 1. Substitute the number for the variable in the equation.

Step 2. Simplify the expressions on both sides of the equation.

Step 3. Determine whether the resulting equation is true.

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

Example 2.15.1: determine the solution

Determine whether each of the following is a solution of $2x-5=-13$:

a. $x=4$
b. $x=-4$
c. $x=-9$

Solution

(a) Substitute 4 for x in the equation to determine if it is true.	2x − 5 = −13
Substitute 4 for x.	$2(4)-5 \stackrel{?}{=} -13$
Multiply.	$8-5 \stackrel{?}{=} -13$
Subtract.	$3 \neq -13$

Since $x=4$ does not result in a true equation, 4 is not a solution to the equation.

(b) Substitute -4 for x in the equation to determine if it is true.	2x − 5 = −13

Substitute -4 for x.	$2(-4)-5 \stackrel{?}{=} -13$
Multiply.	$-8-5 \stackrel{?}{=} -13$
Subtract.	$-13=-13$ ✓

Since $x=-4$ results in a true equation, -4 is a solution to the equation.

(b) Substitute -9 for x in the equation to determine if it is true.	2x − 5 = −13
Substitute -9 for x.	$2(-9)-5 \stackrel{?}{=} -13$
Multiply.	$-18-5 \stackrel{?}{=} -13$
Subtract.	$-23 \neq -13$

Since $x=-9$ does not result in a true equation, -9 is not a solution to the equation.

? Exercise 2.15.1

Determine whether each of the following is a solution of $2x-8=-14$:

a. $x=-11$
b. $x=11$
c. $x=-3$

Answer a

no

Answer b

no

Answer c

yes

? Exercise 2.15.2

Determine whether each of the following is a solution of $2y+3=-11$:

a. $y=4$
b. $y=-4$
c. $y=-7$

Answer a

no

Answer b

no

Answer c

yes

Solve Equations with Integers Using the Addition and Subtraction Properties of Equality

In Solve Equations with the Subtraction and Addition Properties of Equality, we solved equations similar to the two shown here using the Subtraction and Addition Properties of Equality. Now we can use them again with integers.

$$\begin{aligned} x+4 &= 12 \\ x+4-4 &= 12-4 \\ x &= 8 \end{aligned} \qquad \begin{aligned} y-5 &= 9 \\ y-5+5 &= 9+5 \\ y &= 14 \end{aligned}$$

When you add or subtract the same quantity from both sides of an equation, you still have equality.

Definition: Properties of Equalities

Subtraction Property of Equality	Addition Property of Equality
For any numbers a, b, c, if a = b then a − c = b − c.	For any numbers a, b, c, if a = b then a + c = b + c.

Example 2.15.2:

Solve: $y+9=5$.

Solution

Subtract 9 from each side to undo the addition.	$y+9-9=5-9$
Simplify.	$y=-4$

Check the result by substituting -4 into the original equation.

Substitute −4 for y	$-4+9 \stackrel{?}{=} 5$
	$5=5$ ✓

Since $y=-4$ makes $y+9=5$ a true statement, we found the solution to this equation

Exercise 2.15.3

Solve: $y+11=7$

Answer

-4

Exercise 2.15.4

Solve: $y+15=-4$

Answer

-19

Example 2.15.3: solve

Solve: $a-6=-8$

Solution

Add 6 to each side to undo the subtraction.	$a-6+6=-8+6$
Simplify.	$a=-2$
Check the result by substituting −2 into the original equation.	$a-6=-8$
Substitute −2 for a.	$-2-6 \stackrel{?}{=} -8$

	$-8 = -8$ ✓

The solution to $a - 6 = -8$ is -2. Since $a = -2$ makes $a - 6 = -8$ a true statement, we found the solution to this equation.

Exercise 2.15.5

Solve: $a - 2 = -8$

Answer

-6

Exercise 2.15.6

Solve: $n - 4 = -8$

Answer

-4

Model the Division Property of Equality

All of the equations we have solved so far have been of the form $x + a = b$ or $x - a = b$. We were able to isolate the variable by adding or subtracting the constant term. Now we'll see how to solve equations that involve division. We will model an equation with envelopes and counters in Figure 2.15.1.

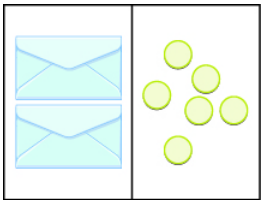


Figure 2.15.1

Here, there are two identical envelopes that contain the same number of counters. Remember, the left side of the workspace must equal the right side, but the counters on the left side are "hidden" in the envelopes. So how many counters are in each envelope?

To determine the number, separate the counters on the right side into 2 groups of the same size. So 6 counters divided into 2 groups means there must be 3 counters in each group (since $6 \div 2 = 3$).

What equation models the situation shown in Figure 2.15.2? There are two envelopes, and each contains x counters. Together, the two envelopes must contain a total of 6 counters. So the equation that models the situation is $2x = 6$.

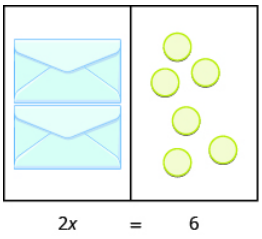


Figure 2.15.2

We can divide both sides of the equation by 2 as we did with the envelopes and counters.

$$\frac{2x}{2} = \frac{6}{2}$$
$$x = 3$$

We found that each envelope contains 3 counters. Does this check? We know $2 \bullet 3 = 6$, so it works. Three counters in each of two envelopes does equal six. Figure 2.15.3 shows another example.

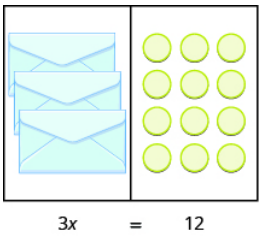


Figure 2.15.3

Now we have 3 identical envelopes and 12 counters. How many counters are in each envelope? We have to separate the 12 counters into 3 groups. Since $12 \div 3 = 4$, there must be 4 counters in each envelope. See Figure 2.15.4

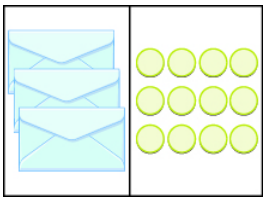


Figure 2.15.4

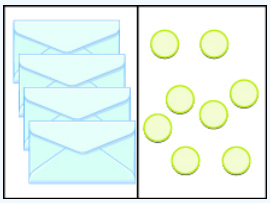
The equation that models the situation is $3x = 12$. We can divide both sides of the equation by 3.

$$\frac{3x}{3} = \frac{12}{3}$$
$$x = 4$$

Does this check? It does because $3 \bullet 4 = 12$.

Example 2.15.4: write an equation

Write an equation modeled by the envelopes and counters, and then solve it.



Solution

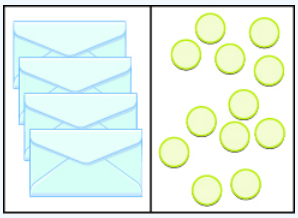
There are 4 envelopes, or 4 unknown values, on the left that match the 8 counters on the right. Let's call the unknown quantity in the envelopes x.

Write the equation.	$4x = 8$
Divide both sides by 4.	$\frac{4x}{4} = \frac{8}{4}$
Simplify.	$x = 2$

There are 2 counters in each envelope.

Exercise 2.15.7

Write the equation modeled by the envelopes and counters. Then solve it.

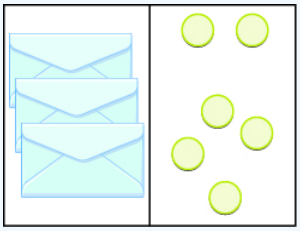


Answer

$4x = 12; x = 3$

Exercise 2.15.8

Write the equation modeled by the envelopes and counters. Then solve it.



Answer

$3x = 6; x = 2$

Solve Equations Using the Division Property of Equality

The previous examples lead to the Division Property of Equality. When you divide both sides of an equation by any nonzero number, you still have equality.

Definition: Division Property of Equality

For any numbers a, b, c, and $c \neq 0$,

$$\text{If } a = b \text{ then } \frac{a}{c} = \frac{b}{c}. \tag{2.15.1}$$

Example 2.15.5: solve

Solve: $7x = -49$.

Solution

To isolate x, we need to undo multiplication.

Divide each side by 7.	$\frac{7x}{7} = \frac{-49}{7}$
Simplify	$x = -7$

Check the solution.

Substitute −7 for x.	$7(-7) \stackrel{?}{=} -49$

	$-49 = -49$ ✓

Therefore, -7 is the solution to the equation.

? Exercise 2.15.9

Solve: $8a = 56$

Answer

7

? Exercise 2.15.10

Solve: $11n = 121$

Answer

11

✓ Example 2.15.6: solve

Solve: $-3y = 63$.

Solution

To isolate y, we need to undo the multiplication.

Divide each side by −3.	$\frac{-3y}{-3} = \frac{63}{-3}$
Simplify.	$y = -21$

Check the solution.

Substitute −21 for y.	$-3(-21) \stackrel{?}{=} 63$
	$63 = 63$ ✓

Since this is a true statement, $y = -21$ is the solution to the equation.

? Exercise 2.15.11

Solve: $-8p = 96$

Answer

-12

? Exercise 2.15.12

Solve: $-12m = 108$

Answer

-9

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2.16: Solve Equations Using Integers; The Division Property of Equality (Part 2)

Translate to an Equation and Solve

In the past several examples, we were given an equation containing a variable. In the next few examples, we'll have to first translate word sentences into equations with variables and then we will solve the equations.

Example 2.16.7: translate

Translate and solve: five more than x is equal to -3.

Solution

Translate.	x + 5 = −3
Subtract 5 from both sides.	x + 5 − 5 = −3 − 5
Simplify.	x = −8

Check the answer by substituting it into the original equation.

$$\begin{aligned} x+5 &= -3 \\ -8+5 &\stackrel{?}{=} -3 \\ -3 &= -3 \checkmark \end{aligned}$$

Exercise 2.16.13

Translate and solve: Seven more than x is equal to -2.

Answer

$x+7=-2$; $x=-9$

Exercise 2.16.14

Translate and solve: Eleven more than y is equal to 2.

Answer

$y+11=2$; $y=-9$

Example 2.16.8: translate

Translate and solve: the difference of n and 6 is -10.

Solution

Translate.	n − 6 = −10
Add 6 to each side.	n − 6 + 6 = −10 + 6
Simplify.	n = −4

Check the answer by substituting it into the original equation.

$$\begin{aligned} n-6 &= -10 \\ -4-6 &\stackrel{?}{=} -10 \\ -10 &= -10 \checkmark \end{aligned}$$

Exercise 2.16.15

Translate and solve: The difference of p and 2 is -4.

Answer

$p-2=-4$; $p=-2$

Exercise 2.16.16

Translate and solve: The difference of q and 7 is -3.

Answer

$q-7=-3$; $q=4$

Example 2.16.9: translate

Translate and solve: the number 108 is the product of -9 and y.

Solution

Translate.	108 = −9y	
Divide by −9.	$\frac{108}{-9}=\frac{-9y}{-9}$	(2.16.1)
Simplify.	−12 = y	

Check the answer by substituting it into the original equation.

$$108=-9y$$
$$108\stackrel{?}{=}-9(-12)$$
$$108=108\ \checkmark$$

Exercise 2.16.17

Translate and solve: The number 132 is the product of -12 and y.

Answer

$132=-12y$; $y=-11$

Exercise 2.16.18

Translate and solve: The number 117 is the product of -13 and z.

Answer

$117=-13z$; $z=-9$

Access Additional Online Resources

- One-Step Equations With Adding Or Subtracting
- One-Step Equations With Multiplying Or Dividing

Key Concepts

- How to determine whether a number is a solution to an equation.
  - Step 1. Substitute the number for the variable in the equation.
  - Step 2. Simplify the expressions on both sides of the equation.
  - Step 3. Determine whether the resulting equation is true.
    - If it is true, the number is a solution.
    - If it is not true, the number is not a solution.
- Properties of Equalities

Subtraction Property of Equality	Addition Property of Equality
For any numbers a,b,c, if a=b then a−c=b−c	For any numbers a,b,c, if a=b then a+c=b+c

- Division Property of Equality
  - For any numbers a, b, c and $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$

Practice Makes Perfect

Determine Whether a Number is a Solution of an Equation

In the following exercises, determine whether each number is a solution of the given equation.

285. $4x - 2 = 6$
   a. $x = -2$
   b. $x = -1$
   c. $x = 2$

286. $4y - 10 = -14$
   a. $y = -6$
   b. $y = -1$
   c. $y = 1$

287. $9a + 27 = -63$
   a. $a = 6$
   b. $a = -6$
   c. $a = -10$

288. $7c + 42 = -56$
   a. $c = 2$
   b. $c = -2$
   c. $c = -14$

Solve Equations Using the Addition and Subtraction Properties of Equality

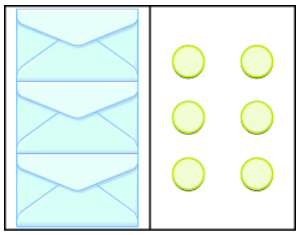
In the following exercises, solve for the unknown.

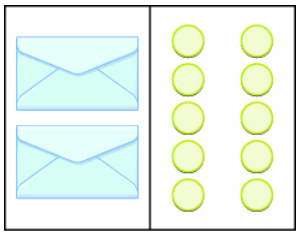
289. $n + 12 = 5$
290. $m + 16 = 2$
291. $p + 9 = -8$
292. $q + 5 = -6$
293. $u - 3 = -7$
294. $v - 7 = -8$
295. $h - 10 = -4$
296. $k - 9 = -5$
297. $x + (-2) = -18$
298. $y + (-3) = -10$
299. $r - (-5) = -9$

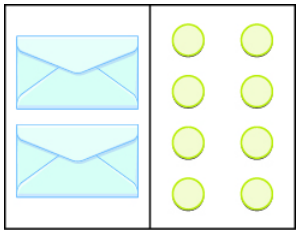
300. $s - (-2) = -11$

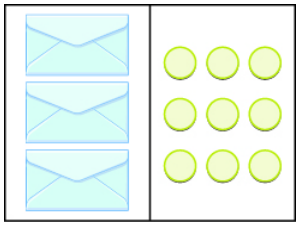
Model the Division Property of Equality

In the following exercises, write the equation modeled by the envelopes and counters and then solve it.

301. 

302. 

303. 

304. 

Solve Equations Using the Division Property of Equality

In the following exercises, solve each equation using the division property of equality and check the solution.

305. $5x = 45$
306. $4p = 64$
307. $-7c = 56$
308. $-9x = 54$
309. $-14p = -42$
310. $-8m = -40$
311. $-120 = 10q$
312. $-75 = 15y$
313. $24x = 480$
314. $18n = 540$
315. $-3z = 0$
316. $4u = 0$

Translate to an Equation and Solve

In the following exercises, translate and solve.

317. Four more than n is equal to 1.
318. Nine more than m is equal to 5.

319. The sum of eight and p is −3.
320. The sum of two and q is −7.
321. The difference of a and three is −14.
322. The difference of b and 5 is −2.
323. The number −42 is the product of −7 and x.
324. The number −54 is the product of −9 and y.
325. The product of f and −15 is 75.
326. The product of g and −18 is 36.
327. −6 plus c is equal to 4.
328. −2 plus d is equal to 1.
329. Nine less than n is −4.
330. Thirteen less than n is −10 .

Mixed Practice

In the following exercises, solve.

331. (a) $x + 2 = 10$ (b) $2x = 10$
332. (a) $y + 6 = 12$ (b) $6y = 12$
333. (a) $-3p = 27$ (b) $p - 3 = 27$
334. (a) $-2q = 34$ (b) $q - 2 = 34$
335. $a - 4 = 16$
336. $b - 1 = 11$
337. $-8m = -56$
338. $-6n = -48$
339. $-39 = u + 13$
340. $-100 = v + 25$
341. $11r = -99$
342. $15s = -300$
343. $100 = 20d$
344. $250 = 25n$
345. $-49 = x - 7$
346. $64 = y - 4$

Everyday Math

347. **Cookie packaging** A package of 51 cookies has 3 equal rows of cookies. Find the number of cookies in each row, c, by solving the equation $3c = 51$.
348. **Kindergarten class** Connie's kindergarten class has 24 children. She wants them to get into 4 equal groups. Find the number of children in each group, g, by solving the equation $4g = 24$.

Writing Exercises

349. Is modeling the Division Property of Equality with envelopes and counters helpful to understanding how to solve the equation $3x = 15$? Explain why or why not.
350. Suppose you are using envelopes and counters to model solving the equations $x + 4 = 12$ and $4x = 12$. Explain how you would solve each equation.
351. Frida started to solve the equation $-3x = 36$ by adding 3 to both sides. Explain why Frida's method will not solve the equation.
352. Raoul started to solve the equation $4y = 40$ by subtracting 4 from both sides. Explain why Raoul's method will not solve the equation.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether an integer is a solution of an equation.			
solve equations with integers using the addition and subtraction properties of equality.			
model division property of equality.			
solve equations using the division property of equality.			
translate to an equation and solve.			

(b) Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

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2.E: Integers (Exercises)

3.1 - Introduction to Integers

Locate Positive and Negative Numbers on the Number Line

In the following exercises, locate and label the integer on the number line.

353. 5
354. −5
355. −3
356. 3
357. −8
358. −7

Order Positive and Negative Numbers

In the following exercises, order each of the following pairs of numbers, using < or >.

359. 4__8
360. −6__3
361. −5__−10
362. −9__−4
363. 2__−7
364. −3__1

Find Opposites

In the following exercises, find the opposite of each number.

365. 6
366. −2
367. −4
368. 3

In the following exercises, simplify.

369. (a) −(8) (b) −(−8)
370. (a) −(9) (b) −(−9)

In the following exercises, evaluate.

371. −x, when (a) x = 32 (b) x = −32
372. −n, when (a) n = 20 (b) n = −20

Simplify Absolute Values

In the following exercises, simplify.

373. |−21|
374. |−42|
375. |36|
376. −|15|
377. |0|
378. −|−75|

In the following exercises, evaluate.

379. |x| when x = −14
380. −|r| when r = 27
381. −|−y| when y = 33
382. |−n| when n = −4

In the following exercises, fill in <, >, or = for each of the following pairs of numbers.

383. $-|-4|\_\_4$
384. $-2\_\_|-2|$
385. $-|-6|\_\_-6$
386. $-|-9|\_\_|-9|$

In the following exercises, simplify.

387. $-(-55)$ and $-|-55|$
388. $-(-48)$ and $-|-48|$
389. $|12-5|$
390. $|9+7|$
391. $6|-9|$
392. $|14-8|-|-2|$
393. $|9-3|-|5-12|$
394. $5+4|15-3|$

Translate Phrases to Expressions with Integers

In the following exercises, translate each of the following phrases into expressions with positive or negative numbers.

395. the opposite of 16
396. the opposite of −8
397. negative 3
398. 19 minus negative 12
399. a temperature of 10 below zero
400. an elevation of 85 feet below sea level

3.2 - Add Integers

Model Addition of Integers

In the following exercises, model the following to find the sum.

401. $3+7$
402. $-2+6$
403. $5+(-4)$
404. $-3+(-6)$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

405. $14+82$
406. $-33+(-67)$
407. $-75+25$
408. $54+(-28)$
409. $11+(-15)+3$
410. $-19+(-42)+12$
411. $-3+6(-1+5)$
412. $10+4(-3+7)$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

413. $n+4$ when (a) $n=-1$ (b) $n=-20$
414. $x+(-9)$ when (a) $x=3$ (b) $x=-3$
415. $(x+y)^3$ when $x=-4$, $y=1$
416. $(u+v)^2$ when $u=-4$, $v=11$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate each phrase into an algebraic expression and then simplify.

417. the sum of −8 and 2
418. 4 more than −12
419. 10 more than the sum of −5 and −6
420. the sum of 3 and −5, increased by 18

Add Integers in Applications

In the following exercises, solve.

421. **Temperature** On Monday, the high temperature in Denver was −4 degrees. Tuesday's high temperature was 20 degrees more. What was the high temperature on Tuesday?
422. **Credit** Frida owed $75 on her credit card. Then she charged $21 more. What was her new balance?

3.3 - Subtract Integers

Model Subtraction of Integers

In the following exercises, model the following.

423. 6 − 1
424. −4 − (−3)
425. 2 − (−5)
426. −1 − 4

Simplify Expressions with Integers

In the following exercises, simplify each expression.

427. 24 − 16
428. 19 − (−9)
429. −31 − 7
430. −40 − (−11)
431. −52 − (−17) − 23
432. 25 − (−3 − 9)
433. (1 − 7) − (3 − 8)
434. $3^2 - 7^2$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

435. x − 7 when (a) x = 5 (b) x = −4
436. 10 − y when (a) y = 15 (b) y = −16
437. $2n^2 - n + 5$ when n = −4
438. $-15 - 3u^2$ when u = −5

Translate Phrases to Algebraic Expressions

In the following exercises, translate each phrase into an algebraic expression and then simplify.

439. the difference of −12 and 5
440. subtract 23 from −50

Subtract Integers in Applications

In the following exercises, solve the given applications.

441. **Temperature** One morning the temperature in Bangor, Maine was 18 degrees. By afternoon, it had dropped 20 degrees. What was the afternoon temperature?

442. **Temperature** On January 4, the high temperature in Laredo, Texas was 78 degrees, and the high in Houlton, Maine was −28 degrees. What was the difference in temperature of Laredo and Houlton?

3.4 - Multiply and Divide Integers

Multiply Integers

In the following exercises, multiply.

443. $-9 \cdot 4$
444. $5(-7)$
445. $(-11)(-11)$
446. $-1 \cdot 6$

Divide Integers

In the following exercises, divide.

447. $56 \div (-8)$
448. $-120 \div (-6)$
449. $-96 \div 12$
450. $96 \div (-16)$
451. $45 \div (-1)$
452. $-162 \div (-1)$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

453. $5(-9) - 3(-12)$
454. $(-2)^5$
455. -3^4
456. $(-3)(4)(-5)(-6)$
457. $42 - 4(6 - 9)$
458. $(8 - 15)(9 - 3)$
459. $-2(-18) \div 9$
460. $45 \div (-3) - 12$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

461. $7x - 3$ when $x = -9$
462. $16 - 2n$ when $n = -8$
463. $5a + 8b$ when $a = -2$, $b = -6$
464. $x^2 + 5x + 4$ when $x = -3$

Translate Word Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

465. the product of −12 and 6
466. the quotient of 3 and the sum of −7 and s

3.5 - Solve Equations using Integers; The Division Property of Equality

Determine Whether a Number is a Solution of an Equation

In the following exercises, determine whether each number is a solution of the given equation.

467. $5x - 10 = -35$
    a. $x = -9$
    b. $x = -5$

c. x = 5

468. 8u + 24 = −32

a. u = −7
b. u = −1
c. u = 7

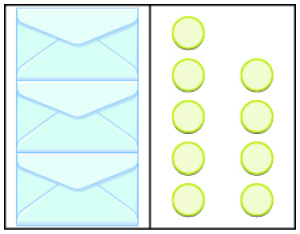
Using the Addition and Subtraction Properties of Equality

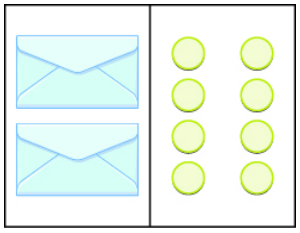
In the following exercises, solve.

469. a + 14 = 2
470. b − 9 = −15
471. c + (−10) = −17
472. d − (−6) = −26

Model the Division Property of Equality

In the following exercises, write the equation modeled by the envelopes and counters. Then solve it.

473. 

474. 

Solve Equations Using the Division Property of Equality

In the following exercises, solve each equation using the division property of equality and check the solution.

475. 8p = 72
476. −12q = 48
477. −16r = −64
478. −5s = −100

Translate to an Equation and Solve.

In the following exercises, translate and solve.

479. The product of −6 and y is −42
480. The difference of z and −13 is −18.
481. Four more than m is −48.
482. The product of −21 and n is 63.

Everyday Math

483. Describe how you have used two topics from this chapter in your life outside of your math class during the past month.

PRACTICE TEST

484. Locate and label 0, 2, −4, and −1 on a number line.

In the following exercises, compare the numbers, using < or > or =.

485. (a) −6__3 (b) −1__−4

486. (a) $-5 \_\_ |-5|$ (b) $-|-2| \_\_ -2$

In the following exercises, find the opposite of each number.

487. (a) -7 (b) 8

In the following exercises, simplify.

488. $-(-22)$
489. $|4 - 9|$
490. $-8 + 6$
491. $-15 + (-12)$
492. $-7 - (-3)$
493. $10 - (5 - 6)$
494. $-3 \cdot 8$
495. $-6(-9)$
496. $70 \div (-7)$
497. $(-2)^3$
498. -4^2
499. $16-3(5-7)$
500. $|21 - 6| - |-8|$

In the following exercises, evaluate.

501. $35 - a$ when $a = -4$
502. $(-2r)^2$ when $r = 3$
503. $3m - 2n$ when $m = 6, n = -8$
504. $-|-y|$ when $y = 17$

In the following exercises, translate each phrase into an algebraic expression and then simplify, if possible.

505. the difference of -7 and -4
506. the quotient of 25 and the sum of m and n.

In the following exercises, solve.

507. Early one morning, the temperature in Syracuse was $-8°$F. By noon, it had risen $12°$. What was the temperature at noon?
508. Collette owed $128 on her credit card. Then she charged $65. What was her new balance?

In the following exercises, solve.

509. $n + 6 = 5$
510. $p - 11 = -4$
511. $-9r = -54$

In the following exercises, translate and solve.

512. The product of 15 and x is 75.
513. Eight less than y is -32.

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2.S: Integers (Summary)

Key Terms

absolute value	A number's distance from 0 on the number line.
integers	Counting numbers, their opposites, and zero $\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$ (2.S.1)
negative number	A number less than zero.
opposites	The number that is the same distance from zero on the number line, but on the opposite side of zero.

Key Concepts

3.1 - Introduction to Integers

- **Opposite Notation**
  - −a means the opposite of the number a
  - The notation −a is read the opposite of a.
- **Absolute Value Notation**
  - The absolute value of a number n is written as |n|.
  - |n| ≥ 0 for all numbers.

3.2 - Add Integers

- **Addition of Positive and Negative Integers**

5 + 3	−5 + (−3)
both positive, sum positive	both negative, sum negative
When the signs are the same, the counters would be all the same color, so add them.	
−5 + 3	5 + (−3)
different signs, more negatives	different signs, more positives
sum negative	sum positive
When the signs are different, some counters would make neutral pairs; subtract to see how many are left.	

3.3 - Subtract Integers

- **Subtraction of Integers**

Table 3.110

5 – 3 = 2	–5 – (–3) = –2
2 positives	2 negatives
When there would be enough counters of the color to take away, subtract.	
–5 – 3 =	5 – (–3) = 8
5 negatives, want to subtract 3 positives	5 positives, want to subtract 3 negatives
When there would not be enough of the counters to take away, add neutral pairs.	

- **Subtraction Property**
  - a − b = a + (−b)

- $a - (-b) = a + b$

- **Solve Application Problems**
  1. Identify what you are asked to find.
  2. Write a phrase that gives the information to find it.
  3. Translate the phrase to an expression.
  4. Simplify the expression.
  5. Answer the question with a complete sentence.

3.4 - Multiply and Divide Integers

- **Multiplication of Signed Numbers**
  - To determine the sign of the product of two signed numbers:

Same Signs	Product
Two positives	Positive
Two negatives	Positive

Different Signs	Product
Positive • negative	Negative
Negative • positive	Negative

- **Division of Signed Numbers**
  - To determine the sign of the quotient of two signed numbers:

Same Signs	Quotient
Two positives	Positive
Two negatives	Positive

Different Signs	Quotient
Positive & negative	Negative
Negative & positive	Negative

- **Multiplication by −1**
  - Multiplying a number by −1 gives its opposite: $-1a = -a$
- **Division by −1**
  - Dividing a number by −1 gives its opposite: $a \div (-1) = -a$

3.5 - Solve Equations Using Integers; The Division Property of Equality

- **How to determine whether a number is a solution to an equation.**
  1. Substitute the number for the variable in the equation.
  2. Simplify the expressions on both sides of the equation.
  3. Determine whether the resulting equation is true.
  - If it is true, the number is a solution.
  - If it is not true, the number is not a solution.
- **Properties of Equalities**

Subtraction Property of Equality	Addition Property of Equality	Division Property of Equality

Subtraction Property of Equality	Addition Property of Equality	Division Property of Equality
For any numbers a, b, c, if a = b then a − c = b − c.	For any numbers a, b, c, if a = b then a + c = b + c.	For any numbers a, b, c, and c ≠ 0 If a = b, then $\frac{a}{c} = \frac{b}{c}$.

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CHAPTER OVERVIEW

3: Algebraic Expressions

You may not realize it, but you already use algebra every day. Perhaps you figure out how much to tip a server in a restaurant. Maybe you calculate the amount of change you should get when you pay for something. It could even be when you compare batting averages of your favorite players. You can describe the algebra you use in specific words, and follow an orderly process. In this chapter, you will explore the words used to describe algebra and start on your path to solving algebraic problems easily, both in class and in your everyday life.

3.1: Use the Language of Algebra (Part 1)
3.2: Use the Language of Algebra (Part 2)
3.3: Evaluate, Simplify, and Translate Expressions (Part 1)
3.4: Evaluate, Simplify, and Translate Expressions (Part 2)
3.5: Solving Equations Using the Subtraction and Addition Properties of Equality (Part 1)
3.6: Solving Equations Using the Subtraction and Addition Properties of Equality (Part 2)
3.E: Introduction to the Language of Algebra (Exercises)
3.S: Introduction to the Language of Algebra (Summary)

Figure 2.1 - Algebra has a language of its own. The picture shows just some of the words you may see and use in your study of Prealgebra.

Contributors

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3.1: Use the Language of Algebra (Part 1)

Learning Objectives

- Use variables and algebraic symbols
- Identify expressions and equations
- Simplify expressions with exponents
- Simplify expressions using the order of operations

Be Prepared!

Before you get started, take this readiness quiz.

a. Add: $43+69$. If you missed this problem, review Example 1.2.8.
b. Multiply: $(896)201$. If you missed this problem, review Example 1.4.11.
c. Divide: $7,263 \div 9$. If you missed this problem, review Example 1.5.8.

Use Variables and Algebraic Symbols

Greg and Alex have the same birthday, but they were born in different years. This year Greg is 20 years old and Alex is 23, so Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more, right?

In the language of algebra, we say that Greg's age and Alex's age are variable and the three is a constant. The ages change, or vary, so age is a variable. The 3 years between them always stays the same, so the age difference is the constant.

In algebra, letters of the alphabet are used to represent variables. Suppose we call Greg's age g. Then we could use $g+3$ to represent Alex's age. See Table 3.1.1.

Table 3.1.1

Greg's age	Alex's age
12	15
20	23
35	38
g	g + 3

Letters are used to represent variables. Letters often used for variables are $x, y, a, b,$ and c.

Definition: Variables and Constants

A variable is a letter that represents a number or quantity whose value may change.

A constant is a number whose value always stays the same.

To write algebraically, we need some symbols as well as numbers and variables. There are several types of symbols we will be using. In Whole Numbers, we introduced the symbols for the four basic arithmetic operations: addition, subtraction, multiplication, and division. We will summarize them here, along with words we use for the operations and the result.

Table 3.1.2

Operation	Notation	Say:	The result is...
Addition	a + b	a plus b	the sum of a and b
Subtraction	a − b	a minus b	the difference of a and b
Multiplication	a • b, (a)(b), (a)b, a(b)	a times b	the product of a and b

Operation	Notation	Say:	The result is...
Division	a ÷ b, a / b, $\frac{a}{b}$, $b\overline{)a}$	a divided by b	the quotient of a and b

In algebra, the cross symbol, $\times$, is not used to show multiplication because that symbol may cause confusion. Does $3xy$ mean $3 \times y$ (three times y) or $3 \bullet x \bullet y$ (three times x times y)? To make it clear, use $\bullet$ or parentheses for multiplication.

We perform these operations on two numbers. When translating from symbolic form to words, or from words to symbolic form, pay attention to the words *of* or *and* to help you find the numbers.

The *sum* ***of*** 5 ***and*** 3 means add 5 plus 3, which we write as $5+3$.

The *difference* ***of*** 9 ***and*** 2 means subtract 9 minus 2, which we write as $9-2$.

The *product* ***of*** 4 ***and*** 8 means multiply 4 times 8, which we can write as $4 \bullet 8$.

The *quotient* ***of*** 20 ***and*** 5 means divide 20 by 5, which we can write as $20 \div 5$.

Example 3.1.1: translate to words

Translate from algebra to words:

a. $12+14$
b. $(30)(5)$
c. $64 \div 8$
d. $x-y$

Solution

a.

12 + 14
12 plus 14
the sum of twelve and fourteen

b.

(30)(5)
30 times 5
the product of thirty and five

c.

64 ÷ 8
64 divided by 8
the quotient of sixty-four and eight

d.

x − y
x minus y
the difference of x and y

exercise 3.1.1

Translate from algebra to words.

a. $18+11$
b. $(27)(9)$
c. $84\div 7$
d. $p-q$

Answer a

18 plus 11; the sum of eighteen and eleven

Answer b

27 times 9; the product of twenty-seven and nine

Answer c

84 divided by 7; the quotient of eighty-four and seven

Answer d

p minus q; the difference of p and q

exercise 3.1.2

Translate from algebra to words.

a. $47-19$
b. $72\div 9$
c. $m+n$
d. $(13)(7)$

Answer a

47 minus 19; the difference of forty-seven and nineteen

Answer b

72 divided by 9; the quotient of seventy-two and nine

Answer c

m plus n; the sum of m and n

Answer d

13 times 7; the product of thirteen and seven

When two quantities have the same value, we say they are equal and connect them with an *equal sign*.

Definition: Equality Symbol

$a=b$ is read a is equal to b

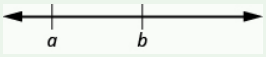
The symbol $=$ is called the equal sign.

An inequality is used in algebra to compare two quantities that may have different values. The number line can help you understand inequalities. Remember that on the number line the numbers get larger as they go from left to right. So if we know that b is greater than a, it means that b is to the right of a on the number line. We use the symbols "$<$" and "$>$" for inequalities.

Definition: Inequality

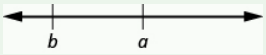
$a < b$ is read a is less than b

a is to the left of b on the number line



$a > b$ is read a is greater than b

a is to the right of b on the number line



The expressions $a < b$ and $a > b$ can be read from left-to-right or right-to-left, though in English we usually read from left-to-right. In general, $a < b$ is equivalent to $b > a$. For example, $7 < 11$ is equivalent to $11 > 7$. $a > b$ is equivalent to $b < a$. For example, $17 > 4$ is equivalent to $4 < 17$.

When we write an inequality symbol with a line under it, such as $a \leq b$, it means $a < b$ or $a = b$. We read this a is less than or equal to b. Also, if we put a slash through an equal sign, $\neq$, it means not equal.

We summarize the symbols of equality and inequality in Table 3.1.3.

Table 3.1.3

Algebraic Notation	Say
a = b	a is equal to b
a ≠ b	a is not equal to b
a < b	a is less than b
a > b	a is greater than b
a ≤ b	a is less than or equal to b
a ≥ b	a is greater than or equal to b

Definition: Symbols $<$ and $>$

The symbols $<$ and $>$ each have a smaller side and a larger side.

$$\text{smaller side} < \text{larger side}$$

$$\text{larger side} > \text{smaller side}$$

The smaller side of the symbol faces the smaller number and the larger faces the larger number.

Example 3.1.2: translate to words

Translate from algebra to words:

a. $20 \leq 35$
b. $11 \neq 15 - 3$
c. $9 > 10 \div 2$
d. $x + 2 < 10$

Solution

a.

$20 \leq 35$
20 is less than or equal to 35

b.

$11 \neq 15 - 3$
11 is not equal to 15 minus 3

c.

$9 > 10 \div 2$
9 is greater than 10 divided by 2

d.

$x + 2 < 10$
x plus 2 is less than 10

exercise 3.1.3

Translate from algebra to words.

a. $14 \leq 27$
b. $19 - 2 \neq 8$
c. $12 > 4 \div 2$
d. $x - 7 < 1$

Answer a

fourteen is less than or equal to twenty-seven

Answer b

nineteen minus two is not equal to eight

Answer c

twelve is greater than four divided by two

Answer d

x minus seven is less than one

exercise 3.1.4

Translate from algebra to words.

a. $19 \geq 15$
b. $7 = 12 - 5$
c. $15 \div 3 < 8$
d. $y - 3 > 6$

Answer a

nineteen is greater than or equal to fifteen

Answer b

seven is equal to twelve minus five

Answer c

fifteen divided by three is less than eight

Answer d

y minus three is greater than six

Example 3.1.3: translate

The information in Figure 3.1.1 compares the fuel economy in miles-per-gallon (mpg) of several cars. Write the appropriate symbol =, in each expression to compare the fuel economy of the cars.

Car	Prius	Mini Cooper	Toyota Corolla	Versa	Honda Fit
Fuel economy (mpg)	48	27	28	26	27

Figure 3.1.1: (credit: modification of work by Bernard Goldbach, Wikimedia Commons)

a. MPG of Prius _____ MPG of Mini Cooper
b. MPG of Versa _____ MPG of Fit
c. MPG of Mini Cooper _____ MPG of Fit
d. MPG of Corolla _____ MPG of Versa
e. MPG of Corolla_____ MPG of Prius

Solution

a.

	MPG of Prius____MPG of Mini Cooper
Find the values in the chart.	48____27
Compare.	48 > 27
	MPG of Prius > MPG of Mini Cooper

b.

	MPG of Versa____MPG of Fit
Find the values in the chart.	26____27
Compare.	26 < 27
	MPG of Versa < MPG of Fit

c.

	MPG of Mini Cooper____MPG of Fit
Find the values in the chart.	27____27
Compare.	27 = 27
	MPG of Mini Cooper = MPG of Fit

d.

	MPG of Corolla____MPG of Versa
Find the values in the chart.	28____26
Compare.	28 > 26

	MPG of Corolla > MPG of Versa

e.

	MPG of Corolla____MPG of Prius
Find the values in the chart.	28____48
Compare.	28 < 48
	MPG of Corolla < MPG of Prius

? exercise 3.1.5

Use Figure 3.1.1 to fill in the appropriate symbol, $=$, $<$, or $>$.

a. MPG of Prius_____MPG of Versa
b. MPG of Mini Cooper_____ MPG of Corolla

Answer a

$>$

Answer b

$<$

? exercise 3.1.6

Use Figure 3.1.1 to fill in the appropriate symbol, $=$, $<$, or $>$.

a. MPG of Fit_____ MPG of Prius
b. MPG of Corolla _____ MPG of Fit

Answer a

$<$

Answer b

$<$

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in written language. They indicate which expressions are to be kept together and separate from other expressions. Table 3.1.4 lists three of the most commonly used grouping symbols in algebra.

Table 3.1.4

Common Grouping Symbols	
parentheses	( )
brackets	[ ]
braces	{ }

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

$$8(14-8) \qquad 21-3[2+4(9-8)] \qquad 24\div 13-2[1(6-5)+4]$$

Identify Expressions and Equations

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. “Running very fast” is a phrase, but “The football player was running very fast” is a sentence. A sentence has a subject and a verb.

In algebra, we have *expressions* and *equations*. An expression is like a phrase. Here are some examples of expressions and how they relate to word phrases:

Table 3.1.5

Expression	Words	Phrase
3 + 5	3 plus 5	the sum of three and five
n - 1	n minus one	the difference of n and one
6 • 7	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Notice that the phrases do not form a complete sentence because the phrase does not have a verb. An equation is two expressions linked with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb. Here are some examples of equations:

Table 3.1.6

Equation	Sentence
3 + 5 = 8	The sum of three and five is equal to eight.
n − 1 = 14	n minus one equals fourteen.
6 • 7 = 42	The product of six and seven is equal to forty-two.
x = 53	x is equal to fifty-three.
y + 9 = 2y − 3	y plus nine is equal to two y minus three.

Definition: Expressions and Equations

An expression is a number, a variable, or a combination of numbers and variables and operation symbols.

An equation is made up of two expressions connected by an equal sign.

Example 3.1.4: expression or equation

Determine if each is an expression or an equation:

a. $16 - 6 = 10$
b. $4 \bullet 2 + 1$
c. $x \div 25$
d. $y + 8 = 40$

Solution

(a) 16 − 6 = 10	This is an equation—two expressions are connected with an equal sign.
(b) 4 • 2 + 1	This is an expression—no equal sign.
(c) x ÷ 25	This is an expression—no equal sign.

(d) y + 8 = 40	This is an equation—two expressions are connected with an equal sign.

? exercise 3.1.7

Determine if each is an expression or an equation:

a. $23+6=29$
b. $7 \bullet 3-7$

Answer a

equation

Answer b

expression

? exercise 3.1.8

Determine if each is an expression or an equation:

a. $y \div 14$
b. $x-6=21$

Answer a

expression

Answer b

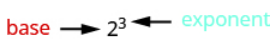
equation

Simplify Expressions with Exponents

To simplify a numerical expression means to do all the math possible. For example, to simplify $4 \bullet 2+1$ we'd first multiply $4 \bullet 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:

$$\begin{aligned} 4 \cdot 2+1 \\ 8+1 \\ 9 \end{aligned}$$

Suppose we have the expression $2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2$. We could write this more compactly using exponential notation. Exponential notation is used in algebra to represent a quantity multiplied by itself several times. We write $2 \bullet 2 \bullet 2$ as 2^3 and $2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2$ as 2^9. In expressions such as 2^3, the 2 is called the **base** and the 3 is called the **exponent**. The exponent tells us how many factors of the base we have to multiply.

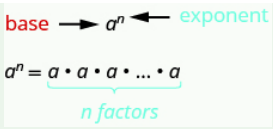


means multiply three factors of 2

We say 2^3 is in exponential notation and $2 \bullet 2 \bullet 2$ is in expanded notation.

Definition: Exponential Notation

For any expression a^n, a is a factor multiplied by itself n times if n is a positive integer.



The expression a^n is read a to the nth power.

For powers of $n = 2$ and $n = 3$, we have special names. a^2 is read as "a squared" a^3 is read as "a cubed" Table 3.1.7 lists some examples of expressions written in exponential notation.

Table 3.1.7

Exponential Notation	In Words
7^2	7 to the second power, or 7 squared
5^3	5 to the third power, or 5 cubed
9^4	9 to the fourth power
12^5	12 to the fifth power

Example 3.1.5: exponential form

Write each expression in exponential form:

a. $16 \bullet 16 \bullet 16 \bullet 16 \bullet 16 \bullet 16 \bullet 16$
b. $9 \bullet 9 \bullet 9 \bullet 9 \bullet 9$
c. $x \bullet x \bullet x \bullet x$
d. $a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a$

Solution

(a) The base 16 is a factor 7 times.	16^7
(b) The base 9 is a factor 5 times.	9^5
(c) The base x is a factor 4 times.	x^4
(d) The base a is a factor 8 times.	a^8

exercise 3.1.9

Write each expression in exponential form: $41 \bullet 41 \bullet 41 \bullet 41 \bullet 41$

Answer

41^5

exercise 3.1.10

Write each expression in exponential form: $7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7$

Answer

7^9

Example 3.1.6: expanded form

Write each exponential expression in expanded form:

a. 8^6
b. x^5

Solution

a. The base is 8 and the exponent is 6, so 8^6 means $8 \bullet 8 \bullet 8 \bullet 8 \bullet 8 \bullet 8$
b. The base is x and the exponent is 5, so x^5 means $x \bullet x \bullet x \bullet x \bullet x$

exercise 3.1.11

Write each exponential expression in expanded form:

a. 4^8
b. a^7

Answer a

$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

Answer b

$a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$

exercise 3.1.12

Write each exponential expression in expanded form:

a. 8^8
b. b^6

Answer a

$8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$

Answer b

$b \cdot b \cdot b \cdot b \cdot b \cdot b$

To simplify an exponential expression without using a calculator, we write it in expanded form and then multiply the factors.

Example 3.1.7: simplify

Simplify: 3^4.

Solution

Expand the expression.	$3^4 = 3 \cdot 3 \cdot 3 \cdot 3$
Multiply left to right.	$9 \cdot 3 \cdot 3 = 27 \cdot 3$
Multiply.	81

exercise 3.1.13

Simplify:

a. 5^3
b. 1^7

Answer a

125

Answer b

1

? exercise 3.1.14

Simplify:

a. 7^2
b. 0^5

Answer a

49

Answer b

0

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3.2: Use the Language of Algebra (Part 2)

Simplify Expressions Using the Order of Operations

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values. For example, consider the expression:

$$4+3\cdot 7$$

Some students say it simplifies to	49	Some students say it simplifies to	25
Since 4 + 3 gives 7.	4 + 3 • 7 = 7 • 7	Since 3 • 7 is 21.	4 + 3 • 7 = 4 + 21
And 7 • 7 is 49.	7 • 7 = 49	And 21 + 4 makes 25.	4 + 21 = 25

Imagine the confusion that could result if every problem had several different correct answers. The same expression should give the same result. So mathematicians established some guidelines called the order of operations, which outlines the order in which parts of an expression must be simplified.

Definition: Order of Operations

When simplifying mathematical expressions perform the operations in the following order:

1. **P**arentheses and other Grouping Symbols
- Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
2. **E**xponents
- Simplify all expressions with exponents.
3. **M**ultiplication and **D**ivision
- Perform all multiplication and division in order from left to right. These operations have equal priority.
4. **A**ddition and **S**ubtraction
- Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, "How will I remember the order?" Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase. **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally.

Table 3.2.8

Order of Operations	
Please	**P**arentheses
Excuse	**E**xponents
My **D**ear	**M**ultiplication and **D**ivision
Aunt **S**ally	**A**ddition and **S**ubtraction

It's good that '**M**y **D**ear' goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, '**A**unt **S**ally' goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

Example 3.2.8: simplify

Simplify the expressions:

a. $4+3 \bullet 7$
b. $(4+3) \bullet 7$

Solution

a.

Are there any parentheses? No. Are there any exponents? No. Is there any multiplication or division? Yes.	$4+3 \cdot 7$
Multiply first.	$4+3 \cdot 7$
Add.	$4+21$
	25

b.

Are there any parentheses? Yes.	$(4+3) \cdot 7$
Simplify inside the parentheses.	$(4+3) \cdot 7$
Are there any exponents? No. Is there any multiplication or division? Yes.	$(7)7$
Multiply.	49

exercise 3.2.15

Simplify the expressions:

a. $12-5 \bullet 2$
b. $(12-5) \bullet 2$

Answer a

2

Answer b

14

exercise 3.2.16

Simplify the expressions:

a. $8+3 \bullet 9$
b. $(8+3) \bullet 9$

Answer a

35

Answer b

99

Example 3.2.9: simplify

Simplify:

a. $18 \div 9 \bullet 2$
b. $18 \bullet 9 \div 2$

Solution

a.

Are there any parentheses? No. Are there any exponents? No. Is there any multiplication or division? Yes.	$18 \div 9 \cdot 2$
Multiply and divide from left to right. Divide.	$2 \cdot 2$
Multiply.	4

b.

Are there any parentheses? No. Are there any exponents? No. Is there any multiplication or division? Yes.	$18 \cdot 9 \div 2$
Multiply and divide from left to right. Multiply.	$162 \div 2$
Divide.	81

exercise 3.2.17

Simplify: $42 \div 7 \bullet 3$

Answer

18

exercise 3.2.18

Simplify: $12 \bullet 3 \div 4$

Answer

9

Example 3.2.10: simplify

Simplify: $18 \div 6 + 4(5 - 2)$.

Solution

Parentheses? Yes, subtract first.	$18 \div 6 + 4(5 - 2)$
Exponents? No. Multiplication or division? Yes.	$18 \div 6 + 4(3)$
Divide first because we multiply and divide left to right.	$3 + 4(3)$
Any other multiplication or division? Yes. Multiply.	$3 + 12$
Any other multiplication or division? No. Any addition or subtraction? Yes	15

exercise 3.2.19

Simplify: $30 \div 5 + 10(3-2)$

Answer

16

exercise 3.2.20

Simplify: $70 \div 10 + 4(6-2)$

Answer

23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

Example 3.2.11: simplify

Simplify: $5 + 2^3 + 3[6 - 3(4-2)]$.

Solution

Are there any parentheses (or other grouping symbol)? Yes.	$5 + 2^3 + 3[6 - 3(4-2)]$
Focus on the parentheses that are inside the brackets.	$5 + 2^3 + 3[6 - 3(4-2)]$
Subtract.	$5 + 2^3 + 3[6 - 3(2)]$
Continue inside the brackets and multiply.	$5 + 2^3 + 3[6 - 6]$
Continue inside the brackets and subtract.	$5 + 2^3 + 3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes.	$5 + 2^3 + 3[0]$
Simplify exponents.	$5 + 8 + 3[0]$
Is there any multiplication or division? Yes.	$5 + 8 + 3[0]$
Multiply.	$5 + 8 + 0$
Is there any addition or subtraction? Yes.	$textcolorred5 + 8 + 0$
Add.	$13 + 0$
Add.	13

exercise 3.2.21

Simplify: $9 + 5^3 - [4(9+3)]$

Answer

86

exercise 3.2.22

Simplify: $7^2 - 2[4(5+1)]$

Answer

1

Example 3.2.12: simplify

Simplify: $2^3 + 34 \div 3 - 5^2$.

Solution

If an expression has several exponents, they may be simplified in the same step.	$2^3 + 3^4 \div 3 - 52$
Simplify exponents.	$2^3 + 3^4 \div 3 - 5^2$
Divide.	$8 + 81 \div 3 - 25$
Add.	$8 + 27 - 25$
Subtract.	$35 - 25$
	10

exercise 3.2.23

Simplify: $3^2 + 2^4 \div 2 + 4^3$

Answer

81

exercise 3.2.24

Simplify: $6^2 - 5^3 \div 5 + 8^2$

Answer

75

Access Additional Online Resources

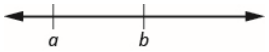
- Order of Operations
- Order of Operations – The Basics
- Ex: Evaluate an Expression Using the Order of Operations
- Example 3: Evaluate an Expression Using The Order of Operations

Key Concepts

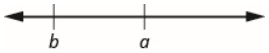
Operation	Notation	Say:	The result is…
Addition	a+b	a plus b	the sum of a and b
Multiplication	a·b,(a)(b),(a)b,a(b)	a times b	The product of a and b
Subtraction	a−b	a minus b	the difference of a and b

Operation	Notation	Say:	The result is…
Division	$a \div b, a/b, \frac{a}{b}, a\overline{)b}$	a divided by b	The quotient of a and b

- **Equality Symbol**
  - $a = b$ is read as a is equal to b
  - The symbol $=$ is called the equal sign.
- **Inequality**
  - $a < b$ is read a is less than b
  - a is to the left of b on the number line

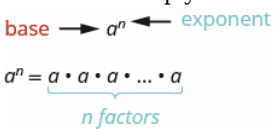


  - $a > b$ is read a is greater than b
  - a is to the right of b on the number line



Algebraic Notation	Say
a=b	a is equal to b
a≠b	a is not equal to b
a<b	a is less than b
a>b	a is greater than b
a≤b	a is less than or equal to b
a≥b	a is greater than or equal to b

- **Exponential Notation**
  - For any expression a^n is a factor multiplied by itself n times, if n is a positive integer.
  - a^n means multiply n factors of a



  - The expression of a^n is read a to the nth power.

Order of Operations When simplifying mathematical expressions perform the operations in the following order:

- Parentheses and other Grouping Symbols: Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
- Exponents: Simplify all expressions with exponents.
- Multiplication and Division: Perform all multiplication and division in order from left to right. These operations have equal priority.
- Addition and Subtraction: Perform all addition and subtraction in order from left to right. These operations have equal priority.

Glossary

expressions

An expression is a number, a variable, or a combination of numbers and variables and operation symbols.

equation

An equation is made up of two expressions connected by an equal sign.

Practice Makes Perfect

Use Variables and Algebraic Symbols

In the following exercises, translate from algebraic notation to words.

1. $16 - 9$
2. $25 - 7$
3. $5 \cdot 6$
4. $3 \cdot 9$
5. $28 \div 4$
6. $45 \div 5$
7. $x + 8$
8. $x + 11$
9. $(2)(7)$
10. $(4)(8)$
11. $14 < 21$
12. $17 < 35$
13. $36 \geq 19$
14. $42 \geq 27$
15. $3n = 24$
16. $6n = 36$
17. $y - 1 > 6$
18. $y - 4 > 8$
19. $2 \leq 18 \div 6$
20. $3 \leq 20 \div 4$
21. $a \neq 7 \cdot 4$
22. $a \neq 1 \cdot 12$

Identify Expressions and Equations

In the following exercises, determine if each is an expression or an equation.

23. $9 \cdot 6 = 54$
24. $7 \cdot 9 = 63$
25. $5 \cdot 4 + 3$
26. $6 \cdot 3 + 5$
27. $x + 7$
28. $x + 9$
29. $y - 5 = 25$
30. $y - 8 = 32$

Simplify Expressions with Exponents

In the following exercises, write in exponential form.

31. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
32. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
33. $x \cdot x \cdot x \cdot x \cdot x$
34. $y \cdot y \cdot y \cdot y \cdot y \cdot y$

In the following exercises, write in expanded form.

35. 5^3
36. 8^3
37. 2^8
38. 10^5

Simplify Expressions Using the Order of Operations

In the following exercises, simplify.

39. (a) 3 + 8 • 5 (b) (3+8) • 5
40. (a) 2 + 6 • 3 (b) (2+6) • 3
41. $2^3 - 12 \div (9 - 5)$
42. $3^2 - 18 \div (11 - 5)$
43. 3 • 8 + 5 • 2
44. 4 • 7 + 3 • 5
45. 2 + 8(6 + 1)
46. 4 + 6(3 + 6)
47. 4 • 12 / 8
48. 2 • 36 / 6
49. 6 + 10 / 2 + 2
50. 9 + 12 / 3 + 4
51. (6 + 10) ÷ (2 + 2)
52. (9 + 12) ÷ (3 + 4)
53. 20 ÷ 4 + 6 • 5
54. 33 ÷ 3 + 8 • 2
55. 20 ÷ (4 + 6) • 5
56. 33 ÷ (3 + 8) • 2
57. $4^2 + 5^2$
58. $3^2 + 7^2$
59. $(4 + 5)^2$
60. $(3 + 7)^2$
61. $3(1 + 9 \cdot 6) - 4^2$
62. $5(2 + 8 \cdot 4) - 7^2$
63. 2[1 + 3(10 − 2)]
64. 5[2 + 4(3 − 2)]

Everyday Math

65. **Basketball** In the 2014 NBA playoffs, the San Antonio Spurs beat the Miami Heat. The table below shows the heights of the starters on each team. Use this table to fill in the appropriate symbol (=, <, >).

Spurs	Height	Heat	Height
Tim Duncan	83″	Rashard Lewis	82"
Boris Diaw	80"	LeBron James	80"
Kawhi Leonard	79"	Chris Bosh	83"
Tony Parker	74"	Dwyane Wade	76"
Danny Green	78"	Ray Allen	77"

a. Height of Tim Duncan____Height of Rashard Lewis
b. Height of Boris Diaw____Height of LeBron James
c. Height of Kawhi Leonard____Height of Chris Bosh
d. Height of Tony Parker____Height of Dwyane Wade
e. Height of Danny Green____Height of Ray Allen

66. **Elevation** In Colorado there are more than 50 mountains with an elevation of over 14,000 feet. The table shows the ten tallest. Use this table to fill in the appropriate inequality symbol.

Mountain	Elevation

Mountain	Elevation
Mt. Elbert	14,433′
Mt. Massive	14,421′
Mt. Harvard	14,420′
Blanca Peak	14,345′
La Plata Peak	14,336′
Uncompahgre Peak	14,309′
Crestone Peak	14,294′
Mt. Lincoln	14,286′
Grays Peak	14,270′
Mt. Antero	14,269′

a. Elevation of La Plata Peak____Elevation of Mt. Antero
b. Elevation of Blanca Peak____Elevation of Mt. Elbert
c. Elevation of Gray's Peak____Elevation of Mt. Lincoln
d. Elevation of Mt. Massive____Elevation of Crestone Peak
e. Elevation of Mt. Harvard____Elevation of Uncompahgre Peak

Writing Exercises

67. Explain the difference between an expression and an equation.
68. Why is it important to use the order of operations to simplify an expression?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use variables and algebraic symbols.			
identify expressions and equations.			
simplify expressions with exponents.			
simplify expressions using the order of operations.			

(b) If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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3.3: Evaluate, Simplify, and Translate Expressions (Part 1)

Learning Objectives

- Evaluate algebraic expressions
- Identify terms, coefficients, and like terms
- Simplify expressions by combining like terms
- Translate word phrases to algebraic expressions

Be prepared!

Before you get started, take this readiness quiz.

1. Is $n \div 5$ an expression or an equation? If you missed this problem, review Example 2.1.4.
2. Simplify 4^5. If you missed this problem, review Example 2.1.6.
3. Simplify $1 + 8 \bullet 9$. If you missed this problem, review Example 2.1.8.

Evaluate Algebraic Expressions

In the last section, we simplified expressions using the order of operations. In this section, we'll evaluate expressions—again following the order of operations.

To **evaluate** an algebraic expression means to find the value of the expression when the variable is replaced by a given number. To evaluate an expression, we substitute the given number for the variable in the expression and then simplify the expression using the order of operations.

Example 3.3.1: evaluate

Evaluate $x + 7$ when

a. $x = 3$
b. $x = 12$

Solution

a. To evaluate, substitute 3 for x in the expression, and then simplify.

	$x + 7$
Substitute.	$3 + 7$
Add.	10

When $x = 3$, the expression $x + 7$ has a value of 10.

b. To evaluate, substitute 12 for x in the expression, and then simplify.

	$x + 7$
Substitute.	$12 + 7$
Add.	19

When $x = 12$, the expression $x + 7$ has a value of 19. Notice that we got different results for parts (a) and (b) even though we started with the same expression. This is because the values used for x were different. When we evaluate an expression, the value varies depending on the value used for the variable.

exercise 3.3.1

Evaluate: $y+4$ when

a. $y=6$
b. $y=15$

Answer a

10

Answer b

19

exercise 3.3.2

Evaluate: $a-5$ when

a. $a=9$
b. $a=17$

Answer a

4

Answer b

12

Example 3.3.2

Evaluate $9x-2$, when

a. $x=5$
b. $x=1$

Solution

Remember ab means a times b, so $9x$ means 9 times x.

a. To evaluate the expression when $x=5$, we substitute 5 for x, and then simplify.

	$9x-2$
Substitute 5 for x.	$9\cdot 5-2$
Multiply.	$45-2$
Subtract.	43

b. To evaluate the expression when $x=1$, we substitute 1 for x, and then simplify.

	$9x-2$
Substitute 1 for x.	$9\cdot 1-2$
Multiply.	$9-2$
Subtract.	7

Notice that in part (a) that we wrote $9 \bullet 5$ and in part (b) we wrote $9(1)$. Both the dot and the parentheses tell us to multiply.

exercise 3.3.3

Evaluate: $8x - 3$, when

a. $x = 2$
b. $x = 1$

Answer a

13

Answer b

5

exercise 3.3.4

Evaluate: $4y - 4$, when

a. $y = 3$
b. $y = 5$

Answer a

8

Answer b

16

Example 3.3.3: evaluate

Evaluate x^2 when $x = 10$.

Solution

We substitute 10 for x, and then simplify the expression.

	x^2
Substitute 10 for x.	10^2
Use the definition of exponent.	$10 \cdot 10$
Multiply	100

When $x = 10$, the expression x^2 has a value of 100.

exercise 3.3.5

Evaluate: x^2 when $x = 8$.

Answer

64

exercise 3.3.6

Evaluate: x^3 when $x = 6$.

Answer

216

Example 3.3.4: evaluate

Evaluate 2^x when $x = 5$.

Solution

In this expression, the variable is an exponent.

	2^x
Substitute 5 for x.	2^5
Use the definition of exponent.	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
Multiply	32

When $x = 5$, the expression 2^x has a value of 32.

exercise 3.3.7

Evaluate: 2^x when $x = 6$.

Answer

64

exercise 3.3.8

Evaluate: 3^x when $x = 4$.

Answer

81

Example 3.3.5: evaluate

Evaluate $3x + 4y - 6$ when $x = 10$ and $y = 2$.

Solution

This expression contains two variables, so we must make two substitutions.

	$3x + 4y - 6$
Substitute 10 for x and 2 for y.	$3(10) + 4(2) - 6$
Multiply.	$30 + 8 - 6$
Add and subtract left to right.	32

When $x = 10$ and $y = 2$, the expression $3x + 4y - 6$ has a value of 32.

exercise 3.3.9

Evaluate: $2x + 5y - 4$ when $x = 11$ and $y = 3$

Answer

33

exercise 3.3.10

Evaluate: $5x - 2y - 9$ when $x = 7$ and $y = 8$

Answer

10

Example 3.3.6: evaluate

Evaluate $2x^2 + 3x + 8$ when $x = 4$.

Solution

We need to be careful when an expression has a variable with an exponent. In this expression, $2x^2$ means $2 \bullet x \bullet x$ and is different from the expression $(2x)^2$, which means $2x \bullet 2x$.

	$2x^2 + 3x + 8$
Substitute 4 for each x.	$2(4)^2 + 3(4) + 8$
Simplify 4^2.	$2(16) + 3(4) + 8$
Multiply.	$32 + 12 + 8$
Add.	52

exercise 3.3.11

Evaluate: $3x^2 + 4x + 1$ when $x = 3$.

Answer

40

exercise 3.3.12

Evaluate: $6x^2 - 4x - 7$ when $x = 2$.

Answer

9

Identify Terms, Coefficients, and Like Terms

Algebraic expressions are made up of *terms*. A **term** is a constant or the product of a constant and one or more variables. Some examples of terms are 7, y, $5x^2$, $9a$, and $13xy$.

The constant that multiplies the variable(s) in a term is called the **coefficient**. We can think of the coefficient as the number *in front of* the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1 \bullet x$. Table 3.3.1 gives the coefficients for each of the terms in the left column.

Table 3.3.1

Term	Coefficient
7	7
9a	9
y	1

Term	Coefficient
$5x^2$	5

An algebraic expression may consist of one or more terms added or subtracted. In this chapter, we will only work with terms that are added together. Table 3.3.2 gives some examples of algebraic expressions with various numbers of terms. Notice that we include the operation before a term with it.

Table 3.3.2

Expression	Terms
7	7
y	y
x + 7	x, 7
2x + 7y + 4	2x, 7y, 4
$3x^2 + 4x^2 + 5y + 3$	$3x^2$, $4x^2$, 5y, 3

Example 3.3.7:

Identify each term in the expression $9b+15x^2+a+6$. Then identify the coefficient of each term.

Solution

The expression has four terms. They are $9b$, $15x^2$, a, and 6.

The coefficient of $9b$ is 9.

The coefficient of $15x^2$ is 15.

Remember that if no number is written before a variable, the coefficient is 1. So the coefficient of a is 1.

The coefficient of a constant is the constant, so the coefficient of 6 is 6.

exercise 3.3.13

Identify all terms in the given expression, and their coefficients: $4x+3b+2$

Answer

The terms are $4x, 3b,$ and 2. The coefficients are $4, 3,$ and 2.

exercise 3.3.14

Identify all terms in the given expression, and their coefficients: $9a+13a^2+a^3$

Answer

The terms are $9a, 13a^2,$ and a^3, The coefficients are $9, 13,$ and 1.

Some terms share common traits. Look at the following terms. Which ones seem to have traits in common?

$5x, 7, n^2, 4, 3x, 9n^2$

Which of these terms are like terms?

- The terms 7 and 4 are both constant terms.
- The terms $5x$ and $3x$ are both terms with x.
- The terms n^2 and $9n^2$ both have n^2.

Terms are called like terms if they have the same variables and exponents. All constant terms are also like terms. So among the terms $5x, 7, n^2, 4, 3x, 9n^2, 7$and 4 are like terms, $5x$ and $3x$ are like terms, and n^2 and $9n^2$ are like terms.

Definition: Like terms

Terms that are either constants or have the same variables with the same exponents are like terms.

Example 3.3.8: identify

Identify the like terms:

a. $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$
b. $4x^2 + 2x + 5x^2 + 6x + 40x + 8xy$

Solution

a. $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$

Look at the variables and exponents. The expression contains y^3, x^2, x, and constants. The terms y^3 and $4y^3$ are like terms because they both have y^3. The terms $7x^2$ and $5x^2$ are like terms because they both have x^2. The terms 14 and 23 are like terms because they are both constants. The term $9x$ does not have any like terms in this list since no other terms have the variable x raised to the power of 1.

b. $4x^2 + 2x + 5x^2 + 6x + 40x + 8xy$

Look at the variables and exponents. The expression contains the terms $4x^2, 2x, 5x^2, 6x, 40x$and $8xy$The terms $4x^2$ and $5x^2$ are like terms because they both have x^2. The terms $2x, 6x$, and $40x$ are like terms because they all have x. The term $8xy$ has no like terms in the given expression because no other terms contain the two variables xy.

exercise 3.3.15

Identify the like terms in the list or the expression: $9, 2x^3, y^2, 8x^3, 15, 9y, 11y^2$

Answer

$9, 15$; $2x^3$ and $8x^3$, y^2, and $11y^2$

exercise 3.3.16

Identify the like terms in the list or the expression: $4x^3 + 8x^2 + 19 + 3x^2 + 24 + 6x^3$

Answer

$4x^3$ and $6x^3$; $8x^2$ and $3x^2$; 19 and 24

Simplify Expressions by Combining Like Terms

We can simplify an expression by combining the like terms. What do you think $3x + 6x$ would simplify to? If you thought $9x$, you would be right!

We can see why this works by writing both terms as addition problems.

$$\overbrace{x+x+x}^{3x} + \overbrace{x+x+x+x+x+x}^{6x}$$
$$+$$
$$9x$$

Add the coefficients and keep the same variable. It doesn't matter what x is. If you have 3 of something and add 6 more of the same thing, the result is 9 of them. For example, 3 oranges plus 6 oranges is 9 oranges. We will discuss the mathematical properties behind this later.

The expression $3x + 6x$ has only two terms. When an expression contains more terms, it may be helpful to rearrange the terms so that like terms are together. The Commutative Property of Addition says that we can change the order of addends without changing the sum. So we could rearrange the following expression before combining like terms.

$$3x + 4y - 2x + 6y$$
$$3x - 2x + 4y + 6y$$

Now it is easier to see the like terms to be combined.

HOW TO: COMBINE LIKE TERMS

Step 1. Identify like terms.

Step 2. Rearrange the expression so like terms are together.

Step 3. Add the coefficients of the like terms.

Example 3.3.9: simplify

Simplify the expression: $3x + 7 + 4x + 5$.

Solution

	$3x + 7 + 4x + 5$
Identify the like terms	$3x + 7 + 4x + 5$
Rearrange the expression, so the like terms are together.	$3x + 4x + 7 + 5$
Add the coefficients of the like terms.	$7x + 12$
The original expression is simplified to...	$7x + 12$

exercise 3.3.17

Simplify: $7x + 9 + 9x + 8$

Answer

$16x + 17$

exercise 3.3.18

Simplify: $5y + 2 + 8y + 4y + 5$

Answer

$17y + 7$

Example 3.3.10: simplify

Simplify the expression: $7x^2 + 8x + x^2 + 4x$.

Solution

	$7x^2 + 8x + x^2 + 4x$
Identify the like terms.	$7x^2 + 8x + x^2 + 4x$
Rearrange the expression so like terms are together.	$7x^2 + x^2 + 8x + 4x$

Add the coefficients of the like terms.	$8x^2+12x$

These are not like terms and cannot be combined. So $8x^2+12x$ is in simplest form.

exercise 3.3.19

Simplify: $3x^2+9x+x^2+5x$

Answer

$4x^2+14x$

exercise 3.3.20

Simplify: $11y^2+8y+y^2+7y$

Answer

$12y^2+15y$

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3.4: Evaluate, Simplify, and Translate Expressions (Part 2)

Translate Words to Algebraic Expressions

In the previous section, we listed many operation symbols that are used in algebra, and then we translated expressions and equations into word phrases and sentences. Now we'll reverse the process and translate word phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. They are summarized in Table 3.4.3.

Table 3.4.3

Operation	Phrase	Expression
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	a + b
Subtraction	a minus b the difference of a and b b subtracted from a a decreased by b b less than a	a - b
Multiplication	a times b the product of a and b	a • b, ab, a(b), (a)(b)
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	a ÷ b, a / b, $\frac{a}{b}$, $b\overline{)a}$

Look closely at these phrases using the four operations:

- the sum of a and b
- the difference of a and b
- the product of a and b
- the quotient of a and b

Each phrase tells you to operate on two numbers. Look for the words ***of*** and ***and*** to find the numbers.

Example 3.4.11: translate

Translate each word phrase into an algebraic expression:

a. the difference of 20 and 4
b. the quotient of $10x$ and 3

Solution

a. The key word is *difference*, which tells us the operation is subtraction. Look for the words *of* and *and* to find the numbers to subtract.

$$\text{the difference of } 20 \text{ and } 4$$

$$20 \text{ minus } 4$$

$$20 - 4$$

b. The key word is *quotient*, which tells us the operation is division.

the quotient of $10x$ and 3

divide $10x$ by 3

$$10x \div 3$$

This can also be written as $10x/3$ or $\frac{10x}{3}$

? exercise 3.4.21

Translate the given word phrase into an algebraic expression:

a. the difference of 47 and 41
b. the quotient of $5x$ and 2

Answer a

$47 - 41$

Answer b

$5x \div 2$

? exercise 3.4.22

Translate the given word phrase into an algebraic expression:

a. the sum of 17 and 19
b. the product of 7 and x

Answer a

$17 + 19$

Answer b

$7x$

How old will you be in eight years? What age is eight more years than your age now? Did you add 8 to your present age? Eight *more than* means eight added to your present age.

How old were you seven years ago? This is seven years less than your age now. You subtract 7 from your present age. Seven *less than* means seven subtracted from your present age.

✓ Example 3.4.12: translate

Translate each word phrase into an algebraic expression:

a. Eight more than y
b. Seven less than $9z$

Solution

a. The key words are more than. They tell us the operation is addition. More than means "added to".

Eight more than y

Eight added to y

$$y + 8$$

b. The key words are less than. They tell us the operation is subtraction. Less than means "subtracted from".

Seven less than $9z$

Seven subtracted from $9z$

$$9z-7$$

exercise 3.4.23

Translate each word phrase into an algebraic expression:

a. Eleven more than x
b. Fourteen less than $11a$

Answer a

$x+11$

Answer b

$11a-14$

exercise 3.4.24

Translate each word phrase into an algebraic expression:

a. 19 more than j
b. 21 less than $2x$

Answer a

$j+19$

Answer b

$2x-21$

Example 3.4.13: translate

Translate each word phrase into an algebraic expression:

a. five times the sum of m and n
b. the sum of five times m and n

Solution

a. There are two operation words: times tells us to multiply and sum tells us to add. Because we are multiplying 5 times the sum, we need parentheses around the sum of m and n.

$$\text{five times the sum of } m \text{ and } n$$
$$5(m+n)$$

b. To take a sum, we look for the words of and and to see what is being added. Here we are taking the sum of five times m and n.

$$\text{the sum of five times } m \text{ and } n$$
$$5m+n$$

Notice how the use of parentheses changes the result. In part (a), we add first and in part (b), we multiply first.

exercise 3.4.25

Translate the word phrase into an algebraic expression:

a. four times the sum of p and q
b. the sum of four times p and q

Answer a

$4(p+q)$

Answer b

$4p+q$

exercise 3.4.26

Translate the word phrase into an algebraic expression:

a. the difference of two times x and 8
b. two times the difference of x and 8

Answer a

$2x-8$

Answer b

$2(x-8)$

Later in this course, we'll apply our skills in algebra to solving equations. We'll usually start by translating a word phrase to an algebraic expression. We'll need to be clear about what the expression will represent. We'll see how to do this in the next two examples.

Example 3.4.14: write an expression

The height of a rectangular window is 6 inches less than the width. Let w represent the width of the window. Write an expression for the height of the window.

Solution

Write a phrase about the height.	6 less than the width
Substitute w for the width.	6 less than w
Rewrite 'less than' as 'subtracted from'.	6 subtracted from w
Translate the phrase into algebra.	w - 6

exercise 3.4.27

The length of a rectangle is 5 inches less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Answer

$w-5$

exercise 3.4.28

The width of a rectangle is 2 meters greater than the length. Let l represent the length of the rectangle. Write an expression for the width of the rectangle.

Answer

$l+2$

Example 3.4.15: write an expression

Blanca has dimes and quarters in her purse. The number of dimes is 2 less than 5 times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution

Write a phrase about the number of dimes.	two less than five times the number of quarters
Substitute q for the number of quarters.	2 less than five times q
Translate 5 *times* q.	2 less than 5q
Translate the phrase into algebra.	5q - 2

exercise 3.4.29

Geoffrey has dimes and quarters in his pocket. The number of dimes is seven less than six times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Answer

$6q-7$

exercise 3.4.30

Lauren has dimes and nickels in her purse. The number of dimes is eight more than four times the number of nickels. Let n represent the number of nickels. Write an expression for the number of dimes.

Answer

$4n+8$

Access Additional Online Resources

- Algebraic Expression Vocabulary

Key Concepts

- Combine like terms.
  - Identify like terms.
  - Rearrange the expression so like terms are together.
  - Add the coefficients of the like terms

Glossary

term

A term is a constant or the product of a constant and one or more variables.

coefficient

The constant that multiplies the variable(s) in a term is called the coefficient.

like terms

Terms that are either constants or have the same variables with the same exponents are like terms.

evaluate

To evaluate an algebraic expression means to find the value of the expression when the variable is replaced by a given number.

Practice Makes Perfect

Evaluate Algebraic Expressions

In the following exercises, evaluate the expression for the given value.

69. $7x + 8$ when $x = 2$
70. $9x + 7$ when $x = 3$
71. $5x - 4$ when $x = 6$
72. $8x - 6$ when $x = 7$
73. x^2 when $x = 12$
74. x^3 when $x = 5$
75. x^5 when $x = 2$
76. x^4 when $x = 3$
77. 3^x when $x = 3$
78. 4^x when $x = 2$
79. $x^2 + 3x - 7$ when $x = 4$
80. $x^2 + 5x - 8$ when $x = 6$
81. $2x + 4y - 5$ when $x = 7, y = 8$
82. $6x + 3y - 9$ when $x = 6, y = 9$
83. $(x - y)^2$ when $x = 10, y = 7$
84. $(x + y)^2$ when $x = 6, y = 9$
85. $a^2 + b^2$ when $a = 3, b = 8$
86. $r^2 - s^2$ when $r = 12, s = 5$
87. $2l + 2w$ when $l = 15, w = 12$
88. $2l + 2w$ when $l = 18, w = 14$

Identify Terms, Coefficients, and Like Terms

In the following exercises, list the terms in the given expression.

89. $15x^2 + 6x + 2$
90. $11x^2 + 8x + 5$
91. $10y^3 + y + 2$
92. $9y^3 + y + 5$

In the following exercises, identify the coefficient of the given term.

93. $8a$
94. $13m$
95. $5r^2$
96. $6x^3$

In the following exercises, identify all sets of like terms.

97. x^3, $8x$, 14, $8y$, 5, $8x^3$
98. $6z$, $3w^2$, 1, $6z^2$, $4z$, w^2
99. $9a$, a^2, $16ab$, $16b^2$, $4ab$, $9b^2$
100. 3, $25r^2$, $10s$, $10r$, $4r^2$, $3s$

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the given expression by combining like terms.

101. $10x + 3x$
102. $15x + 4x$
103. $17a + 9a$
104. $18z + 9z$
105. $4c + 2c + c$
106. $6y + 4y + y$

107. $9x + 3x + 8$
108. $8a + 5a + 9$
109. $7u + 2 + 3u + 1$
110. $8d + 6 + 2d + 5$
111. $7p + 6 + 5p + 4$
112. $8x + 7 + 4x - 5$
113. $10a + 7 + 5a - 2 + 7a - 4$
114. $7c + 4 + 6c - 3 + 9c - 1$
115. $3x^2 + 12x + 11 + 14x^2 + 8x + 5$
116. $5b^2 + 9b + 10 + 2b^2 + 3b - 4$

Translate English Phrases into Algebraic Expressions

In the following exercises, translate the given word phrase into an algebraic expression.

117. The sum of 8 and 12
118. The sum of 9 and 1
119. The difference of 14 and 9
120. 8 less than 19
121. The product of 9 and 7
122. The product of 8 and 7
123. The quotient of 36 and 9
124. The quotient of 42 and 7
125. The difference of x and 4
126. 3 less than x
127. The product of 6 and y
128. The product of 9 and y
129. The sum of 8x and 3x
130. The sum of 13x and 3x
131. The quotient of y and 3
132. The quotient of y and 8
133. Eight times the difference of y and nine
134. Seven times the difference of y and one
135. Five times the sum of x and y
136. Nine times five less than twice x

In the following exercises, write an algebraic expression.

137. Adele bought a skirt and a blouse. The skirt cost $15 more than the blouse. Let b represent the cost of the blouse. Write an expression for the cost of the skirt.
138. Eric has rock and classical CDs in his car. The number of rock CDs is 3 more than the number of classical CDs. Let c represent the number of classical CDs. Write an expression for the number of rock CDs.
139. The number of girls in a second-grade class is 4 less than the number of boys. Let b represent the number of boys. Write an expression for the number of girls.
140. Marcella has 6 fewer male cousins than female cousins. Let f represent the number of female cousins. Write an expression for the number of boy cousins.
141. Greg has nickels and pennies in his pocket. The number of pennies is seven less than twice the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.
142. Jeannette has $5 and $10 bills in her wallet. The number of fives is three more than six times the number of tens. Let t represent the number of tens. Write an expression for the number of fives.

Everyday Math

In the following exercises, use algebraic expressions to solve the problem.

143. **Car insurance** Justin's car insurance has a $750 deductible per incident. This means that he pays $750 and his insurance company will pay all costs beyond $750. If Justin files a claim for $2,100, how much will he pay, and how much will his

insurance company pay?

144. **Home insurance** Pam and Armando's home insurance has a $2,500 deductible per incident. This means that they pay $2,500 and their insurance company will pay all costs beyond $2,500. If Pam and Armando file a claim for $19,400, how much will they pay, and how much will their insurance company pay?

Writing Exercises

145. Explain why "the sum of x and y" is the same as "the sum of y and x," but "the difference of x and y" is not the same as "the difference of y and x." Try substituting two random numbers for x and y to help you explain. 146. Explain the difference between "4 times the sum of x and y" and "the sum of 4 times x and y."

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate algebraic expressions.			
identify terms, coefficients, and like terms.			
simplify expressions by combining like terms.			
translate word phrases to algebraic expressions.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

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3.5: Solving Equations Using the Subtraction and Addition Properties of Equality (Part 1)

Learning Objectives

- Determine whether a number is a solution of an equation
- Model the Subtraction Property of Equality
- Solve equations using the Subtraction Property of Equality
- Solve equations using the Addition Property of Equality
- Translate word phrases to algebraic equations
- Translate to an equation and solve

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $x+8$ when $x=11$. If you missed this problem, review Example 2..2.1.
2. Evaluate $5x-3$ when $x=9$. If you missed this problem, review Example 2.2.2.
3. Translate into algebra: the difference of x and 8. If you missed this problem, review Example 2.2.11.

When some people hear the word *algebra*, they think of solving equations. The applications of solving equations are limitless and extend to all careers and fields. In this section, we will begin solving equations. We will start by solving basic equations, and then as we proceed through the course we will build up our skills to cover many different forms of equations.

Determine Whether a Number is a Solution of an Equation

Solving an equation is like discovering the answer to a puzzle. An algebraic equation states that two algebraic expressions are equal. To solve an equation is to determine the values of the variable that make the equation a true statement. Any number that makes the equation true is called a **solution** of the equation. It is the answer to the puzzle!

Definition: Solution of an Equation

A solution to an equation is a value of a variable that makes a true statement when substituted into the equation. The process of finding the solution to an equation is called solving the equation.

To find the solution to an equation means to find the value of the variable that makes the equation true. Can you recognize the solution of $x+2=7$? If you said 5, you're right! We say 5 is a solution to the equation $x+2=7$ because when we substitute 5 for x the resulting statement is true.

$$\begin{aligned} x+2 &= 7 \\ 5+2 &\stackrel{?}{=} 7 \\ 7 &= 7 \checkmark \end{aligned}$$

Since $5+2=7$ is a true statement, we know that 5 is indeed a solution to the equation. The symbol $\stackrel{?}{=}$ asks whether the left side of the equation is equal to the right side. Once we know, we can change to an equal sign ($=$) or not-equal sign ($\neq$).

HOW TO: DETERMINE WHETHER A NUMBER IS A Solution TO AN EQUATION

Step 1. Substitute the number for the variable in the equation.

Step 2. Simplify the expressions on both sides of the equation.

Step 3. Determine whether the resulting equation is true.

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

Example 3.5.1

Determine whether $x = 5$ is a solution of $6x - 17 = 16$.

Solution

	$6x - 17 = 16$
Substitute 5 for x.	$6 \cdot 5 - 17 \stackrel{?}{=} 16$
Multiply.	$30 - 17 \stackrel{?}{=} 16$
Subtract	$13 \neq 16$

So $x = 5$ is not a solution to the equation $6x - 17 = 16$.

exercise 3.5.1

Is $x = 3$ a solution of $4x - 7 = 16$?

Answer

no

exercise 3.5.2

Is $x = 2$ a solution of $6x - 2 = 10$?

Answer

yes

Example 3.5.2

Determine whether $y = 2$ is a solution of $6y - 4 = 5y - 2$.

Solution

Here, the variable appears on both sides of the equation. We must substitute 2 for each y.

	$6y - 4 = 5y - 2$
Substitute 2 for y.	$6(2) - 4 \stackrel{?}{=} 5(2) - 2$
Multiply.	$12 - 4 \stackrel{?}{=} 10 - 2$
Subtract	$8 = 8$ ✓

Since $y = 2$ results in a true equation, we know that 2 is a solution to the equation $6y - 4 = 5y - 2$.

exercise 3.5.3

Is $y = 3$ a solution of $9y - 2 = 8y + 1$?

Answer

yes

? exercise 3.5.4

Is $y = 4$ a solution of $5y - 3 = 3y + 5$?

Answer

yes

Model the Subtraction Property of Equality

We will use a model to help you understand how the process of solving an equation is like solving a puzzle. An envelope represents the variable – since its contents are unknown – and each counter represents one.

Suppose a desk has an imaginary line dividing it in half. We place three counters and an envelope on the left side of desk, and eight counters on the right side of the desk as in Figure 3.5.1. Both sides of the desk have the same number of counters, but some counters are hidden in the envelope. Can you tell how many counters are in the envelope?

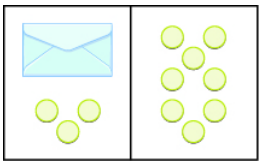

Figure 3.5.1

What steps are you taking in your mind to figure out how many counters are in the envelope? Perhaps you are thinking "I need to remove the 3 counters from the left side to get the envelope by itself. Those 3 counters on the left match with 3 on the right, so I can take them away from both sides. That leaves five counters on the right, so there must be 5 counters in the envelope." Figure 3.5.2 shows this process.

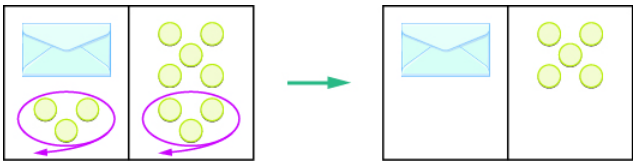

Figure 3.5.2

What algebraic equation is modeled by this situation? Each side of the desk represents an expression and the center line takes the place of the equal sign. We will call the contents of the envelope x, so the number of counters on the left side of the desk is $x + 3$. On the right side of the desk are 8 counters. We are told that $x + 3$ is equal to 8 so our equation is $x + 3 = 8$.

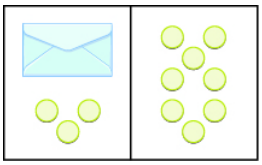

Figure 3.5.3

$x + 3 = 8$

Let's write algebraically the steps we took to discover how many counters were in the envelope.

	$x + 3 = 8$
First, we took away three from each side.	$x + 3 - 3 = 8 - 3$
Then we were left with five.	$x = 5$

Now let's check our solution. We substitute 5 for x in the original equation and see if we get a true statement.

$$x+3=8$$
$$5+3\stackrel{?}{=}8$$
$$8=8\ \checkmark$$

Our solution is correct. Five counters in the envelope plus three more equals eight

Example 3.5.3

Write an equation modeled by the envelopes and counters, and then solve the equation:

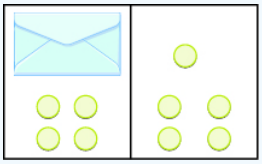

Figure 3.5.4

Solution

On the left, write x for the contents of the envelope, add the 4 counters, so we have x + 4.	x + 4
On the right, there are 5 counters.	5
The two sides are equal.	x + 4 = 5
Solve the equation by subtracting 4 counters from each side.	

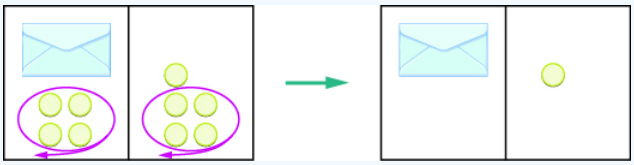

Figure 3.5.5

We can see that there is one counter in the envelope. This can be shown algebraically as:

$$x+4=5$$
$$x+4-4=5-4$$
$$x=1$$

Substitute 1 for x in the equation to check.

$$x+4=5$$
$$1+4\stackrel{?}{=}5$$
$$5=5\ \checkmark$$

Since $x=1$ makes the statement true, we know that 1 is indeed a solution.

Exercise 3.5.5

Write the equation modeled by the envelopes and counters, and then solve the equation:

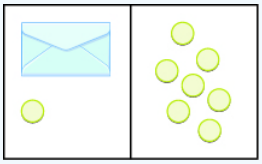

Figure 3.5.6

Answer

$x+1=7; x=6$

Exercise 3.5.6

Write the equation modeled by the envelopes and counters, and then solve the equation:

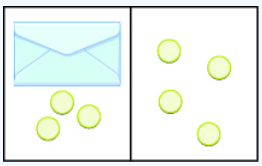

Figure 3.5.7

Answer

$x+3=4; x=1$

Solve Equations Using the Subtraction Property of Equality

Our puzzle has given us an idea of what we need to do to solve an equation. The goal is to isolate the variable by itself on one side of the equations. In the previous examples, we used the Subtraction Property of Equality, which states that when we subtract the same quantity from both sides of an equation, we still have equality.

Definition: Subtraction Property of Equality

For any numbers a, b, and c, if $a=b$ then $a-c=b-c$

Think about twin brothers Andy and Bobby. They are 17 years old. How old was Andy 3 years ago? He was 3 years less than 17, so his age was $17-3$, or 14. What about Bobby's age 3 years ago? Of course, he was 14 also. Their ages are equal now, and subtracting the same quantity from both of them resulted in equal ages 3 years ago.

$$\begin{aligned} a &= b \\ a-3 &= b-3 \end{aligned}$$

HOW TO: SOLVE AN EQUATION USING THE SUBTRACTION PROPERTY OF EQUALITY

Step 1. Use the Subtraction Property of Equality to isolate the variable.

Step 2. Simplify the expressions on both sides of the equation.

Step 3. Check the solution.

Example 3.5.4: solve

Solve: $x+8=17$.

Solution

We will use the Subtraction Property of Equality to isolate x.

	$x+8=17$
Subtract 8 from both sides.	$x+8-8=17-8$
Simplify.	$x=9$

Check the solution.	$9+8=17$
	$17=17$ ✓

Since $x=9$ makes $x+8=17$ a true statement, we know 9 is the solution to the equation.

Exercise 3.5.7

Solve: $x+6=19$

Answer

$x=13$

Exercise 3.5.8

Solve: $x+9=14$

Answer

$x=5$

Example 3.5.5: solve

Solve: $100=y+74$.

Solution

To solve an equation, we must always isolate the variable—it doesn't matter which side it is on. To isolate y, we will subtract 74 from both sides.

	$100=y+74$
Subtract 74 from both sides.	$100-74=y+74-74$
Simplify.	$26=y$
Substitute 26 for y to check	$100\stackrel{?}{=}26+74$
	$100=100$ ✓

Since $y=26$ makes $100=y+74$ a true statement, we have found the solution to this equation.

Exercise 3.5.9

Solve: $95=y+67$

Answer

$y=28$

Exercise 3.5.10

Solve: $91=y+45$

Answer

$y=46$

Solve Equations Using the Addition Property of Equality

In all the equations we have solved so far, a number was added to the variable on one side of the equation. We used subtraction to "undo" the addition in order to isolate the variable.

But suppose we have an equation with a number subtracted from the variable, such as $x-5=8$. We want to isolate the variable, so to "undo" the subtraction we will add the number to both sides.

We use the Addition Property of Equality, which says we can add the same number to both sides of the equation without changing the equality. Notice how it mirrors the Subtraction Property of Equality.

Definition: Addition Property of Equality

For any numbers a, b, and c, if $a=b$ then $a+c=b+c$

Remember the 17-year-old twins, Andy and Bobby? In ten years, Andy's age will still equal Bobby's age. They will both be 27.

$$\begin{aligned} a &= b \\ a+10 &= b+10 \end{aligned}$$

We can add the same number to both sides and still keep the equality.

HOW TO: SOLVE AN EQUATION USING THE ADDITION PROPERTY OF EQUALITY

Step 1. Use the Addition Property of Equality to isolate the variable.

Step 2. Simplify the expressions on both sides of the equation.

Step 3. Check the solution.

Example 3.5.6: solve

Solve: $x-5=8$.

Solution

We will use the Addition Property of Equality to isolate the variable.

	$x-5=8$
Add 5 to both sides.	$x-5+5=8+5$
Simplify	$x=13$
Now we can check. Let x = 13.	$13-5\stackrel{?}{=}8$
	$8=8$ ✓

Exercise 3.5.11

Solve: $x-9=13$

Answer

$x=22$

Exercise 3.5.12

Solve: $y-1=3$

Answer

$y = 4$

Example 3.5.7: solve

Solve: $27 = a - 16$.

Solution

We will add 16 to each side to isolate the variable.

	$27 = a - 16$
Add 16 to each side.	$27 + 16 = a - 16 + 16$
Simplify.	$43 = a$
Now we can check. Let a = 43.	$27 \stackrel{?}{=} 43 - 16$
	$27 = 27$ ✓

The solution to $27 = a - 16$ is $a = 43$.

Exercise 3.5.13

Solve: $19 = a - 18$

Answer

$a = 37$

Exercise 3.5.14

Solve: $27 = n - 14$

Answer

$n = 41$

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3.6: Solving Equations Using the Subtraction and Addition Properties of Equality (Part 2)

Translate Word Phrases to Algebraic Equations

Remember, an equation has an equal sign (=) between two algebraic expressions. So if we have a sentence that tells us that two phrases are equal, we can translate it into an equation. We look for clue words that mean equals. Some words that translate to the equal sign are:

- is equal to
- is the same as
- is
- gives
- was
- will be

It may be helpful to put a box around the equals word(s) in the sentence to help you focus separately on each phrase. Then translate each phrase into an expression, and write them on each side of the equal sign.

We will practice translating word sentences into algebraic equations. Some of the sentences will be basic number facts with no variables to solve for. Some sentences will translate into equations with variables. The focus right now is just to translate the words into algebra.

Example 3.6.8: translate

Translate the sentence into an algebraic equation: The sum of 6 and 9 is 15.

Solution

The word is tells us the equal sign goes between 9 and 15.

Locate the “equals” word(s).	The sum of 6 and 9 is 15
Write the = sign.	The sum of 6 and 9 = 15
Translate the words to the left of the *equals* word into an algebraic expression.	6 + 9 = _____
Translate the words to the right of the *equals* word into an algebraic expression.	6 + 9 = 15

Exercise 3.6.15

Translate the sentence into an algebraic equation: The sum of 7 and 6 gives 13.

Answer

$7+6=13$

Exercise 3.6.16

Translate the sentence into an algebraic equation: The sum of 8 and 6 is 14.

Answer

$8+6=14$

Example 3.6.9: translate

Translate the sentence into an algebraic equation: The product of 8 and 7 is 56.

Solution

The location of the word is tells us that the equal sign goes between 7 and 56.

Locate the "equals" word(s).	The product of 8 and 7 is 56
Write the = sign.	The product of 8 and 7 = 56
Translate the words to the left of the *equals* word into an algebraic expression.	8 • 7 = _____
Translate the words to the right of the *equals* word into an algebraic expression.	8 • 7 = 56

Exercise 3.6.17

Translate the sentence into an algebraic equation: The product of 6 and 9 is 54.

Answer

$6 \cdot 9 = 54$

Exercise 3.6.18

Translate the sentence into an algebraic equation: The product of 21 and 3 gives 63.

Answer

$21 \cdot 3 = 63$

Example 3.6.10: translate

Translate the sentence into an algebraic equation: Twice the difference of x and 3 gives 18.

Solution

Locate the "equals" word(s).	Twice the difference of x and 3 gives 18.
Recognize the key words: *twice*; *difference of … and …*	*Twice* means two times.
Translate.	Twice the difference of x and 3 gives 18. 2 $(x-3)$ = 18

Exercise 3.6.19

Translate the given sentence into an algebraic equation: Twice the difference of x and 5 gives 30.

Answer

$2(x-5) = 30$

Exercise 3.6.20

Translate the given sentence into an algebraic equation: Twice the difference of y and 4 gives 16.

Answer

$2(y-4)=16$

Translate to an Equation and Solve

Now let's practice translating sentences into algebraic equations and then solving them. We will solve the equations by using the Subtraction and Addition Properties of Equality.

Example 3.6.11: translate and solve

Translate and solve: Three more than x is equal to 47.

Solution

	Three more than x is equal to 47.
Translate.	$x+3=47$
Subtract 3 from both sides of the equation.	$x+3-3=47-3$
Simplify	$x=44$
We can check. Let x = 44.	$44+3\stackrel{?}{=}47$
	$47=47$ ✓

So $x=44$ is the solution.

Exercise 3.6.21

Translate and solve: Seven more than x is equal to 37.

Answer

$x+7=37; x=30$

Exercise 3.6.22

Translate and solve: Eleven more than y is equal to 28.

Answer

$y+11=28; y=17$

Example 3.6.12: translate and solve

Translate and solve: The difference of y and 14 is 18.

Solution

	The difference of y and 14 is 18.
Translate.	$y-14=18$
Add 14 to both sides.	$y-14+14=18+14$

Simplify.	$y = 32$
We can check. Let y = 32.	$32 - 14 \stackrel{?}{=} 18$
	$18 = 18$ ✓

So $y = 32$ is the solution.

? Exercise 3.6.23

Translate and solve: The difference of z and 17 is equal to 37.

Answer

$z - 17 = 37; z = 54$

? Exercise 3.6.24

Translate and solve: The difference of x and 19 is equal to 45.

Answer

$x - 19 = 45; x = 64$

Access Additional Online Resources

- Solving One Step Equations By Addition and Subtraction

Key Concepts

- Determine whether a number is a solution to an equation.
  1. Substitute the number for the variable in the equation.
  2. Simplify the expressions on both sides of the equation.
  3. Determine whether the resulting equation is true. If it is true, the number is a solution.

  If it is not true, the number is not a solution.
- Subtraction Property of Equality
  - For any numbers a, b, and c,

if	a=b
then	a−c=b−c

- Solve an equation using the Subtraction Property of Equality.
  1. Use the Subtraction Property of Equality to isolate the variable.
  2. Simplify the expressions on both sides of the equation.
  3. Check the solution.
- Addition Property of Equality
  - For any numbers a, b, and c,

if	a=b
then	a+c=b+c

- Solve an equation using the Addition Property of Equality.
  1. Use the Addition Property of Equality to isolate the variable.
  2. Simplify the expressions on both sides of the equation.

3. Check the solution.

Glossary

solution of an equation

A solution to an equation is a value of a variable that makes a true statement when substituted into the equation. The process of finding the solution to an equation is called solving the equation.

Practice Makes Perfect

Determine Whether a Number is a Solution of an Equation

In the following exercises, determine whether each given value is a solution to the equation.

147. $x + 13 = 21$
   a. $x = 8$
   b. $x = 34$

148. $y + 18 = 25$
   a. $y = 7$
   b. $y = 43$

149. $m - 4 = 13$
   a. $m = 9$
   b. $m = 17$

150. $n - 9 = 6$
   a. $n = 3$
   b. $n = 15$

151. $3p + 6 = 15$
   a. $p = 3$
   b. $p = 7$

152. $8q + 4 = 20$
   a. $q = 2$
   b. $q = 3$

153. $18d - 9 = 27$
   a. $d = 1$
   b. $d = 2$

154. $24\,f - 12 = 60$
   a. $f = 2$
   b. $f = 3$

155. $8u - 4 = 4u + 40$
   a. $u = 3$
   b. $u = 11$

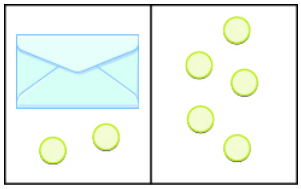
156. $7v - 3 = 4v + 36$
   a. $v = 3$
   b. $v = 11$

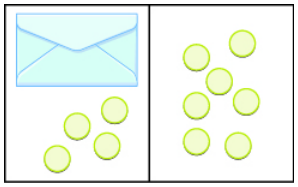
157. $20h - 5 = 15h + 35$
   a. $h = 6$
   b. $h = 8$

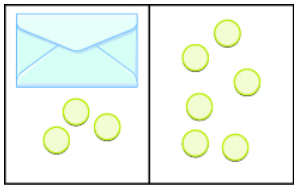
158. $18k - 3 = 12k + 33$
   a. $k = 1$
   b. $k = 6$

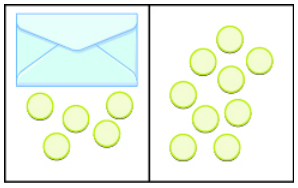
Model the Subtraction Property of Equality

In the following exercises, write the equation modeled by the envelopes and counters and then solve using the subtraction property of equality.

159. 

160. 

161. 

162. 

Solve Equations using the Subtraction Property of Equality

In the following exercises, solve each equation using the subtraction property of equality.

163. $a + 2 = 18$
164. $b + 5 = 13$
165. $p + 18 = 23$
166. $q + 14 = 31$
167. $r + 76 = 100$
168. $s + 62 = 95$
169. $16 = x + 9$
170. $17 = y + 6$
171. $93 = p + 24$
172. $116 = q + 79$
173. $465 = d + 398$
174. $932 = c + 641$

Solve Equations using the Addition Property of Equality

In the following exercises, solve each equation using the addition property of equality.

175. $y - 3 = 19$
176. $x - 4 = 12$
177. $u - 6 = 24$
178. $v - 7 = 35$
179. $f - 55 = 123$
180. $g - 39 = 117$
181. $19 = n - 13$
182. $18 = m - 15$

183. $10 = p - 38$
184. $18 = q - 72$
185. $268 = y - 199$
186. $204 = z - 149$

Translate Word Phrase to Algebraic Equations

In the following exercises, translate the given sentence into an algebraic equation.

187. The sum of 8 and 9 is equal to 17.
188. The sum of 7 and 9 is equal to 16.
189. The difference of 23 and 19 is equal to 4.
190. The difference of 29 and 12 is equal to 17.
191. The product of 3 and 9 is equal to 27.
192. The product of 6 and 8 is equal to 48.
193. The quotient of 54 and 6 is equal to 9.
194. The quotient of 42 and 7 is equal to 6.
195. Twice the difference of n and 10 gives 52.
196. Twice the difference of m and 14 gives 64.
197. The sum of three times y and 10 is 100.
198. The sum of eight times x and 4 is 68.

Translate to an Equation and Solve

In the following exercises, translate the given sentence into an algebraic equation and then solve it.

199. Five more than p is equal to 21.
200. Nine more than q is equal to 40.
201. The sum of r and 18 is 73.
202. The sum of s and 13 is 68.
203. The difference of d and 30 is equal to 52.
204. The difference of c and 25 is equal to 75.
205. 12 less than u is 89.
206. 19 less than w is 56.
207. 325 less than c gives 799.
208. 299 less than d gives 850.

Everyday Math

209. **Insurance** Vince's car insurance has a $500 deductible. Find the amount the insurance company will pay, p, for an $1800 claim by solving the equation $500 + p = 1800$.
210. **Insurance** Marta's homeowner's insurance policy has a $750 deductible. The insurance company paid $5800 to repair damages caused by a storm. Find the total cost of the storm damage, d, by solving the equation $d - 750 = 5800$.
211. **Sale purchase** Arthur bought a suit that was on sale for $120 off. He paid $340 for the suit. Find the original price, p, of the suit by solving the equation $p - 120 = 340$.
212. **Sale purchase** Rita bought a sofa that was on sale for $1299. She paid a total of $1409, including sales tax. Find the amount of the sales tax, t, by solving the equation $1299 + t = 1409$.

Writing Exercises

213. Is $x = 1$ a solution to the equation $8x - 2 = 16 - 6x$? How do you know?
214. Write the equation $y - 5 = 21$ in words. Then make up a word problem for this equation.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether a number is a solution of an equation.			
model the subtraction property of equality.			
solve equations using the subtraction property of equality.			
solve equations using the addition property of equality.			
translate word phrases to algebraic equations.			
translate to an equation and solve.			

(b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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3.E: Introduction to the Language of Algebra (Exercises)

2.1 - Use the Language of Algebra

Use Variables and Algebraic Symbols

In the following exercises, translate from algebra to English.

317. $3 \cdot 8$
318. $12 - x$
319. $24 \div 6$
320. $9 + 2a$
321. $50 \geq 47$
322. $3y < 15$
323. $n + 4 = 13$
324. $32 - k = 7$

Identify Expressions and Equations

In the following exercises, determine if each is an expression or equation.

325. $5 + u = 84$
326. $36 - 6s$
327. $4y - 11$
328. $10x = 120$

Simplify Expressions with Exponents

In the following exercises, write in exponential form.

329. $2 \cdot 2 \cdot 2$
330. $a \cdot a \cdot a \cdot a \cdot a$
331. $x \cdot x \cdot x \cdot x \cdot x \cdot x$
332. $10 \cdot 10 \cdot 10$

In the following exercises, write in expanded form.

333. 8^4
334. 3^6
335. y^5
336. n^4

In the following exercises, simplify each expression.

337. 3^4
338. 10^6
339. 2^7
340. 4^3

Simplify Expressions Using the Order of Operations

In the following exercises, simplify.

341. $10 + 2 \cdot 5$
342. $(10 + 2) \cdot 5$
343. $(30 + 6) \div 2$
344. $30 + 6 \div 2$
345. $7^2 + 5^2$
346. $(7 + 5)^2$
347. $4 + 3(10 - 1)$
348. $(4 + 3)(10 - 1)$

2.2 - Evaluate, Simplify, and Translate Expressions

Evaluate an Expression

In the following exercises, evaluate the following expressions.

349. $9x - 5$ when $x = 7$
350. y^3 when $y = 5$
351. $3a - 4b$ when $a = 10, b = 1$
352. bh when $b = 7, h = 8$

Identify Terms, Coefficients and Like Terms

In the following exercises, identify the terms in each expression.

353. $12n^2 + 3n + 1$
354. $4x^3 + 11x + 3$

In the following exercises, identify the coefficient of each term.

355. $6y$
356. $13x^2$

In the following exercises, identify the like terms.

357. $5x^2$, 3, $5y^2$, $3x$, x, 4
358. 8, $8r^2$, $8r$, $3r$, r^2, $3s$

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the following expressions by combining like terms.

359. $15a + 9a$
360. $12y + 3y + y$
361. $4x + 7x + 3x$
362. $6 + 5c + 3$
363. $8n + 2 + 4n + 9$
364. $19p + 5 + 4p - 1 + 3p$
365. $7y^2 + 2y + 11 + 3y^2 - 8$
366. $13x^2 - x + 6 + 5x^2 + 9x$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate the following phrases into algebraic expressions.

367. the difference of x and 6
368. the sum of 10 and twice a
369. the product of $3n$ and 9
370. the quotient of s and 4
371. 5 times the sum of y and 1
372. 10 less than the product of 5 and z
373. Jack bought a sandwich and a coffee. The cost of the sandwich was $3 more than the cost of the coffee. Call the cost of the coffee c. Write an expression for the cost of the sandwich.
374. The number of poetry books on Brianna's bookshelf is 5 less than twice the number of novels. Call the number of novels n. Write an expression for the number of poetry books.

2.3 - Solve Equations Using the Subtraction and Addition Properties of Equality

Determine Whether a Number is a Solution of an Equation

In the following exercises, determine whether each number is a solution to the equation.

375. $y + 16 = 40$
    a. 24

b. 56

376. d − 6 = 21

a. 15
b. 27

377. 4n + 12 = 36

a. 6
b. 12

378. 20q − 10 = 70

a. 3
b. 4

379. 15x − 5 = 10x + 45

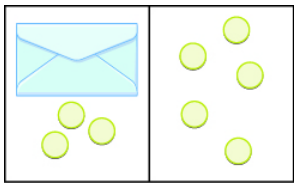
a. 2
b. 10

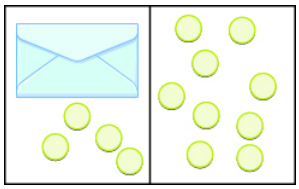
380. 22p − 6 = 18p + 86

a. 4
b. 23

Model the Subtraction Property of Equality

In the following exercises, write the equation modeled by the envelopes and counters and then solve the equation using the subtraction property of equality

381. 

382. 

Solve Equations using the Subtraction Property of Equality

In the following exercises, solve each equation using the subtraction property of equality.

383. c + 8 = 14
384. v + 8 = 150
385. 23 = x + 12
386. 376 = n + 265

Solve Equations using the Addition Property of Equality

In the following exercises, solve each equation using the addition property of equality.

387. y − 7 = 16
388. k − 42 = 113
389. 19 = p − 15
390. 501 = u − 399

Translate English Sentences to Algebraic Equations

In the following exercises, translate each English sentence into an algebraic equation.

391. The sum of 7 and 33 is equal to 40.

392. The difference of 15 and 3 is equal to 12.
393. The product of 4 and 8 is equal to 32.
394. The quotient of 63 and 9 is equal to 7.
395. Twice the difference of n and 3 gives 76.
396. The sum of five times y and 4 is 89.

Translate to an Equation and Solve

In the following exercises, translate each English sentence into an algebraic equation and then solve it.

397. Eight more than x is equal to 35.
398. 21 less than a is 11.
399. The difference of q and 18 is 57.
400. The sum of m and 125 is 240.

Mixed Practice

In the following exercises, solve each equation.

401. $h - 15 = 27$
402. $k - 11 = 34$
403. $z + 52 = 85$
404. $x + 93 = 114$
405. $27 = q + 19$
406. $38 = p + 19$
407. $31 = v - 25$
408. $38 = u - 16$

2.4 - Find Multiples and Factors

Identify Multiples of Numbers

In the following exercises, list all the multiples less than 50 for each of the following.

409. 3
410. 2
411. 8
412. 10

Use Common Divisibility Tests

In the following exercises, using the divisibility tests, determine whether each number is divisible by 2, by 3, by 5, by 6, and by 10.

413. 96
414. 250
415. 420
416. 625

Find All the Factors of a Number

In the following exercises, find all the factors of each number.

417. 30
418. 70
419. 180
420. 378

Identify Prime and Composite Numbers

In the following exercises, identify each number as prime or composite.

421. 19
422. 51

423. 121

2.5 - Prime Factorization and the Least Common Multiple

Find the Prime Factorization of a Composite Number

In the following exercises, find the prime factorization of each number.

425. 84
426. 165
427. 350
428. 572

Find the Least Common Multiple of Two Numbers

In the following exercises, find the least common multiple of each pair of numbers.

429. 9, 15
430. 12, 20
431. 25, 35
432. 18, 40

Everyday Math

433. Describe how you have used two topics from The Language of Algebra chapter in your life outside of your math class during the past month.

PRACTICE TEST

In the following exercises, translate from an algebraic equation to English phrases.

434. $6 \cdot 4$
435. $15 - x$

In the following exercises, identify each as an expression or equation.

436. $5 \cdot 8 + 10$
437. $x + 6 = 9$
438. $3 \cdot 11 = 33$
439. (a) Write $n \cdot n \cdot n \cdot n \cdot n \cdot n$ in exponential form. (b) Write 3^5 in expanded form and then simplify.

In the following exercises, simplify, using the order of operations.

440. $4 + 3 \cdot 5$
441. $(8 + 1) \cdot 4$
442. $1 + 6(3 - 1)$
443. $(8 + 4) \div 3 + 1$
444. $(1 + 4)^2$
445. $5[2 + 7(9 - 8)]$

In the following exercises, evaluate each expression.

446. $8x - 3$ when $x = 4$
447. y^3 when $y = 5$
448. $6a - 2b$ when $a = 5, b = 7$
449. hw when $h = 12, w = 3$
450. Simplify by combining like terms.
    a. $6x + 8x$
    b. $9m + 10 + m + 3$

In the following exercises, translate each phrase into an algebraic expression.

451. 5 more than x

452. the quotient of 12 and y
453. three times the difference of a and b
454. Caroline has 3 fewer earrings on her left ear than on her right ear. Call the number of earrings on her right ear, r. Write an expression for the number of earrings on her left ear.

In the following exercises, solve each equation.

455. n − 6 = 25
456. x + 58 = 71

In the following exercises, translate each English sentence into an algebraic equation and then solve it.

457. 15 less than y is 32.
458. the sum of a and 129 is 164.
459. List all the multiples of 4, that are less than 50.
460. Find all the factors of 90.
461. Find the prime factorization of 1080.
462. Find the LCM (Least Common Multiple) of 24 and 40.

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3.S: Introduction to the Language of Algebra (Summary)

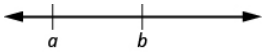
Key Terms

coefficient	The constant that multiplies the variable(s) in a term.
composite number	A composite number is a counting number that is not prime.
divisibility	If a number m is a multiple of n, then we say that m is divisible by n.
equation	An equation is made up of two expressions connected by an equal sign.
evaluate	To evaluate an algebraic expression means to find the value of the expression when the variable is replaced by a given number.
expression	An expression is a number, a variable, or a combination of numbers and variables and operation symbols.
least common multiple (LCM)	The smallest number that is a multiple of two numbers.
like terms	Terms that are either constants or have the same variables with the same exponents.
multiple of a number	A number is a multiple of n if it is the product of a counting number and n.
prime factorization	The product of prime numbers that equals the number.
prime number	A counting number greater than 1 whose only factors are 1 and itself.
solution of an equation	A value of a variable that makes a true statement when substituted into the equation. The process of finding the solution to an equation is called solving the equation.
term	A constant or the product of a constant and one or more variables.

Key Concepts

2.1 - Use the Language of Algebra

Operation	Notation	Say:	The result is…
Addition	a + b	a plus b	The sum of a and b
Multiplication	a • b, (a)(b), (a)b, a(b)	a times b	The product of a and b
Subtraction	a - b	a minus b	The difference of a and b
Division	a ÷ b, a / b, $\frac{a}{b}$, $b\overline{)a}$	a divided by b	The quotient of a and b

- **Equality Symbol**
  - a = b is read as a is equal to b
  - The symbol = is called the equal sign.
- **Inequality**
  - a < b is read a is less than b
  - a is to the left of b on the number line: 

  - a > b is read a is greater than b

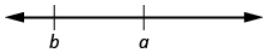
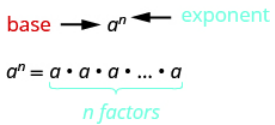
- a is to the right of b on the number line: 

Table 2.77

Algebraic Notation	Say
a = b	a is equal to b
a ≠ b	a is not equal to b
a < b	a is less than b
a > b	a is greater than b
a ≤ b	a is less than or equal to b
a ≥ b	a is greater than or equal to b

- **Exponential Notation**
  - For any expression a n is a factor multiplied by itself n times, if n is a positive integer.
  - a^n means multiply n factors of a
  - The expression of a^n is read a to the nth power



- **Order of Operations**: When simplifying mathematical expressions perform the operations in the following order:

1. Parentheses and other Grouping Symbols: Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
2. Exponents: Simplify all expressions with exponents.
3. Multiplication and Division: Perform all multiplication and division in order from left to right. These operations have equal priority.
4. Addition and Subtraction: Perform all addition and subtraction in order from left to right. These operations have equal priority.

2.2 - Evaluate, Simplify, and Translate Expressions

- **Combine like terms**.

1. Identify like terms.
2. Rearrange the expression so like terms are together.
3. Add the coefficients of the like terms

2.3 - Solving Equations Using the Subtraction and Addition Properties of Equality

- **Determine whether a number is a solution to an equation.**

1. Substitute the number for the variable in the equation.
2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true. If it is true, the number is a solution. If it is not true, the number is not a solution.

- **Subtraction Property of Equality**
  - For any numbers a, b, and c, if a = b, then a - c = b - c.
- Solve an equation using the Subtraction Property of Equality.

1. Use the Subtraction Property of Equality to isolate the variable.
2. Simplify the expressions on both sides of the equation.
3. Check the solution.

- **Addition Property of Equality**

- For any numbers a, b, and c, if a = b, then a + c = b + c.

- Solve an equation using the Addition Property of Equality.

1. Use the Addition Property of Equality to isolate the variable.
2. Simplify the expressions on both sides of the equation.
3. Check the solution.

2.4 - Find Multiples and Factors

Divisibility Tests	
A number is divisible by	
2	if the last digit is 0, 2, 4, 6, or 8
3	if the sum of the digits is divisible by 3
5	if the last digit is 5 or 0
6	if divisible by both 2 and 3
10	if the last digit is 0

- **Factors:** If a • b = m, then a and b are factors of m, and m is the product of a and b.
- **Find all the factors of a counting number.**

1. Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.
   a. If the quotient is a counting number, the divisor and quotient are a pair of factors.
   b. If the quotient is not a counting number, the divisor is not a factor.
2. List all the factor pairs.
3. Write all the factors in order from smallest to largest.

- **Determine if a number is prime**.

1. Test each of the primes, in order, to see if it is a factor of the number.
2. Start with 2 and stop when the quotient is smaller than the divisor or when a prime factor is found.
3. If the number has a prime factor, then it is a composite number. If it has no prime factors, then the number is prime.

2.5 - Prime Factorization and the Least Common Multiple

- **Find the prime factorization of a composite number using the tree method**.

1. Find any factor pair of the given number, and use these numbers to create two branches.
2. If a factor is prime, that branch is complete. Circle the prime.
3. If a factor is not prime, write it as the product of a factor pair and continue the process.
4. Write the composite number as the product of all the circled primes.

- **Find the prime factorization of a composite number using the ladder method**.

1. Divide the number by the smallest prime.
2. Continue dividing by that prime until it no longer divides evenly.
3. Divide by the next prime until it no longer divides evenly.
4. Continue until the quotient is a prime.
5. Write the composite number as the product of all the primes on the sides and top of the ladder.

- **Find the LCM by listing multiples**

1. List the first several multiples of each number.
2. Look for multiples common to both lists. If there are no common multiples in the lists, write out additional multiples for each number.
3. Look for the smallest number that is common to both lists.
4. This number is the LCM.

- **Find the LCM using the prime factors method**.

1. Find the prime factorization of each number.
2. Write each number as a product of primes, matching primes vertically when possible.
3. Bring down the primes in each column.
4. Multiply the factors to get the LCM.

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CHAPTER OVERVIEW

4: Fractions

Often in life, whole amounts are not exactly what we need. A baker must use a little more than a cup of milk or part of a teaspoon of sugar. Similarly a carpenter might need less than a foot of wood and a painter might use part of a gallon of paint. In this chapter, we will learn about numbers that describe parts of a whole. These numbers, called fractions, are very useful both in algebra and in everyday life. You will discover that you are already familiar with many examples of fractions!

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Figure 4.1 - Bakers combine ingredients to make delicious breads and pastries. (credit: Agustín Ruiz, Flickr)

Contributors

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4.1: Visualize Fractions (Part 1)

Learning Objectives

- Understand the meaning of fractions
- Model improper fractions and mixed numbers
- Convert between improper fractions and mixed numbers
- Model equivalent fractions
- Find equivalent fractions
- Locate fractions and mixed numbers on the number line
- Order fractions and mixed numbers

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: $5 \bullet 2+1$. If you missed this problem, review Example 2.1.8.
2. Fill in the blank with $<$ or $>$: $-2\_\_-5$. If you missed this problem, review Example 3.1.2.

Understand the Meaning of Fractions

Andy and Bobby love pizza. On Monday night, they share a pizza equally. How much of the pizza does each one get? Are you thinking that each boy gets half of the pizza? That's right. There is one whole pizza, evenly divided into two parts, so each boy gets one of the two equal parts. In math, we write $\frac{1}{2}$ to mean one out of two parts.

Figure 4.1.1

On Tuesday, Andy and Bobby share a pizza with their parents, Fred and Christy, with each person getting an equal amount of the whole pizza. How much of the pizza does each person get? There is one whole pizza, divided evenly into four equal parts. Each person has one of the four equal parts, so each has $\frac{1}{4}$ of the pizza.

Figure 4.1.2

On Wednesday, the family invites some friends over for a pizza dinner. There are a total of 12 people. If they share the pizza equally, each person would get $\frac{1}{12}$ of the pizza.

Figure 4.1.3

Definition: Fractions

A fraction is written $\frac{a}{b}$, where a and b are integers and $b \neq 0$. In a fraction, a is called the numerator and b is called the denominator.

A fraction is a way to represent parts of a whole. The denominator b represents the number of equal parts the whole has been divided into, and the numerator a represents how many parts are included. The denominator, b, cannot equal zero because division by zero is undefined.

In Figure 4.1.4, the circle has been divided into three parts of equal size. Each part represents $\frac{1}{3}$ of the circle. This type of model is called a fraction circle. Other shapes, such as rectangles, can also be used to model fractions.

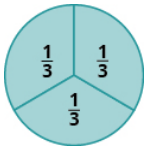


Figure 4.1.4

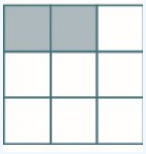
What does the fraction $\frac{2}{3}$ represent? The fraction $\frac{2}{3}$ means two of three equal parts.

Figure 4.1.5

Example 4.1.1: name the fraction

Name the fraction of the shape that is shaded in each of the figures.

a.

b. 

Solution

We need to ask two questions. First, how many equal parts are there? This will be the denominator. Second, of these equal parts, how many are shaded? This will be the numerator.

a.

How many equal parts are there?	There are eight equal parts.
How many are shaded?	Five parts are shaded.

Five out of eight parts are shaded. Therefore, the fraction of the circle that is shaded is $\frac{5}{8}$.

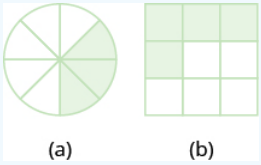
b.

How many equal parts are there?	There are nineequal parts.
How many are shaded?	Two parts are shaded.

Two out of nine parts are shaded. Therefore, the fraction of the square that is shaded is $\frac{2}{9}$.

Exercise 4.1.1

Name the fraction of the shape that is shaded in each figure:



Answer a

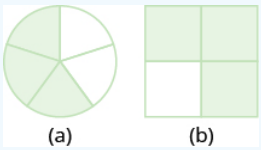
$\frac{3}{8}$

Answer b

$\frac{4}{9}$

Exercise 4.1.2

Name the fraction of the shape that is shaded in each figure:



Answer a

$\frac{3}{5}$

Answer b

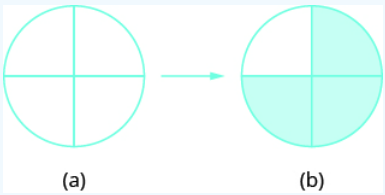
$\frac{3}{4}$

Example 4.1.2:

Shade $\frac{3}{4}$ of the circle.

Solution

The denominator is 4, so we divide the circle into four equal parts (a). The numerator is 3, so we shade three of the four parts (b).



$\frac{3}{4}$ of the circle is shaded.

Exercise 4.1.3

Shade $\frac{6}{8}$ of the circle.

Answer

Exercise 4.1.4

Shade $\frac{2}{5}$ of the rectangle.

Answer

In Example 4.1.1 and Example 4.1.2, we used circles and rectangles to model fractions. Fractions can also be modeled as manipulatives called fraction tiles, as shown in Figure 4.1.6. Here, the whole is modeled as one long, undivided rectangular tile. Beneath it are tiles of equal length divided into different numbers of equally sized parts.

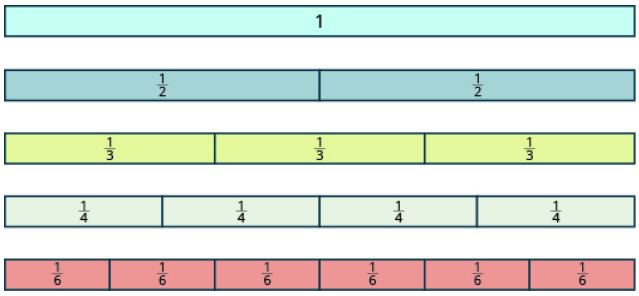


Figure 4.1.6

We'll be using fraction tiles to discover some basic facts about fractions. Refer to Figure 4.1.6 to answer the following questions:

How many $\frac{1}{2}$ tiles does it take to make one whole tile?	It takes two halves to make a whole, so two out of two is $\frac{2}{2} = 1$.
How many $\frac{1}{3}$ tiles does it take to make one whole tile?	It takes three thirds, so three out of three is $\frac{3}{3} = 1$.
How many $\frac{1}{4}$ tiles does it take to make one whole tile?	It takes four fourths, so four out of four is $\frac{4}{4} = 1$.
How many $\frac{1}{5}$ tiles does it take to make one whole tile?	It takes six sixths, so six out of six is $\frac{6}{6} = 1$.
What if the whole were divided into 24 equal parts? (We have not shown fraction tiles to represent this, but try to visualize it in your mind.) How many $\frac{1}{24}$ tiles does it take to make one whole tile?	It takes 24 twenty-fourths, so $\frac{24}{24} = 1$.

It takes 24 twenty-fourths, so $\frac{24}{24} = 1$. This leads us to the Property of One.

Definition: Property of One

Any number, except zero, divided by itself is one.

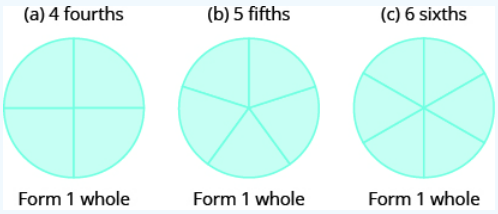
$$\frac{a}{a} = 1 \qquad (a \neq 0) \tag{4.1.1}$$

Example 4.1.3: fraction circles to form wholes

Use fraction circles to make wholes using the following pieces:

a. 4 fourths
b. 5 fifths
c. 6 sixths

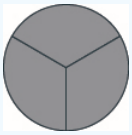
Solution



Exercise 4.1.5

Use fraction circles to make wholes with the following pieces: 3 thirds.

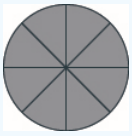
Answer



Exercise 4.1.6

Use fraction circles to make wholes with the following pieces: 8 eighths.

Answer



What if we have more fraction pieces than we need for 1 whole? We'll look at this in the next example.

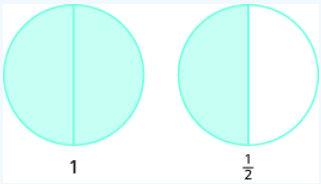
Example 4.1.4: fraction circles to form whole

Use fraction circles to make wholes using the following pieces:

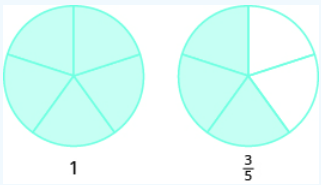
a. 3 halves
b. 8 fifths
c. 7 thirds

Solution

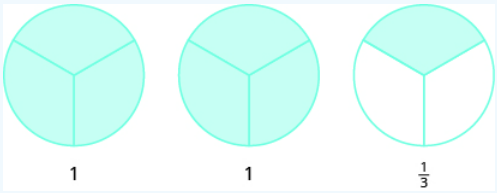
a. 3 halves make 1 whole with 1 half left over.



b. 8 fifths make 1 whole with 3 fifths left over.



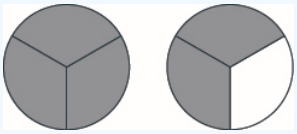
c. 7 thirds make 2 wholes with 1 third left over.



Exercise 4.1.7

Use fraction circles to make wholes with the following pieces: 5 thirds.

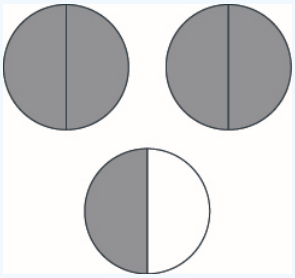
Answer



Exercise 4.1.8

Use fraction circles to make wholes with the following pieces: 5 halves.

Answer



Model Improper Fractions and Mixed Numbers

In Example 4.1.4b, you had eight equal fifth pieces. You used five of them to make one whole, and you had three fifths left over. Let us use fraction notation to show what happened. You had eight pieces, each of them one fifth, $\frac{1}{5}$, so altogether you had eight fifths, which we can write as $\frac{8}{5}$. The fraction $\frac{8}{5}$ is one whole, 1, plus three fifths, $\frac{3}{5}$, or $1\frac{3}{5}$, which is read as *one and three-fifths*. The number $1\frac{3}{5}$ is called a mixed number. A mixed number consists of a whole number and a fraction.

Definition: Mixed Numbers

A mixed number consists of a whole number a and a fraction $\frac{b}{c}$ where $c \neq 0$. It is written as follows.

$$a\frac{b}{c} \qquad c \neq 0 \tag{4.1.2}$$

Fractions such as $\frac{5}{4}$, $\frac{3}{2}$, $\frac{5}{5}$, and $\frac{7}{3}$ are called improper fractions. In an improper fraction, the numerator is greater than or equal to the denominator, so its value is greater than or equal to one. When a fraction has a numerator that is smaller than the denominator, it is called a proper fraction, and its value is less than one. Fractions such as $\frac{1}{2}$, $\frac{3}{7}$, and $\frac{11}{18}$ are proper fractions.

Definition: Proper and Improper Fractions

The fraction $\frac{a}{b}$ is a proper fraction if $a < b$ and an improper fraction if $a \geq b$.

Example 4.1.5: improper fraction

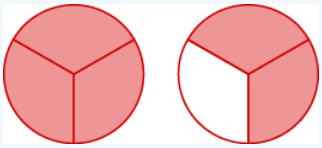
Name the improper fraction modeled. Then write the improper fraction as a mixed number.

Solution

Each circle is divided into three pieces, so each piece is $\frac{1}{3}$ of the circle. There are four pieces shaded, so there are four thirds or $\frac{4}{3}$. The figure shows that we also have one whole circle and one third, which is $1\frac{1}{3}$. So, $\frac{4}{3}=1\frac{1}{3}$.

Exercise 4.1.9

Name the improper fraction. Then write it as a mixed number.

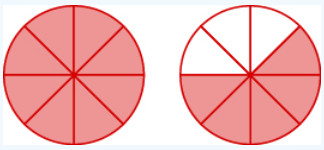


Answer

$\frac{5}{3}=1\frac{2}{3}$

Exercise 4.1.10

Name the improper fraction. Then write it as a mixed number.



Answer

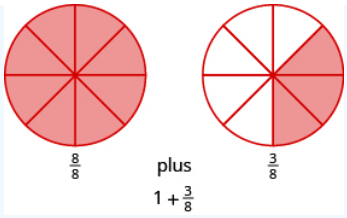
$\frac{13}{8}=1\frac{5}{8}$

Example 4.1.6: model a fraction

Draw a figure to model $\frac{11}{8}$.

Solution

The denominator of the improper fraction is 8. Draw a circle divided into eight pieces and shade all of them. This takes care of eight eighths, but we have 11 eighths. We must shade three of the eight parts of another circle.

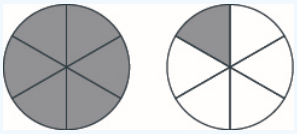


So, $\frac{11}{8}=1\frac{3}{8}$.

? Exercise4.1.11

Draw a figure to model $\frac{7}{6}$.

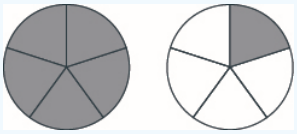
Answer



? Exercise 4.1.12

Draw a figure to model $\frac{6}{5}$.

Answer



✓ Example 4.1.7: model a fraction

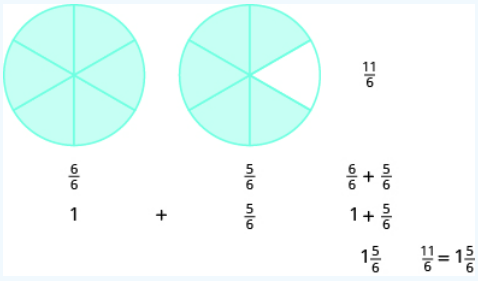
Use a model to rewrite the improper fraction $\frac{11}{6}$ as a mixed number.

Solution

We start with 11 sixths $\left(\frac{11}{6}\right)$. We know that six sixths makes one whole.

$$\frac{6}{6}=1$$

That leaves us with five more sixths, which is $\frac{5}{6}$ (11 sixths minus 6 sixths is 5 sixths). So, $\frac{11}{6}=1\frac{5}{6}$.



Exercise 4.1.13

Use a model to rewrite the improper fraction as a mixed number: $\frac{9}{7}$.

Answer

$1\frac{2}{7}$

Exercise 4.1.14

Use a model to rewrite the improper fraction as a mixed number: $\frac{7}{4}$.

Answer

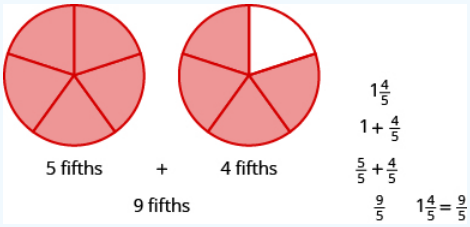
$1\frac{3}{4}$

Example 4.1.8: model a fraction

Use a model to rewrite the mixed number $1\frac{4}{5}$ as an improper fraction.

Solution

The mixed number $1\frac{4}{5}$ means one whole plus four fifths. The denominator is 5, so the whole is $\frac{5}{5}$. Together five fifths and four fifths equals nine fifths. So, $1\frac{4}{5}=\frac{9}{5}$.



Exercise 4.1.15

Use a model to rewrite the mixed number as an improper fraction: $1\frac{3}{8}$.

Answer

$\frac{11}{8}$

Exercise 4.1.16

Use a model to rewrite the mixed number as an improper fraction: $1\frac{5}{6}$.

Answer

$\frac{11}{6}$

Convert between Improper Fractions and Mixed Numbers

In Example 4.1.7, we converted the improper fraction $\frac{11}{6}$ to the mixed number $1\frac{5}{6}$ using fraction circles. We did this by grouping six sixths together to make a whole; then we looked to see how many of the 11 pieces were left. We saw that $\frac{11}{6}$ made one whole group of six sixths plus five more sixths, showing that $\frac{11}{6} = \frac{15}{6}$.

The division expression $\frac{11}{6}$ (which can also be written as $6\overline{)11}$) tells us to find how many groups of 6 are in 11. To convert an improper fraction to a mixed number without fraction circles, we divide.

Example 4.1.9:

Convert $\frac{11}{6}$ to a mixed number.

Solution

Divide the denominator into the numerator.	Remember $\frac{11}{6}$ means $11 \div 6$.
Identify the quotient, remainder and divisor.	$6\overline{)11}$ 1 ← quotient; divisor → 6; $\frac{6}{5}$ ← remainder
Write the mixed number as $quotient\ \frac{remainder}{divisor}$.	$1\frac{5}{6}$

So, $\frac{11}{6} = 1\frac{5}{6}$.

Exercise 4.1.17

Convert the improper fraction to a mixed number: $\frac{13}{7}$.

Answer

$1\frac{6}{7}$

Exercise 4.1.18

Convert the improper fraction to a mixed number: $\frac{14}{9}$.

Answer

$1\frac{5}{9}$

HOW TO: CONVERT AN IMPROPER FRACTION TO A MIXED NUMBER

Step 1. Divide the denominator into the numerator.

Step 2. Identify the quotient, remainder, and divisor.

Step 3. Write the mixed number as $quotient\ \frac{remainder}{divisor}$.

Example 4.1.10:

Convert the improper fraction $\frac{33}{8}$ to a mixed number.

Solution

Divide the denominator into the numerator.	Remember, $\frac{33}{8}$ means $8\overline{)33}$.
Identify the quotient, remainder, and divisor.	$\begin{array}{r} 4 \leftarrow \text{quotient} \\ 8\overline{)33} \leftarrow \text{divisor} \\ \underline{32} \\ 1 \leftarrow \text{remainder} \end{array}$
Write the mixed number as $quotient\,\frac{remainder}{divisor}$.	$4\frac{1}{8}$

So, $\frac{33}{8} = 4\frac{1}{8}$.

Exercise 4.1.19

Convert the improper fraction to a mixed number: $\frac{23}{7}$.

Answer

$3\frac{2}{7}$

Exercise 4.1.20

Convert the improper fraction to a mixed number: $\frac{48}{11}$.

Answer

$4\frac{4}{11}$

In Example 4.1.8, we changed $1\frac{4}{5}$ to an improper fraction by first seeing that the whole is a set of five fifths. So we had five fifths and four more fifths.

$$\frac{5}{5} + \frac{4}{5} = \frac{9}{5}$$

Where did the nine come from? There are nine fifths—one whole (five fifths) plus four fifths. Let us use this idea to see how to convert a mixed number to an improper fraction.

Example 4.1.11: convert

Convert the mixed number $4\frac{2}{3}$ to an improper fraction.

Multiply the whole number by the denominator.	$4\frac{2}{3}$
The whole number is 4 and the denominator is 3.	$\frac{4 \cdot 3 + \square}{\square}$

Simplify.	$\frac{12+\square}{\square}$
Add the numerator to the product.	
The numerator of the mixed number is 2.	$\frac{12+2}{\square}$
Simplify.	$\frac{14}{\square}$
Write the final sum over the original denominator.	
The denominator is 3.	$\frac{14}{3}$

Exercise 4.1.21

Convert the mixed number to an improper fraction: $3\frac{5}{7}$.

Answer

$\frac{26}{7}$

Exercise 4.1.22

Convert the mixed number to an improper fraction: $2\frac{7}{8}$.

Answer

$\frac{23}{8}$

HOW TO: CONVERT A MIXED NUMBER TO AN IMPROPER FRACTION

Step 1. Multiply the whole number by the denominator.

Step 2. Add the numerator to the product found in Step 1.

Step 3. Write the final sum over the original denominator.

Example 4.1.12:

Convert the mixed number $10\frac{2}{7}$ to an improper fraction.

Multiply the whole number by the denominator.	$10\frac{2}{7}$
The whole number is 10 and the denominator is 7.	$\frac{10\cdot 7+\square}{\square}$
Simplify.	$\frac{70+\square}{\square}$
Add the numerator to the product.	
The numerator of the mixed number is 2.	$\frac{70+2}{\square}$

Simplify.	$\frac{72}{\square}$
Write the final sum over the original denominator.	
The denominator is 7.	$\frac{72}{7}$

Exercise 4.1.23

Convert the mixed number to an improper fraction: $4\frac{6}{11}$.

Answer

$\frac{50}{11}$

Exercise 4.1.24

Convert the mixed number to an improper fraction: $11\frac{1}{3}$.

Answer

$\frac{34}{3}$

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4.2: Visualize Fractions (Part 2)

Model Equivalent Fractions

Let's think about Andy and Bobby and their favorite food again. If Andy eats $\frac{1}{2}$ of a pizza and Bobby eats $\frac{2}{4}$ of the pizza, have they eaten the same amount of pizza? In other words, does $\frac{1}{2} = \frac{2}{4}$? We can use fraction tiles to find out whether Andy and Bobby have eaten *equivalent* parts of the pizza.

Definition: Equivalent Fractions

Equivalent fractions are fractions that have the same value.

Fraction tiles serve as a useful model of equivalent fractions. You may want to use fraction tiles to do the following activity. Or you might make a copy of Figure 4.3 and extend it to include eighths, tenths, and twelfths.

Start with a $\frac{1}{2}$ tile. How many fourths equal one-half? How many of the $\frac{1}{4}$ tiles exactly cover the $\frac{1}{2}$ tile?

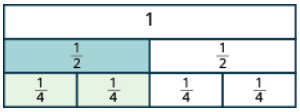


Figure 4.2.7

Since two $\frac{1}{4}$ tiles cover the $\frac{1}{2}$ tile, we see that $\frac{2}{4}$ is the same as $\frac{1}{2}$, or $\frac{2}{4} = \frac{1}{2}$.

How many of the $\frac{1}{6}$ tiles cover the $\frac{1}{2}$ tile?

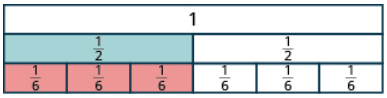


Figure 4.2.8

Since three $\frac{1}{6}$ tiles cover the $\frac{1}{2}$ tile, we see that $\frac{3}{6}$ is the same as $\frac{1}{2}$. So, $\frac{3}{6} = \frac{1}{2}$. The fractions are equivalent fractions.

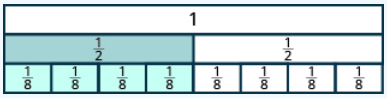
Example 4.2.13: equivalent fractions

Use fraction tiles to find equivalent fractions. Show your result with a figure.

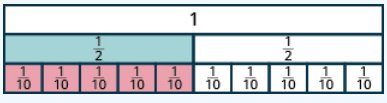
a. How many eighths equal one-half?
b. How many tenths equal one-half?
c. How many twelfths equal one-half?

Solution

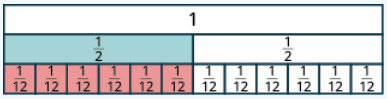
a. It takes four $\frac{1}{8}$ tiles to exactly cover the $\frac{1}{2}$ tile, so $\frac{4}{8} = \frac{1}{2}$.



b. It takes five $\frac{1}{10}$ tiles to exactly cover the $\frac{1}{2}$ tile, so $\frac{5}{10} = \frac{1}{2}$.



c. It takes six $\frac{1}{12}$ tiles to exactly cover the $\frac{1}{2}$ tile, so $\frac{6}{12} = \frac{1}{2}$.



Suppose you had tiles marked $\frac{1}{20}$. How many of them would it take to equal $\frac{1}{2}$? Are you thinking ten tiles? If you are, you're right, because $\frac{10}{20} = \frac{1}{2}$.

We have shown that $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}$, and $\frac{10}{20}$ are all equivalent fractions.

Exercise 4.2.25

Use fraction tiles to find equivalent fractions: How many eighths equal one-fourth?

Answer

2

Exercise 4.2.26

Use fraction tiles to find equivalent fractions: How many twelfths equal one-fourth?

Answer

3

Find Equivalent Fractions

We used fraction tiles to show that there are many fractions equivalent to $\frac{1}{2}$. For example, $\frac{2}{4}, \frac{3}{6}$, and $\frac{4}{8}$ are all equivalent to $\frac{1}{2}$. When we lined up the fraction tiles, it took four of the $\frac{1}{8}$ tiles to make the same length as a $\frac{1}{2}$ tile. This showed that $\frac{4}{8} = \frac{1}{2}$. See Example 4.2.13.

We can show this with pizzas, too. Figure $4.2.9a$ shows a single pizza, cut into two equal pieces with $\frac{1}{2}$ shaded. Figure $4.2.9b$ shows a second pizza of the same size, cut into eight pieces with $\frac{4}{8}$ shaded.

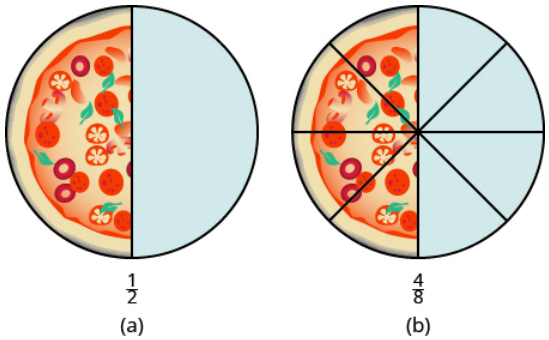


Figure 4.2.9

This is another way to show that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$. How can we use mathematics to change $\frac{1}{2}$ into $frac48$? How could you take a pizza that is cut into two pieces and cut it into eight pieces? You could cut each of the two larger pieces into four smaller

pieces! The whole pizza would then be cut into eight pieces instead of just two. Mathematically, what we've described could be written as:

$$\frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$$

These models lead to the Equivalent Fractions Property, which states that if we multiply the numerator and denominator of a fraction by the same number, the value of the fraction does not change.

Definition: Equivalent Fractions Property

If a, b, and c are numbers where $b \neq 0$ and $c \neq 0$, then

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \tag{4.2.1}$$

When working with fractions, it is often necessary to express the same fraction in different forms. To find equivalent forms of a fraction, we can use the Equivalent Fractions Property. For example, consider the fraction one-half.

$$\frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6} \text{ so } \frac{1}{2} = \frac{3}{6}$$
$$\frac{1 \cdot 2}{2 \cdot 2} = \frac{2}{4} \text{ so } \frac{1}{2} = \frac{2}{4}$$
$$\frac{1 \cdot 10}{2 \cdot 10} = \frac{10}{20} \text{ so } \frac{1}{2} = \frac{10}{20}$$

So, we say that $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{10}{20}$ are equivalent fractions.

Example 4.2.14: equivalent fractions

Find three fractions equivalent to $\frac{2}{5}$.

Solution

To find a fraction equivalent to $\frac{2}{5}$, we multiply the numerator and denominator by the same number (but not zero). Let us multiply them by 2, 3, and 5.

$$\frac{2 \cdot 2}{5 \cdot 2} = \frac{4}{10} \qquad \frac{2 \cdot 3}{5 \cdot 3} = \frac{6}{15} \qquad \frac{2 \cdot 5}{5 \cdot 5} = \frac{10}{25}$$

So, $\frac{4}{10}$, $\frac{6}{15}$, and $\frac{10}{25}$ are equivalent to $\frac{2}{5}$.

Exercise 4.2.27

Find three fractions equivalent to $\frac{3}{5}$.

Answer

Correct answers include $\frac{6}{10}$, $\frac{9}{15}$ and $\frac{12}{20}$

Exercise 4.2.28

Find three fractions equivalent to $\frac{4}{5}$.

Answer

Correct answers include $\frac{8}{10}$, $\frac{12}{15}$ and $\frac{16}{20}$

Example 4.2.15: equivalent fractions

Find a fraction with a denominator of 21 that is equivalent to $\frac{2}{7}$.

Solution

To find equivalent fractions, we multiply the numerator and denominator by the same number. In this case, we need to multiply the denominator by a number that will result in 21.

Since we can multiply 7 by 3 to get 21, we can find the equivalent fraction by multiplying both the numerator and denominator by 3.

$$\frac{2}{7} = \frac{2 \cdot 3}{7 \cdot 3} = \frac{6}{21}$$

Exercise 4.2.29

Find a fraction with a denominator of 21 that is equivalent to $\frac{6}{7}$.

Answer

$\frac{18}{21}$

Exercise 4.2.30

Find a fraction with a denominator of 100 that is equivalent to $\frac{3}{10}$.

Answer

$\frac{30}{100}$

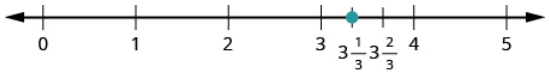
Locate Fractions and Mixed Numbers on the Number Line

Now we are ready to plot fractions on a number line. This will help us visualize fractions and understand their values.

Let us locate $\frac{1}{5}, \frac{4}{5}, 3, 3\frac{1}{3}, \frac{7}{4}, \frac{9}{2}, 5$, and $\frac{8}{3}$ on the number line. We will start with the whole numbers 3 and 5 because they are the easiest to plot.

The proper fractions listed are $\frac{1}{5}$ and $\frac{4}{5}$. We know proper fractions have values less than one, so $\frac{1}{5}$ and $\frac{1}{5}$ are located between the whole numbers 0 and 1. The denominators are both 5, so we need to divide the segment of the number line between 0 and 1 into five equal parts. We can do this by drawing four equally spaced marks on the number line, which we can then label as $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$, and $\frac{4}{5}$. Now plot points at $\frac{1}{5}$ and $\frac{4}{5}$.

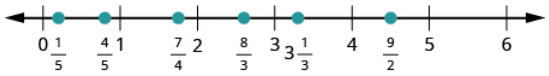
The only mixed number to plot is $3\frac{1}{3}$. Between what two whole numbers is $3\frac{1}{3}$? Remember that a mixed number is a whole number plus a proper fraction, so $3\frac{1}{3} > 3$. Since it is greater than 3, but not a whole unit greater, $3\frac{1}{3}$ is between 3 and 4. We need to divide the portion of the number line between 3 and 4 into three equal pieces (thirds) and plot $3\frac{1}{3}$ at the first mark.



Finally, look at the improper fractions $\frac{7}{4}$, $\frac{9}{2}$, and $\frac{8}{3}$. Locating these points will be easier if you change each of them to a mixed number.

$$\frac{7}{4} = 1\frac{3}{4}, \qquad \frac{9}{2} = 4\frac{1}{2}, \qquad \frac{8}{3} = 2\frac{2}{3}$$

Here is the number line with all the points plotted.

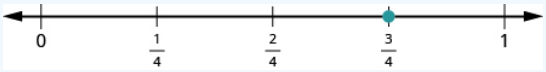


Example 4.2.16: locate and label

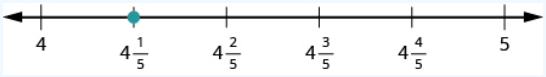
Locate and label the following on a number line: $\frac{3}{4}$, $\frac{4}{3}$, $\frac{5}{3}$, $4\frac{1}{5}$, and $\frac{7}{2}$.

Solution

Start by locating the proper fraction $\frac{3}{4}$. It is between 0 and 1. To do this, divide the distance between 0 and 1 into four equal parts. Then plot $\frac{3}{4}$.



Next, locate the mixed number $4\frac{1}{5}$. It is between 4 and 5 on the number line. Divide the number line between 4 and 5 into five equal parts, and then plot $4\frac{1}{5}$ one-fifth of the way between 4 and 5.

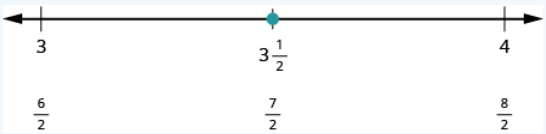


Now locate the improper fractions $\frac{4}{3}$ and $\frac{5}{3}$. It is easier to plot them if we convert them to mixed numbers first.

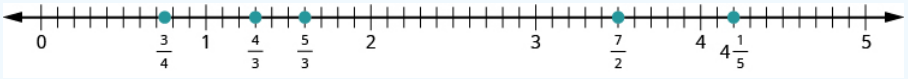
$$\frac{4}{3} = 1\frac{1}{3}, \qquad \frac{5}{3} = 1\frac{2}{3}$$

Divide the distance between 1 and 2 into thirds.

Next let us plot $\frac{7}{2}$. We write it as a mixed number, $\frac{7}{2} = 3\frac{1}{2}$. Plot it between 3 and 4.



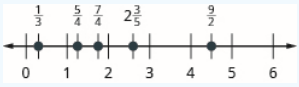
The number line shows all the numbers located on the number line.



Exercise 4.2.31

Locate and label the following on a number line: $\frac{1}{3}, \frac{5}{4}, \frac{7}{4}, 2\frac{3}{5}, \frac{9}{2}$.

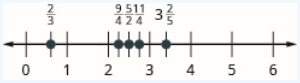
Answer



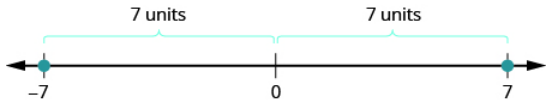
Exercise 4.2.32

Locate and label the following on a number line: $\frac{2}{3}, \frac{5}{2}, \frac{9}{4}, \frac{11}{4}, 3\frac{2}{5}$.

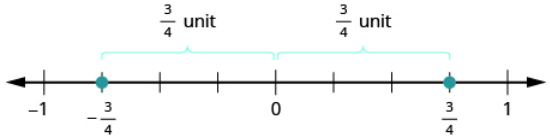
Answer



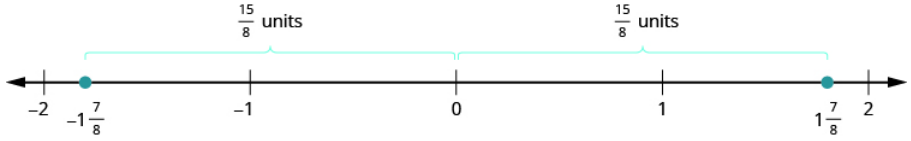
In Introduction to Integers, we defined the opposite of a number. It is the number that is the same distance from zero on the number line but on the opposite side of zero. We saw, for example, that the opposite of 7 is -7 and the opposite of -7 is 7.



Fractions have opposites, too. The opposite of $\frac{3}{4}$ is $-\frac{3}{4}$. It is the same distance from 0 on the number line, but on the opposite side of 0.



Thinking of negative fractions as the opposite of positive fractions will help us locate them on the number line. To locate $-\frac{15}{8}$ on the number line, first think of where $\frac{15}{8}$ is located. It is an improper fraction, so we first convert it to the mixed number $1\frac{7}{8}$ and see that it will be between 1 and 2 on the number line. So its opposite, $-\frac{15}{8}$, will be between -1 and -2 on the number line.



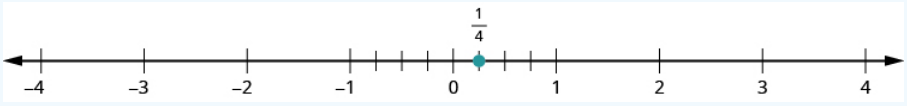
Example 4.2.17: locate and label

Locate and label the following on the number line: $\frac{1}{4}, -\frac{1}{4}, 1\frac{1}{3}, -1\frac{1}{3}, \frac{5}{2}$, and $-\frac{5}{2}$.

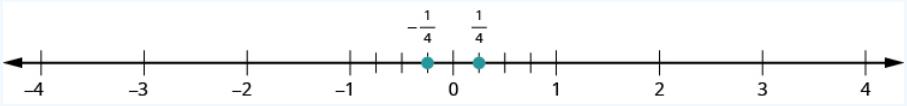
Solution

Draw a number line. Mark 0 in the middle and then mark several units to the left and right.

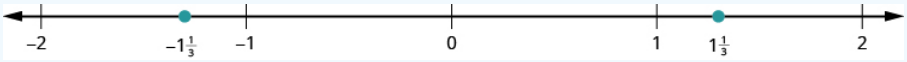
To locate $\frac{1}{4}$, divide the interval between 0 and 1 into four equal parts. Each part represents one-quarter of the distance. So plot $\frac{1}{4}$ at the first mark.



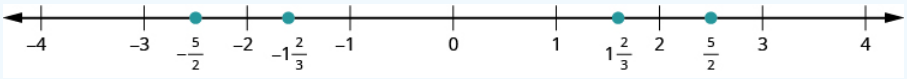
To locate $-\frac{1}{4}$, divide the interval between 0 and -1 into four equal parts. Plot $-\frac{1}{4}$ at the first mark to the left of 0.



Since $1\frac{1}{3}$ is between 1 and 2, divide the interval between 1 and 2 into three equal parts. Plot $1\frac{1}{3}$ at the first mark to the right of 1. Then since $-1\frac{1}{3}$ is the opposite of $1\frac{1}{3}$ it is between -1 and -2. Divide the interval between -1 and -2 into three equal parts. Plot $-1\frac{1}{3}$ at the first mark to the left of -1.



To locate $\frac{5}{2}$ and $-\frac{5}{2}$, it may be helpful to rewrite them as the mixed numbers $2\frac{1}{2}$ and $-2\frac{1}{2}$. Since $2\frac{1}{2}$ is between 2 and 3, divide the interval between 2 and 3 into two equal parts. Plot $\frac{5}{2}$ at the mark. Then since $-2\frac{1}{2}$ is between -2 and -3, divide the interval between -2 and -3 into two equal parts. Plot $-\frac{5}{2}$ at the mark.



Exercise 4.2.33

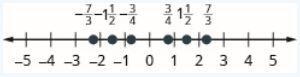
Locate and label each of the given fractions on a number line: $\frac{2}{3}, -\frac{2}{3}, 2\frac{1}{4}, -2\frac{1}{4}, \frac{3}{2}, -\frac{3}{2}$

Answer

Exercise 4.2.34

Locate and label each of the given fractions on a number line: $\frac{3}{4}, -\frac{3}{4}, 1\frac{1}{2}, -1\frac{1}{2}, \frac{7}{3}, -\frac{7}{3}$

Answer



Order Fractions and Mixed Numbers

We can use the inequality symbols to order fractions. Remember that $a > b$ means that a is to the right of b on the number line. As we move from left to right on a number line, the values increase.

Example 4.2.18: order

Order each of the following pairs of numbers, using $<$ or $>$:

a. $-\frac{2}{3}\_\_\_\_-1$

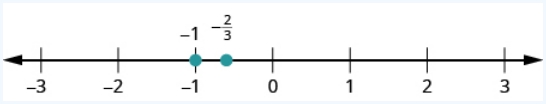
b. $-3\frac{1}{2}\_\_\_\_-3$

c. $-\frac{3}{7}\_\_\_\_-\frac{3}{8}$

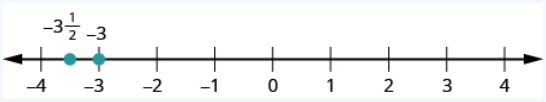
d. $-2\_\_\_\_-\frac{16}{9}$

Solution

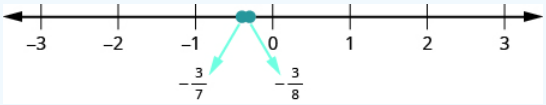
a. $-\frac{2}{3} > -1$



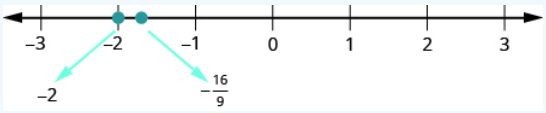
b. $-3\frac{1}{2} < -3$



c. $-\frac{3}{7} < -\frac{3}{8}$



d. $-2 < -\frac{16}{9}$



Exercise 4.2.35

Order each of the following pairs of numbers, using $<$ or $>$:

a. $-\frac{1}{3}\_\_-1$

b. $-1\frac{1}{2}\_\_-2$

c. $-\frac{2}{3}\_\_-\frac{1}{3}$

d. $-3\_\_-\frac{7}{3}$

Answer a

$>$

Answer b

$>$

Answer c

$<$

Answer d

$<$

Exercise 4.2.36

Order each of the following pairs of numbers, using $<$ or $>$:

a. $-3 \_\_ -\frac{17}{5}$

b. $-2\frac{1}{4} \_\_ -2$

c. $-\frac{3}{5} \_\_ -\frac{4}{5}$

d. $-4 \_\_ -\frac{10}{3}$

Answer a

$>$

Answer b

$<$

Answer c

$>$

Answer d

$<$

Access Additional Online Resources

- Introduction to Fractions
- Identify Fractions Using Pattern Blocks

Key Concepts

- Property of One
  - Any number, except zero, divided by itself is one.
    $\frac{a}{a} = 1$, where $a \neq 0$.
- Mixed Numbers
  - A mixed number consists of a whole number a and a fraction $\frac{b}{c}$ where $c \neq 0$.
  - It is written as follows: $a\frac{b}{c}\ c \neq 0$
- Proper and Improper Fractions
  - The fraction $\frac{a}{b}$ is a proper fraction if $a < b$ and an improper fraction if $a \geq b$.
- Convert an improper fraction to a mixed number.
  1. Divide the denominator into the numerator.

2. Identify the quotient, remainder, and divisor.
3. Write the mixed number as $\frac{\text{remainder}}{\text{divisor}}$.

- Convert a mixed number to an improper fraction.
  1. Multiply the whole number by the denominator.
  2. Add the numerator to the product found in Step 1.
  3. Write the final sum over the original denominator.
- Equivalent Fractions Property
  - If a, b and c are numbers where $b \neq 0$, $c \neq 0$, then \(\dfrac{a}{b} = \dfrac{a\cdot c}{b\cdot c}\]).

Glossary

equivalent fractions

Equivalent fractions are two or more fractions that have the same value.

fraction

A fraction is written $\frac{a}{b}$. in a fraction, a is the numerator and b is the denominator. A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

mixed number

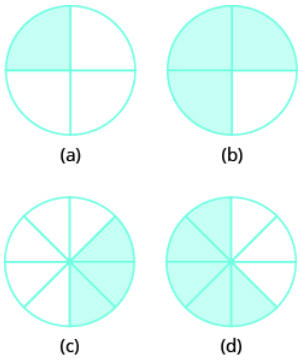
A mixed number consists of a whole number a and a fraction $\frac{b}{c}$ where $c \neq 0$. It is written as $a\frac{b}{c}$, where $c \neq 0$.

proper and improper fractions

The fraction $\frac{a}{b}$ is proper if $a < b$ and improper if $a > b$.

Practice Makes Perfect

1.



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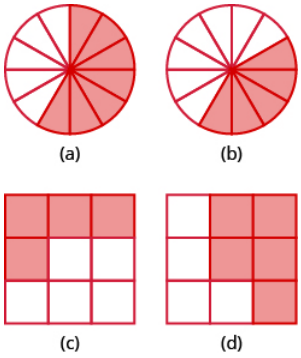


In the following exercises, shade parts of circles or squares to model the following fractions.

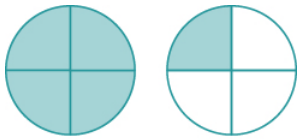
3. $\frac{1}{2}$

4. $\frac{1}{3}$
5. $\frac{3}{4}$
6. $\frac{2}{5}$
7. $\frac{5}{6}$
8. $\frac{7}{8}$
9. $\frac{5}{8}$
10. $\frac{7}{10}$

In the following exercises, use fraction circles to make wholes, if possible, with the following pieces.

11. 3 thirds
12. 8 eighths
13. 7 sixths
14. 4 thirds
15. 7 fifths
16. 7 fourths

In the following exercises, name the improper fractions. Then write each improper fraction as a mixed number.

17. 
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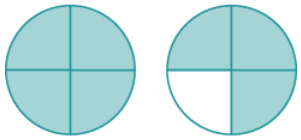
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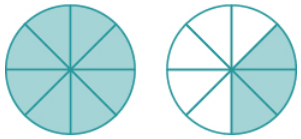
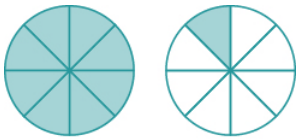

(c)



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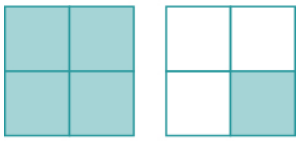

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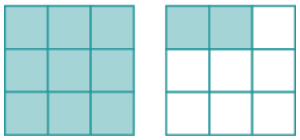

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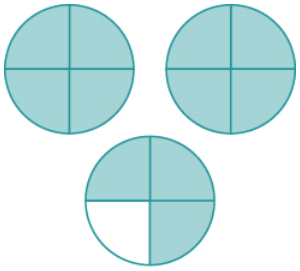

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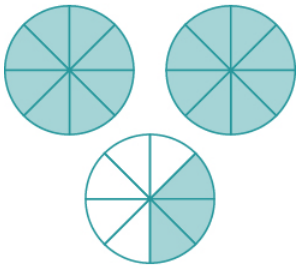

(b)

19.

In the following exercises, draw fraction circles to model the given fraction.

20. $\frac{3}{3}$
21. $\frac{4}{4}$
22. $\frac{7}{4}$
23. $\frac{5}{3}$
24. $\frac{11}{6}$
25. $\frac{13}{8}$
26. $\frac{10}{3}$
27. $\frac{9}{4}$

In the following exercises, rewrite the improper fraction as a mixed number.

28. $\frac{3}{2}$
29. $\frac{5}{3}$
30. $\frac{11}{4}$
31. $\frac{13}{5}$
32. $\frac{25}{6}$
33. $\frac{28}{9}$
34. $\frac{42}{13}$
35. $\frac{47}{15}$

In the following exercises, rewrite the mixed number as an improper fraction.

36. $1\frac{2}{3}$
37. $1\frac{2}{5}$
38. $2\frac{1}{4}$
39. $2\frac{5}{6}$
40. $2\frac{7}{9}$
41. $2\frac{5}{7}$
42. $3\frac{4}{7}$
43. $3\frac{5}{9}$

In the following exercises, use fraction tiles or draw a figure to find equivalent fractions.

44. How many sixths equal one-third?
45. How many twelfths equal one-third?
46. How many eighths equal three-fourths?
47. How many twelfths equal three-fourths?
48. How many fourths equal three-halves?
49. How many sixths equal three-halves?

In the following exercises, find three fractions equivalent to the given fraction. Show your work, using figures or algebra.

50. $\frac{1}{4}$
51. $\frac{1}{3}$
52. $\frac{3}{8}$
53. $\frac{5}{6}$
54. $\frac{2}{7}$
55. $\frac{5}{9}$

In the following exercises, plot the numbers on a number line.

56. $\frac{2}{3}, \frac{5}{4}, \frac{12}{5}$

57. $\frac{1}{3}, \frac{7}{4}, \frac{13}{5}$
58. $\frac{1}{4}, \frac{9}{5}, \frac{11}{3}$
59. $\frac{7}{10}, \frac{5}{2}, \frac{13}{8}, 3$
60. $2\frac{1}{3}, -2\frac{1}{3}$
61. $1\frac{3}{4}, -1\frac{3}{5}$
62. $\frac{3}{4}, -\frac{3}{4}, 1\frac{2}{3}, -1\frac{2}{3}, \frac{5}{2}, -\frac{5}{2}$
63. $\frac{2}{5}, -\frac{2}{5}, 1\frac{3}{4}, -1\frac{3}{4}, \frac{8}{3}, -\frac{8}{3}$

In the following exercises, order each of the following pairs of numbers, using < or >.

64. $-1 \_\_ -\frac{1}{4}$
65. $-1 \_\_ -\frac{1}{3}$
66. $-2\frac{1}{2} \_\_ -3$
67. $-1\frac{3}{4} \_\_ -2$
68. $-\frac{5}{12} \_\_ -\frac{7}{12}$
69. $-\frac{9}{10} \_\_ -\frac{3}{10}$
70. $-3 \_\_ -\frac{13}{5}$
71. $-4 \_\_ -\frac{23}{6}$

Everyday Math

72. **Music Measures** A choreographed dance is broken into counts. A $\frac{1}{1}$ count has one step in a count, a $\frac{1}{2}$ count has two steps in a count and a 1 3 count has three steps in a count. How many steps would be in a $\frac{1}{5}$ count? What type of count has four steps in it?
73. **Music Measures** Fractions are used often in music. In 4 4 time, there are four quarter notes in one measure.
    a. How many measures would eight quarter notes make?
    b. The song “Happy Birthday to You” has 25 quarter notes. How many measures are there in “Happy Birthday to You?”
74. **Baking** Nina is making five pans of fudge to serve after a music recital. For each pan, she needs 1 2 cup of walnuts.
    a. How many cups of walnuts does she need for five pans of fudge?
    b. Do you think it is easier to measure this amount when you use an improper fraction or a mixed number? Why?

Writing Exercises

75. Give an example from your life experience (outside of school) where it was important to understand fractions.
76. Explain how you locate the improper fraction $\frac{21}{4}$ on a number line on which only the whole numbers from 0 through 10 are marked.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
understand the meaning of fractions.			
model improper fractions and mixed numbers.			
convert between improper fractions and mixed numbers.			
model equivalent fractions.			
find equivalent fractions.			
locate fractions and mixed numbers on the number line.			
order fractions and mixed numbers.			

(b) If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Contributors and Attributions

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4.3: Rational Expressions (INTRO)

Learning Objectives

- Determine the restrictions to the domain of a rational expression.
- Simplify rational expressions.
- Simplify expressions with opposite binomial factors.
- Simplify and evaluate rational functions.

Rational Expressions, Evaluation, and Restrictions

A rational number, or fraction $\frac{a}{b}$, is a real number defined as a quotient of two integers ***a*** and ***b***, where $b \neq 0$. Similarly, we define a rational expression, or algebraic fraction $\frac{P}{Q}$, as the quotient of two polynomials ***P*** and ***Q***, where $Q \neq 0$. Some examples of rational expressions follow:

The example $\frac{x+3}{x-5}$ consists of linear expressions in both the numerator and denominator. Because the denominator contains a variable, this expression is not defined for all values of ***x***.

Evaluate $\frac{x+3}{x-5}$ for the set of ***x***-values $\{-3, 4, 5\}$.

Solution:

Substitute the values in for ***x***.

$x=-3$	$x=4$	$x=5$
$\frac{x+3}{x-5}=\frac{(-3)+3}{(-3)-5}$	$\frac{x+3}{x-5}=\frac{(4)+3}{(4)-5}$	$\frac{x+3}{x-5}=\frac{(5)+3}{(5)-5}$
$=\frac{0}{-8}$	$=\frac{7}{-1}$	$=\frac{8}{0}$ *Undefined*
$=0$	$=-7$	

Answer:

When $x=-3$, the value of the rational expression is 0; when $x=4$, the value of the rational expression is -7; and when $x=5$, the value of the rational expression is undefined.

This example illustrates that variables are restricted to values that do not make the denominator equal to 0. The domain of a rational expression is the set of real numbers for which it is defined, and restrictions are the real numbers for which the expression is not defined. We often express the domain of a rational expression in terms of its restrictions.

Example 4.3.2

Find the domain of the following:

$$\frac{x+7}{2x^2+x-6}$$

Solution:

In this example, the numerator $x+7$ is a linear expression and the denominator $2x^2+x-6$ is a quadratic expression. If we factor the denominator, then we will obtain an equivalent expression.

$$\frac{x+7}{2x^2+x-6}=\frac{x+7}{(2x-3)(x+2)}$$

Because rational expressions are undefined when the denominator is 0, we wish to find the values for ***x*** that make it 0. To do this, apply the zero product property. Set each factor in the denominator equal to 0 and solve.

$$(2x-3)(x+2)=0$$

$$\begin{aligned} 2x-3=0 \quad &\text{or} \quad x+2=0 \\ 2x=3 \quad &\quad\quad x=-2 \\ x=\tfrac{3}{2} \quad &\end{aligned}$$

We conclude that the original expression is defined for any real number except $\frac{3}{2}$ and -2. These two values are the restrictions to the domain. It is important to note that -7 is ***not*** a restriction to the domain because the expression is defined as 0 when the numerator is 0.

$$\begin{aligned} \frac{x+7}{2x^2+x-6} &= \frac{(-7)+7}{2(-7)^2+(-7)-6} \\ &= \frac{0}{98-7-6} \\ &= \frac{0}{85} \\ &= 0 \end{aligned}$$

Answer:

The domain consists of any real number ***x***, where $x \neq \frac{3}{2}$ and $x \neq -2$.

We can express the domain of the previous example using notation as follows:

Set − builder notation	*Interval notation*
$\left\{x \middle\| x \neq -2, \frac{3}{2}\right\}$	$(-\infty,-2)\cup\left(-2,\frac{3}{2}\right)\cup\left(\frac{3}{2},\infty\right)$

The restrictions to the domain of a rational expression are determined by the denominator. Ignore the numerator when finding those restrictions.

Example 4.3.3

Determine the domain:

$$\frac{x^4+x^3-2x^2-x}{x^2-1}$$

Solution:

To find the restrictions to the domain, set the denominator equal to 0 and solve:

$$\begin{aligned} x^2-1 &= 0 \\ (x+1)(x-1) &= 0 \\ x+1 = 0 \quad &\text{or} \quad x-1=0 \\ x=-1 \quad &\quad\quad x=1 \end{aligned}$$

These two values cause the denominator to be 0. Hence they are restricted from the domain.

Answer:

The domain consists of any real number ***x***, where $x \neq \pm 1$.

Example 4.3.4

Determine the domain:

$$\frac{x^2-25}{4}$$

Solution:

There is no variable in the denominator and thus no restriction to the domain.

Answer:

The domain consists of all real numbers, **R**.

Simplifying Rational Expressions

When simplifying fractions, look for common factors that cancel. For example,

$$\frac{12}{60}=\frac{1\cdot \cancel{12}}{5\cdot \cancel{12}}=\frac{1}{5}$$

We say that the fraction $\frac{12}{60}$ is equivalent to $\frac{1}{5}$. Fractions are in simplest form if the numerator and denominator share no common factor other than 1. Similarly, when working with rational expressions, look for factors to cancel. For example,

$$\frac{x+4}{(x-3)(x+4)}=\frac{1\cdot \cancel{(x+4)}}{(x-3)\ \cancel{(x+4)}}=\frac{1}{x-3}$$

The resulting rational expression is equivalent if it shares the same domain. Therefore, we must make note of the restrictions and write

$$\frac{x+4}{(x-3)(x+4)}=\frac{1}{x-3},\ \text{where } x\neq 3 \text{ and } x\neq -4$$

In words, $\frac{x+4}{(x-3)(x+4)}$ is equivalent to $\frac{1}{x-3}$, if $x\neq 3$ and $x\neq -4$. We can verify this by choosing a few values with which to evaluate both expressions to see if the results are the same. Here we choose $x=7$ and evaluate as follows:

$$\begin{aligned}\frac{x+4}{(x-3)(x+4)}&=\frac{1}{x-3}\\ \frac{(7)+4}{(7-3)(7+4)}&=\frac{1}{(7)-3}\\ \frac{11}{(4)(11)}&=\frac{1}{4}\\ \frac{1}{4}&=\frac{1}{4}\ \checkmark\end{aligned}$$

It is important to state the restrictions before simplifying rational expressions because the simplified expression may be defined for restrictions of the original. In this case, the expressions are not equivalent. Here −4 is defined for the simplified equivalent but not for the original, as illustrated below:

$$\begin{aligned}\frac{x+4}{(x-3)(x+4)}&=\frac{1}{x-3}\\ \frac{(-4)+4}{(-4-3)(-4+4)}&=\frac{1}{(-4)-3}\\ \frac{0}{(-7)(0)}&=\frac{1}{-7}\\ \frac{0}{0}&=-\frac{1}{7}\ \ x\end{aligned}$$

Example 4.3.5

Simplify and state the restriction:

$$\frac{25x^2}{15x^3}$$

Solution:

In this example, the expression is undefined when ***x*** is 0.

$$\frac{25x^2}{15x^3}=\frac{25(0)^2}{15(0)^3}=\frac{0}{0}\ \ Undefined$$

Therefore, the domain consists of all real numbers ***x***, where $x\neq 0$. With this understanding, we can cancel common factors.

$$\begin{aligned}\frac{25x^2}{15x^3}&=\frac{5\cdot \cancel{5x^2}}{3x\cdot \cancel{5x^2}}\\ &=\frac{5}{3x}\end{aligned}$$

Answer:

$\frac{5}{3x}$, where $x \neq 0$

Example 4.3.6

State the restrictions and simplify:

$$\frac{3x(x-5)}{(2x-1)(x-5)}$$

Solution:

To determine the restrictions, set the denominator equal to 0 and solve.

$$(2x+1)(x-5)=0$$

$$\begin{aligned} 2x+1 &= 0 \quad \text{or} \quad x-5=0 \\ 2x &= -1 \qquad\quad x=5 \\ x &= -\tfrac{1}{2} \end{aligned}$$

The domain consists of all real numbers except for $-\frac{1}{2}$ and 5. Next, we find an equivalent expression by canceling common factors.

$$\begin{aligned} \frac{3x(x-5)}{(2x+1)(x-5)} &= \frac{3x\ \cancel{(x-5)}}{(2x+1)\ \cancel{(x-5)}} \\ &= \frac{3x}{2x+1} \end{aligned}$$

Answer:

$\frac{3x}{2x+1}$, where $x \neq -\frac{1}{2}$ and $x \neq 5$

Typically, rational expressions are not given in factored form. If this is the case, factor first and then cancel. The steps are outlined in the following example.

Example 4.3.7

State the restrictions and simplify:

$$\frac{3x+6}{x^2+x-2}$$

Solution:

Step 1: Completely factor the numerator and denominator.

$$\frac{3x+6}{x^2+x-2} = \frac{3(x+2)}{(x-1)(x+2)}$$

Step 2: Determine the restrictions to the domain. To do this, set the denominator equal to 0 and solve.

$$(x-1)(x+2)=0$$

$$\begin{aligned} x-1 &= 0 \quad \text{or} \quad x+2=0 \\ x &= 1 \qquad\quad x=-2 \end{aligned}$$

The domain consists of all real numbers except -2 and 1.

Step 3: Cancel common factors, if any.

$$\begin{aligned} \frac{3x+6}{x^2+x-2} &= \frac{3\ \cancel{(x+2)}}{(x-1)\ \cancel{(x+2)}} \\ &= \frac{3}{x-1} \end{aligned}$$

Answer:

$\frac{3}{x-1}$, where $x \neq 1$ and $x \neq -2$

Example 4.3.8

State the restrictions and simplify:

Solution:

First, factor the numerator and denominator.

Any value of x that results in a value of 0 in the denominator is a restriction. By inspection, we determine that the domain consists of all real numbers except 4 and 3. Next, cancel common factors.

$$= \frac{\cancel{(x-3)}(x+10)}{(x-4)\ \cancel{(x-3)}}$$
$$= \frac{x+10}{x-4}$$

Answer:

$\frac{x+10}{x-4}$, where $x \neq 3$ and $x \neq 4$

It is important to remember that we can only cancel factors of a product. A common mistake is to cancel terms. For example,

State the restrictions and simplify:

$$\frac{x^2-16}{5x^2-20x}$$

Answer

$\frac{x+4}{5x}$, where $x \neq 0$ and $x \neq 4$

In some examples, we will make a broad assumption that the denominator is nonzero. When we make that assumption, we do not need to determine the restrictions.

Example 4.3.9

Simplify:

(Assume all denominators are nonzero.)

Solution:

Factor the numerator by grouping. Factor the denominator using the formula for a difference of squares.

Next, cancel common factors.

$$= \frac{\cancel{(x+y)}(y-3)}{\cancel{(x+y)}(x-y)}$$
$$= \frac{y-3}{x-y}$$

Answer:

$$\frac{y-3}{x-y}$$

Opposite Binomial Factors

Recall that the opposite of the real number a is $-a$. Similarly, we can define the opposite of a polynomial P to be $-P$. We first consider the opposite of the binomial $a-b$:

$$-(a-b)=-a+b=b-a \tag{4.3.1}$$

This leads us to the opposite binomial property:

$$-(a-b)=(b-a) \tag{4.3.2}$$

This is equivalent to factoring out a -1.

$$(b-a)=-1(a-b) \tag{4.3.3}$$

If $a \neq b$, then we can divide both sides by $(a-b)$ and obtain the following:

$$\frac{b-a}{a-b}=-1$$

Example 4.3.10

State the restrictions and simplify:

$$\frac{3-x}{x-3}$$

Solution:

By inspection, we can see that the denominator is 0 if $x=3$. Therefore, 3 is the restriction to the domain. Apply the opposite binomial property to the numerator and then cancel.

Answer:

$\frac{3-x}{x-3}$, where $x \neq 3$

Since addition is commutative, we have

$$(a+b)=(b+a) \tag{4.3.4}$$

or

$$\frac{b+a}{a+b}=1$$

Take care not to confuse this with the opposite binomial property. Also, it is important to recall that

$$\frac{-a}{b}=-\frac{a}{b}=\frac{a}{-b}$$

In other words, show a negative fraction by placing the negative sign in the numerator, in front of the fraction bar, or in the denominator. Generally, negative denominators are avoided.

Example 4.3.11

Simplify and state the restrictions:

$$\frac{4-x^2}{x^2+3x-10}$$

Solution:

Begin by factoring the numerator and denominator.

$$\begin{aligned}\frac{4-x^2}{x^2+3x-10} &= \frac{(2+x)(2-x)}{(x-2)(x+5)} && \textit{The restrictions are } x \neq 2 \textit{ and } x \neq -5.\\ &= \frac{(2+x)\cdot(-1)\,(x-2)}{(x-2)(x+5)} && \textit{Apply the opposite binomial property, then cancel.}\\ &= \frac{(2+x)\cdot(-1)}{(x+5)}\\ &= -\frac{2+x}{x+5} \quad \text{or} \quad = -\frac{x+2}{x+5}\end{aligned}$$

Answer:

$-\frac{x+2}{x+5}$, where $x \neq 2$ and $x \neq -5$

Exercise 4.3.2

Simplify and state the restrictions:

Answer

$\frac{-2x+3}{x+5}$, **where** $x \neq \pm 5$

Rational Functions

Rational functions have the form

$$r(x) = \frac{p(x)}{q(x)} \tag{4.3.5}$$

,

where *p(x)* and *q(x)* are polynomials and *q(x)≠0*. The domain of a rational function consists of all real numbers ***x*** such that the denominator *q(x)≠0*.

Example 4.3.12

a. Simplify: $r(x) = \frac{2x^2+5x-3}{6x^2+18x}$.
b. State the domain.
c. Calculate $r(-2)$.

Solution:

a. To simplify the rational function, first factor and then cancel.

$$\begin{aligned} r(x) &= \frac{2x^2+5x-3}{6x^2+18x} \\ &= \frac{(2x-1)\,(x+3)}{6x\,(x+3)} \\ &= \frac{2x-1}{6x} \end{aligned}$$

b. To determine the restrictions, set the denominator of the original function equal to 0 and solve.

$$\begin{aligned} 6x^2+18x &= 0 \\ 6x(x+3) &= 0 \\ 6x = 0 \quad &\text{or} \quad x+3 = 0 \\ x = 0 \quad &\quad\;\; x = -3 \end{aligned}$$

The domain consists of all real numbers ***x***, where $x \neq 0$ and $x \neq -3$.

c. Since -2 is not a restriction, substitute it for the variable ***x*** using the simplified form.

$$r(x) = \frac{2x-1}{6x}$$
$$r(-2) = \frac{2(-2)-1}{6(-2)}$$
$$= \frac{-4-1}{-12}$$
$$= \frac{-5}{-12}$$
$$= \frac{5}{12}$$

Answer:

a. $\frac{2x-1}{6x}$ b. The domain is all real numbers except 0 and -3. c. $r(-2) = \frac{5}{12}$

If a cost function $C(x)$ represents the cost of producing ***x*** units, then the average cost $c(x)$ is the cost divided by the number of units produced.

$$\textit{Average cost}: c(x) = \frac{C(x)}{x} \tag{4.3.6}$$

Example 4.3.13

The cost in dollars of producing t-shirts with a company logo is given by $C(x) = 7x + 200$, where ***x*** represents the number of shirts produced. Determine the average cost of producing

a. 40 t-shirts
b. 250 t-shirts
c. 1,000 t-shirts

Solution:

Set up a function representing the average cost.

$$c(x) = \frac{C(x)}{x} = \frac{7x+200}{x}$$

Next, calculate c(40), c(250), and c(1000).

$$c(40) = \frac{7(40)+200}{40} = \frac{280+200}{40} = \frac{480}{40} = 12.00$$
$$c(250) = \frac{7(250)+200}{250} = \frac{1750+200}{250} = \frac{1950}{250} = 7.80$$
$$c(1000) = \frac{7(1000)+200}{1000} = \frac{7000+200}{1000} = \frac{7200}{1000} = 7.20$$

Answer:

a. If 40 t-shirts are produced, then the average cost per t-shirt is $12.00.
b. If 250 t-shirts are produced, then the average cost per t-shirt is $7.80.
c. If 1,000 t-shirts are produced, then the average cost per t-shirt is $7.20.

Key Takeaways

- Rational expressions usually are not defined for all real numbers. The real numbers that give a value of 0 in the denominator are not part of the domain. These values are called restrictions.
- Simplifying rational expressions is similar to simplifying fractions. First, factor the numerator and denominator and then cancel the common factors. Rational expressions are simplified if there are no common factors other than 1 in the numerator and the denominator.
- Simplified rational expressions are equivalent for values in the domain of the original expression. Be sure to state the restrictions if the denominators are not assumed to be nonzero.

- Use the opposite binomial property to cancel binomial factors that involve subtraction. Use $-(a-b)=b-a$ to replace factors that will then cancel. Do not confuse this with factors that involve addition, such as $(a+b)=(b+a)$.

Exercise 4.3.3 Rational Expressions

Evaluate for the given set of *x*-values.

1. $5x; -1, 0, 1$
2. $\frac{4x^3}{x^2}; -1, 0, 1$
3. $\frac{1}{x+9}; -10, -9, 0$
4. $\frac{x+6}{x-5}; -6, 0, 5$
5. $\frac{3x(x-2)}{2x-1}; 0, \frac{1}{2}, 2$
6. $\frac{9x^2-1}{x-7}; 0, \frac{1}{3}, 7$
7. $5x^2-9; -3, 0, 3$
8. $\frac{x^2-2}{5x^2-3x-10}; -5, -4, 5$
9. Fill in the following chart:

x	$\frac{1}{x}$
-3	
-2	
0	
2	
3	

***Figure* 4.3.1**

10. Fill in the following chart:

x	$\frac{1}{x-2}$
-1	
0	
1	
2	
3	

***Figure* 4.3.2**

11. Fill in the following chart:

x	$\frac{1}{x+2}$
-4	
-3	
-2	
-1	
0	

***Figure* 4.3.3**

12. Fill in the following chart:

x	$\frac{1}{x}-2$
-2	
-1	
0	
1	
2	

***Figure* 4.3.4**

Answer

1. -5, undefined, 5
3. -1, undefined, $\frac{1}{9}$
5. 0, undefined, 0
7. Undefined, $-\frac{5}{9}$, undefined
9.

x	$\frac{1}{x}$
-3	$-1/3$
-2	$-1/2$
0	undefined
2	$1/2$
3	$1/3$

Figure 4.3.5

11.

x	$\frac{1}{x+2}$
-4	$-1/2$
-3	-1
-2	undefined
-1	1
0	$1/2$

Figure 4.3.6

? Exercise 4.3.4 Rational Expressions

An object's weight depends on its height above the surface of earth. If an object weighs 120 pounds on the surface of earth, then its weight in pounds, W, x miles above the surface is approximated by the formula

$$W = \frac{120 \cdot 4000^2}{(4000+x)^2}$$

For each problem below, approximate the weight of a 120-pound object at the given height above the surface of earth. (1 mile = 5,280 feet)

1. 100 miles
2. 1,000 miles
3. 44,350 feet
4. 90,000 feet

Answer

1. 114 pounds

3. 119.5 pounds

Exercise 4.3.5 Rational Expressions

The price to earnings ratio (P/E) is a metric used to compare the valuations of similar publicly traded companies. The P/E ratio is calculated using the stock price and the earnings per share (EPS) over the previous 12-month period as follows:

$$\text{P/E}=\frac{\text{price per share}}{\text{earnings per share}}$$

If each share of a company stock is priced at $22.40, then calculate the P/E ratio given the following values for the earnings per share.

1. $1.40
2. $1.21
3. What happens to the P/E ratio when earnings decrease?
4. What happens to the P/E ratio when earnings increase?

Answer

1. 16

3. The P/E ratio increases.

Exercise 4.3.6 Rational Expressions

State the restrictions to the domain.

1. $\frac{1}{3x}$
2. $\frac{3x^2}{7x^5}$
3. $\frac{3x(x+1)}{x+4}$
4. $\frac{2x^2(x-3)}{x-1}$
5. $\frac{1}{5x-1}$
6. $\frac{x-2}{3x-2}$
7. $\frac{x-9}{5x(x-2)}$
8. $\frac{1}{(x-3)(x+6)}$
9. $\frac{x}{1-x^2}$
10. $\frac{x^2-9}{x^2-36}$
11. $\frac{1}{2x(x+3)(2x-1)}$
12. $\frac{x-3}{(3x-1)(2x+3)}$
13. $\frac{4x(2x+1)}{12x^2+x-1}$
14. $\frac{x-5}{3x^2-15x}$

Answer

1. $x \neq 0$

3. $x \neq -4$

5. $x \neq \frac{1}{5}$

7. $x \neq 0$ and $x \neq 2$

9. $x \neq \pm 1$

11. $x \neq 0$, $x \neq -3$, and $x \neq \frac{1}{2}$

13. $x \neq -\frac{1}{3}$ and $x \neq \frac{1}{4}$

Exercise 4.3.7 Simplifying Rational Expressions

State the restrictions and then simplify.

1. $\frac{5x^2}{20x^3}$
2. $\frac{12x^6}{60x}$
3. $\frac{3x^2(x-2)}{9x(x-2)}$
4. $\frac{20(x-3)(x-5)}{6(x-3)(x+1)}$
5. $\frac{6x^2(x-8)}{36x(x+9)(x-8)}$
6. $\frac{16x^2-1}{(4x+1)^2}$
7. $\frac{9x^2-6x+1}{(3x-1)^2}$
8. $\frac{x-7}{x^2-49}$
9. $\frac{x^2-64}{x^2+8x}$
10. $\frac{x+10}{x^2-100}$
11. $\frac{2x^3-12x^2}{5x^2-30x}$
12. $\frac{30x^5+60x^4}{2x^3-8x}$
13. $\frac{2x-12}{x^2+x-6}$
14. $\frac{x^2-x-6}{3x^2-8x-3}$
15. $\frac{6x^2-25x+25}{3x^2+16x-35}$
16. $\frac{3x^2+4x-1}{5x^2-9}$
17. $\frac{x^2-10x+21}{x^2-4x-21}$
18. $\frac{x^3-1}{x^2-1}$
19. $\frac{x^3+8}{x^2-4}$
20. $\frac{x^4-1}{6x^2-4}$

Answer

1. $\frac{1}{4x}; x \neq 0$
3. $\frac{x}{3}; x \neq 0, 2$
5. $\frac{x}{6(x+9)}; x \neq 0, -9, 8$
7. $1; x \neq \frac{1}{3}$
9. $\frac{x-8}{x}; x \neq 0, -8$
11. $\frac{2x}{5}; x \neq 0, 6$
13. $\frac{2x-12}{x^2+x-6}$ $x \neq -2, \frac{3}{2}$
15. $\frac{2x-5}{x+7}; x \neq -7, \frac{5}{3}$
17. $\frac{x-3}{x+3}; x \neq -3, 7$
19. $\frac{x^2-2x+4}{x-2}; x \neq \pm 2$

Exercise 4.3.8 Simplifying Rational Expressions with Opposite Binomial Factors

State the restrictions and then simplify.

1. $\frac{x-9}{9-x}$
2. $\frac{3x-2}{2-3x}$
3. $\frac{x+6}{6+x}$
4. $\frac{3x+1}{1+3x}$
5. $\frac{(2x-5)(x-7)}{(7-x)(2x-1)}$
6. $\frac{(3x+2)(x+5)}{(x-5)(2+3x)}$
7. $\frac{x^2-4}{(2-x)^2}$
8. $\frac{16-9x^2}{(3x+4)^2}$
9. $\frac{4x^2(10-x)}{3x^3-300x}$
10. $-\frac{2x+1}{4x^3-49x}$
11. $\frac{2x^2-7x-4}{1-4x^2}$
12. $\frac{9x^2-4}{4x-6x^2}$
13. $\frac{x^2-5x-14}{7-15x+2x^2}$
14. $\frac{2x^3+x^2-2x-1}{1+x-2x^2}$
15. $\frac{x^3+2x-3x^2-6}{2+x^2}$
16. $\frac{27+x^3}{x^2+6x+9}$
17. $\frac{64-x^3}{x^2-8x+16}$
18. $\frac{x^2+4}{4-x^2}$

Answer

1. $-1; x \neq 9$
3. $1; x \neq -6$
5. $\frac{-2x-5}{2x-1}; x \neq \frac{1}{2}, 7$
7. $\frac{x+2}{x-2}; x \neq 2$
9. $-\frac{4x}{3(x+10)}; x \neq \pm 10, 0$
11. $\frac{x-4}{1-2x}; x \neq \pm\frac{1}{2}$
13. $\frac{x+2}{2x-1}; x \neq \frac{1}{2}, 7$
15. $x-3$; none
17. $\frac{-16+4x+x^2}{x-4}; x \neq 4$

Exercise 4.3.9 Simplifying Rational Expressions with Opposite Binomial Factors

Simplify. (Assume all denominators are nonzero.)

1. $-\frac{15x^3y^2}{5xy^2(x+y)}$
2. $\frac{14x^7y^2(x-2y)^4}{7x^8y(x-2y)^2}$
3. $\frac{y+x}{x^2-y^2}$

4. $\frac{y-x}{x^2-y^2}$
5. $\frac{x^2-y^2}{(x-y)^2}$
6. $\frac{a^2-ab-6b^2}{a^2-6ab+9b^2}$
7. $\frac{2a^2-11a+12}{-32+2a^2}$
8. $\frac{a^2b-3a^2}{3a^2-3ab}$
9. $\frac{xy^2-x+y^3-y}{x-xy^2}$
10. $\frac{x^3-xy^2-x^2y+y}{3x^2-2xy+y^2}$
11. $\frac{x^3-27}{x^2+3x+9}$
12. $\frac{x^2-x+1}{x^3+1}$

Answer

1. $-\frac{3x^2}{x+y}$
3. $\frac{1}{x-y}$
5. $\frac{x+y}{x-y}$
7. $\frac{2a-3}{2(a+4)}$
9. $-\frac{x+y}{x}$
11. $x-3$

Exercise 4.3.10 Rational Functions

Calculate the following.

1. $f(x)=\frac{5x}{x-3}; f(0), f(2), f(4)$
2. $f(x)=\frac{x+7}{x^2+1}; f(-1), f(0), f(1)$
3. $g(x)=\frac{x^3}{(x-2)^2}; g(0), g(2), g(-2)$
4. $g(x)=\frac{x^2-9}{9-x^2}; g(-2), g(0), g(2)$
5. $g(x)=\frac{x^3}{x^2+1}; g(-1), g(0), g(1)$
6. $g(x)=\frac{5x+1}{x^2-25}; g(-\frac{1}{5}), g(-1), g(-5)$

Answer

1. f(0)=0, f(2)=−10, f(4)=20
3. g(0)=0, g(2) undefined, g(−2)=$-\frac{1}{2}$
5. g(−1)=$-\frac{1}{2}$, g(0)=0, g(1)=$\frac{1}{2}$

Exercise 4.3.11 Rational Functions

State the restrictions to the domain and then simplify.

1. $f(x)=-\frac{3x^2-6x}{x^2+4x+4}$
2. $f(x)=\frac{x^2+6x+9}{2x^2+5x-3}$
3. $g(x)=\frac{9-x}{x^2-81}$
4. $g(x)=\frac{x^3-27}{3-x}$

5. $g(x) = \frac{3x-15}{10-2x}$
6. $g(x) = \frac{25-5x}{4x-20}$
7. The cost in dollars of producing coffee mugs with a company logo is given by $C(x) = x + 40$, where x represents the number of mugs produced. Calculate the average cost of producing 100 mugs and the average cost of producing 500 mugs.
8. The cost in dollars of renting a moving truck for the day is given by $C(x) = 0.45x + 90$, where x represents the number of miles driven. Calculate the average cost per mile if the truck is driven 250 miles in one day.
9. The cost in dollars of producing sweat shirts with a custom design on the back is given by $C(x) = 1200 + (12 - 0.05x)x$, where x represents the number of sweat shirts produced. Calculate the average cost of producing 150 custom sweat shirts.
10. The cost in dollars of producing a custom injected molded part is given by $C(x) = 500 + (3 - 0.001x)x$, where x represents the number of parts produced. Calculate the average cost of producing 1,000 custom parts.

Answer

1. $f(x) = -\frac{3x}{x+2}; x \neq -2$
3. $g(x) = -\frac{1}{x+9}; x \neq \pm 9$
5. $g(x) = -\frac{3}{2}; x \neq 5$
7. The average cost of producing 100 mugs is $1.40 per mug. The average cost of producing 500 mugs is $1.08 per mug.
9. $12.50

? Exercise 4.3.12 Discussion Board

1. Explain why $\frac{b-a}{a-b} = -1$ and illustrate this fact by substituting some numbers for the variables.
2. Explain why $\frac{b+a}{a+b} = 1$ and illustrate this fact by substituting some numbers for the variables.
3. Explain why we cannot cancel x in the expression $\frac{x}{x+1}$.

Answer

1. Answers may vary
3. Answers may vary

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4.4: Multiply and Divide Fractions (Part 1)

Learning Objectives

- Simplify fractions
- Multiply fractions
- Find reciprocals
- Divide fractions

be prepared!

Before you get started, take this readiness quiz.

1. Find the prime factorization of 48. If you missed this problem, review Example 2.5.1.
2. Draw a model of the fraction $\frac{3}{4}$. If you missed this problem, review Example 4.1.2.
3. Find two fractions equivalent to $\frac{5}{6}$. Answers may vary. Acceptable answers include $\frac{10}{12}$, $\frac{15}{18}$, $\frac{50}{60}$, etc. If you missed this problem, review Example 4.1.14.

Simplify Fractions

In working with equivalent fractions, you saw that there are many ways to write fractions that have the same value, or represent the same part of the whole. How do you know which one to use? Often, we'll use the fraction that is in *simplified* form.

A fraction is considered simplified if there are no common factors, other than 1, in the numerator and denominator. If a fraction does have common factors in the numerator and denominator, we can reduce the fraction to its simplified form by removing the common factors.

Definition: Simplified Fraction

A fraction is considered simplified if there are no common factors in the numerator and denominator.

For example,

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

The process of simplifying a fraction is often called *reducing the fraction*. In the previous section, we used the Equivalent Fractions Property to find equivalent fractions. We can also use the Equivalent Fractions Property in reverse to simplify fractions. We rewrite the property to show both forms together.

Definition: Equivalent Fractions Property

If a, b, c are numbers where $b \neq 0, c \neq 0$, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

Notice that c is a common factor in the numerator and denominator. Anytime we have a common factor in the numerator and denominator, it can be removed.

HOW TO: SIMPLIFY A FRACTION

Step 1. Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers.

Step 2. Simplify, using the equivalent fractions property, by removing common factors.

Step 3. Multiply any remaining factors.

Example 4.4.1: simplify

Simplify: $\frac{10}{15}$.

Solution

To simplify the fraction, we look for any common factors in the numerator and the denominator.

Notice that 5 is a factor of both 10 and 15.	$\frac{10}{15}$
Factor the numerator and denominator.	$\frac{2 \cdot 5}{3 \cdot 5}$
Remove the common factors.	$\frac{2 \cdot \cancel{5}}{3 \cdot \cancel{5}}$
Simplify.	$\frac{2}{3}$

Exercise 4.4.1

Simplify: $\frac{8}{12}$.

Answer

$\frac{2}{3}$

Exercise 4.4.2

Simplify: $\frac{12}{16}$.

Answer

$\frac{3}{4}$

To simplify a negative fraction, we use the same process as in Example 4.4.1. Remember to keep the negative sign.

Example 4.4.2: simplify

Simplify: $-\frac{18}{24}$.

Solution

We notice that 18 and 24 both have factors of 6.	$-\frac{18}{24}$
Rewrite the numerator and denominator showing the common factor.	$-\frac{3 \cdot 6}{4 \cdot 6}$
Remove common factors.	$-\frac{3 \cdot \cancel{6}}{4 \cdot \cancel{6}}$
Simplify.	$-\frac{3}{4}$

Exercise 4.4.3

Simplify: $-\frac{21}{28}$.

Answer

$-\frac{3}{4}$

Exercise 4.4.4

Simplify: $-\frac{16}{24}$.

Answer

$-\frac{2}{3}$

After simplifying a fraction, it is always important to check the result to make sure that the numerator and denominator do not have any more factors in common. Remember, the definition of a simplified fraction: *a fraction is considered simplified if there are no common factors in the numerator and denominator*.

When we simplify an improper fraction, there is no need to change it to a mixed number.

Example 4.4.3:

Simplify: $-\frac{56}{32}$.

Solution

	$-\frac{56}{32}$
Rewrite the numerator and denominator, showing the common factors, 8.	$-\frac{7\cdot 8}{4\cdot 8}$
Remove common factors.	$-\frac{7\cdot \cancel{8}}{4\cdot \cancel{8}}$
Simplify.	$-\frac{7}{4}$

Exercise 4.4.5

Simplify: $-\frac{54}{42}$.

Answer

$-\frac{9}{7}$

Exercise 4.4.6

Simplify: $-\frac{81}{45}$.

Answer

$-\frac{9}{5}$

HOW TO: SIMPLIFY A FRACTION

Step 1. Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers.

Step 2. Simplify, using the equivalent fractions property, by removing common factors.

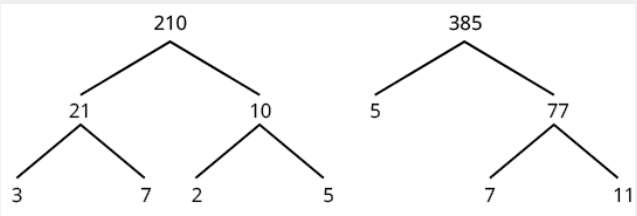
Step 3. Multiply any remaining factors.

Sometimes it may not be easy to find common factors of the numerator and denominator. A good idea, then, is to factor the numerator and the denominator into prime numbers. (You may want to use the factor tree method to identify the prime factors.) Then divide out the common factors using the Equivalent Fractions Property.

Example 4.4.4: simplify

Simplify: $\frac{210}{385}$.

Solution

	$\frac{210}{385}$
Use factor trees to factor the numerator and denominator.	
Rewrite the numerator and denominator as the product of the primes.	$\frac{210}{385} = \frac{2 \cdot 3 \cdot 5 \cdot 7}{5 \cdot 7 \cdot 11}$
Remove the common factors.	$\frac{2 \cdot 3 \cdot \cancel{5} \cdot \cancel{7}}{\cancel{5} \cdot \cancel{7} \cdot 11}$
Simplify.	$\frac{2 \cdot 3}{11}$
Multiply any remaining factors.	$\frac{6}{11}$

Exercise 4.4.7

Simplify: $\frac{69}{120}$.

Answer

$\frac{23}{40}$

Exercise 4.4.8

Simplify: $\frac{120}{192}$.

Answer

$\frac{5}{8}$

We can also simplify fractions containing variables. If a variable is a common factor in the numerator and denominator, we remove it just as we do with an integer factor.

Example 4.4.5: simplify

Simplify: $\frac{5xy}{15x}$.

Solution

	$\frac{5xy}{15x}$
Rewrite numerator and denominator showing common factors.	$\frac{5 \cdot x \cdot y}{3 \cdot 5 \cdot x}$
Remove common factors.	$\frac{\cancel{5} \cdot \cancel{x} \cdot y}{3 \cdot \cancel{5} \cdot \cancel{x}}$
Simplify.	$\frac{y}{3}$

Exercise 4.4.9

Simplify: $\frac{7x}{7y}$.

Answer

$\frac{x}{y}$

Exercise 4.4.10

Simplify: $\frac{9a}{9b}$.

Answer

$\frac{a}{b}$

Multiply Fractions

A model may help you understand multiplication of fractions. We will use fraction tiles to model $\frac{1}{2} \cdot \frac{3}{4}$. To multiply $\frac{1}{2}$ and $\frac{3}{4}$, think $\frac{1}{2}$ of $\frac{3}{4}$.

Start with fraction tiles for three-fourths. To find one-half of three-fourths, we need to divide them into two equal groups. Since we cannot divide the three $\frac{1}{4}$ tiles evenly into two parts, we exchange them for smaller tiles.

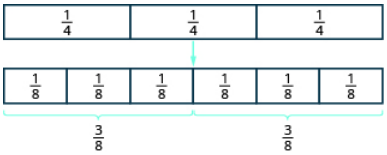


Figure 4.4.1

We see $\frac{6}{8}$ is equivalent to $\frac{3}{4}$. Taking half of the six $\frac{1}{8}$ tiles gives us three $\frac{1}{8}$ tiles, which is $\frac{3}{8}$. Therefore,

$$\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$

Example 4.4.6: model a fraction

Use a diagram to model $\frac{1}{2} \cdot \frac{3}{4}$.

Solution

First shade in $\frac{3}{4}$ of the rectangle.

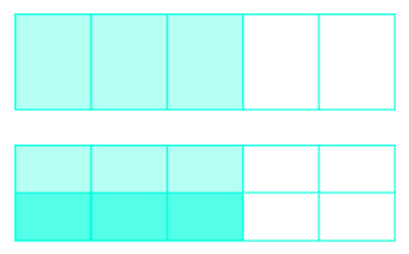
We will take $\frac{1}{2}$ of this $\frac{3}{4}$, so we heavily shade $\frac{1}{2}$ of the shaded region.

Notice that 3 out of the 8 pieces are heavily shaded. This means that $\frac{3}{8}$ of the rectangle is heavily shaded. Therefore, $\frac{1}{2}$ of $\frac{3}{4}$ is $\frac{3}{4}$, or $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$.

Exercise 4.4.11

Use a diagram to model: $\frac{1}{2} \cdot \frac{3}{5}$.

Answer

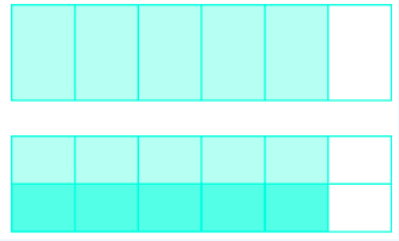


$\frac{3}{10}$

Exercise 4.4.12

Use a diagram to model: $\frac{1}{2} \cdot \frac{5}{6}$.

Answer



$\frac{5}{12}$

Look at the result we got from the model in Example 4.4.6. We found that $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$. Do you notice that we could have gotten the same answer by multiplying the numerators and multiplying the denominators?

	$\frac{1}{2} \cdot \frac{3}{4}$
Multiply the numerators, and multiply the denominators.	$\frac{1}{2} \cdot \frac{3}{4}$
Simplify.	$\frac{3}{8}$

This leads to the definition of fraction multiplication. To multiply fractions, we multiply the numerators and multiply the denominators. Then we write the fraction in simplified form.

Definition: Fraction Multiplication

If a, b, c, and d are numbers where $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \tag{4.4.1}$$

Example 4.4.7: multiply

Multiply, and write the answer in simplified form: $\frac{3}{4} \cdot \frac{1}{5}$.

Solution

	$\frac{3}{4} \cdot \frac{1}{5}$
Multiply the numerators, and multiply the denominators.	$\frac{3 \cdot 1}{4 \cdot 5}$
Simplify.	$\frac{3}{20}$

There are no common factors, so the fraction is simplified.

Exercise 4.4.13

Multiply, and write the answer in simplified form: $\frac{1}{3} \cdot \frac{2}{5}$.

Answer

$\frac{2}{15}$

Exercise 4.4.14

Multiply, and write the answer in simplified form: $\frac{3}{5} \cdot \frac{7}{8}$.

Answer

$\frac{21}{40}$

When multiplying fractions, the properties of positive and negative numbers still apply. It is a good idea to determine the sign of the product as the first step. In Example 4.4.8 we will multiply two negatives, so the product will be positive.

Example 4.4.8: multiply

Multiply, and write the answer in simplified form: $-\frac{5}{8}\left(-\frac{2}{3}\right)$.

Solution

	$-\frac{5}{8}\left(-\frac{2}{3}\right)$
The signs are the same, so the product is positive. Multiply the numerators, multiply the denominators.	$\frac{5 \cdot 2}{8 \cdot 3}$
Simplify.	$\frac{10}{24}$
Look for common factors in the numerator and denominator. Rewrite showing common factors.	$\frac{5 \cdot \cancel{2}}{12 \cdot \cancel{2}}$
Remove common factors.	$\frac{5}{12}$

Another way to find this product involves removing common factors earlier.

	$-\frac{5}{8}\left(-\frac{2}{3}\right)$
Determine the sign of the product. Multiply.	$\frac{5 \cdot 2}{8 \cdot 3}$
Show common factors and then remove them.	$\frac{5 \cdot \cancel{2}}{12 \cdot \cancel{2}}$
Multiply remaining factors.	$\frac{5}{12}$

We get the same result.

Exercise 4.4.15

Multiply, and write the answer in simplified form: $-\frac{4}{7}\left(-\frac{5}{8}\right)$.

Answer

$\frac{5}{14}$

Exercise 4.4.16

Multiply, and write the answer in simplified form: $-\frac{7}{12}\left(-\frac{8}{9}\right)$.

Answer

$\frac{14}{27}$

Example 4.4.9: multiply

Multiply, and write the answer in simplified form: $-\frac{14}{15}\cdot\frac{20}{21}$.

Solution

	$-\frac{14}{15}\cdot\frac{20}{21}$
Determine the sign of the product; multiply.	$-\frac{14}{15}\cdot\frac{20}{21}$
Are there any common factors in the numerator and the denominator? We know that 7 is a factor of 14 and 21, and 5 is a factor of 20 and 15.	
Rewrite showing common factors.	$-\frac{2\cdot\cancel{7}\cdot 4\cdot\cancel{5}}{3\cdot\cancel{5}\cdot 3\cdot\cancel{7}}$
Remove the common factors.	$-\frac{2\cdot 4}{3\cdot 3}$
Multiply the remaining factors.	$-\frac{8}{9}$

Exercise 4.4.17

Multiply, and write the answer in simplified form: $-\frac{10}{28}\cdot\frac{8}{15}$.

Answer

$-\frac{4}{21}$

Exercise 4.4.18

Multiply, and write the answer in simplified form: $-\frac{9}{20}\cdot\frac{5}{12}$.

Answer

$-\frac{3}{16}$

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $\frac{a}{1}$. So, $3=\frac{3}{1}$, for example.

Example 4.4.10:

Multiply, and write the answer in simplified form:

a. $\frac{1}{7} \cdot 56$

b. $\frac{12}{5}(-20x)$

Solution

a.

	$\frac{1}{7} \cdot 56$
Write 56 as a fraction.	$\frac{1}{7} \cdot \frac{56}{1}$
Determine the sign of the product; multiply.	$\frac{56}{7}$
Simplify.	8

b.

	$\frac{12}{5}(-20x)$
Write −20x as a fraction.	$\frac{12}{5}\left(\frac{-20x}{1}\right)$
Determine the sign of the product; multiply.	$-\frac{12 \cdot 20 \cdot x}{5 \cdot 1}$
Show common factors and then remove them.	$-\frac{12 \cdot 4 \cdot \cancel{5}x}{\cancel{5} \cdot 1}$
Multiply remaining factors; simplify.	$-48x$

Exercise 4.4.19

Multiply, and write the answer in simplified form:

a. $\frac{1}{8} \bullet 72$

b. $\frac{11}{3}(-9a)$

Answer a

9

Answer b

$-33a$

Exercise 4.4.20

Multiply, and write the answer in simplified form:

a. $\frac{3}{8} \bullet 64$

b. $16x \bullet \frac{11}{12}$

Answer a

24

Answer b

$\frac{44x}{3}$

Contributors and Attributions

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4.5: Multiply and Divide Fractions (Part 2)

Find Reciprocals

The fractions $\frac{2}{3}$ and $\frac{3}{2}$ are related to each other in a special way. So are $-\frac{10}{7}$ and $-\frac{7}{10}$. Do you see how? Besides looking like upside-down versions of one another, if we were to multiply these pairs of fractions, the product would be 1.

$$\frac{2}{3} \cdot \frac{3}{2} = 1 \quad and \quad -\frac{10}{7}\left(-\frac{7}{10}\right) = 1$$

Such pairs of numbers are called reciprocals.

Definition: Reciprocal

The reciprocal of the fraction $\frac{a}{b}$ is $\frac{b}{a}$, where $a \neq 0$ and $b \neq 0$.

A number and its reciprocal have a product of 1.

$$\frac{a}{b} \cdot \frac{b}{a} = 1 \tag{4.2.54}$$

To find the reciprocal of a fraction, we invert the fraction. This means that we place the numerator in the denominator and the denominator in the numerator.

To get a positive result when multiplying two numbers, the numbers must have the same sign. So reciprocals must have the same sign.

$$\frac{a}{b} \cdot \frac{b}{a} = 1 \text{ positive}$$

$$3 \cdot \frac{1}{3} = 1 \quad \text{and} \quad -3 \cdot \left(-\frac{1}{3}\right) = 1$$

both positive both negative

To find the reciprocal, keep the same sign and invert the fraction. The number zero does not have a reciprocal. Why? A number and its reciprocal multiply to 1. Is there any number r so that $0 \cdot r = 1$? No. So, the number 0 does not have a reciprocal.

Example 4.5.11: reciprocal

Find the reciprocal of each number. Then check that the product of each number and its reciprocal is 1.

a. $\frac{4}{9}$
b. $-\frac{1}{6}$
c. $-\frac{14}{5}$
d. 7

Solution

To find the reciprocals, we keep the sign and invert the fractions.

a.

Find the reciprocal of $\frac{4}{9}$.	The reciprocal of $\frac{4}{9}$ is $\frac{9}{4}$.

Check:

Multiply the number and its reciprocal.	$\frac{4}{9} \cdot \frac{9}{4}$

Multiply numerators and denominators.	$\frac{36}{36}$
Simplify.	1 ✓

b.

Find the reciprocal of $-\frac{1}{6}$.	The reciprocal of $-\frac{1}{6}$ is $\frac{6}{1}$.
Simplify.	-6
Check.	$-\frac{1}{6} \cdot (-6) = 1$ ✓

c.

Find the reciprocal of $-\frac{14}{5}$.	$-\frac{5}{14}$
Check.	$-\frac{14}{5} \cdot \left(-\frac{5}{14}\right) = \frac{70}{70} = 1$ ✓

d.

Find the reciprocal of 7.	
Write 7 as a fraction.	$\frac{7}{1}$
Write the reciprocal of $\frac{7}{1}$.	$\frac{1}{7}$
Check.	$7 \cdot \left(\frac{1}{7}\right) = 1$ ✓

Exercise 4.5.21

Find the reciprocal:

a. $\frac{5}{7}$
b. $-\frac{1}{8}$
c. $-\frac{11}{4}$
d. 14

Answer a

$\frac{7}{5}$

Answer b

-8

Answer c

$-\frac{4}{11}$

Answer d

$\frac{1}{14}$

Exercise 4.5.22

Find the reciprocal:

a. $\frac{3}{7}$

b. $-\frac{1}{12}$

c. $-\frac{14}{9}$

d. 21

Answer a

$\frac{7}{3}$

Answer b

-12

Answer c

$-\frac{9}{14}$

Answer d

$\frac{1}{21}$

In a previous chapter, we worked with opposites and absolute values. Table 4.5.1 compares opposites, absolute values, and reciprocals.

Table 4.5.1

Opposite	Absolute Value	Reciprocal
has opposite sign	is never negative	has same sign, fraction inverts

Example 4.5.12: fractions

Fill in the chart for each fraction in the left column:

Number	Opposite	Absolute Value	Reciprocal
$-\frac{3}{8}$			
$\frac{1}{2}$			
$\frac{9}{5}$			
-5			

Solution

To find the opposite, change the sign. To find the absolute value, leave the positive numbers the same, but take the opposite of the negative numbers. To find the reciprocal, keep the sign the same and invert the fraction.

Number	Opposite	Absolute Value	Reciprocal

Number	Opposite	Absolute Value	Reciprocal
$-\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{8}{3}$
$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	2
$\frac{9}{5}$	$-\frac{9}{5}$	$\frac{9}{5}$	$\frac{5}{9}$
-5	5	5	$-\frac{1}{5}$

Exercise 4.5.23

Fill in the chart for each number given:

Number	Opposite	Absolute Value	Reciprocal
$-\frac{5}{8}$			
$\frac{1}{4}$			
$\frac{8}{3}$			
-8			

Answer

Number	Opposite	Absolute Value	Reciprocal
$-\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$-\frac{8}{5}$
$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	4
$\frac{8}{3}$	$-\frac{8}{3}$	$\frac{8}{3}$	$\frac{3}{8}$
-8	8	8	$-\frac{1}{8}$

Exercise 4.5.24

Fill in the chart for each number given:

Number	Opposite	Absolute Value	Reciprocal
$-\frac{4}{7}$			
$\frac{1}{8}$			
$\frac{9}{4}$			
-1			

Answer

Number	Opposite	Absolute Value	Reciprocal
$-\frac{4}{7}$	$\frac{4}{7}$	$\frac{4}{7}$	$-\frac{7}{4}$
$\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	8
$\frac{9}{4}$	$-\frac{9}{4}$	$\frac{9}{4}$	$\frac{4}{9}$
-1	1	1	$-\frac{1}{1}$

Divide Fractions

Why is $12 \div 3 = 4$? We previously modeled this with counters. How many groups of 3 counters can be made from a group of 12 counters?

Figure 4.5.2

There are 4 groups of 3 counters. In other words, there are four 3s in 12. So, $12 \div 3 = 4$.

What about dividing fractions? Suppose we want to find the quotient: $\frac{1}{2} \div \frac{1}{6}$. We need to figure out how many $\frac{1}{6}$s there are in $\frac{1}{2}$. We can use fraction tiles to model this division. We start by lining up the half and sixth fraction tiles as shown in Figure 4.5.3. Notice, there are three $\frac{1}{6}$ tiles in $\frac{1}{2}$, so $\frac{1}{2} \div \frac{1}{6} = 3$.

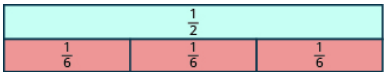


Figure 4.5.3

Example 4.5.13: model

Model: $\frac{1}{4} \div \frac{1}{8}$.

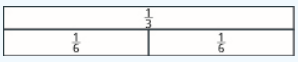
Solution

We want to determine how many $\frac{1}{8}$s are in $\frac{1}{4}$. Start with one $\frac{1}{4}$ tile. Line up $\frac{1}{8}$ tiles underneath the $\frac{1}{4}$ tile.

Exercise 4.5.25

Model: $\frac{1}{3} \div \frac{1}{6}$.

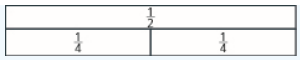
Answer



Exercise 4.5.26

Model: $\frac{1}{2} \div \frac{1}{4}$.

Answer

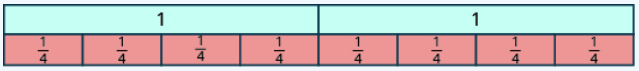


Example 4.5.14: model

Model: $2 \div \frac{1}{4}$.

Solution

We are trying to determine how many $\frac{1}{4}$s there are in 2. We can model this as shown.

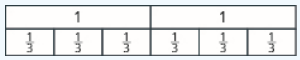


Because there are eight $\frac{1}{4}$s in 2, $2 \div \frac{1}{4} = 8$.

Exercise 4.5.27

Model: $2 \div \frac{1}{3}$

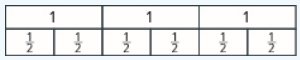
Answer



Exercise 4.5.28

Model: $3 \div \frac{1}{2}$

Answer



Let's use money to model $2 \div \frac{1}{4}$ in another way. We often read $\frac{1}{4}$ as a 'quarter', and we know that a quarter is one-fourth of a dollar as shown in Figure 4.5.4. So we can think of $2 \div \frac{1}{4}$ as, "How many quarters are there in two dollars?" One dollar is 4 quarters, so 2 dollars would be 8 quarters. So again, $2 \div \frac{1}{4} = 8$.

Figure 4.5.4: The U.S. coin called a quarter is worth one-fourth of a dollar.

Using fraction tiles, we showed that $\frac{1}{2} \div \frac{1}{6} = 3$. Notice that $\frac{1}{2} \cdot \frac{6}{1} = 3$ also. How are $\frac{1}{6}$ and $\frac{6}{1}$ related? They are reciprocals. This leads us to the procedure for fraction division.

Definition: Fraction Division

If a, b, c, and d are numbers where $b \neq 0$, $c \neq 0$, and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \tag{4.5.1}$$

To divide fractions, multiply the first fraction by the reciprocal of the second.

We need to say $b \neq 0$, $c \neq 0$ and $d \neq 0$ to be sure we don't divide by zero.

Example 4.5.15: divide

Divide, and write the answer in simplified form: \(\dfrac{2}{5} \div \left(- \dfrac{3}{7}\right).

Solution

Multiply the first fraction by the reciprocal of the second.	$\frac{2}{5}\left(-\frac{7}{3}\right)$
Multiply. The product is negative.	$-\frac{14}{15}$

Exercise 4.5.29

Divide, and write the answer in simplified form: $\frac{3}{7} \div \left(-\frac{2}{3}\right)$.

Answer

$-\frac{9}{14}$

Exercise 4.5.30

Divide, and write the answer in simplified form: $\frac{2}{3} \div \left(-\frac{7}{5}\right)$.

Answer

$-\frac{10}{21}$

Example 4.5.16: divide

Divide, and write the answer in simplified form: $\frac{2}{3} \div \frac{n}{5}$.

Solution

Multiply the first fraction by the reciprocal of the second.	$\frac{2}{3} \div \frac{5}{n}$
Multiply.	$\frac{10}{3n}$

Exercise 4.5.31

Divide, and write the answer in simplified form: $\frac{3}{5} \div \frac{p}{7}$.

Answer

$\frac{21}{5p}$

Exercise 4.5.32

Divide, and write the answer in simplified form: $\frac{5}{8} \div \frac{q}{3}$.

Answer

$\frac{15}{8q}$

Example 4.5.17: divide

Divide, and write the answer in simplified form: $-\frac{3}{4} \div \left(-\frac{7}{8}\right)$.

Solution

Multiply the first fraction by the reciprocal of the second.	$-\frac{3}{4} \cdot \left(-\frac{8}{7}\right)$
Multiply. Remember to determine the sign first.	$\frac{3 \cdot 8}{4 \cdot 7}$
Rewrite to show common factors.	$\frac{3 \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 7}$
Remove common factors and simplify.	$\frac{6}{7}$

Exercise 4.5.33

Divide, and write the answer in simplified form: $-\frac{2}{3} \div \left(-\frac{5}{6}\right)$.

Answer

$\frac{4}{5}$

Exercise 4.5.34

Divide, and write the answer in simplified form: $-\frac{5}{6} \div \left(-\frac{2}{3}\right)$.

Answer

$\frac{5}{4}$

Example 4.5.18: divide

Divide, and write the answer in simplified form: $\frac{7}{18} \div \frac{14}{27}$.

Solution

Multiply the first fraction by the reciprocal of the second.	$\frac{7}{18} \cdot \frac{27}{14}$
Multiply.	$\frac{7 \cdot 27}{18 \cdot 14}$

Rewrite showing common factors.	$\frac{\cancel{7}\cdot\cancel{9}\cdot 3}{\cancel{9}\cdot\cancel{7}\cdot 2}$
Remove common factors.	$\frac{3}{2\cdot 2}$
Simplify.	$\frac{3}{4}$

Exercise 4.5.35

Divide, and write the answer in simplified form: $\frac{7}{27} \div \frac{35}{36}$.

Answer

$\frac{4}{15}$

Exercise 4.5.36

Divide, and write the answer in simplified form: $\frac{5}{14} \div \frac{15}{28}$.

Answer

$\frac{2}{3}$

Access Additional Online Resources

- Simplifying Fractions
- Multiplying Fractions (Positive Only)
- Multiplying Signed Fractions
- Dividing Fractions (Positive Only)
- Dividing Signed Fractions

Key Concepts

- **Equivalent Fractions Property**
- If a, b, c are numbers where $b \neq 0, c \neq 0$, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$
- **Simplify a fraction.**
  1. Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers.
  2. Simplify, using the equivalent fractions property, by removing common factors.
  3. Multiply any remaining factors.
- **Fraction Multiplication**
  - If a, b, c, and dd are numbers where $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
- **Reciprocal**
  - A number and its reciprocal have a product of 1. $\frac{a}{b} \cdot \frac{b}{a} = 1$
  -

Opposite	Absolute Value	Reciprocal
has opposite sign	is never negative	has same sign, fraction inverts

- **Fraction Division**
  - If a, b, c, and d are numbers where $b \neq 0$, $c \neq 0$, and $d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$
  - To divide fractions, multiply the first fraction by the reciprocal of the second.

Glossary

reciprocal

The reciprocal of the fraction $\frac{a}{b}$ is $\frac{b}{a}$ where $a \neq 0$ and $b \neq 0$.

simplified fraction

A fraction is considered simplified if there are no common factors in the numerator and denominator.

Practice Makes Perfect

Simplify Fractions

In the following exercises, simplify each fraction. Do not convert any improper fractions to mixed numbers.

77. $\frac{7}{21}$
78. $\frac{8}{24}$
79. $\frac{15}{20}$
80. $\frac{12}{18}$
81. $-\frac{40}{88}$
82. $-\frac{63}{99}$
83. $-\frac{108}{63}$
84. $-\frac{104}{48}$
85. $\frac{120}{252}$
86. $\frac{182}{294}$
87. $-\frac{168}{192}$
88. $-\frac{140}{224}$
89. $\frac{11x}{11y}$
90. $\frac{15a}{15b}$
91. $-\frac{3x}{12y}$
92. $-\frac{4x}{32y}$
93. $\frac{14x^2}{21y}$
94. $\frac{24a}{32b^2}$

Multiply Fractions

In the following exercises, use a diagram to model.

95. $\frac{1}{2} \cdot \frac{2}{3}$
96. $\frac{1}{2} \cdot \frac{5}{8}$
97. $\frac{1}{3} \cdot \frac{5}{6}$
98. $\frac{1}{3} \cdot \frac{2}{5}$

In the following exercises, multiply, and write the answer in simplified form.

99. $\frac{2}{5} \cdot \frac{1}{3}$
100. $\frac{1}{2} \cdot \frac{3}{8}$
101. $\frac{3}{4} \cdot \frac{9}{10}$
102. $\frac{4}{5} \cdot \frac{2}{7}$
103. $-\frac{2}{3}\left(-\frac{3}{8}\right)$
104. $-\frac{3}{4}\left(-\frac{4}{9}\right)$
105. $-\frac{5}{9} \cdot \frac{3}{10}$
106. $-\frac{3}{8} \cdot \frac{4}{15}$
107. $-\frac{7}{12}\left(-\frac{8}{21}\right)$
108. $\frac{5}{12}\left(-\frac{8}{15}\right)$
109. $\left(-\frac{14}{15}\right)\left(\frac{9}{20}\right)$
110. $\left(-\frac{9}{10}\right)\left(\frac{25}{33}\right)$
111. $\left(-\frac{63}{84}\right)\left(-\frac{44}{90}\right)$
112. $\left(-\frac{33}{60}\right)\left(-\frac{40}{88}\right)$
113. $4 \cdot \frac{5}{11}$
114. $5 \cdot \frac{8}{3}$
115. $\frac{3}{7} \cdot 21n$
116. $\frac{5}{6} \cdot 30m$
117. $-28p\left(-\frac{1}{4}\right)$
118. $-51q\left(-\frac{1}{3}\right)$
119. $-8\left(\frac{17}{4}\right)$
120. $\frac{14}{5}(-15)$
121. $-1\left(-\frac{3}{8}\right)$
122. $(-1)\left(-\frac{6}{7}\right)$

123. $\left(\frac{2}{3}\right)^3$
124. $\left(\frac{4}{5}\right)^2$
125. $\left(\frac{6}{5}\right)^4$
126. $\left(\frac{4}{7}\right)^4$

Find Reciprocals In the following exercises, find the reciprocal.

127. $\frac{3}{4}$
128. $\frac{2}{3}$
129. $-\frac{5}{17}$
130. $-\frac{6}{19}$
131. $\frac{11}{8}$
132. -13
133. -19
134. -1
135. 1
136. Fill in the chart.

	Opposite	Absolute Value	Reciprocal
$-\frac{7}{11}$			
$\frac{4}{5}$			
$\frac{10}{7}$			
-8			

137. Fill in the chart.

	Opposite	Absolute Value	Reciprocal
$-\frac{3}{13}$			
$\frac{9}{14}$			
$\frac{15}{7}$			
-9			

Divide Fractions

In the following exercises, model each fraction division.

138. $\frac{1}{2} \div \frac{1}{4}$
139. $\frac{1}{2} \div \frac{1}{8}$
140. $2 \div \frac{1}{5}$

141. $3 \div \frac{1}{4}$

In the following exercises, divide, and write the answer in simplified form.

142. $\frac{1}{2} \div \frac{1}{4}$
143. $\frac{1}{2} \div \frac{1}{8}$
144. $\frac{3}{4} \div \frac{2}{3}$
145. $\frac{4}{5} \div \frac{3}{4}$
146. $-\frac{4}{5} \div \frac{4}{7}$
147. $-\frac{3}{4} \div \frac{3}{5}$
148. $-\frac{7}{9} \div \left(-\frac{7}{9}\right)$
149. $-\frac{5}{6} \div \left(-\frac{5}{6}\right)$
150. $\frac{3}{4} \div \frac{x}{11}$
151. $\frac{2}{5} \div \frac{y}{9}$
152. $\frac{5}{8} \div \frac{a}{10}$
153. $\frac{5}{6} \div \frac{c}{15}$
154. $\frac{5}{18} \div \left(-\frac{15}{24}\right)$
155. $\frac{7}{18} \div \left(-\frac{14}{27}\right)$
156. $\frac{7p}{12} \div \frac{21p}{8}$
157. $\frac{5q}{12} \div \frac{15q}{8}$
158. $\frac{8u}{15} \div \frac{12v}{25}$
159. $\frac{12r}{25} \div \frac{18s}{35}$
160. $-5 \div \frac{1}{2}$
161. $-3 \div \frac{1}{4}$
162. $\frac{3}{4} \div (-12)$
163. $\frac{2}{5} \div (-10)$
164. $-18 \div \left(-\frac{9}{2}\right)$
165. $-15 \div \left(-\frac{5}{3}\right)$
166. $\frac{1}{2} \div \left(-\frac{3}{4}\right) \div \frac{7}{8}$
167. $\frac{11}{2} \div \frac{7}{8} \cdot \frac{2}{11}$

Everyday Math

168. **Baking** A recipe for chocolate chip cookies calls for 3 4 cup brown sugar. Imelda wants to double the recipe.
    a. How much brown sugar will Imelda need? Show your calculation. Write your result as an improper fraction and as a mixed number.
    b. Measuring cups usually come in sets of $\frac{1}{8}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, and 1 cup. Draw a diagram to show two different ways that Imelda could measure the brown sugar needed to double the recipe.
169. **Baking** Nina is making 4 pans of fudge to serve after a music recital. For each pan, she needs 2 3 cup of condensed milk.
    a. How much condensed milk will Nina need? Show your calculation. Write your result as an improper fraction and as a mixed number.
    b. Measuring cups usually come in sets of $\frac{1}{8}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, and 1 cup. Draw a diagram to show two different ways that Nina could measure the condensed milk she needs.
170. **Portions** Don purchased a bulk package of candy that weighs 5 pounds. He wants to sell the candy in little bags that hold $\frac{1}{4}$ pound. How many little bags of candy can he fill from the bulk package?
171. **Portions** Kristen has $\frac{3}{4}$ yards of ribbon. She wants to cut it into equal parts to make hair ribbons for her daughter's 6 dolls. How long will each doll's hair ribbon be?

Writing Exercises

172. Explain how you find the reciprocal of a fraction.
173. Explain how you find the reciprocal of a negative fraction.
174. Rafael wanted to order half a medium pizza at a restaurant. The waiter told him that a medium pizza could be cut into 6 or 8 slices. Would he prefer 3 out of 6 slices or 4 out of 8 slices? Rafael replied that since he wasn't very hungry, he would prefer 3 out of 6 slices. Explain what is wrong with Rafael's reasoning.
175. Give an example from everyday life that demonstrates how $\frac{1}{2} \cdot \frac{2}{3}$ is $\frac{1}{3}$.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify fractions.			
multiply fractions.			
find reciprocals.			
divide fractions.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

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4.6: Find Multiples and Factors (Part 1)

Learning Objectives

- Identify multiples of numbers
- Use common divisibility tests
- Find all the factors of a number
- Identify prime and composite numbers

Be prepared!

Before you get started, take this readiness quiz.

1. Which of the following numbers are counting numbers (natural numbers)? $0, 4, 215$ If you missed this problem, review Example 1.1.1.
2. Find the sum of 3, 5, and 7. If you missed the problem, review Section 2.1.

Identify Multiples of Numbers

Annie is counting the shoes in her closet. The shoes are matched in pairs, so she doesn't have to count each one. She counts by twos: $2, 4, 6, 8, 10, 12$ She has 12 shoes in her closet.

The numbers $2, 4, 6, 8, 10, 12$ are called multiples of 2. Multiples of 2 can be written as the product of a counting number and 2. The first six multiples of 2 are given below.

$$\begin{aligned} 1 \cdot 2 &= 2 \\ 2 \cdot 2 &= 4 \\ 3 \cdot 2 &= 6 \\ 4 \cdot 2 &= 8 \\ 5 \cdot 2 &= 10 \\ 6 \cdot 2 &= 12 \end{aligned}$$

A **multiple of a number** is the product of the number and a counting number. So a multiple of 3 would be the product of a counting number and 3. Below are the first six multiples of 3.

$$\begin{aligned} 1 \cdot 3 &= 3 \\ 2 \cdot 3 &= 6 \\ 3 \cdot 3 &= 9 \\ 4 \cdot 3 &= 12 \\ 5 \cdot 3 &= 15 \\ 6 \cdot 3 &= 18 \end{aligned}$$

We can find the multiples of any number by continuing this process. Table $4.6.1$ shows the multiples of 2 through 9 for the first twelve counting numbers.

Table 4.6.1

Counting Number	1	2	3	4	5	6	7	8	9	10	11	12
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48

Counting Number	1	2	3	4	5	6	7	8	9	10	11	12
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108

Definition: Multiple of a Number

A number is a multiple of n if it is the product of a counting number and n.

Recognizing the patterns for multiples of 2, 5, 10, and 3 will be helpful to you as you continue in this course.

Figure $4.6.1$ shows the counting numbers from 1 to 50. Multiples of 2 are highlighted. Do you notice a pattern?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Figure $4.6.1$: Multiples of 2 between 1 and 50

The last digit of each highlighted number in Figure $4.6.1$ is either 0, 2, 4, 6, or 8. This is true for the product of 2 and any counting number. So, to tell if any number is a multiple of 2 look at the last digit. If it is 0, 2, 4, 6, or 8, then the number is a multiple of 2.

Example 4.6.1: multiples of 2

Determine whether each of the following is a multiple of 2:

a. 489
b. $3,714$

Solution

a.

Is 489 a multiple of 2?	
Is the last digit 0, 2, 4, 6, or 8?	No.
	489 is not a multiple of 2.

b.

Is 3,714 a multiple of 2?	
Is the last digit 0, 2, 4, 6, or 8?	Yes.
	3,714 is a multiple of 2.

Exercise 4.6.1

Determine whether each number is a multiple of 2:

a. 678
b. 21,493

Answer a

yes

Answer b

no

Exercise 4.6.2

Determine whether each number is a multiple of 2:

a. 979
b. 17,780

Answer a

no

Answer b

yes

Now let's look at multiples of 5. Figure 4.6.2 highlights all of the multiples of 5 between 1 and 50. What do you notice about the multiples of 5?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Figure 4.6.2: Multiples of 5 between 1 and 50

All multiples of 5 end with either 5 or 0. Just like we identify multiples of 2 by looking at the last digit, we can identify multiples of 5 by looking at the last digit.

Example 4.6.2: multiples of 5

Determine whether each of the following is a multiple of 5:

a. 579
b. 880

Solution

a.

Is 579 a multiple of 5?	
Is the last digit 5 or 0?	No.
	579 is not a multiple of 5.

b.

Is 880 a multiple of 5?	

Is the last digit 5 or 0?	Yes.
	880 is not a multiple of 5.

Exercise 4.6.3

Determine whether each number is a multiple of 5.

a. 675
b. 1,578

Answer a

yes

Answer b

no

Exercise 4.6.4

Determine whether each number is a multiple of 5.

a. 421
b. 2,690

Answer a

no

Answer b

yes

Figure 4.6.3 highlights the multiples of 10 between 1 and 50. All multiples of 10 all end with a zero.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Figure 4.6.3: Multiples of 10 between 1 and 50

Example 4.6.3: multiples of 10

Determine whether each of the following is a multiple of 10:

a. 425
b. 350

Solution

a.

Is 425 a multiple of 10?	
Is the last digit zero?	No.
	425 is not a multiple of 10.

b.

Is 350 a multiple of 10?	
Is the last digit zero?	Yes.
	350 is a multiple of 10.

Exercise 4.6.5

Determine whether each number is a multiple of 10:

a. 179
b. 3,540

Answer a

no

Answer b

yes

Exercise 4.6.6

Determine whether each number is a multiple of 10:

a. 110
b. 7,595

Answer a

yes

Answer b

no

Figure 4.6.4 highlights multiples of 3. The pattern for multiples of 3 is not as obvious as the patterns for multiples of 2, 5, and 10.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Figure 4.6.4: Multiples of 3 between 1 and 50

Unlike the other patterns we've examined so far, this pattern does not involve the last digit. The pattern for multiples of 3 is based on the sum of the digits. If the sum of the digits of a number is a multiple of 3, then the number itself is a multiple of 3. See Table 4.6.2.

Table 4.6.2

Multiple of 3	3	6	9	12	15	18	21	24
Sum of digits	3	6	9	1 + 2 3	1 + 5 6	1 + 8 9	2 + 1 3	2 + 4 6

Consider the number 42. The digits are 4 and 2, and their sum is $4+2=6$. Since 6 is a multiple of 3, we know that 42 is also a multiple of 3.

Example 4.6.4: multiples of 3

Determine whether each of the given numbers is a multiple of 3:

a. 645
b. 10,519

Solution

a. Is 645 a multiple of 3?

Find the sum of the digits.	6 + 4 + 5 = 15
Is 15 a multiple of 3?	Yes.
If we're not sure, we could add its digits to find out. We can check it by dividing 645 by 3.	645 ÷ 3
The quotient is 215.	3 • 215 = 645

b. Is 10,519 a multiple of 3?

Find the sum of the digits.	1 + 0 + 5 + 1 + 9 = 16
Is 15 a multiple of 3?	No.
So 10,519 is not a multiple of 3 either..	645 ÷ 3
We can check this by dividing by 10,519 by 3.	$3\overline{)10{,}519}$ $3{,}506\text{R}1$

When we divide 10,519 by 3, we do not get a counting number, so 10,519 is not the product of a counting number and 3. It is not a multiple of 3.

Exercise 4.6.7

Determine whether each number is a multiple of 3:

a. 954
b. 3,742

Answer a

yes

Answer b

no

Exercise 4.6.8

Determine whether each number is a multiple of 3:

a. 643
b. 8,379

Answer a

no

Answer b

yes

Look back at the charts where you highlighted the multiples of 2, of 5, and of 10. Notice that the multiples of 10 are the numbers that are multiples of both 2 and 5. That is because $10 = 2 \cdot 5$. Likewise, since $6 = 2 \cdot 3$, the multiples of 6 are the numbers that are multiples of both 2 and 3.

Use Common Divisibility Tests

Another way to say that 375 is a multiple of 5 is to say that 375 is divisible by 5. In fact, $375 \div 5$ is 75, so 375 is $5 \cdot 75$. Notice in Example 4.6.4 that 10,519 is not a multiple 3. When we divided 10,519 by 3 we did not get a counting number, so 10,519 is not divisible by 3.

Definition: Divisibility

If a number m is a multiple of n, then we say that m is divisible by n.

Since multiplication and division are inverse operations, the patterns of multiples that we found can be used as divisibility tests. Table 4.6.3 summarizes divisibility tests for some of the counting numbers between one and ten.

Table 4.6.3: Divisibility tests

A number is divisible by	
2	if the last digit is 0, 2, 4, 6, or 8
3	if the sum of the digits is divisible by 3
5	if the last digit is 5 or 0
6	if divisible by both 2 and 3
10	if the last digit is 0

Example 4.6.5: divisibility

Determine whether 1,290 is divisible by 2, 3, 5, and 10.

Solution

Table 4.6.4 applies the divisibility tests to 1,290. In the far right column, we check the results of the divisibility tests by seeing if the quotient is a whole number.

Table 4.6.4

Divisible by…?	Test	Divisible?	Check
2	Is last digit 0, 2, 4, 6, or 8?	yes	1290 ÷ 2 = 645
3	Is sum of digits divisible by 3? 1 + 2 + 9 + 0 = 12	yes	1290 ÷ 3 = 430
5	Is last digit 5 or 0?	yes	1290 ÷ 5 = 258
10	Is last digit 0?	yes	1290 ÷ 10 = 129

Thus, 1,290 is divisible by 2, 3, 5, and 10.

Exercise 4.6.9

Determine whether the given number is divisible by 2, 3, 5, and 10.

6240

Answer

Divisible by 2, 3, 5, and 10

Exercise 4.6.10

Determine whether the given number is divisible by 2, 3, 5, and 10.

7248

Answer

Divisible by 2 and 3, not 5 or 10

Example 4.6.6: divisibility

Determine whether 5,625 is divisible by 2, 3, 5, and 10.

Solution

Table 4.6.5 applies the divisibility tests to 5,625 and tests the results by finding the quotients.

Table 4.6.5

Divisible by…?	Test	Divisible?	Check
2	Is last digit 0, 2, 4, 6, or 8?	no	5625 ÷ 2 = 2812.5
3	Is sum of digits divisible by 3? 5 + 6 + 2 + 5 = 18	yes	5625 ÷ 3 = 1875
5	Is last digit 5 or 0?	yes	5625 ÷ 5 = 1125
10	Is last digit 0?	no	5625 ÷ 10 = 562.5

Thus, 5,625 is divisible by 3 and 5, but not 2, or 10.

Exercise 4.6.11

Determine whether the given number is divisible by 2, 3, 5, and 10.

4962

Answer

Divisible by 2, 3, not 5 or 10

Exercise 4.6.12

Determine whether the given number is divisible by 2, 3, 5, and 10.

3765

Answer

Divisible by 3 and 5

Find All the Factors of a Number

There are often several ways to talk about the same idea. So far, we've seen that if m is a multiple of n, we can say that m is divisible by n. We know that 72 is the product of 8 and 9, so we can say 72 is a multiple of 8 and 72 is a multiple of 9. We can also

say 72 is divisible by 8 and by 9. Another way to talk about this is to say that 8 and 9 are factors of 72. When we write $72 = 8 \cdot 9$ we can say that we have factored 72.

$$\underbrace{8 \cdot 9}_{\text{factors}} = \underbrace{72}_{\text{product}}$$

Definition: Factors

If $a \bullet b = m$, then a and b are factors of m, and m is the product of a and b.

In algebra, it can be useful to determine all of the factors of a number. This is called factoring a number, and it can help us solve many kinds of problems.

For example, suppose a choreographer is planning a dance for a ballet recital. There are 24 dancers, and for a certain scene, the choreographer wants to arrange the dancers in groups of equal sizes on stage.

In how many ways can the dancers be put into groups of equal size? Answering this question is the same as identifying the factors of 24. Table 4.6.6 summarizes the different ways that the choreographer can arrange the dancers.

Table 4.6.6

Number of Groups	Dancers per Group	Total Dancers
1	24	1 • 24 = 24
2	12	2 • 12= 24
3	8	3 • 8= 24
4	6	4 • 6= 24
6	4	6 • 4= 24
8	3	8 • 3= 24
12	2	12 • 2= 24
24	1	24 • 1= 24

What patterns do you see in Table 4.6.6? Did you notice that the number of groups times the number of dancers per group is always 24? This makes sense, since there are always 24 dancers.

You may notice another pattern if you look carefully at the first two columns. These two columns contain the exact same set of numbers—but in reverse order. They are mirrors of one another, and in fact, both columns list all of the factors of 24, which are:

$$1, 2, 3, 4, 6, 8, 12, 24$$

We can find all the factors of any counting number by systematically dividing the number by each counting number, starting with 1. If the quotient is also a counting number, then the divisor and the quotient are factors of the number. We can stop when the quotient becomes smaller than the divisor.

HOW TO: FIND ALL THE FACTORS OF A COUNTING NUMBER.

Step 1. Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.

- If the quotient is a counting number, the divisor and quotient are a pair of factors.
- If the quotient is not a counting number, the divisor is not a factor.

Step 2. List all the factor pairs.

Step 3. Write all the factors in order from smallest to largest.

Example 4.6.7: factors

Find all the factors of 72.

Solution

Divide 72 by each of the counting numbers starting with 1. If the quotient is a whole number, the divisor and quotient are a pair of factors.

Dividend	Divisor	Quotient	Factors
72	1	72	1, 72
72	2	36	2, 36
72	3	24	3, 24
72	4	18	4, 18
72	5	14.4	–
72	6	12	6, 12
72	7	~10.29	–
72	8	9	8, 9

The next line would have a divisor of 9 and a quotient of 8. The quotient would be smaller than the divisor, so we stop. If we continued, we would end up only listing the same factors again in reverse order. Listing all the factors from smallest to greatest, we have 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, and 72.

Exercise 4.6.13

Find all the factors of the given number: 96

Answer

$1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96$

Exercise 4.6.14

Find all the factors of the given number: 80

Answer

$1, 2, 4, 5, 8, 10, 16, 20, 40, 80$

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4.7: Find Multiples and Factors (Part 2)

Identify Prime and Composite Numbers

Some numbers, like 72, have many factors. Other numbers, such as 7, have only two factors: 1 and the number. A number with only two factors is called a **prime number**. A number with more than two factors is called a **composite number**. The number 1 is neither prime nor composite. It has only one factor, itself.

Definition: Prime Numbers and Composite Numbers

A prime number is a counting number greater than 1 whose only factors are 1 and itself.

A composite number is a counting number that is not prime.

Figure 4.7.5 lists the counting numbers from 2 through 20 along with their factors. The highlighted numbers are prime, since each has only two factors.

Number	Factors	Prime or Composite?		Number	Factor	Prime or Composite?
2	1,2	Prime		12	1,2,3,4,6,12	Composite
3	1,3	Prime		13	1,13	Prime
4	1,2,4	Composite		14	1,2,7,14	Composite
5	1,5	Prime		15	1,3,5,15	Composite
6	1,2,3,6	Composite		16	1,2,4,8,16	Composite
7	1,7	Prime		17	1,17	Prime
8	1,2,4,8	Composite		18	1,2,3,6,9,18	Composite
9	1,3,9	Composite		19	1,19	Prime
10	1,2,5,10	Composite		20	1,2,4,5,10,20	Composite
11	1,11	Prime				

Figure 4.7.5: Factors of the counting numbers from 2 through 20, with prime numbers highlighted

The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. There are many larger prime numbers too. In order to determine whether a number is prime or composite, we need to see if the number has any factors other than 1 and itself. To do this, we can test each of the smaller prime numbers in order to see if it is a factor of the number. If none of the prime numbers are factors, then that number is also prime.

HOW TO: DETERMINE IF A NUMBER IS PRIME.

- Step 1. Test each of the primes, in order, to see if it is a factor of the number.
- Step 2. Start with 2 and stop when the quotient is smaller than the divisor or when a prime factor is found.
- Step 3. If the number has a prime factor, then it is a composite number. If it has no prime factors, then the number is prime.

Example 4.7.8: prime or composite

Identify each number as prime or composite:

a. 83
b. 77

Solution

a. Test each prime, in order, to see if it is a factor of 83, starting with 2, as shown. We will stop when the quotient is smaller than the divisor.

Prime	Test	Factor of 83?
2	Last digit of 83 is not 0, 2, 4, 6, or 8.	No.
3	8 + 3 = 11, and 11 is not divisible by 3.	No.

Prime	Test	Factor of 83?
5	The last digit of 83 is not 5 or 0.	No.
7	83 ÷ 7 = 11.857….	No.
11	83 ÷ 11 = 7.545…	No.

We can stop when we get to 11 because the quotient (7.545 . . .) is less than the divisor. We did not find any prime numbers that are factors of 83, so we know 83 is prime.

b. Test each prime, in order, to see if it is a factor of 77.

Prime	Test	Factor of 77?
2	Last digit is not 0, 2, 4, 6, or 8.	No.
3	7 + 7 = 14, and 14 is not divisible by 3.	No.
5	The last digit is not 5 or 0.	No.
7	77 ÷ 7 = 11	Yes.

Since 77 is divisible by 7, we know it is not a prime number. It is composite.

? Exercise 4.7.15

Identify the number as prime or composite: 91

Answer

composite

? Exercise 4.7.16

Identify the number as prime or composite: 137

Answer

prime

Access Additional Online Resources

- Divisibility Rules
- Factors
- Ex 1: Determine Factors of a Number
- Ex 2: Determine Factors of a Number
- Ex 3: Determine Factors of a Number

Key Concepts

Divisibility Tests	
A number is divisible by	
2	if the last digit is 0, 2, 4, 6, or 8
3	if the sum of the digits is divisible by 3
5	if the last digit is 5 or 0

Divisibility Tests	
6	if divisible by both 2 and 3
10	if the last digit is 0

- Factors If $a \cdot b = m$, then a and b are factors of m, and m is the product of a and b.
- Find all the factors of a counting number.
  - Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.
    - If the quotient is a counting number, the divisor and quotient are a pair of factors.
    - If the quotient is not a counting number, the divisor is not a factor.
  - List all the factor pairs.
  - Write all the factors in order from smallest to largest.
- Determine if a number is prime.
  - Test each of the primes, in order, to see if it is a factor of the number.
  - Start with 2 and stop when the quotient is smaller than the divisor or when a prime factor is found.
  - If the number has a prime factor, then it is a composite number. If it has no prime factors, then the number is prime.

Glossary

multiple of a number

A number is a multiple of n if it is the product of a counting number and n.

divisibility

If a number m is a multiple of n, then we say that m is divisible by n.

prime number

A prime number is a counting number greater than 1 whose only factors are 1 and itself.

composite number

A composite number is a counting number that is not prime.

Practice Makes Perfect

Identify Multiples of Numbers

In the following exercises, list all the multiples less than 50 for the given number.

215. 2
216. 3
217. 4
218. 5
219. 6
220. 7
221. 8
222. 9
223. 10
224. 12

Use Common Divisibility Tests

In the following exercises, use the divisibility tests to determine whether each number is divisible by 2, 3, 4, 5, 6, and 10.

225. 84
226. 96
227. 75
228. 78

229. 168
230. 264
231. 900
232. 800
233. 896
234. 942
235. 375
236. 750
237. 350
238. 550
239. 1430
240. 1080
241. 22,335
242. 39,075

Find All the Factors of a Number

In the following exercises, find all the factors of the given number.

243. 36
244. 42
245. 60
246. 48
247. 144
248. 200
249. 588
250. 576

Identify Prime and Composite Numbers

In the following exercises, determine if the given number is prime or composite.

251. 43
252. 67
253. 39
254. 53
255. 71
256. 119
257. 481
258. 221
259. 209
260. 359
261. 667
262. 1771

Everyday Math

263. **Banking** Frank's grandmother gave him $100 at his high school graduation. Instead of spending it, Frank opened a bank account. Every week, he added $15 to the account. The table shows how much money Frank had put in the account by the end of each week. Complete the table by filling in the blanks.

Weeks after graduation	Total number of dollars Frank put in the account	Simplified Total
0	100	100
1	100 + 15	115
2	100 + 15 • 2	130

Weeks after graduation	Total number of dollars Frank put in the account	Simplified Total
3	100 + 15 • 3	
4	100 + 15 • []	
5	100 + []	
6		
20		
x		

264. **Banking** In March, Gina opened a Christmas club savings account at her bank. She deposited $75 to open the account. Every week, she added $20 to the account. The table shows how much money Gina had put in the account by the end of each week. Complete the table by filling in the blanks.

Weeks after opening the account	Total number of dollars Gina put in the account	Simplified Total
0	75	75
1	75 + 20	95
2	75 + 20 • 2	115
3	75 + 20 • 3	
4	75 + 20 • []	
5	75 + []	
6		
20		
x		

Writing Exercises

265. If a number is divisible by 2 and by 3, why is it also divisible by 6?
266. What is the difference between prime numbers and composite numbers?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify multiples of numbers.			
use common divisibility tests.			
find all the factors of a number.			
identify prime and composite numbers.			

(b) On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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4.8: Prime Factorization and the Least Common Multiple (Part 1)

Learning Objectives

- Find the prime factorization of a composite number
- Find the least common multiple (LCM) of two numbers

be prepared!

Before you get started, take this readiness quiz.

1. Is 810 divisible by $2, 3, 5, 6,$ or 10? If you missed this problem, review Example 2.4.5.
2. Is 127 prime or composite? If you missed this problem, review Example 2.4.8.
3. Write $2 \cdot 2 \cdot 2 \cdot 2$ in exponential notation. If you missed this problem, review Example 2.1.5.

Find the Prime Factorization of a Composite Number

In the previous section, we found the factors of a number. Prime numbers have only two factors, the number 1 and the prime number itself. Composite numbers have more than two factors, and every composite number can be written as a unique product of primes. This is called the **prime factorization** of a number. When we write the prime factorization of a number, we are rewriting the number as a product of primes. Finding the prime factorization of a composite number will help you later in this course.

Definition: Prime Factorization

The prime factorization of a number is the product of prime numbers that equals the number.

You may want to refer to the following list of prime numbers less than 50 as you work through this section.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$$

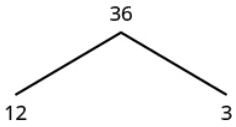
Prime Factorization Using the Factor Tree Method

One way to find the prime factorization of a number is to make a factor tree. We start by writing the number, and then writing it as the product of two factors. We write the factors below the number and connect them to the number with a small line segment—a "branch" of the factor tree.

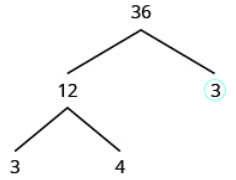
If a factor is prime, we circle it (like a bud on a tree), and do not factor that "branch" any further. If a factor is not prime, we repeat this process, writing it as the product of two factors and adding new branches to the tree.

We continue until all the branches end with a prime. When the factor tree is complete, the circled primes give us the prime factorization.

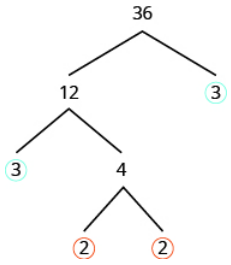
For example, let's find the prime factorization of 36. We can start with any factor pair such as 3 and 12. We write 3 and 12 below 36 with branches connecting them.



The factor 3 is prime, so we circle it. The factor 12 is composite, so we need to find its factors. Let's use 3 and 4. We write these factors on the tree under the 12.



The factor 3 is prime, so we circle it. The factor 4 is composite, and it factors into $2 \bullet 2$. We write these factors under the 4. Since 2 is prime, we circle both $2s$.



The prime factorization is the product of the circled primes. We generally write the prime factorization in order from least to greatest.

$$2 \cdot 2 \cdot 3 \cdot 3$$

In cases like this, where some of the prime factors are repeated, we can write prime factorization in exponential form.

$$2 \cdot 2 \cdot 3 \cdot 3$$

$$2^2 \cdot 3^2$$

Note that we could have started our factor tree with any factor pair of 36. We chose 12 and 3, but the same result would have been the same if we had started with 2 and 18, 4 and 9, or 6 and 6.

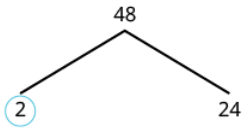
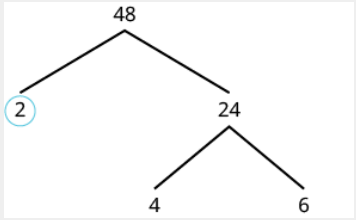
HOW TO: FIND THE PRIME FACTORIZATION OF A COMPOSITE NUMBER USING THE TREE METHOD

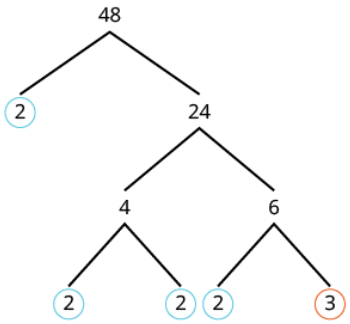
- Step 1. Find any factor pair of the given number, and use these numbers to create two branches.
- Step 2. If a factor is prime, that branch is complete. Circle the prime.
- Step 3. If a factor is not prime, write it as the product of a factor pair and continue the process.
- Step 4. Write the composite number as the product of all the circled primes.

Example 4.8.1: prime factors

Find the prime factorization of 48 using the factor tree method.

Solution

We can start our tree using any factor pair of 48. Let's use 2 and 24. We circle the 2 because it is prime and so that branch is complete.	
Now we will factor 24. Let's use 4 and 6.	

Neither factor is prime, so we do not circle either. We factor the 4, using 2 and 2. We factor 6, using 2 and 3. We circle the 2s and the 3 since they are prime. Now all of the branches end in a prime.	
Write the product of the circled numbers.	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$
Write in exponential form.	$2^4 \cdot 3$

Check this on your own by multiplying all the factors together. The result should be 48.

Exercise 4.8.1

Find the prime factorization using the factor tree method: 80

Answer

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$, or $2^4 \cdot 5$

Exercise 4.8.2

Find the prime factorization using the factor tree method: 60

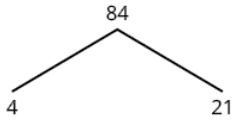
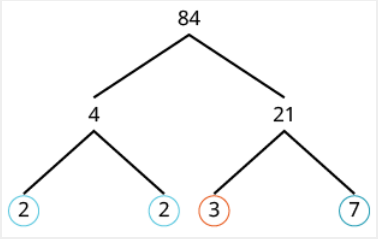
Answer

$2 \cdot 2 \cdot 3 \cdot 5$, or $2^2 \cdot 3 \cdot 5$

Example 4.8.2: prime factors

Find the prime factorization of 84 using the factor tree method.

Solution

We start with the factor pair 4 and 21. Neither factor is prime so we factor them further.	
Now the factors are all prime, so we circle them.	
Then we write 84 as the product of all circled primes.	$2 \cdot 2 \cdot 3 \cdot 7 = 2^2 \cdot 3 \cdot 7$

Draw a factor tree of 84.

Exercise 4.8.3

Find the prime factorization using the factor tree method: 126

Answer

$2 \cdot 3 \cdot 3 \cdot 7$, or $2 \cdot 3^2 \cdot 7$

Exercise 4.8.4

Find the prime factorization using the factor tree method: 294

Answer

$2 \cdot 3 \cdot 7 \cdot 7$, or $2 \cdot 3 \cdot 7^2$

Prime Factorization Using the Ladder Method

The ladder method is another way to find the prime factors of a composite number. It leads to the same result as the factor tree method. Some people prefer the ladder method to the factor tree method, and vice versa.

To begin building the "ladder," divide the given number by its smallest prime factor. For example, to start the ladder for 36, we divide 36 by 2, the smallest prime factor of 36.

$$2\overline{)36}\quad 18$$

To add a "step" to the ladder, we continue dividing by the same prime until it no longer divides evenly.

$$\begin{array}{r} 9 \\ 2\overline{)18} \\ 2\overline{)\ 36} \end{array}$$

Then we divide by the next prime; so we divide 9 by 3.

$$\begin{array}{r} 3 \\ 3\overline{)9} \\ 2\overline{)18} \\ 2\overline{)\ 36} \end{array}$$

We continue dividing up the ladder in this way until the quotient is prime. Since the quotient, 3, is prime, we stop here. Do you see why the ladder method is sometimes called stacked division?

The prime factorization is the product of all the primes on the sides and top of the ladder.

$$2 \cdot 2 \cdot 3 \cdot 3$$

$$2^2 \cdot 3^2$$

Notice that the result is the same as we obtained with the factor tree method.

HOW TO: FIND THE PRIME FACTORIZATION OF A COMPOSITE NUMBER USING THE LADDER METHOD

Step 1. Divide the number by the smallest prime.

Step 2. Continue dividing by that prime until it no longer divides evenly.

Step 3. Divide by the next prime until it no longer divides evenly.

Step 4. Continue until the quotient is a prime.

Step 5. Write the composite number as the product of all the primes on the sides and top of the ladder.

Example 4.8.3: prime factors

Find the prime factorization of 120 using the ladder method.

Solution

Divide the number by the smallest prime, which is 2.	$2\overline{)120}$ (60)
Continue dividing by 2 until it no longer divides evenly.	$3\overline{)15}$ (5); $2\overline{)30}$; $2\overline{)60}$; $2\overline{)120}$
Divide by the next prime, 3.	$2\overline{)30}$ (15); $2\overline{)60}$; $2\overline{)120}$
The quotient, 5, is prime, so the ladder is complete. Write the prime factorization of 120.	$2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$ $2^3 \cdot 3 \cdot 5$

Check this yourself by multiplying the factors. The result should be 120.

Exercise 4.8.5

Find the prime factorization using the ladder method: 80

Answer

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$, or $2^4 \cdot 5$

Exercise 4.8.6

Find the prime factorization using the ladder method: 60

Answer

$2 \cdot 2 \cdot 3 \cdot 5$, or $2^2 \cdot 3 \cdot 5$

Example 4.8.4: prime factors

Find the prime factorization of 48 using the ladder method.

Solution

Divide the number by the smallest prime, 2.	$2\overline{)48}$ (24)
Continue dividing by 2 until it no longer divides evenly.	$3\overline{)6}$ (2); $2\overline{)12}$; $2\overline{)24}$; $2\overline{)48}$
The quotient, 3, is prime, so the ladder is complete. Write the prime factorization of 48.	$2 \bullet 2 \bullet 2 \bullet 2 \bullet 3$ $2^4 \bullet 3$

Exercise 4.8.7

Find the prime factorization using the ladder method: 126

Answer

$2 \cdot 3 \cdot 3 \cdot 7$, or $2 \cdot 3^2 \cdot 7$

Exercise 4.8.8

Find the prime factorization using the ladder method: 294

Answer

$2 \cdot 3 \cdot 7 \cdot 7$, or $2 \cdot 3 \cdot 7^2$

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4.9: Prime Factorization and the Least Common Multiple (Part 2)

Find the Least Common Multiple (LCM) of Two Numbers

One of the reasons we look at multiples and primes is to use these techniques to find the least common multiple of two numbers. This will be useful when we add and subtract fractions with different denominators.

Listing Multiples Method

A common multiple of two numbers is a number that is a multiple of both numbers. Suppose we want to find common multiples of 10 and 25. We can list the first several multiples of each number. Then we look for multiples that are common to both lists—these are the common multiples.

$$10\colon\ 10, 20, 30, 40, \mathbf{50}, 60, 70, 80, 90, \mathbf{100}, 110, \ldots$$
$$25\colon\ 25, \mathbf{50}, 75, \mathbf{100}, 125, \ldots$$

We see that 50 and 100 appear in both lists. They are common multiples of 10 and 25. We would find more common multiples if we continued the list of multiples for each.

The smallest number that is a multiple of two numbers is called the **least common multiple** (LCM). So the least LCM of 10 and 25 is 50.

HOW TO: FIND THE LEAST COMMON MULTIPLE (LCM) OF TWO NUMBERS BY LISTING MULTIPLES

Step 1. List the first several multiples of each number.

Step 2. Look for multiples common to both lists. If there are no common multiples in the lists, write out additional multiples for each number.

Step 3. Look for the smallest number that is common to both lists.

Step 4. This number is the LCM.

Example 4.9.5: lcm

Find the LCM of 15 and 20 by listing multiples.

Solution

List the first several multiples of 15 and of 20. Identify the first common multiple.

$$15\colon\ 15, 30, 45, \mathbf{60}, 75, 90, 105, 120$$
$$20\colon\ 20, 40, \mathbf{60}, 80, 100, 120, 140, 160$$

The smallest number to appear on both lists is 60, so 60 is the least common multiple of 15 and 20. Notice that 120 is on both lists, too. It is a common multiple, but it is not the least common multiple.

Exercise 4.9.9

Find the least common multiple (LCM) of the given numbers: 9 and 12

Answer

36

Exercise 4.9.10

Find the least common multiple (LCM) of the given numbers: 18 and 24

Answer

72

Prime Factors Method

Another way to find the least common multiple of two numbers is to use their prime factors. We'll use this method to find the LCM of 12 and 18.

We start by finding the prime factorization of each number.

$$12 = 2 \cdot 2 \cdot 3 \qquad 18 = 2 \cdot 3 \cdot 3$$

Then we write each number as a product of primes, matching primes vertically when possible.

$$\begin{aligned} 12 &= 2 \cdot 2 \cdot 3 \\ 18 &= 2 \cdot \quad 3 \cdot 3 \end{aligned}$$

Now we bring down the primes in each column. The LCM is the product of these factors.

$$\begin{aligned} 12 &= 2 \cdot 2 \cdot 3 \\ 18 &= 2 \cdot \quad 3 \cdot 3 \\ \hline \text{LCM} &= 2 \cdot 2 \cdot 3 \cdot 3 \\ \text{LCM} &= 2 \cdot 2 \cdot 3 \cdot 3 = 36 \end{aligned}$$

Notice that the prime factors of 12 and the prime factors of 18 are included in the LCM. By matching up the common primes, each common prime factor is used only once. This ensures that 36 is the least common multiple.

HOW TO: FIND THE LCM USING THE PRIME FACTORS METHOD

Step 1. Find the prime factorization of each number.

Step 2. Write each number as a product of primes, matching primes vertically when possible.

Step 3. Bring down the primes in each column.

Step 4. Multiply the factors to get the LCM.

Example 4.9.6: lcm

Find the LCM of 15 and 18 using the prime factors method.

Solution

Write each number as a product of primes.	$15 = 3 \cdot 5 \qquad 18 = 2 \cdot 3 \cdot 3$
Write each number as a product of primes, matching primes vertically when possible.	$\begin{aligned} 15 &= \quad 3 \cdot \quad 5 \\ 18 &= 2 \cdot 3 \cdot 3 \end{aligned}$
Bring down the primes in each column.	$\begin{aligned} 15 &= \quad 3 \cdot \quad 5 \\ 18 &= 2 \cdot 3 \cdot 3 \\ \hline \text{LCM} &= 2 \cdot 3 \cdot 3 \cdot 5 \end{aligned}$
Multiply the factors to get the LCM.	LCM = 2 • 3 • 3 • 5 The LCM of 15 and 18 is 90.

Exercise 4.9.11

Find the LCM using the prime factors method: 15 and 20

Answer

60

Exercise 4.9.12

Find the LCM using the prime factors method: 15 and 35

Answer

105

Example 4.9.7: lcm

Find the LCM of 50 and 100 using the prime factors method.

Solution

Write the prime factorization of each number.	$50 = 2 \cdot 5 \cdot 5 \qquad 100 = 2 \cdot 2 \cdot 5 \cdot 5$
Write each number as a product of primes, matching primes vertically when possible.	$50 = \quad 2 \cdot 5 \cdot 5$ $100 = 2 \cdot 2 \cdot 5 \cdot 5$
Bring down the primes in each column.	$50 = \quad 2 \cdot 5 \cdot 5$ $100 = 2 \cdot 2 \cdot 5 \cdot 5$ $\text{LCM} = 2 \cdot 2 \cdot 5 \cdot 5$
Multiply the factors to get the LCM.	LCM = 2 • 2 • 5 • 5 The LCM of 50 and 100 is 100.

Exercise 4.9.13

Find the LCM using the prime factors method: 55, 88

Answer

440

Exercise 4.9.14

Find the LCM using the prime factors method: 60, 72

Answer

360

Access Additional Online Resources

- Ex 1: Prime Factorization
- Ex 2: Prime Factorization
- Ex 3: Prime Factorization
- Ex 1: Prime Factorization Using Stacked Division
- Ex 2: Prime Factorization Using Stacked Division
- The Least Common Multiple
- Example: Determining the Least Common Multiple Using a List of Multiples
- Example: Determining the Least Common Multiple Using Prime Factorization

Key Concepts

- Find the prime factorization of a composite number using the tree method.
  - Find any factor pair of the given number, and use these numbers to create two branches.
  - If a factor is prime, that branch is complete. Circle the prime.
  - If a factor is not prime, write it as the product of a factor pair and continue the process.
  - Write the composite number as the product of all the circled primes.
- Find the prime factorization of a composite number using the ladder method.
  - Divide the number by the smallest prime.
  - Continue dividing by that prime until it no longer divides evenly.
  - Divide by the next prime until it no longer divides evenly.
  - Continue until the quotient is a prime.
  - Write the composite number as the product of all the primes on the sides and top of the ladder.
- Find the LCM by listing multiples.
  - List the first several multiples of each number.
  - Look for multiples common to both lists. If there are no common multiples in the lists, write out additional multiples for each number.
  - Look for the smallest number that is common to both lists.
  - This number is the LCM.
- Find the LCM using the prime factors method.
  - Find the prime factorization of each number.
  - Write each number as a product of primes, matching primes vertically when possible.
  - Bring down the primes in each column.
  - Multiply the factors to get the LCM.

Glossary

least common multiple

The smallest number that is a multiple of two numbers is called the least common multiple (LCM).

prime factorization

The prime factorization of a number is the product of prime numbers that equals the number.

Practice Makes Perfect

Find the Prime Factorization of a Composite Number

In the following exercises, find the prime factorization of each number using the factor tree method.

267. 86
268. 78
269. 132
270. 455
271. 693
272. 420
273. 115
274. 225
275. 2475
276. 1560

In the following exercises, find the prime factorization of each number using the ladder method.

277. 56
278. 72
279. 168
280. 252

281. 391
282. 400
283. 432
284. 627
285. 2160
286. 2520

In the following exercises, find the prime factorization of each number using any method.

287. 150
288. 180
289. 525
290. 444
291. 36
292. 50
293. 350
294. 144

Find the Least Common Multiple (LCM) of Two Numbers

In the following exercises, find the least common multiple (LCM) by listing multiples.

295. 8, 12
296. 4, 3
297. 6, 15
298. 12, 16
299. 30, 40
300. 20, 30
301. 60, 75
302. 44, 55

In the following exercises, find the least common multiple (LCM) by using the prime factors method.

303. 8, 12
304. 12, 16
305. 24, 30
306. 28, 40
307. 70, 84
308. 84, 90

In the following exercises, find the least common multiple (LCM) using any method.

309. 6, 21
310. 9, 15
311. 24, 30
312. 32, 40

Everyday Math

313. **Grocery shopping** Hot dogs are sold in packages of ten, but hot dog buns come in packs of eight. What is the smallest number of hot dogs and buns that can be purchased if you want to have the same number of hot dogs and buns? (Hint: it is the LCM!)
314. **Grocery shopping** Paper plates are sold in packages of 12 and party cups come in packs of 8. What is the smallest number of plates and cups you can purchase if you want to have the same number of each? (Hint: it is the LCM!)

Writing Exercises

315. Do you prefer to find the prime factorization of a composite number by using the factor tree method or the ladder method? Why?
316. Do you prefer to find the LCM by listing multiples or by using the prime factors method? Why?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the prime factorization of a composite number.			
find the least common multiple (LCM) of two numbers.			

(b) Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

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4.10: Multiply and Divide Mixed Numbers and Complex Fractions (Part 1)

Learning Objectives

- Multiply and divide mixed numbers
- Translate phrases to expressions with fractions
- Simplify complex fractions
- Simplify expressions written with a fraction bar

be prepared!

Before you get started, take this readiness quiz.

1. Divide and reduce, if possible: $(4+5) \div (10-7)$. If you missed this problem, review Example 3.2.8.
2. Multiply and write the answer in simplified form: $\frac{1}{8} \cdot \frac{2}{3}$. If you missed this problem, review Example 4.2.7.
3. Convert $2\frac{3}{5}$ into an improper fraction. If you missed this problem, review Example 4.1.11.

Multiply and Divide Mixed Numbers

In the previous section, you learned how to multiply and divide fractions. All of the examples there used either proper or improper fractions. What happens when you are asked to multiply or divide mixed numbers? Remember that we can convert a mixed number to an improper fraction. And you learned how to do that in Visualize Fractions.

Example 4.10.1: multiply

Multiply: $3\frac{1}{3} \cdot \frac{5}{8}$

Solution

Convert $3\frac{1}{3}$ to an improper fraction.	$\frac{10}{3} \cdot \frac{5}{8}$
Multiply.	$\frac{10 \cdot 5}{3 \cdot 8}$
Look for common factors.	$\frac{\cancel{2} \cdot 5 \cdot 5}{3 \cdot \cancel{2} \cdot 4}$
Remove common factors.	$\frac{5 \cdot 5}{3 \cdot 4}$
Simplify.	$\frac{25}{12}$

Notice that we left the answer as an improper fraction, $\frac{25}{12}$, and did not convert it to a mixed number. In algebra, it is preferable to write answers as improper fractions instead of mixed numbers. This avoids any possible confusion between $2\frac{1}{12}$ and $2 \cdot \frac{1}{12}$.

Exercise 4.10.1

Multiply, and write your answer in simplified form: $5\frac{2}{3} \cdot \frac{6}{17}$.

Answer

2

Exercise 4.10.2

Multiply, and write your answer in simplified form: $\frac{3}{7} \cdot 5\frac{1}{4}$.

Answer

$\frac{9}{4}$

HOW TO: MULTIPLY OR DIVIDE MIXED NUMBERS

Step 1. Convert the mixed numbers to improper fractions.

Step 2. Follow the rules for fraction multiplication or division.

Step 3. Simplify if possible.

Example 4.10.2:

Multiply, and write your answer in simplified form: $2\frac{4}{5}\left(-1\frac{7}{8}\right)$.

Solution

Convert mixed numbers to improper fractions.	$\frac{14}{5}\left(-1\frac{7}{8}\right)$
Multiply.	$-\frac{14 \cdot 15}{5 \cdot 8}$
Look for common factors.	$-\frac{\cancel{2} \cdot 7 \cdot \cancel{5} \cdot 3}{\cancel{5} \cdot \cancel{2} \cdot 4}$
Remove common factors.	$-\frac{7 \cdot 3}{4}$
Simplify.	$-\frac{21}{4}$

Exercise 4.10.3

Multiply, and write your answer in simplified form. $5\frac{5}{7}\left(-2\frac{5}{8}\right)$.

Answer

-15

Exercise 4.10.4

Multiply, and write your answer in simplified form. $-3\frac{2}{5} \cdot 4\frac{1}{6}$.

Answer

$-\frac{85}{6}$

Example 4.10.3: divide

Divide, and write your answer in simplified form: $3\frac{4}{7} \div 5$.

Solution

Convert mixed numbers to improper fractions.	$\frac{25}{7} \div \frac{5}{1}$
Multiply the first fraction by the reciprocal of the second.	$\frac{25}{7} \cdot \frac{1}{5}$
Multiply.	$\frac{25 \cdot 1}{7 \cdot 5}$
Look for common factors.	$\frac{\cancel{5} \cdot 5 \cdot 1}{7 \cdot \cancel{5}}$
Remove common factors.	$\frac{5 \cdot 1}{7}$
Simplify.	$\frac{5}{7}$

Exercise 4.10.5

Divide, and write your answer in simplified form: $4\frac{3}{8} \div 7$.

Answer

$\frac{5}{8}$

Exercise 4.10.6

Divide, and write your answer in simplified form: $2\frac{5}{8} \div 3$.

Answer

$\frac{7}{8}$

Example 4.10.4: divide

Divide: $2\frac{1}{2} \div 1\frac{1}{4}$.

Solution

Convert mixed numbers to improper fractions.	$\frac{5}{2} \div \frac{5}{4}$
Multiply the first fraction by the reciprocal of the second.	$\frac{5}{2} \cdot \frac{4}{5}$
Multiply.	$\frac{5 \cdot 4}{2 \cdot 5}$
Look for common factors.	$\frac{\cancel{5} \cdot \cancel{2} \cdot 2}{\cancel{2} \cdot 1 \cdot \cancel{5}}$

Remove common factors.	$\frac{2}{1}$
Simplify.	2

Exercise 4.10.7

Divide, and write your answer in simplified form: $2\frac{2}{3} \div 1\frac{1}{3}$.

Answer

2

Exercise 4.10.8

Divide, and write your answer in simplified form: $3\frac{3}{4} \div 1\frac{1}{2}$.

Answer

$\frac{5}{2}$

Translate Phrases to Expressions with Fractions

The words *quotient* and *ratio* are often used to describe fractions. In Subtract Whole Numbers, we defined quotient as the result of division. The quotient of a and b is the result you get from dividing a by b, or $\frac{a}{b}$. Let's practice translating some phrases into algebraic expressions using these terms.

Example 4.10.5: translate

Translate the phrase into an algebraic expression: "the quotient of $3x$ and 8."

Solution

The keyword is *quotient*; it tells us that the operation is division. Look for the words *of* and *and* to find the numbers to divide.

The quotient **of** $3x$ **and** 8.

This tells us that we need to divide $3x$ by 8. $\frac{3x}{8}$

Exercise 4.10.9

Translate the phrase into an algebraic expression: the quotient of $9s$ and 14.

Answer

$\frac{9s}{14}$

Exercise 4.10.10

Translate the phrase into an algebraic expression: the quotient of $5y$ and 6.

Answer

$\frac{5y}{6}$

Example 4.10.6:

Translate the phrase into an algebraic expression: the quotient of the difference of m and n, and p.

Solution

We are looking for the *quotient* of the *difference* of m and n, and p. This means we want to divide the difference of m and n by p.

$$\frac{m-n}{p}$$

Exercise 4.10.11

Translate the phrase into an algebraic expression: the quotient of the difference of a and b, and cd.

Answer

$$\frac{a-b}{cd}$$

Exercise 4.10.12

Translate the phrase into an algebraic expression: the quotient of the sum of p and q, and r.

Answer

$$\frac{p+q}{r}$$

Simplify Complex Fractions

Our work with fractions so far has included proper fractions, improper fractions, and mixed numbers. Another kind of fraction is called **complex fraction**, which is a fraction in which the numerator or the denominator contains a fraction. Some examples of complex fractions are:

$$\frac{\frac{6}{7}}{3} \quad \frac{\frac{3}{4}}{\frac{5}{8}} \quad \frac{\frac{x}{2}}{\frac{5}{6}}$$

To simplify a complex fraction, remember that the fraction bar means division. So the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ can be written as $\frac{3}{4} \div \frac{5}{8}$.

Example 4.10.7: simplify

Simplify: $\frac{\frac{3}{4}}{\frac{5}{8}}$.

Solution

Rewrite as division.	$\frac{3}{4} \div \frac{5}{8}$

Multiply the first fraction by the reciprocal of the second.	$\frac{3}{4} \cdot \frac{8}{5}$
Multiply.	$\frac{3 \cdot 8}{4 \cdot 5}$
Look for common factors.	$\frac{3 \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 5}$
Remove common factors and simplify.	$\frac{6}{5}$

Exercise 4.10.13

Simplify: $\frac{\frac{2}{3}}{\frac{5}{6}}$.

Answer

$\frac{4}{5}$

Exercise 4.10.14

Simplify: $\frac{\frac{3}{7}}{\frac{6}{11}}$.

Answer

$\frac{11}{14}$

HOW TO: SIMPLIFY A COMPLEX FRACTION

Step 1. Rewrite the complex fraction as a division problem.

Step 2. Follow the rules for dividing fractions.

Step 3. Simplify if possible.

Example 4.10.8: simplify

Simplify: $\frac{-\frac{6}{7}}{3}$.

Solution

Rewrite as division.	$-\frac{6}{7} \div 3$
Multiply the first fraction by the reciprocal of the second.	$-\frac{6}{7} \cdot \frac{1}{3}$
Multiply; the product will be negative.	$-\frac{6 \cdot 1}{7 \cdot 3}$

Look for common factors.	$-\frac{\cancel{3}\cdot 2\cdot 1}{7\cdot \cancel{3}}$
Remove common factors and simplify.	$-\frac{2}{7}$

Exercise 4.10.15

Simplify: $\frac{-\frac{8}{7}}{4}$.

Answer

$-\frac{2}{7}$

Exercise 4.10.16

Simplify: $-\frac{3}{\frac{9}{10}}$.

Answer

$-\frac{10}{3}$

Example 4.10.9: simplify

Simplify: $\frac{\frac{x}{2}}{\frac{xy}{6}}$.

Solution

Rewrite as division.	$\frac{x}{2}\div\frac{xy}{6}$
Multiply the first fraction by the reciprocal of the second.	$\frac{x}{2}\cdot\frac{6}{xy}$
Multiply.	$\frac{x\cdot 6}{2\cdot xy}$
Look for common factors.	$\frac{\cancel{x}\cdot 3\cdot \cancel{2}}{\cancel{2}\cdot \cancel{x}\cdot y}$
Remove common factors and simplify.	$\frac{3}{y}$

Exercise 4.10.17

Simplify: $\frac{\frac{a}{8}}{\frac{ab}{6}}$.

Answer

$\frac{3}{4b}$

Exercise 4.10.18

Simplify: $\frac{\frac{p}{2}}{\frac{pq}{8}}$.

Answer

$\frac{4}{q}$

Example 4.10.10: simplify

Simplify: $\frac{2\frac{3}{4}}{\frac{1}{8}}$.

Solution

Rewrite as division.	$2\frac{3}{4} \div \frac{1}{8}$
Change the mixed number to an improper fraction.	$\frac{11}{4} \div \frac{1}{8}$
Multiply the first fraction by the reciprocal of the second.	$\frac{11}{4} \cdot \frac{8}{1}$
Multiply.	$\frac{11 \cdot 8}{4 \cdot 1}$
Look for common factors.	$\frac{11 \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 1}$
Remove common factors and simplify.	22

Exercise 4.10.19

Simplify: $\frac{\frac{5}{7}}{1\frac{2}{5}}$.

Answer

$\frac{25}{49}$

Exercise 4.10.20

Simplify: $\frac{\frac{8}{5}}{3\frac{1}{5}}$.

Answer

$\frac{1}{2}$

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4.11: Multiply and Divide Mixed Numbers and Complex Fractions (Part 2)

Simplify Expressions with a Fraction Bar

Where does the negative sign go in a fraction? Usually, the negative sign is placed in front of the fraction, but you will sometimes see a fraction with a negative numerator or denominator. Remember that fractions represent division. The fraction $-\frac{1}{3}$ could be the result of dividing $\frac{-1}{3}$, a negative by a positive, or of dividing $\frac{1}{-3}$, a positive by a negative. When the numerator and denominator have different signs, the quotient is negative.

$$\frac{-1}{3}=-\frac{1}{3} \quad \frac{negative}{positive}=negative \quad \frac{1}{-3}=-\frac{1}{3} \quad \frac{positive}{negative}=negative$$

If *both* the numerator and denominator are negative, then the fraction itself is positive because we are dividing a negative by a negative.

$$\frac{-1}{-3}=\frac{1}{3} \qquad \frac{negative}{negative}=positive$$

Placement of Negative Sign in a Fraction

For any positive numbers a and b,

$$\frac{-a}{b}=\frac{a}{-b}=-\frac{a}{b} \tag{4.11.1}$$

Example 4.11.11: equivalent fractions

Which of the following fractions are equivalent to $\frac{7}{-8}$?

$$\frac{-7}{-8}, \frac{-7}{8}, \frac{7}{8}, -\frac{7}{8}$$

Solution

The quotient of a positive and a negative is a negative, so $\frac{7}{-8}$ is negative. Of the fractions listed, $\frac{-7}{8}$ and $-\frac{7}{8}$ are also negative.

Exercise 4.11.21

Which of the following fractions are equivalent to $\frac{-3}{5}$?

$$\frac{-3}{-5}, \frac{3}{5}, -\frac{3}{5}, \frac{3}{-5}$$

Answer

$$-\frac{3}{5}, \frac{3}{-5}$$

Exercise 4.11.22

Which of the following fractions are equivalent to $-\frac{2}{7}$?

$$\frac{-2}{-7}, \frac{-2}{7}, \frac{2}{7}, \frac{2}{-7}$$

Answer

$\frac{-2}{7}, \frac{2}{-7}$

Fraction bars act as grouping symbols. The expressions above and below the fraction bar should be treated as if they were in parentheses. For example, $\frac{4+8}{5-3}$ means $(4+8) \div (5-3)$. The order of operations tells us to simplify the numerator and the denominator first—as if there were parentheses—before we divide.

We'll add fraction bars to our set of grouping symbols from Use the Language of Algebra to have a more complete set here.

Grouping Symbols

Parentheses	( )
Brackets	[ ]
Braces	{ }
Absolute value	| |
Fraction Bar	$\frac{\square}{\square}$

HOW TO: SIMPLIFY AN EXPRESSION WITH A FRACTION BAR

Step 1. Simplify the numerator.

Step 2. Simplify the denominator.

Step 3. Simplify the fraction.

Example 4.11.12:

Simplify: $\frac{4+8}{5-3}$.

Solution

Simplify the expression in the numerator.	$\frac{12}{5-3}$
Simplify the expression in the denominator.	$\frac{12}{2}$
Simplify the fraction.	6

Exercise 4.11.23

Simplify: $\frac{4+6}{11-2}$.

Answer

$\frac{10}{9}$

Exercise 4.11.24

Simplify: $\frac{3+5}{18-2}$.

Answer

$\frac{1}{2}$

Example 4.11.13:

Simplify: $\frac{4-2(3)}{2^2+2}$.

Solution

Use the order of operations. Multiply in the numerator and use the exponent in the denominator.	$\frac{4-6}{4+2}$
Simplify the numerator and the denominator.	$\frac{-2}{6}$
Simplify the fraction.	$-\frac{1}{3}$

Exercise 4.11.25

Simplify: $\frac{6-3(5)}{3^2+3}$.

Answer

$\frac{-3}{4}$

Exercise 4.11.26

Simplify: $\frac{4-4(6)}{3^3+3}$.

Answer

$-\frac{2}{3}$

Example 4.11.14: simplify

Simplify: $\frac{(8-4)^2}{8^2-4^2}$.

Solution

Use the order of operations (parentheses first, then exponents).	$\frac{(4)^2}{64-16}$
Simplify the numerator and denominator.	$\frac{16}{48}$
Simplify the fraction.	$\frac{1}{3}$

Exercise 4.11.27

Simplify: $\frac{(11-7)^2}{11^2-7^2}$.

Answer

$\frac{2}{9}$

Exercise 4.11.28

Simplify: $\frac{(6+2)^2}{6^2-2^2}$.

Answer

$\frac{8}{5}$

Example 4.11.15: simplify

Simplify: $\frac{4(-3)+6(-2)}{-3(2)-2}$.

Solution

Multiply.	$\frac{-12+(-12)}{-6-2}$
Simplify.	$\frac{-24}{-8}$
Divide.	3

Exercise 4.11.29

Simplify: $\frac{8(-2)+4(-3)}{-5(2)+3}$.

Answer

4

Exercise 4.11.30

Simplify: $\frac{7(-1)+9(-3)}{-5(3)+2}$.

Answer

2

Access Additional Online Resources

- Division Involving Mixed Numbers
- Evaluate a Complex Fraction

Key Concepts

- Multiply or divide mixed numbers.
  1. Convert the mixed numbers to improper fractions.
  2. Follow the rules for fraction multiplication or division.
  3. Simplify if possible.
- Simplify a complex fraction.
  1. Rewrite the complex fraction as a division problem.
  2. Follow the rules for dividing fractions.
  3. Simplify if possible.
- Placement of negative sign in a fraction.
  - For any positive numbers a and b, $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.
- Simplify an expression with a fraction bar.
  1. Simplify the numerator.
  2. Simplify the denominator.
  3. Simplify the fraction.

Glossary

complex fraction

A complex fraction is a fraction in which the numerator or the denominator contains a fraction.

Practice Makes Perfect

Multiply and Divide Mixed Numbers

In the following exercises, multiply and write the answer in simplified form.

176. $4\frac{3}{8} \cdot \frac{7}{10}$
177. $2\frac{4}{9} \cdot \frac{6}{7}$
178. $\frac{15}{22} \cdot 3\frac{3}{5}$
179. $\frac{25}{36} \cdot 6\frac{3}{10}$
180. $4\frac{2}{3}(-1\frac{1}{8})$
181. $2\frac{2}{5}(-2\frac{2}{9})$
182. $-4\frac{4}{9} \cdot 5\frac{13}{16}$
183. $-1\frac{7}{20} \cdot 2\frac{11}{12}$

In the following exercises, divide, and write your answer in simplified form.

184. $5\frac{1}{3} \div 4$
185. $13\frac{1}{2} \div 9$
186. $-12 \div 3\frac{3}{11}$
187. $-7 \div 5\frac{1}{4}$
188. $6\frac{3}{8} \div 2\frac{1}{8}$
189. $2\frac{1}{5} \div 1\frac{1}{10}$

190. $-9\frac{3}{5} \div (-1\frac{3}{5})$
191. $-18\frac{3}{4} \div (-3\frac{3}{4})$

Translate Phrases to Expressions with Fractions

In the following exercises, translate each English phrase into an algebraic expression.

192. the quotient of 5u and 11
193. the quotient of 7v and 13
194. the quotient of p and q
195. the quotient of a and b
196. the quotient of r and the sum of s and 10
197. the quotient of A and the difference of 3 and B

Simplify Complex Fractions

In the following exercises, simplify the complex fraction.

198. $\frac{\frac{2}{3}}{\frac{8}{9}}$
199. $\frac{\frac{4}{5}}{\frac{8}{15}}$
200. $\frac{-\frac{8}{21}}{\frac{12}{35}}$
201. $\frac{-\frac{9}{16}}{\frac{33}{40}}$
202. $\frac{-\frac{4}{5}}{2}$
203. $\frac{-\frac{9}{10}}{3}$
204. $\frac{2}{\frac{5}{8}}$
205. $\frac{\frac{5}{3}}{10}$
206. $\frac{\frac{m}{3}}{n}$
207. $\frac{\frac{2}{r}}{\frac{5}{s}}$
208. $\frac{-\frac{x}{6}}{-\frac{8}{9}}$

209. $\dfrac{-\frac{3}{8}}{-\frac{y}{12}}$

210. $\dfrac{2\frac{4}{5}}{\frac{1}{10}}$

211. $\dfrac{4\frac{2}{3}}{\frac{1}{6}}$

212. $\dfrac{\frac{7}{9}}{-2\frac{4}{5}}$

213. $\dfrac{\frac{3}{8}}{-6\frac{3}{4}}$

Simplify Expressions with a Fraction Bar

In the following exercises, identify the equivalent fractions.

214. Which of the following fractions are equivalent to $\dfrac{5}{-11}$?

$$\frac{-5}{-11}, \frac{-5}{11}, \frac{5}{11}, -\frac{5}{11} \tag{4.11.2}$$

215. Which of the following fractions are equivalent to $\dfrac{-4}{9}$?

$$\frac{-4}{-9}, \frac{-4}{9}, \frac{4}{9}, -\frac{4}{9} \tag{4.11.3}$$

216. Which of the following fractions are equivalent to $-\dfrac{11}{3}$?

$$\frac{-11}{3}, \frac{11}{3}, \frac{-11}{-3}, \frac{11}{-3} \tag{4.11.4}$$

217. Which of the following fractions are equivalent to $-\dfrac{13}{6}$?

$$\frac{13}{6}, \frac{13}{-6}, \frac{-13}{-6}, \frac{-13}{6} \tag{4.11.5}$$

In the following exercises, simplify.

218. $\dfrac{4+11}{8}$

219. $\dfrac{9+3}{7}$

220. $\dfrac{22+3}{10}$

221. $\dfrac{19-4}{6}$

222. $\dfrac{48}{24-15}$

223. $\dfrac{46}{4+4}$

224. $\dfrac{-6+6}{8+4}$

225. $\frac{-6+3}{17-8}$
226. $\frac{22-14}{19-13}$
227. $\frac{15+9}{18+12}$
228. $\frac{5\cdot 8}{-10}$
229. $\frac{3\cdot 4}{-24}$
230. $\frac{4\cdot 3}{6\cdot 6}$
231. $\frac{6\cdot 6}{9\cdot 2}$
232. $\frac{4^2-1}{25}$
233. $\frac{7^2+1}{60}$
234. $\frac{8\cdot 3+2\cdot 9}{14+3}$
235. $\frac{9\cdot 6-4\cdot 7}{22+3}$
236. $\frac{15\cdot 5-5^2}{2\cdot 10}$
237. $\frac{12\cdot 9-3^2}{3\cdot 18}$
238. $\frac{5\cdot 6-3\cdot 4}{4\cdot 5-2\cdot 3}$
239. $\frac{8\cdot 9-7\cdot 6}{5\cdot 6-9\cdot 2}$
240. $\frac{5^2-3^2}{3-5}$
241. $\frac{6^2-4^2}{4-6}$
242. $\frac{2+4(3)}{-3-2^2}$
243. $\frac{7+3(5)}{-2-3^2}$
244. $\frac{7\cdot 4-2(8-5)}{9\cdot 3-3\cdot 5}$
245. $\frac{9\cdot 7-3(12-8)}{8\cdot 7-6\cdot 6}$
246. $\frac{9(8-2)-3(15-7)}{6(7-1)-3(17-9)}$
247. $\frac{8(9-2)-4(14-9)}{7(8-3)-3(16-9)}$

Everyday Math

248. **Baking** A recipe for chocolate chip cookies calls for $2\frac{1}{4}$ cups of flour. Graciela wants to double the recipe.
    a. How much flour will Graciela need? Show your calculation. Write your result as an improper fraction and as a mixed number.
    b. Measuring cups usually come in sets with cups for $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, and 1 cup. Draw a diagram to show two different ways that Graciela could measure out the flour needed to double the recipe.

249. **Baking** A booth at the county fair sells fudge by the pound. Their award winning "Chocolate Overdose" fudge contains $2\frac{2}{3}$ cups of chocolate chips per pound.
    a. How many cups of chocolate chips are in a half-pound of the fudge?
    b. The owners of the booth make the fudge in 10 -pound batches. How many chocolate chips do they need to make a 10 -pound batch? Write your results as improper fractions and as a mixed numbers.

Writing Exercises

250. Explain how to find the reciprocal of a mixed number.
251. Explain how to multiply mixed numbers.
252. Randy thinks that $3\frac{1}{2} \cdot 5\frac{1}{4}$ is $15\frac{1}{8}$. Explain what is wrong with Randy's thinking.
253. Explain why $-\frac{1}{2}$, $\frac{-1}{2}$, and $\frac{1}{-2}$ are equivalent.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
multiply and divide mixed numbers.			
translate phrases to expressions with fractions.			
simplify complex fractions.			
simplify expressions written with a fraction bar.			

(b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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4.12: Add and Subtract Fractions with Common Denominators

Learning Objectives

- Model fraction addition
- Add fractions with a common denominator
- Model fraction subtraction
- Subtract fractions with a common denominator

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: $2x+9+3x-4$. If you missed this problem, review Example 2.2.10.
2. Draw a model of the fraction $\frac{3}{4}$. If you missed this problem, review Example 4.1.2.
3. Simplify: $\frac{3+2}{6}$. If you missed this problem, review Example 4.3.12.

Model Fraction Addition

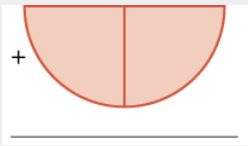
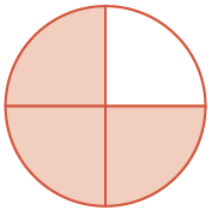
How many quarters are pictured? One quarter plus 2 quarters equals 3 quarters.

Figure 4.12.1

Remember, quarters are really fractions of a dollar. Quarters are another way to say fourths. So the picture of the coins shows that

$$\begin{matrix} \frac{1}{4} & \frac{2}{4} & \frac{3}{4} \\ one\ quarter + two\ quarters & = & three\ quarters \end{matrix}$$

Let's use fraction circles to model the same example, $\frac{1}{4}+\frac{2}{4}$.

Start with one $\frac{1}{4}$ piece.		$\frac{1}{4}$
Add two more $\frac{1}{4}$ pieces.		$+\frac{2}{4}$
The result is $\frac{3}{4}$.		$\frac{3}{4}$

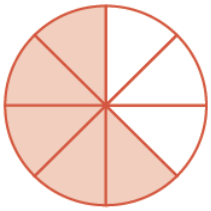
So again, we see that

$$\frac{1}{4}+\frac{2}{4}=\frac{3}{4}$$

Example 4.12.1: addition

Use a model to find the sum $\frac{3}{8}+\frac{2}{8}$.

Solution

Start with three $\frac{1}{8}$ pieces.		$\frac{3}{8}$
Add two $\frac{1}{8}$ pieces.		$+\frac{2}{8}$
How many $\frac{1}{8}$ pieces are there?		$\frac{5}{8}$

There are five $\frac{1}{8}$ pieces, or five-eighths. The model shows that $\frac{3}{8}+\frac{2}{8}=\frac{5}{8}$.

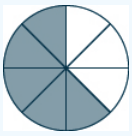
Exercise 4.12.1

Use a model to find each sum. Show a diagram to illustrate your model.

$$\frac{1}{8}+\frac{4}{8}$$

Answer

$\frac{5}{8}$



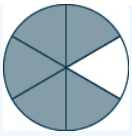
Exercise 4.12.2

Use a model to find each sum. Show a diagram to illustrate your model.

$$\frac{1}{6}+\frac{4}{6}$$

Answer

$\frac{5}{6}$



Add Fractions with a Common Denominator

Example 4.12.1 shows that to add the same-size pieces—meaning that the fractions have the same denominator—we just add the number of pieces.

Definition: Fraction Addition

If a, b, and c are numbers where $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \tag{4.12.1}$$

To add fractions with a common denominator, add the numerators and place the sum over the common denominator.

Example 4.12.2: addition

Find the sum: $\frac{3}{5} + \frac{1}{5}$.

Solution

Add the numerators and place the sum over the common denominator.	$\frac{3+1}{5}$
Simplify.	$\frac{4}{5}$

Exercise 4.12.3

Find each sum: $\frac{3}{6} + \frac{2}{6}$.

Answer

$\frac{5}{6}$

Exercise 4.12.4

Find each sum: $\frac{3}{10} + \frac{7}{10}$.

Answer

1

Example 4.12.3: addition

Find the sum: $\frac{x}{3} + \frac{2}{3}$.

Solution

Add the numerators and place the sum over the common denominator.	$\frac{x+2}{3}$

Note that we cannot simplify this fraction any more. Since x and 2 are not like terms, we cannot combine them.

? Exercise 4.12.5

Find the sum: $\frac{x}{4}+\frac{3}{4}$.

Answer

$\frac{x+3}{4}$

? Exercise 4.12.6

Find the sum: $\frac{y}{8}+\frac{5}{8}$.

Answer

$\frac{y+5}{8}$

✓ Example 4.12.4: addition

Find the sum: $-\frac{9}{d}+\frac{3}{d}$.

Solution

We will begin by rewriting the first fraction with the negative sign in the numerator.

$$-\frac{a}{b}=\frac{-a}{b}$$

Rewrite the first fraction with the negative in the numerator.	$\frac{-9}{d}+\frac{3}{d}$
Add the numerators and place the sum over the common denominator.	$\frac{-9+3}{d}$
Simplify the numerator.	$\frac{-6}{d}$
Rewrite with negative sign in front of the fraction.	$-\frac{6}{d}$

? Exercise 4.12.7

Find the sum: $-\frac{7}{d}+\frac{8}{d}$.

Answer

$\frac{1}{d}$

Exercise 4.12.8

Find the sum: $-\frac{6}{m}+\frac{9}{m}$.

Answer

$\frac{3}{m}$

Example 4.12.5: addition

Find the sum: $\frac{2n}{11}+\frac{5n}{11}$.

Solution

Add the numerators and place the sum over the common denominator.	$\frac{2n+5n}{11}$
Combine like terms.	$\frac{7n}{11}$

Exercise 4.12.9

Find the sum: $\frac{3p}{8}+\frac{6p}{8}$.

Answer

$\frac{9p}{8}$

Exercise 4.12.10

Find the sum: $\frac{2q}{5}+\frac{7q}{5}$.

Answer

$\frac{9q}{5}$

Example 4.12.6: addition

Find the sum: $-\frac{3}{12}+\left(-\frac{5}{12}\right)$.

Solution

Add the numerators and place the sum over the common denominator.	$\frac{-3+(-5)}{12}$
Add.	$\frac{-8}{12}$
Simplify the fraction.	$-\frac{2}{3}$

Exercise 4.12.11

Find each sum: $-\frac{4}{15}+\left(-\frac{6}{15}\right)$.

Answer

$-\frac{2}{3}$

Exercise 4.12.12

Find each sum: $-\frac{5}{21}+\left(-\frac{9}{21}\right)$.

Answer

$-\frac{2}{3}$

Model Fraction Subtraction

Subtracting two fractions with common denominators is much like adding fractions. Think of a pizza that was cut into 12 slices. Suppose five pieces are eaten for dinner. This means that, after dinner, there are seven pieces (or $\frac{7}{12}$ of the pizza) left in the box. If Leonardo eats 2 of these remaining pieces (or $\frac{2}{12}$ of the pizza), how much is left? There would be 5 pieces left (or $\frac{5}{12}$ of the pizza).

$$\frac{7}{12}-\frac{2}{12}=\frac{5}{12}$$

Let's use fraction circles to model the same example, $\frac{7}{12}-\frac{2}{12}$. Start with seven $\frac{1}{12}$ pieces. Take away two $\frac{1}{12}$ pieces. How many twelfths are left?

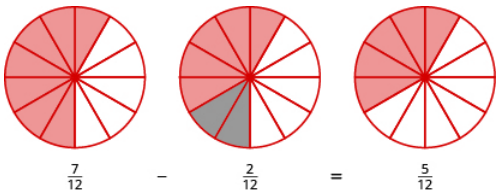


Figure 4.12.2

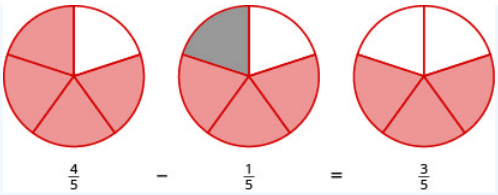
Again, we have five twelfths, $\frac{5}{12}$.

Example 4.12.7: difference

Use fraction circles to find the difference: $\frac{4}{5}-\frac{1}{5}$.

Solution

Start with four $\frac{1}{5}$ pieces. Take away one $\frac{1}{5}$ piece. Count how many fifths are left. There are three $\frac{1}{5}$ pieces left.



Exercise 4.12.13

Use a model to find each difference. Show a diagram to illustrate your model. $\frac{7}{8}-\frac{4}{8}$

Answer

$\frac{3}{8}$, models may differ.

Exercise 4.12.14

Use a model to find each difference. Show a diagram to illustrate your model. $\frac{5}{6}-\frac{4}{6}$

Answer

$\frac{1}{6}$, models may differ.

Subtract Fractions with a Common Denominator

We subtract fractions with a common denominator in much the same way as we add fractions with a common denominator.

Definition: Fraction Subtraction

If a, b, and c are numbers where $c \neq 0$, then

$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c} \tag{4.12.2}$$

To subtract fractions with a common denominator, we subtract the numerators and place the difference over the common denominator.

Example 4.12.8: difference

Find the difference: $\frac{23}{24}-\frac{14}{24}$.

Solution

Subtract the numerators and place the difference over the common denominator.	$\frac{23-14}{24}$
Simplify the numerator.	$\frac{9}{24}$
Simplify the fraction by removing common factors.	$\frac{3}{8}$

Exercise 4.12.15

Find the difference: $\frac{19}{28} - \frac{7}{28}$.

Answer

$\frac{3}{7}$

Exercise 4.12.16

Find the difference: $\frac{27}{32} - \frac{11}{32}$.

Answer

$\frac{1}{2}$

Example 4.12.9: difference

Find the difference: $\frac{y}{6} - \frac{1}{6}$.

Solution

Subtract the numerators and place the difference over the common denominator.	$\frac{y-1}{6}$

The fraction is simplified because we cannot combine the terms in the numerator.

Exercise 4.12.17

Find the difference: $\frac{x}{7} - \frac{2}{7}$.

Answer

$\frac{x-2}{7}$

Exercise 4.12.18

Find the difference: $\frac{y}{14} - \frac{13}{14}$.

Answer

$\frac{y-13}{14}$

Example 4.12.10: difference

Find the difference: $-\frac{10}{x} - \frac{4}{x}$.

Solution

Remember, the fraction $-\frac{10}{x}$ can be written as $\frac{-10}{x}$.

Subtract the numerators.	$\frac{-10-4}{x}$
Simplify.	$\frac{-14}{x}$
Rewrite with the negative sign in front of the fraction.	$-\frac{14}{x}$

Exercise 4.12.19

Find the difference: $-\frac{9}{x}-\frac{7}{x}$.

Answer

$-\frac{16}{x}$

Exercise 4.12.20

Find the difference: $-\frac{17}{a}-\frac{5}{a}$.

Answer

$-\frac{22}{a}$

Now lets do an example that involves both addition and subtraction.

Example 4.12.11: simplify

Simplify: $\frac{3}{8}+\left(-\frac{5}{8}\right)-\frac{1}{8}$.

Solution

Combine the numerators over the common denominator.	$\frac{3+(-5)-1}{8}$
Simplify the numerator, working left to right.	$\frac{-2-1}{8}$
Subtract the terms in the numerator.	$\frac{-3}{8}$
Rewrite with the negative sign in front of the fraction.	$-\frac{3}{8}$

Exercise 4.12.21

Simplify: $\frac{2}{5}+\left(-\frac{4}{5}\right)-\frac{3}{5}$.

Answer

-1

Exercise 4.12.22

Simplify: $\frac{5}{9}+\left(-\frac{4}{9}\right)-\frac{7}{9}$.

Answer

$-\frac{2}{3}$

Access Additional Online Resources

- Adding Fractions With Pattern Blocks
- Adding Fractions With Like Denominators
- Subtracting Fractions With Like Denominators

Key Concepts

- Fraction Addition
  - If $a, b,$, and c are numbers where $c \neq 0$, then $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$
  - To add fractions, add the numerators and place the sum over the common denominator.
- Fraction Subtraction
  - If $a, b,$, and c are numbers where $c \neq 0$, then $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$
  - To subtract fractions, subtract the numerators and place the difference over the common denominator.

Practice Makes Perfect

Model Fraction Addition

In the following exercises, use a model to add the fractions. Show a diagram to illustrate your model.

254. $\frac{2}{5}+\frac{1}{5}$
255. $\frac{3}{10}+\frac{4}{10}$
256. $\frac{1}{6}+\frac{3}{6}$
257. $\frac{3}{8}+\frac{3}{8}$

Add Fractions with a Common Denominator

In the following exercises, find each sum.

258. $\frac{4}{9}+\frac{1}{9}$
259. $\frac{2}{9}+\frac{5}{9}$
260. $\frac{6}{13}+\frac{7}{13}$
261. $\frac{9}{15}+\frac{7}{15}$
262. $\frac{x}{4}+\frac{3}{4}$
263. $\frac{y}{3}+\frac{2}{3}$
264. $\frac{7}{p}+\frac{9}{p}$

265. $\frac{8}{q}+\frac{6}{q}$
266. $\frac{8b}{9}+\frac{3b}{9}$
267. $\frac{5a}{7}+\frac{4a}{7}$
268. $\frac{-12y}{8}+\frac{3y}{8}$
269. $\frac{-11x}{5}+\frac{7x}{5}$
270. $-\frac{1}{8}+\left(-\frac{3}{8}\right)$
271. $-\frac{1}{8}+\left(-\frac{5}{8}\right)$
272. $-\frac{3}{16}+\left(-\frac{7}{16}\right)$
273. $-\frac{5}{16}+\left(-\frac{9}{16}\right)$
274. $-\frac{8}{17}+\frac{15}{17}$
275. $-\frac{9}{19}+\frac{17}{19}$
276. $-\frac{6}{13}+\left(-\frac{10}{13}\right)+\left(-\frac{12}{13}\right)$
277. $-\frac{5}{12}+\left(-\frac{7}{12}\right)+\left(-\frac{11}{12}\right)$

Model Fraction Subtraction

In the following exercises, use a model to subtract the fractions. Show a diagram to illustrate your model.

278. $\frac{5}{8}-\frac{2}{8}$
279. $\frac{5}{6}-\frac{2}{6}$

Subtract Fractions with a Common Denominator

In the following exercises, find the difference.

280. $\frac{4}{5}-\frac{1}{5}$
281. $\frac{4}{5}-\frac{3}{5}$
282. $\frac{11}{15}-\frac{7}{15}$
283. $\frac{9}{13}-\frac{4}{13}$
284. $\frac{11}{12}-\frac{5}{12}$
285. $\frac{7}{12}-\frac{5}{12}$
286. $\frac{4}{21}-\frac{19}{21}$
287. $-\frac{8}{9}-\frac{16}{9}$
288. $\frac{y}{17}-\frac{9}{17}$
289. $\frac{x}{19}-\frac{8}{19}$
290. $\frac{5y}{8}-\frac{7}{8}$

291. $\frac{11z}{13} - \frac{8}{13}$
292. $-\frac{8}{d} - \frac{3}{d}$
293. $-\frac{7}{c} - \frac{7}{c}$
294. $-\frac{23}{u} - \frac{15}{u}$
295. $-\frac{29}{v} - \frac{26}{v}$
296. $-\frac{6c}{7} - \frac{5c}{7}$
297. $-\frac{12d}{11} - \frac{9d}{11}$
298. $\frac{-4r}{13} - \frac{5r}{13}$
299. $\frac{-7s}{3} - \frac{7s}{3}$
300. $-\frac{3}{5} - \left(-\frac{4}{5}\right)$
301. $-\frac{3}{7} - \left(-\frac{5}{7}\right)$
302. $-\frac{7}{9} - \left(-\frac{5}{9}\right)$
303. $-\frac{8}{11} - \left(-\frac{5}{11}\right)$

Mixed Practice

In the following exercises, perform the indicated operation and write your answers in simplified form.

304. $-\frac{5}{18} \cdot \frac{9}{10}$
305. $-\frac{3}{14} \cdot \frac{7}{12}$
306. $\frac{n}{5} - \frac{4}{5}$
307. $\frac{6}{11} - \frac{s}{11}$
308. $-\frac{7}{24} - \frac{2}{24}$
309. $-\frac{5}{18} - \frac{1}{18}$
310. $\frac{8}{15} \div \frac{12}{5}$
311. $\frac{7}{12} \div \frac{9}{28}$

Everyday Math

312. **Trail Mix** Jacob is mixing together nuts and raisins to make trail mix. He has $\frac{6}{10}$ of a pound of nuts and $\frac{3}{10}$ of a pound of raisins. How much trail mix can he make?
313. **Baking** Janet needs $\frac{5}{8}$ of a cup of flour for a recipe she is making. She only has $\frac{3}{8}$ of a cup of flour and will ask to borrow the rest from her next-door neighbor. How much flour does she have to borrow?

Writing Exercises

314. Greg dropped his case of drill bits and three of the bits fell out. The case has slots for the drill bits, and the slots are arranged in order from smallest to largest. Greg needs to put the bits that fell out back in the case in the empty slots. Where do the three bits go? Explain how you know.

Bits in case: $\frac{1}{16}, \frac{1}{8}, \_\_\_, \_\_\_, \frac{5}{16}, \frac{3}{8}, \_\_\_, \frac{1}{2}, \frac{9}{16}, \frac{5}{8}$.

Bits that fell out: $\frac{7}{16}, \frac{3}{16}, \frac{1}{4}$.

315. After a party, Lupe has $\frac{5}{12}$ of a cheese pizza, $\frac{4}{12}$ of a pepperoni pizza, and $\frac{4}{12}$ of a veggie pizza left. Will all the slices fit into 1 pizza box? Explain your reasoning.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
model fraction addition.			
add fractions with a common denominator.			
model fraction subtraction.			
subtract fractions with a common denominator.			
find the least common denominator (LCD).			
convert fractions to equivalent fractions with the LCD.			

(b) On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Contributors and Attributions

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4.13: Add and Subtract Fractions with Different Denominators (Part 1)

Learning Objectives

- Find the least common denominator (LCD)
- Convert fractions to equivalent fractions with the LCD
- Add and subtract fractions with different denominators
- Identify and use fraction operations
- Use the order of operations to simplify complex fractions
- Evaluate variable expressions with fractions

be prepared!

Before you get started, take this readiness quiz.

1. Find two fractions equivalent to $\frac{5}{6}$. If you missed this problem, review Example 4.1.14.
2. Simplify: $\frac{1+5\cdot 3}{2^2+4}$. If you missed this problem, review Example 4.3.13.

Find the Least Common Denominator

In the previous section, we explained how to add and subtract fractions with a common denominator. But how can we add and subtract fractions with unlike denominators?

Let's think about coins again. Can you add one quarter and one dime? You could say there are two coins, but that's not very useful. To find the total value of one quarter plus one dime, you change them to the same kind of unit—cents. One quarter equals 25 cents and one dime equals 10 cents, so the sum is 35 cents. See Figure 4.13.1.

Figure 4.13.1: Together, a quarter and a dime are worth 35 cents, or $\frac{35}{100}$ of a dollar.

Similarly, when we add fractions with different denominators we have to convert them to equivalent fractions with a common denominator. With the coins, when we convert to cents, the denominator is 100. Since there are 100 cents in one dollar, 25 cents is $\frac{25}{100}$ and 10 cents is $\frac{10}{100}$. So we add $\frac{25}{100}+\frac{10}{100}$ to get $\frac{35}{100}$, which is 35 cents.

You have practiced adding and subtracting fractions with common denominators. Now let's see what you need to do with fractions that have different denominators.

First, we will use fraction tiles to model finding the common denominator of $\frac{1}{2}$ and $\frac{1}{3}$. We'll start with one $\frac{1}{2}$ tile and $\frac{1}{3}$ tile. We want to find a common fraction tile that we can use to match *both* $\frac{1}{2}$ and $\frac{1}{3}$ exactly. If we try the $\frac{1}{4}$ pieces, 2 of them exactly match the $\frac{1}{2}$ piece, but they do not exactly match the $\frac{1}{3}$ piece.

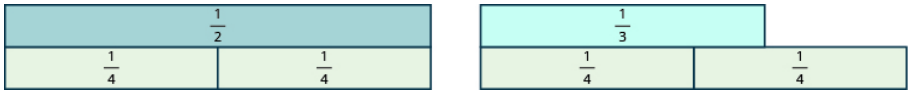


Figure 4.13.2

If we try the $\frac{1}{5}$ pieces, they do not exactly cover the $\frac{1}{2}$ piece or the $\frac{1}{3}$ piece.

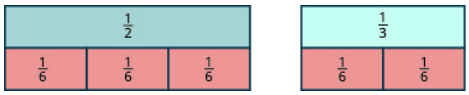


Figure 4.13.3

If we were to try the $\frac{1}{12}$ pieces, they would also work.

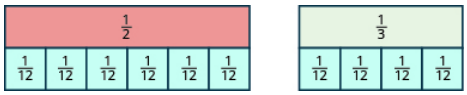


Figure 4.13.4

Even smaller tiles, such as $\frac{1}{24}$ and $\frac{1}{48}$, would also exactly cover the $\frac{1}{2}$ piece and the $\frac{1}{3}$ piece. The denominator of the largest piece that covers both fractions is the **least common denominator** (LCD) of the two fractions. So, the least common denominator of $\frac{1}{2}$ and $\frac{1}{3}$ is 6.

Notice that all of the tiles that cover $\frac{1}{2}$ and $\frac{1}{3}$ have something in common: Their denominators are common multiples of 2 and 3, the denominators of $\frac{1}{2}$ and $\frac{1}{3}$. The least common multiple (LCM) of the denominators is 6, and so we say that 6 is the least common denominator (LCD) of the fractions $\frac{1}{2}$ and $\frac{1}{3}$.

Definition: Least Common Denominator

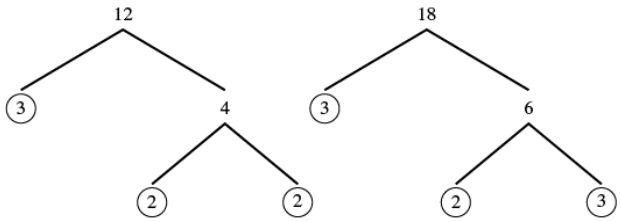
The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

To find the LCD of two fractions, we will find the LCM of their denominators. We follow the procedure we used earlier to find the LCM of two numbers. We only use the denominators of the fractions, not the numerators, when finding the LCD.

Example 4.13.1: lcd

Find the LCD for the fractions $\frac{7}{12}$ and $\frac{5}{18}$.

Solution

Factor each denominator into its primes.	
List the primes of 12 and the primes of 18 lining them up in columns when possible.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$
Bring down the columns.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 3 \cdot 3$
Multiply the factors. The product is the LCM.	LCM = 36
The LCM of 12 and 18 is 36, so the LCD of $\frac{7}{12}$ and $\frac{5}{18}$ is 36.	LCD of $\frac{7}{12}$ and $\frac{5}{18}$ is 36.

Exercise 4.13.1

Find the least common denominator for the fractions: $\frac{7}{12}$ and $\frac{11}{15}$.

Answer

60

Exercise 4.13.2

Find the least common denominator for the fractions: $\frac{13}{15}$ and $\frac{17}{5}$.

Answer

15

To find the LCD of two fractions, find the LCM of their denominators. Notice how the steps shown below are similar to the steps we took to find the LCM.

HOW TO: FIND THE LEAST COMMON DENOMINATOR (LCD) OF TWO FRACTIONS

Step 1. Factor each denominator into its primes.

Step 2. List the primes, matching primes in columns when possible.

Step 3. Bring down the columns.

Step 4. Multiply the factors. The product is the LCM of the denominators.

Step 5. The LCM of the denominators is the LCD of the fractions.

Example 4.13.2:

Find the least common denominator for the fractions $\frac{8}{15}$ and $\frac{11}{24}$.

Solution

To find the LCD, we find the LCM of the denominators. Find the LCM of 15 and 24.

$$\begin{aligned} 15 &= 3 \cdot 5 \\ 24 &= 2 \cdot 2 \cdot 2 \cdot 3 \\ \hline \text{LCD} &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \\ \text{LCD} &= 120 \end{aligned}$$

The LCM of 15 and 24 is 120. So, the LCD of $\frac{8}{15}$ and $\frac{11}{24}$ is 120.

Exercise 4.13.3

Find the least common denominator for the fractions: $\frac{13}{24}$ and $\frac{17}{32}$.

Answer

96

Exercise 4.13.4

Find the least common denominator for the fractions: $\frac{9}{28}$ and $\frac{21}{32}$.

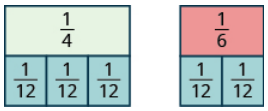
Answer

224

Convert Fractions to Equivalent Fractions with the LCD

Earlier, we used fraction tiles to see that the LCD of $\frac{1}{4}$ and $\frac{1}{6}$ is 12. We saw that three $\frac{1}{12}$ pieces exactly covered $\frac{1}{4}$ and two $\frac{1}{12}$ pieces exactly covered $\frac{1}{6}$, so

$$\frac{1}{4}=\frac{3}{12} \quad and \quad \frac{1}{6}=\frac{2}{12}.$$



We say that $\frac{1}{4}$ and $\frac{3}{12}$ are equivalent fractions and also that $\frac{1}{6}$ and $\frac{2}{12}$ are equivalent fractions.

We can use the Equivalent Fractions Property to algebraically change a fraction to an equivalent one. Remember, two fractions are equivalent if they have the same value. The Equivalent Fractions Property is repeated below for reference.

Definition: Equivalent Fractions Property

If a, b, c are whole numbers where $b \neq 0$, $c \neq 0$, then

$$\frac{a}{b}=\frac{a \cdot c}{b \cdot c} \quad and \quad \frac{a \cdot c}{b \cdot c}=\frac{a}{b} \tag{4.13.1}$$

To add or subtract fractions with different denominators, we will first have to convert each fraction to an equivalent fraction with the LCD. Let's see how to change $\frac{1}{4}$ and $\frac{1}{6}$ to equivalent fractions with denominator 12 without using models.

Example 4.13.3: conversion

Convert $\frac{1}{4}$ and $\frac{1}{6}$ to equivalent fractions with denominator 12, their LCD.

Solution

Find the LCD.	The LCD of $\frac{1}{4}$ and $\frac{1}{6}$ is 12.
Find the number to multiply 4 to get 12.	$4 \cdot 3 = 12$
Find the number to multiply 6 to get 12.	$6 \cdot 2 = 12$
Use the Equivalent Fractions Property to convert each fraction to an equivalent fraction with the LCD, multiplying both the numerator and denominator of each fraction by the same number.	$\frac{1}{4}$ $\quad$ $\frac{1}{6}$ $\frac{1 \cdot 3}{4 \cdot 3}$ $\quad$ $\frac{1 \cdot 2}{6 \cdot 2}$
Simplify the numerators and denominators.	$\frac{3}{12}$ $\quad$ $\frac{2}{12}$

We do not reduce the resulting fractions. If we did, we would get back to our original fractions and lose the common denominator.

Exercise 4.13.5

Change to equivalent fractions with the LCD: $\frac{3}{4}$ and $\frac{5}{6}$, $LCD = 12$

Answer

$\frac{9}{12}, \frac{10}{12}$

Exercise 4.13.6

Change to equivalent fractions with the LCD: $-\frac{7}{12}$ and $\frac{11}{15}$, $LCD = 60$

Answer

$-\frac{35}{60}, \frac{44}{60}$

HOW TO: CONVERT TWO FRACTIONS TO EQUIVALENT FRACTIONS WITH THEIR LCD AS THE COMMON DENOMINATOR

Step 1. Find the LCD.

Step 2. For each fraction, determine the number needed to multiply the denominator to get the LCD.

Step 3. Use the Equivalent Fractions Property to multiply both the numerator and denominator by the number you found in Step 2.

Step 4. Simplify the numerator and denominator.

Example 4.13.4: conversion

Convert $\frac{8}{15}$ and $\frac{11}{24}$ to equivalent fractions with denominator 120, their LCD.

Find the number that must multiply 15 to get 120.	$15 \cdot 8 = 120$
Find the number that must multiply 24 to get 120.	$24 \cdot 5 = 120$
Use the Equivalent Fractions Property.	$\frac{8 \cdot 8}{15 \cdot 8}$ $\frac{11 \cdot 5}{24 \cdot 5}$
Simplify the numerators and denominators.	$\frac{64}{120}$ $\frac{55}{120}$

Exercise 4.13.7

Change to equivalent fractions with the LCD: $\frac{13}{24}$ and $\frac{17}{32}$, LCD 96

Answer

$\frac{52}{96}, \frac{51}{96}$

Exercise 4.13.8

Change to equivalent fractions with the LCD: $\frac{9}{28}$ and $\frac{27}{32}$, LCD 224

Answer

$\frac{72}{224}, \frac{189}{224}$

Add and Subtract Fractions with Different Denominators

Once we have converted two fractions to equivalent forms with common denominators, we can add or subtract them by adding or subtracting the numerators.

HOW TO: ADD OR SUBTRACT FRACTIONS WITH DIFFERENT DENOMINATORS

Step 1. Find the LCD.

Step 2. Convert each fraction to an equivalent form with the LCD as the denominator.

Step 3. Add or subtract the fractions.

Step 4. Write the result in simplified form.

Example 4.13.5: add

Add: $\frac{1}{2}+\frac{1}{3}$.

Solution

Find the LCD of 2, 3.	$2=2$ $3=\quad 3$ $\text{LCD}=2\cdot 3$ $\text{LCD}=6$
Change into equivalent fractions with the LCD 6.	$\frac{1\cdot 3}{2\cdot 3}+\frac{1\cdot 2}{3\cdot 2}$
Simplify the numerators and denominators.	$\frac{3}{6}+\frac{2}{6}$
Add.	$\frac{5}{6}$

Remember, always check to see if the answer can be simplified. Since 5 and 6 have no common factors, the fraction $\frac{5}{6}$ cannot be reduced.

Exercise 4.13.9

Add: $\frac{1}{4}+\frac{1}{3}$.

Answer

$\frac{7}{12}$

Exercise 4.13.10

Add: $\frac{1}{2}+\frac{1}{5}$.

Answer

$\frac{7}{10}$

Example 4.13.6: subtract

Subtract: $\frac{1}{2}-\left(-\frac{1}{4}\right)$.

Solution

Find the LCD of 2 and 4.	$\begin{array}{l} 2=2 \\ 4=2\cdot 2 \\ \hline \text{LCD}=2\cdot 2 \\ \text{LCD}=4 \end{array}$
Rewrite as equivalent fractions using the LCD 4.	$\frac{1\cdot 2}{2\cdot 2}-\left(-\frac{1}{4}\right)$
Simplify the first fraction.	$\frac{2}{4}-\left(-\frac{1}{4}\right)$
Subtract.	$\frac{2-(-1)}{4}$
Simplify.	$\frac{3}{4}$

One of the fractions already had the least common denominator, so we only had to convert the other fraction.

Exercise 4.13.11

Subtract: $\frac{1}{2}-\left(-\frac{1}{8}\right)$.

Answer

$\frac{5}{8}$

Exercise 4.13.12

Subtract: $\frac{1}{3}-\left(-\frac{1}{6}\right)$.

Answer

$\frac{1}{2}$

Example 4.13.7: add

Add: $\frac{7}{12}+\frac{5}{18}$.

Solution

Find the LCD of 12 and 18.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3$ $\text{LCD} = 36$
Rewrite as equivalent fractions with the LCD.	$\frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$
Simplify the numerators and denominators.	$\frac{21}{36} + \frac{10}{36}$
Add.	$\frac{31}{36}$

Because 31 is a prime number, it has no factors in common with 36. The answer is simplified.

Exercise 4.13.13

Add: $\frac{7}{12} + \frac{11}{15}$.

Answer

$\frac{79}{60}$

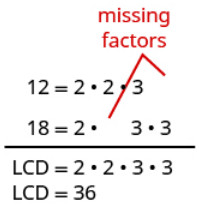
Exercise 4.13.14

Add: $\frac{13}{15} + \frac{17}{20}$.

Answer

$\frac{103}{60}$

When we use the Equivalent Fractions Property, there is a quick way to find the number you need to multiply by to get the LCD. Write the factors of the denominators and the LCD just as you did to find the LCD. The "missing" factors of each denominator are the numbers you need.



The LCD, 36, has 2 factors of 2 and 2 factors of 3. Twelve has two factors of 2, but only one of 3 —so it is 'missing' one 3. We multiplied the numerator and denominator of $\frac{7}{12}$ by 3 to get an equivalent fraction with denominator 36. Eighteen is missing one factor of 2 —so you multiply the numerator and denominator $\frac{5}{18}$ by 2 to get an equivalent fraction with denominator 36. We will apply this method as we subtract the fractions in the next example.

Example 4.13.8: subtract

Subtract: $\frac{7}{15} - \frac{19}{24}$.

Solution

Find the LCD. 15 is 'missing' three factors of 2 24 is 'missing' a factor of 5	$\begin{array}{l} 15 = \quad 3 \cdot 5 \\ 24 = 2 \cdot 2 \cdot 2 \cdot 3 \\ \hline \text{LCD} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \\ \text{LCD} = 120 \end{array}$
Rewrite as equivalent fractions with the LCD.	$\frac{7 \cdot 8}{15 \cdot 8} - \frac{19 \cdot 5}{24 \cdot 5}$
Simplify each numerator and denominator.	$\frac{56}{120} - \frac{95}{120}$
Subtract.	$-\frac{39}{120}$
Rewrite showing the common factor of 3.	$-\frac{13 \cdot 3}{40 \cdot 3}$
Remove the common factor to simplify.	$-\frac{13}{40}$

? Exercise 4.13.15

Subtract: $\frac{13}{24} - \frac{17}{32}$.

Answer

$\frac{1}{96}$

? Exercise 4.13.16

Subtract: $\frac{21}{32} - \frac{9}{28}$.

Answer

$\frac{75}{224}$

✓ Example 4.13.9: add

Add: $-\frac{11}{30} + \frac{23}{42}$.

Solution

Find the LCD.	$\begin{array}{l} 30 = 2 \cdot 3 \cdot 5 \\ 42 = 2 \cdot 3 \cdot \quad 7 \\ \hline \text{LCD} = 2 \cdot 3 \cdot 5 \cdot 7 \\ \text{LCD} = 210 \end{array}$
Rewrite as equivalent fractions with the LCD.	$-\frac{11 \cdot 7}{30 \cdot 7} + \frac{23 \cdot 5}{42 \cdot 5}$
Simplify each numerator and denominator.	$-\frac{77}{210} + \frac{115}{210}$
Add.	$\frac{38}{210}$
Rewrite showing the common factor of 2.	$\frac{19 \cdot 2}{105 \cdot 2}$
Remove the common factor to simplify.	$\frac{19}{105}$

Exercise 4.13.17

Add: $-\frac{13}{42}+\frac{17}{35}$.

Answer

$\frac{37}{210}$

Exercise 4.13.18

Add: $-\frac{19}{24}+\frac{17}{32}$.

Answer

$-\frac{25}{96}$

In the next example, one of the fractions has a variable in its numerator. We follow the same steps as when both numerators are numbers.

Example 4.13.10: add

Add: $\frac{3}{5}+\frac{x}{8}$.

Solution

The fractions have different denominators.

Find the LCD.	$\begin{array}{l} 5= \quad 5 \\ 8=2\cdot 2\cdot 2 \\ \hline \text{LCD}=2\cdot 2\cdot 2\cdot 5 \\ \text{LCD}=40 \end{array}$
Rewrite as equivalent fractions with the LCD.	$\frac{3\cdot 8}{5\cdot 8}+\frac{x\cdot 5}{8\cdot 5}$
Simplify the numerators and denominators.	$\frac{24}{40}+\frac{5x}{40}$
Add.	$\frac{24+5x}{40}$

We cannot add 24 and $5x$ since they are not like terms, so we cannot simplify the expression any further.

Exercise 4.13.19

Add: $\frac{y}{6}+\frac{7}{9}$.

Answer

$\frac{3y+14}{18}$

Exercise 4.13.20

Add: $\frac{x}{6}+\frac{7}{15}$.

Answer

$$\frac{5x+14}{30}$$

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4.14: Add and Subtract Fractions with Different Denominators (Part 2)

Identify and Use Fraction Operations

By now in this chapter, you have practiced multiplying, dividing, adding, and subtracting fractions. The following table summarizes these four fraction operations. Remember: You need a common denominator to add or subtract fractions, but not to multiply or divide fractions.

Summary of Fraction Operations

Fraction multiplication: Multiply the numerators and multiply the denominators.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \tag{4.14.1}$$

Fraction division: Multiply the first fraction by the reciprocal of the second.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \tag{4.14.2}$$

Fraction addition: Add the numerators and place the sum over the common denominator. If the fractions have different denominators, first convert them to equivalent forms with the LCD.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \tag{4.14.3}$$

Fraction subtraction: Subtract the numerators and place the difference over the common denominator. If the fractions have different denominators, first convert them to equivalent forms with the LCD.

$$\frac{a}{c} - \frac{a}{c} = \frac{a-b}{c} \tag{4.14.4}$$

Example 4.14.11: simplify

Simplify:

a. $-\frac{1}{4} + \frac{1}{6}$

b. $-\frac{1}{4} \div \frac{1}{6}$

Solution

First we ask ourselves, “What is the operation?”

a. The operation is addition. Do the fractions have a common denominator? No.

Find the LCD.	$\begin{array}{l} 4 = 2 \cdot 2 \\ 6 = 2 \cdot \quad 3 \\ \hline \text{LCD} = 2 \cdot 2 \cdot 3 \\ \text{LCD} = 12 \end{array}$
Rewrite each fraction as an equivalent fraction with the LCD.	$-\frac{1 \cdot 3}{4 \cdot 3} + \frac{1 \cdot 2}{6 \cdot 2}$
Simplify the numerators and denominators.	$-\frac{3}{12} + \frac{2}{12}$
Add the numerators and place the sum over the common denominator.	$-\frac{1}{12}$
Check to see if the answer can be simplified. It cannot.	

b. The operation is division. We do not need a common denominator.

To divide fractions, multiply the first fraction by the reciprocal of the second.	$-\frac{1}{4} \cdot \frac{6}{1}$
Multiply.	$-\frac{6}{4}$
Simplify.	$-\frac{3}{2}$

Exercise 4.14.21

Simplify:

a. $-\frac{3}{4}-\frac{1}{6}$

b. $-\frac{3}{4} \cdot \frac{1}{6}$

Answer a

$-\frac{11}{12}$

Answer b

$-\frac{1}{8}$

Exercise 4.14.22

Simplify:

a. $\frac{5}{6} \div\left(-\frac{1}{4}\right)$

b. $\frac{5}{6}-\left(-\frac{1}{4}\right)$

Answer a

$-\frac{10}{3}$

Answer b

$\frac{13}{12}$

Example 4.14.12: simplify

Simplify:

a. $\frac{5x}{6}-\frac{3}{10}$

b. $\frac{5x}{6} \cdot \frac{3}{10}$

Solution

a. The operation is subtraction. The fractions do not have a common denominator.

Rewrite each fraction as an equivalent fraction with the LCD, 30.	$\frac{5x \cdot 5}{6 \cdot 5}-\frac{3 \cdot 3}{10 \cdot 3}$

	$\frac{25x}{30}-\frac{9}{30}$
Subtract the numerators and place the difference over the common denominator.	$\frac{25x-9}{30}$

b. The operation is multiplication; no need for a common denominator.

To multiply fractions, multiply the numerators and multiply the denominators.	$\frac{5x\cdot 3}{6\cdot 10}$
Rewrite, showing common factors.	$\frac{\cancel{5}\cdot x\cdot \cancel{3}}{2\cdot \cancel{3}\cdot 2\cdot \cancel{5}}$
Remove common factors to simplify.	$\frac{x}{4}$

? Exercise 4.14.23

Simplify:

a. $\frac{3a}{4}-\frac{8}{9}$

b. $\frac{3a}{4}\cdot\frac{8}{9}$

Answer a

$\frac{27a-32}{36}$

Answer b

$\frac{2a}{3}$

? Exercise 4.14.24

Simplify:

a. $\frac{4k}{5}+\frac{5}{6}$

b. $\frac{4k}{5}\div\frac{5}{6}$

Answer a

$\frac{24k+25}{30}$

Answer b

$\frac{24k}{25}$

Use the Order of Operations to Simplify Complex Fractions

In Multiply and Divide Mixed Numbers and Complex Fractions, we saw that a complex fraction is a fraction in which the numerator or denominator contains a fraction. We simplified complex fractions by rewriting them as division problems. For example,

$$\frac{\frac{3}{4}}{\frac{5}{8}} = \frac{3}{4} \div \frac{5}{8}$$

Now we will look at complex fractions in which the numerator or denominator can be simplified. To follow the order of operations, we simplify the numerator and denominator separately first. Then we divide the numerator by the denominator.

HOW TO: SIMPLIFY COMPLEX FRACTIONS

Step 1. Simplify the numerator.

Step 2. Simplify the denominator.

Step 3. Divide the numerator by the denominator.

Step 4. Simplify if possible.

Example 4.14.13: simplify

Simplify: $\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$.

Solution

Simplify the numerator.	$\frac{\frac{1}{4}}{4+3^2}$
Simplify the term with the exponent in the denominator.	$\frac{\frac{1}{4}}{4+9}$
Add the terms in the denominator.	$\frac{\frac{1}{4}}{13}$
Divide the numerator by the denominator.	$\frac{1}{4} \div 13$
Rewrite as multiplication by the reciprocal.	$\frac{1}{4} \cdot \frac{1}{13}$
Multiply.	$\frac{1}{52}$

Exercise 4.14.25

Simplify: $\frac{\left(\frac{1}{3}\right)^2}{2^3+2}$.

Answer

$\frac{1}{90}$

Exercise 4.14.26

Simplify: $\dfrac{1+4^2}{\left(\frac{1}{4}\right)^2}$.

Answer

272

Example 4.14.14: simplify

Simplify: $\dfrac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$.

Solution

Rewrite numerator with the LCD of 6 and denominator with LCD of 12.	$\dfrac{\frac{3}{6}+\frac{4}{6}}{\frac{9}{12}-\frac{2}{12}}$
Add in the numerator. Subtract in the denominator.	$\dfrac{\frac{7}{6}}{\frac{7}{12}}$
Divide the numerator by the denominator.	$\frac{7}{6} \div \frac{7}{12}$
Rewrite as multiplication by the reciprocal.	$\frac{7}{6} \cdot \frac{12}{7}$
Rewrite, showing common factors.	$\dfrac{\cancel{7} \cdot \cancel{6} \cdot 2}{\cancel{6}\cancel{7} \cdot 1}$
Simplify.	2

Exercise 4.14.27

Simplify: $\dfrac{\frac{1}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$.

Answer

2

Exercise 4.14.28

Simplify: $\dfrac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{4}+\frac{1}{3}}$.

Answer

$\frac{2}{7}$

Evaluate Variable Expressions with Fractions

We have evaluated expressions before, but now we can also evaluate expressions with fractions. Remember, to evaluate an expression, we substitute the value of the variable into the expression and then simplify.

Example 4.14.15: evaluate

Evaluate $x+\frac{1}{3}$ when

a. $x=-\frac{1}{3}$

b. $x=-\frac{3}{4}$

Solution

a. To evaluate $x+\frac{1}{3}$ when $x=-\frac{1}{3}$, substitute $-\frac{1}{3}$ for x in the expression.

Substitute $-\frac{1}{3}$ for x.	$-\frac{1}{3}+\frac{1}{3}$
Simplify.	0

b. To evaluate $x+\frac{1}{3}$ when $x=-\frac{3}{4}$, we substitute $-\frac{3}{4}$ for x in the expression.

Substitute $-\frac{3}{4}$ for x.	$-\frac{1}{3}+\frac{1}{3}$
Rewrite as equivalent fractions with the LCD, 12.	$-\frac{3\cdot 3}{4\cdot 3}+\frac{1\cdot 4}{3\cdot 4}$
Simplify the numerators and denominators.	$-\frac{9}{12}+\frac{4}{12}$
Add.	$-\frac{5}{12}$

Exercise 4.14.29

Evaluate $x+\frac{3}{4}$ when:

a. $x=-\frac{7}{4}$

b. $x=-\frac{5}{4}$

Answer a

-1

Answer b

$-\frac{1}{2}$

Exercise 4.14.30

Evaluate $y+\frac{1}{2}$ when:

a. $y=\frac{2}{3}$

b. $y=-\frac{3}{4}$

Answer a

$\frac{7}{6}$

Answer b

$-\frac{1}{4}$

Example 4.14.16: evaluate

Evaluate $y-\frac{5}{6}$ when $y=-\frac{2}{3}$.

Solution

We substitute $-\frac{2}{3}$ for y in the expression.

Substitute $-\frac{2}{3}$ for y.	$-\frac{2}{3}-\frac{5}{6}$
Rewrite as equivalent fractions with the LCD, 6.	$-\frac{4}{6}-\frac{5}{6}$
Subtract.	$-\frac{9}{6}$
Simplify.	$-\frac{3}{2}$

Exercise 4.14.31

Evaluate $y-\frac{1}{2}$ when $y=-\frac{1}{4}$.

Answer

$-\frac{3}{4}$

Exercise 4.14.32

Evaluate $x-\frac{3}{8}$ when $x=-\frac{5}{2}$.

Answer

$-\frac{23}{8}$

Example 4.14.17:

Evaluate $2x^2y$ when $x = \frac{1}{4}$ and $y = -\frac{2}{3}$.

Solution

Substitute the values into the expression. In $2x^2y$, the exponent applies only to x.

Substitute $\frac{1}{4}$ for x and $-\frac{2}{3}$ for y.	$2\left(\frac{1}{4}\right)^2\left(-\frac{2}{3}\right)$
Simplify exponents first.	$2\left(\frac{1}{16}\right)\left(-\frac{2}{3}\right)$
Multiply. The product will be negative.	$-\frac{2}{1}\cdot\frac{1}{16}\cdot\frac{2}{3}$
Simplify.	$-\frac{4}{48}$
Remove the common factors.	$-\frac{1\cdot\cancel{4}}{\cancel{4}\cdot 12}$
Simplify.	$-\frac{1}{12}$

Exercise 4.14.33

Evaluate: $3ab^2$ when $a = -\frac{2}{3}$ and $b = -\frac{1}{2}$.

Answer

$-\frac{1}{2}$

Exercise 4.14.34

Evaluate: $4c^3d$ when $c = -\frac{1}{2}$ and $d = -\frac{4}{3}$.

Answer

$\frac{2}{3}$

Example 4.14.18: evaluate

Evaluate: $\frac{p+q}{r}$ when $p = -4$, $q = -2$, and $r = 8$.

Solution

We substitute the values into the expression and simplify.

Substitute -4 for p, -2 for q and 8 for r.	$\frac{-4+(-2)}{8}$
Add in the numerator first.	$-\frac{6}{8}$
Simplify.	$-\frac{3}{4}$

Exercise 4.14.35

Evaluate: $\frac{a+b}{c}$ when $a=-8$, $b=-7$, and $c=6$.

Answer

$-\frac{5}{2}$

Exercise 4.14.36

Evaluate: $\frac{x+y}{z}$ when $x=9$, $y=-18$, and $z=-6$.

Answer

$\frac{3}{2}$

Practice Makes Perfect

Find the Least Common Denominator (LCD)

In the following exercises, find the least common denominator (LCD) for each set of fractions.

316. $\frac{2}{3}$ and $\frac{3}{4}$
317. $\frac{3}{4}$ and $\frac{2}{5}$
318. $\frac{7}{12}$ and $\frac{5}{8}$
319. $\frac{9}{16}$ and $\frac{7}{12}$
320. $\frac{13}{30}$ and $\frac{25}{42}$
321. $\frac{23}{30}$ and $\frac{5}{48}$
322. $\frac{21}{35}$ and $\frac{39}{56}$
323. $\frac{18}{35}$ and $\frac{33}{49}$
324. $\frac{2}{3}$, $\frac{1}{6}$ and $\frac{3}{4}$
325. $\frac{2}{3}$, $\frac{1}{4}$ and $\frac{3}{5}$

Convert Fractions to Equivalent Fractions with the LCD

In the following exercises, convert to equivalent fractions using the LCD.

326. $\frac{1}{3}$ and $\frac{1}{4}$, LCD = 12
327. $\frac{1}{4}$ and $\frac{1}{5}$, LCD = 20
328. $\frac{5}{12}$ and $\frac{7}{8}$, LCD = 24
329. $\frac{7}{12}$ and $\frac{5}{8}$, LCD = 24
330. $\frac{13}{16}$ and $-\frac{11}{12}$, LCD = 48
331. $\frac{11}{16}$ and $-\frac{5}{12}$, LCD = 48

332. $\frac{1}{3}$, $\frac{5}{6}$, and $\frac{3}{4}$, LCD = 12
333. $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{3}{5}$, LCD = 60

Add and Subtract Fractions with Different Denominators

In the following exercises, add or subtract. Write the result in simplified form.

334. $\frac{1}{3}+\frac{1}{5}$
335. $\frac{1}{4}+\frac{1}{5}$
336. $\frac{1}{2}+\frac{1}{7}$
337. $\frac{1}{3}+\frac{1}{8}$
338. $\frac{1}{3}-\left(-\frac{1}{9}\right)$
339. $\frac{1}{4}-\left(-\frac{1}{8}\right)$
340. $\frac{1}{5}-\left(-\frac{1}{10}\right)$
341. $\frac{1}{2}-\left(-\frac{1}{6}\right)$
342. $\frac{2}{3}+\frac{3}{4}$
343. $\frac{3}{4}+\frac{2}{5}$
344. $\frac{7}{12}+\frac{5}{8}$
345. $\frac{5}{12}+\frac{3}{8}$
346. $\frac{7}{12}-\frac{9}{16}$
347. $\frac{7}{16}-\frac{5}{12}$
348. $\frac{11}{12}-\frac{3}{8}$
349. $\frac{5}{8}-\frac{7}{12}$
350. $\frac{2}{3}-\frac{3}{8}$
351. $\frac{5}{6}-\frac{3}{4}$
352. $-\frac{11}{30}+\frac{27}{40}$
353. $-\frac{9}{20}+\frac{17}{30}$
354. $-\frac{13}{30}+\frac{25}{42}$
355. $-\frac{23}{30}+\frac{5}{48}$
356. $-\frac{39}{56}-\frac{22}{35}$
357. $-\frac{33}{49}-\frac{18}{35}$
358. $-\frac{2}{3}-\left(-\frac{3}{4}\right)$
359. $-\frac{3}{4}-\left(-\frac{4}{5}\right)$

360. $-\frac{9}{16}-\left(-\frac{4}{5}\right)$
361. $-\frac{7}{20}-\left(-\frac{5}{8}\right)$
362. $1+\frac{7}{8}$
363. $1+\frac{5}{6}$
364. $1-\frac{5}{9}$
365. $1-\frac{3}{10}$
366. $\frac{x}{3}+\frac{1}{4}$
367. $\frac{y}{2}+\frac{2}{3}$
368. $\frac{y}{4}-\frac{3}{5}$
369. $\frac{x}{5}-\frac{1}{4}$

Identify and Use Fraction Operations

In the following exercises, perform the indicated operations. Write your answers in simplified form.

370. (a) $\frac{3}{4}+\frac{1}{6}$ (b) $\frac{3}{4}\div\frac{1}{6}$
371. (a) $\frac{2}{3}+\frac{1}{6}$ (b) $\frac{2}{3}\div\frac{1}{6}$
372. (a) $-\frac{2}{5}-\frac{1}{8}$ (b) $-\frac{2}{5}\cdot\frac{1}{8}$
373. (a) $-\frac{4}{5}-\frac{1}{8}$ (b) $-\frac{4}{5}\cdot\frac{1}{8}$
374. (a) $\frac{5n}{6}\div\frac{8}{15}$ (b) $\frac{5n}{6}-\frac{8}{15}$
375. (a) $\frac{3a}{8}\div\frac{7}{12}$ (b) $\frac{3a}{8}-\frac{7}{12}$
376. (a) $\frac{9}{10}\cdot\left(-\frac{11d}{12}\right)$ (b) $\frac{9}{10}+\left(-\frac{11d}{12}\right)$
377. (a) $\frac{4}{15}\cdot\left(-\frac{5}{q}\right)$ (b) $\frac{4}{15}+\left(-\frac{5}{q}\right)$
378. $-\frac{3}{8}\div\left(-\frac{3}{10}\right)$
379. $-\frac{5}{12}\div\left(-\frac{5}{9}\right)$
380. $-\frac{3}{8}+\frac{5}{12}$
381. $-\frac{1}{8}+\frac{7}{12}$
382. $\frac{5}{6}-\frac{1}{9}$
383. $\frac{5}{9}-\frac{1}{6}$
384. $\frac{3}{8}\cdot\left(-\frac{10}{21}\right)$
385. $\frac{7}{12}\cdot\left(-\frac{8}{35}\right)$
386. $-\frac{7}{15}-\frac{y}{4}$
387. $-\frac{3}{8}-\frac{x}{11}$

388. $\frac{11}{12a} \cdot \frac{9a}{16}$

389. $\frac{10y}{13} \cdot \frac{8}{15y}$

Use the Order of Operations to Simplify Complex Fractions

In the following exercises, simplify.

390. $\frac{\left(\frac{1}{5}\right)^2}{2+3^2}$

391. $\frac{\left(\frac{1}{3}\right)^2}{5+2^2}$

392. $\frac{2^3+4^2}{\left(\frac{2}{3}\right)^2}$

393. $\frac{3^3-3^2}{\left(\frac{3}{4}\right)^2}$

394. $\frac{\left(\frac{3}{5}\right)^2}{\left(\frac{3}{7}\right)^2}$

395. $\frac{\left(\frac{3}{4}\right)^2}{\left(\frac{5}{8}\right)^2}$

396. $\frac{2}{\frac{1}{3}+\frac{1}{5}}$

397. $\frac{5}{\frac{1}{4}+\frac{1}{3}}$

398. $\frac{\frac{2}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{2}{3}}$

399. $\frac{\frac{3}{4}+\frac{1}{2}}{\frac{5}{6}-\frac{2}{3}}$

400. $\frac{\frac{7}{8}-\frac{2}{3}}{\frac{1}{2}+\frac{3}{8}}$

401. $\frac{\frac{3}{4}-\frac{3}{5}}{\frac{1}{4}+\frac{2}{5}}$

Mixed Practice

In the following exercises, simplify.

402. $\frac{1}{2}+\frac{2}{3}\cdot\frac{5}{12}$
403. $\frac{1}{3}+\frac{2}{5}\cdot\frac{3}{4}$
404. $1-\frac{3}{5}\div\frac{1}{10}$
405. $1-\frac{5}{6}\div\frac{1}{12}$
406. $\frac{2}{3}+\frac{1}{6}+\frac{3}{4}$
407. $\frac{2}{3}+\frac{1}{4}+\frac{3}{5}$
408. $\frac{3}{8}-\frac{1}{6}+\frac{3}{4}$
409. $\frac{2}{5}+\frac{5}{8}-\frac{3}{4}$
410. $12\left(\frac{9}{20}-\frac{4}{15}\right)$
411. $8\left(\frac{15}{16}-\frac{5}{6}\right)$
412. $\dfrac{\frac{5}{8}+\frac{1}{6}}{\frac{19}{24}}$
413. $\dfrac{\frac{1}{6}+\frac{3}{10}}{\frac{14}{30}}$
414. $\left(\frac{5}{9}+\frac{1}{6}\right)\div\left(\frac{2}{3}-\frac{1}{2}\right)$
415. $\left(\frac{3}{4}+\frac{1}{6}\right)\div\left(\frac{5}{8}-\frac{1}{3}\right)$

In the following exercises, evaluate the given expression. Express your answers in simplified form, using improper fractions if necessary.

416. $x+\frac{1}{2}$ when
    a. $x=-\frac{1}{8}$
    b. $x=-\frac{1}{2}$
417. $x+\frac{2}{3}$ when
    a. $x=-\frac{1}{6}$
    b. $x=-\frac{5}{3}$
418. $x+\left(-\frac{5}{6}\right)$ when
    a. $x=\frac{1}{3}$
    b. $x=-\frac{1}{6}$
419. $x+\left(-\frac{11}{12}\right)$ when
    a. $x=\frac{11}{12}$

b. $x = \frac{3}{4}$

420. $x - \frac{2}{5}$ when

a. $x = \frac{3}{5}$

b. $x = -\frac{3}{5}$

421. $x - \frac{1}{3}$ when

a. $x = \frac{2}{3}$

b. $x = -\frac{2}{3}$

422. $\frac{7}{10} - w$ when

a. $w = \frac{1}{2}$

b. $w = -\frac{1}{2}$

423. $\frac{5}{12} - w$ when

a. $w = \frac{1}{4}$

b. $w = -\frac{1}{4}$

424. $4p^2q$ when $p = -\frac{1}{2}$ and $q = \frac{5}{9}$

425. $5m^2n$ when $m = -\frac{2}{5}$ and $n = \frac{1}{3}$

426. $2x^2y^3$ when $x = -\frac{2}{3}$ and $y = -\frac{1}{2}$

427. $8u^2v^3$ when $u = -\frac{3}{4}$ and $v = -\frac{1}{2}$

428. $\frac{u+v}{w}$ when $u = -4$, $v = -8$, $w = 2$

429. $\frac{m+n}{p}$ when $m = -6$, $n = -2$, $p = 4$

430. $\frac{a+b}{a-b}$ when $a = -3$, $b = 8$

431. $\frac{r-s}{r+s}$ when $r = 10$, $s = -5$

Everyday Math

432. **Decorating** Laronda is making covers for the throw pillows on her sofa. For each pillow cover, she needs $\frac{3}{16}$ yard of print fabric and $\frac{3}{8}$ yard of solid fabric. What is the total amount of fabric Laronda needs for each pillow cover?

433. **Baking** Vanessa is baking chocolate chip cookies and oatmeal cookies. She needs $1\frac{1}{4}$ cups of sugar for the chocolate chip cookies, and $1\frac{1}{8}$ cups for the oatmeal cookies How much sugar does she need altogether?

Writing Exercises

434. Explain why it is necessary to have a common denominator to add or subtract fractions.

435. Explain how to find the LCD of two fractions.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract fractions with different denominators.			
identify and use fraction operations.			
use the order of operations to simplify complex fractions.			
evaluate variable expressions with fractions.			

(b) After looking at the checklist, do you think you are well prepared for the next section? Why or why not?

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4.15: Add and Subtract Mixed Numbers (Part 1)

Learning Objectives

- Model addition of mixed numbers with a common denominator
- Add mixed numbers with a common denominator
- Model subtraction of mixed numbers
- Subtract mixed numbers with a common denominator
- Add and subtract mixed numbers with different denominators

be prepared!

Before you get started, take this readiness quiz.

1. Draw figure to model $\frac{7}{3}$. If you missed this problem, review Example 4.1.6.
2. Change $\frac{11}{4}$ to a mixed number. If you missed this problem, review Example 4.1.9.
3. Change $3\frac{1}{2}$ to an improper fraction. If you missed this problem, review Example 4.1.11.

Model Addition of Mixed Numbers with a Common Denominator

So far, we've added and subtracted proper and improper fractions, but not mixed numbers. Let's begin by thinking about addition of mixed numbers using money.

If Ron has 1 dollar and 1 quarter, he has $1\frac{1}{4}$ dollars. If Don has 2 dollars and 1 quarter, he has $2\frac{1}{4}$ dollars. What if Ron and Don put their money together? They would have 3 dollars and 2 quarters. They add the dollars and add the quarters. This makes $3\frac{2}{4}$ dollars. Because two quarters is half a dollar, they would have 3 and a half dollars, or $3\frac{1}{2}$ dollars.

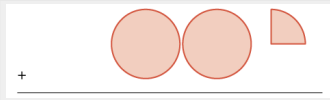
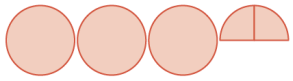
$$\begin{array}{r} 1\frac{1}{4} \\ +2\frac{1}{4} \\ \hline \end{array}$$

$$3\frac{2}{4} = 3\frac{1}{2}$$

When you added the dollars and then added the quarters, you were adding the whole numbers and then adding the fractions.

$$1\frac{1}{4} + 2\frac{1}{4}$$

We can use fraction circles to model this same example:

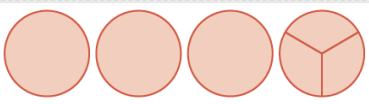
Start with $1\frac{1}{4}$.	one whole and one $\frac{1}{4}$ pieces		$1\frac{1}{4}$
Add $2\frac{1}{4}$ more.	two wholes and one $\frac{1}{4}$ pieces		$+2\frac{1}{4}$
The sum is:	three wholes and two $\frac{1}{4}$'s		$3\frac{2}{4} = 3\frac{1}{2}$

Example 4.15.1: model

Model $2\frac{1}{3} + 1\frac{2}{3}$ and give the sum.

Solution

We will use fraction circles, whole circles for the whole numbers and $\frac{1}{3}$ pieces for the fractions.

two wholes and one $\frac{1}{3}$		$2\frac{1}{3}$
plus one whole and two $\frac{1}{3}$s		$+1\frac{2}{3}$
sum is three wholes and three $\frac{1}{3}$s		$3\frac{3}{3} = 4$

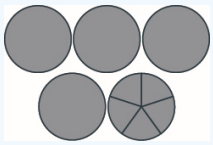
This is the same as 4 wholes. So, $2\frac{1}{3}+1\frac{2}{3}=4$.

Exercise 4.15.1

Use a model to add the following. Draw a picture to illustrate your model. $1\frac{2}{5}+3\frac{3}{5}$

Answer

5

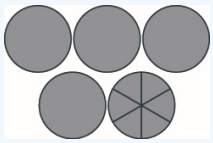


Exercise 4.15.2

Use a model to add the following. Draw a picture to illustrate your model. $2\frac{1}{6}+2\frac{5}{6}$

Answer

5

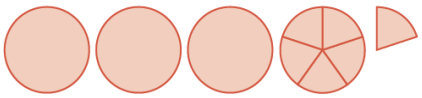


Example 4.15.2: model

Model $1\frac{3}{5}+2\frac{3}{5}$ and give the sum as a mixed number.

Solution

We will use fraction circles, whole circles for the whole numbers and $\frac{1}{5}$ pieces for the fractions.

one whole and three $\frac{1}{5}$s		$1\frac{3}{5}$
plus two wholes and three $\frac{1}{5}$s	+	$+2\frac{3}{5}$
sum is three wholes and six $\frac{1}{5}$s		$3\frac{6}{5}=4\frac{1}{5}$

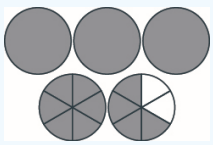
Adding the whole circles and fifth pieces, we got a sum of $3\frac{6}{5}$. We can see that $\frac{6}{5}$ is equivalent to $1\frac{1}{5}$, so we add that to the 3 to get $4\frac{1}{5}$.

Exercise 4.15.3

Model, and give the sum as a mixed number. Draw a picture to illustrate your model. $2\frac{5}{6}+1\frac{5}{6}$

Answer

$4\frac{2}{3}$

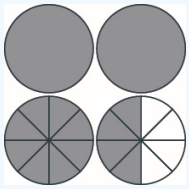


Exercise 4.15.4

Model, and give the sum as a mixed number. Draw a picture to illustrate your model. $1\frac{5}{8}+1\frac{7}{8}$

Answer

$3\frac{1}{2}$



Add Mixed Numbers

Modeling with fraction circles helps illustrate the process for adding mixed numbers: We add the whole numbers and add the fractions, and then we simplify the result, if possible.

HOW TO: ADD MIXED NUMBERS WITH A COMMON DENOMINATOR

Step 1. Add the whole numbers.

Step 2. Add the fractions.

Step 3. Simplify, if possible.

Example 4.15.3: add

Add: $3\frac{4}{9}+2\frac{2}{9}$.

Solution

Add the whole numbers.	$\begin{array}{r} 3\frac{4}{9} \\ +2\frac{2}{9} \\ \hline 5 \end{array}$
Add the fractions.	$\begin{array}{r} 3\frac{4}{9} \\ +2\frac{2}{9} \\ \hline 5\frac{6}{9} \end{array}$
Simplify the fraction.	$\begin{array}{r} 3\frac{4}{9} \\ +2\frac{2}{9} \\ \hline 5\frac{6}{9}=5\frac{2}{3} \end{array}$

Exercise 4.15.5

Find the sum: $4\frac{4}{7}+1\frac{2}{7}$.

Answer

$5\frac{6}{7}$

Exercise 4.15.6

Find the sum: $2\frac{3}{11}+5\frac{6}{11}$.

Answer

$7\frac{9}{11}$

In Example 4.15.3, the sum of the fractions was a proper fraction. Now we will work through an example where the sum is an improper fraction.

Example 4.15.4: add

Find the sum: $9\frac{5}{9}+5\frac{7}{9}$.

Solution

Add the whole numbers and then add the fractions.	$\begin{array}{r} 9\frac{5}{9} \\ +5\frac{7}{9} \\ \hline 14\frac{12}{9} \end{array}$
Rewrite $\frac{12}{9}$ as an improper fraction.	$14+1\frac{3}{9}$
Add.	$15\frac{3}{9}$
Simplify.	$15\frac{1}{3}$

Exercise 4.15.7

Find the sum: $8\frac{7}{8}+7\frac{5}{8}$.

Answer

$16\frac{1}{2}$

Exercise 4.15.8

Find the sum: $6\frac{7}{9}+8\frac{5}{9}$.

Answer

$15\frac{1}{3}$

An alternate method for adding mixed numbers is to convert the mixed numbers to improper fractions and then add the improper fractions. This method is usually written horizontally.

Example 4.15.5: add

Add by converting the mixed numbers to improper fractions: $3\frac{7}{8}+4\frac{3}{8}$.

Solution

Convert to improper fractions.	$\frac{31}{8}+\frac{35}{8}$
Add the fractions.	$\frac{31+35}{8}$
Simplify the numerator.	$\frac{66}{8}$
Rewrite as a mixed number.	$8\frac{2}{8}$
Simplify the fraction.	$8\frac{1}{4}$

Since the problem was given in mixed number form, we will write the sum as a mixed number.

Exercise 4.15.9

Find the sum by converting the mixed numbers to improper fractions: $5\frac{5}{9}+3\frac{7}{9}$

Answer

$9\frac{1}{3}$

Exercise 4.15.10

Find the sum by converting the mixed numbers to improper fractions: $3\frac{7}{10}+2\frac{9}{10}$

Answer

$6\frac{3}{5}$

Table 4.15.1 compares the two methods of addition, using the expression $3\frac{2}{5}+6\frac{4}{5}$ as an example. Which way do you prefer?

Table 4.15.1

Mixed Numbers	Improper Fractions
$\begin{array}{r} 3\frac{2}{5} \\ +6\frac{4}{5} \\ \hline 9\frac{6}{5} \end{array}$	$3\frac{2}{5}+6\frac{4}{5}$
$9+\frac{6}{5}$	$\frac{17}{5}+\frac{34}{5}$
$9+1\frac{1}{5}$	$\frac{51}{5}$
$10\frac{1}{5}$	$10\frac{1}{5}$

Model Subtraction of Mixed Numbers

Let's think of pizzas again to model subtraction of mixed numbers with a common denominator. Suppose you just baked a whole pizza and want to give your brother half of the pizza. What do you have to do to the pizza to give him half? You have to cut it into at least two pieces. Then you can give him half.

We will use fraction circles (pizzas!) to help us visualize the process. Start with one whole.

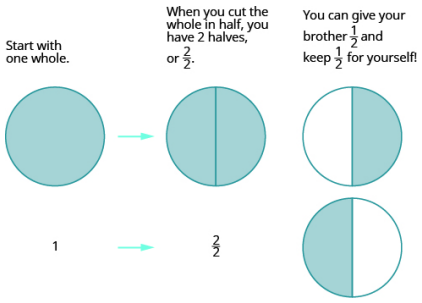


Figure 4.15.1

Algebraically, you would write:

$$\begin{array}{r} 1 \\ -\frac{1}{2} \end{array} \rightarrow \begin{array}{r} \frac{2}{2} \\ -\frac{1}{2} \end{array} \rightarrow \begin{array}{r} \frac{2}{2} \\ -\frac{1}{2} \\ \hline \frac{1}{2} \end{array}$$

Example 4.15.6: subtract

Use a model to subtract: $1-\frac{1}{3}$.

Solution

	Model	Math Notation
Rewrite vertically. Start with one whole.		$\begin{array}{r} 1 \\ -\frac{1}{3} \end{array}$
Since $\frac{1}{3}$ has denominator 3, cut the whole into 3 pieces. The 1 whole becomes $\frac{3}{3}$.		$\begin{array}{r} \frac{3}{3} \\ -\frac{1}{3} \end{array}$
Take away $\frac{1}{3}$. There are $\frac{2}{3}$ left.		$\begin{array}{r} \frac{3}{3} \\ -\frac{1}{3} \\ \hline \frac{2}{3} \end{array}$

Exercise 4.15.11

Use a model to subtract: $1-\frac{1}{4}$.

Answer

$\frac{3}{4}$

Exercise 4.15.12

Use a model to subtract: $1 - \frac{1}{5}$.

Answer

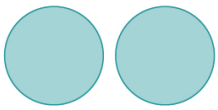
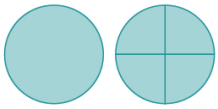
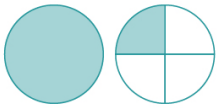
$\frac{4}{5}$

What if we start with more than one whole? Let's find out.

Example 4.15.7: subtract

Use a model to subtract: $2 - \frac{3}{4}$.

Solution

	Model	Math Notation
Rewrite vertically. Start with two wholes.		$\begin{array}{r} 2 \\ -\frac{3}{4} \\ \hline \end{array}$
Since $\frac{3}{4}$ has denominator 4, cut one of the wholes into 4 pieces. You have 1 whole and $\frac{4}{4}$.		$\begin{array}{r} 1\frac{4}{4} \\ -\frac{3}{4} \\ \hline \end{array}$
Take away $\frac{3}{4}$. There is $1\frac{1}{4}$ left.		$\begin{array}{r} 1\frac{4}{4} \\ -\frac{3}{4} \\ \hline 1\frac{1}{4} \end{array}$

Exercise 4.15.13

Use a model to subtract: $2 - \frac{1}{5}$.

Answer

$\frac{9}{5}$

Exercise 4.15.14

Use a model to subtract: $2 - \frac{1}{3}$.

Answer

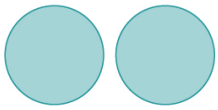
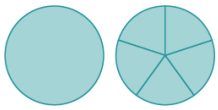
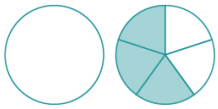
$\frac{5}{3}$

In the next example, we'll subtract more than one whole.

Example 4.15.8: subtract

Use a model to subtract: $2 - 1\frac{2}{5}$.

Solution

	Model	Math Notation
Rewrite vertically. Start with two wholes.		$\begin{array}{r} 2 \\ -1\frac{2}{5} \\ \hline \end{array}$
Since $\frac{2}{5}$ has denominator 5, cut one of the wholes into 5 pieces. You have 1 whole and $\frac{5}{5}$.		$\begin{array}{r} 1\frac{5}{5} \\ -1\frac{2}{5} \\ \hline \end{array}$
Take away $1\frac{2}{5}$. There is $\frac{3}{5}$ left.		$\begin{array}{r} 1\frac{5}{5} \\ -1\frac{2}{5} \\ \hline \frac{3}{5} \end{array}$

Exercise 4.15.15

Use a model to subtract: $2-1\frac{1}{3}$.

Answer

$\frac{2}{3}$

Exercise 4.15.16

Use a model to subtract: $2-1\frac{1}{4}$.

Answer

$\frac{3}{4}$

What if you start with a mixed number and need to subtract a fraction? Think about this situation: You need to put three quarters in a parking meter, but you have only a $1 bill and one quarter. What could you do? You could change the dollar bill into 4 quarters. The value of 4 quarters is the same as one dollar bill, but the 4 quarters are more useful for the parking meter. Now, instead of having a $1 bill and one quarter, you have 5 quarters and can put 3 quarters in the meter.

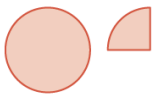
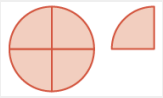
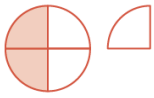
This models what happens when we subtract a fraction from a mixed number. We subtracted three quarters from one dollar and one quarter.

We can also model this using fraction circles, much like we did for addition of mixed numbers.

Example 4.15.9: subtract

Use a model to subtract: $1\frac{1}{4}-\frac{3}{4}$

Solution

Rewrite vertically. Start with one whole and one fourth.		$\begin{array}{r} 1\frac{1}{4} \\ -\frac{3}{4} \\ \hline \end{array}$
Since the fractions have denominator 4, cut the whole into 4 pieces. You now have $\frac{4}{4}$ and $\frac{1}{4}$ which is $\frac{5}{4}$.		$\begin{array}{r} \frac{5}{4} \\ -\frac{3}{4} \\ \hline \end{array}$
Take away $\frac{3}{4}$. There is $\frac{1}{2}$ left.		\begin{split} & \dfrac{5}{4}\\\textcolor{red}{- & \dfrac}3}{4}}\\ \hline \\

Exercise 4.15.17

Use a model to subtract. Draw a picture to illustrate your model. $1\frac{1}{3}-\frac{2}{3}$

Answer

Exercise 4.15.18

Use a model to subtract. Draw a picture to illustrate your model. $1\frac{1}{5}-\frac{4}{5}$

Answer

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4.16: Add and Subtract Mixed Numbers (Part 2)

Subtract Mixed Numbers with a Common Denominator

Now we will subtract mixed numbers without using a model. But it may help to picture the model in your mind as you read the steps.

HOW TO: Subtract mixed numbers with common denominators

Step 1. Rewrite the problem in vertical form.

Step 2. Compare the two fractions.

- If the top fraction is larger than the bottom fraction, go to Step 3.
- If not, in the top mixed number, take one whole and add it to the fraction part, making a mixed number with an improper fraction.

Step 3. Subtract the fractions.

Step 4. Subtract the whole numbers.

Step 5. Simplify, if possible.

Example 4.16.10: subtract

Find the difference: $5\frac{3}{5}-2\frac{4}{5}$.

Solution

Rewrite the problem in vertical form.	$5\frac{3}{5}$ $-2\frac{4}{5}$
Since $\frac{3}{5}$ is less than $\frac{4}{5}$, take 1 from the 5 and add it to the $\frac{3}{5}$: $\left(\frac{5}{5}+\frac{3}{5}=\frac{8}{5}\right)$	$5\frac{3}{5} \rightarrow 4\frac{8}{5}$ $-2\frac{4}{5} \quad -2\frac{4}{5}$
Subtract the fractions.	$4\frac{8}{5}$ $-2\frac{4}{5}$ $\frac{4}{5}$
Subtract the whole parts. The result is in simplest form.	$4\frac{8}{5}$ $-2\frac{4}{5}$ $2\frac{4}{5}$

Since the problem was given with mixed numbers, we leave the result as mixed numbers.

Exercise 4.16.19

Find the difference: $6\frac{4}{9}-3\frac{7}{9}$.

Answer

$2\frac{2}{3}$

Exercise 4.16.20

Find the difference: $4\frac{4}{7}-2\frac{6}{7}$.

Answer

$1\frac{5}{7}$

Just as we did with addition, we could subtract mixed numbers by converting them first to improper fractions. We should write the answer in the form it was given, so if we are given mixed numbers to subtract we will write the answer as a mixed number.

HOW TO: SUBTRACT MIXED NUMBERS WITH COMMON DENOMINATORS AS IMPROPER FRACTIONS

Step 1. Rewrite the mixed numbers as improper fractions.

Step 2. Subtract the numerators.

Step 3. Write the answer as a mixed number, simplifying the fraction part, if possible.

Example 4.16.11: subtract

Find the difference by converting to improper fractions: $9\frac{6}{11}-7\frac{10}{11}$.

Solution

Rewrite as improper fractions.	$\frac{105}{11}-\frac{87}{11}$
Subtract the numerators.	$\frac{18}{11}$
Rewrite as a mixed number.	$1\frac{7}{11}$

Exercise 4.16.21

Find the difference by converting to improper fractions: $6\frac{4}{9}-3\frac{7}{9}$.

Answer

$2\frac{2}{3}$

Exercise 4.16.22

Find the difference by converting to improper fractions: $4\frac{4}{7}-2\frac{6}{7}$.

Answer

$1\frac{5}{7}$

Add and Subtract Mixed Numbers with Different Denominators

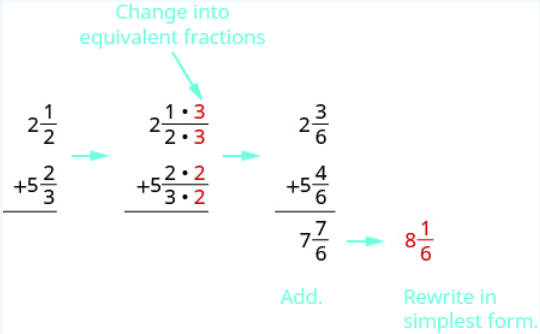
To add or subtract mixed numbers with different denominators, we first convert the fractions to equivalent fractions with the LCD. Then we can follow all the steps we used above for adding or subtracting fractions with like denominators.

Example 4.16.12: add

Add: $2\frac{1}{2}+5\frac{2}{3}$.

Solution

Since the denominators are different, we rewrite the fractions as equivalent fractions with the LCD, 6. Then we will add and simplify.



We write the answer as a mixed number because we were given mixed numbers in the problem.

Exercise 4.16.23

Add: $1\frac{5}{6}+4\frac{3}{4}$.

Answer

$6\frac{7}{12}$

Exercise 4.16.24

Add: $3\frac{4}{5}+8\frac{1}{2}$.

Answer

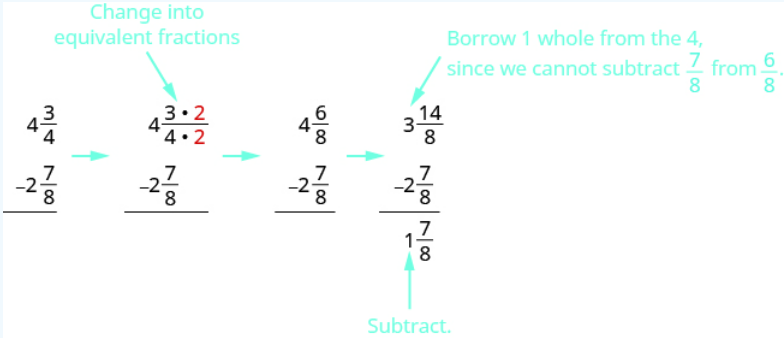
$12\frac{3}{10}$

Example 4.16.13: subtract

Subtract: $4\frac{3}{4}-2\frac{7}{8}$.

Solution

Since the denominators of the fractions are different, we will rewrite them as equivalent fractions with the LCD 8. Once in that form, we will subtract. But we will need to borrow 1 first.



We were given mixed numbers, so we leave the answer as a mixed number.

Exercise 4.16.25

Find the difference: $8\frac{1}{2}-3\frac{4}{5}$.

Answer

$4\frac{7}{10}$

Exercise 4.16.26

Find the difference: $4\frac{3}{4}-1\frac{5}{6}$.

Answer

$2\frac{11}{12}$

Example 4.16.14:

Subtract: $3\frac{5}{11}-4\frac{3}{4}$.

Solution

We can see the answer will be negative since we are subtracting 4 from 3. Generally, when we know the answer will be negative it is easier to subtract with improper fractions rather than mixed numbers.

Change to equivalent fractions with the LCD.	$3\frac{5\cdot 4}{11\cdot 4}-4\frac{3\cdot 11}{4\cdot 11}$ $3\frac{20}{44}-4\frac{33}{44}$
Rewrite as improper fractions.	$\frac{152}{44}-\frac{209}{44}$
Subtract.	$-\frac{57}{44}$

Rewrite as a mixed number.	$-1\frac{13}{44}$

Exercise 4.16.27

Subtract: $1\frac{3}{4}-6\frac{7}{8}$.

Answer

$-\frac{41}{8}$

Exercise 4.16.28

Subtract: $10\frac{3}{7}-22\frac{4}{9}$.

Answer

$-\frac{757}{63}$

Access Additional Online Resources

- Adding Mixed Numbers
- Subtracting Mixed Numbers

Practice Makes Perfect

Model Addition of Mixed Numbers

In the following exercises, use a model to find the sum. Draw a picture to illustrate your model.

436. $1\frac{1}{5}+3\frac{1}{5}$
437. $2\frac{1}{3}+1\frac{1}{3}$
438. $1\frac{3}{8}+1\frac{7}{8}$
439. $1\frac{5}{6}+1\frac{5}{6}$

Add Mixed Numbers with a Common Denominator

In the following exercises, add.

440. $5\frac{1}{3}+6\frac{1}{3}$
441. $2\frac{4}{9}+5\frac{1}{9}$
442. $4\frac{5}{8}+9\frac{3}{8}$
443. $7\frac{9}{10}+3\frac{1}{10}$
444. $3\frac{4}{5}+6\frac{4}{5}$
445. $9\frac{2}{3}+1\frac{2}{3}$
446. $6\frac{9}{10}+8\frac{3}{10}$

447. $8\frac{4}{9}+2\frac{8}{9}$

Model Subtraction of Mixed Numbers

In the following exercises, use a model to find the difference. Draw a picture to illustrate your model.

448. $1\frac{1}{6}-\frac{1}{6}$
449. $1\frac{1}{8}-\frac{1}{8}$

Subtract Mixed Numbers with a Common Denominator

In the following exercises, find the difference.

450. $2\frac{7}{8}-1\frac{3}{8}$
451. $2\frac{7}{12}-1\frac{5}{12}$
452. $8\frac{3}{7}-4\frac{4}{7}$
453. $19\frac{13}{15}-13\frac{7}{15}$
454. $8\frac{3}{7}-4\frac{4}{7}$
455. $5\frac{2}{9}-3\frac{4}{9}$
456. $2\frac{5}{8}-1\frac{7}{8}$
457. $2\frac{5}{12}-1\frac{7}{12}$

Add and Subtract Mixed Numbers with Different Denominators

In the following exercises, write the sum or difference as a mixed number in simplified form.

458. $3\frac{1}{4}+6\frac{1}{3}$
459. $2\frac{1}{6}+5\frac{3}{4}$
460. $1\frac{5}{8}+4\frac{1}{2}$
461. $7\frac{2}{3}+8\frac{1}{2}$
462. $2\frac{5}{12}-1\frac{7}{12}$
463. $6\frac{4}{5}-1\frac{1}{4}$
464. $2\frac{2}{3}-3\frac{1}{2}$
465. $2\frac{7}{8}-4\frac{1}{3}$

Mixed Practice

In the following exercises, perform the indicated operation and write the result as a mixed number in simplified form.

466. $2\frac{5}{8}\cdot 1\frac{3}{4}$
467. $1\frac{2}{3}\cdot 4\frac{1}{6}$
468. $\frac{2}{7}+\frac{4}{7}$
469. $\frac{2}{9}+\frac{5}{9}$

470. $1\frac{5}{12} \div \frac{1}{12}$
471. $2\frac{3}{10} \div \frac{1}{10}$
472. $13\frac{5}{12} - 9\frac{7}{12}$
473. $15\frac{5}{8} - 6\frac{7}{8}$
474. $\frac{5}{9} - \frac{4}{9}$
475. $\frac{11}{15} - \frac{7}{15}$
476. $4 - \frac{3}{4}$
477. $6 - \frac{2}{5}$
478. $\frac{9}{20} \div \frac{3}{4}$
479. $\frac{7}{24} \div \frac{14}{3}$
480. $9\frac{6}{11} + 7\frac{10}{11}$
481. $8\frac{5}{13} + 4\frac{9}{13}$
482. $3\frac{2}{5} + 5\frac{3}{4}$
483. $2\frac{5}{6} + 4\frac{1}{5}$
484. $\frac{8}{15} \cdot \frac{10}{19}$
485. $\frac{5}{12} \cdot \frac{8}{9}$
486. $6\frac{7}{8} - 2\frac{1}{3}$
487. $6\frac{5}{9} - 4\frac{2}{5}$
488. $5\frac{2}{9} - 4\frac{4}{5}$
489. $4\frac{3}{8} - 3\frac{2}{3}$

Everyday Math

490. **Sewing** Renata is sewing matching shirts for her husband and son. According to the patterns she will use, she needs $2\frac{3}{8}$ yards of fabric for her husband's shirt and $1\frac{1}{8}$ yards of fabric for her son's shirt. How much fabric does she need to make both shirts?
491. **Sewing** Pauline has $3\frac{1}{4}$ yards of fabric to make a jacket. The jacket uses $2\frac{2}{3}$ yards. How much fabric will she have left after making the jacket?
492. **Printing** Nishant is printing invitations on his computer. The paper is $8\frac{1}{2}$ inches wide, and he sets the print area to have a $1\frac{1}{2}$-inch border on each side. How wide is the print area on the sheet of paper?
493. **Framing a picture** Tessa bought a picture frame for her son's graduation picture. The picture is 8 inches wide. The picture frame is $2\frac{5}{8}$ inches wide on each side. How wide will the framed picture be?

Writing Exercises

494. Draw a diagram and use it to explain how to add $1\frac{5}{8} + 2\frac{7}{8}$.
495. Edgar will have to pay $3.75 in tolls to drive to the city.

a. Explain how he can make change from a $10 bill before he leaves so that he has the exact amount he needs.

b. How is Edgar's situation similar to how you subtract $10 - 3\frac{3}{4}$?

496. Add $4\frac{5}{12} + 3\frac{7}{8}$ twice, first by leaving them as mixed numbers and then by rewriting as improper fractions. Which method do you prefer, and why?

497. Subtract $3\frac{7}{8} - 4\frac{5}{12}$ twice, first by leaving them as mixed numbers and then by rewriting as improper fractions. Which method do you prefer, and why?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
model addition of mixed numbers common with a denominator.			
add mixed numbers with a common denominator.			
model subtraction of mixed numbers.			
subtract mixed numbers with a common denominator.			
add and subtract mixed numbers with different denominators.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

Contributors and Attributions

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4.17: Complex Rational Expressions

Learning Objectives

- Simplify complex rational expressions by multiplying the numerator by the reciprocal of the divisor.
- Simplify complex rational expressions by multiplying numerator and denominator by the least common denominator (LCD).

Definitions

A complex fraction is a fraction where the numerator or denominator consists of one or more fractions. For example,

$$\frac{\frac{3}{4}}{\frac{1}{2}}$$

Simplifying such a fraction requires us to find an equivalent fraction with integer numerator and denominator. One way to do this is to divide. Recall that dividing fractions involves multiplying by the reciprocal of the divisor.

$$\frac{\frac{3}{4}}{\frac{1}{2}} = \frac{3}{\cancel{4}_{2}} \cdot \frac{\overset{1}{\cancel{2}}}{1} = \frac{3}{2} \qquad Method\ 1:\ using\ division$$

An alternative method for simplifying this complex fraction involves multiplying both the numerator and denominator by the LCD of all the given fractions. In this case, the LCD = 4.

$$\frac{\frac{3}{4} \cdot 4}{\frac{1}{2} \cdot 4} = \frac{3}{2} \qquad Method\ 2:\ using\ the\ LCD$$

A complex rational expression is defined as a rational expression that contains one or more rational expressions in the numerator or denominator or both. For example,

$$\frac{\frac{1}{2} + \frac{1}{x}}{\frac{1}{4} - \frac{1}{x^2}}$$

We simplify a complex rational expression by finding an equivalent fraction where the numerator and denominator are polynomials. As illustrated above, there are two methods for simplifying complex rational expressions, and we will outline the steps for both methods. For the sake of clarity, assume that variable expressions used as denominators are nonzero.

Method 1: Simplify Using Division

We begin our discussion on simplifying complex rational expressions using division. Before we can multiply by the reciprocal of the divisor, we must simplify the numerator and denominator separately. The goal is to first obtain single algebraic fractions in the numerator and the denominator. The steps for simplifying a complex algebraic fraction are illustrated in the following example.

Example 4.17.1

Simplify:

$$\frac{\frac{1}{2} + \frac{1}{x}}{\frac{1}{4} - \frac{1}{x^2}}$$

Solution:

Step 1: Simplify the numerator and denominator. The goal is to obtain a single algebraic fraction divided by another single algebraic fraction. In this example, find equivalent terms with a common denominator in both the numerator and denominator before adding and subtracting.

$$\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}} = \frac{\frac{1}{2}\cdot\frac{x}{x}+\frac{1}{x}\cdot\frac{2}{2}}{\frac{1}{4}\cdot\frac{x^2}{x^2}-\frac{1}{x^2}\cdot\frac{4}{4}}$$

$$= \frac{\frac{x}{2x}+\frac{2}{2x}}{\frac{x^2}{4x^2}-\frac{4}{4x^2}}$$ *Equivalent fractions with common denominators*

$$= \frac{\frac{x+2}{2x}}{\frac{x^2-4}{4x^2}}$$ *Add the fractions in the numerator and denominator.*

At this point we have a single algebraic fraction divided by a single algebraic fraction.

Step 2: Multiply the numerator by the reciprocal of the divisor.

$$\frac{\frac{x+2}{2x}}{\frac{x^2-4}{4x^2}} = \frac{x+2}{2x}\cdot\frac{4x^2}{x^2-4}$$

Step 3: Factor all numerators and denominators completely.

$$= \frac{x+2}{2x}\cdot\frac{4x^2}{(x+2)(x-2)}$$

Step 4: Cancel all common factors.

$$= \frac{4x^2\cdot(x+2)}{2x(x+2)(x-2)}$$

$$= \frac{\overset{2x}{\cancel{4x^2}}\cdot\cancel{(x+2)}}{\cancel{2x}\cancel{(x+2)}(x-2)}$$

$$= \frac{2x}{(x-2)}$$

Answer:

$$\frac{2x}{x-2}$$

Example 4.17.2

Simplify:

Solution:

Answer:

$$-\frac{1}{2}$$

Example 4.17.3

Simplify:

$$\frac{1-\frac{4}{x}-\frac{21}{x^2}}{1-\frac{2}{x}-\frac{15}{x^2}}$$

Solution:

The LCD of the rational expressions in both the numerator and denominator is x^2. Multiply by the appropriate factors to obtain equivalent terms with this as the denominator and then subtract.

$$\begin{aligned}\frac{1-\frac{4}{x}-\frac{21}{x^2}}{1-\frac{2}{x}-\frac{15}{x^2}} &= \frac{\frac{1}{1}\cdot\frac{x^2}{x^2}-\frac{4}{x}\cdot\frac{x}{x}-\frac{21}{x^2}}{\frac{1}{1}\cdot\frac{x^2}{x^2}-\frac{2}{x}\cdot\frac{x}{x}-\frac{15}{x^2}} \\ &= \frac{\frac{x^2}{x^2}-\frac{4x}{x^2}-\frac{21}{x^2}}{\frac{x^2}{x^2}-\frac{2x}{x^2}-\frac{15}{x^2}} \\ &= \frac{\frac{x^2-4x-21}{x^2}}{\frac{x^2-2x-15}{x^2}}\end{aligned}$$

We now have a single rational expression divided by another single rational expression. Next, multiply the numerator by the reciprocal of the divisor and then factor and cancel.

Answer:

$\frac{x-7}{x-5}$

Example 4.17.4

Simplify:

$$\frac{1-\frac{1}{x^2}}{\frac{1}{x}-1}$$

Solution:

$$\begin{aligned}\frac{1-\frac{1}{x^2}}{\frac{1}{x}-1} &= \frac{\frac{1}{1}\cdot\frac{x^2}{x^2}-\frac{1}{x^2}}{\frac{1}{x}-\frac{1}{1}\cdot\frac{x}{x}} \\ &= \frac{\frac{x^2-1}{x^2}}{\frac{1-x}{x}} \\ &= \frac{x^2-1}{x^2}\cdot\frac{x}{1-x} \\ &= \frac{(x+1)\,(x-1)}{x^2}\cdot\frac{x}{-1\cdot\,(x-1)} \\ &= \frac{x+1}{-1\cdot x} \\ &= -\frac{x+1}{x}\end{aligned}$$

Answer:

$$-\frac{x+1}{x}$$

Exercise 4.17.1

Simplify:

$$\frac{\frac{1}{81}-\frac{1}{x^2}}{\frac{1}{9}+\frac{1}{x}}$$

Answer

$\frac{x-9}{9x}$

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Method 2: Simplify Using the LCD

An alternative method for simplifying complex rational expressions involves clearing the fractions by multiplying the expression by a special form of 1. In this method, multiply the numerator and denominator by the least common denominator (LCD) of all given fractions.

Example 4.17.5

Simplify:

$$\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}}$$

Solution:

Step 1: Determine the LCD of all the fractions in the numerator and denominator. In this case, the denominators of the given fractions are $2, x, 4$, and x^2. Therefore, the LCD is $4x^2$.

Step 2: Multiply the numerator and denominator by the LCD. This step should clear the fractions in both the numerator and denominator.

$$\begin{aligned}\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}} &= \frac{\left(\frac{1}{2}+\frac{1}{x}\right)\cdot 4x^2}{\left(\frac{1}{4}-\frac{1}{x^2}\right)\cdot 4x^2} && \textit{Distribute.}\\ &= \frac{\frac{1}{2}\cdot 4x^2+\frac{1}{x}\cdot 4x^2}{\frac{1}{4}\cdot 4x^2-\frac{1}{x^2}\cdot 4x^2} && \textit{Cancel.}\\ &= \frac{2x^2+4x}{x^2-4}\end{aligned}$$

This leaves us with a single algebraic fraction with a polynomial in the numerator and in the denominator.

Step 3: Factor the numerator and denominator completely.

$$\begin{aligned}&= \frac{2x^2+4x}{x^2-4}\\ &= \frac{2x(x+2)}{(x+2)(x-2)}\end{aligned}$$

Step 4: Cancel all common factors.

$$\begin{aligned}&= \frac{2x\,\cancel{(x+2)}}{\cancel{(x+2)}(x-2)}\\ &= \frac{2x}{x-2}\end{aligned}$$

Answer:

$$\frac{2x}{x-2}$$

Note

This was the same problem that we began this section with, and the results here are the same. It is worth taking the time to compare the steps involved using both methods on the same problem.

Example 4.17.6

Simplify:

$$\frac{1-\frac{2}{x}-\frac{15}{x^2}}{3-\frac{14}{x}-\frac{5}{x^2}}$$

Solution:

Considering all of the denominators, we find that the LCD is x^2. Therefore, multiply the numerator and denominator by x^2:

At this point, we have a rational expression that can be simplified by factoring and then canceling the common factors.

$$= \frac{(x+3)\,\cancel{(x-5)}}{(3x+1)\,\cancel{(x-5)}} \qquad \textit{Cancel.}$$

$$= \frac{x+3}{3x+1}$$

Answer:

$$\frac{x+3}{3x+1}$$

It is important to point out that multiplying the numerator and denominator by the same nonzero factor is equivalent to multiplying by 1 and does not change the problem. Because $\frac{x^2}{x^2} = 1$, we can multiply the numerator and denominator by x^2 in the previous example and obtain an equivalent expression.

Example 4.17.7

Simplify:

$$\frac{\frac{1}{x+1}+\frac{3}{x-3}}{\frac{2}{x-3}-\frac{1}{x+1}}$$

Solution:

The LCM of all the denominators is $(x+1)(x-3)$. Begin by multiplying the numerator and denominator by these factors.

$$\begin{aligned}
\frac{\frac{1}{x+1}+\frac{3}{x-3}}{\frac{2}{x-3}-\frac{1}{x+1}} &= \frac{\left(\frac{1}{x+1}+\frac{3}{x-3}\right)\cdot(x+1)(x-3)}{\left(\frac{2}{x-3}-\frac{1}{x+1}\right)\cdot(x+1)(x-3)} && \textit{Distribute.}\\
&= \frac{\frac{1\,\cancel{(x+1)}\,\cancel{(x-3)}}{\cancel{x+1}}+\frac{3(x+1)\,\cancel{(x-3)}}{\cancel{x-3}}}{\frac{2(x+1)\,\cancel{(x-3)}}{\cancel{x-3}}-\frac{1\,\cancel{(x+1)}\,\cancel{(x-3)}}{\cancel{x+1}}} && \textit{Cancel.}\\
&= \frac{(x-3)+3(x+1)}{2(x+1)-1(x-3)} && \textit{Simplify.}\\
&= \frac{x-3+3x+3}{2x+2-x+3}\\
&= \frac{4x}{x+5}
\end{aligned}$$

Answer:

$$\frac{4x}{x+5}$$

Exercise 4.17.2

Simplify:

$$\frac{\frac{1}{y}-\frac{1}{4}}{\frac{1}{16}-\frac{1}{y^2}}$$

Answer

$$-\frac{4y}{y+4}$$

Key Takeaways

- Complex rational expressions can be simplified into equivalent expressions with a polynomial numerator and polynomial denominator.

- One method of simplifying a complex rational expression requires us to first write the numerator and denominator as a single algebraic fraction. Then multiply the numerator by the reciprocal of the divisor and simplify the result.
- Another method for simplifying a complex rational expression requires that we multiply it by a special form of 1. Multiply the numerator and denominator by the LCM of all the denominators as a means to clear the fractions. After doing this, simplify the remaining rational expression.
- An algebraic fraction is reduced to lowest terms if the numerator and denominator are polynomials that share no common factors other than 1.

Exercise 4.17.3 Complex Rational Expressions

Simplify. (Assume all denominators are nonzero.)

1. $\frac{\frac{1}{2}}{\frac{5}{4}}$
2. $\frac{\frac{7}{8}}{\frac{5}{4}}$
3. $\frac{\frac{10}{3}}{\frac{20}{9}}$
4. $-\frac{\frac{4}{21}}{\frac{8}{7}}$
5. $\frac{\frac{2}{3}}{\frac{5}{6}}$
6. $\frac{\frac{7}{4}}{\frac{14}{3}}$
7. $\frac{1-\frac{3}{2}}{\frac{5}{4}-\frac{1}{3}}$
8. $\frac{\frac{1}{2}-5}{\frac{1}{2}+\frac{1}{3}}$
9. $\frac{1+\frac{3}{2}}{1-\frac{1}{4}}$
10. $\frac{2-\frac{1}{2}}{1+\frac{3}{4}}$
11. $\frac{\frac{5x^2}{x+1}}{\frac{25x}{x+1}}$
12. $\frac{\frac{7+x}{7x}}{\frac{x+7}{14x^2}}$
13. $\frac{\frac{3y}{x}}{\frac{y^2}{x-1}}$
14. $\frac{\frac{5a^2}{b-1}}{\frac{15a^3}{(b-1)^2}}$
15. $\frac{1+\frac{1}{x}}{2-\frac{1}{x}}$
16. $\frac{\frac{2}{x}+1}{3-\frac{1}{x}}$
17. $\frac{\frac{2}{3y}-4}{6-\frac{1}{y}}$
18. $\frac{\frac{5}{y-12}}{\frac{10-y}{y^2}}$
19. $\frac{\frac{1}{5}-\frac{1}{x}}{\frac{1}{25}-\frac{1}{x^2}}$
20. $\frac{\frac{1}{x}+\frac{1}{5}}{\frac{1}{25}-\frac{1}{x^2}}$
21. $\frac{\frac{1}{x}-\frac{1}{3}}{\frac{1}{9}-\frac{1}{x^2}}$

22. $\dfrac{\frac{1}{4}+\frac{1}{x}}{\frac{1}{x^2}-\frac{1}{16}}$
23. $\dfrac{16-\frac{1}{x^2}}{\frac{1}{x}-4}$
24. $\dfrac{2-\frac{1}{y}}{1-\frac{1}{4y^2}}$
25. $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{y^2}-\frac{1}{x^2}}$
26. $\dfrac{\frac{1}{2x}-\frac{4}{3}}{\frac{1}{4x^2}-\frac{16}{9}}$
27. $\dfrac{\frac{2}{25}-\frac{1}{2x^2}}{\frac{1}{5}-\frac{1}{2x}}$
28. $\dfrac{\frac{4}{25}-\frac{1}{4x^2}}{\frac{1}{5}+\frac{1}{4x}}$
29. $\dfrac{\frac{1}{y}-\frac{1}{x}}{4-\frac{2}{xy}}$
30. $\dfrac{\frac{1}{ab}+2}{\frac{1}{a}+\frac{1}{b}}$
31. $\dfrac{\frac{1}{y}+\frac{1}{x}}{xy}$
32. $\dfrac{\frac{3}{x}}{\frac{1}{3}-\frac{1}{x}}$
33. $\dfrac{1-\frac{4}{x}-\frac{21}{x^2}}{1-\frac{2}{x}-\frac{15}{x^2}}$
34. $\dfrac{1-\frac{3}{x}-\frac{4}{x^2}}{1-\frac{16}{x^2}}$
35. $\dfrac{3-\frac{1}{2x}-\frac{1}{2x^2}}{2-\frac{2}{x}+\frac{1}{2x^2}}$
36. $\dfrac{\frac{1}{2}-\frac{5}{x}+\frac{12}{x^2}}{\frac{1}{2}-\frac{6}{x}+\frac{18}{x^2}}$
37. $\dfrac{\frac{1}{x}-\frac{4}{3x^2}}{3-\frac{8}{x}+\frac{16}{3x^2}}$
38. $\dfrac{1+\frac{3}{10x}-\frac{1}{10x^2}}{\frac{3}{5}-\frac{1}{10x}-\frac{1}{5x^2}}$
39. $\dfrac{x-1}{1+\frac{4}{x}}-\dfrac{5}{x^2}$
40. $2-\dfrac{5}{2x}-\dfrac{3x}{24x+3}$
41. $\dfrac{\frac{1}{x-3}+\frac{2}{x}}{\frac{1}{x}-\frac{3}{x-3}}$
42. $\dfrac{\frac{14}{x}-5+\frac{1}{x^2}}{\frac{1}{x^2}+\frac{13}{x}-10}$
43. $\dfrac{\frac{1}{x+5}+\frac{4}{x-2}}{\frac{2}{x-2}-\frac{1}{x+5}}$
44. $\dfrac{\frac{3}{x-1}-\frac{2}{x+3}}{\frac{2}{x+3}+\frac{1}{x-3}}$
45. $\dfrac{\frac{x}{x+1}-\frac{2}{x+3}}{\frac{x}{3x+4}+\frac{1}{x+1}}$
46. $\dfrac{\frac{x}{x-9}+\frac{2}{x+1}}{\frac{x}{7x-9}-\frac{1}{x+1}}$
47. $\dfrac{\frac{x}{3x+2}-\frac{1}{x+2}}{\frac{x}{x+2}-\frac{2}{x+2}}$

48. $\frac{\frac{x}{x-4}+\frac{1}{x+2}}{\frac{x}{3x+4}+\frac{1}{x+2}}$
49. $\frac{\frac{a}{3-8b^3}}{\frac{2}{\frac{7}{a}-\frac{2}{b}}}$
50. $\frac{\frac{27a}{3+b^3}}{\frac{\frac{a}{b}}{3a+b}}$
51. $\frac{\frac{1}{b^3}+\frac{1}{a^3}}{\frac{1}{b}+\frac{1}{a}}$
52. $\frac{\frac{1}{b^3}-\frac{1}{a^3}}{\frac{1}{a}-\frac{1}{b}}$
53. $\frac{\frac{x^2+y^2}{\frac{x}{y}+2}}{\frac{x^2-y^2}{\frac{2}{xy}}}$
54. $\frac{x}{\frac{y+4+4yx}{xy+3+2yx}}$
55. $1+\frac{1}{1+\frac{1}{2}}$
56. $2-\frac{1}{1+\frac{1}{3}}$
57. $\frac{1}{1+\frac{1}{1+\frac{1}{1+x}}}$
58. $\frac{x+\frac{1}{x}}{1-\frac{1}{x}+1}$
59. $\frac{1-\frac{1}{x}}{x-\frac{1}{x}}$
60. $\frac{\frac{1}{x}-x}{x-\frac{1}{x^2}}$

Answer

1. $\frac{2}{5}$
3. $\frac{3}{2}$
5. $\frac{4}{5}$
7. $-\frac{6}{11}$
9. $\frac{10}{3}$
11. $\frac{x}{5}$
13. $\frac{3(x-1)}{yx}$
15. $\frac{x+1}{2x-1}$
17. $-\frac{2}{3}$
19. $\frac{5x}{x+5}$
21. $-\frac{3x}{x+3}$
23. $-\frac{4x+1}{x}$
25. $\frac{xy}{x-y}$
27. $\frac{2x+5}{5x}$
29. $\frac{x-y}{4xy-2}$
31. $\frac{x+y}{y^2x^2}$

33. $\frac{x-7}{x-5}$

35. $\frac{3x+1}{2x-1}$

37. $\frac{1}{3x-4}$

39. $\frac{x(x-1)}{x+4}-\frac{5}{x^2}$

41. $\frac{3x-6}{-2x-3}$

43. $\frac{5x+18}{x+12}$

45. $\frac{(x-1)(3x+4)}{(x+2)(x+3)}$

47. $\frac{x+1}{3x+2}$

49. $\frac{7b-2a}{2b\left(3-8b^3\right)}$

51. $\frac{a^2-ba+b^2}{b^2a^2}$

53. $\frac{2\left(x^2+y^2\right)}{x(x+2y)(x^2-y^2)}$

55. $\frac{5}{3}$

57. $\frac{x+1}{x+2}$

59. $\frac{1}{x+1}$

Exercise 4.17.4 Discussion Board Topics

1. Choose a problem from this exercise set and clearly work it out on paper, explaining each step in words. Scan your page and post it on the discussion board.
2. Explain why we need to simplify the numerator and denominator to a single algebraic fraction before multiplying by the reciprocal of the divisor.
3. Two methods for simplifying complex rational expressions have been presented in this section. Which of the two methods do you feel is more efficient, and why?

Answer

1. Answers may vary

3. Answers may vary

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4.18: Solve Equations with Fractions (Part 1)

Learning Objectives

- Determine whether a fraction is a solution of an equation
- Solve equations with fractions using the Addition, Subtraction, and Division Properties of Equality
- Solve equations using the Multiplication Property of Equality
- Translate sentences to equations and solve

be prepared!

Before you get started, take this readiness quiz. If you miss a problem, go back to the section listed and review the material.

1. Evaluate $x+4$ when $x=-3$ If you missed this problem, review Example 3.2.10.
2. Solve: $2y-3=9$. If you missed this problem, review Example 3.5.2.
3. Solve: $y-3=-9$ If you missed this problem, review Example 4.2.10.

Determine Whether a Fraction is a Solution of an Equation

As we saw in Solve Equations with the Subtraction and Addition Properties of Equality and Solve Equations Using Integers; The Division Property of Equality, a solution of an equation is a value that makes a true statement when substituted for the variable in the equation. In those sections, we found whole number and integer solutions to equations. Now that we have worked with fractions, we are ready to find fraction solutions to equations.

The steps we take to determine whether a number is a solution to an equation are the same whether the solution is a whole number, an integer, or a fraction.

HOW TO: DETERMINE WHETHER A NUMBER IS A Solution TO AN EQUATION

Step 1. Substitute the number for the variable in the equation.

Step 2. Simplify the expressions on both sides of the equation.

Step 3. Determine whether the resulting equation is true. If it is true, the number is a solution. If it is not true, the number is not a solution.

Example 4.18.1: find solution

Determine whether each of the following is a solution of $x-\frac{3}{10}=\frac{1}{2}$.

a. $x=1$

b. $x=\frac{4}{5}$

c. $x=-\frac{4}{5}$

Solution

a.

Substitute 1 for x.	$1-\frac{3}{10}\stackrel{?}{=}\frac{1}{2}$
Change to fractions with a LCD of 10.	$\frac{10}{10}-\frac{3}{10}\stackrel{?}{=}\frac{5}{10}$
Subtract.	$\frac{7}{10}\stackrel{?}{=}\frac{5}{10}$

Since $x=1$ does not result in a true equation, 1 is not a solution to the equation.

b.

Substitute $\frac{4}{5}$ for x.	$\frac{4}{5}-\frac{3}{10}\stackrel{?}{=}\frac{1}{2}$
	$\frac{8}{10}-\frac{3}{10}\stackrel{?}{=}\frac{5}{10}$
Subtract.	$\frac{5}{10}=\frac{5}{10}$ ✓

Since $x=\frac{4}{5}$ results in a true equation, $\frac{4}{5}$ is a solution to the equation $x-\frac{3}{10}=\frac{1}{2}$.

c.

Substitute $-\frac{4}{5}$ for x.	$-\frac{4}{5}-\frac{3}{10}\stackrel{?}{=}\frac{1}{2}$
	$-\frac{8}{10}-\frac{3}{10}\stackrel{?}{=}\frac{5}{10}$
Subtract.	$\frac{11}{10}\neq\frac{5}{10}$

Since $x=-\frac{4}{5}$ does not result in a true equation, $-\frac{4}{5}$ is not a solution to the equation.

Exercise 4.18.1

Determine whether each number is a solution of the given equation. $x-\frac{2}{3}=\frac{1}{6}$

a. $x=1$
b. $x=\frac{5}{6}$
c. $x=-\frac{5}{6}$

Answer a

no

Answer b

yes

Answer c

no

Exercise 4.18.2

Determine whether each number is a solution of the given equation. $y-\frac{1}{4}=\frac{3}{8}$

a. $y=1$
b. $y=-\frac{5}{8}$
c. $y=\frac{5}{8}$

Answer a

no

Answer b

no

Answer c

yes

Solve Equations with Fractions using the Addition, Subtraction, and Division Properties of Equality

In Solve Equations with the Subtraction and Addition Properties of Equality and Solve Equations Using Integers; The Division Property of Equality, we solved equations using the Addition, Subtraction, and Division Properties of Equality. We will use these same properties to solve equations with fractions.

Addition, Subtraction, and Division Properties of Equality

For any numbers a, b, and c,

Table 4.18.1

if $a = b$, then $a + c = b + c$.	Addition Property of Equality
if $a = b$, then $a - c = b - c$.	Subtraction Property of Equality
if $a = b$, then $ac = bc$, $c \neq 0$.	Division Property of Equality

In other words, when you add or subtract the same quantity from both sides of an equation, or divide both sides by the same quantity, you still have equality.

Example 4.18.2: solve

Solve: $y + \frac{9}{16} = \frac{5}{16}$.

Solution

Subtract $\frac{9}{16}$ from each side to undo the addition.	$y + \frac{9}{16} - \frac{9}{16} = \frac{5}{16} - \frac{9}{16}$
Simplify on each side of the equation.	$y + 0 = -\frac{4}{16}$
Simplify the fraction.	$y = -\frac{1}{4}$

Check:

Substitute y = (− \dfrac{1}{4}\).	$-\frac{1}{4} + \frac{9}{16} \stackrel{?}{=} \frac{5}{16}$
Rewrite as fractions with the LCD.	\textcolor{red}{- \dfrac{4}{16}} + \dfrac{9}{16} \stackrel{?}{=} \dfrac{5}{16}\)
Add.	$\frac{5}{16} \stackrel{?}{=} \frac{5}{16}$ ✓

Since $y = -\frac{1}{4}$ makes $y + \frac{9}{16} = \frac{5}{16}$ a true statement, we know we have found the solution to this equation.

Exercise 4.18.3

Solve: $y + \frac{11}{12} = \frac{5}{12}$.

Answer

$-\frac{1}{2}$

Exercise 4.18.4

Solve: $y+\frac{8}{15}=\frac{4}{15}$.

Answer

$-\frac{4}{15}$

We used the Subtraction Property of Equality in Example 4.18.2 Now we'll use the Addition Property of Equality.

Example 4.18.3: solve

Solve: a $-\frac{5}{9}=-\frac{8}{9}$.

Solution

Add $\frac{5}{9}$ from each side to undo the addition.	$a-\frac{5}{9}+\frac{5}{9}=-\frac{8}{9}+\frac{5}{9}$
Simplify on each side of the equation.	$a+0=-\frac{3}{9}$
Simplify the fraction.	$a=-\frac{1}{3}$

Check:

Substitute a = $-\frac{1}{3}$.	$-\frac{1}{3}-\frac{5}{9}\stackrel{?}{=}-\frac{8}{9}$
Change to common denominator.	$-\frac{3}{9}-\frac{5}{9}\stackrel{?}{=}-\frac{8}{9}$
Subtract.	$-\frac{8}{9}=-\frac{8}{9}$ ✓

Since $a=-\frac{1}{3}$ makes the equation true, we know that $a=-\frac{1}{3}$ is the solution to the equation.

Exercise 4.18.5

Solve: $a-\frac{3}{5}=-\frac{8}{5}$.

Answer

-1

Exercise 4.18.6

Solve: $n-\frac{3}{7}=-\frac{9}{7}$.

Answer

$-\frac{6}{7}$

The next example may not seem to have a fraction, but let's see what happens when we solve it.

Example 4.18.4: solve

Solve: $10q = 44$.

Solution

Divide both sides by 10 to undo the multiplication.	$\frac{10q}{10} = \frac{44}{10}$
Simplify.	$q = \frac{22}{5}$

Check:

Substitute $q = \frac{22}{5}$ into the original equation.	$10\left(\frac{22}{5}\right) \stackrel{?}{=} 44$
Simplify.	$\cancel{10}^{2}\left(\frac{22}{\cancel{5}}\right) \stackrel{?}{=} 44$
Multiply.	$44 = 44$ ✓

The solution to the equation was the fraction $\frac{22}{5}$. We leave it as an improper fraction.

Exercise 4.18.7

Solve: $12u = -76$.

Answer

$-\frac{19}{3}$

Exercise 4.18.8

Solve: $8m = 92$.

Answer

$\frac{23}{2}$

Solve Equations with Fractions Using the Multiplication Property of Equality

Consider the equation $\frac{x}{4} = 3$. We want to know what number divided by 4 gives 3. So to "undo" the division, we will need to multiply by 4. The *Multiplication Property of Equality* will allow us to do this. This property says that if we start with two equal quantities and multiply both by the same number, the results are equal.

Definition: The Multiplication Property of Equality

For any numbers a, b, and c, if $a = b$, then $ac = bc$. If you multiply both sides of an equation by the same quantity, you still have equality.

Let's use the Multiplication Property of Equality to solve the equation $\frac{x}{7} = -9$.

Example 4.18.5: solve

Solve: $\frac{x}{7} = -9$.

Solution

Use the Multiplication Property of Equality to multiply both sides by 7 . This will isolate the variable.	$7 \cdot \frac{x}{7} = 7(-9)$
Multiply.	$\frac{7x}{7} = -63$
Simplify.	$x = -63$
Check. Substitute -63 for x in the original equation.	$\frac{-63}{7} \stackrel{?}{=} -9$
The equation is true.	$-9 = -9$; ✓

Exercise 4.18.9

Solve: $\frac{f}{5} = -25$.

Answer

-125

Exercise 4.18.10

Solve: $\frac{h}{9} = -27$.

Answer

-243

Example 4.18.6:solve

Solve: $\frac{p}{-8} = -40$.

Solution

Here, p is divided by -8. We must multiply by -8 to isolate p.

Multiply both sides by −8.	$-8\left(\frac{p}{-8}\right) = -8(-40)$
Multiply.	$\frac{-8p}{-8} = 320$
Simplify.	$p = 320$

Check:

Substitute p = 320.	$\frac{320}{-8} \stackrel{?}{=} -40$
The equation is true.	$-40 = -40$ ✓

Exercise 4.18.11

Solve: $\frac{c}{-7} = -35$.

Answer

245

Exercise 4.18.12

Solve: $\frac{x}{-11} = -12$.

Answer

132

Solve Equations with a Coefficient of -1

Look at the equation $-y = 15$. Does it look as if y is already isolated? But there is a negative sign in front of y, so it is not isolated.

There are three different ways to isolate the variable in this type of equation. We will show all three ways in Example 4.18.7.

Example 4.18.7: solve

Solve: $-y = 15$.

Solution

One way to solve the equation is to rewrite $-y$ as $-1y$, and then use the Division Property of Equality to isolate y.

Rewrite −y as −1y.	$-1y = 15$
Divide both sides by −1.	$\frac{-1y}{-1} = \frac{15}{-1}$
Simplify each side.	$y = -15$

Another way to solve this equation is to multiply both sides of the equation by −1.

Multiply both sides by −1.	$-1(-y) = -1(15)$
Simplify each side.	$y = -15$

The third way to solve the equation is to read $-y$ as "the opposite of y." What number has 15 as its opposite? The opposite of 15 is -15. So $y = -15$.

For all three methods, we isolated y is isolated and solved the equation.

Check:

Substitute y = −15.	$-(15) \stackrel{?}{=} (15)$
Simplify. The equation is true.	$15 = 15$ ✓

Exercise 4.18.13

Solve: $-y = 48$.

Answer

-48

Exercise 4.18.14

Solve: $-c = -23$.

Answer

23

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4.19: Solve Equations with Fractions (Part 2)

Solve Equations with a Fraction Coefficient

When we have an equation with a fraction coefficient we can use the Multiplication Property of Equality to make the coefficient equal to 1. For example, in the equation:

$$\frac{3}{4}x = 24$$

The coefficient of x is $\frac{3}{4}$. To solve for x, we need its coefficient to be 1. Since the product of a number and its reciprocal is 1, our strategy here will be to isolate x by multiplying by the reciprocal of $\frac{3}{4}$. We will do this in Example 4.19.1.

Example 4.19.8: solve

Solve: $\frac{3}{4}x = 24$.

Solution

Multiply both sides by the reciprocal of the coefficient.	$\frac{4}{3} \cdot \frac{3}{4}x = \frac{4}{3} \cdot 24$
Simplify.	$1x = \frac{4}{3} \cdot \frac{24}{1}$
Multiply.	$x = 32$

Check:

Substitute x = 32.	$\frac{3}{4} \cdot 32 \stackrel{?}{=} 24$
Rewrite 32 as a fraction.	$\frac{3}{4} \cdot \frac{32}{1} \stackrel{?}{=} 24$
Multiply. The equation is true.	$24 = 24$ ✓

Notice that in the equation $\frac{3}{4}x = 24$, we could have divided both sides by $\frac{3}{4}$ to get x by itself. Dividing is the same as multiplying by the reciprocal, so we would get the same result. But most people agree that multiplying by the reciprocal is easier.

Exercise 4.19.15

Solve: $\frac{2}{5}n = 14$.

Answer

35

Exercise 4.19.16

Solve: $\frac{5}{6}y = 15$.

Answer

18

Example 4.19.9: solve

Solve: $-\frac{3}{8}w = 72$.

Solution

The coefficient is a negative fraction. Remember that a number and its reciprocal have the same sign, so the reciprocal of the coefficient must also be negative.

Multiply both sides by the reciprocal of $-\frac{3}{8}$.	$-\frac{8}{3}\left(-\frac{3}{8}w\right) = \left(-\frac{8}{3}\right)72$
Simplify; reciprocals multiply to one.	$1w = -\frac{8}{3} \cdot \frac{72}{1}$
Multiply.	$w = -192$

Check:

Let w = −192.	$-\frac{3}{8}(-192) \stackrel{?}{=} 72$
Multiply. It checks.	$72 = 72$ ✓

Exercise 4.19.17

Solve: $-\frac{4}{7}a = 52$.

Answer

-91

Exercise 4.19.18

Solve: $-\frac{7}{9}w = 84$.

Answer

-108

Translate Sentences to Equations and Solve

Now we have covered all four properties of equality—subtraction, addition, division, and multiplication. We'll list them all together here for easy reference.

Table 4.19.2

Subtraction Property of Equality: For any real numbers a, b, and c, if a = b, then a − c = b − c.	**Addition Property of Equality**: For any real numbers a, b, and c, if a = b, then a + c = b + c.
Division Property of Equality: For any numbers a, b, and c, where c ≠ 0 if a = b, then $\frac{a}{c} = \frac{b}{c}$.	**Multiplication Property of Equality**: For any real numbers a, b, and c if a = b, then ac = bc.

When you add, subtract, multiply or divide the same quantity from both sides of an equation, you still have equality. In the next few examples, we'll translate sentences into equations and then solve the equations. It might be helpful to review the translation table in Evaluate, Simplify, and Translate Expressions.

Example 4.19.10: solve

Translate and solve: n divided by 6 is -24.

Solution

Translate.	$\underbrace{\frac{n}{6}}_{n \text{ divided by } 6} \underbrace{=}_{\text{is}} \underbrace{-24}_{24}$
Multiply both sides by 6.	$6 \cdot \frac{n}{6} = 6(-24)$
Simplify.	$n = -144$
Check:	Is −144 divided by 6 equal to −24?
Translate.	$\frac{-144}{6} \stackrel{?}{=} -24$
Simplify. It checks.	$-24 = -24$ ✓

Exercise 4.19.19

Translate and solve: n divided by 7 is equal to -21.

Answer

$\frac{n}{7} = -21$; $n = -147$

Exercise 4.19.20

Translate and solve: n divided by 8 is equal to -56.

Answer

$\frac{n}{8} = -56$; $n = -448$

Example 4.19.11: solve

Translate and solve: The quotient of q and -5 is 70.

Solution

Translate.	$\underbrace{\frac{q}{-5}}_{\text{The quotient of } q \text{ and } -5} \underbrace{=}_{\text{is}} \underbrace{70}_{70}$
Multiply both sides by −5.	$-5\left(\frac{q}{-5}\right) = -5(70)$
Simplify.	$q = -350$
Check:	Is the quotient of −350 and −5 equal to 70?
Translate.	$\frac{-350}{-5} \stackrel{?}{=} 70$
Simplify. It checks.	$70 = 70$ ✓

Exercise 4.19.21

Translate and solve: The quotient of q and -8 is 72.

Answer

$\frac{q}{-8} = 72; q = -576$

Exercise 4.19.22

Translate and solve: The quotient of p and -9 is 81.

Answer

$\frac{p}{-9} = 81; p = -729$

Example 4.19.12: solve

Translate and solve: Two-thirds of f is 18.

Solution

Translate.	$\underbrace{\frac{2}{3}f}_{\text{Two-thirds of } f} \quad \underbrace{=}_{\text{is}} \quad \underbrace{18}_{18}$
Multiply both sides by $\frac{3}{2}$.	$\frac{3}{2} \cdot \frac{2}{3}f = \frac{3}{2} \cdot 18$
Simplify.	$f = 27$
Check:	Is two-thirds of 27 equal to 18?
Translate.	$\frac{2}{3}(27) \stackrel{?}{=} 18$
Simplify. It checks.	$18 = 18$ ✓

Exercise 4.19.23

Translate and solve: Two-fifths of f is 16.

Answer

$\frac{2}{5}f = 16; f = 40$

Exercise 4.19.24

Translate and solve: Three-fourths of f is 21.

Answer

$\frac{3}{4}f = 21; f = 28$

Example 4.19.13: solve

Translate and solve: The quotient of m and $\frac{5}{6}$ is $\frac{3}{4}$.

Solution

Translate.	$\frac{\frac{m}{5}}{6} = \frac{3}{4}$
Multiply both sides by $frac56$ to isolate m.	$\frac{5}{6}\left(\frac{\frac{m}{5}}{6}\right) = \frac{5}{6}\left(\frac{3}{4}\right)$
Simplify.	$m = \frac{5 \cdot 3}{6 \cdot 4}$
Remove common factors and multiply.	$m = \frac{5}{8}$

Check:

Is the quotient of $\frac{5}{8}$ and $\frac{5}{6}$ equal to $\frac{3}{4}$?	$\frac{\frac{5}{8}}{\frac{5}{6}} \stackrel{?}{=} \frac{3}{4}$
Rewrite as division.	$\frac{5}{8} \div \frac{5}{6} \stackrel{?}{=} \frac{3}{4}$
Multiply the first fraction by the reciprocal of the second.	$\frac{5}{8} \cdot \frac{6}{5} \stackrel{?}{=} \frac{3}{4}$
Simplify.	$\frac{3}{4} = \frac{3}{4}$ ✓

Our solution checks.

Exercise 4.19.25

Translate and solve. The quotient of n and $\frac{2}{3}$ is $\frac{5}{12}$.

Answer

$\frac{\frac{n}{2}}{3} = \frac{5}{12}$; $n = \frac{5}{18}$

Exercise 4.19.26

Translate and solve. The quotient of c and $\frac{3}{8}$ is $\frac{4}{9}$.

Answer

$\frac{\frac{c}{3}}{8} = \frac{4}{9}$; $c = \frac{1}{6}$

Example 4.19.14: solve

Translate and solve: The sum of three-eighths and x is three and one-half.

Solution

Translate.	The sum of three-eighths and x is three and one-half $\frac{3}{8}+x = 3\frac{1}{2}$
Use the Subtraction Property of Equality to subtract $\frac{3}{8}$ from both sides.	$\frac{3}{8}+x-\frac{3}{8}=3\frac{1}{2}-\frac{3}{8}$
Combine like terms on the left side.	$x=3\frac{1}{2}-\frac{3}{8}$
Convert mixed number to improper fraction.	$x=3\frac{1}{2}-\frac{3}{8}$
Convert to equivalent fractions with LCD of 8.	$x=\frac{7}{2}-\frac{3}{8}$
Subtract.	$x=\frac{25}{8}$
Write as a mixed number.	$x=3\frac{1}{8}$

We write the answer as a mixed number because the original problem used a mixed number. Check: Is the sum of three-eighths and $3\frac{1}{8}$ equal to three and one-half?

Add.	$3\frac{4}{8} \stackrel{?}{=} 3\frac{1}{2}$
Simplify.	$3\frac{1}{2}=3\frac{1}{2}$

The solution checks.

Exercise 4.19.27

Translate and solve: The sum of five-eighths and x is one-fourth.

Answer

$\frac{5}{8}+x=\frac{1}{4}$; $x=-\frac{3}{8}$

Exercise 4.19.28

Translate and solve: The difference of one-and-three-fourths and x is five-sixths.

Answer

$1\frac{3}{4}-x=\frac{5}{6}$; $x=\frac{11}{12}$

Access Additional Online Resources

- Solve One Step Equations With Fractions
- Solve One Step Equations With Fractions by Adding or Subtracting

- Solve One Step Equations With Fraction by Multiplying

Practice Makes Perfect

Determine Whether a Fraction is a Solution of an Equation

In the following exercises, determine whether each number is a solution of the given equation.

498. $x - \frac{2}{5} = \frac{1}{10}$:
   a. $x = 1$
   b. $x = \frac{1}{2}$
   c. $x = -\frac{1}{2}$

499. $y - \frac{1}{2} = \frac{5}{12}$:
   a. $y = 1$
   b. $y = \frac{3}{4}$
   c. $y = -\frac{3}{4}$

500. $h + \frac{3}{4} = \frac{2}{5}$:
   a. $h = 1$
   b. $h = \frac{7}{20}$
   c. $h = -\frac{7}{20}$

501. $k + \frac{2}{5} = \frac{5}{6}$:
   a. $k = 1$
   b. $k = \frac{13}{30}$
   c. $k = -\frac{13}{30}$

Solve Equations with Fractions using the Addition, Subtraction, and Division Properties of Equality

In the following exercises, solve.

502. $y + \frac{1}{3} = \frac{4}{3}$
503. $m + \frac{3}{8} = \frac{7}{8}$
504. $f + \frac{9}{10} = \frac{2}{5}$
505. $h + \frac{5}{6} = \frac{1}{6}$
506. $a - \frac{5}{8} = -\frac{7}{8}$
507. $c - \frac{1}{4} = -\frac{5}{4}$
508. $x - \left(-\frac{3}{20}\right) = -\frac{11}{20}$
509. $z - \left(-\frac{5}{12}\right) = -\frac{7}{12}$
510. $n - \frac{1}{6} = \frac{3}{4}$
511. $p - \frac{3}{10} = \frac{5}{8}$

512. $s + \left(-\frac{1}{2}\right) = -\frac{8}{9}$
513. $k + \left(-\frac{1}{3}\right) = -\frac{4}{5}$
514. $5j = 17$
515. $7k = 18$
516. $-4w = 26$
517. $-9v = 33$

Solve Equations with Fractions Using the Multiplication Property of Equality

In the following exercises, solve.

518. $\frac{f}{4} = -20$
519. $\frac{b}{3} = -9$
520. $\frac{y}{7} = -21$
521. $\frac{x}{8} = -32$
522. $\frac{p}{-5} = -40$
523. $\frac{q}{-4} = -40$
524. $\frac{r}{-12} = -6$
525. $\frac{s}{-15} = -3$
526. $-x = 23$
527. $-y = 42$
528. $-h = -\frac{5}{12}$
529. $-k = -\frac{17}{20}$
530. $\frac{4}{5}n = 20$
531. $\frac{3}{10}p = 30$
532. $\frac{3}{8}q = -48$
533. $\frac{5}{2}m = -40$
534. $-\frac{2}{9}a = 16$
535. $-\frac{3}{7}b = 9$
536. $-\frac{6}{11}u = -24$
537. $-\frac{5}{12}v = -15$

Mixed Practice

In the following exercises, solve.

538. $3x = 0$
539. $8y = 0$
540. $4f = \frac{4}{5}$
541. $7g = \frac{7}{9}$

542. $p + \frac{2}{3} = \frac{1}{12}$
543. $q + \frac{5}{6} = \frac{1}{12}$
544. $\frac{7}{8}m = \frac{1}{10}$
545. $\frac{1}{4}n = \frac{7}{10}$
546. $-\frac{2}{5} = x + \frac{3}{4}$
547. $-\frac{2}{3} = y + \frac{3}{8}$
548. $\frac{11}{20} = -f$
549. $\frac{8}{15} = -d$

Translate Sentences to Equations and Solve

In the following exercises, translate to an algebraic equation and solve.

550. n divided by eight is −16.
551. n divided by six is −24.
552. m divided by −9 is −7.
553. m divided by −7 is −8.
554. The quotient of f and −3 is −18.
555. The quotient of f and −4 is −20.
556. The quotient of g and twelve is 8.
557. The quotient of g and nine is 14.
558. Three-fourths of q is 12.
559. Two-fifths of q is 20.
560. Seven-tenths of p is −63.
561. Four-ninths of p is −28.
562. m divided by 4 equals negative 6.
563. The quotient of h and 2 is 43.
564. Three-fourths of z is the same as 15.
565. The quotient of a and $\frac{2}{3}$ is $\frac{3}{4}$.
566. The sum of five-sixths and x is $\frac{1}{2}$.
567. The sum of three-fourths and x is $\frac{1}{8}$.
568. The difference of y and one-fourth is $-\frac{1}{8}$.
569. The difference of y and one-third is $-\frac{1}{6}$.

Everyday Math

570. **Shopping** Teresa bought a pair of shoes on sale for $48. The sale price was $\frac{2}{3}$ of the regular price. Find the regular price of the shoes by solving the equation $\frac{2}{3}p = 48$
571. **Playhouse** The table in a child's playhouse is $\frac{3}{5}$ of an adult-size table. The playhouse table is 18 inches high. Find the height of an adult-size table by solving the equation $\frac{3}{5}h = 18$.

Writing Exercises

572. Example 4.100 describes three methods to solve the equation $-y = 15$. Which method do you prefer? Why?

573. Richard thinks the solution to the equation $\frac{3}{4}x = 24$ is 16. Explain why Richard is wrong.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether a fraction is a solution of an equation.			
solve equations with fractions using the addition, subtraction, and division properties of equality.			
solve equations using the multiplication property of equality.			
translate sentences to equations and solve.			

(b) Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

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4.E: Fractions (Exercises)

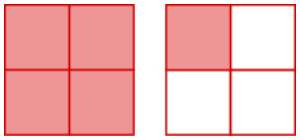
4.1 - Visualize Fractions

In the following exercises, name the fraction of each figure that is shaded.

574.

575.

In the following exercises, name the improper fractions. Then write each improper fraction as a mixed number.

576. 

577.

In the following exercises, convert the improper fraction to a mixed number.

578. $\frac{58}{15}$

579. $\frac{63}{11}$

In the following exercises, convert the mixed number to an improper fraction.

580. $12\frac{1}{4}$

581. $9\frac{4}{5}$

582. Find three fractions equivalent to $\frac{2}{5}$. Show your work, using figures or algebra.

583. Find three fractions equivalent to $-\frac{4}{3}$. Show your work, using figures or algebra.

In the following exercises, locate the numbers on a number line.

584. $\frac{5}{8}, \frac{4}{3}, 3\frac{3}{4}, 4$

585. $\frac{1}{4}, -\frac{1}{4}, 1\frac{1}{3}, -1\frac{1}{3}, \frac{7}{2}, -\frac{7}{2}$

In the following exercises, order each pair of numbers, using < or >.

586. $-1 \_\_\_ -\frac{2}{5}$

587. $-2\frac{1}{2} \_\_\_ -3$

4.2 - Multiply and Divide Fractions

In the following exercises, simplify.

588. $-\frac{63}{84}$
589. $-\frac{90}{120}$
590. $-\frac{14a}{14b}$
591. $-\frac{8x}{8y}$

In the following exercises, multiply.

592. $\frac{2}{5} \cdot \frac{8}{13}$
593. $-\frac{1}{3} \cdot \frac{12}{7}$
594. $\frac{2}{9} \cdot \left(-\frac{45}{32}\right)$
595. $6\text{m} \cdot \frac{4}{11}$
596. $-\frac{1}{4}\ (-32)$
597. $3\frac{1}{5} \cdot 1\frac{7}{8}$

In the following exercises, find the reciprocal.

598. $\frac{2}{9}$
599. $\frac{15}{4}$
600. 3
601. $-\frac{1}{4}$
602. Fill in the chart.

	Opposite	Absolute Value	Reciprocal
$-\frac{5}{13}$			
$\frac{3}{10}$			
$\frac{9}{4}$			
-12			

In the following exercises, divide.

603. $\frac{2}{3} \div \frac{1}{6}$
604. $\left(-\frac{3x}{5}\right) \div \left(-\frac{2y}{3}\right)$
605. $\frac{4}{5} \div 3$
606. $8 \div 2\frac{2}{3}$
607. $8\frac{2}{3} \div 1\frac{1}{12}$

4.3 - Multiply and Divide Mixed Numbers and Complex Fractions

In the following exercises, perform the indicated operation.

608. $3\frac{1}{5} \cdot 1\frac{7}{8}$
609. $-5\frac{7}{12} \cdot 4\frac{4}{11}$
610. $8 \div 2\frac{2}{3}$
611. $8\frac{2}{3} \div 1\frac{1}{12}$

In the following exercises, translate the English phrase into an algebraic expression.

612. the quotient of 8 and y
613. the quotient of V and the difference of h and 6

In the following exercises, simplify the complex fraction.

614. $\dfrac{\frac{5}{8}}{\frac{4}{5}}$
615. $\dfrac{\frac{8}{9}}{-4}$
616. $\dfrac{\frac{n}{4}}{\frac{3}{8}}$
617. $\dfrac{-1\frac{5}{6}}{-\frac{1}{12}}$

In the following exercises, simplify.

618. $\dfrac{5+16}{5}$
619. $\dfrac{8 \cdot 4 - 5^2}{3 \cdot 12}$
620. $\dfrac{8 \cdot 7 + 5(8-10)}{9 \cdot 3 - 6 \cdot 4}$

4.4 - Add and Subtract Fractions with Common Denominators

In the following exercises, add.

621. $\frac{3}{8} + \frac{2}{8}$
622. $\frac{4}{5} + \frac{1}{5}$
623. $\frac{2}{5} + \frac{1}{5}$
624. $\frac{15}{32} + \frac{9}{32}$
625. $\frac{x}{10} + \frac{7}{10}$

In the following exercises, subtract.

626. $\frac{8}{11} - \frac{6}{11}$
627. $\frac{11}{12} - \frac{5}{12}$
628. $\frac{4}{5} - \frac{y}{5}$

629. $-\frac{31}{30}-\frac{7}{30}$
630. $\frac{3}{2}-\left(\frac{3}{2}\right)$
631. $\frac{11}{15}-\frac{5}{15}-\left(-\frac{2}{15}\right)$

4.5 - Add and Subtract Fractions with Different Denominators

In the following exercises, find the least common denominator.

632. $\frac{1}{3}$ and $\frac{1}{12}$
633. $\frac{1}{3}$ and $\frac{4}{5}$
634. $\frac{8}{15}$ and $\frac{11}{20}$
635. $\frac{3}{4}$, $\frac{1}{6}$, and $\frac{5}{10}$

In the following exercises, change to equivalent fractions using the given LCD.

636. $\frac{1}{3}$ and $\frac{1}{5}$, LCD = 15
637. $\frac{3}{8}$ and $\frac{5}{6}$, LCD = 24
638. $-\frac{9}{16}$ and $\frac{5}{12}$, LCD = 48
639. $\frac{1}{3}$, $\frac{3}{4}$ and $\frac{4}{5}$, LCD = 60

In the following exercises, perform the indicated operations and simplify.

640. $\frac{1}{5}+\frac{2}{3}$
641. $\frac{11}{12}-\frac{2}{3}$
642. $-\frac{9}{10}-\frac{3}{4}$
643. $-\frac{11}{36}-\frac{11}{20}$
644. $-\frac{22}{25}+\frac{9}{40}$
645. $\frac{y}{10}-\frac{1}{3}$
646. $\frac{2}{5}+\left(-\frac{5}{9}\right)$
647. $\frac{4}{11}\div\frac{2}{7d}$
648. $\frac{2}{5}+\left(-\frac{3n}{8}\right)\left(-\frac{2}{9n}\right)$
649. $\frac{\left(\frac{2}{3}\right)^2}{\left(\frac{5}{8}\right)^2}$
650. $\left(\frac{11}{12}+\frac{3}{8}\right)\div\left(\frac{5}{6}-\frac{1}{10}\right)$

In the following exercises, evaluate.

651. y − $\frac{4}{5}$ when (a) y = $-\frac{4}{5}$ (b) y = $\frac{1}{4}$

652. $6mn^2$ when m = $\frac{3}{4}$ and n = $-\frac{1}{3}$

4.6 - Add and Subtract Mixed Numbers

In the following exercises, perform the indicated operation.

653. $4\frac{1}{3}+9\frac{1}{3}$

654. $6\frac{2}{5}+7\frac{3}{5}$

655. $5\frac{8}{11}+2\frac{4}{11}$

656. $3\frac{5}{8}+3\frac{7}{8}$

657. $9\frac{13}{20}-4\frac{11}{20}$

658. $2\frac{3}{10}-1\frac{9}{10}$

659. $2\frac{11}{12}-1\frac{7}{12}$

660. $8\frac{6}{11}-2\frac{9}{11}$

4.7 - Solve Equations with Fractions

In the following exercises, determine whether the each number is a solution of the given equation.

661. x − $\frac{1}{2}$ = $\frac{1}{6}$:
   a. x = 1
   b. x = $\frac{2}{3}$
   c. x = $-\frac{1}{3}$

662. y + $\frac{3}{5}$ = $\frac{5}{9}$:
   a. y = $\frac{1}{2}$
   b. y = $\frac{52}{45}$
   c. y = $-\frac{2}{45}$

In the following exercises, solve the equation.

663. n + $\frac{9}{11}$ = $\frac{4}{11}$

664. x − $\frac{1}{6}$ = $\frac{7}{6}$

665. h − $\left(-\frac{7}{8}\right)$ = $-\frac{2}{5}$

666. $\frac{x}{5}$ = −10

667. −z = 23

In the following exercises, translate and solve.

668. The sum of two-thirds and n is $-\frac{3}{5}$.

669. The difference of q and one-tenth is $\frac{1}{2}$.

670. The quotient of p and −4 is −8.

671. Three-eighths of y is 24.

PRACTICE TEST

Convert the improper fraction to a mixed number.

672. $\frac{19}{5}$

Convert the mixed number to an improper fraction.

673. $3\frac{2}{7}$

Locate the numbers on a number line.

674. $\frac{1}{2}, 1\frac{2}{3}, -2\frac{3}{4}$, and $\frac{9}{4}$

In the following exercises, simplify.

675. $\frac{5}{20}$

676. $\frac{18r}{27s}$

677. $\frac{1}{3} \cdot \frac{3}{4}$

678. $\frac{3}{5} \cdot 15$

679. $-36\text{u}\left(-\frac{4}{9}\right)$

680. $-5\frac{7}{12} \cdot 4\frac{4}{11}$

681. $-\frac{5}{6} \div \frac{5}{12}$

682. $\frac{7}{11} \div \left(-\frac{7}{11}\right)$

683. $\frac{9a}{10} \div \frac{15a}{8}$

684. $-6\frac{2}{5} \div 4$

685. $\left(-15\frac{5}{6}\right) \div \left(-3\frac{1}{6}\right)$

686. $\dfrac{\frac{-6}{6}}{\frac{11}{}}$

687. $\dfrac{\frac{p}{2}}{\frac{q}{5}}$

688. $\dfrac{-\frac{4}{15}}{-2\frac{2}{3}}$

689. $\frac{9^2 - 4^2}{9 - 4}$

690. $\frac{2}{d} + \frac{9}{d}$

691. $-\frac{3}{13} + \left(-\frac{4}{13}\right)$

692. $-\frac{22}{25} + \frac{9}{40}$

693. $\frac{2}{5} + \left(-\frac{7}{5}\right)$

694. $-\frac{3}{10}+\left(-\frac{5}{8}\right)$

695. $-\frac{3}{4} \div \frac{x}{3}$

696. $\frac{2^3-2^2}{\left(\frac{3}{4}\right)^2}$

697. $\frac{\frac{5}{14}+\frac{1}{8}}{\frac{9}{56}}$

Evaluate.

698. x + $\frac{1}{3}$ when (a) x = $\frac{2}{3}$ (b) x = $-\frac{5}{6}$

In the following exercises, solve the equation.

699. y + $\frac{3}{5}$ = $\frac{7}{5}$

700. a − $\frac{3}{10}$ = $-\frac{9}{10}$

701. f + $\left(-\frac{2}{3}\right)$ = $\frac{5}{12}$

702. $\frac{m}{-2}$ = −16

703. $-\frac{2}{3}$c = 18

704. Translate and solve: The quotient of p and −4 is −8. Solve for p.

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4.S: Fractions (Summary)

Key Terms

complex fraction	A fraction in which the numerator or the denominator contains a fraction.
equivalent fractions	Two or more fractions that have the same value.
fraction	A fraction is written $\frac{a}{b}$. in a fraction, a is the numerator and b is the denominator. A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.
least common denominator (LCD)	The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.
mixed number	A mixed number consists of a whole number a and a fraction $\frac{b}{c}$ where c ≠ 0. It is written as $a\frac{b}{c}$, where c ≠ 0.
proper and improper fractions	The fraction $\frac{a}{b}$ is proper if a < b and improper if a > b.
reciprocal	The reciprocal of the fraction $\frac{a}{b}$ is $\frac{b}{a}$ where a ≠ 0 and b ≠ 0.
simplified fraction	A fraction is considered simplified if there are no common factors in the numerator and denominator.

Key Concepts

4.1 - Visualize Fractions

- **Property of One**
  - Any number, except zero, divided by itself is one. $\frac{a}{a}$ = 1, where a ≠ 0.
- **Mixed Numbers**
  - A mixed number consists of a whole number a and a fraction $\frac{b}{c}$ where c ≠ 0.
  - It is written as follows: $a\frac{b}{c} \quad c \neq 0$
- **Proper and Improper Fractions**
  - The fraction $\frac{a}{b}$ is a proper fraction if a < b and an improper fraction if a ≥ b .
- **Convert an improper fraction to a mixed number**.
  1. Divide the denominator into the numerator.
  2. Identify the quotient, remainder, and divisor.
  3. Write the mixed number as $quotient\frac{remainder}{divisor}$.
- **Convert a mixed number to an improper fraction**.
  1. Multiply the whole number by the denominator.
  2. Add the numerator to the product found in Step 1.
  3. Write the final sum over the original denominator.
- **Equivalent Fractions Property**: If a, b, and c are numbers where b ≠ 0, c ≠ 0, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$.

4.2 - Multiply and Divide Fractions

- **Equivalent Fractions Property**
  - If a, b, c are numbers where b ≠ 0, c ≠ 0, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.
- **Simplify a fraction**.
  1. Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers.
  2. Simplify, using the equivalent fractions property, by removing common factors.
  3. Multiply any remaining factors.
- **Fraction Multiplication**
  - If a, b, c, and d are numbers where b ≠ 0 and d ≠ 0, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
- **Reciprocal**
  - A number and its reciprocal have a product of 1. $\frac{a}{b} \cdot \frac{b}{a} = 1$.
  - Table 4.98

Opposite	Absolute Value	Reciprocal
has opposite sign	is never negative	has same sign, fraction inverts

- **Fraction Division**
  - If a, b, c, and d are numbers where b ≠ 0, c ≠ 0, and d ≠ 0, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.
  - To divide fractions, multiply the first fraction by the reciprocal of the second.

4.3 - Multiply and Divide Mixed Numbers and Complex Fractions

- **Multiply or divide mixed numbers**.
  1. Convert the mixed numbers to improper fractions.
  2. Follow the rules for fraction multiplication or division.
  3. Simplify if possible.
- **Simplify a complex fraction**.
  1. Rewrite the complex fraction as a division problem.
  2. Follow the rules for dividing fractions.
  3. Simplify if possible.
- **Placement of negative sign in a fraction**.
  - For any positive numbers a and b, $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.
- **Simplify an expression with a fraction bar**.
  1. Simplify the numerator.
  2. Simplify the denominator.
  3. Simplify the fraction.

4.4 - Add and Subtract Fractions with Common Denominators

- **Fraction Addition**
  - If a, b, and c are numbers where c ≠ 0, then $\frac{a}{c} + \frac{b}{c} = \frac{a+c}{c}$.
  - To add fractions, add the numerators and place the sum over the common denominator.
- **Fraction Subtraction**
  - If a, b, and c are numbers where c ≠ 0, then $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$.
  - To subtract fractions, subtract the numerators and place the difference over the common denominator.

4.5 - Add and Subtract Fractions with Different Denominators

- **Find the least common denominator (LCD) of two fractions**.
  1. Factor each denominator into its primes.
  2. List the primes, matching primes in columns when possible.
  3. Bring down the columns.
  4. Multiply the factors. The product is the LCM of the denominators.
  5. The LCM of the denominators is the LCD of the fractions.
- **Equivalent Fractions Property**
  - If a, b, and c are whole numbers where b ≠ 0, c ≠ 0 then $\frac{a}{b}=\frac{a\cdot c}{b\cdot c}$ and $\frac{a\cdot c}{b\cdot c}=\frac{a}{b}$.
- **Convert two fractions to equivalent fractions with their LCD as the common denominator**.
  1. Find the LCD.
  2. For each fraction, determine the number needed to multiply the denominator to get the LCD.
  3. Use the Equivalent Fractions Property to multiply the numerator and denominator by the number from Step 2.
  4. Simplify the numerator and denominator.
- **Add or subtract fractions with different denominators**.
  1. Find the LCD.
  2. Convert each fraction to an equivalent form with the LCD as the denominator.
  3. Add or subtract the fractions.
  4. Write the result in simplified form.
- **Summary of Fraction Operations**
  - *Fraction multiplication*: Multiply the numerators and multiply the denominators. $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$.
  - *Fraction division*: Multiply the first fraction by the reciprocal of the second. $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$.
  - *Fraction addition*: Add the numerators and place the sum over the common denominator. If the fractions have different denominators, first convert them to equivalent forms with the LCD. $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$.
  - *Fraction subtraction*: Subtract the numerators and place the difference over the common denominator. If the fractions have different denominators, first convert them to equivalent forms with the LCD. $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$.
- **Simplify complex fractions**.
  1. Simplify the numerator.
  2. Simplify the denominator.
  3. Divide the numerator by the denominator.
  4. Simplify if possible.

4.6 - Add and Subtract Mixed Numbers

- **Add mixed numbers with a common denominator**.
  1. Add the whole numbers.
  2. Add the fractions.
  3. Simplify, if possible.
- **Subtract mixed numbers with common denominators**.
  1. Rewrite the problem in vertical form.
  2. Compare the two fractions. If the top fraction is larger than the bottom fraction, go to Step 3. If not, in the top mixed number, take one whole and add it to the fraction part, making a mixed number with an improper fraction.
  3. Subtract the fractions.
  4. Subtract the whole numbers.
  5. Simplify, if possible.
- **Subtract mixed numbers with common denominators as improper fractions.**
  1. Rewrite the mixed numbers as improper fractions.

2. Subtract the numerators.
3. Write the answer as a mixed number, simplifying the fraction part, if possible.

4.7 - Solve Equations with Fractions

- Determine whether a number is a solution to an equation.
  1. Substitute the number for the variable in the equation.
  2. Simplify the expressions on both sides of the equation.
  3. Determine whether the resulting equation is true. If it is true, the number is a solution. If it is not true, the number is not a solution.
- Addition, Subtraction, and Division Properties of Equality: For any numbers a, b, and c,
  - if a = b, then a + c = b + c. *Addition Property of Equality*
  - if a = b, then a - c = b - c. *Subtraction Property of Equality*
  - if a = b, then $\frac{a}{c} = \frac{b}{c}$, c ≠ 0. *Division Property of Equality*
- **The Multiplication Property of Equality**
  - For any numbers ab and c, a = b, then ac = bc.
  - If you multiply both sides of an equation by the same quantity, you still have equality.

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CHAPTER OVERVIEW

5: Decimals

Gasoline price changes all the time. They might go down for a period of time, but then they usually rise again. One thing that stays the same is that the price is not usually a whole number. Instead, it is shown using a decimal point to describe the cost in dollars and cents. We use decimal numbers all the time, especially when dealing with money. In this chapter, we will explore decimal numbers and how to perform operations using them.

Topic hierarchy

5.1: Decimals (Part 1)
5.2: Decimals (Part 2)
5.3: Decimal Operations (Part 1)
5.4: Decimal Operations (Part 2)
5.5: Decimals and Fractions (Part 1)
5.6: Decimals and Fractions (Part 2)
5.7: Rational and Irrational Numbers
5.8: Averages and Probability (Part 1)
5.9: Averages and Probability (Part 2)
5.10: Ratios and Rate (Part 1)
5.11: Ratios and Rate (Part 2)
5.12: Simplify and Use Square Roots (Part 1)
5.13: Simplify and Use Square Roots (Part 2)
5.14: Commutative and Associative Properties (Part 1)
5.15: Commutative and Associative Properties (Part 2)
5.16: Distributive Property
5.17: Properties of Identity, Inverses, and Zero
5.18: Solve Equations with Decimals
5.E: Decimals (Exercises)
5.E: The Properties of Real Numbers (Exercises)
5.S: Decimals (Summary)
5.S: The Properties of Real Numbers (Summary)

Figure 5.1 - The price of a gallon of gasoline is written as a decimal number. (credit: Mark Turnauckus, Flickr)

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5.1: Decimals (Part 1)

Learning Objectives

- Name decimals
- Write decimals
- Convert decimals to fractions or mixed numbers
- Locate decimals on the number line
- Order decimals
- Round decimals

be prepared!

Before you get started, take this readiness quiz.

1. Name the number 4,926,015 in words. If you missed this problem, review Example 1.1.4.
2. Round 748 to the nearest ten. If you missed this problem, review Example 1.2.9.
3. Locate $\frac{3}{10}$ on a number line. If you missed this problem, review Example 4.2.16.

Name Decimals

You probably already know quite a bit about decimals based on your experience with money. Suppose you buy a sandwich and a bottle of water for lunch. If the sandwich costs \$3.45, the bottle of water costs \$1.25, and the total sales tax is \$0.33, what is the total cost of your lunch?

$$\begin{array}{rl} \$3.45 & Sandwich \\ \$1.25 & Water \\ +\$0.33 & Tax \\ \hline \$5.03 & Total \end{array}$$

The total is \$5.03. Suppose you pay with a \$5 bill and 3 pennies. Should you wait for change? No, \$5 and 3 pennies is the same as \$5.03.

Because 100 pennies = \$1, each penny is worth $\frac{1}{100}$ of a dollar. We write the value of one penny as \$0.01, since 0.01 = $\frac{1}{100}$.

Writing a number with a decimal is known as decimal notation. It is a way of showing parts of a whole when the whole is a power of ten. In other words, decimals are another way of writing fractions whose denominators are powers of ten. Just as the counting numbers are based on powers of ten, decimals are based on powers of ten. Table 5.1.1 shows the counting numbers.

Table 5.1.1

Counting number	Name
1	One
10 = 10	Ten
10 • 10 = 100	One hundred
10 • 10 • 10 = 1000	One thousand
10 • 10 • 10 • 10 = 10,000	Ten thousand

How are decimals related to fractions? Table 5.1.2 shows the relation.

Table 5.1.2

Decimal	Fraction	Name

Decimal	Fraction	Name
0.1	$\frac{1}{10}$	One tenth
0.01	$\frac{1}{100}$	One hundredth
0.001	$\frac{1}{1000}$	One thousandth
0.0001	$\frac{1}{10,000}$	One ten-thousandth

When we name a whole number, the name corresponds to the place value based on the powers of ten. In Whole Numbers, we learned to read 10,000 as *ten thousand*. Likewise, the names of the decimal places correspond to their fraction values. Notice how the place value names in Figure 5.1.1 relate to the names of the fractions from Table 5.1.2.

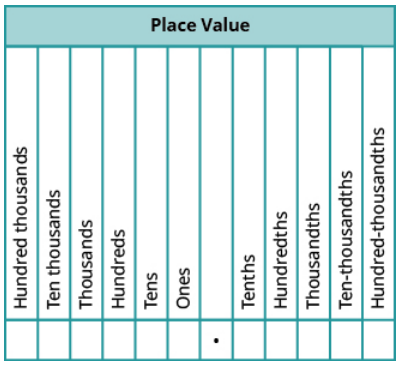

Figure 5.1.1 - This chart illustrates place values to the left and right of the decimal point.

Notice two important facts shown in Figure 5.1.1.

- The “th” at the end of the name means the number is a fraction. “One thousand” is a number larger than one, but “one thousandth” is a number smaller than one.
- The tenths place is the first place to the right of the decimal, but the tens place is two places to the left of the decimal.

Remember that $5.03 lunch? We read $5.03 as *five dollars and three cents*. Naming decimals (those that don’t represent money) is done in a similar way. We read the number 5.03 as *five and three hundredths*.

We sometimes need to translate a number written in decimal notation into words. As shown in Figure 5.1.2, we write the amount on a check in both words and numbers.

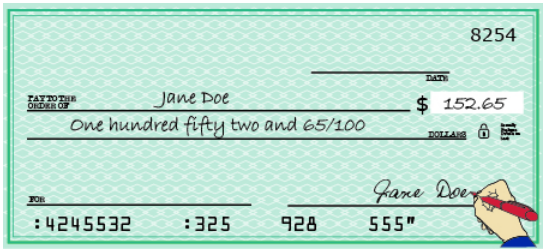

Figure 5.1.2 - When we write a check, we write the amount as a decimal number as well as in words. The bank looks at the check to make sure both numbers match. This helps prevent errors.

Let’s try naming a decimal, such as 15.68.	
We start by naming the number to the left of the decimal.	fifteen_____
We use the word “and” to indicate the decimal point.	fifteen and_____

Then we name the number to the right of the decimal point as if it were a whole number.	fifteen and sixty-eight_____
Last, name the decimal place of the last digit.	fifteen and sixty-eight hundredths

The number 15.68 is read *fifteen and sixty-eight hundredths*.

HOW TO: NAME A DECIMAL NUMBER

- Name the number to the left of the decimal point.
- Write “and” for the decimal point.
- Name the “number” part to the right of the decimal point as if it were a whole number.
- Name the decimal place of the last digit.

Example 5.1.1:

Name each decimal: (a) 4.3 (b) 2.45 (c) 0.009 (d) −15.571.

Solution

(a) 4.3

Name the number to the left of the decimal point.	four_____
Write "and" for the decimal point.	four and_____
Name the number to the right of the decimal point as if it were a whole number.	four and three_____
Name the decimal place of the last digit.	four and three tenths

(b) 2.45

Name the number to the left of the decimal point.	two_____
Write "and" for the decimal point.	two and_____
Name the number to the right of the decimal point as if it were a whole number.	two and forty-five_____
Name the decimal place of the last digit.	two and forty-five hundredths

(c) 0.009

Name the number to the left of the decimal point.	Zero is the number to the left of the decimal; it is not included in the name.
Name the number to the right of the decimal point as if it were a whole number.	nine_____
Name the decimal place of the last digit.	nine thousandths

(d) −15.571

Name the number to the left of the decimal point.	negative fifteen
Write "and" for the decimal point.	negative fifteen and_____
Name the number to the right of the decimal point as if it were a whole number.	negative fifteen and five hundred seventyone_____

Name the decimal place of the last digit.	negative fifteen and five hundred seventyone thousandths

Exercise 5.1.1:

Name each decimal: (a) 6.7 (b) 19.58 (c) 0.018 (d) −2.053

Answer a

six and seven tenths

Answer b

nineteen and fifty-eight hundredths

Answer c

eighteen thousandths

Answer d

negative two and fifty-three thousandths

Exercise 5.1.2:

Name each decimal: (a) 5.8 (b) 3.57 (c) 0.005 (d) −13.461

Answer a

five and eight tenths three and fifty-seven hundredths

Answer b

three and fifty-seven hundredths

Answer c

five thousandths

Answer d

negative thirteen and four hundred sixty-one thousandths

Write Decimals

Now we will translate the name of a decimal number into decimal notation. We will reverse the procedure we just used. Let's start by writing the number six and seventeen hundredths:

The word *and* tells us to place a decimal point.	___.___
The word before *and* is the whole number; write it to the left of the decimal point.	6._____
The decimal part is seventeen hundredths. Mark two places to the right of the decimal point for hundredths.	6._ _
Write the numerals for seventeen in the places marked.	6.17

Example 5.1.2:

Write fourteen and thirty-seven hundredths as a decimal.

Solution

Place a decimal point under the word ‘and’.	_____. _____
Translate the words before ‘and’ into the whole number and place it to the left of the decimal point.	14. _____
Mark two places to the right of the decimal point for “hundredths”.	14.__ __
Translate the words after “and” and write the number to the right of the decimal point.	14.37

Fourteen and thirty-seven hundredths is written 14.37.

Exercise 5.1.3:

Write as a decimal: thirteen and sixty-eight hundredths.

Answer

13.68

Exercise 5.1.4:

Write as a decimal: five and eight hundred ninety-four thousandths.

Answer

5.894

HOW TO: WRITE A DECIMAL NUMBER FROM ITS NAME

Step 1. Look for the word “and”—it locates the decimal point.

Step 2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.

- Place a decimal point under the word “and.” Translate the words before “and” into the whole number and place it to the left of the decimal point.
- If there is no “and,” write a “0” with a decimal point to its right.

Step 3. Translate the words after “and” into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.

Step 4. Fill in zeros for place holders as needed.

The second bullet in Step 2 is needed for decimals that have no whole number part, like ‘nine thousandths’. We recognize them by the words that indicate the place value after the decimal – such as ‘tenths’ or ‘hundredths.’ Since there is no whole number, there is no ‘and.’ We start by placing a zero to the left of the decimal and continue by filling in the numbers to the right, as we did above.

Example 5.1.3:

Write twenty-four thousandths as a decimal.

Solution

Look for the word "and".	There is no "and" so start with 0.
To the right of the decimal point, put three decimal places for thousandths.	0. __ __ __ tenths hundredths thousandths
Write the number 24 with the 4 in the thousandths place.	0. __ 2 4 tenths hundredths thousandths
Put zeros as placeholders in the remaining decimal places.	0.024

So, twenty-four thousandths is written 0.024

Exercise 5.1.5:

Write as a decimal: fifty-eight thousandths.

Answer

0.058

Exercise 5.1.6:

Write as a decimal: sixty-seven thousandths.

Answer

0.067

Before we move on to our next objective, think about money again. We know that $1 is the same as $1.00. The way we write $1 (or $1.00) depends on the context. In the same way, integers can be written as decimals with as many zeros as needed to the right of the decimal.

$$\begin{aligned} 5 &= 5.0 \qquad -2 = -2.0 \\ 5 &= 5.00 \qquad -2 = -2.00 \\ 5 &= 5.000 \qquad -2 = -2.000 \end{aligned}$$

$$and\ so\ on\ldots$$

Convert Decimals to Fractions or Mixed Numbers

We often need to rewrite decimals as fractions or mixed numbers. Let's go back to our lunch order to see how we can convert decimal numbers to fractions. We know that $5.03 means 5 dollars and 3 cents. Since there are 100 cents in one dollar, 3 cents means $\frac{3}{100}$ of a dollar, so 0.03 = $\frac{3}{100}$.

We convert decimals to fractions by identifying the place value of the farthest right digit. In the decimal 0.03, the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03.

$$0.03 = \frac{3}{100}$$

For our $5.03 lunch, we can write the decimal 5.03 as a mixed number.

$$5.03 = 5\frac{3}{100}$$

Notice that when the number to the left of the decimal is zero, we get a proper fraction. When the number to the left of the decimal is not zero, we get a mixed number.

HOW TO: CONVERT A DECIMAL NUMBER TO A FRACTION OR MIXED NUMBER

Step 1. Look at the number to the left of the decimal.

- If it is zero, the decimal converts to a proper fraction.
- If it is not zero, the decimal converts to a mixed number.
  - Write the whole number.

Step 2. Determine the place value of the final digit.

Step 3. Write the fraction.

- numerator—the 'numbers' to the right of the decimal point
- denominator—the place value corresponding to the final digit

Step 4. Simplify the fraction, if possible.

Example 5.1.4:

Write each of the following decimal numbers as a fraction or a mixed number: (a) 4.09 (b) 3.7 (c) −0.286

Solution

(a) 4.09

There is a 4 to the left of the decimal point. Write "4" as the whole number part of the mixed number.	$4\frac{\square}{\square}$
Determine the place value of the final digit.	4. 0 9 tenths hundredths
Write the fraction. Write 9 in the numerator as it is the number to the right of the decimal point.	$4\frac{9}{\square}$
Write 100 in the denominator as the place value of the final digit, 9, is hundredth.	$4\frac{9}{100}$
The fraction is in simplest form.	So, $4.09 = 4\frac{9}{100}$

Did you notice that the number of zeros in the denominator is the same as the number of decimal places?

(b) 3.7

There is a 3 to the left of the decimal point. Write "3" as the whole number part of the mixed number.	$3\frac{\square}{\square}$
Determine the place value of the final digit.	3. 7 tenths
Write the fraction. Write 7 in the numerator as it is the number to the right of the decimal point.	$3\frac{7}{\square}$
Write 10 in the denominator as the place value of the final digit, 7, is tenths.	$3\frac{7}{10}$
The fraction is in simplest form.	So, $3.7 = 3\frac{7}{10}$

(c) −0.286

There is a 0 to the left of the decimal point. Write a negative sign before the fraction.	$-\frac{\square}{\square}$
Determine the place value of the final digit and write it in the denominator.	$-0.\underset{\text{tenths}}{2}\underset{\text{hundredths}}{8}\underset{\text{thousandths}}{6}$
Write the fraction. Write 286 in the numerator as it is the number to the right of the decimal point. Write 1,000 in the denominator as the place value of the final digit, 6, is thousandths.	$-\frac{286}{1000}$
We remove a common factor of 2 to simplify the fraction.	$-\frac{143}{500}$

? Exercise 5.1.7:

Write as a fraction or mixed number. Simplify the answer if possible. (a) 5.3 (b) 6.07 (c) −0.234

Answer a

$5\frac{3}{10}$

Answer b

$6\frac{7}{100}$

Answer c

$-\frac{234}{1000}$

? Exercise 5.1.8:

Write as a fraction or mixed number. Simplify the answer if possible. (a) 8.7 (b) 1.03 (c) −0.024

Answer a

$8\frac{7}{10}$

Answer b

$1\frac{3}{100}$

Answer c

$-\frac{24}{1000}$

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5.2: Decimals (Part 2)

Locate Decimals on the Number Line

Since decimals are forms of fractions, locating decimals on the number line is similar to locating fractions on the number line.

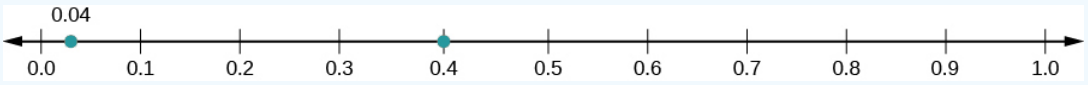
Example 5.2.5:

Locate 0.4 on a number line.

Solution

The decimal 0.4 is equivalent to $\frac{4}{10}$, so 0.4 is located between 0 and 1. On a number line, divide the interval between 0 and 1 into 10 equal parts and place marks to separate the parts.

Label the marks 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. We write 0 as 0.0 and 1 as 1.0, so that the numbers are consistently in tenths. Finally, mark 0.4 on the number line.



Exercise 5.2.9:

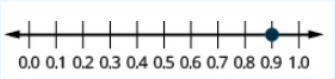
Locate 0.6 on a number line.

Answer

Exercise 5.2.10:

Locate 0.9 on a number line.

Answer



Example 5.2.6:

Locate −0.74 on a number line.

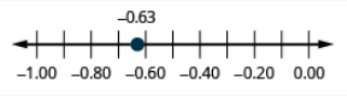
Solution

The decimal −0.74 is equivalent to $-\frac{74}{100}$, so it is located between 0 and −1. On a number line, mark off and label the multiples of -0.10 in the interval between 0 and −1 (−0.10, −0.20, etc.) and mark −0.74 between −0.70 and −0.80, a little closer to −0.70.

Exercise 5.2.11:

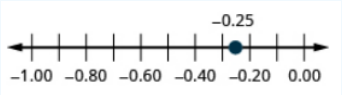
Locate −0.63 on a number line.

Answer



Exercise 5.2.12:

Locate −0.25 on a number line.



Answer

Order Decimals

Which is larger, 0.04 or 0.40?

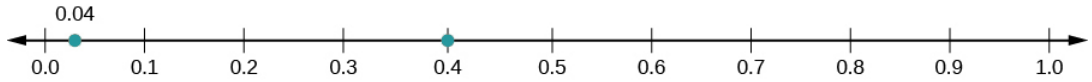
If you think of this as money, you know that $0.40 (forty cents) is greater than $0.04 (four cents). So, 0.40 > 0.04.

In previous chapters, we used the number line to order numbers.

a < b: 'a is less than b' when a is to the left of b on the number line

a > b: 'a is greater than b' when a is to the right of b on the number line

Where are 0.04 and 0.40 located on the number line?



We see that 0.40 is to the right of 0.04. So we know 0.40 > 0.04.

How does 0.31 compare to 0.308? This doesn't translate into money to make the comparison easy. But if we convert 0.31 and 0.308 to fractions, we can tell which is larger.

	0.31		0.308	
Convert to fractions.	$\frac{31}{100}$	(5.2.1)	$\frac{308}{1000}$	(5.2.2)
We need a common denominator to compare them.	$\frac{31 \cdot 10}{100 \cdot 10}$	(5.2.3)	$\frac{308}{1000}$	(5.2.4)
	$\frac{310}{1000}$	(5.2.5)	$\frac{308}{1000}$	(5.2.6)

Because 310 > 308, we know that $\frac{310}{1000} > \frac{308}{1000}$. Therefore, 0.31 > 0.308.

Notice what we did in converting 0.31 to a fraction—we started with the fraction $\frac{31}{100}$ and ended with the equivalent fraction $\frac{310}{1000}$. Converting $\frac{310}{1000}$ back to a decimal gives 0.310. So 0.31 is equivalent to 0.310. Writing zeros at the end of a decimal does

not change its value.

$$\frac{31}{100} = \frac{310}{1000} \quad and \quad 0.31 = 0.310 \tag{5.2.7}$$

If two decimals have the same value, they are said to be equivalent decimals.

$$0.31 = 0.310 \tag{5.2.8}$$

We say 0.31 and 0.310 are equivalent decimals.

HOW TO: ORDER DECIMALS

Step 1. Check to see if both numbers have the same number of decimal places. If not, write zeros at the end of the one with fewer digits to make them match.

Step 2. Compare the numbers to the right of the decimal point as if they were whole numbers.

Step 3. Order the numbers using the appropriate inequality sign.

Example 5.2.7:

Order the following decimals using < or >: (a) 0.64 __0.6 (b) 0.83 __0.803

Solution

(a) 0.64 __0.6

Check to see if both numbers have the same number of decimal places. They do not, so write one zero at the right of 0.6.	0.64 __0.60
Compare the numbers to the right of the decimal point as if they were whole numbers.	64 > 60
Order the numbers using the appropriate inequality sign.	0.64 > 0.60 0.64 > 0.6

(b) 0.83 __0.803

Check to see if both numbers have the same number of decimal places. They do not, so write one zero at the right of 0.83.	0.830 __0.803
Compare the numbers to the right of the decimal point as if they were whole numbers.	830 > 803
Order the numbers using the appropriate inequality sign.	0.830 > 0.803 0.83 > 0.803

Exercise 5.2.13:

Order each of the following pairs of numbers, using < or >: (a) 0.42__0.4 (b) 0.76__0.706

Answer a

>

Answer b

>

Exercise 5.2.14:

Order each of the following pairs of numbers, using < or >: (a) 0.1__0.18 (b) 0.305__0.35

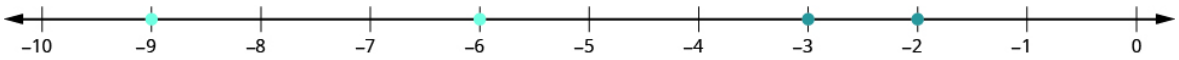
Answer a

<

Answer b

<

When we order negative decimals, it is important to remember how to order negative integers. Recall that larger numbers are to the right on the number line. For example, because −2 lies to the right of −3 on the number line, we know that −2 > −3. Similarly, smaller numbers lie to the left on the number line. For example, because −9 lies to the left of −6 on the number line, we know that −9 < −6.



If we zoomed in on the interval between 0 and −1, we would see in the same way that −0.2 > −0.3 and −0.9 < −0.6.

Example 5.2.8:

Use < or > to order. −0.1__−0.8.

Solution

Write the numbers one under the other, lining up the decimal points.	−0.1 −0.8
They have the same number of digits.	
Since −1 > −8, −1 tenth is greater than −8 tenths.	−0.1 > −0.8

Exercise 5.2.15:

Order each of the following pairs of numbers, using < or >: −0.3___−0.5

Answer

>

Exercise 5.2.16:

Order each of the following pairs of numbers, using < or >: −0.6___−0.7

Answer

>

Round Decimals

In the United States, gasoline prices are usually written with the decimal part as thousandths of a dollar. For example, a gas station might post the price of unleaded gas at $3.279 per gallon. But if you were to buy exactly one gallon of gas at this price, you would pay $3.28 , because the final price would be rounded to the nearest cent. In Whole Numbers, we saw that we round numbers to get an approximate value when the exact value is not needed. Suppose we wanted to round $2.72 to the nearest dollar. Is it closer to $2 or to $3? What if we wanted to round $2.72 to the nearest ten cents; is it closer to $2.70 or to $2.80? The number lines in Figure 5.2.3 can help us answer those questions.

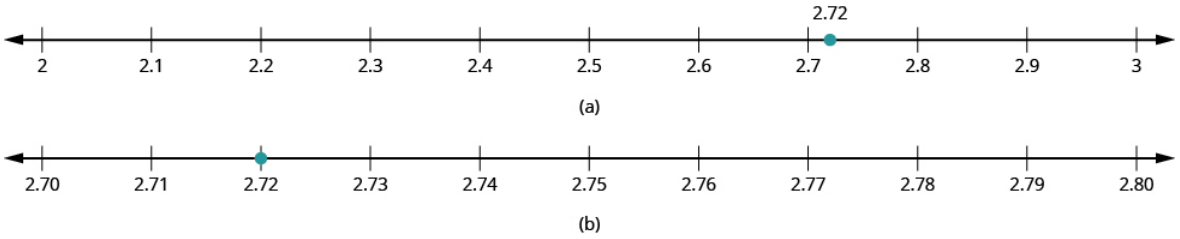


Figure 5.2.3 - (b) We see that 2.72 is closer to 3 than to 2. So, 2.72 rounded to the nearest whole number is 3. (b) We see that 2.72 is closer to 2.70 than 2.80. So we say that 2.72 rounded to the nearest tenth is 2.7.

Can we round decimals without number lines? Yes! We use a method based on the one we used to round whole numbers.

HOW TO: ROUND A DECIMAL

Step 1. Locate the given place value and mark it with an arrow.

Step 2. Underline the digit to the right of the given place value.

Step 3. Is this digit greater than or equal to 5?

- Yes - add 1 to the digit in the given place value.
- No - do not change the digit in the given place value

Step 4. Rewrite the number, removing all digits to the right of the given place value.

Example 5.2.9:

Round 18.379 to the nearest hundredth.

Solution

Locate the hundredths place and mark it with an arrow.	hundredths place ↓ 18.379
Underline the digit to the right of the 7.	hundredths place ↓ 18.379
Because 9 is greater than or equal to 5, add 1 to the 7.	18.379 (add 1 / delete)
Rewrite the number, deleting all digits to the right of the hundredths place.	18.38

18.38 is 18.379 rounded to the nearest hundredth.

Exercise 5.2.17:

Round to the nearest hundredth: 1.047.

Answer

1.05

Exercise 5.2.18:

Round to the nearest hundredth: 9.173.

Answer

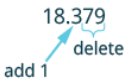
9.17

Example 5.2.10:

Round 18.379 to the nearest (a) tenth (b) whole number.

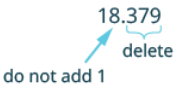
Solution

(a) Round 18.379 to the nearest tenth.

Locate the tenths place and mark it with an arrow.	
Underline the digit to the right of the tenths digit.	
Because 7 is greater than or equal to 5, add 1 to the 3.	
Rewrite the number, deleting all digits to the right of the tenths place.	18.4

So, 18.379 rounded to the nearest tenth is 18.4.

(b) Round 18.379 to the nearest whole number.

Locate the ones place and mark it with an arrow.	
Underline the digit to the right of the ones place.	
Since 3 is not greater than or equal to 5, do not add 1 to the 8.	
Rewrite the number, deleting all digits to the right of the ones place.	18

So 18.379 rounded to the nearest whole number is 18.

Exercise 5.2.19:

Round 6.582 to the nearest (a) hundredth (b) tenth (c) whole number.

Answer a

6.58

Answer b

6.6

Answer c

7

Exercise 5.2.20:

Round 15.2175 to the nearest (a) thousandth (b) hundredth (c) tenth.

Answer a

15.218

Answer b

15.22

Answer c

15.2

ACCESS ADDITIONAL ONLINE RESOURCES

Introduction to Decimal Notation

Write a Number in Decimal Notation from Words

Identify Decimals on the Number Line

Rounding Decimals

Writing a Decimal as a Simplified Fraction

Practice Makes Perfect

Name Decimals In the following exercises, name each decimal.

1. 5.5
2. 7.8
3. 5.01
4. 14.02
5. 8.71
6. 2.64
7. 0.002
8. 0.005
9. 0.381
10. 0.479
11. −17.9
12. −31.4

Write Decimals

In the following exercises, translate the name into a decimal number.

13. Eight and three hundredths
14. Nine and seven hundredths
15. Twenty-nine and eighty-one hundredths
16. Sixty-one and seventy-four hundredths
17. Seven tenths
18. Six tenths

19. One thousandth
20. Nine thousandths
21. Twenty-nine thousandths
22. Thirty-five thousandths
23. Negative eleven and nine ten-thousandths
24. Negative fifty-nine and two ten-thousandths
25. Thirteen and three hundred ninety-five ten thousandths
26. Thirty and two hundred seventy-nine thousandths

Convert Decimals to Fractions or Mixed Numbers

In the following exercises, convert each decimal to a fraction or mixed number.

27. 1.99
28. 5.83
29. 15.7
30. 18.1
31. 0.239
32. 0.373
33. 0.13
34. 0.19
35. 0.011
36. 0.049
37. −0.00007
38. −0.00003
39. 6.4
40. 5.2
41. 7.05
42. 9.04
43. 4.006
44. 2.008
45. 10.25
46. 12.75
47. 1.324
48. 2.482
49. 14.125
50. 20.375

Locate Decimals on the Number Line

In the following exercises, locate each number on a number line.

51. 0.8
52. 0.3
53. −0.2
54. −0.9
55. 3.1
56. 2.7
57. −2.5
58. −1.6

Order Decimals

In the following exercises, order each of the following pairs of numbers, using < or >.

59. 0.9__0.6
60. 0.7__0.8
61. 0.37__0.63

62. 0.86__0.69
63. 0.6__0.59
64. 0.27__0.3
65. 0.91__0.901
66. 0.415__0.41
67. −0.5__−0.3
68. −0.1_−0.4
69. −0.62_−0.619
70. −7.31_−7.3

Round Decimals

In the following exercises, round each number to the nearest tenth.

71. 0.67
72. 0.49
73. 2.84
74. 4.63

In the following exercises, round each number to the nearest hundredth.

75. 0.845
76. 0.761
77. 5.7932
78. 3.6284
79. 0.299
80. 0.697
81. 4.098
82. 7.096

In the following exercises, round each number to the nearest (a) hundredth (b) tenth (c) whole number.

83. 5.781
84. 1.638
85. 63.479
86. 84.281

Everyday Math

87. **Salary Increase** Danny got a raise and now makes $58,965.95 a year. Round this number to the nearest: (a) dollar (b) thousand dollars (c) ten thousand dollars.
88. **New Car Purchase** Selena's new car cost $23,795.95. Round this number to the nearest: (a) dollar (b) thousand dollars (c) ten thousand dollars.
89. **Sales Tax** Hyo Jin lives in San Diego. She bought a refrigerator for $1624.99 and when the clerk calculated the sales tax it came out to exactly $142.186625. Round the sales tax to the nearest (a) penny (b) dollar.
90. **Sales Tax** Jennifer bought a $1,038.99 dining room set for her home in Cincinnati. She calculated the sales tax to be exactly $67.53435. Round the sales tax to the nearest (a) penny (b) dollar.

Writing Exercises

91. How does your knowledge of money help you learn about decimals?
92. Explain how you write "three and nine hundredths" as a decimal.
93. Jim ran a 100-meter race in 12.32 seconds. Tim ran the same race in 12.3 seconds. Who had the faster time, Jim or Tim? How do you know?
94. Gerry saw a sign advertising postcards marked for sale at "10 for 0.99¢." What is wrong with the advertised price?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
name decimals.			
write decimals.			
convert decimals to fractions or mixed numbers.			
locate decimals on the number line.			
order decimals.			
round decimals.			

(b) If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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5.3: Decimal Operations (Part 1)

Learning Objectives

- Add and subtract decimals
- Multiply decimals
- Divide decimals
- Use decimals in money applications

be prepared!

Before you get started, take this readiness quiz.

1. Simplify $\frac{70}{100}$. If you missed this problem, review Example 4.3.1.
2. Multiply $\frac{3}{10} \cdot \frac{9}{10}$. If you missed this problem, review Example 4.3.7.
3. Divide $-36 \div (-9)$. If you missed this problem, review Example 3.7.3.

Add and Subtract Decimals

Let's take one more look at the lunch order from the start of Decimals, this time noticing how the numbers were added together.

$$\begin{array}{rl} \$3.45 & Sandwich \\ \$1.25 & Water \\ +\$0.33 & Tax \\ \hline \$5.03 & Total \end{array}$$

All three items (sandwich, water, tax) were priced in dollars and cents, so we lined up the dollars under the dollars and the cents under the cents, with the decimal points lined up between them. Then we just added each column, as if we were adding whole numbers. By lining up decimals this way, we can add or subtract the corresponding place values just as we did with whole numbers.

HOW TO: ADD OR SUBTRACT DECIMALS

Step 1. Write the numbers vertically so the decimal points line up.

Step 2. Use zeros as place holders, as needed.

Step 3. Add or subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.

Example 5.3.1:

Add: 3.7 + 12.4.

Solution

Write the numbers vertically so the decimal points line up.	$\begin{array}{r} 3.7 \\ +12.4 \\ \hline \end{array}$
Place holders are not needed since both numbers have the same number of decimal places.	

Add the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.	$\begin{array}{r} {\scriptstyle 1} \\ 3.7 \\ +12.4 \\ \hline 16.1 \end{array}$

Exercise 5.3.1:

Add: 5.7 + 11.9.

Answer

17.6

Exercise 5.3.2:

Add: 18.32 + 14.79.

Answer

13.11

Example 5.3.2:

Add: 23.5 + 41.38.

Solution

Write the numbers vertically so the decimal points line up.	$\begin{array}{r} 23.5 \\ +41.38 \\ \hline \end{array}$
Place 0 as a place holder after the 5 in 23.5, so that both numbers have two decimal places.	$\begin{array}{r} 23.50 \\ +41.38 \\ \hline \end{array}$
Add the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.	$\begin{array}{r} 23.50 \\ +41.38 \\ \hline 64.88 \end{array}$

Exercise 5.3.3:

Add: 4.8 + 11.69.

Answer

16.49

Exercise 5.3.4:

Add: 5.123 + 18.47.

Answer

23.593

How much change would you get if you handed the cashier a $20 bill for a $14.65 purchase? We will show the steps to calculate this in the next example.

Example 5.3.3:

Subtract: $20 - 14.65$.

Solution

Write the numbers vertically so the decimal points line up. Remember 20 is a whole number, so place the decimal point after the 0.	$\begin{array}{r} 20. \\ -14.65 \\ \hline \end{array}$
Place two zeros after the decimal point in 20, as place holders so that both numbers have two decimal places.	$\begin{array}{r} 20.00 \\ -14.65 \\ \hline \end{array}$
Subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.	$\begin{array}{r} {\scriptstyle 9}\ \ {\scriptstyle 9}\ \ {\scriptstyle 9} \\ 1\ \cancel{10}\ \cancel{10}\ \cancel{10} \\ \cancel{2}\ \cancel{0}.\ \cancel{0}\ \cancel{0} \\ -1\ 4\ .\ 6\ 5 \\ \hline 5\ .\ 3\ 5 \end{array}$

Exercise 5.3.5:

Subtract: $10 - 9.58$.

Answer

0.42

Exercise 5.3.6:

Subtract: $50 - 37.42$.

Answer

12.58

Example 5.3.4:

Subtract: $2.51 - 7.4$.

Solution

If we subtract 7.4 from 2.51, the answer will be negative since $7.4 > 2.51$. To subtract easily, we can subtract 2.51 from 7.4. Then we will place the negative sign in the result.

Write the numbers vertically so the decimal points line up.	$\begin{array}{r} 7.4 \\ -2.51 \\ \hline \end{array}$
Place zero after the 4 in 7.4 as a place holder, so that both numbers have two decimal places.	$\begin{array}{r} 7.40 \\ -2.51 \\ \hline \end{array}$

Subtract and place the decimal in the answer.	$\begin{array}{r} 7.40 \\ -2.51 \\ \hline 4.89 \end{array}$
Remember that we are really subtracting 2.51 − 7.4 so the answer is negative.	2.51 − 7.4 = − 4.89

Exercise 5.3.7:

Subtract: 4.77 − 6.3.

Answer

-1.53

Exercise 5.3.8:

Subtract: 8.12 − 11.7.

Answer

-3.58

Multiply Decimals

Multiplying decimals is very much like multiplying whole numbers—we just have to determine where to place the decimal point. The procedure for multiplying decimals will make sense if we first review multiplying fractions.

Do you remember how to multiply fractions? To multiply fractions, you multiply the numerators and then multiply the denominators. So let's see what we would get as the product of decimals by converting them to fractions first. We will do two examples side-by-side in Table 5.22. Look for a pattern.

Table 5.3.1

	A	B
	(0.3)(0.7)	(0.2)(0.46)
Convert to fractions.	$\left(\frac{3}{10}\right)\left(\frac{7}{10}\right)$ (5.3.1)	$\left(\frac{2}{10}\right)\left(\frac{46}{100}\right)$ (5.3.2)
Multiply.	$\frac{21}{100}$ (5.3.3)	$\frac{92}{1000}$ (5.3.4)
Convert back to decimals	0.21	0.092

There is a pattern that we can use. In A, we multiplied two numbers that each had one decimal place, and the product had two decimal places. In B, we multiplied a number with one decimal place by a number with two decimal places, and the product had three decimal places.

How many decimal places would you expect for the product of (0.01)(0.004)? If you said "five", you recognized the pattern. When we multiply two numbers with decimals, we count all the decimal places in the factors—in this case two plus three—to get the number of decimal places in the product—in this case five.

$$\underbrace{(0.01)}_{2\text{ places}}\ \underbrace{(0.004)}_{3\text{ places}} = \underbrace{0.00004}_{5\text{ places}}$$

$$\left(\frac{1}{100}\right)\left(\frac{4}{1000}\right) = \frac{4}{100{,}000}$$

Once we know how to determine the number of digits after the decimal point, we can multiply decimal numbers without converting them to fractions first. The number of decimal places in the product is the sum of the number of decimal places in the factors.

The rules for multiplying positive and negative numbers apply to decimals, too, of course.

Definition: Multiplying Two Numbers

When multiplying two numbers,

- if their signs are the same, the product is positive.
- if their signs are different, the product is negative.

When you multiply signed decimals, first determine the sign of the product and then multiply as if the numbers were both positive. Finally, write the product with the appropriate sign.

HOW TO: MULTIPLY DECIMAL NUMBERS

Step 1. Determine the sign of the product.

Step 2. Write the numbers in vertical format, lining up the numbers on the right.

Step 3. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.

Step 4. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors. If needed, use zeros as placeholders.

Step 5. Write the product with the appropriate sign.

Example 5.3.5:

Multiply: (3.9)(4.075).

Solution

Determine the sign of the product. The signs are the same.	The product will be positive.
Write the numbers in vertical format, lining up the numbers on the right.	$\begin{array}{r} 4.075 \\ \times 3.9 \\ \hline \end{array}$
Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.	$\begin{array}{r} 4.075 \\ \times 3.9 \\ \hline 36675 \\ 12225 \\ \hline 158925 \end{array}$
Place the decimal point. Add the number of decimal places in the factors (1 + 3). Place the decimal point 4 places from the right.	$\begin{array}{rl} 4.075 & 3\ places \\ \times 3.9 & 1\ place \\ \hline 36675 & \\ 12225 & \\ \hline 158925 & 4\ places \end{array}$
The product is positive.	(3.9)(4.075) = 15.8925

Exercise 5.3.9:

Multiply: 4.5(6.107).

Answer

27.4815

Exercise 5.3.10:

Multiply: 10.79(8.12).

Answer

87.6148

Example 5.3.6:

Multiply: (−8.2)(5.19).

Solution

The signs are different.	The product will be negative.
Write in vertical format, lining up the numbers on the right.	$\begin{array}{r} 5.19 \\ \times 8.2 \\ \hline \end{array}$
Multiply.	$\begin{array}{r} 5.19 \\ \times 8.2 \\ \hline 1038 \\ 4152 \\ \hline 42558 \end{array}$
Place the decimal point 3 places from the right. (−8.2)(5.19) 1 place 2 places	$\begin{array}{r} 5.19 \\ \times 8.2 \\ \hline 1038 \\ 4152 \\ \hline 42.558 \end{array}$
The product is negative.	(−8.2)(5.19) = −42.558

Exercise 5.3.11:

Multiply: (4.63)(−2.9).

Answer

−13.427

Exercise 5.3.12:

Multiply: (−7.78)(4.9).

Answer

-38.122

In the next example, we'll need to add several placeholder zeros to properly place the decimal point.

Example 5.3.7:

Multiply: (0.03)(0.045).

Solution

The product is positive.	(0.03)(0.045)
Write in vertical format, lining up the numbers on the right.	0.045 $\times 0.03$
Multiply.	0.045 $\times 0.03$ 135
The decimal point must be 5 places from the right. (0.03)(0.045) 2 places 3 places Add zeros as needed to get the 5 places.	0.045 $\times 0.03$ 0.00135
The product is positive.	(0.03)(0.045) = 0.00135

Exercise 5.3.13:

Multiply: (0.04)(0.087).

Answer

0.00348

Exercise 5.3.14:

Multiply: (0.09)(0.067).

Answer

0.00603

Multiply by Powers of 10

In many fields, especially in the sciences, it is common to multiply decimals by powers of 10. Let's see what happens when we multiply 1.9436 by some powers of 10.

1.9436(10)	1.9436(100)	1.9436(1000)
1.9436 × 10 19.4360	1.9436 × 100 194.3600	1.9436 × 1000 1943.6000

Look at the results without the final zeros. Do you notice a pattern?

$$1.9436(10) = 19.436$$
$$1.9436(100) = 194.36$$
$$1.9436(1000) = 1943.6$$

The number of places that the decimal point moved is the same as the number of zeros in the power of ten. Table 5.26 summarizes the results.

Table 5.3.2

Multiply by	Number of zeros	Number of places decimal point moves
10	1	1 place to the right
100	2	2 places to the right
1,000	3	3 places to the right
10,000	4	4 places to the right

We can use this pattern as a shortcut to multiply by powers of ten instead of multiplying using the vertical format. We can count the zeros in the power of 10 and then move the decimal point that same of places to the right. So, for example, to multiply 45.86 by 100, move the decimal point 2 places to the right.

$$45.86 \times 100 = 4586.$$

Sometimes when we need to move the decimal point, there are not enough decimal places. In that case, we use zeros as placeholders. For example, let's multiply 2.4 by 100. We need to move the decimal point 2 places to the right. Since there is only one digit to the right of the decimal point, we must write a 0 in the hundredths place.

$$2.4 \times 100 = 240.$$

HOW TO: MULTIPLY A DECIMAL BY A POWER OF 10

Step 1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.

Step 2. Write zeros at the end of the number as placeholders if needed.

Example 5.3.8:

Multiply 5.63 by factors of (a) 10 (b) 100 (c) 1000.

Solution

By looking at the number of zeros in the multiple of ten, we see the number of places we need to move the decimal to the right.

(a) 5.63(10)

There is 1 zero in 10, so move the decimal point 1 place to the right.	5.63
	56.3

(b) 5.63(100)

There are 2 zeros in 100, so move the decimal point 2 places to the right.	5.63
	563

(c) 5.63(1000)

There are 3 zeros in 1000, so move the decimal point 3 places to the right.	5.63

A zero must be added at the end.	5,630

Exercise 5.3.15:

Multiply 2.58 by factors of (a) 10 (b) 100 (c) 1000.

Answer a

25.8

Answer b

258

Answer c

$2,580$

Exercise 5.3.16:

Multiply 14.2 by factors of (a) 10 (b) 100 (c) 1000.

Answer a

142

Answer b

$1,420$

Answer c

$14,200$

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5.4: Decimal Operations (Part 2)

Divide Decimals

Just as with multiplication, division of decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed.

To understand decimal division, let's consider the multiplication problem

$$(0.2)(4) = 0.8 \tag{5.4.1}$$

Remember, a multiplication problem can be rephrased as a division problem. So we can write 0.8 ÷ 4 = 0.2 We can think of this as "If we divide 8 tenths into four groups, how many are in each group?" Figure 5.4.1 shows that there are four groups of two-tenths in eight-tenths. So 0.8 ÷ 4 = 0.2.

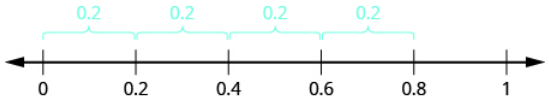


Figure 5.4.1

Using long division notation, we would write

$$4\overline{)0.8}\quad 0.2$$

Notice that the decimal point in the quotient is directly above the decimal point in the dividend.

To divide a decimal by a whole number, we place the decimal point in the quotient above the decimal point in the dividend and then divide as usual. Sometimes we need to use extra zeros at the end of the dividend to keep dividing until there is no remainder.

HOW TO: DIVIDE A DECIMAL BY A WHOLE NUMBER

Step 1. Write as long division, placing the decimal point in the quotient above the decimal point in the dividend.

Step 2. Divide as usual.

Example 5.4.9:

Divide: 0.12 ÷ 3.

Solution

Write as long division, placing the decimal point in the quotient above the decimal point in the dividend.	$3\overline{)0.12}$
Divide as usual. Since 3 does not go into 0 or 1 we use zeros as placeholders.	$3\overline{)0.12}\quad 0.04$ 12 0

0.12 ÷ 3 = 0.04

Exercise 5.4.17:

Divide: 0.28 ÷ 4.

Answer

0.07

Exercise 5.4.18:

Divide: 0.56 ÷ 7.

Answer

0.08

In everyday life, we divide whole numbers into decimals—money—to find the price of one item. For example, suppose a case of 24 water bottles cost $3.99. To find the price per water bottle, we would divide $3.99 by 24, and round the answer to the nearest cent (hundredth).

Example 5.4.10:

Divide: $3.99 ÷ 24.

Solution

Place the decimal point in the quotient above the decimal point in the dividend.	$24\overline{)3.99}$
Divide as usual. When do we stop? Since this division involves money, we round it to the nearest cent (hundredth). To do this, we must carry the division to the thousandths place.	$\begin{array}{r} 0.166 \\ 24\overline{)3.990} \\ \underline{24} \\ 159 \\ \underline{144} \\ 150 \\ \underline{144} \\ 6 \end{array}$
Round to the nearest cent.	$\$0.166 \approx \$0.17$ (5.4.2)

$3.99 ÷ 24 ≈ $0.17

This means the price per bottle is 17 cents.

Exercise 5.4.19:

Divide: $6.99 ÷ 36.

Answer

$0.19

Exercise 5.4.20:

Divide: $4.99 ÷ 12.

Answer

$0.42

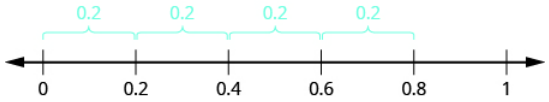
Divide a Decimal by Another Decimal

So far, we have divided a decimal by a whole number. What happens when we divide a decimal by another decimal? Let's look at the same multiplication problem we looked at earlier, but in a different way.

$$(0.2)(4) = 0.8 \quad (5.4.3)$$

Remember, again, that a multiplication problem can be rephrased as a division problem. This time we ask, “Ho w many times does 0.2 go into 0.8?” Because (0.2)(4) = 0.8, we can say that 0.2 goes into 0.8 four times. This means that 0.8 divided by 0.2 is 4.

$$0.8 \div 0.2 = 4 \tag{5.4.4}$$



We would get the same answer, 4, if we divide 8 by 2, both whole numbers. Why is this so? Let’s think about the division problem as a fraction.

$$\frac{0.8}{0.2} \tag{5.4.5}$$

$$\frac{(0.8)10}{(0.2)10} \tag{5.4.6}$$

$$\frac{8}{2} \tag{5.4.7}$$

$$4 \tag{5.4.8}$$

We multiplied the numerator and denominator by 10 and ended up just dividing 8 by 2. To divide decimals, we multiply both the numerator and denominator by the same power of 10 to make the denominator a whole number. Because of the Equivalent Fractions Property, we haven’t changed the value of the fraction. The effect is to move the decimal points in the numerator and denominator the same number of places to the right.

We use the rules for dividing positive and negative numbers with decimals, too. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign. It may help to review the vocabulary for division:

$a \div b$ — a dividend, b divisor

$\frac{a}{b}$ — a dividend, b divisor

$b\overline{)a}$ — b divisor, a dividend

HOW TO: DIVIDE DECIMAL NUMBERS

Step 1. Determine the sign of the quotient.

Step 2. Make the divisor a whole number by moving the decimal point all the way to the right. Move the decimal point in the dividend the same number of places to the right, writing zeros as needed.

Step 3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.

Step 4. Write the quotient with the appropriate sign.

Example 5.4.11:

Divide: −2.89 ÷ (3.4).

Solution

Determine the sign of the quotient.	The quotient will be negative.
Make the divisor the whole number by 'moving' the decimal point all the way to the right. 'Move' the decimal point in the dividend the same number of places to the right.	$3.4\overline{)2.89}$
Divide. Place the decimal point in the quotient above the decimal point in the dividend. Add zeros as needed until the remainder is zero.	$34.\overline{)28.90}$ → 0.85; 272; 170; 170; 0

Write the quotient with the appropriate sign.	$-2.89 \div (3.4) = -0.85$

Exercise 5.4.21:

Divide: $-1.989 \div 5.1$.

Answer

-0.39

Exercise 5.4.22:

Divide: $-2.04 \div 5.1$.

Answer

-0.4

Example 5.4.12:

Divide: $-25.65 \div (-0.06)$.

Solution

The signs are the same.	The quotient is positive.
Make the divisor a whole number by 'moving' the decimal point all the way to the right. 'Move' the decimal point in the dividend the same number of places.	$0.06\overline{)25.65}$
Divide. Place the decimal point in the quotient above the decimal point in the dividend.	$\begin{array}{r} 427.5 \\ 006.\overline{)2565.0} \\ -24 \\ \hline 16 \\ -12 \\ \hline 45 \\ -42 \\ \hline 30 \\ 30 \\ \hline \end{array}$
Write the quotient with the appropriate sign.	$-25.65 \div (-0.06) = 427.5$

Exercise 5.4.23:

Divide: $-23.492 \div (-0.04)$.

Answer

587.3

Exercise 5.4.24:

Divide: $-4.11 \div (-0.12)$.

Answer

34.25

Now we will divide a whole number by a decimal number.

Example 5.4.13

Divide: 4 ÷ (0.05).

Solution

The signs are the same.	The quotient is positive.
Make the divisor a whole number by 'moving' the decimal point all the way to the right. Move the decimal point in the dividend the same number of places, adding zeros as needed.	$0.05\overline{)4.00}$
Divide. Place the decimal point in the quotient above the decimal point in the dividend.	$\begin{array}{r} 80. \\ 5\overline{)400.} \\ \underline{40} \\ 00 \\ \underline{00} \end{array}$
Write the quotient with the appropriate sign.	4 ÷ 0.05 = 80

We can relate this example to money. How many nickels are there in four dollars? Because 4 ÷ 0.05 = 80, there are 80 nickels in $4.

Exercise 5.4.25:

Divide: 6 ÷ 0.03.

Answer

200

Exercise 5.4.26:

Divide: 7 ÷ 0.02

Answer

350

Use Decimals in Money Applications

We often apply decimals in real life, and most of the applications involving money. The Strategy for Applications we used in The Language of Algebra gives us a plan to follow to help find the answer. Take a moment to review that strategy now.

Strategy for Applications

1. Identify what you are asked to find.
2. Write a phrase that gives the information to find it.
3. Translate the phrase to an expression.
4. Simplify the expression.
5. Answer the question with a complete sentence.

Example 5.4.14:

Paul received $50 for his birthday. He spent $31.64 on a video game. How much of Paul's birthday money was left?

Solution

What are you asked to find?	How much did Paul have left?
Write a phrase.	$50 less $31.64
Translate.	$50 - 31.64$
Simplify.	18.36
Write a sentence.	Paul has $18.36 left.

Exercise 5.4.27:

Nicole earned $35 for babysitting her cousins, then went to the bookstore and spent $18.48 on books and coffee. How much of her babysitting money was left?

Answer

$16.52

Exercise 5.4.28:

Amber bought a pair of shoes for $24.75 and a purse for $36.90. The sales tax was $4.32. How much did Amber spend?

Answer

$65.97

Example 5.4.15:

Jessie put 8 gallons of gas in her car. One gallon of gas costs $3.529. How much does Jessie owe for the gas? (Round the answer to the nearest cent.)

Solution

What are you asked to find?	How much did Jessie owe for all the gas?
Write a phrase.	8 times the cost of one gallon of gas
Translate.	8($3.529)
Simplify.	$28.232
Round to the nearest cent.	$28.23
Write a sentence.	Jessie owes $28.23 for her gas purchase.

Exercise 5.4.29:

Hector put 13 gallons of gas into his car. One gallon of gas costs $3.175. How much did Hector owe for the gas? Round to the nearest cent.

Answer

$41.28

Exercise 5.4.30:

Christopher bought 5 pizzas for the team. Each pizza cost $9.75. How much did all the pizzas cost?

Answer

$48.75

Example 5.4.16:

Four friends went out for dinner. They shared a large pizza and a pitcher of soda. The total cost of their dinner was $31.76. If they divide the cost equally, how much should each friend pay?

Solution

What are you asked to find?	How much should each friend pay?
Write a phrase.	$31.76 divided equally among the four friends.
Translate to an expression.	$31.76 ÷ 4
Simplify.	$7.94
Write a sentence.	Each friend should pay $7.94 for his share of the dinner.

Exercise 5.4.31:

Six friends went out for dinner. The total cost of their dinner was $92.82. If they divide the bill equally, how much should each friend pay?

Answer

$15.47

Exercise 5.4.32:

Chad worked 40 hours last week and his paycheck was $570. How much does he earn per hour?

Answer

$14.25

Be careful to follow the order of operations in the next example. Remember to multiply before you add.

Example 5.4.17:

Marla buys 6 bananas that cost $0.22 each and 4 oranges that cost $0.49 each. How much is the total cost of the fruit?

Solution

What are you asked to find?	How much is the total cost of the fruit?
Write a phrase.	6 times the cost of each banana plus 4 times the cost of each orange
Translate to an expression.	6($0.22) + 4($0.49)
Simplify.	$1.32 + $1.96
Add.	$3.28

Write a sentence.	Marla's total cost for the fruit is $3.28.

Exercise 5.4.33:

Suzanne buys 3 cans of beans that cost $0.75 each and 6 cans of corn that cost $0.62 each. How much is the total cost of these groceries?

Answer

$5.97

Exercise 5.4.34:

Lydia bought movie tickets for the family. She bought two adult tickets for $9.50 each and four children's tickets for $6.00 each. How much did the tickets cost Lydia in all?

Answer

$43.00

ACCESS ADDITIONAL ONLINE RESOURCES

Adding and Subtracting Decimals

Multiplying Decimals

Multiplying by Powers of Ten

Dividing Decimals

Dividing by Powers of Ten

Practice Makes Perfect

Add and Subtract Decimals

In the following exercises, add or subtract.

95. 16.92 + 7.56
96. 18.37 + 9.36
97. 256.37 − 85.49
98. 248.25 − 91.29
99. 21.76 − 30.99
100. 15.35 − 20.88
101. 37.5 + 12.23
102. 38.6 + 13.67
103. −16.53 − 24.38
104. −19.47 − 32.58
105. −38.69 + 31.47
106. −29.83 + 19.76
107. −4.2 + (− 9.3)
108. −8.6 + (− 8.6)
109. 100 − 64.2
110. 100 − 65.83
111. 72.5 − 100
112. 86.2 − 100
113. 15 + 0.73

114. 27 + 0.87
115. 2.51 + 40
116. 9.38 + 60
117. 91.75 − (− 10.462)
118. 94.69 − (− 12.678)
119. 55.01 − 3.7
120. 59.08 − 4.6
121. 2.51 − 7.4
122. 3.84 − 6.1

Multiply Decimals

In the following exercises, multiply.

123. (0.3)(0.4)
124. (0.6)(0.7)
125. (0.24)(0.6)
126. (0.81)(0.3)
127. (5.9)(7.12)
128. (2.3)(9.41)
129. (8.52)(3.14)
130. (5.32)(4.86)
131. (−4.3)(2.71)
132. (− 8.5)(1.69)
133. (−5.18)(− 65.23)
134. (− 9.16)(− 68.34)
135. (0.09)(24.78)
136. (0.04)(36.89)
137. (0.06)(21.75)
138. (0.08)(52.45)
139. (9.24)(10)
140. (6.531)(10)
141. (55.2)(1,000)
142. (99.4)(1,000)

Divide Decimals

In the following exercises, divide.

143. 0.15 ÷ 5
144. 0.27 ÷ 3
145. 4.75 ÷ 25
146. 12.04 ÷ 43
147. $8.49 ÷ 12
148. $16.99 ÷ 9
149. $117.25 ÷ 48
150. $109.24 ÷ 36
151. 0.6 ÷ 0.2
152. 0.8 ÷ 0.4
153. 1.44 ÷ (− 0.3)
154. 1.25 ÷ (− 0.5)
155. −1.75 ÷ (− 0.05)
156. −1.15 ÷ (− 0.05)
157. 5.2 ÷ 2.5
158. 6.5 ÷ 3.25
159. 12 ÷ 0.08

160. 5 ÷ 0.04
161. 11 ÷ 0.55
162. 14 ÷ 0.35

Mixed Practice

In the following exercises, simplify.

163. 6(12.4 − 9.2)
164. 3(15.7 − 8.6)
165. 24(0.5) + $(0.3)^2$
166. 35(0.2) + $(0.9)^2$
167. 1.15(26.83 + 1.61)
168. 1.18(46.22 + 3.71)
169. $45 + 0.08($45)
170. $63 + 0.18($63)
171. 18 ÷ (0.75 + 0.15)
172. 27 ÷ (0.55 + 0.35)
173. (1.43 + 0.27) ÷ (0.9 − 0.05)
174. (1.5 − 0.06) ÷ (0.12 + 0.24)
175. [$75.42 + 0.18($75.42)] ÷ 5
176. [$56.31 + 0.22($56.31)] ÷ 4

Use Decimals in Money Applications

In the following exercises, use the strategy for applications to solve.

177. **Spending money** Brenda got $40 from the ATM. She spent $15.11 on a pair of earrings. How much money did she have left?
178. **Spending money** Marissa found $20 in her pocket. She spent $4.82 on a smoothie. How much of the $20 did she have left?
179. **Shopping** Adam bought a t-shirt for $18.49 and a book for $8.92 The sales tax was $1.65. How much did Adam spend?
180. **Restaurant** Roberto's restaurant bill was $20.45 for the entrée and $3.15 for the drink. He left a $4.40 tip. How much did Roberto spend?
181. **Coupon** Emily bought a box of cereal that cost $4.29. She had a coupon for $0.55 off, and the store doubled the coupon. How much did she pay for the box of cereal?
182. **Coupon** Diana bought a can of coffee that cost $7.99. She had a coupon for $0.75 off, and the store doubled the coupon. How much did she pay for the can of coffee?
183. **Diet** Leo took part in a diet program. He weighed 190 pounds at the start of the program. During the first week, he lost 4.3 pounds. During the second week, he had lost 2.8 pounds. The third week, he gained 0.7 pounds. The fourth week, he lost 1.9 pounds. What did Leo weigh at the end of the fourth week?
184. **Snowpack** On April 1, the snowpack at the ski resort was 4 meters deep, but the next few days were very warm. By April 5, the snow depth was 1.6 meters less. On April 8, it snowed and added 2.1 meters of snow. What was the total depth of the snow?
185. **Coffee** Noriko bought 4 coffees for herself and her coworkers. Each coffee was $3.75. How much did she pay for all the coffees?
186. **Subway Fare** Arianna spends $4.50 per day on subway fare. Last week she rode the subway 6 days. How much did she spend for the subway fares? 187. Income Mayra earns $9.25 per hour. Last week she worked 32 hours. How much did she earn?
187. **Income** Peter earns $8.75 per hour. Last week he worked 19 hours. How much did he earn?
188. **Hourly Wage** Alan got his first paycheck from his new job. He worked 30 hours and earned $382.50. How much does he earn per hour?
189. **Hourly Wage** Maria got her first paycheck from her new job. She worked 25 hours and earned $362.50. How much does she earn per hour?
190. **Restaurant** Jeannette and her friends love to order mud pie at their favorite restaurant. They always share just one piece of pie among themselves. With tax and tip, the total cost is $6.00. How much does each girl pay if the total number sharing the mud pie is (a) 2? (b) 3? (c) 4? (d) 5? (e) 6?
191. **Pizza** Alex and his friends go out for pizza and video games once a week. They share the cost of a $15.60 pizza equally. How much does each person pay if the total number sharing the pizza is (a) 2? (b) 3? (c) 4? (d) 5? (e) 6?

192. **Fast Food** At their favorite fast food restaurant, the Carlson family orders 4 burgers that cost $3.29 each and 2 orders of fries at $2.74 each. What is the total cost of the order?
193. **Home Goods** Chelsea needs towels to take with her to college. She buys 2 bath towels that cost $9.99 each and 6 washcloths that cost $2.99 each. What is the total cost for the bath towels and washcloths?
194. **Zoo** The Lewis and Chousmith families are planning to go to the zoo together. Adult tickets cost $29.95 and children's tickets cost $19.95. What will the total cost be for 4 adults and 7 children?
195. **Ice Skating** Jasmine wants to have her birthday party at the local ice skating rink. It will cost $8.25 per child and $12.95 per adult. What will the total cost be for 12 children and 3 adults?

Everyday Math

197. **Paycheck** Annie has two jobs. She gets paid $14.04 per hour for tutoring at City College and $8.75 per hour at a coffee shop. Last week she tutored for 8 hours and worked at the coffee shop for 15 hours. (a) How much did she earn? (b) If she had worked all 23 hours as a tutor instead of working both jobs, how much more would she have earned?
198. **Paycheck** Jake has two jobs. He gets paid $7.95 per hour at the college cafeteria and $20.25 at the art gallery. Last week he worked 12 hours at the cafeteria and 5 hours at the art gallery. (a) How much did he earn? (b) If he had worked all 17 hours at the art gallery instead of working both jobs, how much more would he have earned?

Writing Exercises

199. In the 2010 winter Olympics, two skiers took the silver and bronze medals in the Men's Super-G ski event. The silver medalist's time was 1 minute 30.62 seconds and bronze medalist's time was 1 minute 30.65 seconds. Whose time was faster? Find the difference in their times and then write the name of that decimal.
200. Find the quotient of 0.12 ÷ 0.04 and explain in words all the steps taken.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract decimals.			
multiply decimals.			
divide decimals.			
use decimals in money applications.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

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5.5: Decimals and Fractions (Part 1)

Learning Objectives

- Convert fractions to decimals
- Order decimals and fractions
- Simplify expressions using the order of operations
- Find the circumference and area of circles

be prepared!

Before you get started, take this readiness quiz.

1. Divide: 0.24 ÷ 8. If you missed this problem, review Example 5.4.9.
2. Order 0.64__0.6 using < or >. If you missed this problem, review Example 5.2.7.
3. Order −0.2__−0.1 using < or >. If you missed this problem, review Example 5.2.8.

Convert Fractions to Decimals

In Decimals, we learned to convert decimals to fractions. Now we will do the reverse—convert fractions to decimals. Remember that the fraction bar indicates division. So $\frac{4}{5}$ can be written $4 \div 5$ or $5\overline{)4}$. This means that we can convert a fraction to a decimal by treating it as a division problem.

Note: Convert a Fraction to a Decimal

To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

Example 5.5.1:

Write the fraction $\frac{3}{4}$ as a decimal.

Solution

A fraction bar means division, so we can write the fraction 3 4 using division.	$4\overline{)3}$
Divide.	$\begin{array}{r} 0.75 \\ 4\overline{)3.00} \\ \underline{28} \\ 20 \\ \underline{20} \\ 0 \end{array}$

So the fraction $\frac{3}{4}$ is equal to 0.75.

Exercise 5.5.1:

Write the fraction as a decimal: $\frac{1}{4}$.

Answer

0.25

Exercise 5.5.2:

Write the fraction as a decimal: $\frac{3}{8}$.

Answer

0.375

Example 5.5.2:

Write the fraction $-\frac{7}{2}$ as a decimal.

Solution

The value of this fraction is negative. After dividing, the value of the decimal will be negative. We do the division ignoring the sign, and then write the negative sign in the answer.	$-\frac{7}{2}$ (5.5.1)
Divide 7 by 2.	$2\overline{)7.0}$ $\underline{6}$ 10 $\underline{10}$ 0

So, $-\frac{7}{2} = -3.5$.

Exercise 5.5.3:

Write the fraction as a decimal: $-\frac{9}{4}$.

Answer

-2.25

Exercise 5.5.4:

Write the fraction as a decimal: $-\frac{11}{2}$.

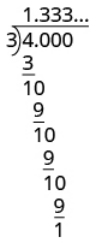
Answer

-5.5

Repeating Decimals

So far, in all the examples converting fractions to decimals the division resulted in a remainder of zero. This is not always the case. Let's see what happens when we convert the fraction $\frac{4}{3}$ to a decimal. First, notice that $\frac{4}{3}$ is an improper fraction. Its value is greater than 1. The equivalent decimal will also be greater than 1.

We divide 4 by 3.



No matter how many more zeros we write, there will always be a remainder of 1, and the threes in the quotient will go on forever. The number 1.333… is called a repeating decimal. Remember that the "…" means that the pattern repeats.

Definition: Repeating Decimal

A repeating decimal is a decimal in which the last digit or group of digits repeats endlessly.

How do you know how many 'repeats' to write? Instead of writing 1.333 … we use a shorthand notation by placing a line over the digits that repeat. The repeating decimal 1.333 … is written $1.\overline{3}$. The line above the 3 tells you that the 3 repeats endlessly. So 1.333… = $1.\overline{3}$. For other decimals, two or more digits might repeat. Table 5.5.1 shows some more examples of repeating decimals.

Table 5.5.1

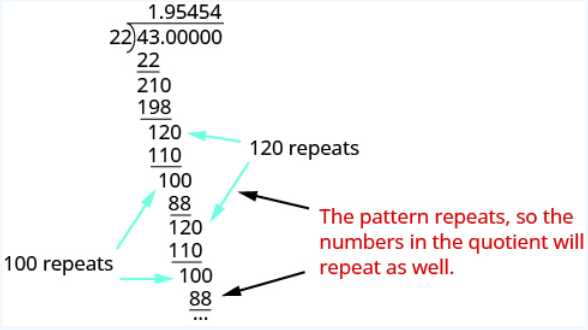
1.333… = $1.\overline{3}$	3 is the repeating digit
4.1666… = $4.1\overline{6}$	6 is the repeating digit
4.161616… = $4.\overline{16}$	16 is the repeating block
0.271271271… = $0.\overline{271}$	271 is the repeating block

Example 5.5.3:

Write $\frac{43}{22}$ as a decimal.

Solution

Divide 43 by 22.



Notice that the differences of 120 and 100 repeat, so there is a repeat in the digits of the quotient; 54 will repeat endlessly. The first decimal place in the quotient, 9, is not part of the pattern. So,

$$\frac{43}{22} = 1.9\overline{54} \tag{5.5.2}$$

Exercise 5.5.5:

Write as a decimal: $\frac{27}{11}$.

Answer

$2.\overline{45}$

Exercise 5.5.6:

Write as a decimal: $\frac{51}{22}$.

Answer

$2.3\overline{18}$

It is useful to convert between fractions and decimals when we need to add or subtract numbers in different forms. To add a fraction and a decimal, for example, we would need to either convert the fraction to a decimal or the decimal to a fraction.

Example 5.5.4:

Simplify: $\frac{7}{8} + 6.4$.

Solution

Change $\frac{7}{8}$ to a decimal.	$\begin{array}{r} 0.875 \\ 8\overline{)7.000} \\ \underline{64} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$	$0.875 + 6.4$
Add.		7.275

Exercise 5.5.7:

Simplify: $\frac{3}{8} + 4.9$.

Answer

5.275

Exercise 5.5.8:

Simplify: $5.7 + \frac{13}{20}$.

Answer

6.35

Order Decimals and Fractions

In Decimals, we compared two decimals and determined which was larger. To compare a decimal to a fraction, we will first convert the fraction to a decimal and then compare the decimals.

Example 5.5.5:

Order $\frac{3}{8}$ __ 0.4 using < or >.

Solution

Convert $\frac{3}{8}$ to a decimal.	0.375__0.4
Compare 0.375 to 0.4	0.375 < 0.4
Rewrite with the original fraction.	$\frac{3}{8}$ < 0.4

Exercise 5.5.9:

Order each of the following pairs of numbers, using < or >.

$$\frac{17}{20} \_\_ 0.82 \tag{5.5.3}$$

Answer

>

Exercise 5.5.10:

Order each of the following pairs of numbers, using < or >.

$$\frac{3}{4} \_\_ 0.785 \tag{5.5.4}$$

Answer

<

When ordering negative numbers, remember that larger numbers are to the right on the number line and any positive number is greater than any negative number.

Example 5.5.6:

Order −0.5___$-\frac{3}{4}$ using < or >.

Solution

Convert $-\frac{3}{4}$ to a decimal.	−0.5___−0.75
Compare −0.5 to −0.75.	−0.5 > −0.75
Rewrite the inequality with the original fraction.	$-0.5 > -\frac{3}{4}$

Exercise 5.5.11:

Order each of the following pairs of numbers, using < or >:

$$-\frac{5}{8}\_\_-0.58 \tag{5.5.5}$$

Answer

$<$

Exercise 5.5.12:

Order each of the following pairs of numbers, using < or >:

$$-0.53\_\_-\frac{11}{20} \tag{5.5.6}$$

Answer

$>$

Example 5.5.7:

Write the numbers $\frac{13}{20}$, 0.61, $\frac{11}{16}$ in order from smallest to largest.

Solution

Convert the fractions to decimals.	0.65, 0.61, 0.6875
Write the smallest decimal number first.	0.61, ____, _____
Write the next larger decimal number in the middle place.	0.61, 0.65, _____
Write the last decimal number (the larger) in the third place.	0.61, 0.65, 0.6875
Rewrite the list with the original fractions.	0.61, $\frac{13}{20}$, $\frac{11}{16}$

Exercise 5.5.13:

Write each set of numbers in order from smallest to largest: $\frac{7}{8}$, $\frac{4}{5}$, 0.82.

Answer

$\frac{4}{5}$, 0.82 $\frac{7}{8}$

Exercise 5.5.14:

Write each set of numbers in order from smallest to largest: 0.835, $\frac{13}{16}$, $\frac{3}{4}$.

Answer

$\frac{3}{4}$, $\frac{13}{16}$, 0.835

Simplify Expressions Using the Order of Operations

The order of operations introduced in Use the Language of Algebra also applies to decimals. Do you remember what the phrase "Please excuse my dear Aunt Sally" stands for?

Example 5.5.8:

Simplify the expressions: (a) $7(18.3 - 21.7)$ (b) $\frac{2}{3}(8.3 - 3.8)$

Solution

(a) $7(18.3 - 21.7)$

Simplify inside parentheses.	$7(-3.4)$
Multiply.	-23.8

(b) $\frac{2}{3}(8.3 - 3.8)$

Simplify inside parentheses.	$\frac{2}{3}(4.5)$	(5.5.7)
Write 4.5 as a fraction.	$\frac{2}{3}\left(\frac{4.5}{1}\right)$	(5.5.8)
Multiply.	$\frac{9}{3}$	(5.5.9)
Simplify.	3	(5.5.10)

Exercise 5.5.15:

Simplify: (a) $8(14.6 - 37.5)$ (b) $\frac{3}{5}(9.6 - 2.1)$.

Answer a

-183.2

Answer b

4.5

Exercise 5.5.16:

Simplify: (a) $25(25.69 - 56.74)$ (b) $\frac{2}{7}(11.9 - 4.2)$.

Answer a

-776.25

Answer b

2.2

Example 5.5.9:

Simplify each expression: (a) $6 \div 0.6 + (0.2)4 - (0.1)^2$ (b) $\left(\frac{1}{10}\right)^2 + (3.5)(0.9)$

Solution

(a) $6 \div 0.6 + (0.2)4 - (0.1)^2$

Simplify exponents.	$6 \div 0.6 + (0.2)4 - 0.01$
Divide.	$10 + (0.2)4 - 0.01$
Multiply.	$10 + 0.8 - 0.01$
Add.	$10.8 - 0.01$
Subtract.	10.79

(b) $\left(\frac{1}{10}\right)^2 + (3.5)(0.9)$

Simplify exponents.	$\frac{1}{100} + (3.5)(0.9)$
Multiply.	$\frac{1}{100} + 3.15$
Convert $\frac{1}{100}$ to a decimal.	$0.01 + 3.15$
Add.	3.16

Exercise 5.5.17:

Simplify: $9 \div 0.9 + (0.4)3 - (0.2)^2$.

Answer

11.16

Exercise 5.5.18:

Simplify: $\left(\frac{1}{2}\right)^2 + (0.3)(4.2)$.

Answer

1.51

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5.6: Decimals and Fractions (Part 2)

Find the Circumference and Area of Circles

The properties of circles have been studied for over 2,000 years. All circles have exactly the same shape, but their sizes are affected by the length of the **radius**, a line segment from the center to any point on the circle. A line segment that passes through a circle's center connecting two points on the circle is called a **diameter**. The diameter is twice as long as the radius. See Figure 5.6.1.

The size of a circle can be measured in two ways. The distance around a circle is called its **circumference**.

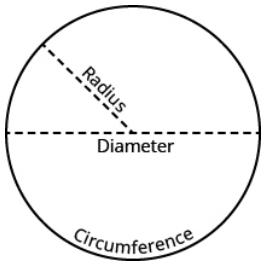
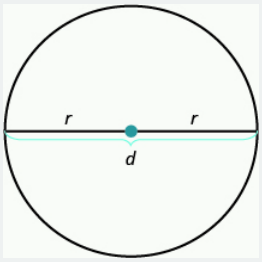


Figure 5.6.1

Archimedes discovered that for circles of all different sizes, dividing the circumference by the diameter always gives the same number. The value of this number is pi, symbolized by Greek letter π (pronounced pie). However, the exact value of π cannot be calculated since the decimal never ends or repeats (we will learn more about numbers like this in The Properties of Real Numbers.)

If we want the exact circumference or area of a circle, we leave the symbol π in the answer. We can get an approximate answer by substituting 3.14 as the value of π. We use the symbol ≈ to show that the result is approximate, not exact.

Note: Properties of Circles



r is the length of the radius.

d is the length of the diameter.

The circumference is $2\pi r$.

$$C = 2\pi r \tag{5.6.1}$$

The area is πr^2.

$$A = \pi r^2 \tag{5.6.2}$$

Since the diameter is twice the radius, another way to find the circumference is to use the formula $C = \pi d$.

Suppose we want to find the exact area of a circle of radius 10 inches. To calculate the area, we would evaluate the formula for the area when r = 10 inches and leave the answer in terms of π.

$$\begin{aligned} A &= \pi r^2 \\ A &= \pi(10^2) \\ A &= \pi \cdot 100 \end{aligned}$$

We write π after the 100. So the exact value of the area is A = 100π square inches. To approximate the area, we would substitute $\pi \approx 3.14$.

$$\begin{aligned} A &= 100\pi \\ &\approx 100 \cdot 3.14 \\ &\approx 314 \ square\ inches \end{aligned}$$

Remember to use square units, such as square inches, when you calculate the area.

Example 5.6.10:

A circle has radius 10 centimeters. Approximate its (a) circumference and (b) area.

Solution

(a) Find the circumference when r = 10.

Write the formula for circumference.	$C = 2\pi r$
Substitute 3.14 for π and 10 for r.	$C \approx 2(3.14)(10)$
Multiply.	C ≈ 62.8 centimeters

(b) Find the area when r = 10.

Write the formula for area.	$A = \pi r^2$
Substitute 3.14 for π and 10 for r.	$A \approx (3.14)(10)^2$
Multiply.	A ≈ 314 square centimeters

Exercise 5.6.19:

A circle has radius 50 inches. Approximate its (a) circumference and (b) area.

Answer a

314 in.

Answer b

7850 sq. in.

Exercise 5.6.20:

A circle has radius 100 feet. Approximate its (a) circumference and (b) area.

Answer a

628 ft.

Answer b

31,400 sq. ft.

Example 5.6.11:

A circle has radius 42.5 centimeters. Approximate its (a) circumference and (b) area.

Solution

(a) Find the circumference when r = 42.5.

Write the formula for circumference.	$C = 2\pi r$
Substitute 3.14 for π and 42.5 for r.	$C \approx 2(3.14)(42.5)$
Multiply.	C ≈ 266.9 centimeters

(b) Find the area when r = 42.5.

Write the formula for area.	$A = \pi r^2$
Substitute 3.14 for π and 42.5 for r.	$A \approx (3.14)(42.5)^2$
Multiply.	A ≈ 5671.625 square centimeters

? Exercise 5.6.21:

A circle has radius 51.8 centimeters. Approximate its (a) circumference and (b) area.

Answer a

325.304 cm

Answer b

8425.3736 sq. cm

? Exercise 5.6.22:

A circle has radius 26.4 meters. Approximate its (a) circumference and (b) area.

Answer a

165.792 m

Answer b

2188.4544 sq. m

Approximate π with a Fraction

Convert the fraction $\frac{22}{7}$ to a decimal. If you use your calculator, the decimal number will fill up the display and show 3.14285714. But if we round that number to two decimal places, we get 3.14, the decimal approximation of π. When we have a circle with radius given as a fraction, we can substitute $\frac{22}{7}$ for π instead of 3.14. And, since $\frac{22}{7}$ is also an approximation of π, we will use the ≈ symbol to show we have an approximate value.

✓ Example 5.6.12:

A circle has radius $\frac{14}{15}$ meter. Approximate its (a) circumference and (b) area.

Solution

(a) Find the circumference when r = $\frac{14}{15}$

Write the formula for circumference.	$C = 2\pi r$

Substitute $\frac{22}{7}$ for π and $\frac{14}{15}$ for r .	$C \approx 2\left(\frac{22}{7}\right)\left(\frac{14}{15}\right)$	(5.6.3)
Multiply.	C ≈ $\frac{88}{15}$ meters	

.(b) Find the area when r = $\frac{14}{15}$.

Write the formula for area.	$A = \pi r^2$	
Substitute $\frac{22}{7}$ for π and $\frac{14}{15}$ for r.	$A \approx \left(\frac{22}{7}\right)\left(\frac{14}{15}\right)^2$	(5.6.4)
Multiply.	A ≈ $\frac{616}{225}$ square meters	

? Exercise 5.6.23:

A circle has radius $\frac{5}{21}$ meters. Approximate its (a) circumference and (b) area.

Answer a

$\frac{220}{147}$ m

Answer b

$\frac{550}{3087}$ sq. m

? Exercise 5.6.24:

A circle has radius $\frac{10}{33}$ inches. Approximate its (a) circumference and (b) area.

Answer a

$\frac{40}{21}$ in.

Answer b

$\frac{200}{693}$ sq. in.

ACCESS ADDITIONAL ONLINE RESOURCES

Converting a Fraction to a Decimal - Part 2

Convert a Fraction to a Decimal (repeating)

Compare Fractions and Decimals using Inequality Symbols

Determine the Area of a Circle

Determine the Circumference of a Circle

Practice Makes Perfect

Convert Fractions to Decimals

In the following exercises, convert each fraction to a decimal.

201. $\frac{2}{5}$
202. $\frac{4}{5}$
203. $-\frac{3}{8}$
204. $-\frac{5}{8}$
205. $\frac{17}{20}$
206. $\frac{13}{20}$
207. $\frac{11}{4}$
208. $\frac{17}{4}$
209. $-\frac{310}{25}$
210. $-\frac{284}{25}$
211. $\frac{5}{9}$
212. $\frac{2}{9}$
213. $\frac{15}{11}$
214. $\frac{18}{11}$
215. $\frac{15}{111}$
216. $\frac{25}{111}$

In the following exercises, simplify the expression.

217. $\frac{1}{2}$ + 6.5
218. $\frac{1}{4}$ + 10.75
219. 2.4 + $\frac{5}{8}$
220. 3.9 + $\frac{9}{20}$
221. 9.73 + $\frac{17}{20}$
222. 6.29 + $\frac{21}{40}$

Order Decimals and Fractions

In the following exercises, order each pair of numbers, using < or >.

223. $\frac{1}{8}$ ___ 0.8
224. $\frac{1}{4}$ ___ 0.4
225. $\frac{2}{5}$ ___ 0.25
226. $\frac{3}{5}$ ___ 0.35

227. $0.725 \_\_\_ \frac{3}{4}$
228. $0.92 \_\_\_ \frac{7}{8}$
229. $0.66 \_\_\_ \frac{2}{3}$
230. $0.83 \_\_\_ \frac{5}{6}$
231. $-0.75 \_\_\_ -\frac{4}{5}$
232. $-0.44 \_\_\_ -\frac{9}{20}$
233. $-\frac{3}{4} \_\_\_ -0.925$
234. $-\frac{2}{3} \_\_\_ -0.632$

In the following exercises, write each set of numbers in order from least to greatest.

235. $\frac{3}{5}, \frac{9}{16}, 0.55$
236. $\frac{3}{8}, \frac{7}{20}, 0.36$
237. $0.702, \frac{13}{20}, \frac{5}{8}$
238. $0.15, \frac{3}{16}, \frac{1}{5}$
239. $-0.3, -\frac{1}{3}, -\frac{7}{20}$
240. $-0.2, -\frac{3}{20}, -\frac{1}{6}$
241. $-\frac{3}{4}, -\frac{7}{9}, -0.7$
242. $-\frac{8}{9}, -\frac{4}{5}, -0.9$

Simplify Expressions Using the Order of Operations

In the following exercises, simplify.

243. $10(25.1 - 43.8)$
244. $30(18.1 - 32.5)$
245. $62(9.75 - 4.99)$
246. $42(8.45 - 5.97)$
247. $\frac{3}{4}(12.4 - 4.2)$
248. $\frac{4}{5}(8.6 + 3.9)$
249. $\frac{5}{12}(30.58 + 17.9)$
250. $\frac{9}{16}(21.96 - 9.8)$
251. $10 \div 0.1 + (1.8)4 - (0.3)^2$
252. $5 \div 0.5 + (3.9)6 - (0.7)^2$
253. $(37.1 + 52.7) \div (12.5 \div 62.5)$
254. $(11.4 + 16.2) \div (18 \div 60)$
255. $\left(\frac{1}{5}\right)^2 + (1.4)(6.5)$
256. $\left(\frac{1}{2}\right)^2 + (2.1)(8.3)$
257. $-\frac{9}{10} \cdot \frac{8}{15} + 0.25$

258. $-\frac{3}{8} \cdot \frac{14}{15} + 0.72$

Mixed Practice

In the following exercises, simplify. Give the answer as a decimal.

259. $3\frac{1}{4} - 6.5$
260. $5\frac{2}{5} - 8.75$
261. $10.86 \div \frac{2}{3}$
262. $5.79 \div \frac{3}{4}$
263. $\frac{7}{8}(103.48) + 1\frac{1}{2}(361)$
264. $\frac{5}{16}(117.6) + 2\frac{1}{3}(699)$
265. $3.6\left(\frac{9}{8} - 2.72\right)$
266. $5.1\left(\frac{12}{5} - 3.91\right)$

Find the Circumference and Area of Circles

In the following exercises, approximate the (a) circumference and (b) area of each circle. If measurements are given in fractions, leave answers in fraction form.

267. radius = 5 in.
268. radius = 20 in.
269. radius = 9 ft.
270. radius = 4 ft.
271. radius = 46 cm
272. radius = 38 cm
273. radius = 18.6 m
274. radius = 57.3 m
275. radius = $\frac{7}{10}$ mile
276. radius = $\frac{7}{11}$ mile
277. radius = $\frac{3}{8}$ yard
278. radius = $\frac{5}{12}$ yard
279. diameter = $\frac{5}{6}$ m
280. diameter = $\frac{3}{4}$ m

Everyday Math

281. Kelly wants to buy a pair of boots that are on sale for $\frac{2}{3}$ of the original price. The original price of the boots is $84.99. What is the sale price of the shoes?
282. An architect is planning to put a circular mosaic in the entry of a new building. The mosaic will be in the shape of a circle with radius of 6 feet. How many square feet of tile will be needed for the mosaic? (Round your answer up to the next whole number.)

Writing Exercises

283. Is it easier for you to convert a decimal to a fraction or a fraction to a decimal? Explain.
284. Describe a situation in your life in which you might need to find the area or circumference of a circle.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section

I can...	Confidently	With some help	No-I don't get it!
convert fractions to decimals.			
order decimals and fractions.			
simplify expressions using the order of operations.			
find the circumference and area of circles.			

(b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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5.7: Rational and Irrational Numbers

Learning Objectives

- Identify rational numbers and irrational numbers
- Classify different types of real numbers

be prepared!

Before you get started, take this readiness quiz.

1. Write 3.19 as an improper fraction. If you missed this problem, review Example 5.1.4.
2. Write $\frac{5}{11}$ as a decimal. If you missed this problem, review Example 5.5.3.
3. Simplify: $\sqrt{144}$. If you missed this problem, review Example 5.12.1.

Identify Rational Numbers and Irrational Numbers

Congratulations! You have completed the first six chapters of this book! It's time to take stock of what you have done so far in this course and think about what is ahead. You have learned how to add, subtract, multiply, and divide whole numbers, fractions, integers, and decimals. You have become familiar with the language and symbols of algebra, and have simplified and evaluated algebraic expressions. You have solved many different types of applications. You have established a good solid foundation that you need so you can be successful in algebra.

In this chapter, we'll make sure your skills are firmly set. We'll take another look at the kinds of numbers we have worked with in all previous chapters. We'll work with properties of numbers that will help you improve your number sense. And we'll practice using them in ways that we'll use when we solve equations and complete other procedures in algebra.

We have already described numbers as counting numbers, whole numbers, and integers. Do you remember what the difference is among these types of numbers?

counting numbers	1, 2, 3, 4…
whole numbers	0, 1, 2, 3, 4…
integers	…−3, −2, −1, 0, 1, 2, 3, 4…

Rational Numbers

What type of numbers would you get if you started with all the integers and then included all the fractions? The numbers you would have form the set of rational numbers. A **rational number** is a number that can be written as a ratio of two integers.

Definition: Rational Numbers

A rational number is a number that can be written in the form $\frac{p}{q}$, where p and q are integers and q ≠ 0.

All fractions, both positive and negative, are rational numbers. A few examples are

$$\frac{4}{5}, -\frac{7}{8}, \frac{13}{4}, \text{ and } -\frac{20}{3} \tag{5.7.1}$$

Each numerator and each denominator is an integer.

We need to look at all the numbers we have used so far and verify that they are rational. The definition of rational numbers tells us that all fractions are rational. We will now look at the counting numbers, whole numbers, integers, and decimals to make sure they are rational.

Are integers rational numbers? To decide if an integer is a rational number, we try to write it as a ratio of two integers. An easy way to do this is to write it as a fraction with denominator one.

$$3 = \frac{3}{1} \quad -8 = \frac{-8}{1} \quad 0 = \frac{0}{1} \tag{5.7.2}$$

Since any integer can be written as the ratio of two integers, all integers are rational numbers. Remember that all the counting numbers and all the whole numbers are also integers, and so they, too, are rational.

What about decimals? Are they rational? Let's look at a few to see if we can write each of them as the ratio of two integers. We've already seen that integers are rational numbers. The integer −8 could be written as the decimal −8.0. So, clearly, some decimals are rational.

Think about the decimal 7.3. Can we write it as a ratio of two integers? Because 7.3 means $7\frac{3}{10}$, we can write it as an improper fraction, $7\frac{3}{10}$. So 7.3 is the ratio of the integers 73 and 10. It is a rational number.

In general, any decimal that ends after a number of digits (such as 7.3 or −1.2684) is a rational number. We can use the reciprocal (or multiplicative inverse) of the place value of the last digit as the denominator when writing the decimal as a fraction.

Example 5.7.1:

Write each as the ratio of two integers: (a) −15 (b) 6.81 (c) $-3\frac{6}{7}$.

Solution

(a) −15

Write the integer as a fraction with denominator 1.	$\frac{-15}{1}$	(5.7.3)

(b) 6.81

Write the decimal as a mixed number.	$6\frac{81}{100}$	(5.7.4)
Then convert it to an improper fraction.	$\frac{681}{100}$	(5.7.5)

(c) $-3\frac{6}{7}$

Convert the mixed number to an improper fraction.	$-\frac{27}{7}$	(5.7.6)

Exercise 5.7.1:

Write each as the ratio of two integers: (a) −24 (b) 3.57.

Answer a

$\frac{-24}{1}$

Answer b

$\frac{357}{100}$

? Exercise 5.7.2:

Write each as the ratio of two integers: (a) −19 (b) 8.41.

Answer a

$\frac{-19}{1}$

Answer b

$\frac{841}{100}$

Let's look at the decimal form of the numbers we know are rational. We have seen that every integer is a rational number, since a = $\frac{a}{1}$ for any integer, a. We can also change any integer to a decimal by adding a decimal point and a zero.

$$\begin{array}{ll} Integer & -2,\quad -1,\quad 0,\quad 1,\quad 2,\ 3 \\ Decimal & -2.0, -1.0, 0.0, 1.0, 2.0, 3.0 \end{array}$$

These decimal numbers stop.

We have also seen that every fraction is a rational number. Look at the decimal form of the fractions we just considered.

$$\begin{array}{ll} Ratio\ of\ Integers & \frac{4}{5},\quad -\frac{7}{8},\quad \frac{13}{4},\quad -\frac{20}{3} \\ Decimal\ forms & 0.8, -0.875, 3.25, -6.666\ldots \\ & \qquad\qquad\qquad\quad -6.\overline{66} \end{array}$$

These decimals either stop or repeat.

What do these examples tell you? Every rational number can be written both as a ratio of integers and as a decimal that either stops or repeats. The table below shows the numbers we looked at expressed as a ratio of integers and as a decimal.

Rational Numbers		
	Fractions	**Integers**
Number	$\frac{4}{5}, -\frac{7}{8}, \frac{13}{4}, \frac{-20}{3}$ (5.7.7)	$-2, -1, 0, 1, 2, 3$ (5.7.8)
Ratio of Integer	$\frac{4}{5}, \frac{-7}{8}, \frac{13}{4}, \frac{-20}{3}$ (5.7.9)	$\frac{-2}{1}, \frac{-1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}$ (5.7.10)
Decimal number	$0.8, -0.875, 3.25, -6.\bar{6}$ (5.7.11)	$-2.0, -1.0, 0.0, 1.0, 2.0, 3.0$ (5.7.1

Irrational Numbers

Are there any decimals that do not stop or repeat? Yes. The number π (the Greek letter pi, pronounced 'pie'), which is very important in describing circles, has a decimal form that does not stop or repeat.

$$\pi = 3.141592654\ldots\ldots \tag{5.7.13}$$

Similarly, the decimal representations of square roots of whole numbers that are not perfect squares never stop and never repeat. For example,

$$\sqrt{5} = 2.236067978\ldots\ldots \tag{5.7.14}$$

A decimal that does not stop and does not repeat cannot be written as the ratio of integers. We call this kind of number an **irrational number**.

Definition: Irrational Number

An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

Let's summarize a method we can use to determine whether a number is rational or irrational.

If the decimal form of a number

- stops or repeats, the number is rational.
- does not stop and does not repeat, the number is irrational.

Example 5.7.2:

Identify each of the following as rational or irrational: (a) $0.58\overline{3}$ (b) 0.475 (c) 3.605551275…

Solution

(a) $0.58\overline{3}$

The bar above the 3 indicates that it repeats. Therefore, 0.583 – is a repeating decimal, and is therefore a rational number.

(b) 0.475

This decimal stops after the 5, so it is a rational number.

(c) 3.605551275…

The ellipsis (…) means that this number does not stop. There is no repeating pattern of digits. Since the number doesn't stop and doesn't repeat, it is irrational.

Exercise 5.7.3:

Identify each of the following as rational or irrational: (a) 0.29 (b) $0.81\overline{6}$ (c) 2.515115111…

Answer a

rational

Answer b

rational

Answer c

irrational

Exercise 5.7.4:

Identify each of the following as rational or irrational: (a) $0.2\overline{3}$ (b) 0.125 (c) 0.418302…

Answer a

rational

Answer b

rational

Answer c

irrational

Let's think about square roots now. Square roots of perfect squares are always whole numbers, so they are rational. But the decimal forms of square roots of numbers that are not perfect squares never stop and never repeat, so these square roots are irrational.

Example 5.7.3:

Identify each of the following as rational or irrational: (a) 36 (b) 44

Solution

(a) The number 36 is a perfect square, since $6^2 = 36$. So $\sqrt{36} = 6$. Therefore $\sqrt{36}$ is rational.

(b)Remember that $6^2 = 36$ and $7^2 = 49$, so 44 is not a perfect square. This means $\sqrt{44}$ is irrational.

Exercise 5.7.5:

Identify each of the following as rational or irrational: (a) $\sqrt{81}$ (b) $\sqrt{17}$

Answer a

rational

Answer b

irrational

Exercise 5.7.6:

Identify each of the following as rational or irrational: (a) $\sqrt{116}$ (b) $\sqrt{121}$

Answer a

irrational

Answer b

rational

Classify Real Numbers

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. Irrational numbers are a separate category of their own. When we put together the rational numbers and the irrational numbers, we get the set of **real numbers**. Figure 5.7.1 illustrates how the number sets are related.

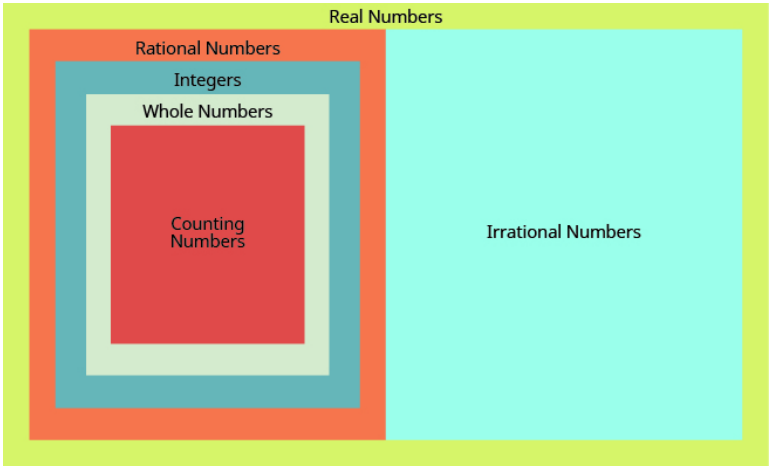

Figure 5.7.1 - This diagram illustrates the relationships between the different types of real numbers.

Definition: Real Numbers

Real numbers are numbers that are either rational or irrational.

Does the term “real numbers” seem strange to you? Are there any numbers that are not “real”, and, if so, what could they be? For centuries, the only numbers people knew about were what we now call the real numbers. Then mathematicians discovered the set of *imaginary numbers*. You won't encounter imaginary numbers in this course, but you will later on in your studies of algebra.

Example 5.7.4:

Determine whether each of the numbers in the following list is a (a) whole number, (b) integer, (c) rational number, (d) irrational number, and (e) real number.

$$-7, \frac{14}{5}, 8, \sqrt{5}, 5.9, -\sqrt{64} \tag{5.7.15}$$

Solution

a. The whole numbers are 0, 1, 2, 3,… The number 8 is the only whole number given.
b. The integers are the whole numbers, their opposites, and 0. From the given numbers, −7 and 8 are integers. Also, notice that 64 is the square of 8 so $-\sqrt{64} = -8$. So the integers are −7, 8, $-\sqrt{64}$.
c. Since all integers are rational, the numbers −7, 8, and $-\sqrt{64}$ are also rational. Rational numbers also include fractions and decimals that terminate or repeat, so $\frac{14}{5}$ and 5.9 are rational.
d. The number 5 is not a perfect square, so $\sqrt{5}$ is irrational.
e. All of the numbers listed are real.

We'll summarize the results in a table.

Number	Whole	Integer	Rational	Irrational	Real
-7		✓	✓		✓
$\frac{14}{5}$			✓		✓
8	✓	✓	✓		✓
$\sqrt{5}$				✓	✓
5.9			✓		✓
$-\sqrt{64}$		✓	✓		✓

Exercise 5.7.7:

Determine whether each number is a (a) whole number,(b) integer,(c) rational number,(d) irrational number, and (e) real number: $-3, -\sqrt{2}, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$.

Answer

Number	Whole	Integer	Rational	Irrational	Real
-3		✓	✓		✓
$-\sqrt{2}$				✓	✓
$0.\overline{3}$			✓		✓
$\frac{9}{5}$			✓		✓
4	✓	✓	✓		✓

Number	Whole	Integer	Rational	Irrational	Real
$\sqrt{49}$	✓	✓	✓		✓
-3		✓	✓		✓
				✓	✓
$0.\overline{3}$			✓		✓
4	✓	✓	✓		✓

? Exercise 5.7.8:

Determine whether each number is a (a) whole number,(b) integer,(c) rational number,(d) irrational number, and (e) real number:

$-\sqrt{25}, -\frac{3}{8}, \frac{9}{5}1, 6, \sqrt{121}, 2.041975\ldots$

Answer

Number	Whole	Integer	Rational	Irrational	Real
$\sqrt{49}$	✓	✓	✓		✓
$-\sqrt{25}$		✓	✓		✓
$-\frac{3}{8}$			✓		✓
-1		✓	✓		✓
6	✓	✓	✓		✓
$\sqrt{121}$	✓	✓	✓		✓
$2.041975\ldots$				✓	✓

ACCESS ADDITIONAL ONLINE RESOURCES

Sets of Real Numbers

Real Numbers

Practice Makes Perfect

Rational Numbers

In the following exercises, write as the ratio of two integers.

1. (a) 5 (b) 3.19
2. (a) 8 (b) −1.61
3. (a) −12 (b) 9.279
4. (a) −16 (b) 4.399

In the following exercises, determine which of the given numbers are rational and which are irrational.

5. 0.75, $0.22\overline{3}$, 1.39174…
6. 0.36, 0.94729…, $2.52\overline{8}$
7. $0.\overline{45}$, 1.919293…, 3.59
8. $0.1\overline{3}$, 0.42982…, 1.875

In the following exercises, identify whether each number is rational or irrational.

9. (a) 25 (b) 30
10. (a) 44 (b) 49
11. (a) 164 (b) 169
12. (a) 225 (b) 216

Classifying Real Numbers

In the following exercises, determine whether each number is whole, integer, rational, irrational, and real.

13. $-8, 0, 1.95286...., \frac{12}{5}, \sqrt{36}, 9$

14. -9 , $-3\frac{4}{9}, -\sqrt{9}, 0.4\overline{09}, \frac{11}{6}, 7$
15. $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$

Everyday Math

16. **Field trip** All the 5th graders at Lincoln Elementary School will go on a field trip to the science museum. Counting all the children, teachers, and chaperones, there will be 147 people. Each bus holds 44 people.
    a. How many buses will be needed?
    b. Why must the answer be a whole number?
    c. Why shouldn't you round the answer the usual way?
17. **Child care** Serena wants to open a licensed child care center. Her state requires that there be no more than 12 children for each teacher. She would like her child care center to serve 40 children.
    a. How many teachers will be needed?
    b. Why must the answer be a whole number?
    c. Why shouldn't you round the answer the usual way?

Writing Exercises

18. In your own words, explain the difference between a rational number and an irrational number.
19. Explain how the sets of numbers (counting, whole, integer, rational, irrationals, reals) are related to each other.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify rational and irrational numbers.			
classify different types of real numbers.			

(b) If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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5.8: Averages and Probability (Part 1)

Learning Objectives

- Calculate the mean of a set of numbers
- Find the median of a set of numbers
- Find the mode of a set of numbers
- Apply the basic definition of probability

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{4+9+2}{3}$. If you missed this problem, review Example 4.6.12.
2. Simplify: 4(8) + 6(3). If you missed this problem, review Example 2.2.8.
3. Convert $\frac{5}{2}$ to a decimal. If you missed this problem, review Example 5.5.1.

One application of decimals that arises often is finding the *average* of a set of numbers. What do you think of when you hear the word *average*? Is it your grade point average, the average rent for an apartment in your city, the batting average of a player on your favorite baseball team? The average is a typical value in a set of numerical data. Calculating an average sometimes involves working with decimal numbers. In this section, we will look at three different ways to calculate an average.

Calculate the Mean of a Set of Numbers

The mean is often called the arithmetic average. It is computed by dividing the sum of the values by the number of values. Students want to know the mean of their test scores. Climatologists report that the mean temperature has, or has not, changed. City planners are interested in the mean household size.

Suppose Ethan's first three test scores were 85, 88, and 94. To find the mean score, he would add them and divide by 3.

$$\frac{85+88+94}{3} \tag{5.8.1}$$

$$\frac{267}{3} \tag{5.8.2}$$

$$89 \tag{5.8.3}$$

His mean test score is 89 points.

Definition: The Mean

The mean of a set of n numbers is the arithmetic average of the numbers.

$$mean = \frac{sum\ of\ values\ in\ data\ set}{n} \tag{5.8.4}$$

HOW TO: CALCULATE THE MEAN OF A SET OF NUMBERS

Step 1. Write the formula for the mean

$$mean = \frac{sum\ of\ values\ in\ data\ set}{n} \tag{5.8.5}$$

Step 2. Find the sum of all the values in the set. Write the sum in the numerator.

Step 3. Count the number, n, of values in the set. Write this number in the denominator.

Step 4. Simplify the fraction.

Step 5. Check to see that the mean is reasonable. It should be greater than the least number and less than the greatest number in the set.

Example 5.8.1:

Find the mean of the numbers 8, 12, 15, 9, and 6.

Solution

Write the formula for the mean.	$$mean = \frac{sum\ of\ values\ in\ data\ set}{n} \quad (5.8.6)$$
Write the sum of the numbers in the numerator.	$$mean = \frac{8+12+15+9+6}{n} \quad (5.8.7)$$
Count how many numbers are in the set. There are 5 numbers in the set, so n = 5.	$$mean = \frac{8+12+15+9+6}{5} \quad (5.8.8)$$
Add the numbers in the numerator.	$$mean = \frac{50}{5} \quad (5.8.9)$$
Then divide.	mean = 10
Check to see that the mean is 'typical': 10 is neither less than 6 nor greater than 15.	The mean is 10.

Exercise 5.8.1:

Find the mean of the numbers: 8, 9, 7, 12, 10, 5.

Answer

8.5

Exercise 5.8.2:

Find the mean of the numbers: 9, 13, 11, 7, 5.

Answer

9

Example 5.8.2:

The ages of the members of a family who got together for a birthday celebration were 16, 26, 53, 56, 65, 70, 93, and 97 years. Find the mean age.

Solution

Write the formula for the mean.	$$mean = \frac{sum\ of\ values\ in\ data\ set}{n} \quad (5.8.10)$$
Write the sum of the numbers in the numerator.	$$mean = \frac{16+26+53+56+65+70+93+97}{n} \quad (5.8.1$$
Count how many numbers are in the set. Call this n and write it in the denominator.	$$mean = \frac{16+26+53+56+65+70+93+97}{8} \quad (5.8.1$$

Simplify the fraction.	$$mean = \frac{476}{5}$$	(5.8.13)
	mean = 59.5	

Is 59.5 ‘typical’? Y es, it is neither less than 16 nor greater than 97. The mean age is 59.5 years.

? Exercise 5.8.3:

The ages of the four students in Ben’s carpool are 25, 18, 21, and 22. Find the mean age of the students.

Answer

21.5 years

? Exercise 5.8.4:

Yen counted the number of emails she received last week. The numbers were 4, 9, 15, 12, 10, 12, and 8. Find the mean number of emails

Answer

10

Did you notice that in the last example, while all the numbers were whole numbers, the mean was 59.5, a number with one decimal place? It is customary to report the mean to one more decimal place than the original numbers. In the next example, all the numbers represent money, and it will make sense to report the mean in dollars and cents.

✓ Example 5.8.3:

For the past four months, Daisy’s cell phone bills were $42.75, $50.12, $41.54, $48.15. Find the mean cost of Daisy’s cell phone bills.

Solution

Write the formula for the mean.	$$mean = \frac{sum\ of\ all\ the\ numbers}{n}$$	(5.8.14)
Write the sum of the numbers in the numerator.	$$mean = \frac{sum\ of\ all\ the\ numbers}{4}$$	(5.8.15)
Count how many numbers are in the set. Call this n and write it in the denominator.	$$mean = \frac{42.75 + 50.12 + 41.54 + 48.15}{4}$$	(5.8.16)
Simplify the fraction.	$$mean = \frac{182.56}{4}$$	(5.8.17)
	mean = 45.64	

Does $45.64 seem ‘typical’ of this set of numbers? Yes, it is neither less than $41.54 nor greater than $50.12. The mean cost of her cell phone bill was $45.64.

? Exercise 5.8.5:

Last week Ray recorded how much he spent for lunch each workday. He spent $6.50, $7.25, $4.90, $5.30, and $12.00. Find the mean of how much he spent each day.

Answer

$7.19

Exercise 5.8.6:

Lisa has kept the receipts from the past four trips to the gas station. The receipts show the following amounts: $34.87, $42.31, $38.04, and $43.26. Find the mean.

Answer

$39.62

Find the Median of a Set of Numbers

When Ann, Bianca, Dora, Eve, and Francine sing together on stage, they line up in order of their heights. Their heights, in inches, are shown in Table 5.70.

Table 5.70

Ann	Bianca	Dora	Eve	Francine
59	60	65	68	70

Dora is in the middle of the group. Her height, 65″, is the *median* of the girls' heights. Half of the heights are less than or equal to Dora's height, and half are greater than or equal. The median is the middle value.

Definition: Median

The median of a set of data values is the middle value.

- Half the data values are less than or equal to the median.
- Half the data values are greater than or equal to the median.

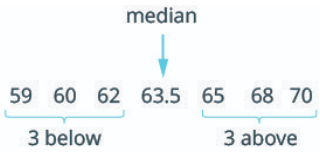
What if Carmen, the pianist, joins the singing group on stage? Carmen is 62 inches tall, so she fits in the height order between Bianca and Dora. Now the data set looks like this:

$$59, 60, 62, 65, 68, 70 \tag{5.8.18}$$

There is no single middle value. The heights of the six girls can be divided into two equal parts.

$$\underbrace{59, 60, 62}\quad\underbrace{65, 68, 70} \tag{5.8.19}$$

Statisticians have agreed that in cases like this the median is the mean of the two values closest to the middle. So the median is the mean of 62 and 65, $\frac{62+65}{2}$. The median height is 63.5 inches.



Notice that when the number of girls was 5, the median was the third height, but when the number of girls was 6, the median was the mean of the third and fourth heights. In general, when the number of values is odd, the median will be the one value in the middle, but when the number is even, the median is the mean of the two middle values.

HOW TO: FIND THE MEDIAN OF A SET OF NUMBERS

Step 1. List the numbers from smallest to largest.

Step 2. Count how many numbers are in the set. Call this n.

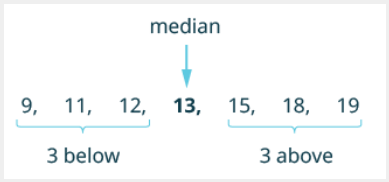
Step 3. Is n odd or even?

- If n is an odd number, the median is the middle value.
- If n is an even number, the median is the mean of the two middle values.

Example 5.8.4:

Find the median of 12, 13, 19, 9, 11, 15, and 18.

Solution

List the numbers in order from smallest to largest.	9, 11, 12, 13, 15, 18, 19
Count how many numbers are in the set. Call this n.	n = 7
Is n odd or even?	odd
The median is the middle value.	
The middle is the number in the 4th position.	So the median of the data is 13.

Exercise 5.8.7:

Find the median of the data set: 43, 38, 51, 40, 46.

Answer

43

Exercise 5.8.8:

Find the median of the data set: 15, 35, 20, 45, 50, 25, 30.

Answer

30

Example 5.8.5:

Kristen received the following scores on her weekly math quizzes: 83, 79, 85, 86, 92, 100, 76, 90, 88, and 64. Find her median score.

Solution

List the numbers in order from smallest to largest.	64, 76, 79, 83, 85, 86, 88, 90, 92, 100
Count the number of data values in the set. Call this n.	n = 10
Is n odd or even?	even

The median is the mean of the two middle values, the 5th and 6th numbers.	64, 76, 79, 83, **85,** **86,** 88, 90, 92, 100 5 numbers 5 numbers
Find the mean of 85 and 86.	mean = 85.5

Kristen's median score is 85.5.

Exercise 5.8.9:

Find the median of the data set: 8, 7, 5, 10, 9, 12.

Answer

8.5

Exercise 5.8.10:

Find the median of the data set: 21, 25, 19, 17, 22, 18, 20, 24.

Answer

20.5

Identify the Mode of a Set of Numbers

The *average* is one number in a set of numbers that is somehow typical of the whole set of numbers. The mean and median are both often called the average. Yes, it can be confusing when the word average refers to two different numbers, the mean and the median! In fact, there is a third number that is also an average. This average is the **mode**. The mode of a set of numbers is the number that occurs the most. The **frequency**, is the number of times a number occurs. So the mode of a set of numbers is the number with the highest frequency.

Definition: mode

The mode of a set of numbers is the number with the highest frequency.

Suppose Jolene kept track of the number of miles she ran since the start of the month, as shown in Figure 5.7.

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
				1 2 mi New Year's day	2	3 15 mi
4 8 mi	5	6 3 mi	7 8 mi	8	9 5 mi	10 8 mi
11	12	13	14	15	16	17

Figure 5.7

If we list the numbers in order it is easier to identify the one with the highest frequency.

$$2, 3, 5, 8, 8, 8, 15 \tag{5.8.20}$$

Jolene ran 8 miles three times, and every other distance is listed only once. So the mode of the data is 8 miles.

HOW TO: IDENTIFY THE MODE OF A SET OF NUMBERS

Step 1. List the data values in numerical order.

Step 2. Count the number of times each value appears.

Step 3. The mode is the value with the highest frequency.

Example 5.8.6:

The ages of students in a college math class are listed below. Identify the mode.

18, 18, 18, 18, 19, 19, 19, 20, 20, 20, 20, 20, 20, 20, 21, 21, 22, 22, 22, 22, 22, 23, 24, 24, 25, 29, 30, 40, 44

Solution

The ages are already listed in order. We will make a table of frequencies to help identify the age with the highest frequency.

Age	18	19	20	21	22	23	24	25	29	30	40	44
Frequency	4	3	7	2	5	1	2	1	1	1	1	1

Now look for the highest frequency. The highest frequency is 7, which corresponds to the age 20. So the mode of the ages in this class is 20 years.

Exercise 5.8.11:

The number of sick days employees used last year: 3, 6, 2, 3, 7, 5, 6, 2, 4, 2. Identify the mode.

Answer

2

Exercise 5.8.12:

The number of handbags owned by women in a book club: 5, 6, 3, 1, 5, 8, 1, 5, 8, 5. Identify the mode.

Answer

5

Example 5.8.7:

The data lists the heights (in inches) of students in a statistics class. Identify the mode.

56	61	63	64	65	66	67	67
60	62	63	64	65	66	67	70
60	63	63	64	66	66	67	74
61	63	64	65	66	67	67	

Solution

List each number with its frequency.

Number	56	60	61	62	63	64	65	66	67	70	74
Frequency	1	2	2	1	5	4	3	5	6	1	1

Now look for the highest frequency. The highest frequency is 6, which corresponds to the height 67 inches. So the mode of this set of heights is 67 inches.

Exercise 5.8.13:

The ages of the students in a statistics class are listed here: 19, 20, 23, 23, 38, 21, 19, 21, 19, 21, 20, 43, 20, 23, 17, 21, 21, 20, 29, 18, 28. What is the mode?

Answer

21

Exercise 5.8.14:

Students listed the number of members in their household as follows: 6, 2, 5, 6, 3, 7, 5, 6, 5, 3, 4, 4, 5, 7, 6, 4, 5, 2, 1, 5. What is the mode?

Answer

5

Some data sets do not have a mode because no value appears more than any other. And some data sets have more than one mode. In a given set, if two or more data values have the same highest frequency, we say they are all modes.

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5.9: Averages and Probability (Part 2)

Use the Basic Definition of Probability

The probability of an event tells us how likely that event is to occur. We usually write probabilities as fractions or decimals. For example, picture a fruit bowl that contains five pieces of fruit - three bananas and two apples.

If you want to choose one piece of fruit to eat for a snack and don't care what it is, there is a $\frac{3}{5}$ probability you will choose a banana, because there are three bananas out of the total of five pieces of fruit. The probability of an event is the number of favorable outcomes divided by the total number of outcomes.

$$\text{Probability of an event} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

$$\text{Probability of choosing a banana} = \frac{3}{5} \begin{array}{l} \leftarrow \text{There are 3 bananas.} \\ \leftarrow \text{There are 5 pieces of fruit.} \end{array}$$

Definition: Probability

The probability of an event is the number of favorable outcomes divided by the total number of outcomes possible.

$$Probability = \frac{number\ of\ favorable\ outcomes}{total\ number\ of\ outcomes} \tag{5.9.1}$$

Converting the fraction $\frac{3}{5}$ to a decimal, we would say there is a 0.6 probability of choosing a banana.

$$\text{Probability of choosing a banana} = \frac{3}{5}$$

$$\text{Probability of choosing a banana} = 0.6$$

This basic definition of probability assumes that all the outcomes are equally likely to occur. If you study probabilities in a later math class, you'll learn about several other ways to calculate probabilities.

Example 5.9.8:

The ski club is holding a raffle to raise money. They sold 100 tickets. All of the tickets are placed in a jar. One ticket will be pulled out of the jar at random, and the winner will receive a prize. Cherie bought one raffle ticket. (a) Find the probability she will win the prize. (b) Convert the fraction to a decimal.

Solution

(a)

What are you asked to find?	The probability Cherie wins the prize.
What is the number of favorable outcomes?	1, because Cherie has 1 ticket.
Use the definition of probability.	$Probability\ of\ an\ event = \frac{number\ of\ favorable\ outcomes}{total\ number\ of\ outcomes}$
Substitute into the numerator and denominator.	Probability Cherie wins = $\frac{1}{100}$

(b)

Write the probability as a fraction.	Probability = $\frac{1}{100}$
Convert the fraction to a decimal.	Probability = 0.01

Exercise 5.9.15:

Ignaly is attending a fashion show where the guests are seated at tables of ten. One guest from each table will be selected at random to receive a door prize. (a) Find the probability Ignaly will win the door prize for her table. (b) Convert the fraction to a decimal.

Answer a

$\frac{1}{10}$

Answer b

0.1

Exercise 5.9.16:

Hoang is among 20 people available to sit on a jury. One person will be chosen at random from the 20. (a) Find the probability Hoang will be chosen. (b) Convert the fraction to a decimal.

Answer a

$\frac{1}{20}$

Answer b

0.05

Example 5.9.9:

Three women and five men interviewed for a job. One of the candidates will be offered the job. (a) Find the probability the job is offered to a woman. (b) Convert the fraction to a decimal.

Solution

What are you asked to find?	The probability the job is offered to a woman.
What is the number of favorable outcomes?	3, because there are three women.
What are the total number of outcomes?	8, because 8 people interviewed.
Use the definition of probability.	$Probability\ of\ an\ event = \frac{number\ of\ favorable\ outcomes}{total\ number\ of\ outcomes}$
Substitute into the numerator and denominator.	Probability = $\frac{3}{8}$

(b)

Write the probability as a fraction.	Probability = $\frac{3}{8}$
Convert the fraction to a decimal.	Probability = 0.375

Exercise 5.9.17:

A bowl of Halloween candy contains 5 chocolate candies and 3 lemon candies. Tanya will choose one piece of candy at random.(a) Find the probability Tanya will choose a chocolate candy.(b) Convert the fraction to a decimal.

Answer a

$\frac{5}{8}$

Answer b

0.625

Exercise 5.9.18:

Dan has 2 pairs of black socks and 6 pairs of blue socks. He will choose one pair at random to wear tomorrow. (a) Find the probability Dan will choose a pair of black socks (b) Convert the fraction to a decimal.

Answer a

$\frac{2}{8}$

Answer b

0.25

ACCESS ADDITIONAL ONLINE RESOURCES

Mean, Median, and Mode

Find the Mean of a Data Set

Find the Median of a Data Set

Find the Mode of a Data Set

Practice Makes Perfect

Calculate the Mean of a Set of Numbers

In the following exercises, find the mean.

357. 3, 8, 2, 2, 5
358. 6, 1, 9, 3, 4, 7
359. 65, 13, 48, 32, 19, 33
360. 34, 45, 29, 61, and 41
361. 202, 241, 265, 274
362. 525, 532, 558, 574
363. 12.45, 12.99, 10.50, 11.25, 9.99, 12.72
364. 28.8, 32.9, 32.5, 27.9, 30.4, 32.5, 31.6, 32.7
365. Four girls leaving a mall were asked how much money they had just spent. The amounts were $0, $14.95, $35.25, and $25.16. Find the mean amount of money spent.
366. Juan bought 5 shirts to wear to his new job. The costs of the shirts were $32.95, $38.50, $30.00, $17.45, and $24.25. Find the mean cost.
367. The number of minutes it took Jim to ride his bike to school for each of the past six days was 21, 18, 16, 19, 24, and 19. Find the mean number of minutes.
368. Norris bought six books for his classes this semester. The costs of the books were $74.28, $120.95, $52.40, $10.59, $35.89, and $59.24. Find the mean cost.
369. The top eight hitters in a softball league have batting averages of .373, .360, .321, .321, .320, .312, .311, and .311. Find the mean of the batting averages. Round your answer to the nearest thousandth.
370. The monthly snowfall at a ski resort over a six-month period was 60.3, 79.7, 50.9, 28.0, 47.4, and 46.1 inches. Find the mean snowfall.

Find the Median of a Set of Numbers

In the following exercises, find the median.

371. 24, 19, 18, 29, 21
372. 48, 51, 46, 42, 50
373. 65, 56, 35, 34, 44, 39, 55, 52, 45
374. 121, 115, 135, 109, 136, 147, 127, 119, 110
375. 4, 8, 1, 5, 14, 3, 1, 12
376. 3, 9, 2, 6, 20, 3, 3, 10
377. 99.2, 101.9, 98.6, 99.5, 100.8, 99.8
378. 28.8, 32.9, 32.5, 27.9, 30.4, 32.5, 31.6, 32.7
379. Last week Ray recorded how much he spent for lunch each workday. He spent $6.50, $7.25, $4.90, $5.30, and $12.00 . Find the median.
380. Michaela is in charge of 6 two-year olds at a daycare center. Their ages, in months, are 25, 24, 28, 32, 29, and 31. Find the median age.
381. Brian is teaching a swim class for 6 three-year olds. Their ages, in months, are 38, 41, 45, 36, 40, and 42. Find the median age.
382. Sal recorded the amount he spent for gas each week for the past 8 weeks. The amounts were $38.65, $32.18, $40.23, $51.50, $43.68, $30.96, $41.37, and $44.72. Find the median amount.

Identify the Mode of a Set of Numbers

In the following exercises, identify the mode.

383. 2, 5, 1, 5, 2, 1, 2, 3, 2, 3, 1
384. 8, 5, 1, 3, 7, 1, 1, 7, 1, 8, 7

385. 18, 22, 17, 20, 19, 20, 22, 19, 29, 18, 23, 25, 22, 24, 23, 22, 18, 20, 22, 20
386. 42, 28, 32, 35, 24, 32, 48, 32, 32, 24, 35, 28, 30, 35, 45, 32, 28, 32, 42, 42, 30
387. The number of children per house on one block: 1, 4, 2, 3, 3, 2, 6, 2, 4, 2, 0, 3, 0.
388. The number of movies watched each month last year: 2, 0, 3, 0, 0, 8, 6, 5, 0, 1, 2, 3.
389. The number of units being taken by students in one class: 12, 5, 11, 10, 10, 11, 5, 11, 11, 11, 10, 12.
390. The number of hours of sleep per night for the past two weeks: 8, 5, 7, 8, 8, 6, 6, 6, 6, 9, 7, 8, 8, 8.

Use the Basic Definition of Probability

In the following exercises, express the probability as both a fraction and a decimal. (Round to three decimal places, if necessary.)

391. Josue is in a book club with 20 members. One member is chosen at random each month to select the next month's book. Find the probability that Josue will be chosen next month.
392. Jessica is one of eight kindergarten teachers at Mandela Elementary School. One of the kindergarten teachers will be selected at random to attend a summer workshop. Find the probability that Jessica will be selected.
393. There are 24 people who work in Dane's department. Next week, one person will be selected at random to bring in doughnuts. Find the probability that Dane will be selected. Round your answer to the nearest thousandth.
394. Monica has two strawberry yogurts and six banana yogurts in her refrigerator. She will choose one yogurt at random to take to work. Find the probability Monica will choose a strawberry yogurt.
395. Michel has four rock CDs and six country CDs in his car. He will pick one CD to play on his way to work. Find the probability Michel will pick a rock CD.
396. Noah is planning his summer camping trip. He can't decide among six campgrounds at the beach and twelve campgrounds in the mountains, so he will choose one campground at random. Find the probability that Noah will choose a campground at the beach.
397. Donovan is considering transferring to a 4-year college. He is considering 10 out-of state colleges and 4 colleges in his state. He will choose one college at random to visit during spring break. Find the probability that Donovan will choose an out-of-state college.
398. There are 258,890,850 number combinations possible in the Mega Millions lottery. One winning jackpot ticket will be chosen at random. Brent chooses his favorite number combination and buys one ticket. Find the probability Brent will win the jackpot. Round the decimal to the first digit that is not zero, then write the name of the decimal.

Everyday Math

399. Joaquin gets paid every Friday. His paychecks for the past 8 Fridays were $315, $236.25, $236.25, $236.25 $315, $315, $236.25, $393.75. Find the (a) mean (b) median, and (c) mode.
400. The cash register receipts each day last week at a coffee shop were $1,845, $1,520, $1,438, $1,682, $1,850, $2,721, $2,539. Find the (a) mean, (b) median, and (c) mode.

Writing Exercises

401. Explain in your own words the difference between the mean, median, and mode of a set of numbers.
402. Make an example of probability that relates to your life. Write your answer as a fraction and explain what the numerator and denominator represent.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
calculate the mean of a set of numbers.			
find the median of a set of numbers.			
find the mode of a set of numbers.			
use the basic definition of probability.			

(b) After looking at the checklist, do you think you are well prepared for the next section? Why or why not?

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5.10: Ratios and Rate (Part 1)

Learning Objectives

- Write a ratio as a fraction
- Write a rate as a fraction
- Find unit rates
- Find unit price
- Translate phrases to expressions with fractions

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{16}{24}$. If you missed this problem, review Example 4.3.1.
2. Divide: 2.76 ÷ 11.5. If you missed this problem, review Example 5.4.9.
3. Simplify: $\frac{1\frac{1}{2}}{2\frac{3}{4}}$. If you missed this problem, review Example 4.5.7.

Write a Ratio as a Fraction

When you apply for a mortgage, the loan officer will compare your total debt to your total income to decide if you qualify for the loan. This comparison is called the debt-to-income ratio. A **ratio** compares two quantities that are measured with the same unit. If we compare a and b , the ratio is written as a to b, $\frac{a}{b}$, or a:b.

Definition: ratios

A ratio compares two numbers or two quantities that are measured with the same unit. The ratio of a to b is written a to b, $\frac{a}{b}$, or a:b.

In this section, we will use the fraction notation. When a ratio is written in fraction form, the fraction should be simplified. If it is an improper fraction, we do not change it to a mixed number. Because a ratio compares two quantities, we would leave a ratio as $\frac{4}{1}$ instead of simplifying it to 4 so that we can see the two parts of the ratio.

Example 5.10.1:

Write each ratio as a fraction: (a) 15 to 27 (b) 45 to 18.

Solution

(a) 15 to 27

Write as a fraction with the first number in the numerator and the second in the denominator.	$\frac{15}{27}$	(5.10.1)
Simplify the fraction.	$\frac{5}{9}$	(5.10.2)

(b) 45 to 18

Write as a fraction with the first number in the numerator and the second in the denominator.	$\frac{45}{18}$	(5.10.3)
Simplify.	$\frac{5}{2}$	(5.10.4)

We leave the ratio in (b) as an improper fraction.

? Exercise 5.10.1:

Write each ratio as a fraction: (a) 21 to 56 (b) 48 to 32.

Answer a

$\frac{3}{8}$

Answer b

$\frac{3}{2}$

? Exercise 5.10.2:

Write each ratio as a fraction: (a) 27 to 72 (b) 51 to 34.

Answer a

$\frac{1}{1}$

Answer b

$\frac{3}{2}$

Ratios Involving Decimals

We will often work with ratios of decimals, especially when we have ratios involving money. In these cases, we can eliminate the decimals by using the Equivalent Fractions Property to convert the ratio to a fraction with whole numbers in the numerator and denominator.

For example, consider the ratio 0.8 to 0.05. We can write it as a fraction with decimals and then multiply the numerator and denominator by 100 to eliminate the decimals.

$$\frac{0.8}{0.05} \tag{5.10.5}$$

$$\frac{(0.8)100}{(0.05)100} \tag{5.10.6}$$

$$\frac{80}{5} \tag{5.10.7}$$

Do you see a shortcut to find the equivalent fraction? Notice that 0.8 = $\frac{8}{10}$ and 0.05 = $\frac{5}{100}$. The least common denominator of $\frac{8}{10}$ and 5 100 is 100. By multiplying the numerator and denominator of $\frac{0.8}{0.05}$ by 100, we 'moved' the decimal two places to the right to get the equivalent fraction with no decimals. Now that we understand the math behind the process, we can find the fraction with no decimals like this:

$$\frac{0.80}{0.05}$$

"Move" the decimal 2 places.	$\frac{80}{5}$	(5.10.8)
Simplify.	$\frac{16}{1}$	(5.10.9)

You do not have to write out every step when you multiply the numerator and denominator by powers of ten. As long as you move both decimal places the same number of places, the ratio will remain the same.

Example 5.10.2:

Write each ratio as a fraction of whole numbers: (a) 4.8 to 11.2 (b) 2.7 to 0.54

Solution

(a) 4.8 to 11.2

Write as a fraction.	$\frac{4.8}{11.2}$	(5.10.10)
Rewrite as an equivalent fraction without decimals, by moving both decimal points 1 place to the right.	$\frac{48}{112}$	(5.10.11)
Simplify.	$\frac{3}{7}$	(5.10.12)

So 4.8 to 11.2 is equivalent to $\frac{3}{7}$.

(b) 2.7 to 0.54

Write as a fraction.	$\frac{2.7}{0.54}$	(5.10.13)
The numerator has one decimal place and the denominator has 2. To clear both decimals we need to move the decimal 2 places to the right.	$\frac{270}{54}$	(5.10.14)
Simplify.	$\frac{5}{1}$	(5.10.15)

So 2.7 to 0.54 is equivalent to $\frac{5}{1}$.

Exercise 5.10.3:

Write each ratio as a fraction: (a) 4.6 to 11.5 (b) 2.3 to 0.69.

Answer a

$\frac{2}{5}$

Answer b

$\frac{10}{3}$

Exercise 5.10.4:

Write each ratio as a fraction: (a) 3.4 to 15.3 (b) 3.4 to 0.68.

Answer a

$\frac{2}{9}$

Answer b

$\frac{5}{1}$

Some ratios compare two mixed numbers. Remember that to divide mixed numbers, you first rewrite them as improper fractions.

Example 5.10.3:

Write the ratio of $1\frac{1}{4}$ to $2\frac{3}{8}$ as a fraction.

Solution

Write as a fraction.	$\frac{1\frac{1}{4}}{2\frac{3}{8}}$	(5.10.16)
Convert the numerator and denominator to improper fractions.	$\frac{\frac{5}{4}}{\frac{19}{8}}$	(5.10.17)
Rewrite as a division of fractions.	$\frac{5}{4} \div \frac{19}{8}$	(5.10.18)
Invert the divisor and multiply.	$\frac{5}{4} \cdot \frac{8}{19}$	(5.10.19)
Simplify.	$\frac{10}{19}$	(5.10.20)

Exercise 5.10.5:

Write each ratio as a fraction: $1\frac{3}{4}$ to $2\frac{5}{8}$.

Answer

\(\dfrac{2}{3}\

Exercise 5.10.6:

Write each ratio as a fraction: $1\frac{1}{8}$ to $2\frac{3}{4}$.

Answer

$\frac{9}{22}$

Applications of Ratios

One real-world application of ratios that affects many people involves measuring cholesterol in blood. The ratio of total cholesterol to HDL cholesterol is one way doctors assess a person's overall health. A ratio of less than 5 to 1 is considered good.

Example 5.10.4:

Hector's total cholesterol is 249 mg/dl and his HDL cholesterol is 39 mg/dl. (a) Find the ratio of his total cholesterol to his HDL cholesterol. (b) Assuming that a ratio less than 5 to 1 is considered good, what would you suggest to Hector?

Solution

(a) First, write the words that express the ratio. We want to know the ratio of Hector's total cholesterol to his HDL cholesterol.

Write as a fraction.	$\frac{total\ cholesterol}{HDL\ cholesterol}$	(5.10.21)
Substitute the values.	$\frac{249}{39}$	(5.10.22)
Simplify.	$\frac{83}{13}$	(5.10.23)

(b) Is Hector's cholesterol ratio ok? If we divide 83 by 13 we obtain approximately 6.4, so $\frac{83}{13} \approx \frac{6.4}{1}$. Hector's cholesterol ratio is high! Hector should either lower his total cholesterol or raise his HDL cholesterol.

Exercise 5.10.7:

Find the patient's ratio of total cholesterol to HDL cholesterol using the given information. Total cholesterol is 185 mg/dL and HDL cholesterol is 40 mg/dL.

Answer

\(\dfrac{37}{8}\

Exercise 5.10.8:

Find the patient’s ratio of total cholesterol to HDL cholesterol using the given information. Total cholesterol is 204 mg/dL and HDL cholesterol is 38 mg/dL.

Answer

\(\dfrac{102}{19}\

Ratios of Two Measurements in Different Units

To find the ratio of two measurements, we must make sure the quantities have been measured with the same unit. If the measurements are not in the same units, we must first convert them to the same units.

We know that to simplify a fraction, we divide out common factors. Similarly in a ratio of measurements, we divide out the common unit.

Example 5.10.5:

The Americans with Disabilities Act (ADA) Guidelines for wheel chair ramps require a maximum vertical rise of 1 inch for every 1 foot of horizontal run. What is the ratio of the rise to the run?

Solution

In a ratio, the measurements must be in the same units. We can change feet to inches, or inches to feet. It is usually easier to convert to the smaller unit, since this avoids introducing more fractions into the problem. Write the words that express the ratio.

Write the ratio as a fraction.	$\frac{rise}{run}$	(5.10.24)
Substitute in the given values.	$\frac{1\ inch}{1\ foot}$	(5.10.25)
Convert 1 foot to inches.	$\frac{1\ inch}{12\ inches}$	(5.10.26)
Simplify, dividing out common factors and units.	$\frac{1}{12}$	(5.10.27)

So the ratio of rise to run is 1 to 12. This means that the ramp should rise 1 inch for every 12 inches of horizontal run to comply with the guidelines.

Exercise 5.10.9:

Find the ratio of the first length to the second length: 32 inches to 1 foot.

Answer

\(\dfrac{8}{3}\

Exercise 5.10.10:

Find the ratio of the first length to the second length: 1 foot to 54 inches.

Answer a

\(\dfrac{2}{9}\

Write a Rate as a Fraction

Frequently we want to compare two different types of measurements, such as miles to gallons. To make this comparison, we use a **rate**. Examples of rates are 120 miles in 2 hours, 160 words in 4 minutes, and $5 dollars per 64 ounces.

Definition: rate

A rate compares two quantities of different units. A rate is usually written as a fraction.

When writing a fraction as a rate, we put the first given amount with its units in the numerator and the second amount with its units in the denominator. When rates are simplified, the units remain in the numerator and denominator.

Example 5.10.6:

Bob drove his car 525 miles in 9 hours. Write this rate as a fraction.

Solution

Write as a fraction, with 525 miles in the numerator and 9 hours in the denominator.	$\frac{525\ miles}{9\ hours}$	(5.10.28)
	$\frac{175\ miles}{3\ hours}$	(5.10.29)

So 525 miles in 9 hours is equivalent to $\frac{175\ miles}{3\ hours}$.

Exercise 5.10.11:

Write the rate as a fraction: 492 miles in 8 hours.

Answer

$\frac{123\ miles}{2\ hours}$

Exercise 5.10.12:

Write the rate as a fraction: 242 miles in 6 hours.

Answer

$\frac{121\ miles}{3\ hours}$

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5.11: Ratios and Rate (Part 2)

Find Unit Rates

In the last example, we calculated that Bob was driving at a rate of $\frac{175\ miles}{3\ hours}$. This tells us that every three hours, Bob will travel 175 miles. This is correct, but not very useful. We usually want the rate to reflect the number of miles in one hour. A rate that has a denominator of 1 unit is referred to as a **unit rate**.

Definition: Unit rate

A unit rate is a rate with denominator of 1 unit.

Unit rates are very common in our lives. For example, when we say that we are driving at a speed of 68 miles per hour we mean that we travel 68 miles in 1 hour. We would write this rate as 68 miles/hour (read 68 miles per hour). The common abbreviation for this is 68 mph. Note that when no number is written before a unit, it is assumed to be 1. So 68 miles/hour really means 68 miles/1 hour.

Two rates we often use when driving can be written in different forms, as shown:

Example	Rate	Write	Abbreviate	Read
68 miles in 1 hour	$\frac{68\ miles}{1\ hour}$ (5.11.1)	68 miles/hour	68 mph	68 miles per hour
36 miles to 1 gallon	$\frac{36\ miles}{1\ gallon}$ (5.11.2)	36 miles/gallon	36 mpg	36 miles per gallon

Another example of unit rate that you may already know about is hourly pay rate. It is usually expressed as the amount of money earned for one hour of work. For example, if you are paid $12.50 for each hour you work, you could write that your hourly (unit) pay rate is $12.50/hour (read $12.50 per hour.)

To convert a rate to a unit rate, we divide the numerator by the denominator. This gives us a denominator of 1.

Example 5.11.7:

Anita was paid $384 last week for working 32 hours. What is Anita's hourly pay rate?

Solution

Start with a rate of dollars to hours. Then divide.	$384 last week for 32 hours	
Write as a rate.	$\frac{\$384}{32\ hours}$	(5.11.3)
Divide the numerator by the denominator.	$\frac{\$12}{1\ hour}$	(5.11.4)
Rewrite as a rate.	$12 / hour	

Anita's hourly pay rate is $12 per hour.

Exercise 5.11.13:

Find the unit rate: $630 for 35 hours.

Answer

$18 / hour

Exercise 5.11.14:

Find the unit rate: $684 for 36 hours.

Answer

$19 / hour

Example 5.11.8:

Sven drives his car 455 miles, using 14 gallons of gasoline. How many miles per gallon does his car get?

Solution

Start with a rate of miles to gallons. Then divide.

Write as a rate.	$\frac{455\ miles}{14\ gallons}$	(5.11.5)
Divide 455 by 14 to get the unit rate.	$\frac{32.5\ miles}{1\ gallon}$	(5.11.6)

Sven's car gets 32.5 miles/gallon, or 32.5 mpg.

Exercise 5.11.15:

Find the unit rate: 423 miles to 18 gallons of gas.

Answer

23.5 mpg

Exercise 5.11.16:

Find the unit rate: 406 miles to 14.5 gallons of gas.

Answer

28 mpg

Find Unit Price

Sometimes we buy common household items 'in bulk', where several items are packaged together and sold for one price. To compare the prices of different sized packages, we need to find the unit price. To find the unit price, divide the total price by the number of items. A **unit price** is a unit rate for one item.

Definition: unit price

A unit price is a unit rate that gives the price of one item.

Example 5.11.9:

The grocery store charges $3.99 for a case of 24 bottles of water. What is the unit price?

Solution

What are we asked to find? We are asked to find the unit price, which is the price per bottle.

Write as a rate.	$\frac{\$3.99}{24\ bottles}$	(5.11.7)
Divide to find the unit price.	$\frac{\$0.16625}{1\ bottle}$	(5.11.8)
Round the result to the nearest penny.	$\frac{\$0.17}{1\ bottle}$	(5.11.9)

The unit price is approximately $0.17 per bottle. Each bottle costs about $0.17.

Exercise 5.11.17:

Find the unit price. Round your answer to the nearest cent if necessary: 24-pack of juice boxes for $6.99

Answer

$\frac{\$0.29}{1\ box}$

Exercise 5.11.18:

Find the unit price. Round your answer to the nearest cent if necessary: 24-pack of bottles of ice tea for $12.72

Answer

$\frac{\$0.53}{1\ bottle}$

Unit prices are very useful if you comparison shop. The *better buy* is the item with the lower unit price. Most grocery stores list the unit price of each item on the shelves.

Example 5.11.10:

Paul is shopping for laundry detergent. At the grocery store, the liquid detergent is priced at $14.99 for 64 loads of laundry and the same brand of powder detergent is priced at $15.99 for 80 loads. Which is the better buy, the liquid or the powder detergent?

Solution

To compare the prices, we first find the unit price for each type of detergent.

	Liquid	Powder

Write as a rate.	$\frac{\$14.99}{64\ loads}$	(5.11.10)	$\frac{\$15.99}{80\ loads}$	(5.11.11)
Find the unit price.	$\frac{\$0.234\ldots}{1\ load}$	(5.11.12)	$\frac{\$0.199\ldots}{1\ load}$	(5.11.13)
Round to the nearest cent.	\$0.23/load (23 cents per load.)		\$0.20/load (20 cents per load)	

Now we compare the unit prices. The unit price of the liquid detergent is about \$0.23 per load and the unit price of the powder detergent is about \$0.20 per load. The powder is the better buy.

? Exercise 5.11.19:

Find each unit price and then determine the better buy. Round to the nearest cent if necessary.

Brand A Storage Bags, \$4.59 for 40 count, or Brand B Storage Bags, \$3.99 for 30 count

Answer

Brand A costs \$0.12 per bag. Brand B costs \$0.13 per bag. Brand A is the better buy

? Exercise 5.11.20:

Find each unit price and then determine the better buy. Round to the nearest cent if necessary.

Brand C Chicken Noodle Soup, \$1.89 for 26 ounces, or Brand D Chicken Noodle Soup, \$0.95 for 10.75 ounces

Answer

Brand C costs \$0.07 per ounce. Brand D costs \$0.09 per ounce. Brand C is the better buy

Notice in Example 5.11.10that we rounded the unit price to the nearest cent. Sometimes we may need to carry the division to one more place to see the difference between the unit prices.

Translate Phrases to Expressions with Fractions

Have you noticed that the examples in this section used the comparison words *ratio of, to, per, in, for, on*, and *from*? When you translate phrases that include these words, you should think either ratio or rate. If the units measure the same quantity (length, time, etc.), you have a ratio. If the units are different, you have a rate. In both cases, you write a fraction.

✓ Example 5.11.11:

Translate the word phrase into an algebraic expression: (a) 427 miles per h hours (b) x students to 3 teachers (c) y dollars for 18 hours

Solution

(a) 427 miles per h hours

Write as a rate.	$\frac{427\ miles}{h\ hours}$	(5.11.14)

(b) x students to 3 teachers

Write as a rate.	$\frac{x\ students}{3\ teachers}$	(5.11.15)

(c) y dollars for 18 hours

Write as a rate.	$\frac{\$y}{18\ hours}$	(5.11.16)

Exercise 5.11.21:

Translate the word phrase into an algebraic expression. (a) 689 miles per h hours (b) y parents to 22 students (c) d dollars for 9 minutes

Answer a

$$\frac{689\ mi}{h\ hours}$$

Answer b

$$\frac{y\ parents}{22\ students}$$

Answer c

$$\frac{\$d}{9\ min}$$

Exercise 5.11.22:

Translate the word phrase into an algebraic expression. (a) m miles per 9 hours (b) x students to 8 buses (c) y dollars for 40 hours

Answer a

$$\frac{m\ mi}{9\ h}$$

Answer b

$$\frac{x\ students}{8\ buses}$$

Answer c

$$\frac{\$y}{40\ h}$$

ACCESS ADDITIONAL ONLINE RESOURCES

Ratios

Write Ratios as a Simplified Fractions Involving Decimals and Fractions

Write a Ratio as a Simplified Fraction

Rates and Unit Rates

Unit Rate for Cell Phone Plan

Practice Makes Perfect

Write a Ratio as a Fraction

In the following exercises, write each ratio as a fraction.

403. 20 to 36
404. 20 to 32
405. 42 to 48
406. 45 to 54
407. 49 to 21
408. 56 to 16
409. 84 to 36
410. 6.4 to 0.8
411. 0.56 to 2.8
412. 1.26 to 4.2
413. $1\frac{2}{3}$ to $2\frac{5}{6}$
414. $1\frac{3}{4}$ to $2\frac{5}{8}$
415. $4\frac{1}{6}$ to $3\frac{1}{3}$
416. $5\frac{3}{5}$ to $3\frac{3}{5}$
417. \$18 to \$63
418. \$16 to \$72
419. \$1.21 to \$0.44
420. \$1.38 to \$0.69
421. 28 ounces to 84 ounces
422. 32 ounces to 128 ounces
423. 12 feet to 46 feet
424. 15 feet to 57 feet
425. 246 milligrams to 45 milligrams
426. 304 milligrams to 48 milligrams
427. total cholesterol of 175 to HDL cholesterol of 45
428. total cholesterol of 215 to HDL cholesterol of 55
429. 27 inches to 1 foot 430. 28 inches to 1 foot

Write a Rate as a Fraction

In the following exercises, write each rate as a fraction.

431. 140 calories per 12 ounces
432. 180 calories per 16 ounces
433. 8.2 pounds per 3 square inches
434. 9.5 pounds per 4 square inches
435. 488 miles in 7 hours
436. 527 miles in 9 hours
437. \$595 for 40 hours
438. \$798 for 40 hours

Find Unit Rates

In the following exercises, find the unit rate. Round to two decimal places, if necessary.

439. 140 calories per 12 ounces
440. 180 calories per 16 ounces
441. 8.2 pounds per 3 square inches
442. 9.5 pounds per 4 square inches

443. 488 miles in 7 hours
444. 527 miles in 9 hours
445. $595 for 40 hours
446. $798 for 40 hours
447. 576 miles on 18 gallons of gas
448. 435 miles on 15 gallons of gas
449. 43 pounds in 16 weeks
450. 57 pounds in 24 weeks
451. 46 beats in 0.5 minute
452. 54 beats in 0.5 minute
453. The bindery at a printing plant assembles 96,000 magazines in 12 hours. How many magazines are assembled in one hour?
454. The pressroom at a printing plant prints 540,000 sections in 12 hours. How many sections are printed per hour?

Find Unit Price

In the following exercises, find the unit price. Round to the nearest cent.

455. Soap bars at 8 for $8.69
456. Soap bars at 4 for $3.39
457. Women's sports socks at 6 pairs for $7.99
458. Men's dress socks at 3 pairs for $8.49
459. Snack packs of cookies at 12 for $5.79
460. Granola bars at 5 for $3.69
461. CD-RW discs at 25 for $14.99
462. CDs at 50 for $4.49
463. The grocery store has a special on macaroni and cheese. The price is $3.87 for 3 boxes. How much does each box cost?
464. The pet store has a special on cat food. The price is $4.32 for 12 cans. How much does each can cost?

In the following exercises, find each unit price and then identify the better buy. Round to three decimal places.

465. Mouthwash, 50.7-ounce size for $6.99 or 33.8-ounce size for $4.79
466. Toothpaste, 6 ounce size for $3.19 or 7.8-ounce size for $5.19
467. Breakfast cereal, 18 ounces for $3.99 or 14 ounces for $3.29
468. Breakfast Cereal, 10.7 ounces for $2.69 or 14.8 ounces for $3.69
469. Ketchup, 40-ounce regular bottle for $2.99 or 64-ounce squeeze bottle for $4.39
470. Mayonnaise, 15-ounce regular bottle for $3.49 or 22-ounce squeeze bottle for $4.99
471. Cheese, $6.49 for 1 lb. block or $3.39 for $\frac{1}{2}$ lb. block
472. Candy, $10.99 for a 1 lb. bag or $2.89 for $\frac{1}{4}$ lb. of loose candy

Translate Phrases to Expressions with Fractions

In the following exercises, translate the English phrase into an algebraic expression.

473. 793 miles per p hours
474. 78 feet per r seconds
475. $3 for 0.5 lbs.
476. j beats in 0.5 minutes
477. 105 calories in x ounces
478. 400 minutes for m dollars
479. the ratio of y and 5x
480. the ratio of 12x and y

Everyday Math

481. One elementary school in Ohio has 684 students and 45 teachers. Write the student-to-teacher ratio as a unit rate.
482. The average American produces about 1,600 pounds of paper trash per year (365 days). How many pounds of paper trash does the average American produce each day? (Round to the nearest tenth of a pound.)

483. A popular fast food burger weighs 7.5 ounces and contains 540 calories, 29 grams of fat, 43 grams of carbohydrates, and 25 grams of protein. Find the unit rate of (a) calories per ounce (b) grams of fat per ounce (c) grams of carbohydrates per ounce (d) grams of protein per ounce. Round to two decimal places.
484. A 16-ounce chocolate mocha coffee with whipped cream contains 470 calories, 18 grams of fat, 63 grams of carbohydrates, and 15 grams of protein. Find the unit rate of (a) calories per ounce (b) grams of fat per ounce (c) grams of carbohydrates per ounce (d) grams of protein per ounce.

Writing Exercises

485. Would you prefer the ratio of your income to your friend's income to be 3/1 or 1/3? Explain your reasoning.
486. The parking lot at the airport charges $0.75 for every 15 minutes. (a) How much does it cost to park for 1 hour? (b) Explain how you got your answer to part (a). Was your reasoning based on the unit cost or did you use another method?
487. Kathryn ate a 4-ounce cup of frozen yogurt and then went for a swim. The frozen yogurt had 115 calories. Swimming burns 422 calories per hour. For how many minutes should Kathryn swim to burn off the calories in the frozen yogurt? Explain your reasoning.
488. Mollie had a 16-ounce cappuccino at her neighborhood coffee shop. The cappuccino had 110 calories. If Mollie walks for one hour, she burns 246 calories. For how many minutes must Mollie walk to burn off the calories in the cappuccino? Explain your reasoning.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
write a ratio as a fraction.			
write a rate as a fraction.			
find unit rates.			
find unit price.			
translate phrases to expressions with fractions.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

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5.12: Simplify and Use Square Roots (Part 1)

Learning Objectives

- Simplify expressions with square roots
- Estimate square roots
- Approximate square roots
- Simplify variable expressions with square roots
- Use square roots in applications

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: $(-9)^2$. If you missed this problem, review Example 3.7.6.
2. Round 3.846 to the nearest hundredth. If you missed this problem, review Example 5.2.9.
3. Evaluate 12d for d = 80. If you missed this problem, review Example 2.3.2.

Simplify Expressions with Square Roots

To start this section, we need to review some important vocabulary and notation. Remember that when a number n is multiplied by itself, we can write this as n^2, which we read aloud as "n squared." For example, 8^2 is read as "8 squared." We call 64 the *square* of 8 because $8^2 = 64$. Similarly, 121 is the square of 11, because $11^2 = 121$.

Definition: square of a number

If $n^2 = m$, then m is the square of n.

Modeling Squares

Do you know why we use the word *square*? If we construct a square with three tiles on each side, the total number of tiles would be nine.

This is why we say that the square of three is nine.

$$3^2 = 9 \tag{5.12.1}$$

The number 9 is called a perfect square because it is the square of a whole number.

The chart shows the squares of the counting numbers 1 through 15. You can refer to it to help you identify the perfect squares.

Number	n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Square	n^2	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

Definition: perfect squares

A perfect square is the square of a whole number.

What happens when you square a negative number?

$$\begin{aligned}(-8)^2 &= (-8)(-8)\\ &= 64\end{aligned}$$

When we multiply two negative numbers, the product is always positive. So, the square of a negative number is always positive. The chart shows the squares of the negative integers from −1 to −15.

Number	n	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15
Square	n^2	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

Did you notice that these squares are the same as the squares of the positive numbers?

Square Roots

Sometimes we will need to look at the relationship between numbers and their squares in reverse. Because $10^2 = 100$, we say 100 is the square of 10. We can also say that 10 is a square root of 100.

Definition: Square Root of a Number

A number whose square is m is called a square root of m. If $n^2 = m$, then n is a square root of m.

Notice $(-10)^2 = 100$ also, so −10 is also a square root of 100. Therefore, both 10 and −10 are square roots of 100. So, every positive number has two square roots: one positive and one negative.

What if we only want the positive square root of a positive number? The *radical sign*, $\sqrt{}$, stands for the positive square root. The positive square root is also called the **principal square root**.

Definition: Square Root Notation

$\sqrt{m}$ is read as "the square root of m." If $m = n^2$, then $m = n$ for $n \geq 0$.

radical sign → $\sqrt{m}$ ← radicand

We can also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root. The chart shows the square roots of the first 15 perfect square numbers.

$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Example 5.12.1:

Simplify: (a) $\sqrt{25}$ (b) $\sqrt{121}$.

Solution

(a) $\sqrt{25}$

Since $5^2 = 25$	5

(b) $\sqrt{121}$

Since $11^2 = 121$	11

Exercise 5.12.1:

Simplify: (a) $\sqrt{36}$ (b) $\sqrt{169}$.

Answer a

6

Answer b

13

Exercise 5.12.2:

Simplify: (a) $\sqrt{16}$ (b) $\sqrt{196}$.

Answer a

4

Answer b

14

Every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{100} = 10$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{100} = -10$.

Example 5.12.2:

Simplify. (a) $-\sqrt{9}$ (b) $-\sqrt{144}$.

Solution

(a) $-\sqrt{9}$

The negative is in front of the radical sign.	-3

(b) $-\sqrt{144}$

The negative is in front of the radical sign.	-12

Exercise 5.12.3:

Simplify: (a) $-\sqrt{4}$ (b) $-\sqrt{225}$.

Answer a

-2

Answer b

-15

Exercise 5.12.4:

Simplify: (a) $-\sqrt{81}$ (b) $-\sqrt{64}$.

Answer a

-9

Answer b

-8

Square Root of a Negative Number

Can we simplify $\sqrt{-25}$? Is there a number whose square is -25?

$$(\ )^2 = -25? \tag{5.12.2}$$

None of the numbers that we have dealt with so far have a square that is −25. Why? Any positive number squared is positive, and any negative number squared is also positive. In the next chapter we will see that all the numbers we work with are called the real numbers. So we say there is no real number equal to $\sqrt{-25}$. If we are asked to find the square root of any negative number, we say that the solution is not a real number.

Example 5.12.3:

Simplify: (a) $\sqrt{-169}$ (b) $-\sqrt{121}$.

Solution

(a) There is no real number whose square is −169. Therefore, $\sqrt{-169}$ is not a real number.

(b) The negative is in front of the radical sign, so we find the opposite of the square root of 121.

The negative is in front of the radical.	-11

Exercise 5.12.5:

Simplify: (a) $\sqrt{-196}$ (b) $-\sqrt{81}$.

Answer a

not a real number

Answer b

-9

Exercise 5.12.6:

Simplify: (a) $-\sqrt{49}$ (b) $\sqrt{-121}$.

Answer a

-7

Answer b

not a real number

Square Roots and the Order of Operations

When using the order of operations to simplify an expression that has square roots, we treat the radical sign as a grouping symbol. We simplify any expressions under the radical sign before performing other operations.

Example 5.12.4:

Simplify: (a) $\sqrt{25}+\sqrt{144}$ (b) $\sqrt{25+144}$.

Solution

(a)

Use the order of operations.	$\sqrt{25}+\sqrt{144}$	(5.12.3)
Simplify each radical.	5 + 12	
Add.	17	

(b)

Use the order of operations.	$\sqrt{25+144}$	(5.12.4)
Add under the radical sign.	$\sqrt{169}$	(5.12.5)
Simplify.	13	

Exercise 5.12.7:

Simplify: (a) $\sqrt{9}+\sqrt{16}$ (b) $\sqrt{9+16}$.

Answer a

7

Answer b

5

Exercise 5.12.8:

Simplify: (a) $\sqrt{64+225}$ (b) $\sqrt{64}+\sqrt{225}$.

Answer a

17

Answer b

23

Notice the different answers in parts (a) and (b) of Example 5.12.4 It is important to follow the order of operations correctly. In (a), we took each square root first and then added them. In (b), we added under the radical sign first and then found the square root.

Estimate Square Roots

So far we have only worked with square roots of perfect squares. The square roots of other numbers are not whole numbers.

Number	Square root
4	$\sqrt{4}=2$
5	$\sqrt{5}$
6	$\sqrt{6}$
7	$\sqrt{7}$
8	$\sqrt{8}$
9	$\sqrt{9}=3$

We might conclude that the square roots of numbers between 4 and 9 will be between 2 and 3, and they will not be whole numbers. Based on the pattern in the table above, we could say that $\sqrt{5}$ is between 2 and 3. Using inequality symbols, we write

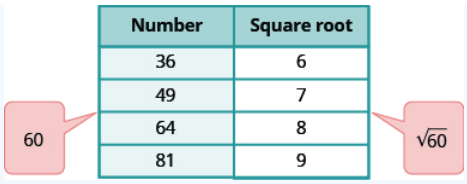
$$2<\sqrt{5}<3 \tag{5.12.6}$$

Example 5.12.5:

Estimate $\sqrt{60}$ between two consecutive whole numbers.

Solution

Think of the perfect squares closest to 60. Make a small table of these perfect squares and their squares roots.



Locate 60 between two consecutive perfect squares.	$49 < 60 < 64$	
$\sqrt{60}$ is between their square roots.	$7 < \sqrt{60} < 8$	$(5.12.7)$

? Exercise 5.12.9:

Estimate $\sqrt{38}$ between two consecutive whole numbers.

Answer

$6 < \sqrt{38} < 7$

? Exercise 5.12.10:

Estimate $\sqrt{84}$ between two consecutive whole numbers.

Answer

$9 < \sqrt{84} < 10$

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5.13: Simplify and Use Square Roots (Part 2)

Approximate Square Roots with a Calculator

There are mathematical methods to approximate square roots, but it is much more convenient to use a calculator to find square roots. Find the $\sqrt{\ }$ or $\sqrt{x}$ key on your calculator. You will to use this key to approximate square roots. When you use your calculator to find the square root of a number that is not a perfect square, the answer that you see is not the exact number. It is an approximation, to the number of digits shown on your calculator's display. The symbol for an approximation is ≈ and it is read *approximately*.

Suppose your calculator has a 10-digit display. Using it to find the square root of 5 will give 2.236067977. This is the approximate square root of 5. When we report the answer, we should use the "approximately equal to" sign instead of an equal sign.

$$\sqrt{5} \approx 2.236067978 \tag{5.13.1}$$

You will seldom use this many digits for applications in algebra. So, if you wanted to round $\sqrt{5}$ to two decimal places, you would write

$$\sqrt{5} \approx 2.24 \tag{5.13.2}$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them.

$$2.236067978^2 = 5.000000002$$
$$2.24^2 = 5.0176$$

The squares are close, but not exactly equal, to 5.

Example 5.13.6:

Round $\sqrt{17}$ to two decimal places using a calculator.

Solution

Use the calculator square root key.	4.123105626	
Round to two decimal places.	4.12	
	$\sqrt{17} \approx 4.12$	(5.13.3)

Exercise 5.13.11:

Round $\sqrt{11}$ to two decimal places.

Answer

≈ 3.32

Exercise 5.13.12:

Round $\sqrt{13}$ to two decimal places

Answer

≈ 3.61

Simplify Variable Expressions with Square Roots

Expressions with square root that we have looked at so far have not had any variables. What happens when we have to find a square root of a variable expression?

Consider $\sqrt{9x^2}$, where x ≥ 0. Can you think of an expression whose square is $9x^2$?

$$(?)^2 = 9x^2$$
$$(3x)^2 = 9x^2$$
$$so\ \sqrt{9x^2} = 3x$$

When we use a variable in a square root expression, for our work, we will assume that the variable represents a nonnegative number. In every example and exercise that follows, each variable in a square root expression is greater than or equal to zero.

Example 5.13.7:

Simplify: x^2.

Solution

Think about what we would have to square to get x^2. Algebraically, $(?)^2 = x^2$

Since $(x)^2 = x^2$	x

Exercise 5.13.13:

Simplify: $\sqrt{y^2}$.

Answer

y

Exercise 5.13.14:

Simplify: $\sqrt{m^2}$.

Answer

m

Example 5.13.8:

Simplify: $\sqrt{16x^2}$.

Solution

Since $(4x)^2 = 16x^2$	4x

Exercise 5.13.15:

Simplify: $\sqrt{64x^2}$.

Answer

$8x$

Exercise 5.13.16:

Simplify: $\sqrt{169y^2}$.

Answer

$13y$

Example 5.13.9:

Simplify: $-\sqrt{81y^2}$.

Solution

Since $(9y)^2 = 81y^2$	-9y

Exercise 5.13.17:

Simplify: $-\sqrt{121y^2}$.

Answer

$-11y$

Exercise 5.13.18:

Simplify: $-\sqrt{100p^2}$.

Answer

$-10p$

Example 5.13.10:

Simplify: $\sqrt{36x^2y^2}$.

Solution

Since $(6xy)^2 = 36x^2 y^2$	6xy

Exercise 5.13.19:

Simplify: $\sqrt{100a^2b^2}$.

Answer

$10ab$

Exercise 5.13.20:

Simplify: $\sqrt{225m^2n^2}$.

Answer

$10mn$

Use Square Roots in Applications

As you progress through your college courses, you'll encounter several applications of square roots. Once again, if we use our strategy for applications, it will give us a plan for finding the answer!

HOW TO: USE A STRATEGY FOR APPLICATIONS WITH SQUARE ROOTS

Step 1. Identify what you are asked to find.

Step 2. Write a phrase that gives the information to find it.

Step 3. Translate the phrase to an expression.

Step 4. Simplify the expression.

Step 5. Write a complete sentence that answers the question.

Square Roots and Area

We have solved applications with area before. If we were given the length of the sides of a square, we could find its area by squaring the length of its sides. Now we can find the length of the sides of a square if we are given the area, by finding the square root of the area.

If the area of the square is A square units, the length of a side is A units. See Table 5.13.1.

Table 5.13.1

Area (square units)	Length of side (units)	
9	$\sqrt{9}=3$	(5.13.4)
144	$\sqrt{144}=12$	(5.13.5)
A	$\sqrt{A}$	(5.13.6)

Example 5.13.11:

Mike and Lychelle want to make a square patio. They have enough concrete for an area of 200 square feet. To the nearest tenth of a foot, how long can a side of their square patio be?

Solution

We know the area of the square is 200 square feet and want to find the length of the side. If the area of the square is A square units, the length of a side is $\sqrt{A}$ units.

What are you asked to find?	The length of each side of a square patio	
Write a phrase.	The length of a side	
Translate to an expression.	$\sqrt{A}$	(5.13.7)
Evaluate $\sqrt{A}$ when A = 200.	$\sqrt{200}$	(5.13.8)
Use your calculator.	14.142135...	
Round to one decimal place.	14.1 feet	
Write a sentence.	Each side of the patio should be 14.1 feet.	

Exercise 5.13.21:

Katie wants to plant a square lawn in her front yard. She has enough sod to cover an area of 370 square feet. To the nearest tenth of a foot, how long can a side of her square lawn be?

Answer

19.2 feet

Exercise 5.13.22:

Sergio wants to make a square mosaic as an inlay for a table he is building. He has enough tile to cover an area of 2704 square centimeters. How long can a side of his mosaic be?

Answer

52 centimeters

Square Roots and Gravity

Another application of square roots involves gravity. On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by evaluating the expression $\frac{\sqrt{h}}{4}$. For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by evaluating $\frac{\sqrt{64}}{4}$.

Take the square root of 64.	$\frac{8}{4}$	(5.13.9)
Simplify the fraction.	2	

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example 5.13.12:

Christy dropped her sunglasses from a bridge 400 feet above a river. How many seconds does it take for the sunglasses to reach the river?

Solution

What are you asked to find?	The number of seconds it takes for the sunglasses to reach the river	
Write a phrase.	The time it will take to reach the river	
Translate to an expression.	$\frac{\sqrt{h}}{4}$	(5.13.10)
Evaluate $\frac{\sqrt{h}}{4}$ when h = 400.	$\frac{\sqrt{400}}{4}$	(5.13.11)
Find the square root of 400.	14.142135...	
Simplify.	14.1 feet	
Write a sentence.	Each side of the patio should be 14.1 feet.	

Exercise 5.13.23:

A helicopter drops a rescue package from a height of 1296 feet. How many seconds does it take for the package to reach the ground?

Answer

9 seconds

Exercise 5.13.24:

A window washer drops a squeegee from a platform 196 feet above the sidewalk. How many seconds does it take for the squeegee to reach the sidewalk?

Answer

3.5 seconds

Square Roots and Accident Investigations

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes. According to some formulas, if the length of the skid marks is d feet, then the speed of the car can be found by evaluating $\sqrt{24d}$.

Example 5.13.13:

After a car accident, the skid marks for one car measured 190 feet. To the nearest tenth, what was the speed of the car (in mph) before the brakes were applied?

Solution

What are you asked to find?	The speed of the car before the brakes were applied	
Write a phrase.	The speed of the car	
Translate to an expression.	$\sqrt{24d}$	(5.13.12)
Evaluate $\sqrt{24d}$ when d = 190.	$\sqrt{24 \cdot 190}$	(5.13.13)
Multiply.	$\sqrt{4,560}$	(5.13.14)
Use your calculator.	67.527772...	
Round to tenths.	67.5	
Write a sentence.	The speed of the car was approximately 67.5 miles per hour.	

Exercise 5.13.25:

An accident investigator measured the skid marks of a car and found their length was 76 feet. To the nearest tenth, what was the speed of the car before the brakes were applied?

Answer

42.7 mph

Exercise 5.13.26:

The skid marks of a vehicle involved in an accident were 122 feet long. To the nearest tenth, how fast had the vehicle been going before the brakes were applied?

Answer

54.1 mph

ACCESS ADDITIONAL ONLINE RESOURCES

Introduction to Square Roots

Estimating Square Roots with a Calculator

Practice Makes Perfect

Simplify Expressions with Square Roots

In the following exercises, simplify.

489. $\sqrt{36}$
490. $\sqrt{4}$
491. $\sqrt{64}$
492. $\sqrt{144}$
493. $-\sqrt{4}$
494. $-\sqrt{100}$
495. $-\sqrt{1}$
496. $-\sqrt{121}$
497. $\sqrt{-121}$
498. $\sqrt{-36}$
499. $\sqrt{-9}$
500. $\sqrt{-49}$
501. $\sqrt{9+16}$
502. $\sqrt{25+144}$
503. $\sqrt{9}+\sqrt{16}$
504. $\sqrt{25}+\sqrt{144}$

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

505. $\sqrt{70}$
506. $\sqrt{5}$
507. $\sqrt{200}$
508. $\sqrt{172}$

Approximate Square Roots with a Calculator

In the following exercises, use a calculator to approximate each square root and round to two decimal places.

509. $\sqrt{19}$
510. $\sqrt{21}$
511. $\sqrt{53}$
512. $\sqrt{47}$

Simplify Variable Expressions with Square Roots

In the following exercises, simplify. (Assume all variables are greater than or equal to zero.)

513. $\sqrt{y^2}$
514. $\sqrt{b^2}$
515. $\sqrt{49x^2}$
516. $\sqrt{100y^2}$
517. $-\sqrt{64a^2}$
518. $-\sqrt{25x^2}$
519. $\sqrt{144x^2y^2}$
520. $\sqrt{196a^2b^2}$

Use Square Roots in Applications

In the following exercises, solve. Round to one decimal place.

521. **Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. How long can a side of his garden be?
522. **Landscaping** Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. How long can a side of his patio be?
523. **Gravity** An airplane dropped a flare from a height of 1,024 feet above a lake. How many seconds did it take for the flare to reach the water?
524. **Gravity** A hang glider dropped his cell phone from a height of 350 feet. How many seconds did it take for the cell phone to reach the ground?
525. **Gravity** A construction worker dropped a hammer while building the Grand Canyon skywalk, 4,000 feet above the Colorado River. How many seconds did it take for the hammer to reach the river?
526. **Accident investigation** The skid marks from a car involved in an accident measured 54 feet. What was the speed of the car before the brakes were applied?
527. **Accident investigation** The skid marks from a car involved in an accident measured 216 feet. What was the speed of the car before the brakes were applied?
528. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. What was the speed of the vehicle before the brakes were applied?
529. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 117 feet. What was the speed of the vehicle before the brakes were applied?

Everyday Math

530. **Decorating** Denise wants to install a square accent of designer tiles in her new shower. She can afford to buy 625 square centimeters of the designer tiles. How long can a side of the accent be?
531. **Decorating** Morris wants to have a square mosaic inlaid in his new patio. His budget allows for 2,025 tiles. Each tile is square with an area of one square inch. How long can a side of the mosaic be?

Writing Exercises

532. Why is there no real number equal to $\sqrt{-64}$?
533. What is the difference between 9^2 and $\sqrt{9}$?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with square roots.			
estimate square roots.			
approximate square roots.			
simplify variable expressions with square roots.			
use square roots in applications.			

(b) Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

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5.14: Commutative and Associative Properties (Part 1)

Learning Objectives

- Use the commutative and associative properties
- Evaluate expressions using the commutative and associative properties
- Simplify expressions using the commutative and associative properties

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: 7y + 2 + y + 13. If you missed this problem, review Example 2.3.10.
2. Multiply: $\frac{2}{3} \cdot 18$. If you missed this problem, review Example 4.3.10.
3. Find the opposite of 15. If you missed this problem, review Example 3.1.3.

In the next few sections, we will take a look at the properties of real numbers. Many of these properties will describe things you already know, but it will help to give names to the properties and define them formally. This way we'll be able to refer to them and use them as we solve equations in the next chapter.

Use the Commutative and Associative Properties

Think about adding two numbers, such as 5 and 3.

$$\begin{array}{cc} 5+3 & 3+5 \\ 8 & 8 \end{array}$$

The results are the same. 5 + 3 = 3 + 5

Notice, the order in which we add does not matter. The same is true when multiplying 5 and 3.

$$\begin{array}{cc} 5 \cdot 3 & 3 \cdot 5 \\ 15 & 15 \end{array}$$

Again, the results are the same! 5 • 3 = 3 • 5. The order in which we multiply does not matter. These examples illustrate the commutative properties of addition and multiplication.

Definition: Commutative Properties

Commutative Property of Addition: if a and b are real numbers, then a + b = b + a

Commutative Property of Multiplication: if a and b are real numbers, then a • b = b • a

The commutative properties have to do with order. If you change the order of the numbers when adding or multiplying, the result is the same.

Example 5.14.1:

Use the commutative properties to rewrite the following expressions: (a) −1 + 3 = _____ (b) 4 • 9 = _____

Solution

(a) −1 + 3 = _____

Use the commutative property of addition to change the order.	−1 + 3 = 3 + (−1)

(b) 4 • 9 = _____

Use the commutative property of multiplication to change the order.	4 • 9 = 9 • 4

Exercise 5.14.1:

Use the commutative properties to rewrite the following expressions: (a) −4 + 7 = _____ (b) 6 • 12 = _____

Answer a

$$-4+7=7+(-4)$$

Answer b

$$6 \cdot 12 = 12 \cdot 6$$

Exercise 5.14.2:

Use the commutative properties to rewrite the following expressions: (a) 14 + (-2) = _____ (b) 3(-5) = _____

Answer a

$$14+(-2)=-2+14$$

Answer b

$$3(-5)=(-5)3$$

What about subtraction? Does order matter when we subtract numbers? Does 7 − 3 give the same result as 3 − 7?

$$\begin{array}{cc} 7-3 & 3-7 \\ 4 & -4 \\ & 4 \neq -4 \end{array}$$

The results are not the same. 7 − 3 ≠ 3 − 7

Since changing the order of the subtraction did not give the same result, we can say that subtraction is not commutative. Let's see what happens when we divide two numbers. Is division commutative?

$$\begin{array}{cc} 12 \div 4 & 4 \div 12 \\ \frac{12}{4} & \frac{4}{12} \\ 3 & \frac{1}{3} \\ & 3 \neq \frac{1}{3} \end{array}$$

The results are not the same. So 12 ÷ 4 ≠ 4 ÷ 12

Since changing the order of the division did not give the same result, division is not commutative.

Addition and multiplication are commutative. Subtraction and division are not commutative.

Suppose you were asked to simplify this expression.

$$7+8+2 \tag{5.14.1}$$

How would you do it and what would your answer be?

Some people would think 7 + 8 is 15 and then 15 + 2 is 17. Others might start with 8 + 2 makes 10 and then 7 + 10 makes 17.

Both ways give the same result, as shown in Figure 5.14.1. (Remember that parentheses are grouping symbols that indicate which operations should be done first.)

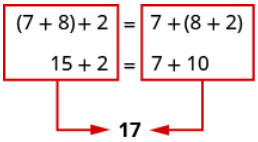

Figure 5.14.1

When adding three numbers, changing the grouping of the numbers does not change the result. This is known as the Associative Property of Addition.

The same principle holds true for multiplication as well. Suppose we want to find the value of the following expression:

$$5 \cdot \frac{1}{3} \cdot 3 \tag{5.14.2}$$

Changing the grouping of the numbers gives the same result, as shown in Figure 5.14.2

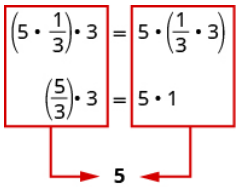

Figure 5.14.2

When multiplying three numbers, changing the grouping of the numbers does not change the result. This is known as the Associative Property of Multiplication.

If we multiply three numbers, changing the grouping does not affect the product. You probably know this, but the terminology may be new to you. These examples illustrate the *Associative Properties*.

Definition: Associative Properties

Associative Property of Addition: if a, b, and c are real numbers, then (a + b) + c = a + (b + c)

Associative Property of Multiplication: if a, b, and c are real numbers, then (a • b) • c = a • (b • c)

Example 5.14.2:

Use the associative properties to rewrite the following: (a) (3 + 0.6) + 0.4 = _____ (b) $\left(-4 \cdot \frac{2}{5}\right) \cdot 15$ = _____

Solution

(a) (3 + 0.6) + 0.4 = _____

Change the grouping.	(3 + 0.6) + 0.4 = 3 + (0.6 + 0.4)

Notice that 0.6 + 0.4 is 1, so the addition will be easier if we group as shown on the right.

(b) $\left(-4 \cdot \frac{2}{5}\right) \cdot 15$ = _____

Change the grouping.	(3 + 0.6) + 0.4 = 3 + (0.6 + 0.4)

Notice that $\frac{2}{5} \cdot 15$ is 6. The multiplication will be easier if we group as shown on the right.

Exercise 5.14.3:

Use the associative properties to rewrite the following: (a) (1 + 0.7) + 0.3 = _____ (b) (−9 • 8) • $\frac{3}{4}$ = _____

Answer a

$(1+0.7)+0.3=1+(0.7+0.3)$

Answer b

$(-9\cdot 8)\cdot \frac{3}{4}=-9\left(8\cdot \frac{3}{4}\right)$

Exercise 5.14.4:

Use the associative properties to rewrite the following: (a) (4 + 0.6) + 0.4 = _____ (b) (−2 • 12) • $\frac{5}{6}$ = _____

Answer a

$(4+0.6)+0.4=4+(0.6+0.4)$

Answer b

$(-2\cdot 12)\cdot \frac{5}{6}=-2\left(12\cdot \frac{5}{6}\right)$

Besides using the associative properties to make calculations easier, we will often use it to simplify expressions with variables.

Example 5.14.3:

Use the Associative Property of Multiplication to simplify: 6(3x).

Solution

Change the grouping.	(6 • 3)x
Multiply in the parentheses.	18x

Notice that we can multiply 6 • 3, but we could not multiply 3 • x without having a value for x.

Exercise 5.14.5:

Use the Associative Property of Multiplication to simplify the given expression: 8(4x).

Answer

$32x$

Exercise 5.14.6:

Use the Associative Property of Multiplication to simplify the given expression: −9(7y).

Answer

$-63y$

Evaluate Expressions using the Commutative and Associative Properties

The commutative and associative properties can make it easier to evaluate some algebraic expressions. Since order does not matter when adding or multiplying three or more terms, we can rearrange and re-group terms to make our work easier, as the next several

examples illustrate.

Example 5.14.4:

Evaluate each expression when $x = \frac{7}{8}$. (a) x + 0.37 + (− x) (b) x + (− x) + 0.37

Solution

(a) x + 0.37 + (− x)

Substitute $\frac{7}{8}$ for x.	$\frac{7}{8} + 0.37 + \left(-\frac{7}{8}\right)$	(5.14.3)
Convert fractions to decimals.	0.875 + 0.37 + (-0.875)	
Add left to right.	1.245 - 0.875	
Subtract.	0.37	

(b) x + (− x) + 0.37

Substitute $\frac{7}{8}$ for x.	$\frac{7}{8} + \left(-\frac{7}{8}\right) + 0.37$	(5.14.4)
Add opposites first.	0.37	

What was the difference between part (a) and part (b)? Only the order changed. By the Commutative Property of Addition, x + 0.37 + (− x) = x + (− x) + 0.37. But wasn't part (b) much easier?

Exercise 5.14.7:

Evaluate each expression when $y = \frac{3}{8}$: (a) y + 0.84 + (− y) (b) y + (− y) + 0.84.

Answer a

0.84

Answer b

0.84

Exercise 5.14.8:

Evaluate each expression when $f = \frac{17}{20}$: (a) f + 0.975 + (− f) (b) f + (− f) + 0.975.

Answer a

0.975

Answer b

0.975

Let's do one more, this time with multiplication.

Example 5.14.5:

Evaluate each expression when n = 17. (a) $\frac{4}{3}\left(\frac{3}{4}n\right)$ (b) $\left(\frac{4}{3}\cdot\frac{3}{4}\right)n$

Solution

(a) $\frac{4}{3}\left(\frac{3}{4}n\right)$

Substitute 17 for n.	$\frac{4}{3}\left(\frac{3}{4}\cdot 17\right)$	(5.14.5)
Multiply in the parentheses first.	$\frac{4}{3}\left(\frac{51}{4}\right)$	(5.14.6)
Multiply again.	17	(5.14.7)

(b) $\left(\frac{4}{3}\cdot\frac{3}{4}\right)n$

Substitute 17 for n.	$\left(\frac{4}{3}\cdot\frac{3}{4}\right)\cdot 17$	(5.14.8)
Multiply. The product of reciprocals is 1.	$(1)\cdot 17$	(5.14.9)
Multiply again.	17	(5.14.10)

What was the difference between part (a) and part (b) here? Only the grouping changed. By the Associative Property of Multiplication, $\frac{4}{3}\left(\frac{3}{4}n\right)=\left(\frac{4}{3}\cdot\frac{3}{4}\right)n$. By carefully choosing how to group the factors, we can make the work easier.

Exercise 5.14.9:

Evaluate each expression when p = 24. (a) $\frac{5}{9}\left(\frac{9}{5}p\right)$ (b) $\left(\frac{5}{9}\cdot\frac{9}{5}\right)p$

Answer a

24

Answer b

24

Exercise 5.14.10:

Evaluate each expression when q = 15. (a) $\frac{7}{11}\left(\frac{11}{7}q\right)$ (b) $\left(\frac{7}{11}\cdot\frac{11}{7}\right)q$

Answer a

15

Answer b

15

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5.15: Commutative and Associative Properties (Part 2)

Simplify Expressions Using the Commutative and Associative Properties

When we have to simplify algebraic expressions, we can often make the work easier by applying the Commutative or Associative Property first instead of automatically following the order of operations. Notice that in Example 7.2.4 part (b) was easier to simplify than part (a) because the opposites were next to each other and their sum is 0. Likewise, part (b) in Example 7.2.5 was easier, with the reciprocals grouped together, because their product is 1. In the next few examples, we'll use our number sense to look for ways to apply these properties to make our work easier.

Example 5.15.6:

Simplify: −84n + (−73n) + 84n.

Solution

Notice the first and third terms are opposites, so we can use the commutative property of addition to reorder the terms.

Re-order the terms.	−84n + 84n + (−73n)
Add left to right.	0 + (−73n)
Add.	−73n

Exercise 5.15.11:

Simplify: −27a + (−48a) + 27a.

Answer

$-48a$

Exercise 5.15.12:

Simplify: 39x + (−92x) + (−39x).

Answer

$-92x$

Now we will see how recognizing reciprocals is helpful. Before multiplying left to right, look for reciprocals—their product is 1.

Example 5.15.7:

Simplify: $\frac{7}{15} \cdot \frac{8}{23} \cdot \frac{15}{7}$.

Solution

Notice the first and third terms are reciprocals, so we can use the Commutative Property of Multiplication to reorder the factors.

Re-order the terms.	$\frac{7}{15} \cdot \frac{15}{7} \cdot \frac{8}{23}$	(5.15.1)
Multiply left to right.	$1 \cdot \frac{8}{23}$	(5.15.2)

Multiply.	$\frac{8}{23}$	(5.15.3)

Exercise 5.15.13:

Simplify: $\frac{9}{16} \cdot \frac{5}{49} \cdot \frac{16}{9}$.

Answer

$\frac{5}{49}$

Exercise 5.15.14:

Simplify: $\frac{6}{17} \cdot \frac{11}{25} \cdot \frac{17}{6}$.

Answer

$\frac{11}{25}$

In expressions where we need to add or subtract three or more fractions, combine those with a common denominator first.

Example 5.15.8:

Simplify: $\left(\frac{5}{13} + \frac{3}{4}\right) + \frac{1}{4}$.

Solution

Notice that the second and third terms have a common denominator, so this work will be easier if we change the grouping.

Group the terms with a common denominator.	$\frac{5}{13} + \left(\frac{3}{4} + \frac{1}{4}\right)$	(5.15.4)
Add in the parentheses first.	$\frac{5}{13} + \left(\frac{4}{4}\right)$	(5.15.5)
Simplify the fraction.	$\frac{5}{13} + 1$	(5.15.6)
Add.	$1\frac{5}{13}$	(5.15.7)
Convert to an improper fraction.	$\frac{18}{13}$	(5.15.8)

Exercise 5.15.15:

Simplify: $\left(\frac{7}{15} + \frac{5}{8}\right) + \frac{3}{8}$.

Answer

$\frac{22}{15}$

Exercise 5.15.16:

Simplify: $\left(\frac{2}{9}+\frac{7}{12}\right)+\frac{5}{12}$.

Answer

$\frac{11}{9}$

When adding and subtracting three or more terms involving decimals, look for terms that combine to give whole numbers.

Example 5.15.9:

Simplify: (6.47q + 9.99q) + 1.01q.

Solution

Notice that the sum of the second and third coefficients is a whole number.

Change the grouping.	6.47q + (9.99q + 1.01q)
Add in the parentheses first.	6.47q + (11.00q)
Add.	17.47q

Many people have good number sense when they deal with money. Think about adding 99 cents and 1 cent. Do you see how this applies to adding 9.99 + 1.01?

Exercise 5.15.17:

Simplify: (5.58c + 8.75c) + 1.25c.

Answer

$15.58c$

Exercise 5.15.18:

Simplify: (8.79d + 3.55d) + 5.45d.

Answer

$17.79d$

No matter what you are doing, it is always a good idea to think ahead. When simplifying an expression, think about what your steps will be. The next example will show you how using the Associative Property of Multiplication can make your work easier if you plan ahead.

Example 5.15.10:

Simplify the expression: [1.67(8)](0.25).

Solution

Notice that multiplying (8)(0.25) is easier than multiplying 1.67(8) because it gives a whole number. (Think about having 8 quarters—that makes $2.)

Regroup.	1.67[(8)(0.25)]
Multiply in the brackets first.	1.67[2]
Multiply.	3.34

? Exercise 5.15.19:

Simplify: [1.17(4)](2.25).

Answer

10.53

? Exercise 5.15.20:

Simplify: [3.52(8)](2.5).

Answer

70.4

When simplifying expressions that contain variables, we can use the commutative and associative properties to re-order or regroup terms, as shown in the next pair of examples.

✓ Example 5.15.11:

Simplify: 6(9x).

Solution

Use the associative property of multiplication to re-group.	(6 • 9)x
Multiply in the parentheses.	54x

? Exercise 5.15.21:

Simplify: 8(3y).

Answer

$24y$

? Exercise 5.15.22:

Simplify: 12(5z).

Answer

$60z$

In The Language of Algebra, we learned to combine like terms by rearranging an expression so the like terms were together. We simplified the expression 3x + 7 + 4x + 5 by rewriting it as 3x + 4x + 7 + 5 and then simplified it to 7x + 12. We were using the Commutative Property of Addition.

Example 5.15.12:

Simplify: 18p + 6q + (−15p) + 5q.

Solution

Use the Commutative Property of Addition to re-order so that like terms are together.

Re-order terms.	18p + (−15p) + 6q + 5q
Combine like terms.	3p + 11q

Exercise 5.15.23:

Simplify: 23r + 14s + 9r + (−15s).

Answer

$32r - s$

Exercise 5.15.24:

Simplify: 37m + 21n + 4m + (−15n).

Answer

$41m + 6n$

Practice Makes Perfect

Use the Commutative and Associative Properties

In the following exercises, use the commutative properties to rewrite the given expression.

20. 8 + 9 = ___
21. 7 + 6 = ___
22. 8(−12) = ___
23. 7(−13) = ___
24. (−19)(−14) = ___
25. (−12)(−18) = ___
26. −11 + 8 = ___
27. −15 + 7 = ___
28. x + 4 = ___
29. y + 1 = ___
30. −2a = ___
31. −3m = ___

In the following exercises, use the associative properties to rewrite the given expression.

32. (11 + 9) + 14 = ___
33. (21 + 14) + 9 = ___
34. (12 · 5) • 7 = ___
35. (14 · 6) • 9 = ___
36. (−7 + 9) + 8 = ___
37. (−2 + 6) + 7 = ___
38. \(\left(16 \cdot \dfrac{4}{5}\right) • 15 = ___
39. \(\left(13 \cdot \dfrac{2}{3}\right) • 18 = ___
40. 3(4x) = ___

41. 4(7x) = ___
42. (12 + x) + 28 = ___
43. (17 + y) + 33 = ___

Evaluate Expressions using the Commutative and Associative Properties

In the following exercises, evaluate each expression for the given value.

44. If y = $\frac{5}{8}$, evaluate:
    a. y + 0.49 + (− y)
    b. y + (− y) + 0.49
45. If z = $\frac{7}{8}$, evaluate:
    a. z + 0.97 + (− z)
    b. z + (− z) + 0.97
46. If c = $-\frac{11}{4}$, evaluate:
    a. c + 3.125 + (− c)
    b. c + (− c) + 3.125
47. If d = $-\frac{9}{4}$, evaluate:
    a. d + 2.375 + (− d)
    b. d + (− d) + 2.375
48. If j = 11, evaluate:
    a. $\frac{5}{6}\left(\frac{6}{5}j\right)$
    b. $\left(\frac{5}{6}\cdot\frac{6}{5}\right)j$
49. If k = 21, evaluate:
    a. $\frac{4}{13}\left(\frac{13}{4}k\right)$
    b. $\left(\frac{4}{13}\cdot\frac{13}{4}\right)k$
50. If m = −25, evaluate:
    a. $-\frac{3}{7}\left(\frac{7}{3}m\right)$
    b. $\left(-\frac{3}{7}\cdot\frac{7}{3}\right)m$
51. If n = −8, evaluate:
    a. $-\frac{5}{21}\left(\frac{21}{5}n\right)$
    b. $\left(-\frac{5}{21}\cdot\frac{21}{5}\right)n$

Simplify Expressions Using the Commutative and Associative Properties

In the following exercises, simplify.

52. −45a + 15 + 45a
53. 9y + 23 + (−9y)
54. $\frac{1}{2}+\frac{7}{8}+\left(-\frac{1}{2}\right)$
55. $\frac{2}{5}+\frac{5}{12}+\left(-\frac{2}{5}\right)$

56. $\frac{3}{20} \cdot \frac{49}{11} \cdot \frac{20}{3}$
57. $\frac{13}{18} \cdot \frac{25}{7} \cdot \frac{18}{13}$
58. $\frac{7}{12} \cdot \frac{9}{17} \cdot \frac{24}{7}$
59. $\frac{3}{10} \cdot \frac{13}{23} \cdot \frac{50}{3}$
60. $-24 \cdot 7 \cdot \frac{3}{8}$
61. $-36 \cdot 11 \cdot \frac{4}{9}$
62. $\left(\frac{5}{6} + \frac{8}{15}\right) + \frac{7}{15}$
63. $\left(\frac{1}{12} + \frac{4}{9}\right) + \frac{5}{9}$
64. $\frac{5}{13} + \frac{3}{4} + \frac{1}{4}$
65. $\frac{8}{15} + \frac{5}{7} + \frac{2}{7}$
66. (4.33p + 1.09p) + 3.91p
67. (5.89d + 2.75d) + 1.25d
68. 17(0.25)(4)
69. 36(0.2)(5)
70. [2.48(12)](0.5)
71. [9.731(4)](0.75)
72. 7(4a)
73. 9(8w)
74. −15(5m)
75. −23(2n)
76. $12\left(\frac{5}{6}p\right)$
77. $20\left(\frac{3}{5}q\right)$
78. 14x + 19y + 25x + 3y
79. 15u + 11v + 27u + 19v
80. 43m + (−12n) + (−16m) + (−9n)
81. −22p + 17q + (−35p) + (−27q)
82. $\frac{3}{8}g + \frac{1}{12}h + \frac{7}{8}g + \frac{5}{12}h$
83. $\frac{5}{6}a + \frac{3}{10}b + \frac{1}{6}a + \frac{9}{10}b$
84. 6.8p + 9.14q + (−4.37p) + (−0.88q)
85. 9.6m + 7.22n + (−2.19m) + (−0.65n)

Everyday Math

86. Stamps Allie and Loren need to buy stamps. Allie needs four \$0.49 stamps and nine \$0.02 stamps. Loren needs eight \$0.49 stamps and three \$0.02 stamps.
    a. How much will Allie's stamps cost?
    b. How much will Loren's stamps cost?
    c. What is the total cost of the girls' stamps?
    d. How many \$0.49 stamps do the girls need altogether? How much will they cost?
    e. How many \$0.02 stamps do the girls need altogether? How much will they cost?
87. Counting Cash Grant is totaling up the cash from a fundraising dinner. In one envelope, he has twenty-three \$5 bills, eighteen \$10 bills, and thirty-four \$20 bills. In another envelope, he has fourteen \$5 bills, nine \$10 bills, and twenty-seven \$20 bills.

a. How much money is in the first envelope?
b. How much money is in the second envelope?
c. What is the total value of all the cash?
d. What is the value of all the $5 bills?
e. What is the value of all $10 bills?
f. What is the value of all $20 bills?

Writing Exercises

88. In your own words, state the Commutative Property of Addition and explain why it is useful.
89. In your own words, state the Associative Property of Multiplication and explain why it is useful.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the Commutative and Associative Properties.			
evaluate expressions using the Commutative and Associative Properties.			
simplify expressions using the Commutative and Associative Properties.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

Contributors and Attributions

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5.16: Distributive Property

Learning Objectives

- Simplify expressions using the distributive property
- Evaluate expressions using the distributive property

be prepared!

Before you get started, take this readiness quiz.

1. Multiply: 3(0.25). If you missed this problem, review Example 5.3.5
2. Simplify: 10 − (−2)(3). If you missed this problem, review Example 3.7.5.
3. Combine like terms: 9y + 17 + 3y − 2. If you missed this problem, review Example 2.3.10.

Simplify Expressions Using the Distributive Property

Suppose three friends are going to the movies. They each need $9.25; that is, 9 dollars and 1 quarter. How much money do they need all together? You can think about the dollars separately from the quarters.

$3 \times 9 = 27$

$3 \times \$0.25 = \$0.75$

They need 3 times $9, so $27, and 3 times 1 quarter, so 75 cents. In total, they need $27.75. If you think about doing the math in this way, you are using the Distributive Property.

Definition: Distributive Property

If a, b, c are real numbers, then a(b + c) = ab + ac.

Back to our friends at the movies, we could show the math steps we take to find the total amount of money they need like this:

$$\begin{aligned} &3(9.25) \\ &3(9+0.25) \\ &3(9)+3(0.25) \\ &27+0.75 \\ &27.75 \end{aligned}$$

In algebra, we use the Distributive Property to remove parentheses as we simplify expressions. For example, if we are asked to simplify the expression 3(x + 4), the order of operations says to work in the parentheses first. But we cannot add x and 4, since they are not like terms. So we use the Distributive Property, as shown in Example 5.16.1.

Example 5.16.1:

Simplify: 3(x + 4).

Solution

Distribute.	3 • x + 3 • 4
Multiply.	3x + 12

Exercise 5.16.1:

Simplify: 4(x + 2).

Answer

$4x + 8$

Exercise 5.16.2:

Simplify: 6(x + 7).

Answer

6x + 42

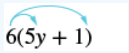
Some students find it helpful to draw in arrows to remind them how to use the Distributive Property. Then the first step in Example 7.17 would look like this:

$$3 \cdot x + 3 \cdot 4 \tag{5.16.1}$$

Example 5.16.2:

Simplify: 6(5y + 1).

Solution



Distribute.	$6 \cdot 5y + 6 \cdot 1$	(5.16.2)
Multiply.	$30y + 6$	(5.16.3)

Exercise 5.16.3:

Simplify: 9(3y + 8).

Answer

27y + 72

Exercise 5.16.4:

Simplify: 5(5w + 9).

Answer

25w + 45

The distributive property can be used to simplify expressions that look slightly different from a(b + c). Here are two other forms.

Definition: Distributive Property

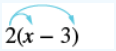
If a, b, c are real numbers, then

$$a(b+c)=ab+ac\$\$Other forms\$\$a(b-c)=ab-ac\$\$\$\$(b+c)a=ba+ca \tag{5.16.4}$$

Example 5.16.3:

Simplify: 2(x − 3).

Solution



Distribute.	$2\cdot x+2\cdot 3$	(5.16.5)
Multiply.	$2x-6$	(5.16.6)

Exercise 5.16.5:

Simplify: 7(x − 6).

Answer

7x - 42

Exercise 5.16.6:

Simplify: 8(x − 5).

Answer

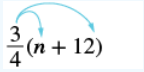
8x - 40

Do you remember how to multiply a fraction by a whole number? We'll need to do that in the next two examples.

Example 5.16.4:

Simplify: $\frac{3}{4}$(n + 12).

Solution



Distribute.	$\frac{3}{4} \cdot n + \frac{3}{4} \cdot 12$	(5.16.7)
Multiply.	$\frac{3}{4}n + 9$	(5.16.8)

Exercise 5.16.7:

Simplify: $\frac{2}{5}$(p + 10).

Answer

$\frac{2}{5}p + 4$

Exercise 5.16.8:

Simplify: $\frac{3}{7}$(u + 21).

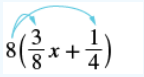
Answer

$\frac{3}{7}u + 9$

Example 5.16.5:

Simplify: $8\left(\frac{3}{8}x + \frac{1}{4}\right)$.

Solution



Distribute.	$8 \cdot \frac{3}{8}x + 8 \cdot \frac{1}{4}$	(5.16.9)
Multiply.	$3x + 2$	(5.16.10)

Exercise 5.16.9:

Simplify: $6\left(\frac{5}{6}y + \frac{1}{2}\right)$.

Answer

5y + 3

Exercise 5.16.10:

Simplify: $12\left(\frac{1}{3}n + \frac{3}{4}\right)$.

Answer

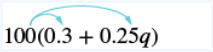
4n + 9

Using the Distributive Property as shown in the next example will be very useful when we solve money applications later.

Example 5.16.6:

Simplify: 100(0.3 + 0.25q).

Solution



Distribute.	$100(0.3)+100(0.25q)$	(5.16.11)
Multiply.	$30+25q$	(5.16.12)

Exercise 5.16.11:

Simplify: 100(0.7 + 0.15p).

Answer

70 + 15p

Exercise 5.16.12:

Simplify: 100(0.04 + 0.35d).

Answer

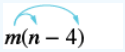
4 + 35d

In the next example we'll multiply by a variable. We'll need to do this in a later chapter.

Example 5.16.7:

Simplify: $m(n-4)$.

Solution



Distribute.	$m\cdot n-m\cdot 4$	(5.16.13)
Multiply.	$mn-4m$	(5.16.14)

Notice that we wrote m • 4 as 4m. We can do this because of the Commutative Property of Multiplication. When a term is the product of a number and a variable, we write the number first.

Exercise 5.16.13:

Simplify: r(s − 2).

Answer

rs - 2r

Exercise 5.16.14:

Simplify: y(z − 8).

Answer

yz - 8y

The next example will use the 'backwards' form of the Distributive Property, (b + c)a = ba + ca.

Example 5.16.8:

Simplify: (x + 8)p.

Solution

$$(x+8)p$$

Distribute.	$px+8p$	(5.16.15)

Exercise 5.16.15:

Simplify: (x + 2)p.

Answer

xp + 2p

Exercise 5.16.16:

Simplify: (y + 4)q.

Answer

yq + 4q

When you distribute a negative number, you need to be extra careful to get the signs correct.

Example 5.16.9:

Simplify: −2(4y + 1).

Solution

$$-2(4y+1)$$

Distribute.	$-2\cdot 4y+(-2)\cdot 1$	(5.16.16)

Simplify.	$-8y-2$	(5.16.17)

Exercise 5.16.17:

Simplify: −3(6m + 5).

Answer

-18m - 15

Exercise 5.16.18:

Simplify: −6(8n + 11).

Answer

-48n - 66

Example 5.16.10:

Simplify: −11(4 − 3a).

Solution

Distribute.	$-11\cdot 4-(-11)\cdot 3a$	(5.16.18)
Multiply.	$-44-(-33a)$	(5.16.19)
Simplify.	$-44+33a$	(5.16.20)

You could also write the result as 33a − 44. Do you know why?

Exercise 5.16.19:

Simplify: −5(2 − 3a).

Answer

-10 + 15a

Exercise 5.16.20:

Simplify: −7(8 − 15y).

Answer

-56 + 105y

In the next example, we will show how to use the Distributive Property to find the opposite of an expression. Remember, −a = −1 • a.

Example 5.16.11:

Simplify: −(y + 5).

Solution

Multiplying by −1 results in the opposite.	$-1(y+5)$	(5.16.21)
Distribute.	$-1\cdot y+(-1)\cdot 5$	(5.16.22)
Simplify.	$-y+(-5)$	(5.16.23)
Simplify.	$-y-5$	(5.16.24)

Exercise 5.16.21:

Simplify: −(z − 11).

Answer

-z + 11

Exercise 5.16.22:

Simplify: −(x − 4).

Answer

-x + 4

Sometimes we need to use the Distributive Property as part of the order of operations. Start by looking at the parentheses. If the expression inside the parentheses cannot be simplified, the next step would be multiply using the distributive property, which removes the parentheses. The next two examples will illustrate this.

Example 5.16.12:

Simplify: 8 − 2(x + 3).

Solution

Distribute.	$8-2\cdot x-2\cdot 3$	(5.16.25)
Multiply.	$8-2x-6$	(5.16.26)
Combine like terms.	$-2x+2$	(5.16.27)

Exercise 5.16.23:

Simplify: 9 − 3(x + 2).

Answer

-3x + 3

Exercise 5.16.24:

Simplify: 7x − 5(x + 4).

Answer

2x - 20

Example 5.16.13:

Simplify: 4(x − 8) − (x + 3).

Solution

Distribute.	$4x - 32 - x - 3$	(5.16.28)
Combine like terms.	$3x - 35$	(5.16.29)

Exercise 5.16.25:

Simplify: 6(x − 9) − (x + 12).

Answer

5x - 66

Exercise 5.16.26:

Simplify: 8(x − 1) − (x + 5).

Answer

7x - 13

Evaluate Expressions Using the Distributive Property

Some students need to be convinced that the Distributive Property always works. In the examples below, we will practice evaluating some of the expressions from previous examples; in part (a), we will evaluate the form with parentheses, and in part (b) we will evaluate the form we got after distributing. If we evaluate both expressions correctly, this will show that they are indeed equal.

Example 5.16.14:

When y = 10 evaluate: (a) 6(5y + 1) (b) 6 • 5y + 6 • 1.

Solution

(a) 6(5y + 1)

Substitute 10 for y.	$6(5 \cdot 10 + 1)$	(5.16.30)
Simplify in the parentheses.	$6(51)$	(5.16.31)
Multiply.	306	(5.16.32)

(b) 6 • 5y + 6 • 1

Substitute 10 for y.	$6 \cdot 5 \cdot 10 + 6 \cdot 1$	(5.16.33)
Simplify.	$300 + 6$	(5.16.34)
Add.	306	(5.16.35)

Notice, the answers are the same. When y = 10, 6(5y + 1) = 6 • 5y + 6 • 1. Try it yourself for a different value of y.

? Exercise 5.16.27:

Evaluate when w = 3: (a) 5(5w + 9) (b) 5 • 5w + 5 • 9.

Answer a

120

Answer b

120

? Exercise 5.16.28:

Evaluate when y = 2: (a) 9(3y + 8) (b) 9 • 3y + 9 • 8.

Answer a

126

Answer b

126

✓ Example 5.16.15:

When y = 3, evaluate (a) −2(4y + 1) (b) −2 • 4y + (−2) • 1.

Solution

(a) −2(4y + 1)

Substitute 3 for y.	$-2(4 \cdot 3 + 1)$	(5.16.36)
Simplify in the parentheses.	$-2(13)$	(5.16.37)
Multiply.	-26	(5.16.38)

(b) −2 • 4y + (−2) • 1

Substitute 3 for y.	$-2 \cdot 4 \cdot 3 + (-2) \cdot 1$	(5.16.39)
Multiply.	$-24 - 2$	(5.16.40)

Subtract.	-26	(5.16.41)
The answers are the same when y = 3.	$-2(4y+1)=-8y-2$	(5.16.42)

? Exercise 5.16.29:

Evaluate when n = −2: (a) −6(8n + 11) (b) −6 • 8n + (−6) • 11.

Answer a

30

Answer b

30

? Exercise 5.16.30:

Evaluate when m = −1: (a) −3(6m + 5) (b) −3 • 6m + (−3) • 5.

Answer a

3

Answer b

3

✓ Example 5.16.16:

When y = 35 evaluate (a) −(y + 5) and (b) −y − 5 to show that −(y + 5) = −y − 5.

Solution

(a) −(y + 5)

Substitute 35 for y.	$-(35+5)$	(5.16.43)
Add in the parentheses.	$-(40)$	(5.16.44)
Simplify.	-40	(5.16.45)

(b) −y − 5

Substitute 35 for y.	$-35-5$	(5.16.46)
Simplify.	-40	(5.16.47)
The answers are the same when y = 35, demonstrating that	$-(y+5)=-y-5$	(5.16.48)

Exercise 5.16.31:

Evaluate when x = 36: (a) −(x − 4) (b) −x + 4 to show that −(x − 4) = − x + 4.

Answer a

-32

Answer b

-32

Exercise 5.16.32:

Evaluate when z = 55: (a) −(z − 10) (b) −z + 10 to show that −(z − 10) = − z + 10.

Answer a

-45

Answer b

-45

ACCESS ADDITIONAL ONLINE RESOURCES

Model Distribution

The Distributive Property

Practice Makes Perfect

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the distributive property.

90. 4(x + 8)
91. 3(a + 9)
92. 8(4y + 9)
93. 9(3w + 7)
94. 6(c − 13)
95. 7(y − 13)
96. 7(3p − 8)
97. 5(7u − 4)
98. $\frac{1}{2}$(n + 8)
99. $\frac{1}{3}$(u + 9)
100. $\frac{1}{4}$(3q + 12)
101. $\frac{1}{5}$(4m + 20)
102. $9\left(\frac{5}{9}y-\frac{1}{3}\right)$
103. $10\left(\frac{3}{10}x-\frac{2}{5}\right)$
104. $12\left(\frac{1}{4}+\frac{2}{3}r\right)$
105. $12\left(\frac{1}{6}+\frac{3}{4}s\right)$

106. r(s − 18)
107. u(v − 10)
108. (y + 4)p
109. (a + 7)x
110. −2(y + 13)
111. −3(a + 11)
112. −7(4p + 1)
113. −9(9a + 4)
114. −3(x − 6)
115. −4(q − 7)
116. −9(3a − 7)
117. −6(7x − 8)
118. −(r + 7)
119. −(q + 11)
120. −(3x − 7)
121. −(5p − 4)
122. 5 + 9(n − 6)
123. 12 + 8(u − 1)
124. 16 − 3(y + 8)
125. 18 − 4(x + 2)
126. 4 − 11(3c − 2)
127. 9 − 6(7n − 5)
128. 22 − (a + 3)
129. 8 − (r − 7)
130. −12 − (u + 10)
131. −4 − (c − 10)
132. (5m − 3) − (m + 7)
133. (4y − 1) − (y − 2)
134. 5(2n + 9) + 12(n − 3)
135. 9(5u + 8) + 2(u − 6)
136. 9(8x − 3) − (−2)
137. 4(6x − 1) − (−8)
138. 14(c − 1) − 8(c − 6)
139. 11(n − 7) − 5(n − 1)
140. 6(7y + 8) − (30y − 15)
141. 7(3n + 9) − (4n − 13)

Evaluate Expressions Using the Distributive Property

In the following exercises, evaluate both expressions for the given value.

142. If v = −2, evaluate
    a. 6(4v + 7)
    b. 6 · 4v + 6 · 7
143. If u = −1, evaluate
    a. 8(5u + 12)
    b. 8 · 5u + 8 · 12
144. If $n = \frac{2}{3}$, evaluate
    a. $3\left(n+\frac{5}{6}\right)$
    b. $3 \cdot n + 3 \cdot \frac{5}{6}$
145. If y = 3 4 , evaluate

a. $4\left(y+3\frac{8}{}\right)$

b. $4 \cdot y + 4 \cdot \frac{3}{8}$

146. If $y = \frac{7}{12}$, evaluate

a. $-3(4y + 15)$

b. $3 \cdot 4y + (-3) \cdot 15$

147. If $p = \frac{23}{30}$, evaluate

a. $-6(5p + 11)$

b. $-6 \cdot 5p + (-6) \cdot 11$

148. If m = 0.4, evaluate

a. $-10(3m - 0.9)$

b. $-10 \cdot 3m - (-10)(0.9)$

149. If n = 0.75, evaluate

a. $-100(5n + 1.5)$

b. $-100 \cdot 5n + (-100)(1.5)$

150. If y = −25, evaluate

a. $-(y - 25)$

b. $-y + 25$

151. If w = −80, evaluate

a. $-(w - 80)$

b. $-w + 80$

152. If p = 0.19, evaluate

a. $-(p + 0.72)$

b. $-p - 0.72$

153. If q = 0.55, evaluate

a. $-(q + 0.48)$

b. $-q - 0.48$

Everyday Math

154. **Buying by the case** Joe can buy his favorite ice tea at a convenience store for $1.99 per bottle. At the grocery store, he can buy a case of 12 bottles for $23.88.

a. Use the distributive property to find the cost of 12 bottles bought individually at the convenience store. (Hint: notice that $1.99 is $2 − $0.01.)

b. Is it a bargain to buy the iced tea at the grocery store by the case?

155. **Multi-pack purchase** Adele's shampoo sells for $3.97 per bottle at the drug store. At the warehouse store, the same shampoo is sold as a 3-pack for $10.49.

a. Show how you can use the distributive property to find the cost of 3 bottles bought individually at the drug store.

b. How much would Adele save by buying the 3-pack at the warehouse store?

Writing Exercises

156. Simplify $8\left(x-\frac{1}{4}\right)$ using the distributive property and explain each step.

157. Explain how you can multiply 4($5.97) without paper or a calculator by thinking of $5.97 as 6 − 0.03 and then using the distributive property.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions using the Distributive Property.			
evaluate expressions using the Distributive Property.			

(b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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5.17: Properties of Identity, Inverses, and Zero

Learning Objectives

- Recognize the identity properties of addition and multiplication
- Use the inverse properties of addition and multiplication
- Use the properties of zero
- Simplify expressions using the properties of identities, inverses, and zero

be prepared!

Before you get started, take this readiness quiz.

1. Find the opposite of −4. If you missed this problem, review Example 3.1.3.
2. Find the reciprocal of $\frac{5}{2}$. If you missed this problem, review Example 4.4.11.
3. Multiply: $\frac{3a}{5} \cdot \frac{9}{2a}$. If you missed this problem, review Example 4.3.9.

Recognize the Identity Properties of Addition and Multiplication

What happens when we add zero to any number? Adding zero doesn't change the value. For this reason, we call 0 the **additive identity**. For example,

$$\begin{array}{lll} 13+0 & -14\ +0 & 0+(-3x) \\ 13 & -14 & -3x \end{array}$$

Definition: Identity Properties

The identity property of addition: for any real number a,

$$a+0=a \qquad 0+a=a \tag{5.17.1}$$

0 is called the additive identity

The identity property of multiplication: for any real number a

$$a \cdot 1=a \qquad 1 \cdot a=a \tag{5.17.2}$$

1 is called the multiplicative identity

Example 5.17.1:

Identify whether each equation demonstrates the identity property of addition or multiplication. (a) 7 + 0 = 7 (b) −16(1) = −16

Solution

(a) 7 + 0 = 7

We are adding 0.	We are using the identity property of addition.

(b) −16(1) = −16

We are multiplying by 1.	We are using the identity property of multiplication.

Exercise 5.17.1:

Identify whether each equation demonstrates the identity property of addition or multiplication: (a) 23 + 0 = 23 (b) −37(1) = −37.

Answer a

identity property of addition

Answer b

identity property of multiplication

Exercise 5.17.2:

Identify whether each equation demonstrates the identity property of addition or multiplication: (a) 1 • 29 = 29 (b) 14 + 0 = 14.

Answer a

identity property of multiplication

Answer b

additive identity

Use the Inverse Properties of Addition and Multiplication

What number added to 5 gives the additive identity, 0?

5 + _____ = 0	We know $5+(-5)=0$

What number added to −6 gives the additive identity, 0?

-6 + _____ = 0	We know $-6+(6)=0$

Notice that in each case, the missing number was the opposite of the number. We call −a the **additive inverse** of a. The opposite of a number is its additive inverse. A number and its opposite add to 0, which is the additive identity.

What number multiplied by $\frac{2}{3}$ gives the multiplicative identity, 1? In other words, two-thirds times what results in 1?

$\frac{2}{3}\cdot$ _____ = 1	We know $\frac{2}{3}\cdot\frac{3}{2}=1$

What number multiplied by 2 gives the multiplicative identity, 1? In other words two times what results in 1?

2 • _____ = 1	We know $2\cdot\frac{1}{2}=1$

Notice that in each case, the missing number was the reciprocal of the number.

We call $\frac{1}{a}$ the **multiplicative inverse** of a ($a \neq 0$). The reciprocal of a number is its multiplicative inverse. A number and its reciprocal multiply to 1, which is the multiplicative identity.

We'll formally state the Inverse Properties here:

Definition: Inverse Properties

Inverse Property of Addition for any real number a,

$$a+(-a)=0 \tag{5.17.3}$$

−a is the additive inverse of a.

Inverse Property of Multiplication for any real number a ≠ 0,

$$a\cdot\frac{1}{a}=1 \tag{5.17.4}$$

$\frac{1}{a}$ is the multiplicative inverse of a.

Example 5.17.2:

Find the additive inverse of each expression: (a) 13 (b) $-\frac{5}{8}$ (c) 0.6.

Solution

To find the additive inverse, we find the opposite.

a. The additive inverse of 13 is its opposite, −13.
b. The additive inverse of $-\frac{5}{8}$ is its opposite, $\frac{5}{8}$.
c. The additive inverse of 0.6 is its opposite, −0.6.

Exercise 5.17.3:

Find the additive inverse of each expression: (a) 18 (b) $\frac{7}{9}$ (c) 1.2.

Answer a

-18

Answer b

$-\frac{7}{9}$

Answer c

-1.2

Exercise 5.17.4:

Find the additive inverse of each expression: (a) 47(b) $\frac{7}{13}$ (c) 8.4.

Answer a

-47

Answer b

$-\frac{7}{13}$

Answer c

-8.4

Example 5.17.3:

Find the multiplicative inverse: (a) 9 (b) $-\frac{1}{9}$ (c) 0.9.

Solution

To find the additive inverse, we find the opposite.

a. The multiplicative inverse of 9 is its reciprocal, $\frac{1}{9}$.

b. The multiplicative inverse of $-\frac{1}{9}$ is its reciprocal, -9.

c. To find the multiplicative inverse of 0.9, we first convert 0.9 to a fraction, $\frac{9}{10}$. Then we find the reciprocal, $\frac{10}{9}$.

Exercise 5.17.5:

Find the multiplicative inverse: (a) 5 (b) $-\frac{1}{7}$ (c) 0.3.

Answer a

$\frac{1}{5}$

Answer b

-7

Answer c

$\frac{10}{3}$

Exercise 5.17.6:

Find the multiplicative inverse: (a) 18 (b) $-\frac{4}{5}$ (c) 0.6.

Answer a

$\frac{1}{18}$

Answer b

$-\frac{5}{4}$

Answer c

$\frac{5}{3}$

Use the Properties of Zero

We have already learned that zero is the additive identity, since it can be added to any number without changing the number's identity. But zero also has some special properties when it comes to multiplication and division.

Multiplication by Zero

What happens when you multiply a number by 0? Multiplying by 0 makes the product equal zero. The product of any real number and 0 is 0.

Definition: Multiplication by Zero

For any real number a,

$$a \cdot 0 = 0 \qquad 0 \cdot a = 0 \tag{5.17.5}$$

Example 5.17.4:

Simplify: (a) $-8 \cdot 0$ (b) $\frac{5}{12} \cdot 0$ (c) $0(2.94)$.

Solution

(a) $-8 \cdot 0$

The product of any real number and 0 is 0.	0

(b) $\frac{5}{12} \cdot 0$

The product of any real number and 0 is 0.	0

(c) $0(2.94)$

The product of any real number and 0 is 0.	0

Exercise 5.17.7:

Simplify: (a) $-14 \cdot 0$ (b) $0 \cdot \frac{2}{3}$ (c) $(16.5) \cdot 0$.

Answer a

0

Answer b

0

Answer c

0

Exercise 5.17.8:

Simplify: (a) $(1.95) \cdot 0$ (b) $0(-17)$ (c) $0 \cdot \frac{5}{4}$.

Answer a

0

Answer b

0

Answer c

0

Dividing with Zero

What about dividing with 0? Think about a real example: if there are no cookies in the cookie jar and three people want to share them, how many cookies would each person get? There are 0 cookies to share, so each person gets 0 cookies.

$$0 \div 3 = 0 \tag{5.17.6}$$

Remember that we can always check division with the related multiplication fact. So, we know that

$$0 \div 3 = 0 \; because \; 0 \cdot 3 = 0. \tag{5.17.7}$$

Definition: Division of Zero

For any real number a, except 0, $\frac{0}{a}$ = 0 and 0 ÷ a = 0.

Zero divided by any real number except zero is zero.

Example 5.17.5:

Simplify: (a) 0 ÷ 5 (b) $\frac{0}{-2}$ (c) $0 \div \frac{7}{8}$.

Solution

(a) 0 ÷ 5

Zero divided by any real number, except 0, is zero.	0

(b) $\frac{0}{-2}$

Zero divided by any real number, except 0, is zero.	0

(c) $0 \div \frac{7}{8}$

Zero divided by any real number, except 0, is zero.	0

Exercise 5.17.9:

Simplify: (a) 0 ÷ 11 (b) $\frac{0}{-6}$ (c) $0 \div \frac{3}{10}$.

Answer a

0

Answer b

0

Answer c

0

Exercise 5.17.10:

Simplify: (a) $0 \div \frac{8}{3}$ (b) 0 ÷ (-10) (c) 0 ÷ 12.75.

Answer a

0

Answer b

0

Answer c

0

Now let's think about dividing a number by zero. What is the result of dividing 4 by 0? Think about the related multiplication fact. Is there a number that multiplied by 0 gives 4?

$$4 \div 0 = \_\_\_ \text{ means } \_\_\_ \cdot 0 = 4$$

Since any real number multiplied by 0 equals 0, there is no real number that can be multiplied by 0 to obtain 4. We can conclude that there is no answer to $4 \div 0$, and so we say that division by zero is undefined.

Definition: Division by Zero

For any real number a, $\frac{a}{0}$, and $a \div 0$ are undefined.

Division *by* zero is undefined.

Example 5.17.6:

Simplify: (a) $7.5 \div 0$ (b) $\frac{-32}{0}$ (c) $\frac{4}{9} \div 0$.

Solution

(a) $7.5 \div 0$

Division by zero is undefined.	undefined

(b) $\frac{-32}{0}$

Division by zero is undefined.	undefined

(c) $\frac{4}{9} \div 0$

Division by zero is undefined.	undefined

Exercise 5.17.11:

Simplify: (a) $16.4 \div 0$ (b) $\frac{-2}{0}$ (c) $\frac{1}{5} \div 0$.

Answer a

undefined

Answer b

undefined

Answer c

undefined

Exercise 5.17.12:

Simplify: (a) $\frac{-5}{0}$ (b) $96.9 \div 0$ (c) $\frac{4}{15} \div 0$.

Answer a

undefined

Answer b

undefined

Answer c

undefined

We summarize the properties of zero.

Definition: properties of zero

Multiplication by Zero: For any real number a,

$$a \cdot 0 = 0 \qquad 0 \cdot a = 0 \tag{5.17.8}$$

The product of any number and 0 is 0.

Division by Zero: For any real number a, a ≠ 0

$$\frac{0}{a} = 0 \tag{5.17.9}$$

Zero divided by any real number, except itself, is zero.

$$\frac{a}{0} \; is \; undefined. \tag{5.17.10}$$

Division by zero is undefined.

Simplify Expressions using the Properties of Identities, Inverses, and Zero

We will now practice using the properties of identities, inverses, and zero to simplify expressions.

Example 5.17.7:

Simplify: 3x + 15 − 3x.

Solution

Notice the additive inverses, 3x and −3x.	0 + 15
Add.	15

Exercise 5.17.13:

Simplify: −12z + 9 + 12z.

Answer

9

Exercise 5.17.14:

Simplify: −25u − 18 + 25u.

Answer

-18

Example 5.17.8:

Simplify: 4(0.25q).

Solution

Regroup, using the associative property.	[4(0.25)]q
Multiply.	1.00q
Simplify; 1 is the multiplicative identity.	q

Exercise 5.17.15:

Simplify: 2(0.5p).

Answer

p

Exercise 5.17.16:

Simplify: 25(0.04r).

Answer

r

Example 5.17.9:

Simplify: $\frac{0}{n+5}$, where n ≠ −5.

Solution

Zero divided by any real number except itself is zero.	0

Exercise 5.17.17:

Simplify: $\frac{0}{m+7}$, where m ≠ −7.

Answer

0

Exercise 5.17.18:

Simplify: $\frac{0}{d-4}$, where d ≠ 4.

Answer

0

Example 5.17.10:

Simplify: $\frac{10-3p}{0}$.

Solution

Division by zero is undefined.	undefined

Exercise 5.17.19:

Simplify: $\frac{18-6c}{0}$.

Answer

undefined

Exercise 5.17.20:

Simplify: $\frac{15-4q}{0}$.

Answer

undefined

Example 5.17.11:

Simplify: $\frac{3}{4} \cdot \frac{4}{3}$ (6x + 12).

Solution

We cannot combine the terms in parentheses, so we multiply the two fractions first.

Multiply; the product of reciprocals is 1.	1(6x + 12)
Simplify by recognizing the multiplicative identity.	6x + 12

Exercise 5.17.21:

Simplify: $\frac{2}{5} \cdot \frac{5}{2}$ (20y+ 50).

Answer

20y + 50

Exercise 5.17.22:

Simplify: $\frac{3}{8} \cdot \frac{8}{3}(12z+16)$.

Answer

12z + 16

All the properties of real numbers we have used in this chapter are summarized in Table 5.17.1.

Table 5.17.1: Properties of Real Numbers

Property	Of Addition	Of Multiplication
Commutative Property		
If a and b are real numbers then…	a + b = b + a	a • b = b • a
Associative Property		
If a, b, and c are real numbers then…	(a + b) + c = a + (b + c)	(a • b) • c = (b • c)
Identity Property	0 is the additive identity	1 is the multiplicative identity
For any real number a,	a + 0 = a 0 + a = a	a • 1 = a 1 • a = a
Inverse Property	−a is the additive inverse of a	a, a ≠ 0 1/a is the multiplicative inverse of a
For any real number a,	a + (−a) = 0	a • 1/a = 1
Distributive Property		
If a, b, c are real numbers, then a(b + c) = ab + ac		
Properties of Zero		
For any real number a,	a • 0 = 0 0 • a = 0	
For any real number a where a ≠ 0	$\frac{0}{a} = 0$ $\frac{a}{0}$ is undefined	

ACCESS ADDITIONAL ONLINE RESOURCES

Multiplying and Dividing Involving Zero

Practice Makes Perfect

Recognize the Identity Properties of Addition and Multiplication

In the following exercises, identify whether each example is using the identity property of addition or multiplication.

158. 101 + 0 = 101
159. $\frac{3}{5}(1) = \frac{3}{5}$
160. −9 • 1 = −9
161. 0 + 64 = 64

Use the Inverse Properties of Addition and Multiplication

In the following exercises, find the multiplicative inverse.

162. 8
163. 14
164. −17
165. −19
166. $\frac{7}{12}$
167. $\frac{8}{13}$
168. $-\frac{3}{10}$
169. $-\frac{5}{12}$
170. 0.8
171. 0.4
172. −0.2
173. −0.5

Use the Properties of Zero

In the following exercises, simplify using the properties of zero.

174. 48 • 0
175. $\frac{0}{6}$
176. $\frac{3}{0}$
177. 22 • 0
178. $0 \div \frac{11}{12}$
179. $\frac{6}{0}$
180. $\frac{0}{3}$
181. $0 \div \frac{7}{15}$
182. $0 \cdot \frac{8}{15}$
183. (−3.14)(0)
184. 5.72 ÷ 0
185. $\frac{\frac{1}{10}}{0}$

Simplify Expressions using the Properties of Identities, Inverses, and Zero

In the following exercises, simplify using the properties of identities, inverses, and zero.

186. 19a + 44 − 19a
187. 27c + 16 − 27c
188. 38 + 11r − 38
189. 92 + 31s − 92
190. 10(0.1d)
191. 100(0.01p)
192. 5(0.6q)
193. 40(0.05n)
194. $\frac{0}{r+20}$, where r ≠ −20

195. $\frac{0}{s+13}$, where s ≠ −13

196. $\frac{0}{u-4.99}$, where u ≠ 4.99

197. $\frac{0}{v-65.1}$, where v ≠ 65.1

198. $0 \div \left(x-\frac{1}{2}\right)$, where x ≠ $\frac{1}{2}$

199. $0 \div \left(y-\frac{1}{6}\right)$, where y ≠ $\frac{1}{6}$

200. $\frac{32-5a}{0}$, where 32 − 5a ≠ 0

201. $\frac{28-9b}{0}$, where 28 − 9b ≠ 0

202. $\frac{2.1+0.4c}{0}$, where 2.1 + 0.4c ≠ 0

203. $\frac{1.75+9f}{0}$, where 1.75 + 9 f ≠ 0

204. \left\dfrac{3}{4} + \dfrac{9}{10}m \right) \div 0, where $\frac{3}{4}+\frac{9}{10}m \neq 0$

205. $\left(\frac{5}{16}n-\frac{3}{7}\right) \div 0$, where $\frac{5}{16}n-\frac{3}{7} \neq 0$

206. $\frac{9}{10} \cdot \frac{10}{9}$ (18p − 21)

207. $\frac{5}{7} \cdot \frac{7}{5}$ (20q − 35)

208. $15 \cdot \frac{3}{5}$(4d + 10)

209. $18 \cdot \frac{5}{6}$(15h + 24)

Everyday Math

210. **Insurance copayment** Carrie had to have 5 fillings done. Each filling cost $80. Her dental insurance required her to pay 20% of the cost. Calculate Carrie's cost
    a. by finding her copay for each filling, then finding her total cost for 5 fillings, and
    b. by multiplying 5(0.20)(80).
    c. Which of the Properties of Real Numbers did you use for part (b)?

211. **Cooking time** Helen bought a 24-pound turkey for her family's Thanksgiving dinner and wants to know what time to put the turkey in the oven. She wants to allow 20 minutes per pound cooking time.
    a. Calculate the length of time needed to roast the turkey by multiplying 24 · 20 to find the number of minutes and then multiplying the product by 1 60 to convert minutes into hours.
    b. Multiply $24\left(20 \cdot \frac{1}{60}\right)$.
    c. Which of the Properties of Real Numbers allows you to multiply $24\left(20 \cdot \frac{1}{60}\right)$ instead of $(24 \cdot 20)\frac{1}{60}$?

Writing Exercises

212. In your own words, describe the difference between the additive inverse and the multiplicative inverse of a number.

213. How can the use of the properties of real numbers make it easier to simplify expressions?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize the Identity Properties of Addition and Multiplication.			
use the Inverse Properties of Addition and Multiplication.			
use the Properties of Zero.			
simplify expressions using the properties of identities, inverses, and zero.			

(b) On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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5.18: Solve Equations with Decimals

Learning Objectives

- Determine whether a decimal is a solution of an equation
- Solve equations with decimals
- Translate to an equation and solve

be prepared!

Before you get started, take this readiness quiz.

1. Evaluate $x + \frac{2}{3}$ when $x = -\frac{1}{4}$. If you missed this problem, review Example 4.9.15.
2. Evaluate $15 - y$ when $y = -5$. If you missed this problem, review Example 3.6.12.
3. Solve $\frac{n}{-7} = 42$. If you missed this problem, review Example 4.12.5.

Determine Whether a Decimal is a Solution of an Equation

Solving equations with decimals is important in our everyday lives because money is usually written with decimals. When applications involve money, such as shopping for yourself, making your family's budget, or planning for the future of your business, you'll be solving equations with decimals.

Now that we've worked with decimals, we are ready to find solutions to equations involving decimals. The steps we take to determine whether a number is a solution to an equation are the same whether the solution is a whole number, an integer, a fraction, or a decimal. We'll list these steps here again for easy reference.

HOW TO: DETERMINE WHETHER A NUMBER IS A Solution TO AN EQUATION

- Step 1. Substitute the number for the variable in the equation.
- Step 2. Simplify the expressions on both sides of the equation.
- Step 3. Determine whether the resulting equation is true.
  - If so, the number is a solution.
  - If not, the number is not a solution.

Example 5.18.1:

Determine whether each of the following is a solution of $x - 0.7 = 1.5$: (a) $x = 1$ (b) $x = -0.8$ (c) $x = 2.2$

Solution

(a) $x - 0.7 = 1.5$

Substitute 1 for x.	$1 - 0.7 \stackrel{?}{=} 1.5$	(5.18.1)
Subtract.	$0.3 \neq 1.5$	(5.18.2)

Since $x = 1$ does not result in a true equation, 1 is not a solution to the equation.

(b) $x - 0.7 = 1.5$

Substitute -0.8 for x.	$-0.8 - 0.7 \stackrel{?}{=} 1.5$	(5.18.3)
Subtract.	$-1.5 \neq 1.5$	(5.18.4)

Since x = −0.8 does not result in a true equation, −0.8 is not a solution to the equation.

(c) x - 0.7 = 1.5

Substitute 2.2 for x.	$2.2 - 0.7 \stackrel{?}{=} 1.5$	(5.18.5)
Subtract.	$1.5 = 1.5$ ✓	(5.18.6)

Since x = 2.2 results in a true equation, 2.2 is a solution to the equation.

Exercise 5.18.1:

Determine whether each value is a solution of the given equation. x − 0.6 = 1.3: (a) x = 0.7 (b) x = 1.9 (c) x = −0.7

Answer a

no

Answer b

yes

Answer c

no

Exercise 5.18.2:

Determine whether each value is a solution of the given equation. y − 0.4 = 1.7: (a) y = 2.1 (b) y = 1.3 (c) −1.3

Answer a

yes

Answer b

no

Answer c

no

Solve Equations with Decimals

In previous chapters, we solved equations using the Properties of Equality. We will use these same properties to solve equations with decimals.

Properties of Equality

Subtraction Property of Equality For any numbers a, b, and c, If a = b, then a − c = b − c.	**Addition Property of Equality** For any numbers a, b, and c, If a = b, then a + c = b + c.
The Division Property of Equality For any numbers a, b, and c, and c ≠ 0 If a = b, then $\frac{a}{c} = \frac{b}{c}$	**The Multiplication Property of Equality** For any numbers a, b, and c, If a = b, then ac = bc

When you add, subtract, multiply or divide the same quantity from both sides of an equation, you still have equality.

Example 5.18.2:

Solve: y + 2.3 = −4.7.

Solution

We will use the Subtraction Property of Equality to isolate the variable.

Subtract 2.3 from each side, to undo the addition.	$y+2.3-2.3=-4.7-2.3$	(5.18.7)
Simplify.	$y=-7$	(5.18.8)

Check:

Substitute y = −7.	$-7+2.3\stackrel{?}{=}-4.7$	(5.18.9)
Simplify.	$-4.7=-4.7$ ✓	(5.18.10)

Since y = −7 makes y + 2.3 = −4.7 a true statement, we know we have found a solution to this equation.

Exercise 5.18.3:

Solve: y + 2.7 = −5.3.

Answer

$y=-8$

Exercise 5.18.4:

Solve: y + 3.6 = −4.8.

Answer

$y=-8.4$

Example 5.18.3:

Solve: a − 4.75 = −1.39.

Solution

We will use the Addition Property of Equality.

Add 4.75 to each side, to undo the subtraction.	$a-4.75+4.75=-1.39+4.75$	(5.18.11)
Simplify.	$a=3.36$	(5.18.12)

Check:

Substitute a = 3.36.	$3.36-4.75\stackrel{?}{=}-1.39$	(5.18.13)

	$-1.39 = -1.39$ ✓	(5.18.14)

Since the result is a true statement, a = 3.36 is a solution to the equation.

? Exercise 5.18.5:

Solve: a − 3.93 = −2.86.

Answer

$a = 1.07$

? Exercise 5.18.6:

Solve: n − 3.47 = −2.64.

Answer

$n = 0.83$

✓ Example 5.18.4:

Solve: −4.8 = 0.8n.

Solution

We will use the Division Property of Equality. Use the Properties of Equality to find a value for n.

We must divide both sides by 0.8 to isolate n.	$\frac{-4.8}{0.8} = \frac{0.8n}{08}$	(5.18.15)
Simplify.	$-6 = n$	(5.18.16)

Check:

Substitute n = -6.	$-4.8 \stackrel{?}{=} 0.8(-6)$	(5.18.17)
	$-4.8 = -4.8$ ✓	(5.18.18)

Since n = −6 makes −4.8 = 0.8n a true statement, we know we have a solution.

? Exercise 5.18.7:

Solve: −8.4 = 0.7b.

Answer

$b = -12$

Exercise 5.18.8:

Solve: −5.6 = 0.7c.

Answer

$c = -8$

Example 5.18.5:

Solve: $\frac{p}{-1.8} = -6.5$.

Solution

We will use the Multiplication Property of Equality.

Here, p is divided by −1.8. We must multiply by −1.8 to isolate p.	$-1.8\left(\frac{p}{-1.8}\right) = -1.8(-6.5)$	(5.18.19)
Multiply.	$p = 11.7$	(5.18.20)

Check:

Substitute p = 11.7.	$\frac{11.7}{-1.8} \stackrel{?}{=} -6.5$	(5.18.21)
	$-6.5 = -6.5$ ✓	(5.18.22)

A solution to $\frac{p}{-1.8} = -6.5$ is p = 11.7.

Exercise 5.18.9:

Solve: $\frac{c}{-2.6} = -4.5$.

Answer

$c = -11.7$

Exercise 5.18.10:

Solve: $\frac{b}{-1.2} = -5.4$.

Answer

$b = 6.48$

Translate to an Equation and Solve

Now that we have solved equations with decimals, we are ready to translate word sentences to equations and solve. Remember to look for words and phrases that indicate the operations to use.

Example 5.18.6:

Translate and solve: The difference of n and 4.3 is 2.1.

Solution

Translate.	$\underbrace{\text{The difference of } n \text{ and } 4.3}_{n-4.3}\ \underbrace{\text{is}}_{=}\ \underbrace{2.1.}_{2.1}$	
Add 4.3 to both sides of the equation.	$n-4.3+4.3=2.1+4.3$	(5.18.23)
Simplify.	$n=6.4$	(5.18.24)

Check:

Let n = 6.4.	Is the difference of 6.4 and 4.3 equal to 2.1?	
Translate.	$6.4-4.3\stackrel{?}{=}2.1$	(5.18.25)
Simplify.	$2.1=2.1$ ✓	(5.18.26)

Exercise 5.18.11:

Translate and solve: The difference of y and 4.9 is 2.8.

Answer

$y-4.9=2.8;\ y=7.7$

Exercise 5.18.12:

Translate and solve: The difference of z and 5.7 is 3.4.

Answer

$z-5.7=3.4;\ z=9.1$

Example 5.18.7:

Translate and solve: The product of −3.1 and x is 5.27.

Solution

Translate.	$\underbrace{\text{The product of } 3.1 \text{ and } x}_{-3.1x}\ \underbrace{\text{is}}_{=}\ \underbrace{5.27.}_{5.27}$	
Divide both sides by −3.1.	$\frac{-3.1x}{-3.1}=\frac{5.27}{-3.1}$	(5.18.27)
Simplify.	$x=-1.7$	(5.18.28)

Check:

Let x = −1.7.	Is the product of −3.1 and −1.7 equal to 5.27?	
Translate.	$-3.1(-1.7) \stackrel{?}{=} 5.27$	(5.18.29)
Simplify.	$5.27 = 5.27$ ✓	(5.18.30)

? Exercise 5.18.13:

Translate and solve: The product of −4.3 and x is 12.04.

Answer

$-4.3x = 12.04; x = -2.8$

? Exercise 5.18.14:

Translate and solve: The product of −3.1 and m is 26.66.

Answer

$-3.1m = 26.66; m = -8.6$

✓ Example 5.18.8:

Translate and solve: The quotient of p and −2.4 is 6.5.

Solution

Translate.	$\underbrace{\frac{p}{-2.4}}_{\text{The quotient of } p \text{ and } -2.4} \underbrace{=}_{\text{is}} \underbrace{6.5}_{6.5.}$	
Multiply both sides by −2.4.	$-2.4\left(\frac{p}{-2.4}\right) = -2.4(6.5)$	(5.18.31)
Simplify.	$p = -15.6$	(5.18.32)

Check:

Let p = −15.6.	Is the quotient of −15.6 and −2.4 equal to 6.5?	
Translate.	$\frac{-15.6}{-2.4} \stackrel{?}{=} 6.5$	(5.18.33)
Simplify.	$6.5 = 6.5$ ✓	(5.18.34)

? Exercise 5.18.15:

Translate and solve: The quotient of q and −3.4 is 4.5.

Answer

$\frac{1}{-3.4} = 4.5; q = -15.3$

Exercise 5.18.16:

Translate and solve: The quotient of r and −2.6 is 2.5.

Answer

$\frac{r}{-2.6} = 2.5; r = -6.5$

Example 5.18.9:

Translate and solve: The sum of n and 2.9 is 1.7.

Solution

Translate.	$\underbrace{n + 2.9}_{\text{The sum of } n \text{ and } 2.9} \underbrace{=}_{\text{is}} \underbrace{1.7}_{1.7.}$	
Subtract 2.9 from each side.	$n + 2.9 - 2.9 = 1.7 - 2.9$	(5.18.35)
Simplify.	$n = -1.2$	(5.18.36)

Check:

Let n = −1.2.	Is the sum −1.2 and 2.9 equal to 1.7?	
Translate.	$-1.2 + 2.9 \stackrel{?}{=} 1.7$	(5.18.37)
Simplify.	$1.7 = 1.7$ ✓	(5.18.38)

Exercise 5.18.17:

Translate and solve: The sum of j and 3.8 is 2.6.

Answer

$j + 3.8 = 2.6; j = -1.2$

Exercise 5.18.18:

Translate and solve: The sum of k and 4.7 is 0.3.

Answer

$k + 4.7 = 0.3; k = -4.4$

ACCESS ADDITIONAL ONLINE RESOURCES

Solving One Step Equations Involving Decimals

Solve a One Step Equation With Decimals by Adding and Subtracting

Solve a One Step Equation With Decimals by Multiplying

Solve a One Step Equation With Decimals by Dividing

Practice Makes Perfect

Determine Whether a Decimal is a Solution of an Equation

In the following exercises, determine whether each number is a solution of the given equation.

285. $x - 0.8 = 2.3$
   a. $x = 2$
   b. $x = -1.5$
   c. $x = 3.1$

286. $y + 0.6 = -3.4$
   a. $y = -4$
   b. $y = -2.8$
   c. $y = 2.6$

287. $\frac{h}{1.5} = -4.3$
   a. $h = 6.45$
   b. $h = -6.45$
   c. $h = -2.1$

288. $0.75k = -3.6$
   a. $k = -0.48$
   b. $k = -4.8$
   c. $k = -2.7$

Solve Equations with Decimals

In the following exercises, solve the equation.

289. $y + 2.9 = 5.7$
290. $m + 4.6 = 6.5$
291. $f + 3.45 = 2.6$
292. $h + 4.37 = 3.5$
293. $a + 6.2 = -1.7$
294. $b + 5.8 = -2.3$
295. $c + 1.15 = -3.5$
296. $d + 2.35 = -4.8$
297. $n - 2.6 = 1.8$
298. $p - 3.6 = 1.7$
299. $x - 0.4 = -3.9$
300. $y - 0.6 = -4.5$
301. $j - 1.82 = -6.5$
302. $k - 3.19 = -4.6$
303. $m - 0.25 = -1.67$
304. $q - 0.47 = -1.53$
305. $0.5x = 3.5$
306. $0.4p = 9.2$
307. $-1.7c = 8.5$
308. $-2.9x = 5.8$
309. $-1.4p = -4.2$
310. $-2.8m = -8.4$
311. $-120 = 1.5q$
312. $-75 = 1.5y$
313. $0.24x = 4.8$
314. $0.18n = 5.4$
315. $-3.4z = -9.18$

316. −2.7u = −9.72
317. $\frac{a}{0.4}$ = −20
318. $\frac{b}{0.3}$ = −9
319. $\frac{x}{0.7}$ = −0.4
320. $\frac{y}{0.8}$ = −0.7
321. $\frac{p}{-5}$ = −1.65
322. $\frac{q}{-4}$ = −5.92
323. $\frac{r}{-1.2}$ = −6
324. $\frac{s}{-1.5}$ = −3

Mixed Practice

In the following exercises, solve the equation. Then check your solution.

325. x − 5 = −11
326. $-\frac{2}{5}$ = x + (\dfrac{3}{4}\)
327. p + 8 = −2
328. p + $\frac{2}{3} = \frac{1}{12}$
329. −4.2m = −33.6
330. q + 9.5 = −14
331. q + $\frac{5}{6} = \frac{1}{12}$
332. $\frac{8.6}{15}$ = − d
333. $\frac{7}{8}m = \frac{1}{10}$
334. $\frac{j}{-6.2}$ = −3
335. $-\frac{2}{3}$ = y + $\frac{3}{8}$
336. s − 1.75 = −3.2
337. $\frac{11}{20}$ = − f
338. −3.6b = 2.52
339. −4.2a = 3.36
340. −9.1n = −63.7
341. r − 1.25 = −2.7
342. $\frac{1}{4}n = \frac{7}{10}$
343. $\frac{h}{-3}$ = −8
344. y − 7.82 = −16

Translate to an Equation and Solve

In the following exercises, translate and solve.

345. The difference of n and 1.9 is 3.4.
346. The difference n and 1.5 is 0.8.
347. The product of −6.2 and x is −4.96.
348. The product of −4.6 and x is −3.22.
349. The quotient of y and −1.7 is −5.

350. The quotient of z and −3.6 is 3.
351. The sum of n and −7.3 is 2.4.
352. The sum of n and −5.1 is 3.8.

Everyday Math

353. Shawn bought a pair of shoes on sale for $78. Solve the equation 0.75p = 78 to find the original price of the shoes, p.
354. Mary bought a new refrigerator. The total price including sales tax was $1,350. Find the retail price, r, of the refrigerator before tax by solving the equation 1.08r = 1,350.

Writing Exercises

355. Think about solving the equation 1.2y = 60, but do not actually solve it. Do you think the solution should be greater than 60 or less than 60? Explain your reasoning. Then solve the equation to see if your thinking was correct.
356. Think about solving the equation 0.8x = 200, but do not actually solve it. Do you think the solution should be greater than 200 or less than 200? Explain your reasoning. Then solve the equation to see if your thinking was correct.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether a decimal is a solution of an equation.			
solve equations with decimals.			
translate to an equation and solve.			

(b) On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Contributors and Attributions

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5.E: Decimals (Exercises)

5.1 - Decimals

Name Decimals

In the following exercises, name each decimal.

534. 0.8
535. 0.375
536. 0.007
537. 5.24
538. −12.5632
539. −4.09

Write Decimals

In the following exercises, write as a decimal.

540. three tenths
541. nine hundredths
542. twenty-seven hundredths
543. ten and thirty-five thousandths
544. negative twenty and three tenths
545. negative five hundredths

Convert Decimals to Fractions or Mixed Numbers

In the following exercises, convert each decimal to a fraction. Simplify the answer if possible.

546. 0.43
547. 0.825
548. 9.7
549. 3.64

Locate Decimals on the Number Line

550. (a) 0.6 (b) −0.9 (c) 2.2 (d) −1.3

Order Decimals

In the following exercises, order each of the following pairs of numbers, using < or >.

551. 0.6___0.8
552. 0.2___0.15
553. 0.803____0.83
554. −0.56____−0.562

Round Decimals

In the following exercises, round each number to the nearest: (a) hundredth (b) tenth (c) whole number.

555. 12.529
556. 4.8447
557. 5.897

5.2 - Decimal Operations

Add and Subtract Decimals

In the following exercises, add or subtract.

558. 5.75 + 8.46
559. 32.89 − 8.22

560. 24 − 19.31
561. 10.2 + 14.631
562. −6.4 + (−2.9)
563. 1.83 − 4.2

Multiply Decimals

In the following exercises, multiply.

564. (0.3)(0.7)
565. (−6.4)(0.25)
566. (−3.35)(−12.7)
567. (15.4)(1000)

Divide Decimals

In the following exercises, divide.

568. 0.48 ÷ 6
569. 4.32 ÷ 24
570. $6.29 ÷ 12
571. (−0.8) ÷ (−0.2)
572. 1.65 ÷ 0.15
573. 9 ÷ 0.045

Use Decimals in Money Applications

In the following exercises, use the strategy for applications to solve.

574. Miranda got $40 from her ATM. She spent $9.32 on lunch and $16.99 on a book. How much money did she have left? Round to the nearest cent if necessary.
575. Jessie put 8 gallons of gas in her car. One gallon of gas costs $3.528. How much did Jessie owe for all the gas?
576. A pack of 16 water bottles cost $6.72. How much did each bottle cost?
577. Alice bought a roll of paper towels that cost $2.49. She had a coupon for $0.35 off, and the store doubled the coupon. How much did Alice pay for the paper towels?

5.3 - Decimals and Fractions

Convert Fractions to Decimals

In the following exercises, convert each fraction to a decimal.

578. $\frac{3}{5}$
579. $\frac{7}{8}$
580. $-\frac{19}{20}$
581. $-\frac{21}{4}$
582. $\frac{1}{3}$
583. $\frac{6}{11}$

Order Decimals and Fractions

In the following exercises, order each pair of numbers, using < or >.

584. $\frac{1}{2}$ ___0.2
585. $\frac{3}{5}$ ___0.

586. $-\frac{7}{8}$ ___ −0.84

587. $-\frac{5}{12}$ ___ −0.42

588. 0.625___$\frac{13}{20}$

589. 0.33___$\frac{5}{16}$

In the following exercises, write each set of numbers in order from least to greatest.

590. $\frac{2}{3}$, $\frac{17}{20}$, 0.65

591. $\frac{7}{9}$, 0.75, $\frac{11}{15}$

Simplify Expressions Using the Order of Operations

In the following exercises, simplify.

592. 4(10.3 − 5.8)

593. $\frac{3}{4}$(15.44 − 7.4)

594. 30 ÷ (0.45 + 0.15)

595. 1.6 + $\frac{3}{8}$

596. 52(0.5) + $(0.4)^2$

597. $-\frac{2}{5} \cdot \frac{9}{10}$ + 0.14

Find the Circumference and Area of Circles

In the following exercises, approximate the (a) circumference and (b) area of each circle.

598. radius = 6 in.

599. radius = 3.5 ft.

600. radius = 7 33 m

601. diameter = 11 cm

5.4 - Solve Equations with Decimals

Determine Whether a Decimal is a Solution of an Equation

In the following exercises, determine whether the each number is a solution of the given equation.

602. x − 0.4 = 2.1

a. x = 1.7

b. x = 2.5

603. y + 3.2 = −1.5

a. y = 1.7

b. y = −4.7

604. $\frac{u}{2.5}$ = −12.5

a. u = −5

b. u = −31.25

605. 0.45v = −40.5

a. v = −18.225

b. v = −90

Solve Equations with Decimals

In the following exercises, solve.

606. m + 3.8 = 7.5
607. h + 5.91 = 2.4
608. a + 2.26 = −1.1
609. p − 4.3 = −1.65
610. x − 0.24 = −8.6
611. j − 7.42 = −3.7
612. 0.6p = 13.2
613. −8.6x = 34.4
614. −22.32 = −2.4z
615. $\frac{a}{0.3} = -24$
616. $\frac{p}{-7} = -4.2$
617. $\frac{s}{-2.5} = -10$

Translate to an Equation and Solve

In the following exercises, translate and solve.

618. The difference of n and 15.2 is 4.4.
619. The product of −5.9 and x is −3.54.
620. The quotient of y and −1.8 is −9.
621. The sum of m and −4.03 is 6.8.

5.5 - Averages and Probability

Find the Mean of a Set of Numbers

In the following exercises, find the mean of the numbers.

622. 2, 4, 1, 0, 1, and 1
623. $270, $310.50, $243.75, and $252.15
624. Each workday last week, Yoshie kept track of the number of minutes she had to wait for the bus. She waited 3, 0, 8, 1, and 8 minutes. Find the mean
625. In the last three months, Raul's water bills were $31.45, $48.76, and $42.60. Find the mean.

Find the Median of a Set of Numbers

In the following exercises, find the median.

626. 41, 45, 32, 60, 58
627. 25, 23, 24, 26, 29, 19, 18, 32
628. The ages of the eight men in Jerry's model train club are 52, 63, 45, 51, 55, 75, 60, and 59. Find the median age.
629. The number of clients at Miranda's beauty salon each weekday last week were 18, 7, 12, 16, and 20. Find the median number of clients.

Find the Mode of a Set of Numbers

In the following exercises, identify the mode of the numbers.

630. 6, 4, 4, 5, 6, 6, 4, 4, 4, 3, 5
631. The number of siblings of a group of students: 2, 0, 3, 2, 4, 1, 6, 5, 4, 1, 2, 3

Use the Basic Definition of Probability

In the following exercises, solve. (Round decimals to three places.)

632. The Sustainability Club sells 200 tickets to a raffle, and Albert buys one ticket. One ticket will be selected at random to win the grand prize. Find the probability Albert will win the grand prize. Express your answer as a fraction and as a decimal.

633. Luc has to read 3 novels and 12 short stories for his literature class. The professor will choose one reading at random for the final exam. Find the probability that the professor will choose a novel for the final exam. Express your answer as a fraction and as a decimal.

5.6 - Ratios and Rate

Write a Ratio as a Fraction

In the following exercises, write each ratio as a fraction. Simplify the answer if possible.

634. 28 to 40
635. 56 to 32
636. 3.5 to 0.5
637. 1.2 to 1.8
638. $1\frac{3}{4}$ to $1\frac{5}{8}$
639. $2\frac{1}{3}$ to $5\frac{1}{4}$
640. 64 ounces to 30 ounces
641. 28 inches to 3 feet

Write a Rate as a Fraction

In the following exercises, write each rate as a fraction. Simplify the answer if possible.

642. 180 calories per 8 ounces 643. 90 pounds per 7.5 square inches
643. 126 miles in 4 hours 645. $612.50 for 35 hours

Find Unit Rates

In the following exercises, find the unit rate.

646. 180 calories per 8 ounces
647. 90 pounds per 7.5 square inches
648. 126 miles in 4 hours
649. $612.50 for 35 hours

Find Unit Price

In the following exercises, find the unit price.

650. T-shirts: 3 for $8.97
651. Highlighters: 6 for $2.52
652. An office supply store sells a box of pens for $11. The box contains 12 pens. How much does each pen cost?
653. Anna bought a pack of 8 kitchen towels for $13.20. How much did each towel cost? Round to the nearest cent if necessary.

In the following exercises, find each unit price and then determine the better buy.

654. Shampoo: 12 ounces for $4.29 or 22 ounces for $7.29?
655. Vitamins: 60 tablets for $6.49 or 100 for $11.99?

Translate Phrases to Expressions with Fractions

In the following exercises, translate the English phrase into an algebraic expression.

656. 535 miles per h hours
657. a adults to 45 children
658. the ratio of 4y and the difference of x and 10
659. the ratio of 19 and the sum of 3 and n

5.7 - Simplify and Use Square Roots

Simplify Expressions with Square Roots

In the following exercises, simplify.

660. $\sqrt{64}$
661. $\sqrt{144}$
662. $-\sqrt{25}$
663. $-\sqrt{81}$
664. $-\sqrt{9}$
665. $\sqrt{-36}$
666. $\sqrt{64}+\sqrt{225}$
667. $\sqrt{64+225}$

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

668. $\sqrt{28}$
669. $\sqrt{155}$

Approximate Square Roots

In the following exercises, approximate each square root and round to two decimal places.

670. $\sqrt{15}$
671. $\sqrt{57}$

Simplify Variable Expressions with Square Roots

In the following exercises, simplify. (Assume all variables are greater than or equal to zero.)

672. $\sqrt{q^2}$
673. $\sqrt{64b^2}$
674. $-\sqrt{121a^2}$
675. $\sqrt{225m^2n^2}$
676. $-\sqrt{100q^2}$
677. $\sqrt{49y^2}$
678. $\sqrt{4a^2b^2}$
679. $\sqrt{121c^2d^2}$

Use Square Roots in Applications

In the following exercises, solve. Round to one decimal place.

680. **Art** Diego has 225 square inch tiles. He wants to use them to make a square mosaic. How long can each side of the mosaic be?
681. **Landscaping** Janet wants to plant a square flower garden in her yard. She has enough topsoil to cover an area of 30 square feet. How long can a side of the flower garden be?
682. **Gravity** A hiker dropped a granola bar from a lookout spot 576 feet above a valley. How long did it take the granola bar to reach the valley floor?
683. **Accident investigation** The skid marks of a car involved in an accident were 216 feet. How fast had the car been going before applying the brakes?

PRACTICE TEST

684. Write six and thirty-four thousandths as a decimal.
685. Write 1.73 as a fraction.
686. Write 5 8 as a decimal.
687. Round 16.749 to the nearest (a) tenth (b) hundredth (c) whole number
688. Write the numbers $\frac{4}{5}$, −0.1, 0.804, $\frac{2}{9}$, −7.4, 0.21 in order from smallest to largest.

In the following exercises, simplify each expression.

689. 15.4 + 3.02
690. 20 − 5.71
691. (0.64)(0.3)
692. (−4.2)(100)
693. 0.96 ÷ (−12)
694. −5 ÷ 0.025
695. −0.6 ÷ (−0.3)
696. (0.7) 2
697. 24 ÷ (0.1 + 0.02)
698. 4(10.3 − 5.8)
699. 1.6 + $\frac{3}{8}$
700. $\frac{2}{3}$(14.65 − 4.6)

In the following exercises, solve.

701. m + 3.7 = 2.5
702. $\frac{h}{0.5}$ = 4.38
703. −6.5y = −57.2
704. 1.94 = a − 2.6
705. Three friends went out to dinner and agreed to split the bill evenly. The bill was $79.35. How much should each person pay?
706. A circle has radius 12. Find the (a) circumference and (b) area. [Use 3.14 for π.]
707. The ages, in months, of 10 children in a preschool class are: 55, 55, 50, 51, 52, 50, 53, 51, 55, 49. Find the (a) mean (b) median (c) mode
708. Of the 16 nurses in Doreen's department, 12 are women and 4 are men. One of the nurses will be assigned at random to work an extra shift next week. (a) Find the probability a woman nurse will be assigned the extra shift. (b) Convert the fraction to a decimal.
709. Find each unit price and then the better buy. Laundry detergent: 64 ounces for $10.99 or 48 ounces for $8.49

In the following exercises, simplify.

710. $\sqrt{36+64}$
711. $\sqrt{144n^2}$
712. Estimate $\sqrt{54}$ to between two whole numbers.
713. Yanet wants a square patio in her backyard. She has 225 square feet of tile. How long can a side of the patio be?

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5.E: The Properties of Real Numbers (Exercises)

7.1 - Rational and Irrational Numbers

In the following exercises, write as the ratio of two integers.

302. 6
303. −5
304. 2.9
305. 1.8

In the following exercises, determine which of the numbers is rational.

306. 0.42, $0.\overline{3}$, 2.56813…
307. 0.75319…, $0.\overline{16}$, 1.95

In the following exercises, identify whether each given number is rational or irrational.

308. (a) 49 (b) 55
309. (a) 72 (b) 64

In the following exercises, list the (a) whole numbers, (b) integers, (c) rational numbers, (d) irrational numbers, (e) real numbers for each set of numbers.

310. −9, 0, 0.361...., $\frac{8}{9}$, $\sqrt{16}$, 9
311. −5, $-2\frac{1}{4}$, $-\sqrt{4}$, $0.\overline{25}$, $\frac{13}{5}$, 4

7.2 - Commutative and Associative Properties

In the following exercises, use the commutative property to rewrite the given expression.

312. 6 + 4 = ____
313. −14 • 5 = ____
314. 3n = ____
315. a + 8 = ____

In the following exercises, use the associative property to rewrite the given expression.

316. (13 • 5) • 2 = _____
317. (22 + 7) + 3 = _____
318. (4 + 9x) + x = _____
319. $\frac{1}{2}$(22y) = _____

In the following exercises, evaluate each expression for the given value.

320. If y = $\frac{11}{12}$, evaluate:
    a. y + 0.7 + (− y)
    b. y + (− y) + 0.7
321. If z = $-\frac{5}{3}$, evaluate:
    a. z + 5.39 + (− z)
    b. z + (− z) + 5.39
322. If k = 65, evaluate:
    a. $\frac{4}{9}\left(\frac{9}{4}k\right)$
    b. $\left(\frac{4}{9}\cdot\frac{9}{4}\right)k$

323. If m = −13, evaluate:
   a. $-\frac{2}{5}\left(\frac{5}{2}m\right)$
   b. $\left(-\frac{2}{5}\cdot\frac{5}{2}\right)m$

In the following exercises, simplify using the commutative and associative properties.

324. 6y + 37 + (−6y)
325. $\frac{1}{4}+\frac{11}{15}+\left(-\frac{1}{4}\right)$
326. $\frac{14}{11}\cdot\frac{35}{9}\cdot\frac{14}{11}$
327. −18 • 15 • $\frac{2}{9}$
328. $\left(\frac{7}{12}+\frac{4}{5}\right)+\frac{1}{5}$
329. (3.98d + 0.75d) + 1.25d
330. −12(4m)
331. $30\left(\frac{5}{6}q\right)$
332. 11x + 8y + 16x + 15y
333. 52m + (−20n) + (−18m) + (−5n)

7.3 - Distributive Property

In the following exercises, simplify using the distributive property.

334. 7(x + 9)
335. 9(u − 4)
336. −3(6m − 1)
337. −8(−7a − 12)
338. $\frac{1}{3}$(15n − 6)
339. (y + 10) • p
340. (a − 4) − (6a + 9)
341. 4(x + 3) − 8(x − 7)

In the following exercises, evaluate using the distributive property.

342. If u = 2, evaluate
   a. 3(8u + 9) and
   b. 3 • 8u + 3 • 9 to show that 3(8u + 9) = 3 • 8u + 3 • 9
343. If n = 7 8 , evaluate
   a. $8\left(n+\frac{1}{4}\right)$ and
   b. 8 • n + 8 • $\frac{1}{4}$ to show that $8\left(n+\frac{1}{4}\right)$ = 8 • n + 8 • $\frac{1}{4}$
344. If d = 14, evaluate
   a. −100(0.1d + 0.35) and
   b. −100 • (0.1d) + (−100)(0.35) to show that −100(0.1d + 0.35) = −100 • (0.1d) + (−100)(0.35)
345. If y = −18, evaluate
   a. −(y − 18) and
   b. −y + 18 to show that −(y − 18) = − y + 18

7.4 - Properties of Identities, Inverses, and Zero

In the following exercises, identify whether each example is using the identity property of addition or multiplication.

346. $-35(1) = -35$
347. $29 + 0 = 29$
348. $(6x + 0) + 4x = 6x + 4x$
349. $9 \cdot 1 + (-3) = 9 + (-3)$

In the following exercises, find the additive inverse.

350. -32
351. 19.4
352. $\frac{3}{5}$
353. $-\frac{7}{15}$

In the following exercises, find the multiplicative inverse.

354. $\frac{9}{2}$
355. -5
356. $\frac{1}{10}$
357. $-\frac{4}{9}$

In the following exercises, simplify.

358. $83 \cdot 0$
359. $\frac{0}{9}$
360. $\frac{5}{0}$
361. $0 \div \frac{2}{3}$
362. $43 + 39 + (-43)$
363. $(n + 6.75) + 0.25$
364. $\frac{5}{13} \cdot 57 \cdot \frac{13}{5}$
365. $\frac{1}{6} \cdot 17 \cdot 12$
366. $\frac{2}{3} \cdot 28 \cdot \frac{3}{7}$
367. $9(6x - 11) + 15$

7.5 - Systems of Measurement

In the following exercises, convert between U.S. units. Round to the nearest tenth.

368. A floral arbor is 7 feet tall. Convert the height to inches.
369. A picture frame is 42 inches wide. Convert the width to feet.
370. Kelly is 5 feet 4 inches tall. Convert her height to inches.
371. A playground is 45 feet wide. Convert the width to yards.
372. The height of Mount Shasta is 14,179 feet. Convert the height to miles.
373. Shamu weighs 4.5 tons. Convert the weight to pounds.
374. The play lasted $1\frac{3}{4}$ hours. Convert the time to minutes.
375. How many tablespoons are in a quart?
376. Naomi's baby weighed 5 pounds 14 ounces at birth. Convert the weight to ounces.
377. Trinh needs 30 cups of paint for her class art project. Convert the volume to gallons.

In the following exercises, solve, and state your answer in mixed units.

378. John caught 4 lobsters. The weights of the lobsters were 1 pound 9 ounces, 1 pound 12 ounces, 4 pounds 2 ounces, and 2 pounds 15 ounces. What was the total weight of the lobsters?
379. Every day last week, Pedro recorded the amount of time he spent reading. He read for 50, 25, 83, 45, 32, 60, and 135 minutes. How much time, in hours and minutes, did Pedro spend reading?
380. Fouad is 6 feet 2 inches tall. If he stands on a rung of a ladder 8 feet 10 inches high, how high off the ground is the top of Fouad's head?
381. Dalila wants to make pillow covers. Each cover takes 30 inches of fabric. How many yards and inches of fabric does she need for 4 pillow covers?

In the following exercises, convert between metric units.

382. Donna is 1.7 meters tall. Convert her height to centimeters.
383. Mount Everest is 8,850 meters tall. Convert the height to kilometers.
384. One cup of yogurt contains 488 milligrams of calcium. Convert this to grams.
385. One cup of yogurt contains 13 grams of protein. Convert this to milligrams.
386. Sergio weighed 2.9 kilograms at birth. Convert this to grams.
387. A bottle of water contained 650 milliliters. Convert this to liters.

In the following exercises, solve.

388. Minh is 2 meters tall. His daughter is 88 centimeters tall. How much taller, in meters, is Minh than his daughter?
389. Selma had a 1-liter bottle of water. If she drank 145 milliliters, how much water, in milliliters, was left in the bottle?
390. One serving of cranberry juice contains 30 grams of sugar. How many kilograms of sugar are in 30 servings of cranberry juice?
391. One ounce of tofu provides 2 grams of protein. How many milligrams of protein are provided by 5 ounces of tofu?

In the following exercises, convert between U.S. and metric units. Round to the nearest tenth.

392. Majid is 69 inches tall. Convert his height to centimeters.
393. A college basketball court is 84 feet long. Convert this length to meters.
394. Caroline walked 2.5 kilometers. Convert this length to miles.
395. Lucas weighs 78 kilograms. Convert his weight to pounds.
396. Steve's car holds 55 liters of gas. Convert this to gallons.
397. A box of books weighs 25 pounds. Convert this weight to kilograms.

In the following exercises, convert the Fahrenheit temperatures to degrees Celsius. Round to the nearest tenth.

398. 95°F
399. 23°F
400. 20°F
401. 64°F

In the following exercises, convert the Celsius temperatures to degrees Fahrenheit. Round to the nearest tenth.

402. 30°C
403. −5°C
404. −12°C
405. 24°C

PRACTICE TEST

406. For the numbers 0.18349…, $0.\overline{2}$, 1.67, list the (a) rational numbers and (b) irrational numbers.
407. Is $\sqrt{144}$ rational or irrational?
408. From the numbers −4, $-1\frac{1}{2}$, 0, $\frac{5}{8}$, $\sqrt{2}$, 7, which are (a) integers (b) rational (c) irrational (d) real numbers?
409. Rewrite using the commutative property: x • 14 = _____
410. Rewrite the expression using the associative property: (y + 6) + 3 = _____
411. Rewrite the expression using the associative property: (8 · 2) · 5 = _____

412. Evaluate $\frac{3}{16}\left(\frac{16}{3}n\right)$ when n = 42.

413. For the number $\frac{2}{5}$ find the (a) additive inverse (b) multiplicative inverse.

In the following exercises, simplify the given expression.

414. $\frac{3}{4}(-29)\left(\frac{4}{3}\right)$

415. −3 + 15y + 3

416. (1.27q + 0.25q) + 0.75q

417. $\left(\frac{8}{15}+\frac{2}{9}\right)+\frac{7}{9}$

418. $-18\left(\frac{3}{2}n\right)$

419. 14y + (−6z) + 16y + 2z

420. 9(q + 9)

421. 6(5x − 4)

422. −10(0.4n + 0.7)

423. $\frac{1}{4}$(8a + 12)

424. m(n + 2)

425. 8(6p − 1) + 2(9p + 3)

426. (12a + 4) − (9a + 6)

427. $\frac{0}{8}$

428. $\frac{4.5}{0}$

429. $0 \div \left(\frac{2}{3}\right)$

In the following exercises, solve using the appropriate unit conversions.

430. Azize walked $4\frac{1}{2}$ miles. Convert this distance to feet. (1 mile = 5,280 feet).

431. One cup of milk contains 276 milligrams of calcium. Convert this to grams. (1 milligram = 0.001 gram)

432. Larry had 5 phone customer phone calls yesterday. The calls lasted 28, 44, 9, 75, and 55 minutes. How much time, in hours and minutes, did Larry spend on the phone? (1 hour = 60 minutes)

433. Janice ran 15 kilometers. Convert this distance to miles. Round to the nearest hundredth of a mile. (1 mile = 1.61 kilometers)

434. Yolie is 63 inches tall. Convert her height to centimeters. Round to the nearest centimeter. (1 inch = 2.54 centimeters)

435. Use the formula F = $\frac{9}{5}$C + 32 to convert 35°C to degrees F.

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5.S: Decimals (Summary)

Key Terms

circumference of a circle	The distance around a circle
diameter of a circle	A line segment that passes through a circle's center connecting two points on the circle.
equivalent decimals	Two decimals are equivalent decimals if they convert to equivalent fractions.
mean	The mean of a set of n numbers is the arithmetic average of the numbers. $$mean = \frac{sum\ of\ values\ in\ data\ set}{n} \quad (5.S.1)$$
median	The median of a set of data values is the middle value. Half the data values are less than or equal to the median. Half the data values are greater than or equal to the median.
mode	The mode of a set of numbers is the number with the highest frequency.
radius of a circle	A line segment from the center to any point on the circle.
rate	A comparison of two quantities with different units. A rate is usually written as a fraction.
ratio	A comparison of two numbers or two quantities that are measured with the same unit. The ratio of a to b is written a to b, $\frac{a}{b}$, or a : b.
repeating decimal	A decimal in which the last digit or group of digits repeats endlessly.
unit price	A unit rate that gives the price of one item.
unit rate	A rate with denominator of 1 unit.

Key Concepts

5.1 - Decimals

- **Name a decimal number.**
  1. Name the number to the left of the decimal point.
  2. Write "and" for the decimal point.
  3. Name the "number" part to the right of the decimal point as if it were a whole number.
  4. Name the decimal place of the last digit.
- **Write a decimal number from its name.**
  1. Look for the word "and"—it locates the decimal point. Place a decimal point under the word "and." Translate the words before "and" into the whole number and place it to the left of the decimal point. If there is no "and," write a "0" with a decimal point to its right.
  2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
  3. Translate the words after "and" into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.
  4. Fill in zeros for place holders as needed.
- **Convert a decimal number to a fraction or mixed number.**

1. Look at the number to the left of the decimal. If it is zero, the decimal converts to a proper fraction. If it is not zero, the decimal converts to a mixed number. Write the whole number.
2. Determine the place value of the final digit.
3. Write the fraction. numerator—the ‘numbers’ to the right of the decimal point denominator—the place value corresponding to the final digit
4. Simplify the fraction, if possible.

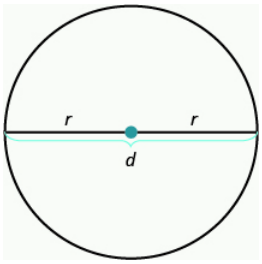
- **Order decimals.**
  1. Check to see if both numbers have the same number of decimal places. If not, write zeros at the end of the one with fewer digits to make them match.
  2. Compare the numbers to the right of the decimal point as if they were whole numbers.
  3. Order the numbers using the appropriate inequality sign.
- Round a decimal.
  1. Locate the given place value and mark it with an arrow.
  2. Underline the digit to the right of the given place value.
  3. Is this digit greater than or equal to 5? Yes - add 1 to the digit in the given place value. No - do not change the digit in the given place value
  4. Rewrite the number, removing all digits to the right of the given place value.

5.2 - Decimal Operations

- **Add or subtract decimals.**
  1. Write the numbers vertically so the decimal points line up.
  2. Use zeros as place holders, as needed.
  3. Add or subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.
- **Multiply decimal numbers.**
  1. Determine the sign of the product.
  2. Write the numbers in vertical format, lining up the numbers on the right.
  3. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
  4. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors. If needed, use zeros as placeholders.
  5. Write the product with the appropriate sign.
- **Multiply a decimal by a power of 10.**
  1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
  2. Write zeros at the end of the number as placeholders if needed.
- **Divide a decimal by a whole number.**
  1. Write as long division, placing the decimal point in the quotient above the decimal point in the dividend.
  2. Divide as usual.
- **Divide decimal numbers.**
  1. Determine the sign of the quotient.
  2. Make the divisor a whole number by moving the decimal point all the way to the right. Move the decimal point in the dividend the same number of places to the right, writing zeros as needed.
  3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
  4. Write the quotient with the appropriate sign.
- **Strategy for Applications**
  1. Identify what you are asked to find.
  2. Write a phrase that gives the information to find it.
  3. Translate the phrase to an expression.
  4. Simplify the expression.
  5. Answer the question with a complete sentence.

5.3 - Decimals and Fractions

- **Convert a Fraction to a Decimal**: To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.
- **Properties of Circles**



r is the length of the radius

d is the length of the diameter

The circumference is $2\pi r$.

$$C = 2\pi r \tag{5.S.2}$$

The area is πr^2.

$$A = \pi r^2 \tag{5.S.3}$$

5.4 - Solve Equations with Decimals

- **Determine whether a number is a solution to an equation.**
  - Substitute the number for the variable in the equation.
  - Simplify the expressions on both sides of the equation.
  - Determine whether the resulting equation is true. If so, the number is a solution. If not, the number is not a solution.
- **Properties of Equality**

Table $5.S.1$

Subtraction Property of Equality For any numbers a, b, and c, if a = b then a − c = b − c.	**Addition Property of Equality** For any numbers a, b, and c, if a = b then a + c = b + c.
Division of Property of Equality For any numbers a, b, and c ≠ 0, if a = b then \(\dfrac{a}{c} = \dfrac{b}{c}).	**Multiplication Property of Equality** For any numbers a, b, and c, if a = b then a • c = b • c.

5.5 - Averages and Probability

- **Calculate the mean of a set of numbers.**
  1. Write the formula for the mean

$$mean = \frac{sum\ of\ values\ in\ data\ set}{n} \tag{5.S.4}$$

  2. Find the sum of all the values in the set. Write the sum in the numerator.
  3. Count the number, n, of values in the set. Write this number in the denominator.
  4. Simplify the fraction.
  5. Check to see that the mean is reasonable. It should be greater than the least number and less than the greatest number in the set.
- **Find the median of a set of numbers.**
  1. List the numbers from least to greatest.
  2. Count how many numbers are in the set. Call this n.
  3. Is n odd or even? If n is an odd number, the median is the middle value. If n is an even number, the median is the mean of the two middle values.

- **Identify the mode of a set of numbers.**
  1. List the data values in numerical order.
  2. Count the number of times each value appears.
  3. The mode is the value with the highest frequency.

5.7 - Simplify and Use Square Roots

- **Square Root Notation**: $\sqrt{m}$ is read 'the square root of m '
  - If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

- **Use a strategy for applications with square roots.**
  - Identify what you are asked to find.
  - Write a phrase that gives the information to find it.
  - Translate the phrase to an expression.
  - Simplify the expression.
  - Write a complete sentence that answers the question.

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5.S: The Properties of Real Numbers (Summary)

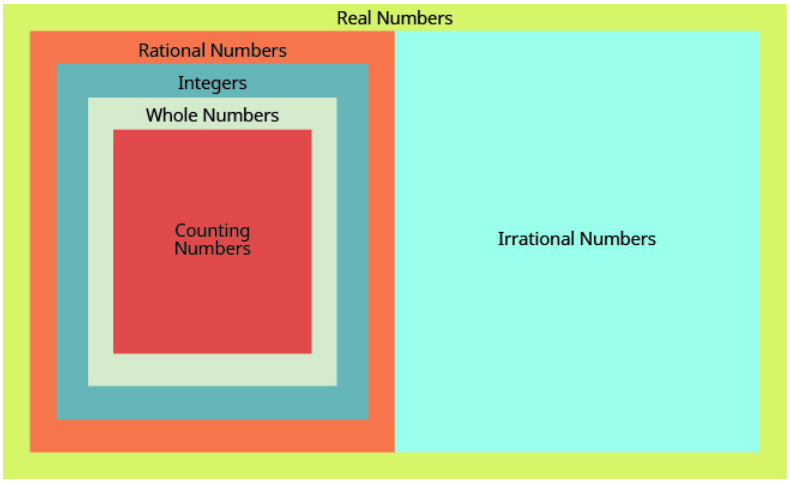
Key Terms

Additive Identity	The additive identity is 0. When zero is added to any number, it does not change the value.
Additive Inverse	The opposite of a number is its additive inverse. The additive inverse of a is $-a$.
Irrational number	A number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.
Multiplicative Identity	The multiplicative identity is 1. When one multiplies any number, it does not change the value.
Multiplicative Inverse	The reciprocal of a number is its multiplicative inverse. The multiplicative inverse of a is $\frac{1}{a}$.
Rational number	A number that can be written in the form $\frac{p}{q}$, where p and q are integers and q ≠ 0. Its decimal form stops or repeats.
Real number	A number that is either rational or irrational.

Key Concepts

7.1 - Rational and Irrational Numbers

- Real numbers



7.2 - Commutative and Associative Properties

- Commutative Properties
  - **Commutative Property of Addition**: If a, b are real numbers, then a + b = b + a
  - **Commutative Property of Multiplication**: If a, b are real numbers, then a • b = b • a
- Associative Properties
  - **Associative Property of Addition**: If a, b, c are real numbers then (a + b) + c = a + (b + c)
  - **Associative Property of Multiplication**: If a, b, c are real numbers then (a • b) • c = a • (b • c)

7.3 - Distributive Property

- **Distributive Property**:
  - If a, b, c are real numbers then

- a(b + c) = ab + ac
- (b + c)a = ba + ca
- a(b - c) = ab - ac

7.4 - Properties of Identity, Inverses, and Zero

- Identity Properties
  - **Identity Property of Addition**: For any real number a: a + 0 = a, 0 + a = a
    - 0 is the additive identity
  - **Identity Property of Multiplication**: For any real number a: a • 1 = a, 1 • a = a
    - 1 is the multiplicative identity
- Inverse Properties
  - **Inverse Property of Addition**: For any real number a: a + (- a) = 0
    - - a is the additive inverse of a
  - **Inverse Property of Multiplication**: For any real number a: (a ≠ 0) a • $\frac{1}{a}$ = 1
    - $\frac{1}{a}$ is the multiplicative inverse of a
- Properties of Zero
  - **Multiplication by Zero**: For any real number a, a • 0 = 0, 0 • a = 0
    - The product of any number and 0 is 0.
  - **Division of Zero**: For any real number a, $\frac{0}{a} = 0$, $0 \div a = 0$
    - Zero divided by any real number, except itself, is zero.
  - **Division by Zero**: For any real number a, $\frac{a}{0}$ is undefined and a ÷ 0 is undefined.
    - Division by zero is undefined.

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CHAPTER OVERVIEW

6: Ratio, Proportion, and Measurement

6.1: Ratio and Proportion
6.2: Introduction to Ratios and Rates
6.3: Introduction to Proportion
6.4: Unit Conversion - American System
6.5: Unit Conversion- Metric System
6.6: American Units to Metric Units and Vice-Versa
6.7: Use a Problem Solving Strategy (Part 1)
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6.S: Math Models and Geometry (Summary)

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6.1: Ratio and Proportion

From the beginnings of the human race, we've long compared one quantity with another, a comparison that is called a ratio in mathematics. "Their tribe has twice as many cattle as ours" or "Two baskets of wheat cost 12 ducats" are examples of ratios that ring from distant times. Indeed, the concept of a ratio cannot be assigned to any one individual or class of individual. In his *History of Mathematics*, D. E. Smith writes:

> *It is rather profitless to speculate as to the domain in which the concept of ratio first appeared. The idea that one tribe is twice as large as another and the idea that one leather strap is only half as long as another both involve the notion of ratio; both are such as would develop early in the history of the race, and yet one has to do with ratio of numbers and the other with the ratio of geometric magnitudes. Indeed, when we come to the Greek writers we find Nicomachus including ratio in his arithmetic, Eudoxus in his geometry, and Theon of Smyrna in his chapter on music.*

Examples and applications of ratios are limitless: speed is a ratio that compares changes in distance with respect to time, acceleration is a ratio that compares changes in speed with respect to time, and percentages compare the part with the whole. We've already studied one classic ratio, the ratio of the circumference of a circle to its diameter, which gives us the definition of π.

One of the most famous ratios in history involves the division of a line segment AB into two segments AC and CB by selecting a point C on the segment AB.

The idea is to select a point C on the segment AB so that

$$\frac{AB}{AC} = \frac{AC}{CB}.$$

This ratio has a special name, the **Golden Ratio**, and has an exact value equal to $(1+\sqrt{5})/2$. The Golden Ratio has been known since the time of Euclid. Ancient and modern architects have long held that the most pleasing rectangular shape is the one whose ratio of length to width is equal to the Golden Ratio.

The comparison of two ratios, such as $AB/AC = AC/CB$, is called a *proportion*. Proportions are used in a number of practical ways. For example, if 5 cans of tomato sauce cost 2 dollars, we can find the number of cans that can be purchased with 10 dollars by comparing two ratios in a proportion:

$$\frac{5 \text{ cans of tomato sauce}}{2 \text{ dollars}} = \frac{\text{x cans of tomato sauce}}{10 \text{ dollars}}$$

Any discussion of ratio involves comparing two quantities, so the units of each quantity become extremely important. Two different systems of units are used when measuring length, capacity, and time: the *American* system of units and the *metric* system of units. In this chapter we will discuss both systems and explain how to convert quantities measured in one system to quantities measured in the other system.

Let's begin the journey.

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6.2: Introduction to Ratios and Rates

We use *ratios* to compare two numeric quantities or quantities with the same units.

Ratio

A *ratio* is the quotient of two numerical quantities or two quantities with the same physical units.

For example, ancient Greek geometers believed that the most pleasing rectangle to the eye had length and width such that the ratio of length to width was a specific number, called the *Golden Ratio*, approximately equal to 1.6180339887.... Architects to this day use this ratio in their designs.

There are a number of equivalent ways of expressing ratios, three of which we will use in this text: fraction notation, “to” notation, and “colon” notation.

- 3/4 is a ratio, read as “the ratio of 3 to 4.”
- 3 to 4 is a ratio, read as “the ratio of 3 to 4.”
- 3:4 is a ratio, read as “the ratio of 3 to 4.”

Example 1

Express each of the following ratios as a fraction reduced to lowest terms: (a) 36 to 24, and (b) 0.12 : 0.18.

Solution

(a) To express the ratio “36 to 24” as a fraction, place 36 over 24 and reduce.

$$\begin{aligned}\frac{36}{24} &= \frac{3\cdot 12}{2\cdot 12} && \text{Factor.}\\ &= \frac{3\cdot \cancel{12}}{2\cdot \cancel{12}} && \text{Cancel common factor.}\\ &= \frac{3}{2}\end{aligned}$$

Thus, the ratio 36 to 24 equals 3/2.

(b) To express the ratio “0.12:0.18” as a fraction, place 0.12 over 0.18 and reduce.

$$\begin{aligned}\frac{0.12}{0.18} &= \frac{(0.18)(100)}{(0.18)(100)} && \text{Multiply numerator and denominator by 100.}\\ &= \frac{12}{18} && \text{Move each decimal 2 places right.}\\ &= \frac{2\cdot 6}{3\cdot 6} && \text{Factor.}\\ &= \frac{2\cdot \cancel{6}}{3\cdot \cancel{6}} && \text{Cancel.}\\ &= \frac{2}{3}\end{aligned}$$

Thus, the ratio 0.12:0.18 equals 2/3.

Exercise

Express 0.12 : 0.3 as a fraction reduced to lowest terms.

Answer

2/5

Example 2

For the rectangle that follows, express the ratio of length to width as a fraction reduced to lowest terms.

Solution.

The ratio length to width can be expressed as a fraction and reduced as follows.

$$\frac{\text{length}}{\text{width}} = \frac{3\frac{1}{4}\text{ ft}}{2\frac{1}{2}\text{ ft}} \quad \text{Length to width as a fraction.}$$

$$= \frac{3\frac{1}{4}\ \cancel{\text{ft}}}{2\frac{1}{2}\ \cancel{\text{ft}}} \quad \text{Cancel common units.}$$

$$= \frac{\frac{13}{4}}{\frac{5}{2}} \quad \text{Mixed to improper fractions.}$$

Invert and multiply, factor, and cancel common factors.

$$= \frac{13}{4} \cdot \frac{2}{5} \quad \text{Invert and multiply.}$$

$$= \frac{26}{20} \quad \text{Multiply numerators and denominators.}$$

$$= \frac{13 \cdot 2}{10 \cdot 2} \quad \text{Factor numerator and denominator.}$$

$$= \frac{13 \cdot \cancel{2}}{10 \cdot \cancel{2}} \quad \text{Cancel common factors.}$$

$$= \frac{13}{10}$$

Hence, the ratio length to width is 13/10.

Exercise

A rectangle has length $8\frac{1}{4}$ inches and width $3\frac{1}{2}$ inches. Express the ratio of length to width as a fraction reduced to lowest terms.

Answer

33/14

Rates

We now introduce the concept of *rate*, a special type of ratio.

Rate

A *rate* is a quotient of two measurements with different units.

The physical interpretation of a rate in terms of its units is an important skill.

Example 3

An automobile travels 224 miles on 12 gallons of gasoline. Express the ratio distance traveled to gas consumption as a fraction reduced to lowest terms. Write a short sentence explaining the physical significance of your solution. Include units in your description.

Solution

Place miles traveled over gallons of gasoline consumed and reduce.

$$\begin{aligned} \frac{224 \text{ mi}}{12 \text{ gal}} &= \frac{56 \cdot 4 \text{ mi}}{3 \cdot 4 \text{ gal}} && \text{Factor.} \\ &= \frac{56 \cdot \cancel{4} \text{ mi}}{3 \cdot \cancel{4} \text{ gal}} && \text{Cancel common factor.} \\ &= \frac{56 \text{ mi}}{3 \text{ gal}} \end{aligned}$$

Thus, the rate is 56 miles to 3 gallons of gasoline. In plain-speak, this means that the automobile travels 56 miles on 3 gallons of gasoline.

Exercise

Lanny travels 180 kilometers on 14 liters of gasoline. Express the ratio distance traveled to gas consumption as a fraction reduced to lowest terms.

Answer

90/7 kilometers per litre

Unit Rates

When making comparisons, it is helpful to have a rate in a form where the denominator is 1. Such rates are given a special name.

Unit Rate

A *unit rate* is a rate whose denominator is 1.

Example 4

Herman drives 120 miles in 4 hours. Find his average rate of speed.

Solution

Place the distance traveled over the time it takes to drive that distance.

$$\begin{aligned} \frac{120 \text{ miles}}{4 \text{ hours}} &= \frac{30 \text{ miles}}{1 \text{ hour}} && \text{Divide: } 120/4 = 30. \\ &= 30 \text{ miles/hour} \end{aligned}$$

Hence, Herman's average rate of speed is 30 miles per hour.

Exercise

Jacob drives 120 kilometers in 3 hours. Find his average rate of speed.

Answer

40 kilometers per hour

Example 5

Aditya works 8.5 hours and receives $95 for his efforts. What is his hourly salary rate?

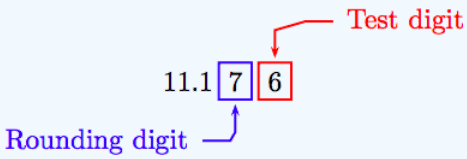
Solution

Let's place money earned over hours worked to get the following rate:

$$\frac{95 \text{ dollars}}{8.5 \text{ hours}}$$

We will get a much better idea of Aditya's salary rate if we express the rate with a denominator of 1. To do so, divide. Push the decimal in the divisor to the far right, then move the decimal an equal number of places in the dividend. As we are dealing with dollars and cents, we will round our answer to the nearest hundredth.

$$\begin{array}{r} 11.176 \\ 85\overline{)950.000} \\ 85 \\ \hline 100 \\ 85 \\ \hline 15\,0 \\ 8\,5 \\ \hline 6\,50 \\ 5\,95 \\ \hline 550 \\ 510 \\ \hline 40 \end{array}$$



Because the test digit is greater than or equal to 5, we add 1 to the rounding digit and truncate; i.e., 95/8.5 ≈ 11.18. Hence,

$$\begin{aligned} \frac{95 \text{ dollars}}{8.5 \text{ hours}} &= \frac{11.18 \text{ dollars}}{1 \text{ hour}} \quad \text{Divide: } 95/8.5 \approx 11.18. \\ &= 11.18 \text{ dollars/hour}. \end{aligned}$$

That is, his salary rate is 11.18 dollars per hour.

Exercise

Frannie works 5.5 hours and receives $120 for her efforts. What is her hourly salary rate? Round your answer to the nearest penny.

Answer

$21.82 per hour

Example 6

One automobile travels 422 miles on 15 gallons of gasoline. A second automobile travels 354 miles on 13 gallons of gasoline. Which automobile gets the better gas mileage?

Solution

Decimal division (rounded to the nearest tenth) reveals the better gas mileage.

In the case of the first automobile, we get the following rate:

$$\frac{422\text{ mi}}{15\text{gal}}$$

Divide.

$$\begin{array}{r} 28.13 \\ 15\overline{)422.00} \\ \underline{30} \\ 122 \\ \underline{120} \\ 20 \\ \underline{15} \\ 50 \\ \underline{45} \\ 5 \end{array}$$

To the nearest tenth, 28.1.

In the case of the second automobile, we get the following rate:

$$\frac{354\text{ mi}}{13\text{ gal}}$$

Divide.

$$\begin{array}{r} 27.23 \\ 13\overline{)354.00} \\ \underline{26} \\ 94 \\ \underline{91} \\ 30 \\ \underline{26} \\ 40 \\ \underline{39} \\ 1 \end{array}$$

To the nearest tenth, 27.2.

In the case of the first automobile, the mileage rate is 28.1 mi/1 gal, which can be read "28.1 miles per gallon." In the case of the second automobile, the mileage rate is 27.2 mi/1 gal, which can be read "27.2 miles per gallon." Therefore, the first automobile gets the better gas mileage.

? Exercise

Alicia works 8 hours and makes $100. Connie works 10 hours and makes $122. Which woman works at the larger hourly rate?

Answer

Alicia

Exercises

In Exercises 1-24, express the given ratio as a fraction reduced to lowest terms.

1. 0.14 : 0.44
2. 0.74 : 0.2
3. 0.05 : 0.75
4. 0.78 : 0.4
5. 0.1:0.95
6. 0.93 : 0.39
7. $2\frac{2}{9} : 1\frac{1}{3}$
8. $3\frac{2}{3} : 2\frac{4}{9}$
9. 0.36 : 0.6
10. 0.58 : 0.42
11. 15 : 21
12. 77 : 121
13. $2\frac{8}{9} : 2\frac{2}{3}$
14. $1\frac{2}{3} : 3\frac{8}{9}$

15. $3\frac{8}{9} : 2\frac{1}{3}$

16. $1\frac{5}{9} : 1\frac{1}{3}$

17. $2\frac{5}{8} : 1\frac{3}{4}$

18. $2\frac{4}{9} : 1\frac{1}{3}$

19. 10 : 35

20. 132 : 84

21. 9 : 33

22. 35 : 10

23. 27 : 99

24. 12 : 28

---

25. One automobile travels 271.8 miles on 10.1 gallons of gasoline. A second automobile travels 257.9 miles on 11.1 gallons of gasoline. Which automobile gets the better gas mileage?

26. One automobile travels 202.9 miles on 13.9 gallons of gasoline. A second automobile travels 221.6 miles on 11.8 gallons of gasoline. Which automobile gets the better gas mileage?

27. Todd is paid 183 dollars for 8.25 hours work. What is his hourly salary rate, rounded to the nearest penny?

28. David is paid 105 dollars for 8.5 hours work. What is his hourly salary rate, rounded to the nearest penny?

29. An automobile travels 140 miles in 4 hours. Find the average rate of speed.

30. An automobile travels 120 miles in 5 hours. Find the average rate of speed.

31. Judah is paid 187 dollars for 8 hours work. What is his hourly salary rate, rounded to the nearest penny?

32. Judah is paid 181 dollars for 8.75 hours work. What is his hourly salary rate, rounded to the nearest penny?

33. One automobile travels 234.2 miles on 10.8 gallons of gasoline. A second automobile travels 270.5 miles on 10.8 gallons of gasoline. Which automobile gets the better gas mileage?

34. One automobile travels 297.6 miles on 10.7 gallons of gasoline. A second automobile travels 298.1 miles on 12.6 gallons of gasoline. Which automobile gets the better gas mileage?

35. An automobile travels 180 miles in 5 hours. Find the average rate of speed.

36. An automobile travels 220 miles in 5 hours. Find the average rate of speed.

---

37. **Antarctic trek.** Seven women on a 562-mile Antarctic ski trek reached the South Pole 38 days after they began their adventure. What was the ladies' average rate of speed per day? Round your result to the nearest tenth of a mile. *Associated Press-Times-Standard 12/31/09 After 562-mile ski trek, seven women reach the South Pole.*

Answers

1. $\frac{7}{22}$
3. $\frac{1}{15}$
5. $\frac{2}{19}$
7. $\frac{5}{3}$
9. $\frac{3}{5}$
11. $\frac{5}{7}$
13. $\frac{13}{12}$
15. $\frac{5}{3}$

17. $\frac{3}{2}$
19. $\frac{2}{7}$
21. $\frac{3}{11}$
23. $\frac{3}{11}$
25. The first automobile has the better mileage per gallon.
27. 22.18 dollars/hr
29. 35 mi/hr
31. 23.38 dollars/hr
33. The second automobile has the better mileage per gallon.
35. 36 mi/hr
37. 14.8 miles per day

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6.3: Introduction to Proportion

In Section 6.1, we introduced the concepts or ratio and rate. In this section, we equate these ratios in a construct called a *proportion*.

Proportions

A *proportion* is a statement that equates two ratios or rates.

For example, each of the equations

$$\frac{1}{3}=\frac{2}{6},\ \frac{15\text{ miles}}{2\text{ hours}}=\frac{30\text{ miles}}{4\text{ hours}},\ \text{and}\ \frac{a}{b}=\frac{c}{d}$$

compare two ratios or rates and is a proportion.

The proportion

$$\frac{1}{3}=\frac{2}{6}$$

is read "one is to three as two is to six." The four numbers that make up this proportion are called the *terms* of the proportion and are ordered in a natural manner.

$$\begin{array}{rcl} \text{First term} \longrightarrow & \frac{1}{3} = \frac{2}{6} & \longleftarrow \text{Third term} \\ \text{Second term} \longrightarrow & & \longleftarrow \text{Fourth term} \end{array}$$

Extremes and Means

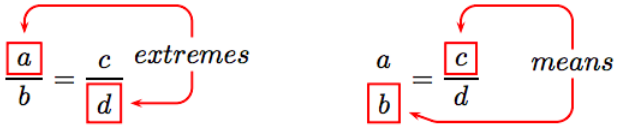
The first and fourth terms are called the *extremes* of the proportion. The second and third terms are called the *means* of the proportion.

In the proportion

$$\frac{a}{b}=\frac{c}{d},$$

the terms a and d are the extremes; the terms b and c are the means.

$$\frac{\boxed{a}}{b}=\frac{c}{\boxed{d}}\quad \textit{extremes} \qquad\qquad \frac{a}{\boxed{b}}=\frac{\boxed{c}}{d}\quad \textit{means}$$



If we multiply both sides of the proportion by the common denominator,

$$bd\left(\frac{a}{b}\right)=bd\left(\frac{c}{d}\right)$$

then cancel,

$$\cancel{b}d\left(\frac{a}{\cancel{b}}\right)=b\,\cancel{d}\left(\frac{c}{\cancel{d}}\right)$$

we get the following result.

$$ad=bc$$

This leads to the following observation.

Product of Extremes and Means

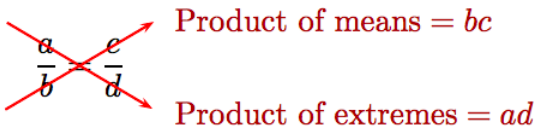
In the proportion

$$\frac{a}{b} = \frac{c}{d}$$

the product of the means equals the product of the extremes. That is,

$$ad = bc.$$

We can get an equivalent result using a technique called *cross multiplication*.

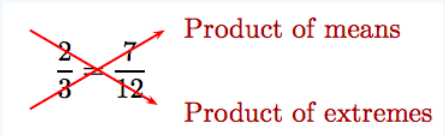


Example 1

Which of the following is a valid proportion: (a) $\frac{2}{3} = \frac{7}{12}$, or (b) $\frac{4}{9} = \frac{12}{27}$.

Solution

(a) Cross multiply

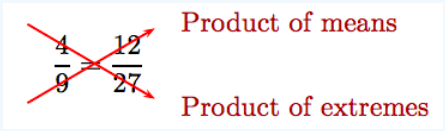


to get

$$24 = 21.$$

Hence, the product of the extremes does not equal the product of the means, so 2/3=7/12 is **not** a valid proportion.

(b) Cross multiply



to get

$$108 = 108.$$

Hence, the product of the extremes equals the product of the means, so 4/9 = 12/27 **is** a valid proportion.

Exercise

Is the following a valid proportion? $\frac{4}{3} = \frac{16}{11}$

Answer

No

Solving Proportions

We already have all the tools needed to solve proportions. Let's begin with the first example.

Example 2

Solve the proportion for x: $\frac{3}{4} = \frac{x}{12}$.

Solution

Cross multiply, then solve the resulting equation.

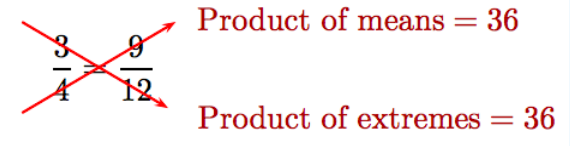
$$\begin{aligned} \frac{3}{4} &= \frac{x}{12} && \text{Original proportion.} \\ 4 \cdot x &= 3 \cdot 12 && \text{Products of means and extremes are equal.} \\ 4x &= 36 && \text{Simplify.} \\ \frac{4x}{4} &= \frac{36}{4} && \text{Divide both sides by 4.} \\ x &= 9 && \text{Simplify.} \end{aligned}$$

Check

Substitute 9 for x into the original proportion and check.

$$\begin{aligned} \frac{3}{4} &= \frac{x}{12} && \text{Original proportion.} \\ \frac{3}{4} &= \frac{9}{12} && \text{Substitute 9 for } x. \end{aligned}$$

Cross multiply.



Thus, the solution 9 checks.

Exercise

Solve the proportion for n: $\frac{2}{3} = \frac{n}{9}$

Answer

6

Example 3

Solve the proportion for n: $\frac{3}{2} = \frac{24}{n}$.

Solution

Cross multiply, then solve the resulting equation.

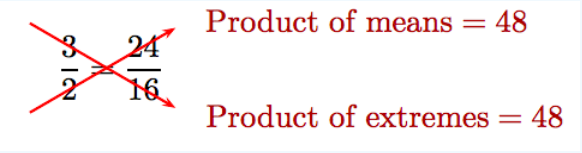
$$\begin{aligned} \frac{3}{2} &= \frac{24}{n} && \text{Original proportion.} \\ 3 \cdot n &= 2 \cdot 24 && \text{Products of means and extremes are equal.} \\ 3n &= 48 && \text{Simplify.} \\ \frac{3n}{3} &= \frac{48}{3} && \text{Divide both sides by 3.} \\ n &= 16 && \text{Simplify.} \end{aligned}$$

Check

Substitute 16 for n into the original proportion and check.

$$\frac{3}{2} = \frac{24}{n} \quad \text{Original proportion.}$$
$$\frac{3}{2} = \frac{24}{16} \quad \text{Substitute 16 for } n.$$

Cross multiply.



Thus, the solution 16 checks.

? Exercise

Solve the proportion for m: $\frac{9}{6} = \frac{m}{4}$

Answer

6

✔ Example 4

Solve the proportion for x: $\frac{2x+1}{15} = \frac{1}{3}$.

Solution

Cross multiply, then solve the resulting equation.

$$\frac{2x+1}{15} = \frac{1}{3} \quad \text{Original proportion.}$$
$$3(2x+1) = 15(1) \quad \text{Products f means and extremes are equal.}$$
$$6x+3 = 15 \quad \text{On the left, distribute. On the right, multiply.}$$
$$6x+3-3 = 15-3 \quad \text{Subtract 3 from both sides.}$$
$$6x = 12 \quad \text{Simplify.}$$
$$\frac{6x}{6} = \frac{12}{6} \quad \text{Divide both sides by 6.}$$
$$x = 2 \quad \text{Simplify both sides.}$$

Check

We'll leave it to our readers to check this solution.

? Exercise

Solve the proportion for y: $\frac{6+2y}{18} = \frac{8}{9}$

Answer

5

Applications

A number of practical applications involve solving a proportion.

Example 5

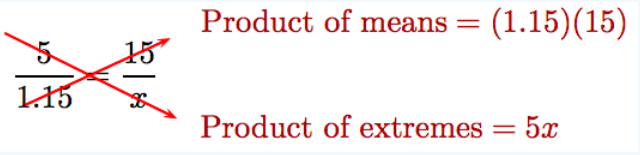
If 5 oranges cost \$1.15, what will be the cost for 15 oranges (assuming an equal rate)?

Solution

Let x represent the cost for 15 oranges. Assuming the rate for 5 oranges at \$1.15 equals the rate for 15 oranges at an unknown cost x, we set up the following proportion.

$$\frac{5}{1.15} = \frac{15}{x}$$

Cross multiply



to get

$$5x = 17.25.$$

Solve for x.

$$\frac{5x}{5} = \frac{17.25}{5}$$
$$x = 3.45$$

Thus, 15 oranges cost \$3.45.

Exercise

If 7 apples cost \$3.15, how much will 10 apples cost (assuming an equal rate)?

Answer

\$4.50

Checking Units is Extremely Important

When setting up a proportion, check to make sure that both numerators have the same units and both denominators have the same units.

For example, in Example 5, both numerators have “oranges” as units and both denominators have “dollars” as units.

$$\begin{array}{rcl} \text{Oranges} \longrightarrow & \frac{5}{1.15} = \frac{15}{x} & \longleftarrow \text{Oranges} \\ \text{Dollars} \longrightarrow & & \longleftarrow \text{Dollars} \end{array}$$

This proportion is set up correctly, because both numerators have the same units and both denominators have the same units. On the other hand, if we had set the proportion up incorrectly as follows,

$$\begin{array}{rcl} \text{Oranges} \longrightarrow & \frac{5}{1.15} = \frac{x}{15} & \longleftarrow \text{Dollars} \\ \text{Dollars} \longrightarrow & & \longleftarrow \text{Oranges} \end{array}$$

a quick check of the units reveals the error; i.e., the numerators have different units and the denominators have different units. Checking units helps us avoid errors!

Example 6

Dylan and David are planning a backpacking trip in Yosemite National Park. On their map, the legend indicates that 1.2 centimeters represents 2 miles. How long is their trip if the route measures 10.6 centimeters on the map? Round your answer to the nearest tenth of a mile.

Solution

Let's set up the proportion with units.

$$\frac{1.2 \text{ cm}}{2 \text{ mi}} = \frac{10.6 \text{ cm}}{x \text{ mi}}$$

Note how including the units aids in the setup of the proportion. Now, let's drop the units and solve for x.

$$\begin{aligned}
\frac{1.2}{2} &= \frac{10.6}{x} && \text{Original proportion.} \\
1.2x &= (2)(10.6) && \text{Cross multiply.} \\
1.2x &= 21.2 && \text{Simplify right-side hand.} \\
\frac{1.2x}{1.2} &= \frac{21.2}{1.2} && \text{Divide both sides by 1.2.} \\
x &\approx 17.66 && \text{On the right: Divide.}
\end{aligned}$$

We carried the division on the right one decimal place past the tenths place. The rounding digit is a 6 and the following test digit is a 6. Add 1 to the rounding digit and truncate.

To the nearest tenth of a mile, the backpacking route is approximately 17.7 miles.

Exercise

Eloise and Susannah are planning a trip is Sequoia National Park. On their map, 3 inches represents 50 miles. How long is their trip if the route measures $4\frac{1}{2}$ inches on the map?

Answer

75 miles

Example 7

A recipe making 2 dozen cookies requires $1\frac{3}{4}$ cups of flour, among other ingredients. If the baker wishes to make twice that number of cookies, how much flour is required?

Solution

Twice 2 dozen is 4 dozen cookies. Let x represent the amount of flour needed for 4 dozen cookies. Assuming an equal rate for 2 dozen cookies (2 dozen requires 1 3 4 cups of flour), we set up the following proportion. Again, using units helps us craft the correct proportion.

$$\frac{2 \text{ dozen}}{1\frac{3}{4} \text{ cups}} = \frac{4 \text{ dozen}}{x \text{ cups}}$$

Note how including the units aids in the setup of the proportion. Now, let's drop the units and solve for x.

$$\begin{aligned}
\frac{2}{1\frac{3}{4}} &= \frac{4}{x} && \text{Original proportion.} \\
2x &= 1\frac{3}{4} \cdot 4 && \text{Cross multiply.} \\
2x &= \frac{7}{4} \cdot 4 && \text{Change to improper fraction.} \\
2x &= 7 && \text{Multiply.}
\end{aligned}$$

Divide both sides of the equation by 2 and finish.

$$\frac{2x}{2} = \frac{7}{2} \quad \text{Divide both sides by 2.}$$
$$x = \frac{7}{2}$$

Change the improper fraction to a mixed fraction. Thus, it will take $3\frac{1}{2}$ cups of flour to make 4 dozen cookies.

? Exercise

Dough for 3 pizzas requires $8\frac{1}{2}$ cups of flour. If the baker wishes to make 9 pizzas, how many cups of flour are required?

Answer

$25\frac{1}{2}$ cups

Exercises

In Exercises 1-12, which of the following is a true proportion?

1. $\frac{9}{7} = \frac{27}{21}$, $\frac{4}{3} = \frac{9}{7}$, $\frac{7}{2} = \frac{8}{9}$, $\frac{4}{8} = \frac{9}{6}$
2. $\frac{6}{7} = \frac{18}{21}$, $\frac{2}{3} = \frac{8}{6}$, $\frac{4}{3} = \frac{3}{2}$, $\frac{8}{9} = \frac{3}{8}$
3. $\frac{7}{6} = \frac{28}{24}$, $\frac{5}{6} = \frac{5}{4}$, $\frac{9}{5} = \frac{7}{3}$, $\frac{9}{2} = \frac{8}{9}$
4. $\frac{7}{6} = \frac{2}{8}$, $\frac{4}{5} = \frac{5}{7}$, $\frac{3}{4} = \frac{15}{20}$, $\frac{8}{4} = \frac{8}{7}$
5. $\frac{6}{5} = \frac{24}{20}$, $\frac{7}{3} = \frac{2}{4}$, $\frac{2}{4} = \frac{2}{6}$, $\frac{5}{2} = \frac{2}{8}$
6. $\frac{9}{8} = \frac{4}{3}$, $\frac{5}{7} = \frac{10}{14}$, $\frac{8}{6} = \frac{5}{4}$, $\frac{8}{5} = \frac{2}{6}$
7. $\frac{3}{5} = \frac{2}{8}$, $\frac{3}{7} = \frac{6}{14}$, $\frac{5}{6} = \frac{2}{4}$, $\frac{7}{4} = \frac{5}{9}$
8. $\frac{7}{3} = \frac{7}{6}$, $\frac{4}{7} = \frac{8}{14}$, $\frac{5}{3} = \frac{7}{8}$, $\frac{5}{7} = \frac{6}{9}$
9. $\frac{5}{4} = \frac{25}{20}$, $\frac{9}{3} = \frac{9}{6}$, $\frac{7}{4} = \frac{3}{6}$, $\frac{3}{5} = \frac{9}{4}$
10. $\frac{7}{6} = \frac{6}{9}$, $\frac{7}{3} = \frac{2}{5}$, $\frac{6}{7} = \frac{30}{35}$, $\frac{4}{7} = \frac{2}{8}$
11. $\frac{9}{7} = \frac{4}{3}$, $\frac{9}{4} = \frac{7}{9}$, $\frac{3}{5} = \frac{6}{10}$, $\frac{3}{9} = \frac{9}{5}$
12. $\frac{4}{3} = \frac{8}{7}$, $\frac{2}{6} = \frac{5}{8}$, $\frac{7}{2} = \frac{3}{6}$, $\frac{9}{4} = \frac{36}{16}$

In Exercises 13-36, solve the given proportion.

13. $\frac{17}{3} = \frac{x}{18}$
14. $\frac{16}{5} = \frac{x}{20}$
15. $\frac{6x+10}{6} = \frac{11}{3}$
16. $\frac{4x+8}{12} = \frac{5}{3}$
17. $\frac{17}{9} = \frac{x}{18}$
18. $\frac{8}{9} = \frac{x}{18}$
19. $\frac{11}{2} = \frac{x}{8}$
20. $\frac{11}{4} = \frac{x}{8}$
21. $\frac{7x+15}{15} = \frac{10}{3}$
22. $\frac{7x+3}{8} = \frac{5}{4}$

23. $\frac{11}{2} = \frac{x}{10}$

24. $\frac{19}{6} = \frac{x}{18}$

25. $\frac{5x+8}{12} = \frac{2}{3}$

26. $\frac{3x+12}{6} = \frac{3}{2}$

27. $\frac{2}{15} = \frac{24}{x}$

28. $\frac{7}{8} = \frac{14}{x}$

29. $\frac{3}{16} = \frac{6}{x}$

30. $\frac{4}{21} = \frac{12}{x}$

31. $\frac{5}{22} = \frac{20}{x}$

32. $\frac{3}{22} = \frac{21}{x}$

33. $\frac{2x+10}{6} = \frac{14}{3}$

34. $\frac{2x+9}{9} = \frac{13}{3}$

35. $\frac{7}{2} = \frac{21}{x}$

36. $\frac{2}{15} = \frac{18}{x}$

---

37. If 13 dog bones cost $1.97, what will be the cost for 7 dog bones (assuming an equal rate)? Round your answer to the nearest penny.

38. If 2 watermelons cost $3.89, what will be the cost for 11 watermelons (assuming an equal rate)? Round your answer to the nearest penny.

39. If 7 bananas cost $2.55, what will be the cost for 14 bananas (assuming an equal rate)? Round your answer to the nearest penny.

40. If 2 apples cost $2.05, what will be the cost for 8 apples (assuming an equal rate)? Round your answer to the nearest penny.

41. If 13 oranges cost $3.61, what will be the cost for 11 oranges (assuming an equal rate)? Round your answer to the nearest penny.

42. If 3 watermelons cost $1.05, what will be the cost for 9 watermelons (assuming an equal rate)? Round your answer to the nearest penny.

43. If 3 dog bones cost $1.23, what will be the cost for 13 dog bones (assuming an equal rate)? Round your answer to the nearest penny.

44. If 3 watermelons cost $4.41, what will be the cost for 7 watermelons (assuming an equal rate)? Round your answer to the nearest penny.

45. If 3 apples cost $3.24, what will be the cost for 13 apples (assuming an equal rate)? Round your answer to the nearest penny.

46. If 6 apples cost $3.43, what will be the cost for 7 apples (assuming an equal rate)? Round your answer to the nearest penny.

47. If 4 dog bones cost $1.03, what will be the cost for 8 dog bones (assuming an equal rate)? Round your answer to the nearest penny.

48. If 4 oranges cost $4.28, what will be the cost for 3 oranges (assuming an equal rate)? Round your answer to the nearest penny.

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49. **Two rolls.** In Haiti, two flat rolls cost 5 gourdes, about 12 cents. How many cents would 20 rolls cost? Associated Press-Times-Standard 02/18/10 Haiti's earthquake camps turning into shanty towns.

50. **Turbines.** As proposed, the Shell Wind Energy project consists of 25 ridge-top turbines that can generate up to 50 megawatts, or enough to supply electricity to about 1, 000 homes. Estimate the number of ridge-top turbines that would be needed to supply electricity to 70, 000 homes, the approximate number of properties in Humboldt County, CA. *John Driscoll Times-Standard 12/24/09 Wind power project goes under analysis.*

51. **Dumptrucks.** U.S. Highway 199 had a landslide where as much as 3, 000 cubic yards of material fell on the road, reportedly requiring about 200 large dumptrucks to remove. Only a week earlier, 40, 000 cubic yards of material fell on Highway 96. Estimate the number of dumptrucks needed for that slide rounded to the nearest whole number. *Associated Press-Times-Standard 03/09/10 Another highway closed by slide.*

52. **Timber sales.** Alaska's 26, 000 square mile Tongass National Forest plan allows for timber sales of up to 267 million board-feet per year – enough for nearly 27, 000 two-bedroom homes, but demand for timber is far short of that. Less than 25 million board-feet was logged in the forest in 2009. Forest Service officials have said they hope to increase logging in the Tongass to about 100 million board-feet per year. *Associated Press-Times-Standard 02/18/10 Industry loses lawsuit over logging in Alaska.*

i) Estimate the number of 2-bedroom homes that 25 million board-feet of timber would build.

ii) How many 2-bedroom homes would 100 million board-feet of timber build?

53. **Costly spill.** In Australia, penalties on ships causing oil spills are approximately $1.75 million Australian dollars, equivalent to about $1.64 million US dollars. After an oil tanker was grounded onto a coral reef, Australian officials are considering raising the fine to $10 million Australian dollars. What will the new fine be in US dollars? Round your answer to the nearest hundredth of a million dollars. *Associated Press-Times-Standard 04/13/10 Ship that leaked oil on Great Barrier Reef removed.*

Answers

1. $\frac{9}{7} = \frac{27}{21}$ is a proportion
3. $\frac{7}{6} = \frac{28}{24}$ is a proportion
5. $\frac{6}{5} = \frac{24}{20}$ is a proportion
7. $\frac{3}{7} = \frac{6}{14}$ is a proportion
9. $\frac{5}{4} = \frac{25}{20}$ is a proportion
11. $\frac{3}{5} = \frac{6}{10}$ is a proportion
13. 102
15. 2
17. 34
19. 44
21. 5
23. 55
25. 0
27. 180
29. 32
31. 88
33. 9
35. 6
37. $1.06
39. $5.10

41. $3.05

43. $5.33

45. $14.04

47. $2.06

49. $0.48

51. 2, 667 loads

53. $9.37 million US dollars

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6.4: Unit Conversion - American System

In this section we will develop a technique for converting units used in the American system. We begin with a discussion of common measurements of length in the United States.

Units of Length

The most common units of length are the inch, foot, yard, and mile. Our focus will be on the technique used to convert from one unit of length to another.

American Units of Length

Facts relating common units of length.

- 1 foot (ft) = 12 inches (in)
- 1 yard (yd) = 3 feet (ft)
- 1 mile (mi) = 5280 feet (ft)

Take for example, the fact that there are 3 feet in 1 yard, which can be stated as an equation, using the common abbreviations for feet (ft) and yards (yd).

$$3 \text{ ft} = 1 \text{ yd}$$

If we divide both sides of the equation by 3 ft,

$$\frac{3 \text{ ft}}{3 \text{ ft}} = \frac{1 \text{ yd}}{3 \text{ ft}},$$

or equivalently,

$$1 = \frac{1 \text{ yd}}{3 \text{ ft}}.$$

The key observation is the fact that the ratio 1 yd/3 ft equals the number 1. Consequently, multiplying by the "conversion factor" 1 yd/3 ft is equivalent to multiplying by 1. This can be used to change a measurement in feet to yards.

Example 1

Change 36 feet to yards.

Solution

Multiply by the conversion factor 1 yd/3 ft.

$$\begin{aligned} 36 \text{ ft} &= 36 \text{ ft} \cdot 1 && \text{Multiplicative Identity Property.} \\ &= 36 \text{ ft} \cdot \frac{1 \text{ yd}}{3 \text{ ft}} && \text{Replace 1 with 1 yd/3 ft.} \\ &= 36 \ \cancel{\text{ft}} \cdot \frac{1 \text{ yd}}{3 \ \cancel{\text{ft}}} && \text{Cancel common unit.} \\ &= \frac{36 \cdot 1}{3} \text{ yd} && \text{Multiply fractions.} \\ &= \frac{36}{3} \text{ yd} && \text{Simplify.} \\ &= 12 \text{ yd} && \text{Divide.} \end{aligned}$$

Hence, 36 feet equals 12 yards.

Exercise

Change 81 feet to yards.

Answer

27 yards

On the other hand, we can start again with

$$3 \text{ ft} = 1 \text{ yd}$$

and divide both sides of the equation by 1 yd.

$$\frac{3 \text{ ft}}{1 \text{ yd}} = \frac{1 \text{ yd}}{1 \text{ yd}}$$

This gives the conversion factor

$$\frac{3 \text{ ft}}{1 \text{ yd}} = 1.$$

The key observation is the fact that the ratio 3 ft/1 yd equals the number 1. Consequently, multiplying by the "conversion factor" 3 ft/1 yd is equivalent to multiplying by 1. This can be used to change a measurement in yards to feet.

Example 2

Change 18 yards to feet.

Solution

Multiply by the conversion factor 3 ft/1 yd.

\[\begin{aligned} 18 \text{ yd} = 18 \text{ yd} \cdot 1 ~ & \textcolor{red}{ \text{ Multiplicative Identity Property.}} \\ = 18 \text{ yd} \cdot \frac{1 \text{ ft}}{1 \text{ yd}} ~ & \textcolor{red}{ \text{ Replace 1 with 3 ft/1 yd.} } \\ = 18 \cancel{ \text{ yd}} \cdot \frac{3 \text{ ft}}{1 \cancel{ \text{ yd}}} ~ & \textcolor{red}{ \text{ Cancel common unit.}} \\ = \frac{18 \cdot 3}{1} \text{ ft} ~ & \textcolor{red}{ \text{ Multiply fractions.}} \\ = 54 \text{ ft} ~ & \textcolor{red}{ \text{ Simplify.}} \end{aligned}\nonumber \]

Hence, 18 yards equals 54 feet.

Exercise

Change 15 yards to feet.

Answer

45 feet

Another common comparison is the fact that there are 12 inches in 1 foot. This can be represented as an equation using the common abbreviation for inches (in) and feet (ft).

$$12 \text{ in} = 1 \text{ ft}$$

Dividing both sides by 12 in

$$\frac{12 \text{ in}}{12 \text{ in}} = \frac{1 \text{ ft}}{12 \text{ in}},$$

yields the conversion factor

$$1 = \frac{1 \text{ ft}}{12 \text{ in}}.$$

The key observation is the fact that the ratio 1 ft/12 in equals the number 1. Consequently, multiplying by the "conversion factor" 1 ft/12 in is equivalent to multiplying by 1. This can be used to change a measurement in inches to feet.

Example 3

Change 24 inches to feet.

Solution

Multiply by the conversion factor 1 ft/12 in.

$$\begin{aligned} 24 \text{ in} &= 24 \text{ in} \cdot 1 && \text{Multiplicative Identity Property.} \\ &= 24 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} && \text{Replace 1 with 1 ft/12 in.} \\ &= 24 \ \cancel{\text{in}} \cdot \frac{1 \text{ in}}{12 \ \cancel{\text{in}}} && \text{Cancel common unit.} \\ &= \frac{24 \cdot 1}{12} \text{ ft} && \text{Multiply fractions.} \\ &= 2 \text{ ft} && \text{Simplify.} \end{aligned}$$

Hence, 24 inches equals 2 feet.

Exercise

Change 48 inches to feet.

Answer

4 feet

We provide a summary of conversion factors for units of length in Table 6.1.

Table 6.1: Conversion factors for units of length.

Convert	Conversion Factor	Convert	Conversion Factor
feet to inches	12 in/1 ft	inches to feet	1 ft/12 in
yards to feet	3 ft/1 yd	feet to yards	1 yd/3 ft
miles to feet	5280 ft/1 mi	feet to miles	1 mi/5280 ft

Some conversions require more than one application of a conversion factor.

Example 4

Change 4 yards to inches.

Solution

We multiply by a chain of conversion factors, the first to change yards to feet, the second to change feet to inches.

$$\begin{aligned} 4 \text{ yd} &= 4 \text{ yd} \cdot \frac{3 \text{ ft}}{1 \text{ yd}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} && \text{Multiply by conversion factors.} \\ &= 4 \ \cancel{\text{yd}} \cdot \frac{3 \ \cancel{\text{ft}}}{1 \ \cancel{\text{yd}}} \cdot \frac{12 \text{ in}}{1 \ \cancel{\text{ft}}} && \text{Cancel common units.} \\ &= \frac{4 \cdot 3 \cdot 12}{1 \cdot 1} \text{ in} && \text{Multiply fractions.} \\ &= 144 \text{ in} && \text{Simplify.} \end{aligned}$$

Hence, 4 yards equals 144 inches.

Exercise

Change 8 yards to inches.

Answer

288 inches

Example 5

Change 2 miles to yards.

Solution

We multiply by a chain of conversion factors, the first to change miles to feet, the second to change feet to yards.

$$2\text{ mi} = 2\text{ mi} \cdot \frac{5280\text{ ft}}{1\text{ mi}} \cdot \frac{1\text{ yd}}{3\text{ ft}} \quad \text{Multiply by conversion factors.}$$

$$= 2\ \cancel{\text{mi}} \cdot \frac{5280\ \cancel{\text{ft}}}{1\ \cancel{\text{mi}}} \cdot \frac{1\text{ yd}}{3\ \cancel{\text{ft}}} \quad \text{Cancel common units.}$$

$$= \frac{2 \cdot 5280 \cdot 1}{1 \cdot 3}\text{ yd} \quad \text{Multiply fractions.}$$

$$= 3520\text{ yd} \quad \text{Simplify.}$$

Hence, 2 miles equals 3,520 yards.

Exercise

Change 5 miles to yards.

Answer

8,800 yards

Units of Weight

The most common units of weight are the ounce, pound, and ton. Our focus will remain on how to convert from one unit to another.

American Units of Weight

Facts relating common units of weight.

- 1 pound (lb) = 16 ounces (oz)
- 1 ton = 2000 pounds (lb)

The above facts lead to the conversion factors in Table 6.2.

Table 6.2: Conversion factors for units of weight.

Convert	Conversion Factor	Convert	Conversion Factor
pounds to ounces	16 oz/1 lb	ounces to pounds	1 lb/16 oz
tons to pounds	2000 lb/1 ton	pounds to tons	1 ton/2000 lb

Example 6

Change $2\frac{1}{2}$ pounds to ounces.

Solution

Multiply by the appropriate conversion factor.

$$
\begin{aligned}
2\frac{1}{2}\ \text{lb} &= 2\frac{1}{2}\ \text{lb} \cdot \frac{16\ \text{oz}}{1\ \text{lb}} && \text{Multiply by conversion factor.}\\
&= 2\frac{1}{2}\ \cancel{\text{lb}} \cdot \frac{16\ \text{oz}}{1\ \cancel{\text{lb}}} && \text{Cancel common units.}\\
&= \left(2\frac{1}{2} \cdot 16\right)\ \text{oz} && \text{Multiply fractions.}\\
&= \left(\frac{5}{2} \cdot 16\right)\ \text{oz} && \text{Mixed to improper fraction.}\\
&= \frac{80}{2}\ \text{oz} && \text{Multiply.}\\
&= 40\ \text{oz} && \text{Divide.}
\end{aligned}
$$

Hence, $2\frac{1}{2}$ pounds equals 40 ounces.

Exercise

Change $6\frac{1}{4}$ pounds to ounces.

Answer

100 ounces

Example 7

Change 3.2 tons to ounces.

Solution

This exercise requires multiplying by a chain of conversion factors.

$$
\begin{aligned}
3.2\ \text{ton} &= 3.2\ \text{ton} \cdot \frac{2000\ \text{lb}}{1\ \text{ton}} \cdot \frac{16\ \text{oz}}{1\ \text{lb}} && \text{Multiply by conversion factors.}\\
&= 3.2\ \cancel{\text{ton}} \cdot \frac{2000\ \cancel{\text{lb}}}{1\ \cancel{\text{ton}}} \cdot \frac{16\ \text{oz}}{1\ \cancel{\text{lb}}} && \text{Cancel common units.}\\
&= \frac{3.2 \cdot 2000 \cdot 16}{1 \cdot 1}\ \text{oz} && \text{Multiply fractions.}\\
&= 102,400\ \text{oz} && \text{Simplify.}
\end{aligned}
$$

Hence, 3.2 tons equals 102,400 ounces.

Exercise

Change 4.1 tons to ounces.

Answer

128,000 ounces

Units of Volume

The most common units of volume are fluid ounces, cups, pints, quarts, and gallons. We will focus on converting from one unit to another.

American Units of Volume

Facts relating common units of volume.

- 1 cup (c) = 8 fluid ounces (fl oz)
- 1 pint (pt) = 2 cups (c)
- 1 quart (qt) = 2 pints (pt)
- 1 gallon (gal) = 4 quarts (qt)

These facts lead to the conversion factors listed in Table 6.3.

Table 6.3: Conversion factors for units of volume.

Convert	Conversion Factor	Convert	Conversion Factor
cups to ounces	8 fl oz/1 c	ounces to cups	1c/8 fl oz
pints to cups	2 c/1 pt	cups to pints	1 pt/2 c
quarts to pints	2 pt/1 qt	pints to quarts	1 qt/2 pt
gallons to quarts	4 qt/1 gal	quarts to gallons	1 gal/4 qt

Example 8

Change 5.6 gallons to pints.

Solution

This exercise requires multiplying by a chain of conversion factors.

$$5.6\ \text{gal} = 5.6\ \text{gal} \cdot \frac{4\ \text{qt}}{1\ \text{gal}} \cdot \frac{2\ \text{pt}}{1\ \text{qt}} \quad \text{Multiply by conversion factors.}$$

$$= 5.6\ \cancel{\text{gal}} \cdot \frac{4\ \cancel{\text{qt}}}{1\ \cancel{\text{gal}}} \cdot \frac{2\ \text{pt}}{1\ \cancel{\text{qt}}} \quad \text{Cancel common units.}$$

$$= \frac{5.6 \cdot 4 \cdot 2}{1 \cdot 1}\ \text{pt} \quad \text{Multiply fractions.}$$

$$= 44.8\ \text{pt} \quad \text{Simplify.}$$

Hence, 5.6 gallons equals 44.8 pints.

Exercise

Change 3.2 gallons to pints.

Answer

25.6 pints

Units of Time

The most common units of time are seconds, minutes, hours, days, and years.

American Units of Time

Facts relating common units of time.

- 1 minute (min) = 60 seconds (s)
- 1 hour (hr) = 60 minutes (min)
- 1 day (day) = 24 hours (hr)
- 1 year (yr) = 365 days (day)

These facts lead to the conversion factors in Table 6.4.

Table 6.4: Conversion factors for units of time.

Convert	Conversion Factor	Convert	Conversion Factor
minutes to seconds	60 s/1 min	seconds to minutes	1 min/60 s
hours to minutes	60 min/1 hr	minutes to hours	1 hr/60 min
days to hours	24 hr/1 day	hours to days	1 day/24 hr
years to days	365 day/1 yr	days to years	1 yr/365 day

Example 9

How many seconds in a year?

Solution

A chain of conversion factors is needed.

$$1\text{ yr} = 1\text{ yr} \cdot \frac{365\text{ day}}{1\text{ yr}} \cdot \frac{24\text{ hr}}{1\text{ day}} \cdot \frac{60\text{ min}}{1\text{ hr}} \cdot \frac{60\text{ s}}{1\text{ min}} \quad \text{Conversion factors.}$$

$$= 1\ \cancel{\text{yr}} \cdot \frac{365\ \cancel{\text{day}}}{1\ \cancel{\text{yr}}} \cdot \frac{24\ \cancel{\text{hr}}}{1\ \cancel{\text{day}}} \cdot \frac{60\ \cancel{\text{min}}}{1\ \cancel{\text{hr}}} \cdot \frac{60\text{ s}}{1\ \cancel{\text{min}}} \quad \text{Cancel common units.}$$

$$= \frac{1 \cdot 365 \cdot 24 \cdot 60 \cdot 60}{1 \cdot 1 \cdot 1 \cdot 1}\text{ s} \quad \text{Multiply fractions.}$$

$$= 31,536,000\text{ s} \quad \text{Simplify.}$$

Thus, 1 year equals 31,536,000 seconds.

Exercise

How many seconds in a day?

Answer

86,400 seconds

Converting Units of Speed

Ever wonder how fast a baseball is moving?

Example 10

A professional pitcher can throw a baseball at 95 miles per hour. How fast is this in feet per second? Round your answer to the nearest foot per second.

Solution

There are 5280 feet in a mile, 60 minutes in an hour, and 60 seconds in a minute.

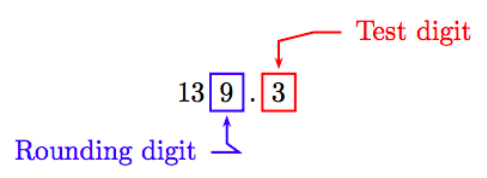
$$95\frac{\text{mi}}{\text{h}} \approx 95\frac{\text{mi}}{\text{h}} \cdot \frac{5280\ \text{ft}}{1\ \text{mi}} \cdot \frac{1\ \text{h}}{60\ \text{min}} \cdot \frac{1\ \text{min}}{60\ \text{s}} \quad \text{Conversion factors.}$$

$$\approx 95\frac{\cancel{\text{mi}}}{\cancel{\text{h}}} \cdot \frac{5280\ \text{ft}}{1\ \cancel{\text{mi}}} \cdot \frac{1\ \cancel{\text{h}}}{60\ \cancel{\text{min}}} \cdot \frac{1\ \cancel{\text{min}}}{60\ \text{s}} \quad \text{Cancel common units.}$$

$$\approx \frac{95 \cdot 5280 \cdot 1 \cdot 1}{1 \cdot 60 \cdot 60}\frac{\text{ft}}{\text{s}} \quad \text{Multiply fractions.}$$

$$\approx 139.3\frac{\text{ft}}{\text{s}} \quad \text{Multiply and divide.}$$

To round to the nearest foot per second, identify the rounding and test digits.



Because the test digit is less than 5, leave the rounding digit alone and truncate. Thus, to the nearest foot per second, the speed is approximately 139 feet per second.

Whew! Since the batter stands at home plate, which is about 60 feet from where the pitch is delivered, the batter has less than 1/2 a second to react to the pitch!

? Exercise

A women's softball pitcher can throw her fastball at 60 miles per hour. How fast is this in feet per second? Round your answer to the nearest foot per second.

Answer

88 feet per second

Exercises

1. Change 8 yards to feet.
2. Change 60 yards to feet.
3. Change 261 feet to yards.
4. Change 126 feet to yards.
5. Change 235 inches to yards. Round your answer to the nearest tenth of a yard.
6. Change 244 inches to yards. Round your answer to the nearest tenth of a yard.
7. Change 141 feet to yards.
8. Change 78 feet to yards.
9. Change 2.8 miles to feet.
10. Change 4.9 miles to feet.
11. Change 104 inches to yards. Round your answer to the nearest tenth of a yard.
12. Change 101 inches to yards. Round your answer to the nearest tenth of a yard.
13. Change 168,372 inches to miles, correct to the nearest tenth of a mile.
14. Change 198,550 inches to miles, correct to the nearest tenth of a mile.
15. Change 82 feet to inches.
16. Change 80 feet to inches.

17. Change 2.9 yards to inches. Round your answer to the nearest inch.
18. Change 4.5 yards to inches. Round your answer to the nearest inch.
19. Change 25,756 feet to miles. Round your answer to the nearest tenth of a mile.
20. Change 19,257 feet to miles. Round your answer to the nearest tenth of a mile.
21. Change 5 yards to feet.
22. Change 20 yards to feet.
23. Change 169,312 inches to miles, correct to the nearest tenth of a mile.
24. Change 162,211 inches to miles, correct to the nearest tenth of a mile.
25. Change 1.5 yards to inches. Round your answer to the nearest inch.
26. Change 2.1 yards to inches. Round your answer to the nearest inch.
27. Change 360 inches to feet.
28. Change 768 inches to feet.
29. Change 48 inches to feet.
30. Change 528 inches to feet.
31. Change 15,363 feet to miles. Round your answer to the nearest tenth of a mile.
32. Change 8,540 feet to miles. Round your answer to the nearest tenth of a mile.
33. Change 1.7 miles to inches.
34. Change 4.7 miles to inches.
35. Change 3.1 miles to inches.
36. Change 1.8 miles to inches.
37. Change 3.6 miles to feet.
38. Change 3.1 miles to feet.
39. Change 18 feet to inches.
40. Change 33 feet to inches.

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41. Change $5\frac{1}{8}$ pounds to ounces.
42. Change $3\frac{1}{16}$ pounds to ounces.
43. Change 2.4 tons to ounces.
44. Change 3.4 tons to ounces.
45. Change 34 ounces to pounds. Express your answer as a fraction reduced to lowest terms.
46. Change 78 ounces to pounds. Express your answer as a fraction reduced to lowest terms.
47. Change 2.2 tons to pounds.
48. Change 4.8 tons to pounds.
49. Change 70 ounces to pounds. Express your answer as a fraction reduced to lowest terms.
50. Change 20 ounces to pounds. Express your answer as a fraction reduced to lowest terms.
51. Change 9,560 pounds to tons. Round your answer to the nearest tenth of a ton.
52. Change 9,499 pounds to tons. Round your answer to the nearest tenth of a ton.
53. Change $2\frac{1}{2}$ pounds to ounces.
54. Change $7\frac{1}{16}$ pounds to ounces.

55. Change 5.9 tons to pounds.
56. Change 2.1 tons to pounds.
57. Change 2.5 tons to ounces.
58. Change 5.3 tons to ounces.
59. Change 8,111 pounds to tons. Round your answer to the nearest tenth of a ton.
60. Change 8,273 pounds to tons. Round your answer to the nearest tenth of a ton.

---

61. Change 4.5625 pints to fluid ounces.
62. Change 2.9375 pints to fluid ounces.
63. Change 32 fluid ounces to pints.
64. Change 160 fluid ounces to pints.
65. Change 3.7 gallons to pints.
66. Change 2.4 gallons to pints.
67. Change 216 pints to gallons.
68. Change 96 pints to gallons.
69. Change 544 fluid ounces to pints.
70. Change 432 fluid ounces to pints.
71. Change 112 pints to gallons.
72. Change 200 pints to gallons.
73. Change 7.7 gallons to pints.
74. Change 5.7 gallons to pints.
75. Change 3.875 pints to fluid ounces.
76. Change 3 pints to fluid ounces.

---

77. Change 7.8 years to hours.
78. Change 4.7 years to hours.
79. Change 7.6 years to hours.
80. Change 6.6 years to hours.
81. Change 4,025,005 seconds to days. Round your answer to the nearest tenth of a day.
82. Change 4,672,133 seconds to days. Round your answer to the nearest tenth of a day.
83. Change 37,668 hours to years.
84. Change 40,296 hours to years.
85. Change 22,776 hours to years.
86. Change 29,784 hours to years.
87. Change 96 days to seconds.
88. Change 50 days to seconds.
89. Change 40 days to seconds.
90. Change 10 days to seconds.
91. Change 3,750,580 seconds to days. Round your answer to the nearest tenth of a day.
92. Change 4,493,469 seconds to days. Round your answer to the nearest tenth of a day.

93. Change 367 feet per second to miles per hour. Round your answer to the nearest mile per hour.

94. Change 354 feet per second to miles per hour. Round your answer to the nearest mile per hour.

95. Change 442 feet per second to miles per hour. Round your answer to the nearest mile per hour.

96. Change 388 feet per second to miles per hour. Round your answer to the nearest mile per hour.

97. Change 30 miles per hour to feet per second. Round your answer to the nearest foot per second.

98. Change 99 miles per hour to feet per second. Round your answer to the nearest foot per second.

99. Change 106 miles per hour to feet per second. Round your answer to the nearest foot per second.

100. Change 119 miles per hour to feet per second. Round your answer to the nearest foot per second.

101. **Strong man.** Famed strongman Joe Rollino, who was still bending quarters with his fingers at age 104, once lifted 3, 200 pounds at Coney Island Amusement Park. How many tons did Joe lift that day? *Associated Press-Times-Standard 01/12/10 NYC amusement park strongman, 104, killed by van.*

102. **Earth day.** The amount of time it takes the Earth to rotate once around its axis is one day. How many seconds is that?

103. **Water break.** "The average age of Washington, DC's water pipes is 76 years, and they are not alone. Every two minutes, somewhere in the country, a pipe breaks." How many pipes break each year in the US? *New York Times 03/14/10 Saving U.S. water and sewer systems could be costly.*

Answers

1. 24 feet

3. 87 yards

5. 6.5 yards

7. 47 yards

9. 14,784 feet

11. 2.9 yards

13. 2.7 miles

15. 984 inches

17. 104 inches

19. 4.9 miles

21. 15 feet

23. 2.7 miles

25. 54 inches

27. 30 feet

29. 4 feet

31. 2.9 miles

33. 107,712 inches

35. 196,416 inches

37. 19,008 feet

39. 216 inches

41. 82 ounces

43. 76,800 ounces

45. $2\frac{1}{8}$ pounds

47. 4,400 pounds

49. $4\frac{3}{8}$ pounds

51. 4.8 tons

53. 40 ounces

55. 11,800 pounds

57. 80,000 ounces

59. 4.1 tons

61. 73 fluid ounces

63. 2 pints

65. 29.6 pints

67. 27 gallons

69. 34 pints

71. 14 gallons

73. 61.6 pints

75. 62 fluid ounces

77. 68,328 hours

79. 66,576 hours

81. 46.6 days

83. 4.3 years

85. 2.6 years

87. 8,294,400 seconds

89. 3,456,000 seconds

91. 43.4 days

93. 250 mi/hr

95. 301 mi/hr

97. 44 ft/s

99. 155 ft/s

101. 1.6 tons

103. 262, 800

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6.5: Unit Conversion- Metric System

The metric system of units is the standard system of units preferred by scientists. It is based on the base ten number system and its decimal format is more friendly to users of this system. There is a common set of prefixes adopted by the metric system to indicate a power of ten to apply to the base unit.

Metric System Preferences

This is a list of standard prefixes for the metric system and their meanings.

$$\begin{array}{ll} \text{deka} = 10 & \text{deci} = 1/10 \\ \text{hecto} = 100 & \text{centi} = 1/100 \\ \text{kilo} = 1000 & \text{milli} = 1/1000 \end{array}$$

Thus, for example, a decameter is 10 meters, a hectoliter is 100 liters, and a kilogram is 1000 grams. Similarly, a decimeter is 1/10 of a meter, a centiliter is 1/100 of a liter, and a milligram is 1/1000 of a gram.

Units of Length

The standard measure of length in the metric system is the meter.

> *Historically, the meter was defined by the French Academy of Sciences as the length between two marks on a platinum-iridium bar, which was designed to represent 110,000,000 of the distance from the equator to the north pole through Paris. In 1983, it was redefined by the International Bureau of Weights and Measures (BIPM) as the distance travelled by light in free space in 1299,792,458 of a second. (Wikipedia)*

We can apply the standard prefixes to get the following result.

Metric Units of Length

These units of length are used in the metric system.

Unit Length	Unit Abbreviation
1 kilometer = 1000 meters	km
1 hectometer = 100 meters	hm
1 dekameter = 10 meters	dam
1 meter	m
1 decimeter = $\frac{1}{10}$ meter	dm
1 centimeter = $\frac{1}{100}$ meter	cm
1 millimeter = $\frac{1}{1000}$ meter	mm

We can use these facts to build conversion factors as we did in Section 6.3. For example, because

$$1 \text{ km} = 1000 \text{ m},$$

we can divide both sides by 1000 m to produce the conversion factor

$$1 = \frac{1 \text{ km}}{1000 \text{ m}}.$$

This conversion factor can help change meters into kilometers.

Before using this conversion factor in an example, we repeat here the rules for multiplying and dividing by powers of ten. We will be making heavy use of these rules in this section.

Multiplying and Dividing by Powers of Ten

- Multiplying a decimal number by 10^n will move the decimal point n places to the right. For example, $3.2567 \cdot 10^2 = 3.2567 \cdot 100 = 325.67$.
- Dividing a decimal number by 10^n will move the decimal point n places to the left. For example, $3.2567/10^2 = 3.2567/100 = 0.032567$.

And now the example.

Example 1

Change 2,326 meters to kilometers.

Solution

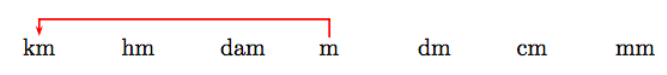
Multiply by the conversion factor 1 km/1000m.

$$\begin{aligned} 2326\ \text{m} &= 2326\ \text{m} \cdot \frac{1\ \text{km}}{1000\ \text{m}} && \text{Apply conversion factor.} \\ &= 2326\ \cancel{\text{m}} \cdot \frac{1\ \text{km}}{1000\ \cancel{\text{m}}} && \text{Cancel common units.} \\ &= \frac{2326 \cdot 1}{1000}\ \text{km} && \text{Multiply fractions.} \\ &= 2.326\ \text{km} && \text{Simplify.} \end{aligned}$$

In the last step, note that dividing by 1000 moves the decimal point three places to the left. Thus, 2326 meters is equal to 2.326 kilometers.

Alternate Solution

A second solution depends upon the fact that multiplying or dividing by a power of ten will move the decimal point right or left a number of places equal to the number of zeros present in the multiplier or divisor. Thus, as we saw above, dividing by 1000 moved the decimal point 3 places to the left. Suppose that we arrange the metric units of length in order, from largest to smallest, as shown below.



Note that we must move 3 places left to move from the meters (m) abbreviation to the kilometers (km) abbreviation. In like manner, if we write 2,326 meters as 2,326.0meters, then we can convert to kilometers by moving the decimal 3 places to the left.

$$2,326.0\ \text{m} = 2.236\ 0\ \text{km} = 2.326\ \text{km}$$

Exercise

Change 1,156 meters to kilometers

Answer

1.156 kilometers

Example 2

Change 537 centimeters to meters.

Solution

We know that

$$1\text{ cm} = \frac{1}{100}\text{ m},$$

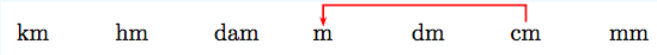
or multiplying both sides of this result by 100,

$$100\text{ cm} = 1\text{ m}.$$

Dividing both sides of this last result by 100 cm, we obtain the conversion factor 1 m/100 cm.

$$\begin{aligned} 537\text{ cm} &= 537\text{ cm} \cdot \frac{1\text{ m}}{100\text{ cm}} && \text{Apply conversion factor.} \\ &= 537\ \cancel{\text{cm}} \cdot \frac{1\text{ m}}{100\ \cancel{\text{cm}}} && \text{Cancel common units.} \\ &= \frac{537 \cdot 1}{100}\text{ m} && \text{Multiply fractions.} \\ &= 5.37\text{ m} \end{aligned}$$

In the last step, note that dividing by 100 moves the decimal point two places to the left. Alternately, we can set up our ordered list of units.



Note that we must move 2 places left to move from the centimeters (cm) abbreviation to the meters (m) abbreviation. In like manner, if we write 537 centimeters as 537.0 centimeters, then we can convert to meters by moving the decimal 2 places to the left.

Sometimes more than one conversion factor is needed.

Exercise

Change 276 centimeters to meters.

Answer

2.76 meters

Example 3

Change 10.2 dekameters to centimeters.

Solution

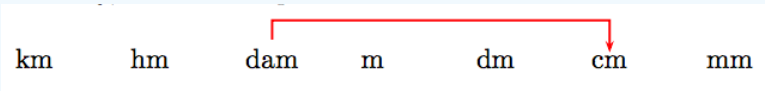
We have two facts:

- 1 dam=10 m, which yields the conversion factor 10 m/1 dam.
- 1 cm=(1/100) m or 100 cm=1 m, which yields the conversion factor 100 cm/1 m.

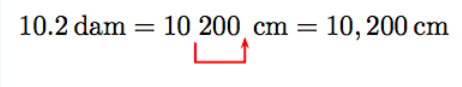
$$
\begin{aligned}
10.2\ \text{dam} &= 10.2\ \text{dam} \cdot \frac{10\ \text{m}}{1\ \text{dam}} \cdot \frac{100\ \text{cm}}{1\ \text{m}} && \text{Apply conversion factor.} \\
&= 10.2\ \cancel{\text{dam}} \cdot \frac{10\ \cancel{\text{m}}}{1\ \cancel{\text{dam}}} \cdot \frac{100\ \text{cm}}{1\ \cancel{\text{m}}} && \text{Cancel common units.} \\
&= \frac{10.2 \cdot 10 \cdot 100}{1}\ \text{cm} && \text{Multiply fractions.} \\
&= 10,200\ \text{cm}
\end{aligned}
$$

In the last step, note that multiplying by 10, then by 100, moves the decimal point three places to the right.

Alternately, we can set up our ordered list of units.



Note that we must move 3 places right to move from the dekameters (dam) abbreviation to the centimeters (cm) abbreviation. In like manner, we can convert 10.2 dekameters to centimeters by moving the decimal 3 places to the right.



Exercise

Change 13.5 dekameters to centimeters

Answer

13,500 centimeters

Units of Mass

The fundamental unit of mass in the metric system is called a gram. Originally, it was defined to be equal to one cubic centimeter of water measured at the temperature of melting ice. Now it is simply defined as 1/1000 of a kilogram, which is defined by a physical prototype preserved by the International Bureau of Weights and Measures (Wikipedia). The mass of an object is not the same as an object's weight, but rather a resistance to motion when an external force is applied.

The same metric system prefixes apply.

Metric Units of Mass

These units of mass are used in the metric system.

Units of Mass	Unit Abbreviation
1 kilogram = 1000 grams	kg
1 hectogram = 100 grams	hg
1 dekagram = 10 grams	dag
1 gram	g
1 decigram = $\frac{1}{10}$ gram	dg
1 centigram = $\frac{1}{100}$ gram	cg
1 milligram = $\frac{1}{1000}$ gram	mg

Example 4

Convert 0.025 dekagrams to milligrams.

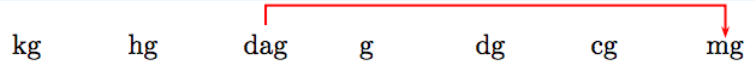
Solution

We'll use two conversion factors:

- 1 dag=10 g, which yields the conversion factor 10 g/1 dag.
- 1 mg=(1/1000) g, which yields the conversion factor 1000 mg/1 g.

$$\begin{aligned} 0.025\ \text{dag} &= 0.025\ \text{dag} \cdot \frac{10\ \text{g}}{1\ \text{dag}} \cdot \frac{1000\ \text{mg}}{1\ \text{g}} && \text{Apply conversion factor.} \\ &= 0.025\ \cancel{\text{dag}} \cdot \frac{10\ \cancel{\text{g}}}{1\ \cancel{\text{dag}}} \cdot \frac{1000\ \text{mg}}{1\ \cancel{\text{g}}} && \text{Cancel common units.} \\ &= \frac{0.025 \cdot 10 \cdot 1000}{1}\ \text{mg} && \text{Multiply fractions.} \\ &= 250\ \text{mg} \end{aligned}$$

Alternately, we can set up our ordered list of units.



Note that we must move 4 places right to move from the dekagrams (dag) abbreviation to the milligrams (mg) abbreviation. In like manner, we can convert 0.025 dekagrams to milligrams by moving the decimal 4 places to the right.

$$0.0250\ \text{dag} = 0\ 0250.\ \text{mg} = 250\ \text{mg}$$

Exercise

Convert 0.05 dekagrams to milligrams.

Answer

500 milligrams

Units of Volume

The fundamental unit of volume in the metric system is called a litre. Originally, one litre was defined as the volume of one kilogram of water measured at 4° C at 760 millimeters of mercury (Wikipedia). Currently, 1 litre is defined as 1 cubic decimeter (imagine a cube with each edge 1/10 of a meter).

The same metric system prefixes apply.

Metric Units of Volume

These units of volume are used in the metric system.

Unit of Volume	Unit Abbreviation
1 kilolitre = 1000 litres	kL
1 hectolitre = 100 litres	hL
1 dekalitre = 10 litres	daL
1 litre	L
1 decilitre = $\frac{1}{10}$ litre	dL

1 centilitre = $\frac{1}{100}$ litre	cL
1 millilitre = $\frac{1}{1000}$ litre	mL

Example 5

Convert 11,725 millilitres to dekalitres.

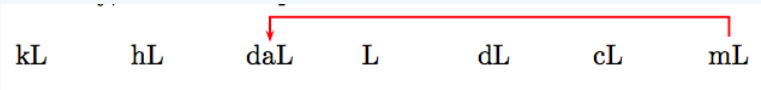
Solution

We'll use two conversion factors:

- 1 daL=10 L, which yields the conversion factor 1 daL/10 L.
- 1 mL=(1/1000) L, which yields the conversion factor 1 L/1000 mL.

$$\begin{aligned} 11,725\ \text{mL} &= 11,725\ \text{mL} \cdot \frac{1\ \text{L}}{1000\ \text{mL}} \cdot \frac{1\ \text{daL}}{10\ \text{L}} && \text{Apply conversion factors.} \\ &= 11,725\ \cancel{\text{mL}} \cdot \frac{1\ \cancel{\text{L}}}{1000\ \cancel{\text{mL}}} \cdot \frac{1\ \text{daL}}{10\ \cancel{\text{L}}} && \text{Cancel common units.} \\ &= \frac{11,725 \cdot 1 \cdot 1}{1000 \cdot 10}\ \text{daL} && \text{Multiply fractions.} \\ &= 1.1725\ \text{daL} \end{aligned}$$

Alternately, we can set up our ordered list of units.



Note that we must move 4 places left to move from the millitres (mL) abbreviation to the dekalitres (daL) abbreviation. In like manner, we can convert 11,725 millilitres to dekalitres by moving the decimal 4 places to the left.

$$11,725.0\ \text{mL} = 1.1725\ 0\ \text{daL} = 1.1725\ \text{daL}$$

Exercise

Convert 5,763 millilitres to dekalitres.

Answer

0.5763 dekalitres

Exercises

1. What is the meaning of the metric system prefix centi?
2. What is the meaning of the metric system prefix deka?
3. What is the meaning of the metric system prefix hecto?
4. What is the meaning of the metric system prefix kilo?
5. What is the meaning of the metric system prefix deci?
6. What is the meaning of the metric system prefix milli?

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7. What is the meaning of the metric system abbreviation mg?
8. What is the meaning of the metric system abbreviation g?
9. What is the meaning of the metric system abbreviation m?

10. What is the meaning of the metric system abbreviation km?
11. What is the meaning of the metric system abbreviation kL?
12. What is the meaning of the metric system abbreviation daL?
13. What is the meaning of the metric system abbreviation hm?
14. What is the meaning of the metric system abbreviation dm?
15. What is the meaning of the metric system abbreviation dam?
16. What is the meaning of the metric system abbreviation cm?
17. What is the meaning of the metric system abbreviation dL?
18. What is the meaning of the metric system abbreviation L?
19. What is the meaning of the metric system abbreviation hg?
20. What is the meaning of the metric system abbreviation kg?
21. What is the meaning of the metric system abbreviation dg?
22. What is the meaning of the metric system abbreviation dag?
23. What is the meaning of the metric system abbreviation hL?
24. What is the meaning of the metric system abbreviation cL?

---

25. Change 5,490 millimeters to meters.
26. Change 8,528 millimeters to meters.
27. Change 64 meters to millimeters.
28. Change 65 meters to millimeters.
29. Change 4,571 millimeters to meters.
30. Change 8,209 millimeters to meters.
31. Change 15 meters to centimeters.
32. Change 12 meters to centimeters.
33. Change 569 centimeters to meters.
34. Change 380 centimeters to meters.
35. Change 79 meters to centimeters.
36. Change 60 meters to centimeters.
37. Change 7.6 kilometers to meters.
38. Change 4.9 kilometers to meters.
39. Change 861 centimeters to meters.
40. Change 427 centimeters to meters.
41. Change 4,826 meters to kilometers.
42. Change 1,929 meters to kilometers.
43. Change 4,724 meters to kilometers.
44. Change 1,629 meters to kilometers.
45. Change 6.5 kilometers to meters.
46. Change 7.9 kilometers to meters.
47. Change 17 meters to millimeters.

48. Change 53 meters to millimeters.

---

49. Change 512 milligrams to centigrams.
50. Change 516 milligrams to centigrams.
51. Change 541 milligrams to centigrams.
52. Change 223 milligrams to centigrams.
53. Change 70 grams to centigrams.
54. Change 76 grams to centigrams.
55. Change 53 centigrams to milligrams.
56. Change 30 centigrams to milligrams.
57. Change 83 kilograms to grams.
58. Change 70 kilograms to grams.
59. Change 8,196 grams to kilograms.
60. Change 6,693 grams to kilograms.
61. Change 564 centigrams to grams.
62. Change 884 centigrams to grams.
63. Change 38 grams to centigrams.
64. Change 88 grams to centigrams.
65. Change 77 centigrams to milligrams.
66. Change 61 centigrams to milligrams.
67. Change 5,337 grams to kilograms.
68. Change 4,002 grams to kilograms.
69. Change 15 kilograms to grams.
70. Change 45 kilograms to grams.
71. Change 833 centigrams to grams.
72. Change 247 centigrams to grams.

---

73. Change 619,560 centilitres to kilolitres.
74. Change 678,962 centilitres to kilolitres.
75. Change 15.2 litres to millilitres.
76. Change 9.7 litres to millilitres.
77. Change 10,850 centilitres to litres.
78. Change 15,198 centilitres to litres.
79. Change 10.7 litres to millilitres.
80. Change 17.3 litres to millilitres.
81. Change 15,665 millilitres to litres.
82. Change 12,157 millilitres to litres.
83. Change 6.3 kilolitres to centilitres.
84. Change 8.3 kilolitres to centilitres.
85. Change 4.5 kilolitres to centilitres.

86. Change 6.2 kilolitres to centilitres.
87. Change 10.6 litres to centilitres.
88. Change 16.6 litres to centilitres.
89. Change 14,383 centilitres to litres.
90. Change 11,557 centilitres to litres.
91. Change 9.9 litres to centilitres.
92. Change 19.5 litres to centilitres.
93. Change 407,331 centilitres to kilolitres.
94. Change 827,348 centilitres to kilolitres.
95. Change 14,968 millilitres to litres.
96. Change 18,439 millilitres to litres.

Answers

1. 1/100
3. 100
5. 1/10
7. milligram
9. meter
11. kilolitre or kiloliter
13. hectometer
15. dekameter
17. decilitre or deciliter
19. hectogram
21. decigram
23. hectolitre or hectoliter
25. 5.49 meters
27. 64,000 millimeters
29. 4.571 meters
31. 1,500 centimeters
33. 5.69 meters
35. 7,900 centimeters
37. 7,600 meters
39. 8.61 meters
41. 4.826 kilometers
43. 4.724 kilometers
45. 6,500 meters
47. 17,000 millimeters
49. 51.2 centigrams
51. 54.1 centigrams

53. 7,000 centigrams
55. 530 milligrams
57. 83,000 grams
59. 8.196 kilograms
61. 5.64 grams
63. 3,800 centigrams
65. 770 milligrams
67. 5.337 kilograms
69. 15,000 grams
71. 8.33 grams
73. 6.1956 kilolitres
75. 15,200 millilitres
77. 108.5 litres
79. 10,700 millilitres
81. 15.665 litres
83. 630,000 centilitres
85. 450,000 centilitres
87. 1,060 centilitres
89. 143.83 litres
91. 990 centilitres
93. 4.07331 kilolitres
95. 14.968 litres

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6.6: American Units to Metric Units and Vice-Versa

We often need to convert from the American system of units to the metric system of units or vice-versa (imagine traveling to a European country using the metric system). That will be our focus in this section.

Converting Units of Length

One meter is slightly longer than one yard. Indeed,

$$1\text{ m} \approx 1.0936\text{ yd}. \tag{6.6.1}$$

If we divide both sides of this equation by 1.0936, then

$$\frac{1\text{ m}}{1.0936} \approx \frac{1.0936\text{ yd}}{1.0936}$$
$$0.9144\text{ m} \approx 1\text{ yd}$$

Further conversions can be made. For example, to change meters to feet, we make the following conversions.

$$\begin{aligned} 1\text{ m} &\approx 1\text{ m} \cdot 1\frac{\text{yd}}{0.9144\text{ m}} \cdot \frac{3\text{ ft}}{1\text{ yd}} && \text{Apply conversion factors.} \\ &\approx 1\ \cancel{\text{m}} \cdot \frac{1\ \cancel{\text{yd}}}{0.9144\ \cancel{\text{m}}} \cdot \frac{3\text{ ft}}{1\ \cancel{\text{yd}}} && \text{Cancel common units.} \\ &\approx \frac{1 \cdot 1 \cdot 3}{0.9144 \cdot 1}\text{ ft} && \text{Multiply fractions.} \\ &\approx 3.2808\text{ ft} \end{aligned}$$

Table 6.5 shows some of the most common conversions between American units of length and metric units of length.

Table 6.5: Length conversions: American — Metric.

American to Metric	Metric to American
1 in ≈ 2.54 cm	1 cm ≈ 0.3937 in
1 ft ≈ 0.3048 m	1 m ≈ 3.2808 ft
1 yd ≈ 0.9144 m	1 m ≈ 1.0936 yd
1 mi ≈ 1.6093 km	1 km ≈ 0.6214 mi

Example 1

A car's speedometer shows that a family has currently traveled 154 miles in route to their vacation destination. Convert this distance to kilometers.

Solution

Choose 1 mi = 1.6093 km from Table 6.5.

$$\begin{aligned} 154\text{ mi} &\approx 154\text{ mi} \cdot \frac{1.6093\text{ km}}{1\text{ mi}} && \text{Apply conversion factor.} \\ &\approx 154\ \cancel{\text{mi}} \cdot \frac{1.6093\text{ km}}{1\ \cancel{\text{mi}}} && \text{Cancel common units.} \\ &\approx \frac{154 \cdot 1.6093}{1}\text{ km} && \text{Multiply fractions.} \\ &\approx 247.8\text{ km} \end{aligned}$$

Hence, 154 miles is approximately 247.8 kilometers.

Alternate Solution

Note that this would work equally well if we chose 1 km ≈ 0.6214 mi from Table 6.5.

$$\begin{aligned} 154\text{ mi} &\approx 154\text{ mi} \cdot \frac{1\text{ km}}{0.6214\text{ mi}} && \text{Apply conversion factor.} \\ &\approx 154\ \cancel{\text{mi}} \cdot \frac{1\text{ km}}{0.6214\ \cancel{\text{mi}}} && \text{Cancel common units.} \\ &\approx \frac{154 \cdot 1}{0.6214}\text{ km} && \text{Multiply fractions.} \\ &\approx 247.8\text{ km} \end{aligned}$$

? Exercise

Change 227 miles to kilometers. Round to the nearest tenth of a kilometer.

Answer

365.3 kilometers

Converting Units of Weight and Mass

It is known that

$$1\text{ kg} \approx 2.2\text{ lb}. \tag{6.6.2}$$

Dividing both sides of this equation by 2.2,

$$\frac{1\text{ kg}}{2.2} \approx \frac{2.2\text{ lb}}{2.2}$$

$$0.454\text{ kg} \approx 1\text{lb}$$

A summary of the more common conversion factors regarding mass and weight are given in Table 6.6.

Table 6.6: Mass—Weight conversions: American — Metric.

American to Metric	Metric to American
1 oz ≈ 28.35 g	1 g ≈ 0.035 oz
1 lb ≈ 0.454 kg	1 kg ≈ 2.2 lb

✓ Example 2

Change 2.3 kilograms to ounces.

Solution

One kilogram weighs 2.2 pounds and there are 16 ounces in a pound.

$$\begin{aligned} 2.3\text{ kg} &\approx 2.3\text{ kg} \cdot \frac{2.2\text{ lb}}{1\text{ kg}} \cdot \frac{16\text{ oz}}{1\text{ lb}} && \text{Apply conversion factors.} \\ &\approx 2.3\ \cancel{\text{kg}} \cdot \frac{2.2\ \cancel{\text{lb}}}{1\ \cancel{\text{kg}}} \cdot \frac{16\text{ oz}}{1\ \cancel{\text{lb}}} && \text{Cancel common units.} \\ &\approx \frac{2.3 \cdot 2.2 \cdot 16}{1 \cdot 1}\text{ oz} && \text{Multiply fractions.} \\ &\approx 80.96\text{ oz} \end{aligned}$$

Hence, 2.3 kilograms weighs 80.96 ounces.

Alternate Solution

Another approach uses the facts that 1 kilogram equals 1000 grams and 1 ounce equals 28.35 grams.

$$\begin{aligned} 2.3\text{ kg} &\approx 2.3\text{ kg} \cdot \frac{1000\text{ g}}{1\text{ kg}} \cdot \frac{1\text{ oz}}{28.35\text{ g}} && \text{Apply conversion factors.} \\ &\approx 2.3\ \cancel{\text{kg}} \cdot \frac{1000\ \cancel{\text{g}}}{1\ \cancel{\text{kg}}} \cdot \frac{1\text{ oz}}{28.35\ \cancel{\text{g}}} && \text{Cancel common units.} \\ &\approx \frac{2.3 \cdot 1000 \cdot 1}{1 \cdot 28.35}\text{ oz} && \text{Multiply fractions.} \\ &\approx 81.13\text{ oz} \end{aligned}$$

Roundoff Error

Why the discrepancy in answers? This difference in approximations is due to something called *round-off error*. Indeed, in the first calculation, we used a conversion factor that is rounded to the nearest tenth; i.e., 1 kg ≈ 2.2 lb. If we use a more accurate conversion factor to change kilograms to pounds, namely 1 kg ≈ 2.2046 lb, we get the following result.

$$\begin{aligned} 2.3\text{ kg} &\approx 2.3\text{ kg} \cdot \frac{2.2046\text{ lb}}{1\text{ kg}} \cdot \frac{16\text{ oz}}{1\text{ lb}} && \text{Apply conversion factors.} \\ &\approx 2.3\ \cancel{\text{kg}} \cdot \frac{2.2046\ \cancel{\text{lb}}}{1\ \cancel{\text{kg}}} \cdot \frac{16\text{oz}}{1\ \cancel{\text{lb}}} && \text{Cancel common units.} \\ &\approx \frac{2.3 \cdot 2.2046 \cdot 16}{1 \cdot 1}\text{ oz} && \text{Multiply fractions.} \\ &\approx 81.13\text{ oz} \end{aligned}$$

Note that this result is in better agreement with the second result above.

? Exercise

Change 5.7 kilograms to ounces. Round to the nearest ounce.

Answer

201 ounces

Important Observation

To obtain better approximations, you need to use a conversion factor that is more accurate. Any time you feel you need more accuracy, you might try an online conversion utility, such as the one at: www.france-property-and-information.com/metric_conversion_table.htm

Converting Units of Volume

It is known that

$$1\text{ qt} \approx 0.946\text{ L}.$$

Dividing both sides of this equation by 0.946,

$$1\text{ L} \approx 1.06\text{ qt}.$$

Again, these conversion factors for volume have been well worked out by scientists. The more common conversion factors for volume are shown in Table 6.7.

American to Metric	Metric to American
1 fl oz ≈ 0.030 L	1 L ≈ 33.8 fl oz
1 pt ≈ 0.473 L	1 L ≈ 2.1 pt
1 qt ≈ 0.946 L	1 L ≈ 1.06 qt
1 gal ≈ 3.785 L	1 L ≈ 0.264 gal

Table 6.7: Volume conversions: American — Metric.

Example 3

Change 2.5 dekalitres to gallons.

Solution

Recall that 1 daL = 10 L and 1 L = 0.264 gal.

$$\begin{aligned} 2.5 \text{ daL} &\approx 2.5 \text{ daL} \cdot \frac{10 \text{ L}}{1 \text{ daL}} \cdot \frac{0.264 \text{ gal}}{1 \text{ L}} && \text{Apply conversion factors.} \\ &\approx 2.5 \ \cancel{\text{daL}} \cdot \frac{10 \ \cancel{\text{L}}}{1 \ \cancel{\text{daL}}} \cdot \frac{0.264 \text{ gal}}{1 \ \cancel{\text{L}}} && \text{Cancel common units.} \\ &\approx \frac{2.5 \cdot 10 \cdot 0.264}{1 \cdot 1} \text{ gal} && \text{Multiply fractions.} \\ &\approx 6.6 \text{ gal} \end{aligned}$$

Hence, 2.5 dekalitres is approximately equal to 6.6 gallons.

Exercise

Change 3.2 dekalitres to gallons.Round your answer to the nearest tenth of a gallon.

Answer

0.8 gallons

Converting Units of Speed

Modern speedometers often show a car's speed in both miles per hour and kilometers per hour.

Example 4

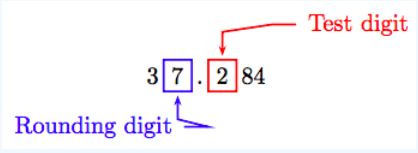
A car's speedometer is showing it speeding along at 60 kilometers per hour. How fast is it traveling in miles per hour? Round your answer to the nearest mile per hour.

Solution

From Table 6.5, 1 km ≈ 0.6214 mi.

$$\begin{aligned} 60 \frac{\text{km}}{\text{h}} &\approx 60 \frac{\text{km}}{\text{h}} \cdot \frac{0.6214 \text{ mi}}{1 \text{ km}} && \text{Apply conversion factor.} \\ &\approx 60 \frac{\cancel{\text{km}}}{\text{h}} \cdot \frac{0.6214 \text{ mi}}{1 \ \cancel{\text{km}}} && \text{Cancel common units.} \\ &\approx \frac{60 \cdot 0.6214}{1} \frac{\text{mi}}{\text{h}} && \text{Multiply fractions.} \\ &\approx 37.284 \frac{\text{mi}}{\text{h}} \end{aligned}$$

To round to the nearest mile per hour, identify the rounding and test digits.



Because the test digit is less than 5, leave the rounding digit alone and truncate. Thus, to the nearest mile per hour, the speed is approximately 37 miles per hour.

Exercise

A car's speedometer registers 45 kilometers per hour. Change this speed to miles per hour. Round your answer to the nearest mile per hour.

Answer

28 miles per hour

Exercises

1. Use the conversion 1 in = 2.54 cm to convert 68 inches to centimeters, rounded to the nearest tenth of a centimeter.
2. Use the conversion 1 in = 2.54 cm to convert 42 inches to centimeters, rounded to the nearest tenth of a centimeter.
3. Use the conversion 1 in = 2.54 cm to convert 44 centimeters to inches, rounded to the nearest tenth of a inch.
4. Use the conversion 1 in = 2.54 cm to convert 22 centimeters to inches, rounded to the nearest tenth of a inch.
5. Use the conversion 1 mi = 1.6093 km to convert 79 miles to kilometers, rounded to the nearest tenth of a kilometer.
6. Use the conversion 1 mi = 1.6093 km to convert 39 miles to kilometers, rounded to the nearest tenth of a kilometer.
7. Use the conversion 1 yd = 0.9144 m to convert 1489 centimeters to yards, rounded to the nearest tenth of a yard.
8. Use the conversion 1 yd = 0.9144 m to convert 1522 centimeters to yards, rounded to the nearest tenth of a yard.
9. Use the conversion 1 yd = 0.9144 m to convert 28 yards to centimeters, rounded to the nearest tenth of a centimeter.
10. Use the conversion 1 yd = 0.9144 m to convert 34 yards to centimeters, rounded to the nearest tenth of a centimeter.
11. Use the conversion 1 m = 3.2808 ft to convert 8.6 meters to inches, rounded to the nearest tenth of a inch.
12. Use the conversion 1 m = 3.2808 ft to convert 8.3 meters to inches, rounded to the nearest tenth of a inch.
13. Use the conversion 1 in = 2.54 cm to convert 60 inches to centimeters, rounded to the nearest tenth of a centimeter.
14. Use the conversion 1 in = 2.54 cm to convert 75 inches to centimeters, rounded to the nearest tenth of a centimeter.
15. Use the conversion 1 m = 3.2808 ft to convert 208 inches to meters, rounded to the nearest tenth of a inch.
16. Use the conversion 1 m = 3.2808 ft to convert 228 inches to meters, rounded to the nearest tenth of a inch.
17. Use the conversion 1 m = 1.0936 yd to convert 20 yards to meters, rounded to the nearest tenth of a meter.
18. Use the conversion 1 m = 1.0936 yd to convert 44 yards to meters, rounded to the nearest tenth of a meter.
19. Use the conversion 1 mi = 1.6093 km to convert 29 miles to kilometers, rounded to the nearest tenth of a kilometer.
20. Use the conversion 1 mi = 1.6093 km to convert 15 miles to kilometers, rounded to the nearest tenth of a kilometer.
21. Use the conversion 1 m = 1.0936 yd to convert 8.2 meters to yards, rounded to the nearest tenth of a yard.
22. Use the conversion 1 m = 1.0936 yd to convert 6.9 meters to yards, rounded to the nearest tenth of a yard.
23. Use the conversion 1 mi = 1.6093 km to convert 4.9 kilometers to miles, rounded to the nearest tenth of a mile.
24. Use the conversion 1 mi = 1.6093 km to convert 4.2 kilometers to miles, rounded to the nearest tenth of a mile.
25. Use the conversion 1 m = 1.0936 yd to convert 25 yards to meters, rounded to the nearest tenth of a meter.
26. Use the conversion 1 m = 1.0936 yd to convert 2 yards to meters, rounded to the nearest tenth of a meter.
27. Use the conversion 1 in = 2.54 cm to convert 47 centimeters to inches, rounded to the nearest tenth of a inch.
28. Use the conversion 1 in = 2.54 cm to convert 19 centimeters to inches, rounded to the nearest tenth of a inch.
29. Use the conversion 1 mi = 1.6093 km to convert 8.3 kilometers to miles, rounded to the nearest tenth of a mile.
30. Use the conversion 1 mi = 1.6093 km to convert 4.8 kilometers to miles, rounded to the nearest tenth of a mile.

31. Use the conversion 1 yd = 0.9144 m to convert 41 yards to centimeters, rounded to the nearest tenth of a centimeter.
32. Use the conversion 1 yd = 0.9144 m to convert 20 yards to centimeters, rounded to the nearest tenth of a centimeter.
33. Use the conversion 1 m = 3.2808 ft to convert 3.7 meters to inches, rounded to the nearest tenth of a inch.
34. Use the conversion 1 m = 3.2808 ft to convert 7.9 meters to inches, rounded to the nearest tenth of a inch.
35. Use the conversion 1 yd = 0.9144 m to convert 1323 centimeters to yards, rounded to the nearest tenth of a yard.
36. Use the conversion 1 yd = 0.9144 m to convert 1715 centimeters to yards, rounded to the nearest tenth of a yard.
37. Use the conversion 1 m = 1.0936 yd to convert 8.4 meters to yards, rounded to the nearest tenth of a yard.
38. Use the conversion 1 m = 1.0936 yd to convert 7.3 meters to yards, rounded to the nearest tenth of a yard.
39. Use the conversion 1 m = 3.2808 ft to convert 289 inches to meters, rounded to the nearest tenth of a inch.
40. Use the conversion 1 m = 3.2808 ft to convert 251 inches to meters, rounded to the nearest tenth of a inch.

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41. Use the conversion 1 kg = 2.2 lb to convert 15.8 kilograms to pounds, rounded to the nearest tenth of a pound.
42. Use the conversion 1 kg = 2.2 lb to convert 19.4 kilograms to pounds, rounded to the nearest tenth of a pound.
43. Use the conversion 1 oz = 28.35 g to convert 35 ounces to grams, rounded to the nearest tenth of a gram.
44. Use the conversion 1 oz = 28.35 g to convert 33 ounces to grams, rounded to the nearest tenth of a gram.
45. Use the conversion 1 kg = 2.2 lb to convert 2.48 kilograms to ounces, rounded to the nearest tenth of a ounce.
46. Use the conversion 1 kg = 2.2 lb to convert 3.74 kilograms to ounces, rounded to the nearest tenth of a ounce.
47. Use the conversion 1 kg = 2.2 lb to convert 2.35 kilograms to ounces, rounded to the nearest tenth of a ounce.
48. Use the conversion 1 kg = 2.2 lb to convert 3.57 kilograms to ounces, rounded to the nearest tenth of a ounce.
49. Use the conversion 1 kg = 2.2 lb to convert 15 pounds to kilograms, rounded to the nearest tenth of a kilogram.
50. Use the conversion 1 kg = 2.2 lb to convert 27 pounds to kilograms, rounded to the nearest tenth of a kilogram.
51. Use the conversion 1 kg = 2.2 lb to convert 10.4 kilograms to pounds, rounded to the nearest tenth of a pound.
52. Use the conversion 1 kg = 2.2 lb to convert 17.7 kilograms to pounds, rounded to the nearest tenth of a pound.
53. Use the conversion 1 kg = 2.2 lb to convert 352 ounces to kilograms, rounded to the nearest tenth of a kilogram.
54. Use the conversion 1 kg = 2.2 lb to convert 326 ounces to kilograms, rounded to the nearest tenth of a kilogram.
55. Use the conversion 1 oz = 28.35 g to convert 96 grams to ounces, rounded to the nearest tenth of a ounce.
56. Use the conversion 1 oz = 28.35 g to convert 100 grams to ounces, rounded to the nearest tenth of a ounce.
57. Use the conversion 1 oz = 28.35 g to convert 14 ounces to grams, rounded to the nearest tenth of a gram.
58. Use the conversion 1 oz = 28.35 g to convert 29 ounces to grams, rounded to the nearest tenth of a gram.
59. Use the conversion 1 kg = 2.2 lb to convert 54 pounds to kilograms, rounded to the nearest tenth of a kilogram.
60. Use the conversion 1 kg = 2.2 lb to convert 82 pounds to kilograms, rounded to the nearest tenth of a kilogram.
61. Use the conversion 1 oz = 28.35 g to convert 92 grams to ounces, rounded to the nearest tenth of a ounce.
62. Use the conversion 1 oz = 28.35 g to convert 103 grams to ounces, rounded to the nearest tenth of a ounce.
63. Use the conversion 1 kg = 2.2 lb to convert 388 ounces to kilograms, rounded to the nearest tenth of a kilogram.
64. Use the conversion 1 kg = 2.2 lb to convert 395 ounces to kilograms, rounded to the nearest tenth of a kilogram.

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65. Use the conversion 1 qt = 0.946 L to convert 55.1 litres to quarts, rounded to the nearest tenth of a quart.
66. Use the conversion 1 qt = 0.946 L to convert 50.3 litres to quarts, rounded to the nearest tenth of a quart.
67. Use the conversion 1 L = 33.8 fl oz to convert 72073 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.
68. Use the conversion 1 L = 33.8 fl oz to convert 56279 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.

69. Use the conversion 1 L = 33.8 fl oz to convert 2.5 kilolitres to fluid ounces.

70. Use the conversion 1 L = 33.8 fl oz to convert 4.5 kilolitres to fluid ounces.

71. Use the conversion 1 qt = 0.946 L to convert 24 quarts to litres, rounded to the nearest tenth of a litre.

72. Use the conversion 1 qt = 0.946 L to convert 21 quarts to litres, rounded to the nearest tenth of a litre.

73. Use the conversion 1 qt = 0.946 L to convert 30 quarts to litres, rounded to the nearest tenth of a litre.

74. Use the conversion 1 qt = 0.946 L to convert 22 quarts to litres, rounded to the nearest tenth of a litre.

75. Use the conversion 1 gal = 3.785 L to convert 11.8 gallons to litres, rounded to the nearest tenth of a litre.

76. Use the conversion 1 gal = 3.785 L to convert 13.5 gallons to litres, rounded to the nearest tenth of a litre.

77. Use the conversion 1 gal = 3.785 L to convert 50.5 litres to gallons, rounded to the nearest tenth of a gallon.

78. Use the conversion 1 gal = 3.785 L to convert 55.9 litres to gallons, rounded to the nearest tenth of a gallon.

79. Use the conversion 1 L = 33.8 fl oz to convert 8.3 kilolitres to fluid ounces.

80. Use the conversion 1 L = 33.8 fl oz to convert 5.3 kilolitres to fluid ounces.

81. Use the conversion 1 qt = 0.946 L to convert 42.4 litres to quarts, rounded to the nearest tenth of a quart.

82. Use the conversion 1 qt = 0.946 L to convert 55.4 litres to quarts, rounded to the nearest tenth of a quart.

83. Use the conversion 1 gal = 3.785 L to convert 17.2 gallons to litres, rounded to the nearest tenth of a litre.

84. Use the conversion 1 gal = 3.785 L to convert 19.6 gallons to litres, rounded to the nearest tenth of a litre.

85. Use the conversion 1 L = 33.8 fl oz to convert 51274 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.

86. Use the conversion 1 L = 33.8 fl oz to convert 82164 fluid ounces to kilolitres, rounded to the nearest tenth of a kilolitre.

87. Use the conversion 1 gal = 3.785 L to convert 55.6 litres to gallons, rounded to the nearest tenth of a gallon.

88. Use the conversion 1 gal = 3.785 L to convert 59.2 litres to gallons, rounded to the nearest tenth of a gallon.

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89. Change 60 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

90. Change 56 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

91. Change 77 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

92. Change 57 miles per hour to kilometers per hour. Round your answer to the nearest kilometer per hour.

93. Change 42 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

94. Change 56 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

95. Change 62 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

96. Change 63 kilometers per hour to miles per hour. Round your answer to the nearest mile per hour.

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97. **Tallest tower.** The world's tallest tower in Dubai has 160 floors at a height of 2, 717 feet. Convert the height of the tower to the nearest tenth of a meter. *Associated Press-Times-Standard 02/09/10 World's tallest tower closed a month after opening.*

98. **High peaks.** For the first time, foreigners will be allowed to climb nearly 100 high-altitude Himalayan peaks on the Indian side of Kashmir, peaks ranging from 9, 840 feet to nearly 26, 246 feet. Convert the highest of the peaks to the nearest tenth of a meter. *Associated Press-TimesStandard 04/11/10 India opens Himalayan peaks to foreigners.*

99. **Ancient find.** In the southern Egyptian town of Luxor, a 3, 400-year-old 4-meter statue of Thoth, the ancient Egyptian god of Wisdom and Magic, was unearthed. Convert the height of the statue to the nearest tenth of a foot. *Associated Press-Times-Standard 03/17/10 3,400-year-old statues unearthed in Egypt.*

100. **Arctic wind.** Blizzard condition winds in the Arctic blew 80 miles per hour. Find the wind speed to the nearest kilometer per hour. *Associated Press-Times-Standard 12/31/09 After 562-mile ski trek, seven women reach the South Pole.*

101. **Solar plane.** The Solar Impulse lifted off from a military airport at a speed no faster than 28 miles per hour. Convert the speed of the solar-powered plane to the nearest kilometer per hour. *Associated Press-Times-Standard 04/09/10 Solar-powered plane makes successful maiden flight.*

Answers

1. 172.7 centimeters
3. 17.3 inches
5. 127.1 kilometers
7. 16.3 yards
9. 2560.3 centimeters
11. 338.6 inches
13. 152.4 centimeters
15. 5.3 meters
17. 18.3 meters
19. 46.7 kilometers
21. 9.0 yards
23. 3.0 miles
25. 22.9 meters
27. 18.5 inches
29. 5.2 miles
31. 3749.0 centimeters
33. 145.7 inches
35. 14.5 yards
37. 9.2 yards
39. 7.3 meters
41. 34.8 pounds
43. 992.3 grams
45. 87.3 ounces
47. 82.7 ounces
49. 6.8 kilograms
51. 22.9 pounds
53. 10.0 kilograms
55. 3.4 ounces
57. 396.9 grams
59. 24.5 kilograms
61. 3.2 ounces
63. 11.0 kilograms
65. 58.2 quarts
67. 2.1 kilolitres

69. 84500 ounces

71. 22.7 litres

73. 28.4 litres

75. 44.7 litres

77. 13.3 gallons

79. 280540 ounces

81. 44.8 quarts

83. 65.1 litres

85. 1.5 kilolitres

87. 14.7 gallons

89. 97 km/hr

91. 124 km/hr

93. 26 mi/hr

95. 39 mi/hr

97. 828.2 meters

99. 13.1 feet

101. 45 km/hr

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6.7: Use a Problem Solving Strategy (Part 1)

Learning Objectives

- Approach word problems with a positive attitude
- Use a problem solving strategy for word problems
- Solve number problems

be prepared!

Before you get started, take this readiness quiz.

1. Translate “6 less than twice x” into an algebraic expression. If you missed this problem, review Example 2.4.13.
2. Solve: $\frac{2}{3}x = 24$. If you missed this problem, review Example 8.3.4.
3. Solve: $3x + 8 = 14$. If you missed this problem, review Example 8.4.1.

Approach Word Problems with a Positive Attitude

The world is full of word problems. How much money do I need to fill the car with gas? How much should I tip the server at a restaurant? How many socks should I pack for vacation? How big a turkey do I need to buy for Thanksgiving dinner, and what time do I need to put it in the oven? If my sister and I buy our mother a present, how much will each of us pay?

Now that we can solve equations, we are ready to apply our new skills to word problems. Do you know anyone who has had negative experiences in the past with word problems? Have you ever had thoughts like the student in Figure 6.7.1?

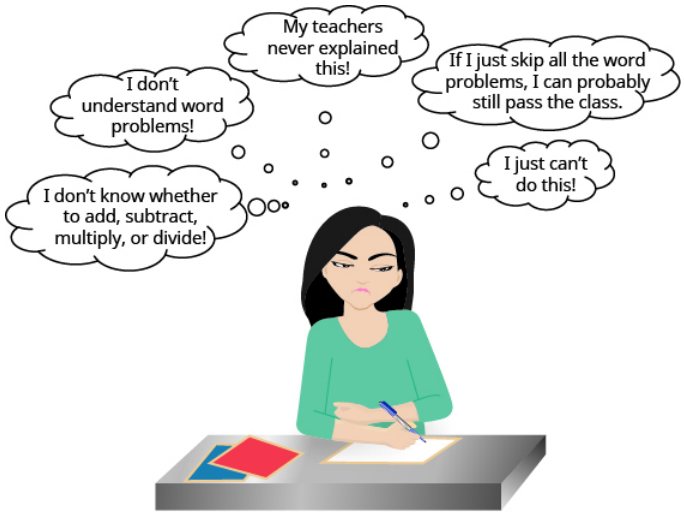

Figure 6.7.1 - Negative thoughts about word problems can be barriers to success.

When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. We need to calm our fears and change our negative feelings.

Start with a fresh slate and begin to think positive thoughts like the student in Figure 6.7.2. Read the positive thoughts and say them out loud.

Figure 6.7.2 - When it comes to word problems, a positive attitude is a big step toward success.

If we take control and believe we can be successful, we will be able to master word problems.

Think of something that you can do now but couldn't do three years ago. Whether it's driving a car, snowboarding, cooking a gourmet meal, or speaking a new language, you have been able to learn and master a new skill. Word problems are no different. Even if you have struggled with word problems in the past, you have acquired many new math skills that will help you succeed now!

Use a Problem-solving Strategy for Word Problems

In earlier chapters, you translated word phrases into algebraic expressions, using some basic mathematical vocabulary and symbols. Since then you've increased your math vocabulary as you learned about more algebraic procedures, and you've had more practice translating from words into algebra.

You have also translated word sentences into algebraic equations and solved some word problems. The word problems applied math to everyday situations. You had to restate the situation in one sentence, assign a variable, and then write an equation to solve. This method works as long as the situation is familiar to you and the math is not too complicated.

Now we'll develop a strategy you can use to solve any word problem. This strategy will help you become successful with word problems. We'll demonstrate the strategy as we solve the following problem.

Example 6.7.1:

Pete bought a shirt on sale for $18, which is one-half the original price. What was the original price of the shirt?

Solution

Step 1. **Read** the problem. Make sure you understand all the words and ideas. You may need to read the problem two or more times. If there are words you don't understand, look them up in a dictionary or on the Internet.

- *In this problem, do you understand what is being discussed? Do you understand every word?*

Step 2. **Identify** what you are looking for. It's hard to find something if you are not sure what it is! Read the problem again and look for words that tell you what you are looking for!

- *In this problem, the words "what was the original price of the shirt" tell you that what you are looking for: the original price of the shirt.*

Step 3. **Name** what you are looking for. Choose a variable to represent that quantity. You can use any letter for the variable, but it may help to choose one that helps you remember what it represents.

- *Let p = the original price of the shirt.*

Step 4. **Translate** into an equation. It may help to first restate the problem in one sentence, with all the important information. Then translate the sentence into an equation.

18	is	one-half	of	the original price .
18	=	$\frac{1}{2}$	·	p

Step 5. **Solve** the equation using good algebra techniques. Even if you know the answer right away, using algebra will better prepare you to solve problems that do not have obvious answers.

Write the equation.	$18 = \frac{1}{2}p$
Multiply both sides by 2.	$2 \cdot 18 = 2 \cdot \frac{1}{2}p$
Simplify.	$36 = p$

Step 6. **Check** the answer in the problem and make sure it makes sense.

- *We found that p = 36, which means the original price was $36. Does $36 make sense in the problem? Yes, because 18 is one-half of 36, and the shirt was on sale at half the original price.*

Step 7. **Answer** the question with a complete sentence.

- *The problem asked "What was the original price of the shirt?" The answer to the question is: "The original price of the shirt was $36."*

If this were a homework exercise, our work might look like this:

Let p = the original price.	18 is one-half the original price. $18 = \frac{1}{2}p$ $2 \cdot 18 = 2 \cdot \frac{1}{2}p$ $36 = p$
Check: Is $36 a reasonable price for a shirt? *Yes.* Is 18 one-half of 36? *Yes.*	The original price of the shirt was $36.

? Exercise 6.7.1:

Joaquin bought a bookcase on sale for $120, which was two-thirds the original price. What was the original price of the bookcase?

Answer

$180

? Exercise 6.7.2:

Two-fifths of the people in the senior center dining room are men. If there are 16 men, what is the total number of people in the dining room?

Answer

40

We list the steps we took to solve the previous example.

Problem-Solving Strategy

- Step 1. **Read** the word problem. Make sure you understand all the words and ideas. You may need to read the problem two or more times. If there are words you don't understand, look them up in a dictionary or on the internet.
- Step 2. **Identify** what you are looking for.
- Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.
- Step 4. **Translate** into an equation. It may be helpful to first restate the problem in one sentence before translating.
- Step 5. **Solve** the equation using good algebra techniques.
- Step 6. **Check** the answer in the problem. Make sure it makes sense.
- Step 7. **Answer** the question with a complete sentence.

Let's use this approach with another example.

Example 6.7.2:

Yash brought apples and bananas to a picnic. The number of apples was three more than twice the number of bananas. Yash brought 11 apples to the picnic. How many bananas did he bring?

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	How many bananas did he bring?
Step 3. **Name** what you are looking for. Choose a variable to represent the number of bananas.	Let b = number of bananas
Step 4. **Translate**. Restate the problem in one sentence with all the important information. Translate into an equation.	$\underbrace{\text{The number of apples}}_{11}\ \underbrace{\text{was}}_{=}\ \underbrace{3}_{3}\ \underbrace{\text{more than}}_{+}\ \underbrace{\text{twice the number of bananas}}_{2b}.$
Step 5. **Solve** the equation.	$11 = 3b + 3 \quad (6.7.1)$
Subtract 3 from each side.	$11 - 3 = 2b + 3 - 3$
Simplify.	$8 = 2b$
Divide each side by 2.	$\frac{8}{2} = \frac{2b}{2}$
Simplify.	$4 = b$
Step 6. **Check**: First, is our answer reasonable? Yes, bringing four bananas to a picnic seems reasonable. The problem says the number of apples was three more than twice the number of bananas. If there are four bananas, does that make eleven apples? Twice 4 bananas is 8. Three more than 8 is 11.	
Step 7. **Answer** the question.	Yash brought 4 bananas to the picnic.

Exercise 6.7.3:

Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was 3 more than the number of notebooks. He bought 5 textbooks. How many notebooks did he buy?

Answer

2

Exercise 6.7.4:

Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is seven more than the number of crossword puzzles. He completed 14 Sudoku puzzles. How many crossword puzzles did he complete?

Answer

7

In Solve Sales Tax, Commission, and Discount Applications, we learned how to translate and solve basic percent equations and used them to solve sales tax and commission applications. In the next example, we will apply our Problem Solving Strategy to more applications of percent.

Example 6.7.3:

Nga's car insurance premium increased by $60, which was 8% of the original cost. What was the original cost of the premium?

Solution

Step 1. **Read** the problem. Remember, if there are words you don't understand, look them up.		
Step 2. **Identify** what you are looking for.	the original cost of the premium	
Step 3. **Name**. Choose a variable to represent the original cost of premium.	Let c = the original cost	
Step 4. **Translate**. Restate as one sentence. Translate into an equation.	$\underbrace{\$60}_{60}$ $\underbrace{\text{was}}_{=}$ $\underbrace{8\%}_{0.08}$ $\underbrace{\text{of}}_{\cdot}$ $\underbrace{\text{the original cost}}_{c}$.	
Step 5. **Solve** the equation.	$60 = 0.08c$	(6.7.2)
Divide both sides by 0.08.	$\frac{60}{0.08} = \frac{0.08c}{0.08}$	(6.7.3)
Simplify.	$c = 750$	(6.7.4)
Step 6. **Check**: Is our answer reasonable? Yes, a $750 premium on auto insurance is reasonable. Now let's check our algebra. Is 8% of 750 equal to 60?	$750 = c$ $0.08(750) = 60$ $60 = 60$ ✓	
Step 7. **Answer** the question.	The original cost of Nga's premium was $750.	

Exercise 6.7.5:

Pilar's rent increased by 4%. The increase was $38. What was the original amount of Pilar's rent?

Answer

$950

Exercise 6.7.6:

Steve saves 12% of his paycheck each month. If he saved $504 last month, how much was his paycheck?

Answer

$4200

Solve Number Problems

Now we will translate and solve **number problems**. In number problems, you are given some clues about one or more numbers, and you use these clues to build an equation. Number problems don't usually arise on an everyday basis, but they provide a good introduction to practicing the Problem Solving Strategy. Remember to look for clue words such as *difference*, *of*, and *and*.

Example 6.7.4:

The difference of a number and six is 13. Find the number.

Solution

Step 1. **Read** the problem. Do you understand all the words?	
Step 2. **Identify** what you are looking for.	the number
Step 3. **Name**. Choose a variable to represent the number.	Let n = the number
Step 4. **Translate**. Restate as one sentence. Translate into an equation.	$\underbrace{\text{The difference of a number and 6}}\ \underbrace{\text{is}}\ \underbrace{\text{13.}}$ $n-6 \quad = \quad 13$
Step 5. **Solve** the equation. Add 6 to both sides. Simplify.	$n-6=13$ $n-+6=13+6$ $n=19$
Step 6. **Check**: The difference of 19 and 6 is 13. It checks.	
Step 7. **Answer** the question.	The number is 19.

Exercise 6.7.7:

The difference of a number and eight is 17. Find the number.

Answer

25

Exercise 6.7.8:

The difference of a number and eleven is −7. Find the number.

Answer

4

Example 6.7.5:

The sum of twice a number and seven is 15. Find the number.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the number
Step 3. **Name**. Choose a variable to represent the number.	Let n = the number
Step 4. **Translate**. Restate the problem as one sentence. Translate into an equation.	$\underbrace{\text{The sum of twice a number}}_{2 \cdot n}\ \underbrace{\text{and}}_{+}\ \underbrace{7}_{7}\ \underbrace{\text{is}}_{=}\ \underbrace{15.}_{15}$
Step 5. **Solve** the equation.	$2n + 7 = 15$ (6.7.5)
Subtract 7 from each side and simplify.	$2n = 8$ (6.7.6)
Divide each side by 2 and simplify.	$n = 4$ (6.7.7)
Step 6. **Check**: is the sum of twice 4 and 7 equal to 15?	$2 \cdot 4 + 7 = 15$ $8 + 7 = 15$ $15 = 15$ ✓
Step 7. **Answer** the question.	The number is 4.

? Exercise 6.7.9:

The sum of four times a number and two is 14. Find the number.

Answer

3

? Exercise 6.7.10:

The sum of three times a number and seven is 25. Find the number.

Answer

6

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6.8: Use a Problem Solving Strategy (Part 2)

Some number word problems ask you to find two or more numbers. It may be tempting to name them all with different variables, but so far we have only solved equations with one variable. We will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

Example 6.8.6:

One number is five more than another. The sum of the numbers is twenty-one. Find the numbers.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	You are looking for two numbers.
Step 3. **Name**. Choose a variable to represent the first number. What do you know about the second number? Translate.	Let n = 1st number. One number is five more than another. x + 5 = 2nd number
Step 4. **Translate**. Restate the problem as one sentence with all the important information. Translate into an equation. Substitute the variable expressions.	The sum of the numbers is 21. The sum of the 1st number and the 2nd number is 21. $\underbrace{1^{st}\text{ number}}_{n} + \underbrace{2^{nd}\text{ number}}_{n+5} = 21$
Step 5. **Solve** the equation.	$n+n+5=21$ (9.1.15)
Combine like terms.	$2n+5=21$ (9.1.16)
Subtract five from both sides and simplify.	$2n=16$ (9.1.17)
Divide by two and simplify.	$n=8$ $1^{st}\ number$ (9.1.18)
Find the second number too.	$n+5$ $2^{nd}\ number$ (9.1.19)
Substitute n = 8.	$8+5$ (9.1.20)
Step 6. **Check**: Do these numbers check in the problem? Is one number 5 more than the other? Is thirteen, 5 more than 8? Yes.	$13 \stackrel{?}{=} 8+5$ $13=13$ ✓
Is the sum of the two numbers 21?	$8+13=21$ $21=21$ ✓
Step 7. **Answer** the question.	The numbers are 8 and 13.

Exercise 6.8.11:

One number is six more than another. The sum of the numbers is twenty-four. Find the numbers.

Answer

9, 15

Exercise 6.8.12:

The sum of two numbers is fifty-eight. One number is four more than the other. Find the numbers.

Answer

27, 31

Example 6.8.7:

The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	two numbers
Step 3. **Name**. Choose a variable. What do you know about the second number? Translate.	Let n = 1st number One number is 4 less than the other. n - 4 = 2nd number
Step 4. **Translate**. Write as one sentence. Translate into an equation. Substitute the variable expressions.	The sum of two numbers is negative fourteen. $\underbrace{1^{st}\text{ number}}_{n} + \underbrace{2^{nd}\text{ number}}_{n-4} = -14$ $n + n - 4 = -14$
Step 5. **Solve** the equation.	$n + n - 4 = -14$ (9.1.21)
Combine like terms.	$2n - 4 = -14$ (9.1.22)
Add 4 to each side and simplify.	$2n = -10$ (9.1.23)
Divide by 2.	$n = -5$ 1st number (9.1.24)
Substitute n = −5 to find the 2nd number.	$n-4$ 2nd number $-5-4$ -9
Step 6. **Check**: Is −9 four less than −5?	$-5-4 \stackrel{?}{=} -9$ $-9 = -9$ ✓
Is their sum −14?	$-5+(-9) \stackrel{?}{=} -14$ $-14 = -14$ ✓
Step 7. **Answer** the question.	The numbers are −5 and −9.

Exercise 6.8.13:

The sum of two numbers is negative twenty-three. One number is 7 less than the other. Find the numbers.

Answer

-8, -15

Exercise 6.8.14:

The sum of two numbers is negative eighteen. One number is 40 more than the other. Find the numbers.

Answer

-29, 11

Example 6.8.8:

One number is ten more than twice another. Their sum is one. Find the numbers.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	two numbers
Step 3. **Name**. Choose a variable. One number is ten more than twice another.	Let x = 1st number 2x + 10 = 2nd number
Step 4. **Translate**. Restate as one sentence.	Their sum is one.
Translate into an equation.	$\underbrace{\text{The sum of the two numbers}}\ \underbrace{\text{is}}\ \underbrace{1.}$ $x+(2x+10) = 1$
Step 5. **Solve** the equation.	$x+2x+10=1$ (9.1.25)
Combine like terms.	$3x+10=1$ (9.1.26)
Subtract 10 from each side.	$3x=-9$ (9.1.27)
Divide each side by 3 to get the first number.	$x=-3$ (9.1.28)
Substitute to get the second number.	$2x+10$ $2(-3)+10$ 4
Step 6. **Check**: Is 4 ten more than twice −3?	$2(-3)+10 \stackrel{?}{=} 4$ $-6+10=4$ $4=4$ ✓
Is their sum 1?	$-3+4 \stackrel{?}{=} 1$ $1=1$ ✓
Step 7. **Answer** the question.	The numbers are −3 and 4.

Exercise 6.8.15:

One number is eight more than twice another. Their sum is negative four. Find the numbers.

Answer

-4, 0

Exercise 6.8.16:

One number is three more than three times another. Their sum is negative five. Find the numbers.

Answer

-2, -3

Consecutive integers are integers that immediately follow each other. Some examples of consecutive integers are:

$$\ldots 1, 2, 3, 4, \ldots \tag{9.1.29}$$

$$\ldots -10, -9, -8, -7, \ldots \tag{9.1.30}$$

$$\ldots 150, 151, 152, 153, \ldots \tag{9.1.31}$$

Notice that each number is one more than the number preceding it. So if we define the first integer as n, the next consecutive integer is n + 1. The one after that is one more than n + 1, so it is n + 1 + 1, or n + 2.

$$\begin{aligned} n &\quad 1^{st}\ integer \\ n+1 &\quad 2^{nd}\ consecutive\ integer \\ n+2 &\quad 3^{rd}\ consecutive\ integer \end{aligned}$$

Example 6.8.9:

The sum of two consecutive integers is 47. Find the numbers.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	two consecutive integers
Step 3. **Name**.	Let n = 1st integer n + 1 = next consecutive integer
Step 4. **Translate**. Restate as one sentence. Translate into an equation.	The sum of the integers is 47. $n + n + 1 = 47$
Step 5. **Solve** the equation.	$n + n + 1 = 47$ (9.1.32)
Combine like terms.	$2n + 1 = 47$ (9.1.33)
Subtract 1 from each side.	$2n = 46$ (9.1.34)
Divide each side by 2.	$n = 23$ $1^{st}\ integer$ (9.1.35)
Substitute to get the second number.	$n+1$ $2^{nd}\ integer$ $23+1$ 24
Step 6. **Check**.	$23 + 24 \stackrel{?}{=} 47$ $47 = 47$ ✓
Step 7. **Answer** the question.	The two consecutive integers are 23 and 24.

Exercise 6.8.17:

The sum of two consecutive integers is 95. Find the numbers.

Answer

47, 48

Exercise 6.8.18:

The sum of two consecutive integers is −31. Find the numbers.

Answer

-15, -16

Example 6.8.10:

Find three consecutive integers whose sum is 42.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	three consecutive integers
Step 3. **Name**.	Let n = 1st integer n + 1 = 2nd consecutive integer n + 2 = 3rd consecutive integer
Step 4. **Translate**. Restate as one sentence. Translate into an equation.	$\underbrace{\text{The sum of the three integers}}\ \underbrace{\text{is}}\ \underbrace{42.}$ $n + n+1 + n+2 = 42$
Step 5. **Solve** the equation.	$n+n+1+n+2=42$ (9.1.36)
Combine like terms.	$3n+3=42$ (9.1.37)
Subtract 3 from each side.	$3n=39$ (9.1.38)
Divide each side by 3.	$n=13$ $1^{st}\ integer$ (9.1.39)
Substitute to get the second number.	$n+1$ $2^{nd}\ integer$ $13+1$ 24
Substitute to get the third number.	$n+2$ $3^{rd}\ integer$ $13+2$ 15
Step 6. **Check**.	$13+14+15 \stackrel{?}{=} 42$ $42=42$ ✓
Step 7. **Answer** the question.	The three consecutive integers are 13, 14, and 15.

Exercise 6.8.19:

Find three consecutive integers whose sum is 96.

Answer

31, 32, 33

Exercise 6.8.20:

Find three consecutive integers whose sum is −36.

Answer

-11, -12, -13

Practice Makes Perfect

Use a Problem-solving Strategy for Word Problems

In the following exercises, use the problem-solving strategy for word problems to solve. Answer in complete sentences.

1. Two-thirds of the children in the fourth-grade class are girls. If there are 20 girls, what is the total number of children in the class?
2. Three-fifths of the members of the school choir are women. If there are 24 women, what is the total number of choir members?
3. Zachary has 25 country music CDs, which is one-fifth of his CD collection. How many CDs does Zachary have?
4. One-fourth of the candies in a bag of are red. If there are 23 red candies, how many candies are in the bag?
5. There are 16 girls in a school club. The number of girls is 4 more than twice the number of boys. Find the number of boys in the club.
6. There are 18 Cub Scouts in Troop 645. The number of scouts is 3 more than five times the number of adult leaders. Find the number of adult leaders.
7. Lee is emptying dishes and glasses from the dishwasher. The number of dishes is 8 less than the number of glasses. If there are 9 dishes, what is the number of glasses?
8. The number of puppies in the pet store window is twelve less than the number of dogs in the store. If there are 6 puppies in the window, what is the number of dogs in the store?
9. After 3 months on a diet, Lisa had lost 12% of her original weight. She lost 21 pounds. What was Lisa's original weight?
10. Tricia got a 6% raise on her weekly salary. The raise was $30 per week. What was her original weekly salary?
11. Tim left a $9 tip for a $50 restaurant bill. What percent tip did he leave?
12. Rashid left a $15 tip for a $75 restaurant bill. What percent tip did he leave?
13. Yuki bought a dress on sale for $72. The sale price was 60% of the original price. What was the original price of the dress?
14. Kim bought a pair of shoes on sale for $40.50. The sale price was 45% of the original price. What was the original price of the shoes?

Solve Number Problems

In the following exercises, solve each number word problem.

15. The sum of a number and eight is 12. Find the number.
16. The sum of a number and nine is 17. Find the number.
17. The difference of a number and twelve is 3. Find the number.
18. The difference of a number and eight is 4. Find the number.
19. The sum of three times a number and eight is 23. Find the number.
20. The sum of twice a number and six is 14. Find the number.
21. The difference of twice a number and seven is 17. Find the number.
22. The difference of four times a number and seven is 21. Find the number.
23. Three times the sum of a number and nine is 12. Find the number.
24. Six times the sum of a number and eight is 30. Find the number.

25. One number is six more than the other. Their sum is forty-two. Find the numbers.
26. One number is five more than the other. Their sum is thirty-three. Find the numbers.
27. The sum of two numbers is twenty. One number is four less than the other. Find the numbers.
28. The sum of two numbers is twenty-seven. One number is seven less than the other. Find the numbers.
29. A number is one more than twice another number. Their sum is negative five. Find the numbers.
30. One number is six more than five times another. Their sum is six. Find the numbers.
31. The sum of two numbers is fourteen. One number is two less than three times the other. Find the numbers.
32. The sum of two numbers is zero. One number is nine less than twice the other. Find the numbers.
33. One number is fourteen less than another. If their sum is increased by seven, the result is 85. Find the numbers.
34. One number is eleven less than another. If their sum is increased by eight, the result is 71. Find the numbers.
35. The sum of two consecutive integers is 77. Find the integers.
36. The sum of two consecutive integers is 89. Find the integers.
37. The sum of two consecutive integers is −23. Find the integers.
38. The sum of two consecutive integers is −37. Find the integers.
39. The sum of three consecutive integers is 78. Find the integers.
40. The sum of three consecutive integers is 60. Find the integers.
41. Find three consecutive integers whose sum is −36.
42. Find three consecutive integers whose sum is −3.

Everyday Math

43. **Shopping** Patty paid $35 for a purse on sale for $10 off the original price. What was the original price of the purse?
44. **Shopping** Travis bought a pair of boots on sale for $25 off the original price. He paid $60 for the boots. What was the original price of the boots?
45. **Shopping** Minh spent $6.25 on 5 sticker books to give his nephews. Find the cost of each sticker book.
46. **Shopping** Alicia bought a package of 8 peaches for $3.20. Find the cost of each peach.
47. **Shopping** Tom paid $1,166.40 for a new refrigerator, including $86.40 tax. What was the price of the refrigerator before tax?
48. **Shopping** Kenji paid $2,279 for a new living room set, including $129 tax. What was the price of the living room set before tax?

Writing Exercises

49. Write a few sentences about your thoughts and opinions of word problems. Are these thoughts positive, negative, or neutral? If they are negative, how might you change your way of thinking in order to do better?
50. When you start to solve a word problem, how do you decide what to let the variable represent?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
approach word problems with a positive attitude.			
use a problem solving strategy for word problems.			
solve number problems.			

(b) If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don’t get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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6.9: Solve a Formula for a Specific Variable

Learning Objectives

- Use the distance, rate, and time formula
- Solve a formula for a specific variable

Be Prepared!

Before you get started, take this readiness quiz.

1. Write 35 miles per gallon as a unit rate. If you missed this problem, review Example 5.11.8.
2. Solve 6x + 24 = 96. If you missed this problem, review Example 8.4.1.
3. Find the simple interest earned after 5 years on $1,000 at an interest rate of 4%. If you missed this problem, review Example 6.4.1.

Use the Distance, Rate, and Time Formula

One formula you'll use often in algebra and in everyday life is the formula for distance traveled by an object moving at a constant speed. The basic idea is probably already familiar to you. Do you know what distance you travel if you drove at a steady rate of 60 miles per hour for 2 hours? (This might happen if you use your car's cruise control while driving on the Interstate.) If you said 120 miles, you already know how to use this formula!

The math to calculate the distance might look like this:

$$distance = \left(\frac{60\ miles}{1\ hour}\right)(2\ hours)$$
$$distance = 120\ miles$$

In general, the formula relating distance, rate, and time is

$$distance = rate \cdot time \tag{6.9.1}$$

Definition: Distance, Rate and Time

For an object moving in at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula

$$d = rt \tag{6.9.2}$$

where d = distance, r = rate, and t = time.

Notice that the units we used above for the rate were miles per hour, which we can write as a ratio $\frac{miles}{hour}$. Then when we multiplied by the time, in hours, the common units 'hour' divided out. The answer was in miles.

Example 6.9.1:

Jamal rides his bike at a uniform rate of 12 miles per hour for $3\frac{1}{2}$ hours. How much distance has he traveled?

Solution

Step 1. **Read** the problem. You may want to create a mini-chart to summarize the information in the problem.	$d = ?$ $r = 12\ mph$ $t = 3\frac{1}{2}\ hours$
Step 2. **Identify** what you are looking for.	distance traveled

Step 3. **Name**. Choose a variable to represent it.	let d = distance
Step 4. **Translate**. Write the appropriate formula for the situation. Substitute in the given information.	$d = rt$ $d = 12 \cdot 3\frac{1}{2}$
Step 5. **Solve** the equation.	d = 42 miles
Step 6. **Check**: Does 42 miles make sense?	Jamal rides 12 miles in 1 hour, 24 miles in 2 hours, 36 miles in 3 hours, 48 miles in 4 hours, 42 miles in $3\frac{1}{2}$ hours is reasonable
Step 7. **Answer** the question with a complete sentence.	Jamal rode 42 miles.

? Exercise 6.9.1:

Lindsay drove for $5\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Answer

330 mi

? Exercise 6.9.2:

Trinh walked for $2\frac{1}{3}$ hours at 3 miles per hour. How far did she walk?

Answer

7 mi

✓ Example 6.9.2:

Rey is planning to drive from his house in San Diego to visit his grandmother in Sacramento, a distance of 520 miles. If he can drive at a steady rate of 65 miles per hour, how many hours will the trip take?

Solution

Step 1. **Read** the problem. Summarize the information in the problem.	$d = 520\ miles$ $r = 65\ mph$ $t = ?$
Step 2. **Identify** what you are looking for.	how many hours (time)
Step 3. **Name**. Choose a variable to represent it.	let t = time
Step 4. **Translate**. Write the appropriate formula for the situation. Substitute in the given information.	$d = rt$ $520 = 65t$
Step 5. **Solve** the equation.	t = 8

Step 6. **Check**: Substitute the numbers into the formula and make sure the result is a true statement.	$d = rt$ $520 \stackrel{?}{=} 65 \cdot 8$ $520 = 520$ ✓
Step 7. **Answer** the question with a complete sentence. We know the units of time will be hours because we divided miles by miles per hour.	Rey's trip will take 8 hours.

? Exercise 6.9.3:

Lee wants to drive from Phoenix to his brother's apartment in San Francisco, a distance of 770 miles. If he drives at a steady rate of 70 miles per hour, how many hours will the trip take?

Answer

11 hours

? Exercise 6.9.4:

Yesenia is 168 miles from Chicago. If she needs to be in Chicago in 3 hours, at what rate does she need to drive?

Answer

56 mph

Solve a Formula for a Specific Variable

In this chapter, you became familiar with some formulas used in geometry. Formulas are also very useful in the sciences and social sciences—fields such as chemistry, physics, biology, psychology, sociology, and criminal justice. Healthcare workers use formulas, too, even for something as routine as dispensing medicine. The widely used spreadsheet program Microsoft Excel™ relies on formulas to do its calculations. Many teachers use spreadsheets to apply formulas to compute student grades. It is important to be familiar with formulas and be able to manipulate them easily.

In Example 9.57 and Example 9.58, we used the formula d = rt. This formula gives the value of d when you substitute in the values of r and t. But in Example 9.58, we had to find the value of t. We substituted in values of d and r and then used algebra to solve to t. If you had to do this often, you might wonder why there isn't a formula that gives the value of t when you substitute in the values of d and r. We can get a formula like this by solving the formula d = rt for t.

To solve a formula for a specific variable means to get that variable by itself with a coefficient of 1 on one side of the equation and all the other variables and constants on the other side. We will call this solving an equation for a specific variable in general. This process is also called solving a literal equation. The result is another formula, made up only of variables. The formula contains letters, or literals.

Let's try a few examples, starting with the distance, rate, and time formula we used above.

✓ Example 6.9.3:

Solve the formula d = rt for t: (a) when d = 520 and r = 65 (b) in general.

Solution

We'll write the solutions side-by-side so you can see that solving a formula in general uses the same steps as when we have numbers to substitute.

	(a) when d = 520 and r = 65	(b) in general
Write the formula.	d = rt	d = rt

Substitute any given values.	520 = 65t	
Divide to isolate t.	$\frac{520}{65} = \frac{65t}{65}$ (6.9.3)	$\frac{d}{r} = \frac{rt}{r}$ (6.9.4)
Simplify.	$8 = t$ $t = 8$	$\frac{d}{r} = t$ $t = \frac{d}{r}$

Notice that the solution for (a) is the same as that in Example 9.58. We say the formula t = $\frac{d}{r}$ is solved for t. We can use this version of the formula anytime we are given the distance and rate and need to find the time.

? Exercise 6.9.5:

Solve the formula d = rt for r: (a) when d = 180 and t = 4 (b) in general

Answer a

$r = 45$

Answer b

$r = \frac{d}{t}$

? Exercise 6.9.6:

Solve the formula d = rt for r: (a) when d = 780 and t = 12 (b) in general

Answer a

$r = 65$

Answer b

$r = \frac{d}{t}$

We used the formula A = $\frac{1}{2}$bh in Use Properties of Rectangles, Triangles, and Trapezoids to find the area of a triangle when we were given the base and height. In the next example, we will solve this formula for the height.

✓ Example 6.9.4:

The formula for area of a triangle is A = $\frac{1}{2}$bh. Solve this formula for h: (a) when A = 90 and b = 15 (b) in general

Solution

	(a) when A = 90 and b = 15	(b) in general
Write the formula.	A = $\frac{1}{2}$bh	A = $\frac{1}{2}$bh
Substitute any given values.	$90 = \frac{1}{2} \cdot 15 \cdot h$ (6.9.5)	

Clear the fractions.	$2 \cdot 90 = 2 \cdot \frac{1}{2} \cdot 15 \cdot h$ (6.9.6)	$2 \cdot A = 2 \cdot \frac{1}{2} \cdot b \cdot h$ (6.9.7)
Simplify.	180 = 15h	2A = bh
Solve for h.	12 = h	$\frac{2A}{b}$ = h

We can now find the height of a triangle, if we know the area and the base, by using the formula

$$h = \frac{2A}{b} \tag{6.9.8}$$

Exercise 6.9.7:

Use the formula A = $\frac{1}{2}$bh to solve for h: (a) when A = 170 and b = 17 (b) in general

Answer a

$h = 20$

Answer b

$h = \frac{2A}{b}$

Exercise 6.9.8:

Use the formula A = $\frac{1}{2}$bh to solve for b: (a) when A = 62 and h = 31 (b) in general

Answer a

$b = 4$

Answer b

$b = \frac{2A}{h}$

In Solve Simple Interest Applications, we used the formula I = Prt to calculate simple interest, where I is interest, P is principal, r is rate as a decimal, and t is time in years.

Example 6.9.5:

Solve the formula I = Prt to find the principal, P: (a) when I = \$5,600, r = 4%, t = 7 years (b) in general

Solution

	(a) when I = \$5,600, r = 4%, t = 7 years	(b) in general
Write the forumla.	I = Prt	I = Prt
Substitute any given values.	5600 = P(0.04)(7)	I = Prt
Multiply r • t.	5600 = P(0.28)	I = P(rt)
Divide to isolate P.	$\frac{5600}{0.28} = \frac{P(0.28)}{0.28}$ (6.9.9)	$\frac{I}{rt} = \frac{P(rt)}{rt}$ (6.9.10)
Simplify.	20,000 = P	$\frac{I}{rt}$ = P

State the answer.	The principal is $20,000.	$P = \frac{I}{rt}$ (6.9.11)

Exercise 6.9.9:

Use the formula I = Prt. Find t: (a) when I = $2,160, r = 6%, P = $12,000; (b) in general

Answer a

$t = 3$ years

Answer b

$t = \frac{I}{Pr}$

Exercise 6.9.10:

Use the formula I = Prt. Find r: (a) when I = $5,400, P = $9,000, t = 5 years; (b) in general

Answer a

$r = 0.12 = 12\%$

Answer b

$t = \frac{I}{Pt}$

Later in this course, and in future algebra classes, you'll encounter equations that relate two variables, usually x and y. You might be given an equation that is solved for y and need to solve it for x, or vice versa. In the following example, we're given an equation with both x and y on the same side and we'll solve it for y. To do this, we will follow the same steps that we used to solve a formula for a specific variable.

Example 6.9.6:

Solve the formula 3x + 2y = 18 for y: (a) when x = 4 (b) in general

Solution

	(a) when x = 4	(b) in general
Write the equation.	3x + 2y = 18	3x + 2y = 18
Substitute any given values.	3(4) + 2y = 18	3x + 2y = 18
Simplify if possible.	12 + 2y = 18	3x + 2y = 18
Subtract to isolate the y-term.	$12 - 12 + 2y = 18 - 12$ (6.9.12)	$3x - 3x + 2y = 18 - 3x$ (6.9.13)
Simplify.	2y = 6	2y = 18 - 3x
Divide.	$\frac{2y}{2} = \frac{6}{2}$ (6.9.14)	$\frac{2y}{2} = \frac{18 - 3x}{2}$ (6.9.15)
Simplify.	y = 3	$y = \frac{18 - 3x}{2}$ (6.9.16)

Exercise 6.9.11:

Solve the formula 3x + 4y = 10 for y: (a) when x = 2 (b) in general

Answer a

$y = 1$

Answer b

$y = \frac{10-3x}{4}$

Exercise 6.9.12:

Solve the formula 5x + 2y = 18 for y: (a) when x = 4 (b) in general

Answer a

$y = -1$

Answer b

$y = \frac{18-5x}{2}$

In the previous examples, we used the numbers in part (a) as a guide to solving in general in part (b). Do you think you're ready to solve a formula in general without using numbers as a guide?

Example 6.9.7:

Solve the formula P = a + b + c for a.

Solution

We will isolate a on one side of the equation.

We will isolate a on one side of the equation.	
Write the equation.	P = a + b + c
Subtract b and c from both sides to isolate a.	$P - b - c = a + b + c - b - c$ (6.9.17)
Simplify.	P − b − c = a

So, a = P − b − c.

Exercise 6.9.13:

Solve the formula P = a + b + c for b.

Answer

b = P - a - c

Exercise 6.9.14:

Solve the formula P = a + b + c for c.

Answer

c = P - a - b

Example 6.9.8:

Solve the equation 3x + y = 10 for y.

Solution

We will isolate y on one side of the equation.

We will isolate y on one side of the equation.		
Write the equation.	3x + y = 10	
Subtract 3x from both sides to isolate y.	$3x - 3x + y = 10 - 3x$	(6.9.18)
Simplify.	y = 10 - 3x	

Exercise 6.9.15:

Solve the formula 7x + y = 11 for y

Answer

y = 11 - 7x

Exercise 6.9.16:

Solve the formula 11x + y = 8 for y.

Answer

y = 8 - 11x

Example 6.9.9:

Solve the equation 6x + 5y = 13 for y.

Solution

We will isolate y on one side of the equation.

We will isolate y on one side of the equation.		
Write the equation.	6x + 5y = 13	
Subtract to isolate the term with y.	$6x + 5y - 6x = 13 - 6x$	(6.9.19)
Simplify.	5y = 13 - 6x	
Divide 5 to make the coefficient 1.	$\frac{5y}{5} = \frac{13 - 6x}{5}$	(6.9.20)
Simplify.	$y = \frac{13 - 6x}{5}$	(6.9.21)

Exercise 6.9.17:

Solve the formula 4x + 7y = 9 for y.

Answer

$y = \frac{9-4x}{7}$

Exercise 6.9.18:

Solve the formula 5x + 8y = 1 for y.

Answer

$y = \frac{1-5x}{8}$

ACCESS ADDITIONAL ONLINE RESOURCES

Distance = Rate x Time

Distance, Rate, Time

Simple Interest

Solving a Formula for a Specific Variable

Solving a Formula for a Specific Variable

Practice Makes Perfect

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

307. Steve drove for $8\frac{1}{2}$ hours at 72 miles per hour. How much distance did he travel?
308. Socorro drove for $4\frac{5}{6}$ hours at 60 miles per hour. How much distance did she travel?
309. Yuki walked for $1\frac{3}{4}$ hours at 4 miles per hour. How far did she walk?
310. Francie rode her bike for $2\frac{1}{2}$ hours at 12 miles per hour. How far did she ride?
311. Connor wants to drive from Tucson to the Grand Canyon, a distance of 338 miles. If he drives at a steady rate of 52 miles per hour, how many hours will the trip take?
312. Megan is taking the bus from New York City to Montreal. The distance is 384 miles and the bus travels at a steady rate of 64 miles per hour. How long will the bus ride be?
313. Aurelia is driving from Miami to Orlando at a rate of 65 miles per hour. The distance is 235 miles. To the nearest tenth of an hour, how long will the trip take?
314. Kareem wants to ride his bike from St. Louis, Missouri to Champaign, Illinois. The distance is 180 miles. If he rides at a steady rate of 16 miles per hour, how many hours will the trip take?
315. Javier is driving to Bangor, Maine, which is 240 miles away from his current location. If he needs to be in Bangor in 4 hours, at what rate does he need to drive?
316. Alejandra is driving to Cincinnati, Ohio, 450 miles away. If she wants to be there in 6 hours, at what rate does she need to drive?
317. Aisha took the train from Spokane to Seattle. The distance is 280 miles, and the trip took 3.5 hours. What was the speed of the train?
318. Philip got a ride with a friend from Denver to Las Vegas, a distance of 750 miles. If the trip took 10 hours, how fast was the friend driving?

Solve a Formula for a Specific Variable

In the following exercises, use the formula. d = rt.

319. Solve for t: (a) when d = 350 and r = 70 (b) in general
320. Solve for t: (a) when d = 240 and r = 60 (b) in general
321. Solve for t: (a) when d = 510 and r = 60 (b) in general
322. Solve for t: (a) when d = 175 and r = 50 (b) in general
323. Solve for r: (a) when d = 204 and t = 3 (b) in general
324. Solve for r: (a) when d = 420 and t = 6 (b) in general
325. Solve for r: (a) when d = 160 and t = 2.5 (b) in general
326. Solve for r: (a) when d = 180 and t = 4.5 (b) in general.

In the following exercises, use the formula $A = \frac{1}{2}bh$.

327. Solve for b: (a) when A = 126 and h = 18 (b) in general
328. Solve for h: (a) when A = 176 and b = 22 (b) in general
329. Solve for h: (a) when A = 375 and b = 25 (b) in general
330. Solve for b: (a) when A = 65 and h = 13 (b) in general

In the following exercises, use the formula I = Prt.

331. Solve for the principal, P for: (a) I = $5,480, r = 4%, t = 7 years (b) in general
332. Solve for the principal, P for: (a) I = $3,950, r = 6%, t = 5 years (b) in general
333. Solve for the time, t for: (a) I = $2,376, P = $9,000, r = 4.4% (b) in general
334. Solve for the time, t for: (a) I = $624, P = $6,000, r = 5.2% (b) in general

In the following exercises, solve.

335. Solve the formula 2x + 3y = 12 for y: (a) when x = 3 (b) in general
336. Solve the formula 5x + 2y = 10 for y: (a) when x = 4 (b) in general
337. Solve the formula 3x + y = 7 for y: (a) when x = −2 (b) in general
338. Solve the formula 4x + y = 5 for y: (a) when x = −3 (b) in general
339. Solve a + b = 90 for b.
340. Solve a + b = 90 for a.
341. Solve 180 = a + b + c for a.
342. Solve 180 = a + b + c for c.
343. Solve the formula 8x + y = 15 for y.
344. Solve the formula 9x + y = 13 for y.
345. Solve the formula −4x + y = −6 for y.
346. Solve the formula −5x + y = −1 for y.
347. Solve the formula 4x + 3y = 7 for y.
348. Solve the formula 3x + 2y = 11 for y.
349. Solve the formula x − y = −4 for y.
350. Solve the formula x − y = −3 for y.
351. Solve the formula P = 2L + 2W for L.
352. Solve the formula P = 2L + 2W for W.
353. Solve the formula C = πd for d.
354. Solve the formula C = πd for π.
355. Solve the formula V = LWH for L.
356. Solve the formula V = LWH for H.

Everyday Math

357. **Converting temperature** While on a tour in Greece, Tatyana saw that the temperature was 40° Celsius. Solve for F in the formula $C = \frac{5}{9}(F - 32)$ to find the temperature in Fahrenheit.

358. **Converting temperature** Yon was visiting the United States and he saw that the temperature in Seattle was 50° Fahrenheit. Solve for C in the formula $F = \frac{9}{5}C + 32$ to find the temperature in Celsius.

Writing Exercises

359. Solve the equation $2x + 3y = 6$ for y: (a) when $x = -3$ (b) in general (c) Which solution is easier for you? Explain why.
360. Solve the equation $5x - 2y = 10$ for x: (a) when $y = 10$ (b) in general (c) Which solution is easier for you? Explain why.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the distance, rate, and time formula.			
solve a formula for a specific variable.			

(b) Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

Contributors and Attributions

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6.E: Math Models and Geometry (Exercises)

9.1 - Use a Problem Solving Strategy

Approach Word Problems with a Positive Attitude

In the following exercises, solve.

361. How has your attitude towards solving word problems changed as a result of working through this chapter? Explain.
362. Did the Problem Solving Strategy help you solve word problems in this chapter? Explain.

Use a Problem Solving Strategy for Word Problems

In the following exercises, solve using the problem-solving strategy for word problems. Remember to write a complete sentence to answer each question.

363. Three-fourths of the people at a concert are children. If there are 87 children, what is the total number of people at the concert?
364. There are 9 saxophone players in the band. The number of saxophone players is one less than twice the number of tuba players. Find the number of tuba players
365. Reza was very sick and lost 15% of his original weight. He lost 27 pounds. What was his original weight?
366. Dolores bought a crib on sale for $350. The sale price was 40% of the original price. What was the original price of the crib?

Solve Number Problems

In the following exercises, solve each number word problem.

367. The sum of a number and three is forty-one. Find the number.
368. Twice the difference of a number and ten is fifty-four. Find the number.
369. One number is nine less than another. Their sum is twenty-seven. Find the numbers.
370. The sum of two consecutive integers is −135. Find the numbers.

9.2 - Solve Money Applications

Solve Coin Word Problems

In the following exercises, solve each coin word problem.

371. Francie has $4.35 in dimes and quarters. The number of dimes is 5 more than the number of quarters. How many of each coin does she have?
372. Scott has $0.39 in pennies and nickels. The number of pennies is 8 times the number of nickels. How many of each coin does he have?
373. Paulette has $140 in $5 and $10 bills. The number of $10 bills is one less than twice the number of $5 bills. How many of each does she have?
374. Lenny has $3.69 in pennies, dimes, and quarters. The number of pennies is 3 more than the number of dimes. The number of quarters is twice the number of dimes. How many of each coin does he have?

Solve Ticket and Stamp Word Problems

In the following exercises, solve each ticket or stamp word problem.

375. A church luncheon made $842. Adult tickets cost $10 each and children’s tickets cost $6 each. The number of children was 12 more than twice the number of adults. How many of each ticket were sold?
376. Tickets for a basketball game cost $2 for students and $5 for adults. The number of students was 3 less than 10 times the number of adults. The total amount of money from ticket sales was $619. How many of each ticket were sold?
377. Ana spent $4.06 buying stamps. The number of $0.41 stamps she bought was 5 more than the number of $0.26 stamps. How many of each did she buy?
378. Yumi spent $34.15 buying stamps. The number of $0.56 stamps she bought was 10 less than 4 times the number of $0.41 stamps. How many of each did she buy?

9.3 - Use Properties of Angles, Triangles, and the Pythagorean Theorem

Use Properties of Angles

In the following exercises, solve using properties of angles.

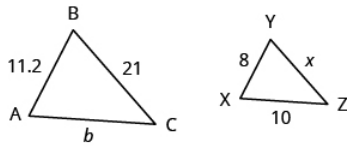
379. What is the supplement of a 48° angle?
380. What is the complement of a 61° angle?
381. Two angles are complementary. The smaller angle is 24° less than the larger angle. Find the measures of both angles.
382. Two angles are supplementary. The larger angle is 45° more than the smaller angle. Find the measures of both angles.

Use Properties of Triangles

In the following exercises, solve using properties of triangles.

383. The measures of two angles of a triangle are 22 and 85 degrees. Find the measure of the third angle.
384. One angle of a right triangle measures 41.5 degrees. What is the measure of the other small angle?
385. One angle of a triangle is 30° more than the smallest angle. The largest angle is the sum of the other angles. Find the measures of all three angles.
386. One angle of a triangle is twice the measure of the smallest angle. The third angle is 60° more than the measure of the smallest angle. Find the measures of all three angles.

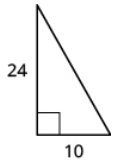
In the following exercises, ΔABC is similar to ΔXYZ. Find the length of the indicated side.

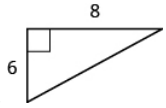


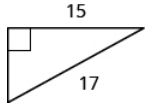
387. side x
388. side b

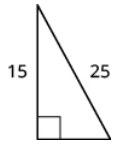
Use the Pythagorean Theorem

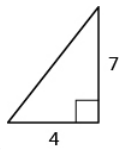
In the following exercises, use the Pythagorean Theorem to find the length of the missing side. Round to the nearest tenth, if necessary

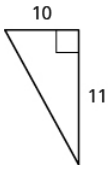
389. 

390. 

391. 

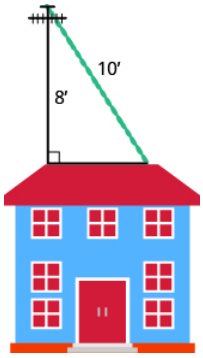
392. 

393. 

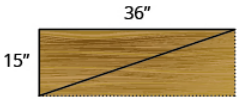
394. 

In the following exercises, solve. Approximate to the nearest tenth, if necessary.

395. Sergio needs to attach a wire to hold the antenna to the roof of his house, as shown in the figure. The antenna is 8 feet tall and Sergio has 10 feet of wire. How far from the base of the antenna can he attach the wire?



396. Seong is building shelving in his garage. The shelves are 36 inches wide and 15 inches tall. He wants to put a diagonal brace across the back to stabilize the shelves, as shown. How long should the brace be?



9.4 - Use Properties of Rectangles, Triangles, and Trapezoids

Understand Linear, Square, Cubic Measure

In the following exercises, would you measure each item using linear, square, or cubic measure?

397. amount of sand in a sandbag
398. height of a tree
399. size of a patio
400. length of a highway

In the following exercises, find (a) the perimeter (b) the area of each figure

401.

402.

Use Properties of Rectangles

In the following exercises, find the (a) perimeter (b) area of each rectangle

403. The length of a rectangle is 42 meters and the width is 28 meters.
404. The length of a rectangle is 36 feet and the width is 19 feet.
405. A sidewalk in front of Kathy's house is in the shape of a rectangle 4 feet wide by 45 feet long.
406. A rectangular room is 16 feet wide by 12 feet long.

In the following exercises, solve.

407. Find the length of a rectangle with perimeter of 220 centimeters and width of 85 centimeters.
408. Find the width of a rectangle with perimeter 39 and length 11.
409. The area of a rectangle is 2356 square meters. The length is 38 meters. What is the width?
410. The width of a rectangle is 45 centimeters. The area is 2700 square centimeters. What is the length?
411. The length of a rectangle is 12 centimeters more than the width. The perimeter is 74 centimeters. Find the length and the width.
412. The width of a rectangle is 3 more than twice the length. The perimeter is 96 inches. Find the length and the width.

Use Properties of Triangles

In the following exercises, solve using the properties of triangles.

413. Find the area of a triangle with base 18 inches and height 15 inches.
414. Find the area of a triangle with base 33 centimeters and height 21 centimeters.
415. A triangular road sign has base 30 inches and height 40 inches. What is its area?
416. If a triangular courtyard has sides 9 feet and 12 feet and the perimeter is 32 feet, how long is the third side?
417. A tile in the shape of an isosceles triangle has a base of 6 inches. If the perimeter is 20 inches, find the length of each of the other sides.
418. Find the length of each side of an equilateral triangle with perimeter of 81 yards.
419. The perimeter of a triangle is 59 feet. One side of the triangle is 3 feet longer than the shortest side. The third side is 5 feet longer than the shortest side. Find the length of each side.
420. One side of a triangle is three times the smallest side. The third side is 9 feet more than the shortest side. The perimeter is 39 feet. Find the lengths of all three sides.

Use Properties of Trapezoids

In the following exercises, solve using the properties of trapezoids.

421. The height of a trapezoid is 8 feet and the bases are 11 and 14 feet. What is the area?
422. The height of a trapezoid is 5 yards and the bases are 7 and 10 yards. What is the area?
423. Find the area of the trapezoid with height 25 meters and bases 32.5 and 21.5 meters.
424. A flag is shaped like a trapezoid with height 62 centimeters and the bases are 91.5 and 78.1 centimeters. What is the area of the flag?

9.5 - Solve Geometry Applications: Circles and Irregular Figures

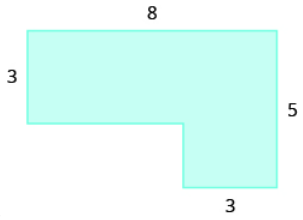
Use Properties of Circle

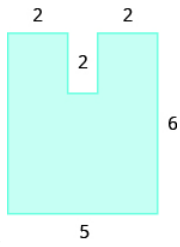
s In the following exercises, solve using the properties of circles. Round answers to the nearest hundredth.

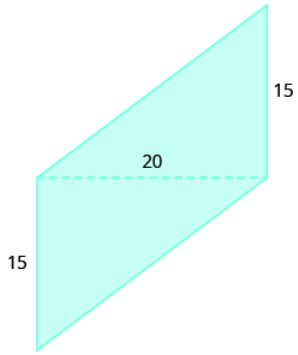
425. A circular mosaic has radius 3 meters. Find the (a) circumference (b) area of the mosaic
426. A circular fountain has radius 8 feet. Find the (a) circumference (b) area of the fountain
427. Find the diameter of a circle with circumference 150.72 inches.
428. Find the radius of a circle with circumference 345.4 centimeters

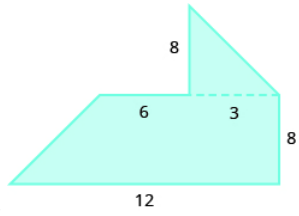
Find the Area of Irregular Figures

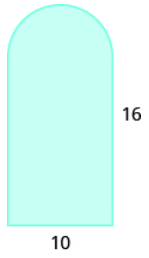
In the following exercises, find the area of each shaded region.

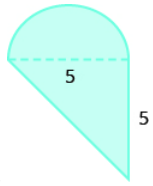
429. 

430. 

431. 

432. 

433. 

434. 

9.6 - Solve Geometry Applications: Volume and Surface Area

Find Volume and Surface Area of Rectangular Solids

In the following exercises, find the (a) volume (b) surface area of the rectangular solid

435. a rectangular solid with length 14 centimeters, width 4.5 centimeters, and height 10 centimeters
436. a cube with sides that are 3 feet long
437. a cube of tofu with sides 2.5 inches

438. a rectangular carton with length 32 inches, width 18 inches, and height 10 inches

Find Volume and Surface Area of Spheres

In the following exercises, find the (a) volume (b) surface area of the sphere.

439. a sphere with radius 4 yards
440. a sphere with radius 12 meters
441. a baseball with radius 1.45 inches
442. a soccer ball with radius 22 centimeters

Find Volume and Surface Area of Cylinders

In the following exercises, find the (a) volume (b) surface area of the cylinder

443. a cylinder with radius 2 yards and height 6 yards
444. a cylinder with diameter 18 inches and height 40 inches
445. a juice can with diameter 8 centimeters and height 15 centimeters
446. a cylindrical pylon with diameter 0.8 feet and height 2.5 feet

Find Volume of Cones

In the following exercises, find the volume of the cone.

447. a cone with height 5 meters and radius 1 meter
448. a cone with height 24 feet and radius 8 feet
449. a cone-shaped water cup with diameter 2.6 inches and height 2.6 inches
450. a cone-shaped pile of gravel with diameter 6 yards and height 5 yards

9.7 - Solve a Formula for a Specific Variable

Use the Distance, Rate, and Time Formula

In the following exercises, solve using the formula for distance, rate, and time.

451. A plane flew 4 hours at 380 miles per hour. What distance was covered?
452. Gus rode his bike for $1\frac{1}{2}$ hours at 8 miles per hour. How far did he ride?
453. Jack is driving from Bangor to Portland at a rate of 68 miles per hour. The distance is 107 miles. To the nearest tenth of an hour, how long will the trip take?
454. Jasmine took the bus from Pittsburgh to Philadelphia. The distance is 305 miles and the trip took 5 hours. What was the speed of the bus?

Solve a Formula for a Specific Variable

In the following exercises, use the formula d = rt.

455. Solve for t: (a) when d = 403 and r = 65 (b) in general
456. Solve for r: (a) when d = 750 and t = 15 (b) in general

In the following exercises, use the formula $A = \frac{1}{2}bh$.

457. Solve for b: (a) when A = 416 and h = 32 (b) in general
458. Solve for h: (a) when A = 48 and b = 8 (b) in general

In the following exercises, use the formula I = Prt.

459. Solve for the principal, P, for: (a) I = \$720, r = 4%, t = 3 years (b) in general
460. Solve for the time, t for: (a) I = \$3630, P = \$11,000, r = 5.5% (b) in general

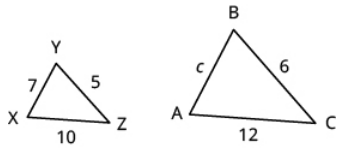
In the following exercises, solve.

461. Solve the formula 6x + 5y = 20 for y: (a) when x = 0 (b) in general
462. Solve the formula 2x + y = 15 for y: (a) when x = −5 (b) in general
463. Solve a + b = 90 for a.

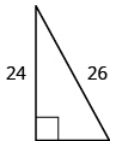
464. Solve 180 = a + b + c for a.
465. Solve the formula 4x + y = 17 for y.
466. Solve the formula −3x + y = −6 for y.
467. Solve the formula P = 2L + 2W for W.
468. Solve the formula V = LWH for H.
469. Describe how you have used two topics from this chapter in your life outside of math class during the past month.

PRACTICE TEST

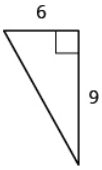
470. Four-fifths of the people on a hike are children. If there are 12 children, what is the total number of people on the hike?
471. The sum of 13 and twice a number is −19. Find the number.
472. One number is 3 less than another number. Their sum is 65. Find the numbers.
473. Bonita has $2.95 in dimes and quarters in her pocket. If she has 5 more dimes than quarters, how many of each coin does she have?
474. At a concert, $1600 in tickets were sold. Adult tickets were $9 each and children's tickets were $4 each. If the number of adult tickets was 30 fewer than twice the number of children's tickets, how many of each kind were sold?
475. Find the complement of a 52° angle.
476. The measure of one angle of a triangle is twice the measure of the smallest angle. The measure of the third angle is 14 more than the measure of the smallest angle. Find the measures of all three angles.
477. The perimeter of an equilateral triangle is 145 feet. Find the length of each side.
478. ΔABC is similar to ΔXYZ. Find the length of side c.



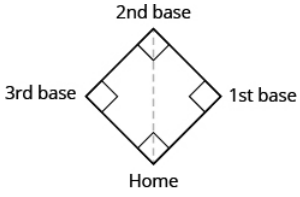
479. Find the length of the missing side. Round to the nearest tenth, if necessary.



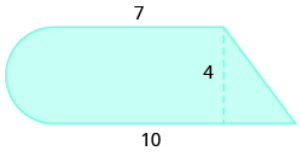
480. Find the length of the missing side. Round to the nearest tenth, if necessary.



481. A baseball diamond is shaped like a square with sides 90 feet long. How far is it from home plate to second base, as shown?



482. The length of a rectangle is 2 feet more than five times the width. The perimeter is 40 feet. Find the dimensions of the rectangle.
483. A triangular poster has base 80 centimeters and height 55 centimeters. Find the area of the poster.
484. A trapezoid has height 14 inches and bases 20 inches and 23 inches. Find the area of the trapezoid.
485. A circular pool has diameter 90 inches. What is its circumference? Round to the nearest tenth.
486. Find the area of the shaded region. Round to the nearest tenth.



487. Find the volume of a rectangular room with width 12 feet, length 15 feet, and height 8 feet.
488. A coffee can is shaped like a cylinder with height 7 inches and radius 5 inches. Find (a) the surface area and (b) the volume of the can. Round to the nearest tenth.
489. A traffic cone has height 75 centimeters. The radius of the base is 20 centimeters. Find the volume of the cone. Round to the nearest tenth.
490. Leon drove from his house in Cincinnati to his sister's house in Cleveland. He drove at a uniform rate of 63 miles per hour and the trip took 4 hours. What was the distance?
491. The Catalina Express takes $1\frac{1}{2}$ hours to travel from Long Beach to Catalina Island, a distance of 22 miles. To the nearest tenth, what is the speed of the boat?
492. Use the formula I = Prt to solve for the principal, P, for: (a) I = $1380, r = 5%, t = 3 years (b) in general
493. Solve the formula A = $\frac{1}{2}$bh for h: (a) when A = 1716 and b = 66 (b) in general
494. Solve x + 5y = 14 for y.

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6.S: Math Models and Geometry (Summary)

Key Terms

angle	An angle is formed by two rays that share a common endpoint. Each ray is called a side of the angle.
area	The area is a measure of the surface covered by a figure.
complementary angles	If the sum of the measures of two angles is 90°, then they are called complementary angles.
cone	A cone is a solid figure with one circular base and a vertex.
cube	A cube is a rectangular solid whose length, width, and height are equal.
cylinder	A cylinder is a solid figure with two parallel circles of the same size at the top and bottom.
equilateral triangle	A triangle with all three sides of equal length is called an equilateral triangle.
hypotenuse	The side of the triangle opposite the 90° angle is called the hypotenuse.
irregular figure	A figure that is not a standard geometric shape. Its area cannot be calculated using any of the standard area formulas.
isosceles triangle	A triangle with two sides of equal length is called an isosceles triangle.
legs of a right triangle	The sides of a right triangle adjacent to the right angle
perimeter	The perimeter is a measure of the distance around a figure.
rectangle	A geometric figure that has four sides and four right angles.
right triangle	A triangle that has one 90° angle.
similar figures	In geometry, if two figures have exactly the same shape but different sizes, we say they are similar figures.
supplementary angles	If the sum of the measures of two angles is 180°, then they are called supplementary angles.
trapezoid	A four-sided figure, a quadrilateral, with two sides that are parallel and two sides that are not.
triangle	A geometric figure with three sides and three angles.
vertex of an angle	When two rays meet to form an angle, the common endpoint is called the vertex of the angle.

Key Concepts

9.1 - Use a Problem Solving Strategy

- **Problem Solving Strategy**
  1. Read the word problem. Make sure you understand all the words and ideas. You may need to read the problem two or more times. If there are words you don't understand, look them up in a dictionary or on the internet.
  2. Identify what you are looking for.
  3. Name what you are looking for. Choose a variable to represent that quantity.
  4. Translate into an equation. It may be helpful to first restate the problem in one sentence before translating.

5. Solve the equation using good algebra techniques.
6. Check the answer in the problem. Make sure it makes sense.
7. Answer the question with a complete sentence.

9.2 - Solve Money Applications

- **Finding the Total Value for Coins of the Same Type**
  - For coins of the same type, the total value can be found as follows:

$$number \cdot value = total\ value \tag{6.S.1}$$

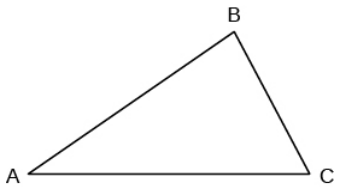
    where number is the number of coins, value is the value of each coin, and total value is the total value of all the coins.
- **Solve a Coin Word Problem**
  1. **Read** the problem. Make sure you understand all the words and ideas, and create a table to organize the information.
  2. **Identify** what you are looking for.
  3. **Name** what you are looking for. Choose a variable to represent that quantity.
     - Use variable expressions to represent the number of each type of coin and write them in the table.
     - Multiply the number times the value to get the total value of each type of coin.
  4. **Translate** into an equation. Write the equation by adding the total values of all the types of coins.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Table 9.16

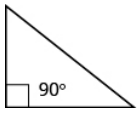
Type	Number	Value ($)	Total Value ($)

9.3 - Use Properties of Angles, Triangles, and the Pythagorean Theorem

- **Supplementary and Complementary Angles**
  - If the sum of the measures of two angles is 180°, then the angles are supplementary.
  - If $\angle A$ and $\angle B$ are supplementary, then $m\angle A + m\angle B = 180$.
  - If the sum of the measures of two angles is 90°, then the angles are complementary.
  - If $\angle A$ and $\angle B$ are complementary, then $m\angle A + m\angle B = 90$.
- **Solve Geometry Applications**
  1. Read the problem and make sure you understand all the words and ideas. Draw a figure and label it with the given information.
  2. Identify what you are looking for.
  3. Name what you are looking for and choose a variable to represent it.
  4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.
- **Sum of the Measures of the Angles of a Triangle**
  - For any ΔABC, the sum of the measures is 180°
  - $m\angle A + m\angle B = 180$



- **Right Triangle**
  - A right triangle is a triangle that has one 90° angle, which is often marked with a ⌞ symbol.



- **Properties of Similar Triangles**
  - If two triangles are similar, then their corresponding angle measures are equal and their corresponding side lengths have the same ratio.

9.4 - Use Properties of Rectangles, Triangles, and Trapezoids

- **Properties of Rectangles**
  - Rectangles have four sides and four right (90°) angles.
  - The lengths of opposite sides are equal.
  - The perimeter, P , of a rectangle is the sum of twice the length and twice the width.

$$P = 2L + 2W \tag{6.S.2}$$

  - The area, A, of a rectangle is the length times the width.$$A = L \cdot W$
- **Triangle Properties**
  - For any triangle ΔABC , the sum of the measures of the angles is 180°.

$$m\angle A + m\angle B + m\angle C = 180^\circ \tag{6.S.3}$$

  - The perimeter of a triangle is the sum of the lengths of the sides.

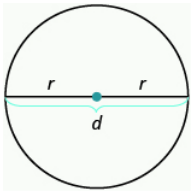
$$P = a + b + c \tag{6.S.4}$$

  - The area of a triangle is one-half the base, b, times the height, h.

$$A = \frac{1}{2}bh \tag{6.S.5}$$

9.5 - Solve Geometry Applications: Circles and Irregular Figures

- **Problem Solving Strategy for Geometry Applications**
  1. Read the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
  2. Identify what you are looking for.
  3. Name what you are looking for. Choose a variable to represent that quantity.
  4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.
- **Properties of Circles**
  - d = 2r
  - Circumference: C = 2πr or C = πd
  - Area: $A = \pi r^2$



9.6 - Solve Geometry Applications: Volume and Surface Area

- **Volume and Surface Area of a Rectangular Solid**
  - V = LWH
  - S = 2LH + 2LW + 2WH
- **Volume and Surface Area of a Cube**
  - $V = s^3$
  - $S = 6s^2$
- **Volume and Surface Area of a Sphere**
  - $V = \frac{4}{3}\pi r^3$
  - $S = 4\pi r^2$
- **Volume and Surface Area of a Cylinder**
  - $V = \pi r^2 h$
  - $S = 2\pi r^2 + 2\pi rh$
- **Volume of a Cone**
  - For a cone with radius r and height h: Volume: $V = \frac{1}{3}\pi r^2 h$

9.7 - Solve a Formula for a Specific Variable

- **Distance, Rate, and Time**
  - d = rt

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CHAPTER OVERVIEW

7: Percents

When you deposit money in a savings account at a bank, it earns additional money. Figuring out how your money will grow involves understanding and applying concepts of percents. In this chapter, we will find out what percents are and how we can use them to solve problems.

7.1: Understand Percent
7.2: Solve General Applications of Percent
7.3: Solve Sales Tax, Commission, and Discount Applications
7.4: Solve Simple Interest Applications
7.5: Percent Increase or Decrease
7.6: Pie Charts
7.7: Solve Proportions and their Applications (Part 1)
7.8: Solve Proportions and their Applications (Part 2)
7.E: Percents (Exercises)
7.S: Percents (Summary)

Figure 6.1 - Banks provide money for savings and charge money for loans. The interest on savings and loans is usually given as a percent. (credit: Mike Mozart, Flickr)

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7.1: Understand Percent

Learning Objectives

- Use the definition of percent
- Convert percents to fractions and decimals
- Convert decimals and fractions to percents

be prepared!

Before you get started, take this readiness quiz.

1. Translate “the ratio of 33 to 5” into an algebraic expression. If you missed this problem, review Table 2.4.3.
2. Write 3 5 as a decimal. If you missed this problem, review Example 5.5.1.
3. Write 0.62 as a fraction. If you missed this problem, review Example 5.1.4.

Use the Definition of Percent

How many cents are in one dollar? There are 100 cents in a dollar. How many years are in a century? There are 100 years in a century. Does this give you a clue about what the word “per cent” means? It is really two words, “per cent,” and means per one hundred. A **percent** is a ratio whose denominator is 100. We use the percent symbol %, to show percent.

Definition: Percent

A percent is a ratio whose denominator is 100.

According to data from the American Association of Community Colleges (2015), about 57% of community college students are female. This means 57 out of every 100 community college students are female, as Figure 7.1.1 shows. Out of the 100 squares on the grid, 57 are shaded, which we write as the ratio $\frac{57}{100}$.

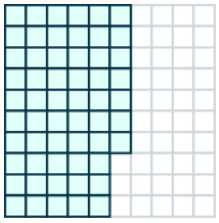

Figure 7.1.1 - Among every 100 community college students, 57 are female.

Similarly, 25% means a ratio of $\frac{25}{100}$, 3% means a ratio of $\frac{3}{100}$ and 100% means a ratio of $\frac{100}{100}$. In words, "one hundred percent" means the total 100% is $\frac{100}{100}$, and since $\frac{100}{100} = 1$, we see that 100% means 1 whole.

Example 7.1.1:

According to the Public Policy Institute of California (2010), 44% of parents of public school children would like their youngest child to earn a graduate degree. Write this percent as a ratio.

Solution

The amount we want to convert is 44%.	44%	
Write the percent as a ratio. Remember that percent means per 100.	$\frac{44}{100}$	(7.1.1)

Exercise 7.1.1:

Write the percent as a ratio. According to a survey, 89% of college students have a smartphone.

Answer

$\frac{89}{100}$

Exercise 7.1.2:

Write the percent as a ratio. A study found that 72% of U.S. teens send text messages regularly.

Answer

$\frac{72}{100}$

Example 7.1.2:

In 2007, according to a U.S. Department of Education report, 21 out of every 100 first-time freshmen college students at 4-year public institutions took at least one remedial course. Write this as a ratio and then as a percent.

Solution

The amount we want to convert is 21 out of 100.	21 out of 100	
Write as a ratio.	$\frac{21}{100}$	(7.1.2)
Convert the 21 per 100 to percent.	21%	

Exercise 7.1.3:

Write as a ratio and then as a percent: The American Association of Community Colleges reported that 62 out of 100 full-time community college students balance their studies with full-time or part time employment.

Answer

$\frac{62}{100}$, 62%

Exercise 7.1.4:

Write as a ratio and then as a percent: In response to a student survey, 41 out of 100 Santa Ana College students expressed a goal of earning an Associate's degree or transferring to a four-year college.

Answer

$\frac{41}{100}$, 41%

Convert Percents to Fractions and Decimals

Since percents are ratios, they can easily be expressed as fractions. Remember that percent means per 100, so the denominator of the fraction is 100.

HOW TO: CONVERT A PERCENT TO A FRACTION

Step 1. Write the percent as a ratio with the denominator 100.

Step 2. Simplify the fraction if possible.

Example 7.1.3:

Convert each percent to a fraction: (a) 36% (b) 125%

Solution

(a) 36%

Write as a ratio with denominator 100.	$\frac{36}{100}$	(7.1.3)
Simplify.	$\frac{9}{25}$	(7.1.4)

(b) 125%

Write as a ratio with denominator 100.	$\frac{125}{100}$	(7.1.5)
Simplify.	$\frac{5}{4}$	(7.1.6)

Exercise 7.1.5:

Convert each percent to a fraction: (a) 48% (b) 110%

Answer a

$\frac{12}{25}$

Answer b

$\frac{11}{10}$

Exercise 7.1.6:

Convert each percent to a fraction: (a) 64% (b) 150%

Answer a

$\frac{16}{25}$

Answer b

$\frac{3}{2}$

The previous example shows that a percent can be greater than 1. We saw that 125% means $\frac{125}{100}$, or $\frac{5}{4}$. These are improper fractions, and their values are greater than one.

Example 7.1.4:

Convert each percent to a fraction: (a) 24.5% (b) $33\frac{1}{3}\%$

Solution

(a) 24.5%

Write as a ratio with denominator 100.	$\frac{24.5}{100}$	(7.1.7)
Clear the decimal by multiplying numerator and denominator by 10.	$\frac{24.5(10)}{100(10)}$	(7.1.8)
Multiply.	$\frac{245}{1000}$	(7.1.9)
Rewrite showing common factors.	$\frac{5 \cdot 49}{5 \cdot 200}$	(7.1.10)
Simplify.	$\frac{49}{200}$	(7.1.11)

(b) $33\frac{1}{3}\%$

Write as a ratio with denominator 100.	$\frac{33\frac{1}{3}}{100}$	(7.1.12)
Write the numerator as an improper fraction.	$\frac{\frac{100}{3}}{100}$	(7.1.13)
Rewrite as fraction division, replacing 100 with $\frac{100}{1}$.	$\frac{100}{3} \div \frac{100}{1}$	(7.1.14)
Multiply by the reciprocal.	$\frac{100}{3} \cdot \frac{1}{100}$	(7.1.15)
Simplify.	$\frac{1}{3}$	(7.1.16)

Exercise 7.1.7:

Convert each percent to a fraction: (a) 64.4% (b) $66\frac{2}{3}\%$

Answer a

$\frac{161}{250}$

Answer b

$\frac{2}{3}$

Exercise 7.1.8:

Convert each percent to a fraction: (a) 42.5% (b) $8\frac{3}{4}\%$

Answer a

$\frac{113}{250}$

Answer b

$\frac{7}{80}$

In Decimals, we learned how to convert fractions to decimals. To convert a percent to a decimal, we first convert it to a fraction and then change the fraction to a decimal.

HOW TO: CONVERT A PERCENT TO A DECIMAL

Step 1. Write the percent as a ratio with the denominator 100.

Step 2. Convert the fraction to a decimal by dividing the numerator by the denominator.

Example 7.1.5:

Convert each percent to a decimal: (a) 6% (b) 78%

Solution

Because we want to change to a decimal, we will leave the fractions with denominator 100 instead of removing common factors.

(a) 6%

Write as a ratio with denominator 100.	$\frac{6}{100}$	(7.1.17)
Change the fraction to a decimal by dividing the numerator by the denominator.	0.06	

(b) 78%

Write as a ratio with denominator 100.	$\frac{78}{100}$	(7.1.18)
Change the fraction to a decimal by dividing the numerator by the denominator.	0.78	

Exercise 7.1.9:

Convert each percent to a decimal: (a) 9% (b) 87%

Answer a

0.09

Answer b

0.87

Exercise 7.1.10:

Convert each percent to a decimal: (a) 3% (b) 91%

Answer a

0.03

Answer b

0.91

Example 7.1.6:

Convert each percent to a decimal: (a) 135% (b) 12.5%

Solution

Write as a ratio with denominator 100.	$\frac{135}{100}$	(7.1.19)
Change the fraction to a decimal by dividing the numerator by the denominator.	1.35	

(b) 12.5%

Write as a ratio with denominator 100.	$\frac{12.5}{100}$	(7.1.20)
Change the fraction to a decimal by dividing the numerator by the denominator.	0.125	

Exercise 7.1.11:

Convert each percent to a decimal: (a) 115% (b) 23.5%

Answer a

1.15

Answer b

0.235

Exercise 7.1.12:

Convert each percent to a decimal: (a) 123% (b) 16.8%

Answer a

1.23

Answer b

0.168

Let's summarize the results from the previous examples in Table 7.1.1, and look for a pattern we could use to quickly convert a percent number to a decimal number.

Table 7.1.1

Percent	Decimal
6%	0.06
78%	0.78
135%	1.35
12.5%	0.125

Do you see the pattern? To convert a percent number to a decimal number, we move the decimal point two places to the left and remove the % sign. (Sometimes the decimal point does not appear in the percent number, but just like we can think of the integer 6 as 6.0, we can think of 6% as 6.0%.) Notice that we may need to add zeros in front of the number when moving the decimal to the left.

Figure 7.1.2 uses the percents in Table 7.1.1 and shows visually how to convert them to decimals by moving the decimal point two places to the left.

Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

Figure 7.1.2

Example 7.1.7:

Among a group of business leaders, 77% believe that poor math and science education in the U.S. will lead to higher unemployment rates. Convert the percent to: (a) a fraction (b) a decimal

Solution

(a)

Write as a ratio with denominator 100.	$\frac{77}{100}$	(7.1.21)

(b)

Change the fraction to a decimal by dividing the numerator by the denominator.	0.77

Exercise 7.1.13:

Twitter's share of web traffic jumped 24% when one celebrity tweeted live on air. Convert the percent to: (a) a fraction and (b) a decimal.

Answer a

$\frac{6}{25}$

Answer b

0.24

Exercise 7.1.14:

The U.S. Census estimated that in 2013, 44% of the population of Boston age 25 or older have a bachelor's or higher degrees. Convert the percent to: (a) a fraction and (b) a decimal.

Answer a

$\frac{22}{50}$

Answer b

0.44

Example 7.1.8:

There are four suits of cards in a deck of cards—hearts, diamonds, clubs, and spades. The probability of randomly choosing a heart from a shuffled deck of cards is 25%. Convert the percent to: (a) a fraction (b) a decimal.

Figure 7.1.3 - (credit: Riles32807, Wikimedia Commons)

Solution

(a)

Write as a ratio with denominator 100.	$\frac{25}{100}$	(7.1.22)
Simplify.	$\frac{1}{4}$	(7.1.23)

(b)

Change the fraction to a decimal by dividing the numerator by the denominator.	0.25

Exercise 7.1.15:

The probability that it will rain Monday is 30%. Convert the percent to: (a) a fraction, and (b) a decimal.

Answer a

$\frac{3}{10}$

Answer b

0.3

Exercise 7.1.16:

The probability of getting heads three times when tossing a coin three times is 12.5%. Convert the percent to: (a) a fraction, and (b) a decimal.

Answer a

$\frac{12.5}{100}$

Answer b

0.125

Convert Decimals and Fractions to Percents

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent.

HOW TO: CONVERT A DECIMAL TO A PERCENT

Step 1. Write the decimal as a fraction.

Step 2. If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.

Step 3. Write this ratio as a percent.

Example 7.1.9:

Convert each decimal to a percent: (a) 0.05 (b) 0.83

Solution

(a) 0.05

Write as a fraction. The denominator is 100.	$\frac{5}{100}$	(7.1.24)
Write this ratio as a percent.	5%	

(b)

The denominator is 100.	$\frac{83}{100}$	(7.1.25)
Write this ratio as a percent.	83%	

Exercise 7.1.17:

Convert each decimal to a percent: (a) 0.01 (b) 0.17.

Answer a

1%

Answer b

17%

Exercise 7.1.18:

Convert each decimal to a percent: (a) 0.04 (b) 0.41.

Answer a

4%

Answer b

41%

To convert a mixed number to a percent, we first write it as an improper fraction.

Example 7.1.10:

Convert each decimal to a percent: (a) 1.05 (b) 0.075

Solution

(a) 1.05

Write as a fraction.	$1\frac{5}{100}$	(7.1.26)
Write as an improper fraction. The denominator is 100.	$\frac{105}{100}$	(7.1.27)
Write this ratio as a percent.	105%	

Notice that since 1.05 > 1, the result is more than 100%.

(b) 0.075

Write as a fraction.	$\frac{75}{1,000}$	(7.1.28)
Divide the numerator and denominator by 10, so that the denominator is 100.	$\frac{7.5}{100}$	(7.1.29)
Write this ratio as a percent.	7.5%	

Exercise 7.1.19:

Convert each decimal to a percent: (a) 1.75 (b) 0.0825

Answer a

175%

Answer b

8.25%

Exercise 7.1.20:

Convert each decimal to a percent: (a) 2.25 (b) 0.0925

Answer a

225%

Answer b

9.25%

Let's summarize the results from the previous examples in Table 6.20 so we can look for a pattern.

Table 7.1.2

Decimal	Percent
0.05	5%
0.83	83%
1.05	105%
0.075	7.5%

Do you see the pattern? To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.

Figure 7.1.4 uses the decimal numbers in Table 7.1.2 and shows visually to convert them to percents by moving the decimal point two places to the right and then writing the % sign.

Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

Figure 7.1.4

In Decimals, we learned how to convert fractions to decimals. Now we also know how to change decimals to percents. So to convert a fraction to a percent, we first change it to a decimal and then convert that decimal to a percent.

HOW TO: CONVERT A FRACTION TO A PERCENT

Step 1. Convert the fraction to a decimal.

Step 2. Convert the decimal to a percent.

Example 7.1.11:

Convert each fraction or mixed number to a percent: (a) $\frac{3}{4}$ (b) $\frac{11}{8}$ (c) $2\frac{1}{5}$

Solution

To convert a fraction to a decimal, divide the numerator by the denominator.

(a)

Change to a decimal.	$\frac{3}{4}$	(7.1.30)

Write as a percent by moving the decimal two places.	0.75	
	75%	

(b)

Change to a decimal.	$\frac{11}{8}$	(7.1.31)
Write as a percent by moving the decimal two places.	1.375	
	137.5%	

(c)

Write as an improper fraction.	$2\frac{1}{5}$	(7.1.32)
Change to a decimal.	$\frac{11}{5}$	(7.1.33)
Write as a percent.	2.20	
	220%	

Notice that we needed to add zeros at the end of the number when moving the decimal two places to the right.

? Exercise 7.1.21:

Convert each fraction or mixed number to a percent: (a) $\frac{5}{8}$ (b) $\frac{11}{4}$ (c) $3\frac{2}{5}$

Answer a

62.5%

Answer b

275%

Answer c

340%

? Exercise 7.1.22:

Convert each fraction or mixed number to a percent: (a) $\frac{7}{8}$ (b) $\frac{9}{4}$ (c) $1\frac{3}{5}$

Answer a

87.5%

Answer b

225%

Answer c

160%

Sometimes when changing a fraction to a decimal, the division continues for many decimal places and we will round off the quotient. The number of decimal places we round to will depend on the situation. If the decimal involves money, we round to the hundredths place. For most other cases in this book we will round the number to the nearest thousandth, so the percent will be rounded to the nearest tenth.

Example 7.1.12:

Convert $\frac{5}{7}$ to a percent.

Solution

To change a fraction to a decimal, we divide the numerator by the denominator.

Change to a decimal—rounding to the nearest thousandth.	0.714
Write as a percent.	71.4%

Exercise 7.1.23:

Convert the fraction to a percent: $\frac{3}{7}$

Answer

42.9%

Exercise 7.1.24:

Convert the fraction to a percent: $\frac{4}{7}$

Answer

57.1%

When we first looked at fractions and decimals, we saw that fractions converted to a repeating decimal. When we converted the fraction $\frac{4}{3}$ to a decimal, we wrote the answer as $1.\overline{3}$. We will use this same notation, as well as fraction notation, when we convert fractions to percents in the next example.

Example 7.1.13:

An article in a medical journal claimed that approximately $\frac{1}{3}$ of American adults are obese. Convert the fraction $\frac{1}{3}$ to a percent.

Solution

Change to a decimal.	$\begin{array}{r} 0.33\ldots \\ 3\overline{)1.00} \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \end{array}$
Write as a repeating decimal.	0.333 …
Write as a percent.	$33\frac{1}{3}\%$

We could also write the percent as $33.\overline{3}\%$.

Exercise 7.1.25:

According to the U.S. Census Bureau, about $\frac{1}{9}$ of United States housing units have just 1 bedroom. Convert the fraction to a percent.

Answer

$11.\overline{1}\%$ or $11\frac{1}{9}\%$

Exercise 7.1.26:

According to the U.S. Census Bureau, about $\frac{1}{6}$ of Colorado residents speak a language other than English at home. Convert the fraction to a percent.

Answer

$16.\overline{6}\%$ or $16\frac{2}{3}\%$

Practice Makes Perfect

Use the Definition of Percents

In the following exercises, write each percent as a ratio.

1. In 2014, the unemployment rate for those with only a high school degree was 6.0%.
2. In 2015, among the unemployed, 29% were long-term unemployed.
3. The unemployment rate for those with Bachelor's degrees was 3.2% in 2014.
4. The unemployment rate in Michigan in 2014 was 7.3%.

In the following exercises, write as (a) a ratio and (b) a percent.

5. 57 out of 100 nursing candidates received their degree at a community college.
6. 80 out of 100 firefighters and law enforcement officers were educated at a community college.
7. 42 out of 100 first-time freshmen students attend a community college.
8. 71 out of 100 full-time community college faculty have a master's degree.

Convert Percents to Fractions and Decimals

In the following exercises, convert each percent to a fraction and simplify all fractions.

9. 4%
10. 8%
11. 17%
12. 19%
13. 52%
14. 78%
15. 125%
16. 135%
17. 37.5%
18. 42.5%
19. 18.4%
20. 46.4%
21. $9\frac{1}{2}\%$

22. $8\frac{1}{2}$ %
23. $5\frac{1}{3}$ %
24. $6\frac{2}{3}$ %

In the following exercises, convert each percent to a decimal.

25. 5%
26. 9%
27. 1%
28. 2%
29. 63%
30. 71%
31. 40%
32. 50%
33. 115%
34. 125%
35. 150%
36. 250%
37. 21.4%
38. 39.3%
39. 7.8%
40. 6.4%

In the following exercises, convert each percent to (a) a simplified fraction and (b) a decimal.

41. In 2010, 1.5% of home sales had owner financing. (Source: Bloomberg Businessweek, 5/ 23–29/2011)
42. In 2000, 4.2% of the United States population was of Asian descent. (Source: www.census.gov)
43. According to government data, in 2013 the number of cell phones in India was 70.23% of the population.
44. According to the U.S. Census Bureau, among Americans age 25 or older who had doctorate degrees in 2014, 37.1% are women.
45. A couple plans to have two children. The probability they will have two girls is 25%.
46. Javier will choose one digit at random from 0 through 9. The probability he will choose 3 is 10%.
47. According to the local weather report, the probability of thunderstorms in New York City on July 15 is 60%.
48. A club sells 50 tickets to a raffle. Osbaldo bought one ticket. The probability he will win the raffle is 2%.

Convert Decimals and Fractions to Percents

In the following exercises, convert each decimal to a percent.

49. 0.01
50. 0.03
51. 0.18
52. 0.15
53. 1.35
54. 1.56
55. 3
56. 4
57. 0.009
58. 0.008
59. 0.0875
60. 0.0625
61. 1.5
62. 2.2
63. 2.254
64. 2.317

In the following exercises, convert each fraction to a percent.

65. $\frac{1}{4}$
66. $\frac{1}{5}$
67. $\frac{3}{8}$
68. $\frac{5}{8}$
69. $\frac{7}{4}$
70. $\frac{9}{8}$
71. $6\frac{4}{5}$
72. $5\frac{1}{4}$
73. $\frac{5}{12}$
74. $\frac{11}{12}$
75. $2\frac{2}{3}$
76. $1\frac{2}{3}$
77. $\frac{3}{7}$
78. $\frac{6}{7}$
79. $\frac{5}{9}$
80. $\frac{4}{9}$

In the following exercises, convert each fraction to a percent.

81. $\frac{1}{4}$ of washing machines needed repair.
82. $\frac{1}{5}$ of dishwashers needed repair.

In the following exercises, convert each fraction to a percent.

83. According to the National Center for Health Statistics, in 2012, 7 20 of American adults were obese.
84. The U.S. Census Bureau estimated that in 2013, 85% of Americans lived in the same house as they did 1 year before.

In the following exercises, complete the table.

85. Table 6.26

Fraction	Decimal	Percent
$\frac{1}{2}$ (7.1.34)		
	0.45	
		18%
$\frac{1}{3}$ (7.1.35)		
	0.0008	

Fraction	Decimal	Percent
2		

86. Table 6.27

Fraction	Decimal	Percent
$\frac{1}{4}$ (7.1.36)		
	0.65	
		22%
$\frac{2}{3}$ (7.1.37)		
	0.0004	
3		

Everyday Math

87. Sales tax Felipa says she has an easy way to estimate the sales tax when she makes a purchase. The sales tax in her city is 9.05%. She knows this is a little less than 10%.
    a. Convert 10% to a fraction.
    b. Use your answer from (a) to estimate the sales tax Felipa would pay on a $95 dress.
88. Savings Ryan has 25% of each paycheck automatically deposited in his savings account.
    a. Write 25% as a fraction.
    b. Use your answer from (a) to find the amount that goes to savings from Ryan's $2,400 paycheck.
89. Amelio is shopping for textbooks online. He found three sellers that are offering a book he needs for the same price, including shipping. To decide which seller to buy from he is comparing their customer satisfaction ratings. The ratings are given in the chart.

Seller	Rating
A	4/5
B	3.5/4
C	85%

90. Write seller C's rating as a fraction and a decimal.
91. Write seller B's rating as a percent and a decimal.
92. Write seller A's rating as a percent and a decimal.
93. Which seller should Amelio buy from and why?

Writing Exercises

94. Convert 25%, 50%, 75%, and 100% to fractions. Do you notice a pattern? Explain what the pattern is.
95. Convert $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}$, and $\frac{9}{10}$ to percents. Do you notice a pattern? Explain what the pattern is.
96. When the Szetos sold their home, the selling price was 500% of what they had paid for the house 30 years ago. Explain what 500% means in this context.
97. According to cnn.com, cell phone use in 2008 was 600% of what it had been in 2001. Explain what 600% means in this context.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the definition of percent.			
convert percents to fractions and decimals.			
convert decimals and fractions to percents.			

(b) If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Contributors and Attributions

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7.2: Solve General Applications of Percent

Learning Objectives

- Translate and solve basic percent equations
- Solve applications of percent
- Find percent increase and percent decrease

be prepared!

Before you get started, take this readiness quiz.

1. Translate and solve: $\frac{3}{4}$ of x is 24. If you missed this problem, review Example 4.13.11.
2. Simplify: (4.5)(2.38). If you missed this problem, review Example 5.3.5.
3. Solve: 3.5 = 0.7n. If you missed this problem, review Example 5.7.4.

Translate and Solve Basic Percent Equations

We will solve percent equations by using the methods we used to solve equations with fractions or decimals. In the past, you may have solved percent problems by setting them up as proportions. That was the best method available when you did not have the tools of algebra. Now as a prealgebra student, you can translate word sentences into algebraic equations, and then solve the equations.

We'll look at a common application of percent—tips to a server at a restaurant—to see how to set up a basic percent application.

When Aolani and her friends ate dinner at a restaurant, the bill came to $80. They wanted to leave a 20% tip. What amount would the tip be? To solve this, we want to find what *amount* is 20% of $80. The $80 is called the *base*. The amount of the tip would be 0.20(80), or $16 See Figure 7.2.1. To find the amount of the tip, we multiplied the percent by the base.

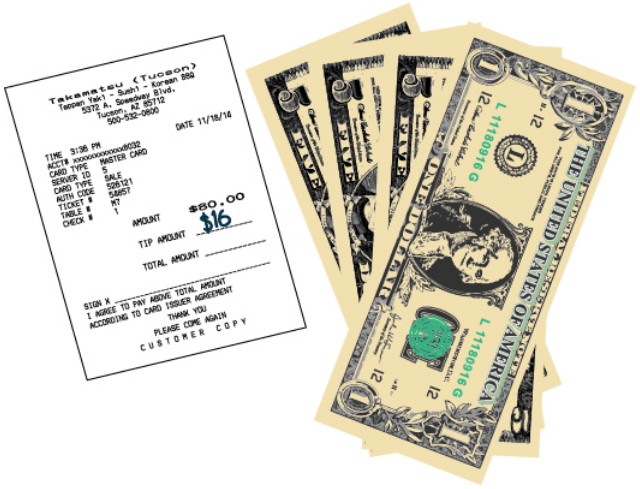

Figure 7.2.1 - A 20% tip for an $80 restaurant bill comes out to $16.

In the next examples, we will find the amount. We must be sure to change the given percent to a decimal when we translate the words into an equation.

Example 7.2.1:

What number is 35% of 90?

Solution

Translate into algebra. Let n = the number.	$\underbrace{\text{What number}}_{n} \quad \underbrace{\text{is}}_{=} \quad \underbrace{35\%}_{0.35} \quad \underbrace{\text{of}}_{\cdot} \quad \underbrace{90?}_{90}$
Multiply.	n = 31.5

31.5 is 35% of 90

? Exercise 7.2.1:

What number is 45% of 80?

Answer

36

? Exercise 7.2.2:

What number is 55% of 60?

Answer

33

✓ Example 7.2.2:

125% of 28 is what number?

Solution

Translate into algebra. Let a = the number.	$\underbrace{125\%}_{1.25} \quad \underbrace{\text{of}}_{\cdot} \quad \underbrace{28}_{28} \quad \underbrace{\text{is}}_{=} \quad \underbrace{\text{what number?}}_{a}$
Multiply.	35 = a

125% of 28 is 35

Remember that a percent over 100 is a number greater than 1. We found that 125% of 28 is 35, which is greater than 28.

? Exercise 7.2.3:

150% of 78 is what number?

Answer

117

? Exercise 7.2.4:

175% of 72 is what number?

Answer

126

In the next examples, we are asked to find the base.

Example 7.2.3:

Translate and solve: 36 is 75% of what number?

Solution

Translate. Let b = the number.	$\underbrace{36}_{36}\ \underbrace{\text{is}}_{=}\ \underbrace{75\%}_{0.75}\ \underbrace{\text{of}}_{.}\ \underbrace{\text{what number?}}_{b}$
Divide both sides by 0.75.	$\frac{36}{0.75}=\frac{0.75b}{0.75}$ (7.2.1)
Simplify.	48 = b

36 is 75% of 48

Exercise 7.2.5:

17 is 25% of what number?

Answer

68

Exercise 7.2.6:

40 is 62.5% of what number?

Answer

64

Example 7.2.4:

6.5% of what number is $1.17?

Solution

Translate. Let b = the number.	$\underbrace{6.5\%}_{0.065}\ \underbrace{\text{of}}_{.}\ \underbrace{\text{what number}}_{b}\ \underbrace{\text{is}}_{=}\ \underbrace{\$1.17?}_{1.17}$
Divide both sides by 0.065.	$\frac{0.065n}{0.065}=\frac{1.17}{0.065}$ (7.2.2)
Simplify.	n = 18

6.5% of $18 is $1.17.

Exercise 7.2.7:

7.5% of what number is $1.95?

Answer

$26

Exercise 7.2.8:

8.5% of what number is $3.06?

Answer

$36

In the next examples, we will solve for the percent.

Example 7.2.5:

What percent of 36 is 9?

Solution

Translate into algebra. Let p = the percent.	$\underbrace{p}_{\text{What percent}} \quad \underbrace{\cdot}_{\text{of}} \quad \underbrace{36}_{36} \quad \underbrace{=}_{\text{is}} \quad \underbrace{9}_{9?}$
Divide by 36.	$\frac{36p}{36} = \frac{9}{36}$ (7.2.3)
Simplify.	$p = \frac{1}{4}$ (7.2.4)
Convert to decimal form.	p = 0.25
Convert to percent.	p = 25%

25% of 36 is 9.

Exercise 7.2.9:

What percent of 76 is 57?

Answer

75%

Exercise 7.2.10:

What percent of 120 is 96?

Answer

80%

Example 7.2.6:

144 is what percent of 96?

Solution

Translate into algebra. Let p = the percent.	$\underbrace{144}_{144} \quad \underbrace{=}_{\text{is}} \quad \underbrace{p}_{\text{what percent}} \quad \underbrace{\cdot}_{\text{of}} \quad \underbrace{96}_{96?}$
Divide by 96.	$\frac{144}{96} = \frac{96p}{96}$ (7.2.5)

Simplify.	1.5 = p
Convert to percent.	150% = p

144 is 150% of 96.

Exercise 7.2.11:

110 is what percent of 88?

Answer

125%

Exercise 7.2.12:

126 is what percent of 72?

Answer

175%

Solve Applications of Percent

Many applications of percent occur in our daily lives, such as tips, sales tax, discount, and interest. To solve these applications we'll translate to a basic percent equation, just like those we solved in the previous examples in this section. Once you translate the sentence into a percent equation, you know how to solve it.

We will update the strategy we used in our earlier applications to include equations now. Notice that we will translate a sentence into an equation.

HOW TO: SOLVE AN APPLICATION

- Step 1. Identify what you are asked to find and choose a variable to represent it.
- Step 2. Write a sentence that gives the information to find it.
- Step 3. Translate the sentence into an equation.
- Step 4. Solve the equation using good algebra techniques.
- Step 5. Check the answer in the problem and make sure it makes sense.
- Step 6. Write a complete sentence that answers the question.

Now that we have the strategy to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications we'll solve involve everyday situations, you can rely on your own experience.

Example 7.2.7:

Dezohn and his girlfriend enjoyed a dinner at a restaurant, and the bill was $68.50. They want to leave an 18% tip. If the tip will be 18% of the total bill, how much should the tip be?

Solution

What are you asked to find?	the amount of the tip
Choose a variable to represent it.	Let t = amount of tip.
Write a sentence that give the information to find it.	The tip is 18% of the total bill.

Translate the sentence into an equation.	$\underbrace{\text{The tip}}_{t}\ \underbrace{\text{is}}_{=}\ \underbrace{18\%}_{0.18}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{\$68.50.}_{\$68.50}$
Multiply.	t = 12.33
Check. Is this answer reasonable?	If we approximate the bill to \$70 and the percent to 20%, we would have a tip of \$14. So a tip of \$12.33 seems reasonable.
Write a complete sentence that answers the question.	The couple should leave a tip of \$12.33.

? Exercise 7.2.13:

Cierra and her sister enjoyed a special dinner in a restaurant, and the bill was \$81.50. If she wants to leave 18% of the total bill as her tip, how much should she leave?

Answer

\$14.67

? Exercise 7.2.14:

Kimngoc had lunch at her favorite restaurant. She wants to leave 15% of the total bill as her tip. If her bill was \$14.40, how much will she leave for the tip?

Answer

\$2.16

✔ Example 7.2.8:

The label on Masao's breakfast cereal said that one serving of cereal provides 85 milligrams (mg) of potassium, which is 2% of the recommended daily amount. What is the total recommended daily amount of potassium?

Nutrition Facts

Serving Size: 1 cup (47g)
Servings Per Container: About 7

Amount Per Serving	Cereal	With Milk
Calories	180	230
Calories from Fat	10	20
	% Daily Value*	
Total Fat 1g	**2%**	**2%**
Saturated Fat 0g	**0%**	**0%**
Trans Fat 0g		
Polyunsaturated Fat 0.5g		
Monounsaturated Fat 0.5g		
Cholesterol 0mg	**0%**	**0%**
Sodium 190mg	**8%**	**11%**
Potassium 85mg	**2%**	**8%**
Total Carbohydrate 40g	**13%**	**15%**
Dietary Fiber 1g	**4%**	**4%**
Sugars 8g		
Protein 3g		

Solution

What are you asked to find?	the total amount of potassium recommended
Choose a variable to represent it.	Let a = total amount of potassium.

Write a sentence that gives the information to find it.	85 mg is 2% of the total amount.
Translate the sentence into an equation.	$\underbrace{85}_{85}\ \underbrace{\text{mg}}\ \underbrace{\text{is}}_{=}\ \underbrace{2\%}_{0.02}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{a}_{a}\ ?$
Divide both sides by 0.02.	$\frac{85}{0.02} = \frac{0.02a}{0.02}$ (7.2.6)
Simplify.	4,250 =a
Check: Is this answer reasonable?	Yes. 2% is a small percent and 85 is a small part of 4,250.
Write a complete sentence that answers the question.	The amount of potassium that is recommended is 4250 mg.

? Exercise 7.2.15:

One serving of wheat square cereal has 7 grams of fiber, which is 29% of the recommended daily amount. What is the total recommended daily amount of fiber?

Answer

24.1 grams

? Exercise 7.2.16:

One serving of rice cereal has 190 mg of sodium, which is 8% of the recommended daily amount. What is the total recommended daily amount of sodium?

Answer

2,375 mg

✓ Example 7.2.9:

Mitzi received some gourmet brownies as a gift. The wrapper said each brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat?

Solution

What are you asked to find?	the percent of the total calories from fat
Choose a variable to represent it.	Let p = percent from fat.
Write a sentence that gives the information to find it.	What percent of 480 is 240?
Translate the sentence into an equation.	$\underbrace{\text{What percent}}_{p}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{480}_{480}\ \underbrace{\text{is}}_{=}\ \underbrace{240?}_{240}$
Divide both sides by 480.	$\frac{p \cdot 480}{480} = \frac{240}{480}$ (7.2.7)
Simplify.	p = 0.5
Convert to percent form.	p = 50%
Check. Is this answer reasonable?	Yes. 240 is half of 480, so 50% makes sense.
Write a complete sentence that answers the question.	Of the total calories in each brownie, 50% is fat.

Exercise 7.2.17:

Veronica is planning to make muffins from a mix. The package says each muffin will be 230 calories and 60 calories will be from fat. What percent of the total calories is from fat? (Round to the nearest whole percent.)

Answer

26%

Exercise 7.2.18:

The brownie mix Ricardo plans to use says that each brownie will be 190 calories, and 70 calories are from fat. What percent of the total calories are from fat?

Answer

37%

Find Percent Increase and Percent Decrease

People in the media often talk about how much an amount has increased or decreased over a certain period of time. They usually express this increase or decrease as a percent.

To find the **percent increase**, first we find the amount of increase, which is the difference between the new amount and the original amount. Then we find what percent the amount of increase is of the original amount.

how to: find percent increase

Step 1. Find the amount of increase.

- increase = new amount − original amount

Step 2. Find the percent increase as a percent of the original amount.

Example 7.2.10:

In 2011, the California governor proposed raising community college fees from $26 per unit to $36 per unit. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution

What are you asked to find?	the percent increase
Choose a variable to represent it.	Let p = percent.
Find the amount of increase.	$\underbrace{36}_{\text{new amount}} - \underbrace{26}_{\text{original amount}} = \underbrace{10}_{\text{increase}}$
Find the percent increase.	The increase is what percent of the original amount?
Translate to an equation.	$\underbrace{10}_{10} \underbrace{\text{ is }}_{=} \underbrace{\text{ what percent }}_{p} \underbrace{\text{ of }}_{\cdot} \underbrace{\text{ 26?}}_{26}$
Divide both sides by 26.	$\frac{10}{26} = \frac{26p}{26}$ (7.2.8)
Round to the nearest thousandth.	0.384 = p
Convert to percent form.	38.4% = p

Write a complete sentence.	The new fees represent a 38.4% increase over the old fees.

Exercise 7.2.19:

In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents. Find the percent increase. (Round to the nearest tenth of a percent.)

Answer

8.8%

Exercise 7.2.20:

In 1995, the standard bus fare in Chicago was $1.50. In 2008, the standard bus fare was $2.25. Find the percent increase. (Round to the nearest tenth of a percent.)

Answer

50%

Finding the **percent decrease** is very similar to finding the percent increase, but now the amount of decrease is the difference between the original amount and the final amount. Then we find what percent the amount of decrease is of the original amount.

HOW TO: FIND PERCENT DECREASE

Step 1. Find the amount of decrease.

- decrease = original amount − new amount

Step 2. Find the percent decrease as a percent of the original amount.

Example 7.2.11:

The average price of a gallon of gas in one city in June 2014 was $3.71. The average price in that city in July was $3.64. Find the percent decrease.

Solution

The average price of a gallon of gas in one city in June 2014 was $3.71. The average price in that city in July was $3.64. Find the percent decrease.

What are you asked to find?	the percent decrease
Choose a variable to represent it.	Let p = percent.
Find the amount of decrease.	$\underbrace{3.71}_{\text{original amount}} - \underbrace{3.64}_{\text{new amount}} = \underbrace{0.07}_{\text{increase}}$
Find the percent of decrease.	The decrease is what percent of the original amount?
Translate to an equation.	$\underbrace{0.07}_{0.07} \underbrace{\text{is}}_{=} \underbrace{\text{what percent}}_{p} \underbrace{\text{of}}_{\cdot} \underbrace{3.71}_{3.71}$
Divide both sides by 3.71.	$\frac{0.07}{3.71} = \frac{3.71p}{3.71}$ (7.2.9)
Round to the nearest thousandth.	0.019 = p

Convert to percent form.	1.9% = p
Write a complete sentence.	The price of gas decreased 1.9%.

Exercise 7.2.21:

The population of one city was about 672,000 in 2010. The population of the city is projected to be about 630,000 in 2020. Find the percent decrease. (Round to the nearest tenth of a percent.)

Answer

6.3%

Exercise 7.2.22:

Last year Sheila's salary was $42,000. Because of furlough days, this year her salary was $37,800. Find the percent decrease. (Round to the nearest tenth of a percent.)

Answer

10%

ACCESS ADDITIONAL ONLINE RESOURCES

Percent Increase and Percent Decrease Visualization

Practice Makes Perfect

Translate and Solve Basic Percent Equations

In the following exercises, translate and solve.

98. What number is 45% of 120?
99. What number is 65% of 100?
100. What number is 24% of 112?
101. What number is 36% of 124?
102. 250% of 65 is what number?
103. 150% of 90 is what number?
104. 800% of 2,250 is what number?
105. 600% of 1,740 is what number?
106. 28 is 25% of what number?
107. 36 is 25% of what number?
108. 81 is 75% of what number?
109. 93 is 75% of what number?
110. 8.2% of what number is $2.87?
111. 6.4% of what number is $2.88?
112. 11.5% of what number is $108.10?
113. 12.3% of what number is $92.25?
114. What percent of 260 is 78?
115. What percent of 215 is 86?
116. What percent of 1,500 is 540?
117. What percent of 1,800 is 846?
118. 30 is what percent of 20?
119. 50 is what percent of 40?
120. 840 is what percent of 480?

121. 790 is what percent of 395?

Solve Applications of Percents

In the following exercises, solve the applications of percents.

122. Geneva treated her parents to dinner at their favorite restaurant. The bill was $74.25. She wants to leave 16% of the total bill as a tip. How much should the tip be?
123. When Hiro and his co-workers had lunch at a restaurant the bill was $90.50. They want to leave 18% of the total bill as a tip. How much should the tip be?
124. Trong has 12% of each paycheck automatically deposited to his savings account. His last paycheck was $2,165. How much money was deposited to Trong's savings account?
125. Cherise deposits 8% of each paycheck into her retirement account. Her last paycheck was $1,485. How much did Cherise deposit into her retirement account?
126. One serving of oatmeal has 8 grams of fiber, which is 33% of the recommended daily amount. What is the total recommended daily amount of fiber?
127. One serving of trail mix has 67 grams of carbohydrates, which is 22% of the recommended daily amount. What is the total recommended daily amount of carbohydrates?
128. A bacon cheeseburger at a popular fast food restaurant contains 2,070 milligrams (mg) of sodium, which is 86% of the recommended daily amount. What is the total recommended daily amount of sodium?
129. A grilled chicken salad at a popular fast food restaurant contains 650 milligrams (mg) of sodium, which is 27% of the recommended daily amount. What is the total recommended daily amount of sodium?
130. The nutrition fact sheet at a fast food restaurant says the fish sandwich has 380 calories, and 171 calories are from fat. What percent of the total calories is from fat?
131. The nutrition fact sheet at a fast food restaurant says a small portion of chicken nuggets has 190 calories, and 114 calories are from fat. What percent of the total calories is from fat?
132. Emma gets paid $3,000 per month. She pays $750 a month for rent. What percent of her monthly pay goes to rent?
133. Dimple gets paid $3,200 per month. She pays $960 a month for rent. What percent of her monthly pay goes to rent?

Find Percent Increase and Percent Decrease

In the following exercises, find the percent increase or percent decrease.

134. Tamanika got a raise in her hourly pay, from $15.50 to $17.55. Find the percent increase.
135. Ayodele got a raise in her hourly pay, from $24.50 to $25.48. Find the percent increase.
136. Annual student fees at the University of California rose from about $4,000 in 2000 to about $9,000 in 2014. Find the percent increase.
137. The price of a share of one stock rose from $12.50 to $50. Find the percent increase.
138. According to Time magazine (7/19/2011) annual global seafood consumption rose from 22 pounds per person in 1960 to 38 pounds per person today. Find the percent increase. (Round to the nearest tenth of a percent.)
139. In one month, the median home price in the Northeast rose from $225,400 to $241,500. Find the percent increase. (Round to the nearest tenth of a percent.)
140. A grocery store reduced the price of a loaf of bread from $2.80 to $2.73. Find the percent decrease.
141. The price of a share of one stock fell from $8.75 to $8.54. Find the percent decrease.
142. Hernando's salary was $49,500 last year. This year his salary was cut to $44,055. Find the percent decrease.
143. From 2000 to 2010, the population of Detroit fell from about 951,000 to about 714,000. Find the percent decrease. (Round to the nearest tenth of a percent.)
144. In one month, the median home price in the West fell from $203,400 to $192,300. Find the percent decrease. (Round to the nearest tenth of a percent.)
145. Sales of video games and consoles fell from $1,150 million to $1,030 million in one year. Find the percent decrease. (Round to the nearest tenth of a percent.)

Everyday Math

146. **Tipping** At the campus coffee cart, a medium coffee costs $1.65. MaryAnne brings $2.00 with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

147. **Late Fees** Alison was late paying her credit card bill of $249. She was charged a 5% late fee. What was the amount of the late fee?

Writing Exercises

148. Without solving the problem “44 is 80% of what number”, think about what the solution might be. Should it be a number that is greater than 44 or less than 44? Explain your reasoning.
149. Without solving the problem “What is 20% of 300?” think about what the solution might be. Should it be a number that is greater than 300 or less than 300? Explain your reasoning.
150. After returning from vacation, Alex said he should have packed 50% fewer shorts and 200% more shirts. Explain what Alex meant.
151. Because of road construction in one city, commuters were advised to plan their Monday morning commute to take 150% of their usual commuting time. Explain what this means.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
translate and solve basic percent equations.			
solve applications of percent.			
find percent increase and percent decrease.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

Contributors and Attributions

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7.3: Solve Sales Tax, Commission, and Discount Applications

Learning Objectives

- Solve sales tax applications
- Solve commission applications
- Solve discount applications
- Solve mark-up applications

be prepared!

Before you get started, take this readiness quiz.

1. Solve 0.0875(720) through multiplication. If you missed this problem, review Example 5.3.7.
2. Solve 12.96 ÷ 0.04 through division. If you missed this problem, review Example 5.4.12.

Solve Sales Tax Applications

Sales tax and commissions are applications of percent in our everyday lives. To solve these applications, we will follow the same strategy we used in the section on decimal operations. We show it again here for easy reference.

HOW TO: Solve an application

- Step 1. Identify what you are asked to find and choose a variable to represent it.
- Step 2. Write a sentence that gives the information to find it.
- Step 3. Translate the sentence into an equation.
- Step 4. Solve the equation using good algebra techniques.
- Step 5. Check the answer in the problem and make sure it makes sense.
- Step 6. Write a complete sentence that answers the question.

Remember that whatever the application, once we write the sentence with the given information (Step 2), we can translate it to a percent equation and then solve it.

Do you pay a tax when you shop in your city or state? In many parts of the United States, sales tax is added to the purchase price of an item. See Figure 7.3.1. The sales tax is determined by computing a percent of the purchase price.

To find the sales tax multiply the purchase price by the sales tax rate. Remember to convert the sales tax rate from a percent to a decimal number. Once the sales tax is calculated, it is added to the purchase price. The result is the total cost—this is what the customer pays.

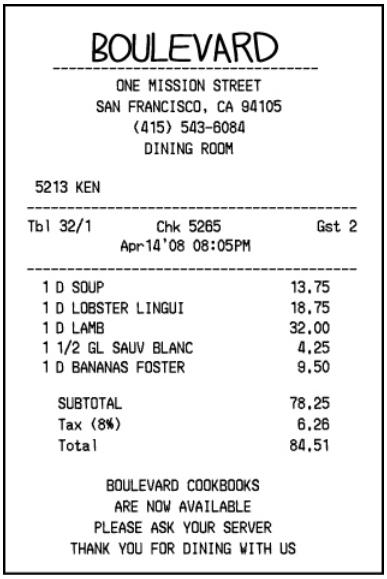

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Figure 7.3.1 - The sales tax is calculated as a percent of the purchase price.

Definition: sales tax

The sales tax is a percent of the purchase price.

$$\text{Sales Tax} = \text{Tax Rate} \cdot \text{Purchase Price}$$

$$\text{Total Cost} = \text{Purchase Price} + \text{Sales Tax}$$

Example 7.3.1:

Cathy bought a bicycle in Washington, where the sales tax rate was 6.5% of the purchase price. What was (a) the sales tax and (b) the total cost of a bicycle if the purchase price of the bicycle was $392?

Solution

(a)

Identify what you are asked to find.	What is the sales tax?
Choose a variable to represent it.	Let t = sales tax.
Write a sentence that gives the information to find it.	The sales tax is 6.5% of the purchase price.
Translate into an equation. (Remember to change the percent to a decimal).	$\underbrace{\text{The sales tax}}_{t} \quad \underbrace{\text{is}}_{=} \quad \underbrace{6.5\%}_{0.065} \quad \underbrace{\text{of}}_{\cdot} \quad \underbrace{\text{the \$392 purchase price.}}_{392}$
Simplify.	t = 25.48
Check: Is this answer reasonable?	Yes, because the sales tax amount is less than 10% of the purchase price.
Write a complete sentence that answers the question.	The sales tax is $25.48.

(b)

Identify what you are asked to find.	What is the total cost of the bicycle?
Choose a variable to represent it.	Let c = total cost of bicycle.
Write a sentence that gives the information to find it.	The total cost is the purchase price plus the sales tax.

Translate into an equation. (Remember to change the percent to a decimal).	$\underbrace{c}_{\text{The total cost}} \quad \underbrace{=}_{\text{is}} \quad \underbrace{392}_{\$392} \quad \underbrace{+}_{\text{plus}} \quad \underbrace{25.48}_{\$25.48}$
Simplify.	c = 417.48
Check: Is this answer reasonable?	Yes, because the total cost is a little more than the purchase price.
Write a complete sentence that answers the question.	The total cost of the bicycle is $417.48.

Exercise 7.3.1:

Find (a) the sales tax and (b) the total cost: Alexandra bought a television set for $724 in Boston, where the sales tax rate was 6.25% of the purchase price.

Answer a

$45.25

Answer b

$769.25

Exercise 7.3.2:

Find (a) the sales tax and (b) the total cost: Kim bought a winter coat for $250 in St. Louis, where the sales tax rate was 8.2% of the purchase price.

Answer a

$20.50

Answer b

$270.50

Example 7.3.2:

Evelyn bought a new smartphone for $499 plus tax. She was surprised when she got the receipt and saw that the tax was $42.42. What was the sales tax rate for this purchase?

Solution

Identify what you are asked to find.	What is the sales tax rate?
Choose a variable to represent it.	Let r = sales tax.
Write a sentence that gives the information to find it.	What percent of the price is the sales tax?
Translate into an equation.	$\underbrace{r}_{\text{What percent}} \quad \underbrace{\cdot}_{\text{of}} \quad \underbrace{499}_{\text{the \$499 price}} \quad \underbrace{=}_{\text{is}} \quad \underbrace{42.42}_{\text{the \$42.42 tax?}}$
Divide.	$\frac{499r}{499} = \frac{42.42}{499}$ (7.3.1)
Simplify	r = 0.085

Check. Is this answer reasonable?	Yes, because 8.5% is close to 10%. 10% of $500 is $50, which is close to $42.42.
Write a complete sentence that answers the question.	The sales tax rate is 8.5%.

Exercise 7.3.3:

Diego bought a new car for $26,525. He was surprised that the dealer then added $2,387.25. What was the sales tax rate for this purchase?

Answer

9%

Exercise 7.3.4:

What is the sales tax rate if a $7,594 purchase will have $569.55 of sales tax added to it?

Answer

7.5%

Solve Commission Applications

Sales people often receive a **commission**, or percent of total sales, for their sales. Their income may be just the commission they earn, or it may be their commission added to their hourly wages or salary. The commission they earn is calculated as a certain percent of the price of each item they sell. That percent is called the **rate of commission**.

Definition: commission

A commission is a percentage of total sales as determined by the rate of commission.

$$\text{commission} = \text{rate of commission} \cdot \text{total sales}$$

To find the commission on a sale, multiply the rate of commission by the total sales. Just as we did for computing sales tax, remember to first convert the rate of commission from a percent to a decimal.

Example 7.3.3:

Helene is a realtor. She receives 3% commission when she sells a house. How much commission will she receive for selling a house that costs $260,000?

Solution

Identify what you are asked to find.	What is the commission?
Choose a variable to represent it.	Let c = the commission.
Write a sentence that gives the information to find it.	The commission is 3% of the price.
Translate into an equation.	$\underbrace{\text{The commission}}_{c} \underbrace{\text{is}}_{=} \underbrace{3\%}_{0.03} \underbrace{\text{of}}_{\cdot} \underbrace{\text{the \$260,000 price.}}_{260{,}000}$
Simplify.	c = 7800
Change to percent form.	r = 8.5%
Check. Is this answer reasonable?	Yes. 1% of $260,000 is $2,600, and $7,800 is three times $2,600.

Write a complete sentence that answers the question.	Helene will receive a commission of $7,800.

Exercise 7.3.5:

Bob is a travel agent. He receives 7% commission when he books a cruise for a customer. How much commission will he receive for booking a $3,900 cruise?

Answer

$273

Exercise 7.3.6:

Fernando receives 18% commission when he makes a computer sale. How much commission will he receive for selling a computer for $2,190?

Answer

$394.20

Example 7.3.4:

Rikki earned $87 commission when she sold a $1,450 stove. What rate of commission did she get?

Solution

Identify what you are asked to find.	What is the rate of commission?
Choose a variable to represent it.	Let r = the rate of commission.
Write a sentence that gives the information to find it.	The commission is what percent of the sale?
Translate into an equation.	$\underbrace{87}_{\text{The \$87 commission}} \underbrace{=}_{\text{is}} \underbrace{r}_{\text{what percent}} \underbrace{\cdot}_{\text{of}} \underbrace{1450}_{\text{the \$1450 sale?}}$
Divide.	$\frac{87}{1450} = \frac{1450r}{1450}$ (7.3.2)
Simplify.	0.06 = r
Change to percent form.	r = 6%
Check if this answer is reasonable.	Yes. A 10% commission would have been $145. The 6% commission, $87, is a little more than half of that.
Write a complete sentence that answers the question.	The commission was 6% of the price of the stove.

Exercise 7.3.7:

Homer received $1,140 commission when he sold a car for $28,500. What rate of commission did he get?

Answer

4%

Exercise 7.3.8:

Bernice earned $451 commission when she sold an $8,200 living room set. What rate of commission did she get?

Answer

5.5%

Solve Discount Applications

Applications of discount are very common in retail settings Figure 6.8. When you buy an item on sale, the **original price** of the item has been reduced by some dollar amount. The **discount rate**, usually given as a percent, is used to determine the amount of the discount. To determine the **amount of discount**, we multiply the discount rate by the original price. We summarize the discount model in the box below.

Figure 6.8 - Applications of discounts are common in everyday life. (credit: Charleston's TheDigitel, Flickr)

Definition: discount

An amount of discount is a percent off the original price.

$$\text{amount of discount} = \text{discount rate} \cdot \text{original price}$$

$$\text{sale price} = \text{original price} - \text{discount}$$

The sale price should always be less than the original price. In some cases, the amount of discount is a fixed dollar amount. Then we just find the sale price by subtracting the amount of discount from the original price.

Example 7.3.5:

Jason bought a pair of sunglasses that were on sale for $10 off. The original price of the sunglasses was $39. What was the sale price of the sunglasses?

Solution

Identify what you are asked to find.	What is the sale price?
Choose a variable to represent it.	Let s = the sale price.
Write a sentence that gives the information to find it.	The sale price is the original price minus the discount.
Translate into an equation.	$\underbrace{s}_{\text{The sale price}} \quad \underbrace{=}_{\text{is}} \quad \underbrace{39}_{\text{the original \$39 price}} \quad \underbrace{-}_{\text{minus}} \quad \underbrace{10}_{\text{the \$10 discount.}}$
Simplify.	s = 29
Check if this answer is reasonable.	Yes. The sale price, $29, is less than the original price, $39.

Write a complete sentence that answers the question.	The sale price of the sunglasses was $29.

? Exercise 7.3.9:

Marta bought a dishwasher that was on sale for $75 off. The original price of the dishwasher was $525. What was the sale price of the dishwasher?

Answer

$450

? Exercise 7.3.10:

Orlando bought a pair of shoes that was on sale for $30 off. The original price of the shoes was $112. What was the sale price of the shoes?

Answer

$82

In Example 6.29, the amount of discount was a set amount, $10. In Example 6.30 the discount is given as a percent of the original price.

✓ Example 7.3.6:

Elise bought a dress that was discounted 35% off of the original price of $140. What was (a) the amount of discount and (b) the sale price of the dress?

Solution

(a) Before beginning, you may find it helpful to organize the information in a list. Original price = $140, Discount rate = 35%, Amount of discount = ?

Identify what you are asked to find.	What is the amount of discount?
Choose a variable to represent it.	Let d = the amount of discount.
Write a sentence that gives the information to find it.	The discount is 35% of the original price.
Translate into an equation.	$\underbrace{\text{The discount}}_{s}\ \underbrace{\text{is}}_{=}\ \underbrace{35\%}_{0.35}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{\text{the \$140 original price.}}_{140}$
Simplify.	d = 49
Check if this answer is reasonable.	Yes. A $49 discount is reasonable for a $140 dress.
Write a complete sentence that answers the question.	The amount of discount was $49.

(b) Original price = $140, Amount of discount = $49, Sale price = ?

Identify what you are asked to find.	What is the sale price of the dress?
Choose a variable to represent it.	Let s = the sale price.
Write a sentence that gives the information to find it.	The sale price is the original price minus the discount.
Translate into an equation.	$\underbrace{\text{The sale price}}_{s}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{the \$140}}_{140}\ \underbrace{\text{minus}}_{-}\ \underbrace{\text{the \$49 discount.}}_{49}$

Simplify.	s = 91
Check if this answer is reasonable.	Yes. The sale price, $91, is less than the original price, $140.
Write a complete sentence that answers the question.	The sale price of the dress was $91.

Exercise 7.3.11:

Find (a) the amount of discount and (b) the sale price: Sergio bought a belt that was discounted 40% from an original price of $29.

Answer a

$11.60

Answer b

$17.40

Exercise 7.3.12:

Find (a) the amount of discount and (b) the sale price: Oscar bought a barbecue grill that was discounted 65% from an original price of

$395.

Answer a

$256.75

Answer b

$138.25

There may be times when you buy something on sale and want to know the discount rate. The next example will show this case.

Example 7.3.7:

Jeannette bought a swimsuit at a sale price of $13.95. The original price of the swimsuit was $31. Find the (a) amount of discount and (b) discount rate.

Solution

(a) Before beginning, you may find it helpful to organize the information in a list. Original price = $31, Amount of discount = ?, Sale price = $13.95

Identify what you are asked to find.	What is the amount of discount?
Choose a variable to represent it.	Let d = the amount of discount.
Write a sentence that gives the information to find it.	The discount is the original price minus the sale price.
Translate into an equation.	$\underbrace{\text{The discount}}_{s}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{the \$31 original price}}_{31}\ \underbrace{\text{minus}}_{-}\ \underbrace{\text{the \$13.95 sale price.}}_{13.95}$
Simplify.	d = 17.05
Check if this answer is reasonable.	Yes. The $17.05 discount is less than the original price.
Write a complete sentence that answers the question.	The amount of discount was $17.05.

(b) Before beginning, you may find it helpful to organize the information in a list. Original price = $31, Amount of discount = $17.05, Discount rate = ?

Identify what you are asked to find.	What is the amount of discount?
Choose a variable to represent it.	Let r = the discount rate.
Write a sentence that gives the information to find it.	The discount is what percent of the original price?
Translate into an equation.	$\underbrace{\text{The discount of \$17.05}}_{s}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{what percent}}_{r}\ \underbrace{\text{of}}_{\cdot}\ \underbrace{\text{the \$31 original price.}}_{31}$
Divide.	$\frac{17.05}{31} = \frac{r(31)}{31}$ (7.3.3)
Simplify.	0.55 = r
Check if this answer is reasonable.	The rate of discount was a little more than 50% and the amount of discount is a little more than half of $31.
Write a complete sentence that answers the question.	The rate of discount was 55%.

? Exercise 7.3.13:

Find (a) the amount of discount and (b) the discount rate: Lena bought a kitchen table at the sale price of $375.20. The original price of the table was $560.

Answer a

$184.80

Answer b

33%

? Exercise 7.3.14:

Find (a) the amount of discount and (b) the discount rate: Nick bought a multi-room air conditioner at a sale price of $340. The original price of the air conditioner was $400.

Answer a

$60

Answer b

15%

Solve Mark-up Applications

Applications of mark-up are very common in retail settings. The price a retailer pays for an item is called the **wholesale price**. The retailer then adds a **mark-up** to the wholesale price to get the **list price**, the price he sells the item for. The mark-up is usually calculated as a percent of the wholesale price. The percent is called the **mark-up rate**. To determine the amount of mark-up, multiply the mark-up rate by the wholesale price. We summarize the mark-up model in the box below.

Definition: mark-up

The mark-up is the amount added to the wholesale price.

$$\text{amount of mark-up} = \text{mark-up rate} \cdot \text{wholesale price}$$

$$\text{list price} = \text{wholesale price} + \text{mark up}$$

The list price should always be more than the wholesale price.

Example 7.3.8:

Adam's art gallery bought a photograph at the wholesale price of $250. Adam marked the price up 40%. Find the (a) amount of mark-up and (b) the list price of the photograph.

Solution

(a)

Identify what you are asked to find.	What is the amount of mark-up?
Choose a variable to represent it.	Let m = the amount of each mark-up.
Write a sentence that gives the information to find it.	The mark-up is 40% of the wholesale price.
Translate into an equation.	$\underbrace{\text{The mark-up}}_{m} \ \underbrace{\text{is}}_{=} \ \underbrace{40\%}_{0.40} \ \underbrace{\text{of}}_{\cdot} \ \underbrace{\text{the \$250 wholesale price.}}_{250}$
Simplify.	m = 100
Check if this answer is reasonable.	Yes. The markup rate is less than 50% and $100 is less than half of $250.
Write a complete sentence that answers the question.	The mark-up on the photograph was $100.

(b)

Identify what you are asked to find.	What is the list price?
Choose a variable to represent it.	Let p = the list price.
Write a sentence that gives the information to find it.	The list price is the wholesale price plus the mark-up.
Translate into an equation.	$\underbrace{\text{The list price}}_{p} \ \underbrace{\text{is}}_{=} \ \underbrace{\text{the \$250 wholesale price}}_{250} \ \underbrace{\text{plus}}_{+} \ \underbrace{\text{the \$100 mark-up.}}_{100}$
Simplify.	p = 350
Check if this answer is reasonable.	Yes. The list price, $350, is more than the wholesale price, $250.
Write a complete sentence that answers the question.	The list price of the photograph was $350.

Exercise 7.3.15:

Jim's music store bought a guitar at wholesale price $1,200. Jim marked the price up 50%. Find the (a) amount of mark-up and (b) the list price.

Answer a

600%

Answer b

$1,800

Exercise 7.3.16:

The Auto Resale Store bought Pablo's Toyota for $8,500. They marked the price up 35%. Find the (a) amount of mark-up and (b) the list price.

Answer a

$2,975

Answer b

$11,475

Practice Makes Perfect

Solve Sales Tax Applications

In the following exercises, find (a) the sales tax and (b) the total cost.

152. The cost of a pair of boots was $84. The sales tax rate is 5% of the purchase price.
153. The cost of a refrigerator was $1,242. The sales tax rate is 8% of the purchase price.
154. The cost of a microwave oven was $129. The sales tax rate is 7.5% of the purchase price.
155. The cost of a tablet computer is $350. The sales tax rate is 8.5% of the purchase price.
156. The cost of a file cabinet is $250. The sales tax rate is 6.85% of the purchase price.
157. The cost of a luggage set $400. The sales tax rate is 5.75% of the purchase price.
158. The cost of a 6 -drawer dresser $1,199. The sales tax rate is 5.125% of the purchase price.
159. The cost of a sofa is $1,350. The sales tax rate is 4.225% of the purchase price.

In the following exercises, find the sales tax rate.

160. Shawna bought a mixer for $300. The sales tax on the purchase was $19.50.
161. Orphia bought a coffee table for $400. The sales tax on the purchase was $38.
162. Bopha bought a bedroom set for $3,600. The sales tax on the purchase was $246.60.
163. Ruth bought a washer and dryer set for $2,100. The sales tax on the purchase was $152.25.

Solve Commission Applications

In the following exercises, find the commission.

164. Christopher sold his dinette set for $225 through an online site, which charged him 9% of the selling price as commission. How much was the commission?
165. Michele rented a booth at a craft fair, which charged her 8% commission on her sales. One day her total sales were $193. How much was the commission?
166. Farrah works in a jewelry store and receives 12% commission when she makes a sale. How much commission will she receive for selling a $8,125 ring?
167. Jamal works at a car dealership and receives 9% commission when he sells a car. How much commission will he receive for selling a $32,575 car?
168. Hector receives 17.5% commission when he sells an insurance policy. How much commission will he receive for selling a policy for $4,910?
169. Denise receives 10.5% commission when she books a tour at the travel agency. How much commission will she receive for booking a tour with total cost $7,420?

In the following exercises, find the rate of commission.

170. Dontay is a realtor and earned $11,250 commission on the sale of a $375,000 house. What is his rate of commission?
171. Nevaeh is a cruise specialist and earned $364 commission after booking a cruise that cost $5,200. What is her rate of commission?

172. As a waitress, Emily earned $420 in tips on sales of $2,625 last Saturday night. What was her rate of commission?
173. Alejandra earned $1,393.74 commission on weekly sales of $15,486 as a salesperson at the computer store. What is her rate of commission?
174. Maureen earned $7,052.50 commission when she sold a $45,500 car. What was the rate of commission?
175. Lucas earned $4,487.50 commission when he brought a $35,900 job to his office. What was the rate of commission?

Solve Discount Applications

In the following exercises, find the sale price.

176. Perla bought a cellphone that was on sale for $50 off. The original price of the cellphone was $189.
177. Sophie saw a dress she liked on sale for $15 off. The original price of the dress was $96.
178. Rick wants to buy a tool set with original price $165. Next week the tool set will be on sale for 40% off.
179. Angelo's store is having a sale on TV sets. One set, with an original price of $859, is selling for $125 off.

In the following exercises, find (a) the amount of discount and (b) the sale price.

180. Janelle bought a beach chair on sale at 60% off. The original price was $44.95
181. Errol bought a skateboard helmet on sale at 40% off. The original price was $49.95.
182. Kathy wants to buy a camera that lists for $389. The camera is on sale with a 33% discount.
183. Colleen bought a suit that was discounted 25% from an original price of $245.
184. Erys bought a treadmill on sale at 35% off. The original price was $949.95.
185. Jay bought a guitar on sale at 45% off. The original price was $514.75.

In the following exercises, find (a) the amount of discount and (b) the discount rate. (Round to the nearest tenth of a percent if needed.)

186. Larry and Donna bought a sofa at the sale price of $1,344. The original price of the sofa was $1,920.
187. Hiroshi bought a lawnmower at the sale price of $240. The original price of the lawnmower is $300.
188. Patty bought a baby stroller on sale for $301.75. The original price of the stroller was $355.
189. Bill found a book he wanted on sale for $20.80. The original price of the book was $32.
190. Nikki bought a patio set on sale for $480. The original price was $850.
191. Stella bought a dinette set on sale for $725. The original price was $1,299.

Solve Mark-up Applications

In the following exercises, find (a) the amount of the mark-up and (b) the list price.

192. Daria bought a bracelet at wholesale cost $16 to sell in her handicraft store. She marked the price up 45%.
193. Regina bought a handmade quilt at wholesale cost $120 to sell in her quilt store. She marked the price up 55%.
194. Tom paid $0.60 a pound for tomatoes to sell at his produce store. He added a 33% mark-up.
195. Flora paid her supplier $0.74 a stem for roses to sell at her flower shop. She added an 85% mark-up.
196. Alan bought a used bicycle for $115. After re-conditioning it, he added 225% mark-up and then advertised it for sale.
197. Michael bought a classic car for $8,500. He restored it, then added 150% mark-up before advertising it for sale.

Everyday Math

198. Coupons Yvonne can use two coupons for the same purchase at her favorite department store. One coupon gives her $20 off and the other gives her 25% off. She wants to buy a bedspread that sells for $195.
    a. Calculate the discount price if Yvonne uses the $20 coupon first and then takes 25% off.
    b. Calculate the discount price if Yvonne uses the 25% off coupon first and then uses the $20 coupon.
    c. In which order should Yvonne use the coupons?
199. Cash Back Jason can buy a bag of dog food for $35 at two different stores. One store offers 6% cash back on the purchase plus $5 off his next purchase. The other store offers 20% cash back.
    a. Calculate the total savings from the first store, including the savings on the next purchase.
    b. Calculate the total savings from the second store.
    c. Which store should Jason buy the dog food from? Why?

Writing Exercises

200. Priam bought a jacket that was on sale for 40% off. The original price of the jacket was $150. While the sales clerk figured the price by calculating the amount of discount and then subtracting that amount from $150, Priam found the price faster by calculating 60% of $150.
    a. Explain why Priam was correct.
    b. Will Priam's method work for any original price?
201. Roxy bought a scarf on sale for 50% off. The original price of the scarf was $32.90. Roxy claimed that the price she paid for the scarf was the same as the amount she saved. Was Roxy correct? Explain.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve sales tax applications.			
solve commission applications.			
solve discount applications.			
solve mark-up applications.			

(b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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7.4: Solve Simple Interest Applications

Learning Objectives

- Use the simple interest formula
- Solve simple interest applications

Be prepared!

Before you get started, take this readiness quiz.

1. Solve 0.6y = 45. If you missed this problem, review Example 5.7.4.
2. Solve $\frac{n}{1.45}$ = 4.6. If you missed this problem, review Example 5.7.5.

Use the Simple Interest Formula

Do you know that banks pay you to let them keep your money? The money you put in the bank is called the **principal**, P, and the bank pays you **interest**, I. The interest is computed as a certain percent of the principal; called the **rate of interest**, r. The rate of interest is usually expressed as a percent per year, and is calculated by using the decimal equivalent of the percent. The variable for time, t, represents the number of years the money is left in the account.

Definition: simple interest

If an amount of money, P, the principal, is invested for a period of t years at an annual interest rate r, the amount of interest, I, earned is

$$I = Prt$$

where

- I = interest
- P = principal
- r = rate
- t = time

Interest earned according to this formula is called simple interest.

The formula we use to calculate simple interest is I = Prt. To use the simple interest formula we substitute in the values for variables that are given, and then solve for the unknown variable. It may be helpful to organize the information by listing all four variables and filling in the given information.

Example 7.4.1

Find the simple interest earned after 3 years on $500 at an interest rate of 6%.

Solution

Organize the given information in a list.

I = ?, P = $500, r = 6%, t = 3 years

We will use the simple interest formula to find the interest.

Write the formula.	I = Prt
Substitute the given information. Remember to write the percent in decimal form.	I = (500)(0.06)(3)
Simplify.	I = 90

Check your answer. Is $90 a reasonable interest earned on $500 in 3 years?	In 3 years the money earned 18%. If we rounded to 20%, the interest would have been 500(0.20) or $100. Yes, $90 is reasonable.
Write a complete sentence that answers the question.	The simple interest is $90.

Exercise 7.4.1:

Find the simple interest earned after 4 years on $800 at an interest rate of 5%.

Answer

$160

Exercise 7.4.2:

Find the simple interest earned after 2 years on $700 at an interest rate of 4%.

Answer

$56

In the next example, we will use the simple interest formula to find the principal.

Example 7.4.2:

Find the principal invested if $178 interest was earned in 2 years at an interest rate of 4%.

Solution

Organize the given information in a list.

$$I = \$178,\ P = ?,\ r = 4\%,\ t = 2 \text{ years}$$

We will use the simple interest formula to find the principal.

Write the formula.	$I = Prt$	
Substitute the given information.	$178 = P(0.04)(2)$	
Divide.	$\frac{178}{0.08} = \frac{0.08P}{0.08}$	(7.4.1)
Simplify.	$2{,}225 = P$	
Check your answer. Is it reasonable that $2,225 would earn $178 in 2 years?	$178 \stackrel{?}{=} 2,225(0.04)(2)$	(7.4.2)
	$178 = 178$ ✓	(7.4.3)
Write a complete sentence that answers the question.	The principal is $2,225.	

Exercise 7.4.3:

Find the principal invested if $495 interest was earned in 3 years at an interest rate of 6%.

Answer

$2,750

Exercise 7.4.4

Find the principal invested if $1,246 interest was earned in 5 years at an interest rate of 7%.

Answer

$3,560

Now we will solve for the rate of interest.

Example 7.4.3

Find the rate if a principal of $8,200 earned $3,772 interest in 4 years.

Solution

Organize the given information.

$$I = \$3{,}772,\ P = \$8{,}200,\ r = ?,\ t = 4 \text{ years}$$

We will use the simple interest formula to find the rate.

Write the formula.	$I = Prt$	
Substitute the given information.	$3{,}772 = 8{,}200r(4)$	
Multiply.	$3{,}772 = 32{,}800r$	
Divide.	$\frac{3,772}{32,800} = \frac{32,800r}{32,800}$	(7.4.4)
Simplify.	$0.115 = r$	
Write as a percent.	$11.5\% = r$	
Check your answer. Is 11.5% a reasonable rate if $3,772 was earned in 4 years?	$3,772 \stackrel{?}{=} 8,200(0.115)(4)$	(7.4.5)
	$3,772 = 3,772$ ✓	(7.4.6)
Write a complete sentence that answers the question.	The rate was 11.5%.	

Exercise 7.4.5:

Find the rate if a principal of $5,000 earned $1,350 interest in 6 years.

Answer

4.5%

Exercise 7.4.6:

Find the rate if a principal of $9,000 earned $1,755 interest in 3 years.

Answer

6.5%

Solve Simple Interest Applications

Applications with simple interest usually involve either investing money or borrowing money. To solve these applications, we continue to use the same strategy for applications that we have used earlier in this chapter. The only difference is that in place of translating to get an equation, we can use the simple interest formula.

We will start by solving a simple interest application to find the interest.

Example 7.4.4:

Nathaly deposited $12,500 in her bank account where it will earn 4% interest. How much interest will Nathaly earn in 5 years?

Solution

We are asked to find the Interest, I. Organize the given information in a list.

$$I = ?,\ P = \$12{,}500,\ r = 4\%,\ t = 5 \text{ years}$$

Write the formula.	$I = Prt$
Substitute the given information.	$I = (12{,}500)(0.04)(5)$
Simplify.	$I = 2{,}500$
Check your answer. Is $2,500 a reasonable interest on $12,500 over 5 years?	At 4% interest per year, in 5 years the interest would be 20% of the principal. Is 20% of $12,500 equal to $2,500? Yes.
Write a complete sentence that answers the question.	The interest is $2,500.

Exercise 7.4.7:

Areli invested a principal of $950 in her bank account with interest rate 3%. How much interest did she earn in 5 years?

Answer

$142.50

Exercise 7.4.8:

Susana invested a principal of $36,000 in her bank account with interest rate 6.5%. How much interest did she earn in 3 years?

Answer

$7,020

There may be times when you know the amount of interest earned on a given principal over a certain length of time, but you don't know the rate. For instance, this might happen when family members lend or borrow money among themselves instead of dealing with a bank. In the next example, we'll show how to solve for the rate.

Example 7.4.5:

Loren lent his brother $3,000 to help him buy a car. In 4 years his brother paid him back the $3,000 plus $660 in interest. What was the rate of interest?

Solution

We are asked to find the rate of interest, r. Organize the given information.

$$I = 660,\ P = \$3{,}000,\ r = ?,\ t = 4 \text{ years}$$

Write the formula.	$I = Prt$

Substitute the given information.	660 = (3,000)r(4)	
Multiply.	660 = (12,000)r	
Divide.	$\frac{660}{12,000} = \frac{(12,000)r}{12,000}$	(7.4.7)
Simplify.	0.055 = r	
Change to percent form.	5.5% = r	
Check your answer. Is 5.5% a reasonable interest rate to pay your brother?	$660 \stackrel{?}{=} (3,000)(0.055)(4)$	(7.4.8)
	$660 = 660$ ✓	(7.4.9)
Write a complete sentence that answers the question.	The rate of interest was 5.5%.	

? Exercise 7.4.9:

Jim lent his sister $5,000 to help her buy a house. In 3 years, she paid him the $5,000, plus $900 interest. What was the rate of interest?

Answer

6%

? Exercise 7.4.10:

Hang borrowed $7,500 from her parents to pay her tuition. In 5 years, she paid them $1,500 interest in addition to the $7,500 she borrowed. What was the rate of interest?

Answer

4%

There may be times when you take a loan for a large purchase and the amount of the principal is not clear. This might happen, for instance, in making a car purchase when the dealer adds the cost of a warranty to the price of the car. In the next example, we will solve a simple interest application for the principal.

✓ Example 7.4.6:

Eduardo noticed that his new car loan papers stated that with an interest rate of 7.5%, he would pay $6,596.25 in interest over 5 years. How much did he borrow to pay for his car?

Solution

We are asked to find the principal, P. Organize the given information.

I = 6,596.25, P = ?, r = 7.5%, t = 5 years

Write the formula.	I = Prt
Substitute the given information.	6,596.25 = P(0.075)(5)
Multiply.	6,596.25 = 0.375P

Divide.	$\frac{6,596.25}{0.375} = \frac{0.375P}{0.375}$	(7.4.10)
Simplify.	17,590 = P	
Check your answer. Is $17,590 a reasonable amount to borrow to buy a car?	$6,596.25 \stackrel{?}{=} (17,590)(0.075)(5)$	(7.4.11)
	$6,596.25 = 6,596.25$ ✓	(7.4.12)
Write a complete sentence that answers the question.	The amount borrowed was $17,590.	

? Exercise 7.4.11:

Sean's new car loan statement said he would pay $4,866.25 in interest from an interest rate of 8.5% over 5 years. How much did he borrow to buy his new car?

Answer

$11,450

? Exercise 7.4.12:

In 5 years, Gloria's bank account earned $2,400 interest at 5%. How much had she deposited in the account?

Answer

$9,600

In the simple interest formula, the rate of interest is given as an annual rate, the rate for one year. So the units of time must be in years. If the time is given in months, we convert it to years.

✓ Example 7.4.7:

Caroline got $900 as graduation gifts and invested it in a 10-month certificate of deposit that earned 2.1% interest. How much interest did this investment earn?

Solution

We are asked to find the interest, I. Organize the given information.

I = ?, P = $900, r = 2.1%, t = 10 months

Write the formula.	I = Prt	
Substitute the given information, converting 10 months to $\frac{10}{12}$ of a year.	$I = \$900(0.021)\left(\frac{10}{12}\right)$	(7.4.13)
Multiply.	I = 15.75	
Check your answer. Is $15.75 a reasonable amount of interest?	If Caroline had invested the $900 for a full year at 2% interest, the amount of interest would have been $18. Yes, $15.75 is reasonable.	
Write a complete sentence that answers the question.	The interest earned was $15.75.	

Exercise 7.4.13:

Adriana invested $4,500 for 8 months in an account that paid 1.9% interest. How much interest did she earn?

Answer

$57.00

Exercise 7.4.14:

Milton invested $2,460 for 20 months in an account that paid 3.5% interest How much interest did he earn?

Answer

$143.50

Practice Makes Perfect

Use the Simple Interest Formula

In the following exercises, use the simple interest formula to fill in the missing information.

202.

Table 7.4.1

Interest	Principal	Rate	Time (years)
	$1200	3%	5

203.

Table 7.4.2

Interest	Principal	Rate	Time (years)
	$1500	2%	4

204.

Table 7.4.3

Interest	Principal	Rate	Time (years)
$4410		4.5%	7

205.

Table 7.4.4

Interest	Principal	Rate	Time (years)
$2212		3.2%	6

206.

Table 7.4.5

Interest	Principal	Rate	Time (years)
$577.08	$4580		2

207.

Table 7.4.6

Interest	Principal	Rate	Time (years)
$528.12	$3260		3

In the following exercises, solve the problem using the simple interest formula.

208. Find the simple interest earned after 5 years on $600 at an interest rate of 3%.
209. Find the simple interest earned after 4 years on $900 at an interest rate of 6%.
210. Find the simple interest earned after 2 years on $8,950 at an interest rate of 3.24%.
211. Find the simple interest earned after 3 years on $6,510 at an interest rate of 2.85%.
212. Find the simple interest earned after 8 years on $15,500 at an interest rate of 11.425%.
213. Find the simple interest earned after 6 years on $23,900 at an interest rate of 12.175%.
214. Find the principal invested if $656 interest was earned in 5 years at an interest rate of 4%.
215. Find the principal invested if $177 interest was earned in 2 years at an interest rate of 3%.
216. Find the principal invested if $70.95 interest was earned in 3 years at an interest rate of 2.75%.
217. Find the principal invested if $636.84 interest was earned in 6 years at an interest rate of 4.35%.
218. Find the principal invested if $15,222.57 interest was earned in 6 years at an interest rate of 10.28%.
219. Find the principal invested if $10,953.70 interest was earned in 5 years at an interest rate of 11.04%.
220. Find the rate if a principal of $5,400 earned $432 interest in 2 years.
221. Find the rate if a principal of $2,600 earned $468 interest in 6 years.
222. Find the rate if a principal of $11,000 earned $1,815 interest in 3 years.
223. Find the rate if a principal of $8,500 earned $3,230 interest in 4 years.

Solve Simple Interest Applications

In the following exercises, solve the problem using the simple interest formula.

224. Casey deposited $1,450 in a bank account with interest rate 4%. How much interest was earned in 2 years?
225. Terrence deposited $5,720 in a bank account with interest rate 6%. How much interest was earned in 4 years?
226. Robin deposited $31,000 in a bank account with interest rate 5.2%. How much interest was earned in 3 years?
227. Carleen deposited $16,400 in a bank account with interest rate 3.9%. How much interest was earned in 8 years?
228. Hilaria borrowed $8,000 from her grandfather to pay for college. Five years later, she paid him back the $8,000, plus $1,200 interest. What was the rate of interest?
229. Kenneth lent his niece $1,200 to buy a computer. Two years later, she paid him back the $1,200, plus $96 interest. What was the rate of interest?
230. Lebron lent his daughter $20,000 to help her buy a condominium. When she sold the condominium four years later, she paid him the $20,000, plus $3,000 interest. What was the rate of interest?
231. Pablo borrowed $50,000 to start a business. Three years later, he repaid the $50,000, plus $9,375 interest. What was the rate of interest?
232. In 10 years, a bank account that paid 5.25% earned $18,375 interest. What was the principal of the account?
233. In 25 years, a bond that paid 4.75% earned $2,375 interest. What was the principal of the bond?
234. Joshua's computer loan statement said he would pay $1,244.34 in interest for a 3 year loan at 12.4%. How much did Joshua borrow to buy the computer?
235. Margaret's car loan statement said she would pay $7,683.20 in interest for a 5 year loan at 9.8%. How much did Margaret borrow to buy the car?
236. Caitlin invested $8,200 in an 18-month certificate of deposit paying 2.7% interest. How much interest did she earn form this investment?
237. Diego invested $6,100 in a 9-month certificate of deposit paying 1.8% interest. How much interest did he earn form this investment?
238. Airin borrowed $3,900 from her parents for the down payment on a car and promised to pay them back in 15 months at a 4% rate of interest. How much interest did she owe her parents?
239. Yuta borrowed $840 from his brother to pay for his textbooks and promised to pay him back in 5 months at a 6% rate of interest. How much interest did Yuta owe his brother?

Everyday Math

240. **Interest on savings** Find the interest rate your local bank pays on savings accounts.
    a. What is the interest rate?
    b. Calculate the amount of interest you would earn on a principal of $8,000 for 5 years.

241. **Interest on a loan** Find the interest rate your local bank charges for a car loan.
    a. What is the interest rate?
    b. Calculate the amount of interest you would pay on a loan of $8,000 for 5 years.

Writing Exercises

242. Why do banks pay interest on money deposited in savings accounts?
243. Why do banks charge interest for lending money?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the simple interest formula.			
solve simple interest applications.			

(b) On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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7.5: Percent Increase or Decrease

A person's salary can increase by a percentage. A town's population can decrease by a percentage. A clothing firm can discount its apparel. These are the types of applications we will investigate in this section.

Percent Increase

Example 1

Add text here.

Solution

Let *x* represent the salesperson's salary increase. Then we can translate the problem into words and symbols.

Salary increase	is	5%	of	original salary
x	$=$	5%	$\cdot$	4000

Solve for *x*.

$$x = 0.05 \cdot 4000 \quad 5\% = 0.05.$$
$$x = 200 \quad \text{Multiply: } 0.05 \cdot 4000 = 200.$$

Therefore, the salary increase is $200. To compute the new salary *N*, we must add this increase to the original salary.

New salary	is	original salary	plus	increase
N	$=$	4000	$+$	200

Thus, the new salary is N = $4, 200 per month.

Alternative Solution

If the salesperson is to receive a 5% increase in his salary, then his new salary will be 105% of his original salary. Let N represent his new monthly salary. Then,

New salary	is	105%	of	original salary
N	$=$	105%	$\cdot$	4000

Solve for *N*.

$$N = 1.05 \cdot 4000 \quad 105\% = 1.05.$$
$$N = 4200 \quad \text{Multiply: } 1.05 \cdot 4000 = 4200.$$

Same answer.

Exercise

A computer technician is granted a 4% salary increase. If the salesperson's current salary is $2,800 per month, what will be his new salary?

Answer

$2,912

Example 2

A salesperson making a salary of $4,500 per month has his salary increased to $5,000 per month. What is the percent increase?

Solution

To find the increase in salary, first subtract the original salary from the new salary.

$$\begin{aligned}\text{Salary increase} &= \text{new salary} - \text{original salary}\\ &= 5000 - 4500\\ &= 500\end{aligned}$$

Hence, the salesperson sees an increase in salary of $500.

Next, let p represent the salesperson's percent salary increase. Then we can translate the problem into words and symbols.

Salary increase	is	what percent	of	original salary
500	=	p	·	4500

The commutative property of multiplication allows us to change the order of multiplication on the right-hand side of this last equation.

$$500 = 4500p$$

Solve for p.

$$\begin{aligned}\frac{500}{4500} &= \frac{4500p}{4500} && \text{Divide both sides by 4500.}\\ \frac{1}{9} &= p && \text{Reduce by dividing numerator and denominator of } 500/4500 \text{ by 500.}\end{aligned}$$

We need to change p = 1/9 to a percent. We can find an exact answer by creating an equivalent fraction with a denominator of 100.

$$\begin{aligned}\frac{1}{9} &= \frac{n}{100} && \text{Make equivalent fraction.}\\ 9n &= 100 && \text{Cross multiply.}\\ \frac{9n}{9} &= \frac{100}{9} && \text{Divide both sides by 9.}\\ n &= 11\frac{1}{9} && \text{Convert } 100/9 \text{ to mixed fraction.}\end{aligned}$$

Hence, the percent increase is

$$p = \frac{1}{9} = \frac{11\frac{1}{9}}{100} = 11\frac{1}{9}\%.$$

Alternative Solution

An alternative approach is to ask what percent of the original salary equals the new salary. In this approach, let p represent the percent of the original salary that equals the new salary.

New salary	is	what percent	of	original salary
5000	=	p	·	4500

Solve for p.

$$\begin{aligned}5000 &= 4500p && \text{Change the order of multiplication.}\\ \frac{5000}{4500} &= \frac{4500p}{4500} && \text{Divide both sides by 4500.}\\ \frac{10}{9} &= p && \text{Reduce: Divide numerator and denominator of } 5000/4500 \text{ by 500.}\end{aligned}$$

We need to change 10/9 to a percent. Again, create an equivalent fraction with a denominator of 100.

$$\begin{aligned} \frac{10}{9} &= \frac{n}{100} && \text{Make equivalent fraction.} \\ 9n &= 1000 && \text{Cross multiply.} \\ \frac{9n}{9} &= \frac{1000}{9} && \text{Divide both sides by 9.} \\ n &= 111\frac{1}{9} && \text{Convert 1000/9 to a mixed fraction.} \end{aligned}$$

Thus,

$$p = \frac{10}{9} = \frac{111\frac{1}{9}}{100} = 111\frac{1}{9}\%.$$

Hence, the new salary is $111\frac{1}{9}\%$ of the original salary. To find the percent increase, subtract 100% from $111\frac{1}{9}\%$.

$$111\frac{1}{9}\% - 100\% = 11\frac{1}{9}\%$$

This represents an $11\frac{1}{9}\%$ increase in salary, which is the same answer garnered by the first solution technique.

? Exercise

A statistician making a salary of $3,200 per month has his salary increased to $3,368 per month. What is the percent increase?

Answer

$5\frac{1}{4}\%$

Percent Decrease

✓ Example 3

Due to a mill closure, the population of Silvertown decreases by 8.5%. If the original population was 10,200 hardy souls, what is the new population?

Solution

Let *x* represent the population decrease. Then we can translate the problem into words and symbols.

Population decrease	is	8.5%	of	original population
x	$=$	8.5%	$\cdot$	10200

Solve for *x*.

$$\begin{aligned} x &= 0.085 \cdot 10200 && 8.5\% = 0.085. \\ x &= 867 && \text{Multiply: } 0.085 \cdot 10200 = 867. \end{aligned}$$

Therefore, the population decrease is 867. To compute the new population P, we must subtract this decrease from the original population.

New population	is	original population	minus	population decrease
P	$=$	10200	$-$	867

Thus, the new population is P = 9,333 hardy souls.

Alternative Solution

Subtract 8.5% from 100% to obtain

$$100\% - 8.5\% = 91.5\%.$$

Thus, if 8.5% of the population leaves town, then 91.5% of the population stays. Thus, the new population P is calculated from the original as follows:

New population	is	91.5%	of	original population
P	$=$	91.5%	$\cdot$	10200

Solve for P.

$$\begin{aligned} P &= 0.915 \cdot 10200 && 91.5\% = 0.915. \\ P &= 9333 && \text{Multiply: } 0.915 \cdot 10200 = 9333. \end{aligned}$$

Same answer.

Exercise

Several retail stores close and the population of Athens decreases by 7.2% as a result. If the original population was 12,500, what is the new population?

Answer

11,600

Example 4

Millertown falls on hard times and its population decreases from 11,256 to 10,923 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

Solution

To find the decrease in population, first subtract the current population from the original population.

$$\begin{aligned} \text{Population decrease} &= \text{original population} - \text{current population} \\ &= 11256 - 10923 \\ &= 333 \end{aligned}$$

Hence, the population has decreased by 333 people.

Next, let p represent the percent population decrease. Then we can translate the problem into words and symbols.

Population decrease	is	what percent	of	original population
333	$=$	p	$\cdot$	11256

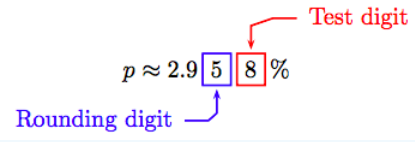
Solve for p.

$$\begin{aligned} \frac{333}{11256} &= \frac{11256p}{11256} && \text{Divide both sides by } 11256. \\ 0.02958 &\approx p && \text{Divide: } 333/11256 \approx 0.02958. \end{aligned}$$

To change p to a percent, move the decimal point two places to the right and append a percent symbol.

$$p \approx 0.02958 \approx 0\,02.958\% \approx 2.958\%.$$

We are asked to round to the nearest hundredth of a percent.



Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. That is,

$$p \approx 2.96\%.$$

Thus, the population of Millertown decreases approximately 2.96%.

Alternative Solution

An alternative approach is to ask what percent of the original population equals the new population.

New population	is	what percent	of	original population
10923	=	p	·	11256

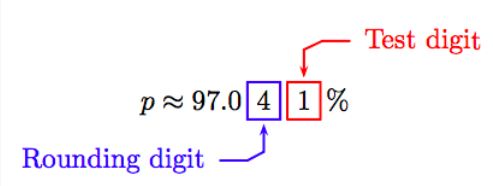
Solve for p.

$$\begin{aligned} 10923 &= 11256p && \text{Change the order of multiplication.} \\ \frac{10923}{11256} &= \frac{11256p}{11256} && \text{Divide both sides by } 11256. \\ 0.97041 &\approx p && \text{Divide: } 10923/11256 \approx 0.97041. \end{aligned}$$

To change p to a percent, move the decimal two places to the right and append a percent symbol.

$$p \approx 0.97041 \approx 0\,97.041\% \approx 97.041\%.$$

We are asked to round to the nearest hundredth of a percent.



Because the test digit is less than 5, leave the rounding digit alone and truncate. That is,

$$p \approx 97.04\%.$$

Thus, 97.04% of the Millertown population remains. To find the percent decrease (the percent who left), subtract 97.04% from 100%.

$$100\% - 97.04\% = 2.96\%$$

Hence, the population of Millertown decreases by 2.96%. Same answer.

? Exercise

A textile mill closure results in the population of the adjacent town decreasing from 8,956 to 7,800. What is the percent decrease in the population, rounded to the nearest tenth of a percent?

Answer

12.9%

Discount

Another important application of percent is the discounting of goods.

✔ Example 5

A pair of skis is marked at $310. However, a sign in the shop indicates that skis are being discounted at 15%. What will be the new selling price of the skis?

Solution

Let D represent the discount (in dollars) given for our pair of skis. Then, in words and symbols:

Discount	is	15%	of	original marked price
D	$=$	15%	$\cdot$	310

Solve for D.

$$D = 0.15 \cdot 310 \quad 15\% = 0.15.$$
$$D = 46.50 \quad \text{Multiply: } 0.15 \cdot 310 = 46.50.$$

Hence, the discount is \$46.50. To find the new selling price, subtract this discount from the original selling price.

$$\begin{aligned} \text{New selling price} &= \text{original selling price} - \text{discount} \\ &= 310 - 46.50 \\ &= 263.50 \end{aligned}$$

Hence, the new selling price is \$263.50.

Alternate Solution

Subtract 15% from 100% to obtain

$$100\% - 15\% = 85\%.$$

That is, if an item is discounted 15%, then its new selling price S is 85% of its original marked price.

New selling price	is	85%	of	original marked price
S	$=$	85%	$\cdot$	310

Solve for S.

$$S = 0.85 \cdot 310 \quad 85\% = 0.85.$$
$$S = 263.50 \quad \text{Multiply: } 0.85 \cdot 310 = 263.50.$$

Thus, the new selling price is \$263.50. Same answer.

? Exercise

A pair of hiking boots is marked at \$200. During a sale, the boots are discounted by 8%. What is the new price of the boots?

Answer

\$184

✔ Example 6

A pair of ski boots marked at \$210 is sold for \$180. Find the percent discount, correct to the nearest tenth of a percent.

Solution

We can find the discount (in dollars) by subtracting the sale price from the original marked price.

$$\begin{aligned} \text{Discount} &= \text{original marked price} - \text{sale price} \\ &= 210 - 180 \\ &= 30 \end{aligned}$$

Hence, the boots are discounted \$30.

Let p represent the percent discount. Then, in words and symbols:

Discount	is	percent discount	of	original marked price
30	$=$	p	$\cdot$	210

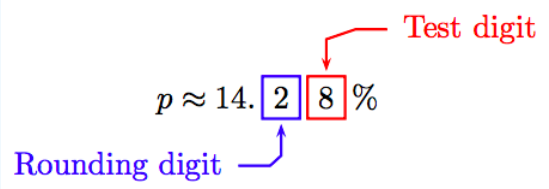
Solve for p.

$$\begin{aligned} 30 &= 210p && \text{Change order of multiplication.} \\ \frac{30}{210} &= \frac{210p}{210} && \text{Divide both sides by 210.} \\ \frac{1}{7} &= p && \text{Reduce: Divide numerator and denominator of } 30/210 \text{ by 30.} \\ p &\approx && \text{Divide: } 1/7 \approx 0.1428. \end{aligned}$$

To change p to a percent, move the decimal point two places to the right and append a percent symbol.

$$p \approx 0.1428 \approx 0\,14.28\% \approx 14.28\%.$$

Round to the nearest tenth of a percent.



Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. Thus, correct to the nearest tenth of a percent, the percent discount is

$$p \approx 14.3\%.$$

Alternate Solution

An alternative approach is to ask what percent p of the original marked price equals the selling price.

New selling price	is	what percent	of	original marked price
180	=	p	·	210

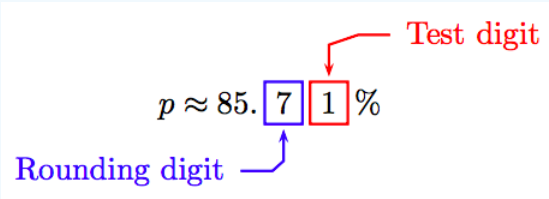
Solve for p.

$$\begin{aligned} 180 &= 210p && \text{Change the order of multiplication.} \\ \frac{180}{210} &= \frac{210p}{210} && \text{Divide both sides by 210.} \\ \frac{6}{7} &= p && \text{Reduce: Divide numerator and denominator of } 180/210 \text{ by 30.} \\ p &\approx && \text{Divide: } 6/7 \approx 0.8571. \end{aligned}$$

To change p to a percent, move the decimal point two places to the right and append a percent symbol.

$$p \approx 0.8571 \approx 0\,85.71\% \approx 85.71\%.$$

Round to the nearest tenth of a percent.



Because the test digit is less than 5, do nothing to the rounding digit and truncate. Thus, correct to the nearest tenth of a percent,

$$p \approx 85.7\%.$$

Thus, the new selling price is 85.7% of the original marked price. Subtract 85.7% from 100%.

$$100\% - 85.7\% = 14.3\%.$$

That is, if the new selling price is 85.7% of the original price, then the percent discount is 14.3%. This is the same answer found with the first method.

? Exercise

A computer marked at $2,000 is sold at a discount for $1,850. Find the percent discount, correct to the nearest tenth of a percent.

Answer

7.5%

Exercises

1. A television set is marked at $447. However, a sign in the shop indicates that the television set is being discounted at 20.5%. What will be the new selling price of the television set? Round your answer to the nearest penny.

2. A stereo set is marked at $380. However, a sign in the shop indicates that the stereo set is being discounted at 7.5%. What will be the new selling price of the stereo set? Round your answer to the nearest penny.

3. Due to a ball bearing plant closure, Anselm falls on hard times and its population decreases from 10,794 to 8,925 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

4. Due to a logging mill closure, Carlytown falls on hard times and its population decreases from 12,113 to 10,833 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

5. A car rack is marked at $500. However, a sign in the shop indicates that the car rack is being discounted at 3.5%. What will be the new selling price of the car rack? Round your answer to the nearest penny.

6. A car rack is marked at $405. However, a sign in the shop indicates that the car rack is being discounted at 17.5%. What will be the new selling price of the car rack? Round your answer to the nearest penny.

7. Due to a textile mill closure, the population of Silvertown decreases by 4.1%. If the original population was 14,678 hardy souls, what is the new population, correct to the nearest person?

8. Due to a department store closure, the population of Petroria decreases by 5.3%. If the original population was 14,034 hardy souls, what is the new population, correct to the nearest person?

9. A bartender is granted a 4.6% salary increase. If the bartender's current salary is $2,500 per month, find the bartender's new monthly salary, rounded to the nearest dollar.

10. A bartender is granted a 5.5% salary increase. If the bartender's current salary is $2,900 per month, find the bartender's new monthly salary, rounded to the nearest dollar.

11. A car rack marked at $358 is sold for $292. Find the percent discount, correct to the nearest tenth of a percent.

12. A bicycle marked at $328 is sold for $264. Find the percent discount, correct to the nearest tenth of a percent.

13. Due to a auto manufacturing plant closure, Carlytown falls on hard times and its population decreases from 14,393 to 12,623 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

14. Due to a ball bearing plant closure, Mayville falls on hard times and its population decreases from 8,494 to 6,609 in the space of one year. What is the percent decrease, rounded to the nearest hundredth of a percent?

15. Due to a auto manufacturing plant closure, the population of Silvertown decreases by 2.4%. If the original population was 8,780 hardy souls, what is the new population, correct to the nearest person?

16. Due to a textile mill closure, the population of Ghosttown decreases by 6.1%. If the original population was 14,320 hardy souls, what is the new population, correct to the nearest person?

17. A clerk making a salary of $2,600 per month has her salary increased to $2,950 per month. Find the percent increase correct to the nearest tenth of a percent.

18. A clerk making a salary of $3,600 per month has her salary increased to $4,100 per month. Find the percent increase correct to the nearest tenth of a percent.

19. A bartender making a salary of $4,200 per month has her salary increased to $4,300 per month. Find the percent increase correct to the nearest tenth of a percent.

20. A bartender making a salary of $3,200 per month has her salary increased to $3,550 per month. Find the percent increase correct to the nearest tenth of a percent.

21. A gardener is granted a 5.1% salary increase. If the gardener's current salary is $3,200 per month, find the gardener's new monthly salary, rounded to the nearest dollar.

22. A secretary is granted a 2.8% salary increase. If the secretary's current salary is $3,600 per month, find the secretary's new monthly salary, rounded to the nearest dollar.

23. A television set marked at $437 is sold for $347. Find the percent discount, correct to the nearest tenth of a percent.

24. A camera marked at $390 is sold for $328. Find the percent discount, correct to the nearest tenth of a percent.

---

25. Suppose that the price of an 8-ounce can of tomato sauce increased from $0.20 to $0.28.

    a) What was the amount of increase?

    b) What was the percent increase?

26. The following table summarizes summertime gasoline prices in San Francisco, CA. The price is the number of dollars required to purchase one gallon of unleaded gasoline. *Data from gasbuddy.com.*

Year	Price per gallon
2003	1.80
2004	2.28
2005	2.57
2006	3.20
2007	3.28
2008	4.61
2009	3.01

What is the percent increase or decrease from 2003 to 2005? Round your answer to the nearest whole percent.

27. Refer to the table of gas prices in Exercise 26. What is the percent increase or decrease from 2005 to 2007? Round your answer to the nearest whole percent.

28. Refer to the table of gas prices in Exercise 26. What is the percent increase or decrease from 2007 to 2009? Round your answer to the nearest whole percent.

29. **Rate hike.** A family paying $858 monthly for health coverage is faced with a 39% hike in rates. What will be their new monthly cost after the increase? *Associated Press-Times-Standard 02/09/10 HHS secretary asks Anthem Blue Cross to justify rate hike.*

30. **Parking fine.** San Francisco's Metropolitan Transportation Agency was expected to consider raising fines for the use of fake, stolen, or borrowed disabled parking tags from $100 to $825. What is the percent increase for this fine? *Associated Press-Times-Standard 01/06/10 Fines for fake disabled parking tags may go up in San Francisco.*

31. **Industrial move.** Regulations in California have caused factory farmers to move out of state. Idaho's industrial dairy went from 180, 000 cows in 1990 to 530, 000 cows in 2009. What is the percent increase for Idaho industrial dairy cows, rounded to the nearest percent? *Associated Press-Times-Standard 02/09/10 Idaho, others prepare for California egg farm exodus.*

32. **Whooping crane.** Whooping crane populations made a remarkable comeback from just 15 birds in 1941 to about 400 birds worldwide in 2010. What is the percent increase for the whooping crane population over the past seventy years, rounded to the nearest percent? *Associated Press-Times-Standard 03/17/10 Plucky whooping crane gives wildlife experts hope.*

33. **Underwater.** The loss of Arctic sea ice will allow for an underwater fiber optic cable that will cut the time it takes to send a message from London to Tokyo from a current 140 milliseconds down to 88 milliseconds. *Associated Press-Times-Standard 01/22/10 Global warming opens up Northwest Passage for underwater fiber optic cable.*

a) What is the estimated percent decrease in communication time?

b) At a cost of $1.2 billion, what is the cost per millisecond of saving (rounded to the nearest dollar)?

34. **Chinook salmon.** During the 2009 season in the Sacramento river basin, a record low 39,500 chinook salmon were recorded, way down from the more than 750,000 counted in 2002. What is the percent decrease in the salmon count from 2002 to 2009, rounded to the nearest percent? *Associated Press-Times-Standard 02/25/10 Feds predict better year for California salmon.*

35. **Daylight hours.** In middle latitudes, summer days can have as many as 14 hours of daylight, while winter days can have a few as 10 hours of daylight. What percent more daylight is there in summer than in winter?

36. **Cyber-experts.** Defense Secretary Robert Gates said the Pentagon will increase the number of cyber-experts it can train each year from 80 to 250 by 2011. What percent increase is this? Round your answer to the nearest percent. *Associated Press Times-Standard 4/19/09*

37. **Home prices.** Real estate data for Humboldt County, California, is given below. *Associated Press-Times-Standard 01/17/10 How is our local real estate market compared to other regions?*

Year	Number of homes sold	Average home price
2000	1,358	$152,257
2005	1,432	$344,500
2009	833	$285,000

a) What percent change in average home price occurred between 2000 and 2009?

b) What percent change in homes sold occurred between 2000 and 2009?

Answers

1. $355.36

3. 17.32%

5. $482.50

7. 14,076 people

9. $2,615

11. 18.4%

13. 12.30%

15. 8,569

17. 13.5%

19. 2.4%

21. $3,363

23. 20.6%

25.

a) $0.08

b) 40%

27. 28% increase

29. $1,193

31. 194% increase

33.

a) 37%

b) About $23,076,923 per millisecond

35. 40% more daylight

37.

a) 87% increase in average home price

b) 39% decrease in home sales

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7.6: Pie Charts

In this section we turn our attention to pie charts, but before we do, we need to establish some fundamentals regarding measurement of angles.

If you take a circle and divide it into 360 equal increments, then each increment is called one degree (1°). See Figure 7.2.

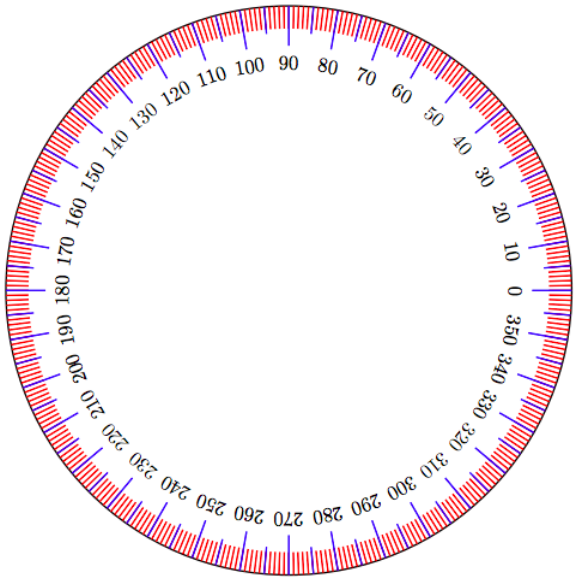


Figure 7.2: There are 360 degrees (360°) in a circle.

A *ray* is a line that starts at a point and then extends indefinitely in one direction. The starting point of the ray is called its *vertex*.

Figure 7.3: A ray with vertex V *extends indefinitely in one direction.*

If two rays have a common vertex, they form what is called an *angle*. In Figure 7.4 we've labeled the first ray as the "Initial Side" of the angle, and the second as the "Terminal Side" of the angle.

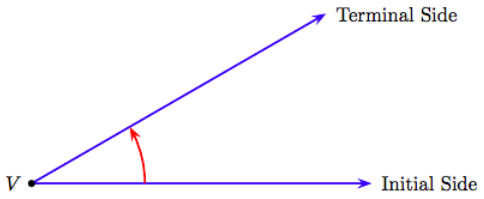


Figure 7.4: Two rays with a common vertex V *form an angle.*

We can find the degree measure of the angle by using a device called a *protractor*. Align the notch in the center of the base of the protractor with the vertex of the angle, then align the base of the protractor with the initial side of the angle. The terminal side of the angle will intersect the protractor edge where we can read the degree measure of the angle (see Figure 7.5). In Figure 7.5, note that the terminal side of the angle passes through the tick mark at the number 30, indicating that the degree measure of this angle is 30°.

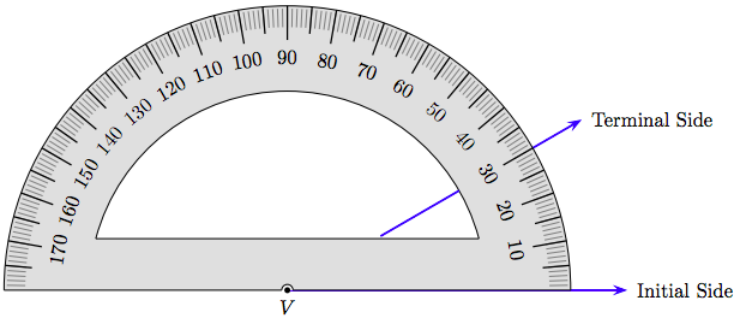


Figure 7.5: The degree measure is 30°.

Pie Charts

Now that we can measure angles, we can turn our attention to constructing *pie charts*.

Pie Chart

A pie chart is a circular chart that is divided into sectors, each sector representing a particular quantity. The area of each sector is a percentage of the area of the whole circle and is determined by calculating the quantity it represents as a percentage of the whole.

Example 1

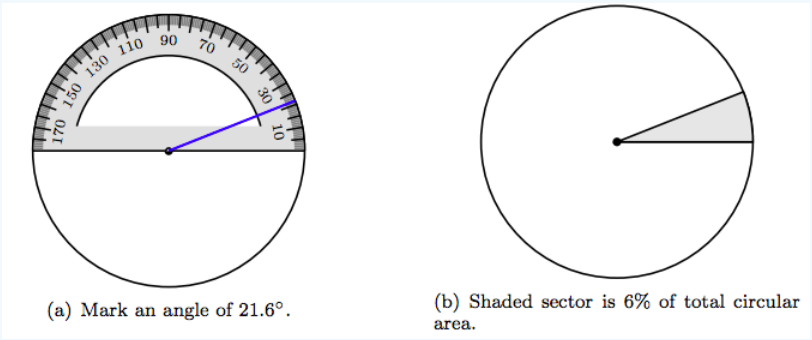
In a recent Gallup poll, 66% of those polled approved of the President's job performance, 28% disapproved, and 6% were undecided. Create a pie chart that displays these percentages.

Solution

Let's begin with 6% undecided. We want to create a sector that is 6% of the area of the whole circle. There are 360 degrees in a full circle, so 6% of this number is

$$\begin{aligned} 6\% \cdot 360^\circ &= 0.06 \cdot 360^\circ \\ &= 21.6^\circ. \end{aligned}$$

Start with a circle, set the baseline notch of the protractor on the center of the circle, then mark an angle of 21.6°, as seen in Figure 7.6(a). Shade the resulting region as shown in Figure 7.6(b), called a *sector*, which represents 6% of the total area of the circle.



(a) Mark an angle of 21.6°.

(b) Shaded sector is 6% of total circular area.

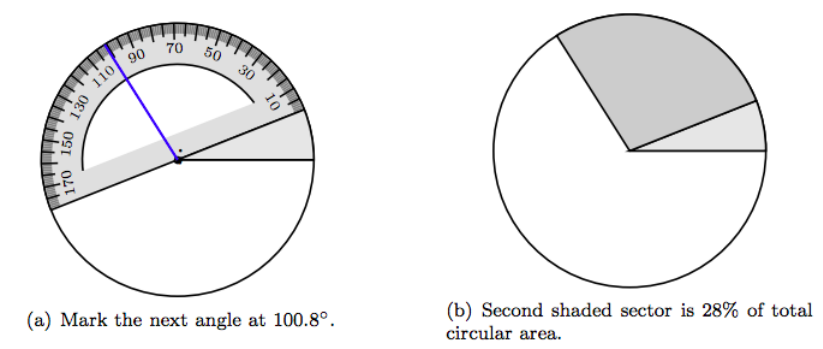
Figure 7.6: Sector with central angle 21.6° represents the 6% of the polling sample that were undecided about the president's performance.

Next, 28% disapproved of the President's job performance. Thus,

$$\begin{aligned} 28\% \cdot 360^\circ &= 0.28 \cdot 360^\circ \\ &= 100.8^\circ. \end{aligned}$$

Therefore, a sector with a central angle of 100.8° will represent 28% job disapproval.

Place the notch on the baseline of your protractor on the center of the circle, then align the baseline of the protractor with the terminal side of the first angle, as shown in Figure 7.7(a). Mark a central angle of 100.8°, as shown in Figure 7.7(a). Shade the resulting second sector with a darker shade of gray, as shown in Figure 7.7(b). This sector contains 28% of the total area of the circle and represents the portion of the polling sample that disapproved of the president's job performance.



(a) Mark the next angle at 100.8°.

(b) Second shaded sector is 28% of total circular area.

Figure 7.7: Sector with central angle 100.8° represents the 28% of the polling sample that disapproved of the president's performance.

Finally, as we've shaded the sectors representing 6% and 28% of the polling data in Figure 7.7(b), the remaining sector in Figure 7.7(b), shaded in white, represents the 66% of the polling sample who approved of the president's job performance (and 66% of the area of the whole circle).

Once you have computed and plotted the correct central angles for each of the sectors, you will want to label your pie chart. One possible annotation method is shown in Figure 7.8.

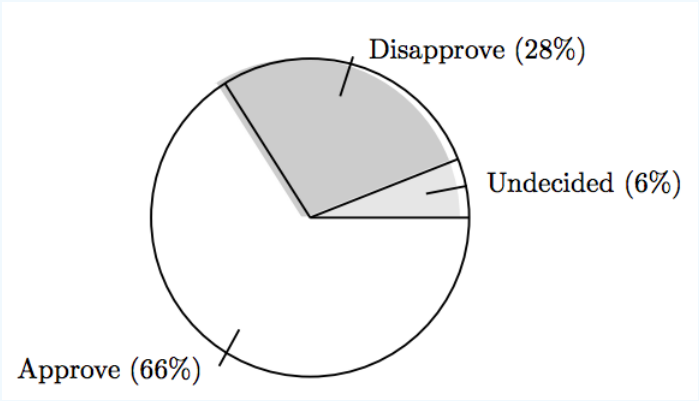
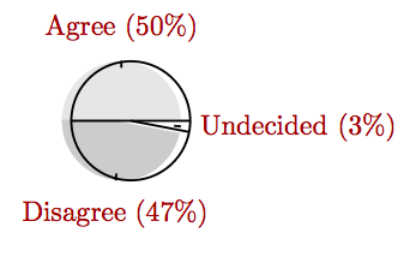


Figure 7.8: Annotating the final pie chart.

? Exercise

In a recent Gallup poll, 50% of the American public say it is the federal government's responsibility to make sure all Americans have health care, 47% disagree, and 3% were undecided. Create a pie chart that displays these percentages.

Answer



Example 2

One thousand people were polled with the question "Where does your dog sleep during the night?" The responses are shown in the following table.

Location	Number
Outside	30
Another Room	220
On the Bedroom Floor	330
On the Bed	420
Totals	1000

Create a pie chart showing the distribution of these responses.

Solution

The first step is to express the number in each location as a percentage of the totals. For example,

Outside	is	what percent	of	total
30	=	p	·	1000

Solving for p,

$$\frac{1000p}{1000} = \frac{30}{1000} \quad \text{Divide both sides by 1000.}$$
$$p = 0.03 \quad \text{Divide: } 30/1000 = 0.03.$$

Thus, p = 3%. In similar fashion, divide the number in each location by 1000 to find the following percents.

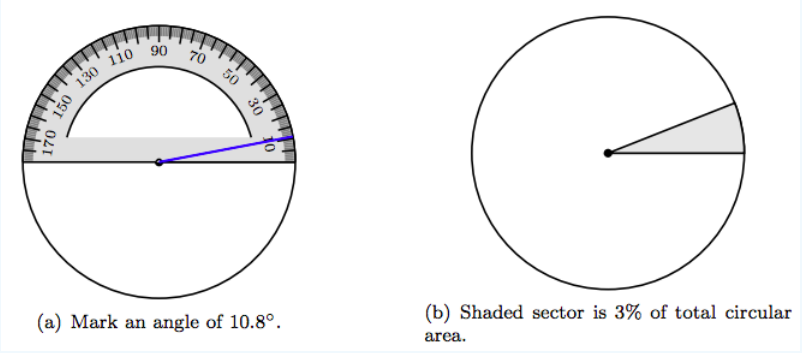
Location	Number	Percent
Outside	30	3%
Another Room	220	22%
On the Bedroom Floor	330	33%
On the Bed	420	42%
Totals	1000	100%

Note that the individual percents must total 100%.

Let's begin with the fact that 3% of the dog owners have their dogs sleep outside. To find the portion of the full circle that represents 3%, we take 3% of 360°.

$$3\% \cdot 360^\circ = 0.03 \cdot 360^\circ$$
$$= 10.8^\circ$$

Start with a circle, set the baseline notch of the protractor on the center of the circle, then mark an angle of 10.8°, as seen in Figure 7.9(a).



(a) Mark an angle of 10.8°.

(b) Shaded sector is 3% of total circular area.

Figure 7.9: Sector with central angle 10.8° represents the 3% of the polling sample that have their dogs sleep outside.

Shade the resulting sector as shown in Figure 7.9(b), which represents 3% of the total area of the circle.

Next, 22% have their dog sleep in another room.

$$\begin{aligned} 22\% \cdot 360^\circ &= 0.22 \cdot 360^\circ \\ &= 79.2^\circ. \end{aligned}$$

Therefore, a sector with a central angle of 79.2° will represent the fact that 22% of the dog owners have their dog sleep in another room.

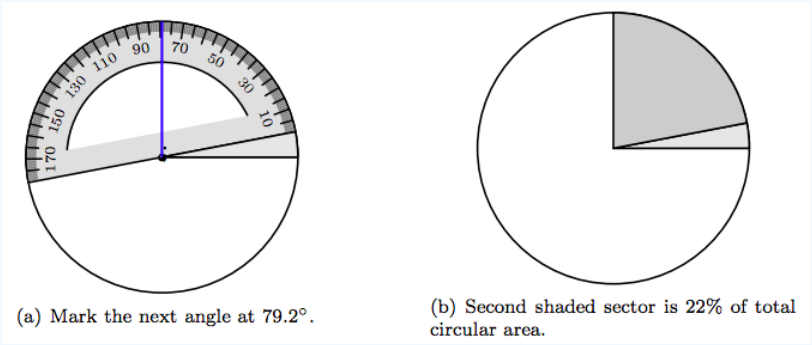
Place the notch on the baseline of your protractor on the center of the circle, then align the baseline of the protractor with the terminal side of the first angle, as shown in Figure 7.10(a). Mark a central angle of $79.2\circ$, as shown in Figure 7.7(a). Shade the resulting second sector with a darker shade of gray, as shown in Figure 7.10(b). This sector contains 22% of the total area of the circle and represents the portion of the polling sample whose dog sleeps in another room.

Next, 33% allow their dog sleep on the bedroom floor.

$$\begin{aligned} 33\% \cdot 360^\circ &= 0.33 \cdot 360^\circ \\ &= 118.8^\circ. \end{aligned}$$

Therefore, a sector with a central angle of 118.8° will represent the fact that 33% of the dog owners allow their dog to sleep on the bedroom floor.

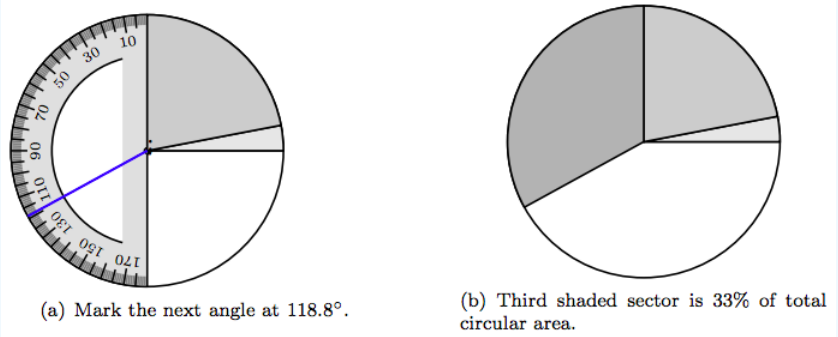
Place the notch on the baseline of your protractor on the center of the circle, then align the baseline of the protractor with the terminal side of the second sector, as shown in Figure 7.11(a). Mark a central angle of 118.8°, as shown in Figure 7.11(a). Shade the resulting second sector with a darker shade of gray, as shown in Figure 7.11(b). This sector contains 33% of the total area of the circle and represents the portion of the polling sample whose dog sleeps on the bedroom floor.



(a) Mark the next angle at 79.2°.

(b) Second shaded sector is 22% of total circular area.

Figure 7.10: Sector with central angle 79.2° represents the 22% of the polling sample whose dogs sleep in another room.



(a) Mark the next angle at 118.8°.

(b) Third shaded sector is 33% of total circular area.

Figure 7.11: Sector with central angle 118.8° represents the 33% of the polling sample whose dogs sleep on the bedroom floor.

Because the first three sectors, shaded in various levels of gray, represent 3%, 22%, and 33% of the total circular area, respectively, the remaining sector (shaded in white) automatically represents

$$100\% - (3\% + 22\% + 33\%) = 42\%$$

of the total circular area. This region represents the percent of dog owners who allow their dogs to sleep on the bed. The final result, with annotations, is shown in Figure 7.1

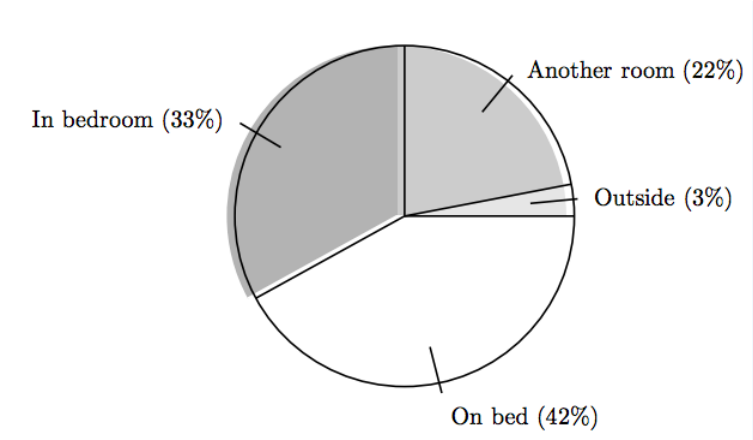
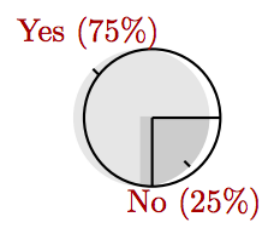


Figure 7.12: Annotating the final pie chart.

? Exercise

Two hundred people were asked whether they vote “Yes” or “No” on Proposition 8. There were 150 “Yes” votes and 50 “No” votes. Create a pie chart showing the distribution of these responses.

Answer



Exercises

1. In an election for class president, Raven received 21% of the votes, Anita received 27%, Jamal received 24% of the votes, and 28% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Raven’s share of the vote? Round your answer to the nearest degree.

2. In an election for class president, Fernando received 26% of the votes, Luisa received 26%, Ali received 26% of the votes, and 22% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Fernando’s share of the vote? Round your answer to the nearest degree.

3. In an election for class president, Akbar received 23% of the votes, Ali received 27%, Juanita received 30% of the votes, and 20% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Akbar’s share of the vote? Round your answer to the nearest degree.

4. In an election for class president, Kamili received 21% of the votes, Bernardo received 22%, Fernando received 30% of the votes, and 27% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Kamili’s share of the vote? Round your answer to the nearest degree.

5. In an election for class president, Jamal received 30% of the votes, Luisa received 20%, Kamili received 28% of the votes, and 22% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree measure of the central angle of the sector representing Jamal’s share of the vote? Round your answer to the nearest degree.

6. In an election for class president, Juanita received 30% of the votes, Ali received 24%, Estevan received 24% of the votes, and 22% of the votes were received by other candidates. If a pie chart is created for this data, what will be the degree

measure of the central angle of the sector representing Juanita's share of the vote? Round your answer to the nearest degree.

---

7. In an election for class president, Chin received 5 votes, Mabel received 13 votes, and Juanita received the remaining 32 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Chin's share of the vote? Round your answer to the nearest degree.

8. In an election for class president, Anita received 11 votes, Jose received 9 votes, and Bernardo received the remaining 30 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Anita's share of the vote? Round your answer to the nearest degree.

9. In an election for class president, Kamili received 14 votes, Jamal received 9 votes, and Jose received the remaining 27 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Kamili's share of the vote? Round your answer to the nearest degree.

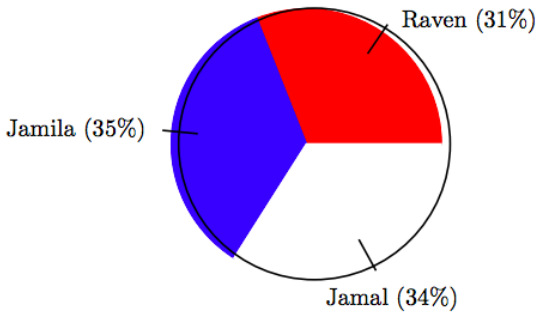
10. In an election for class president, Jun received 13 votes, Abdul received 15 votes, and Raven received the remaining 22 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Jun's share of the vote? Round your answer to the nearest degree.

11. In an election for class president, Hue received 13 votes, Ali received 6 votes, and Henry received the remaining 31 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Hue's share of the vote? Round your answer to the nearest degree.

12. In an election for class president, Mercy received 9 votes, Bernardo received 7 votes, and Hans received the remaining 34 votes cast. If a pie chart is created for this voting data, what will be the degree measure of the central angle of the sector representing Mercy's share of the vote? Round your answer to the nearest degree.

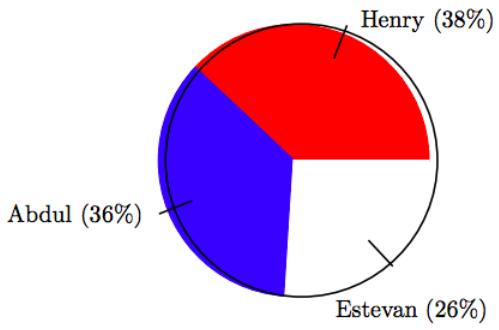
---

13. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



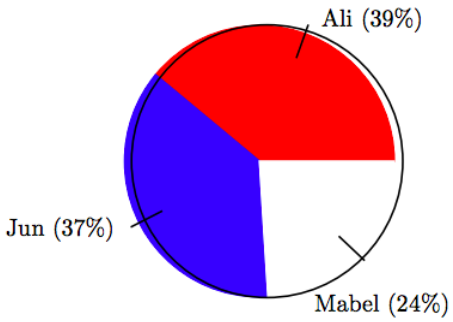
If there were a total of 95 votes cast in the election, find the number of votes that Raven received, correct to the nearest vote.

14. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



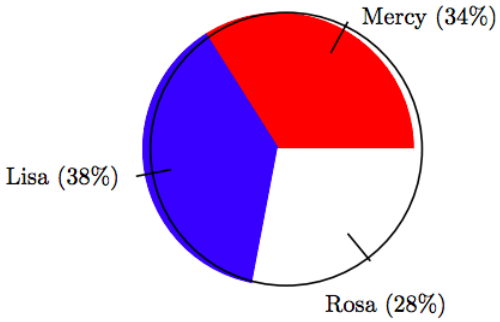
If there were a total of 79 votes cast in the election, find the number of votes that Henry received, correct to the nearest vote.

15. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



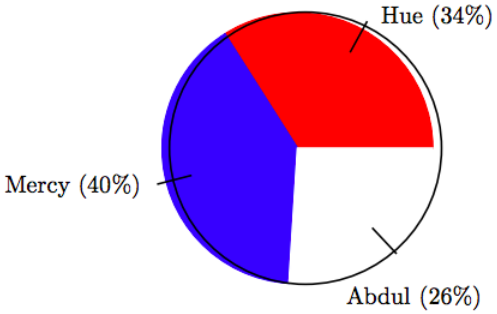
If there were a total of 58 votes cast in the election, find the number of votes that Ali received, correct to the nearest vote.

16. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



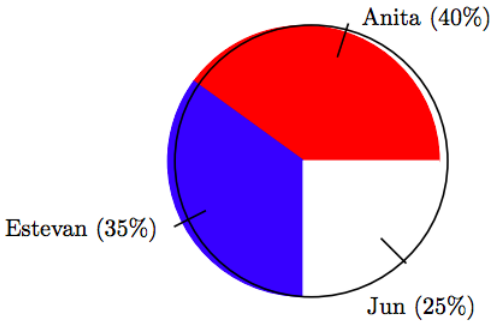
If there were a total of 65 votes cast in the election, find the number of votes that Mercy received, correct to the nearest vote.

17. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



If there were a total of 95 votes cast in the election, find the number of votes that Hue received, correct to the nearest vote.

18. In an election for class president, the vote distribution among three candidates is shown in the following pie chart.



If there were a total of 75 votes cast in the election, find the number of votes that Anita received, correct to the nearest vote.

---

19. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Ali	45
Jamal	34
Jun	52

Use a protractor to help create a pie chart showing the distribution of votes.

20. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Aisha	39
Akbar	31
Fernando	36

Use a protractor to help create a pie chart showing the distribution of votes.

21. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Bernardo	44
Rosa	40
Abdul	58

Use a protractor to help create a pie chart showing the distribution of votes.

22. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Estevan	46
Ali	58
Henry	49

Use a protractor to help create a pie chart showing the distribution of votes.

23. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Mercy	56
Hans	53
Lisa	41

Use a protractor to help create a pie chart showing the distribution of votes.

24. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Estevan	60
Hue	33
Aisha	31

Use a protractor to help create a pie chart showing the distribution of votes.

25. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Raven	43
Mabel	40
Bernardo	52

Use a protractor to help create a pie chart showing the distribution of votes.

26. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Hue	48
Lisa	48
Akbar	47

Use a protractor to help create a pie chart showing the distribution of votes.

27. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Jun	57
Lisa	30
Aisha	58

Use a protractor to help create a pie chart showing the distribution of votes.

28. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Bernardo	54
Mabel	38
Henry	49

Use a protractor to help create a pie chart showing the distribution of votes.

29. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Henry	35
Bernardo	32
Estevan	47

Use a protractor to help create a pie chart showing the distribution of votes.

30. In an election for class president, the vote distribution among three candidates is shown in the following table.

Candidate	Votes
Bernardo	38
Fernando	49
Aisha	44

Use a protractor to help create a pie chart showing the distribution of votes.

31. **Guard deployment.** The table shows the number of guard troop services since Sept. 11, 2001 (as of Dec. 2008; some troops have activated multiple times). *Associated Press-Times-Standard 02/18/10 Guard troops wait for promised pay.*

Mission	Troops
Operation Iraqi Freedom	193,598
Operation Enduring Freedom	29,212
Other missions	35,849

Use a protractor to help create a pie chart showing the distribution of National Guard troops.

Answers

1. 76°

3. 83°

5. 108°

7. 36°

9. 101°

11. 94°

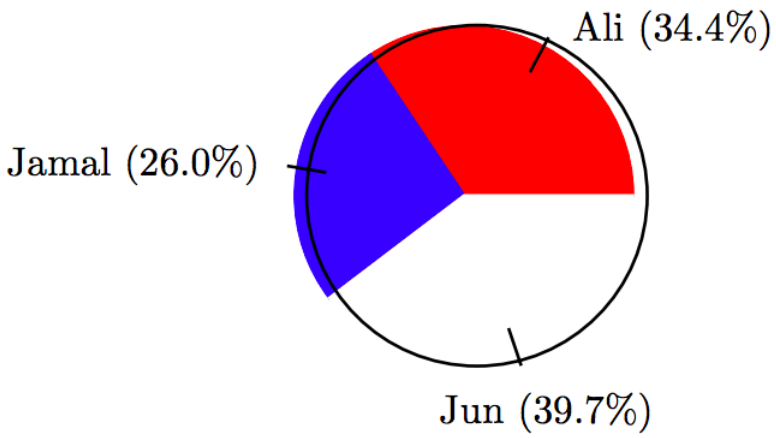
13. 29 votes

15. 23 votes

17. 32 votes

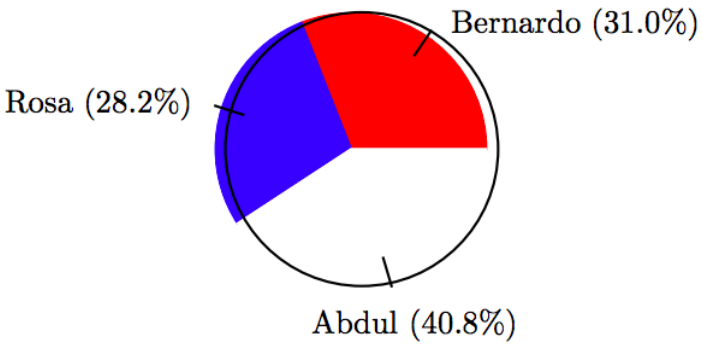
19.

Candidate	Votes	Percent	Degrees
Ali	45	34.4%	123.84°
Jamal	34	26.0%	93.6°
Jun	52	39.7%	142.92



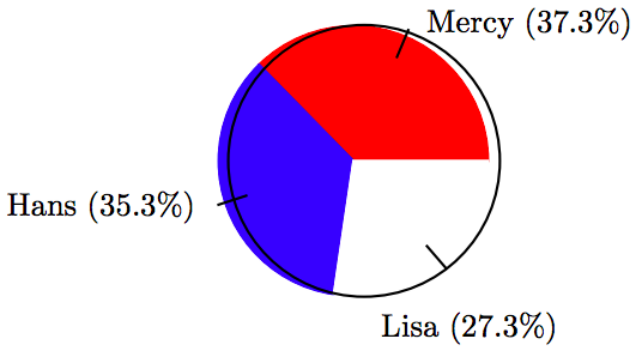
21.

Candidate	Votes	Percent	Degrees
Bernardo	44	31.0%	111.6°
Rosa	40	28.2%	101.52°
Abdul	58	40.8%	146.88°



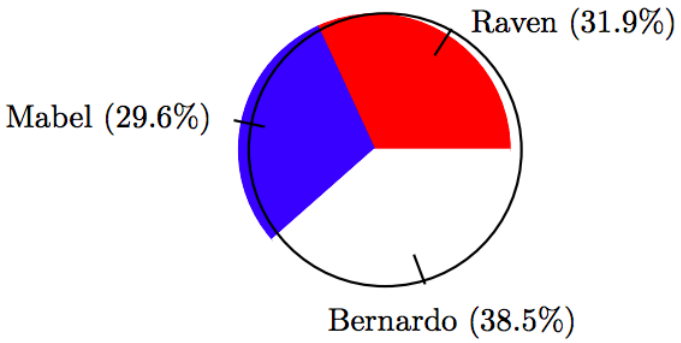
23.

Candidate	Votes	Percent	Degrees
Mercy	56	37.3%	134.28°
Hans	53	35.3%	127.08°
Lisa	41	27.3%	98.28°



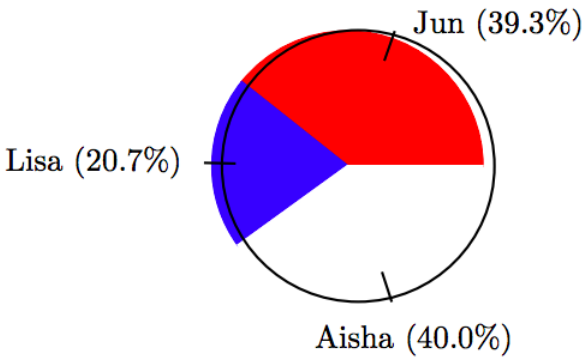
25.

Candidate	Votes	Percent	Degrees
Raven	43	31.9%	114.84°
Mabel	40	29.6%	106.56°
Bernardo	52	38.5%	138.6°



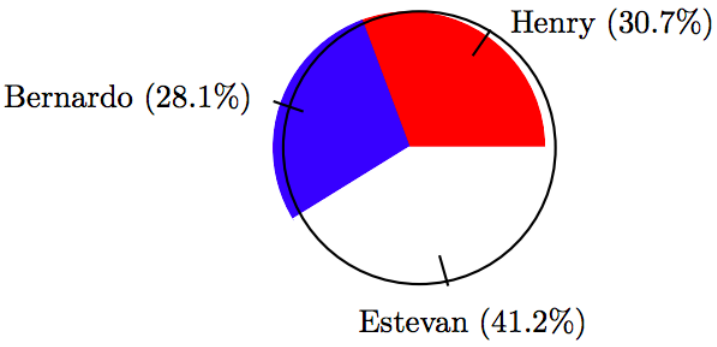
27.

Candidate	Votes	Percent	Degrees
Jun	57	39.3%	141.48°
Lisa	30	20.7%	74.52°
Aisha	58	40.0%	144°



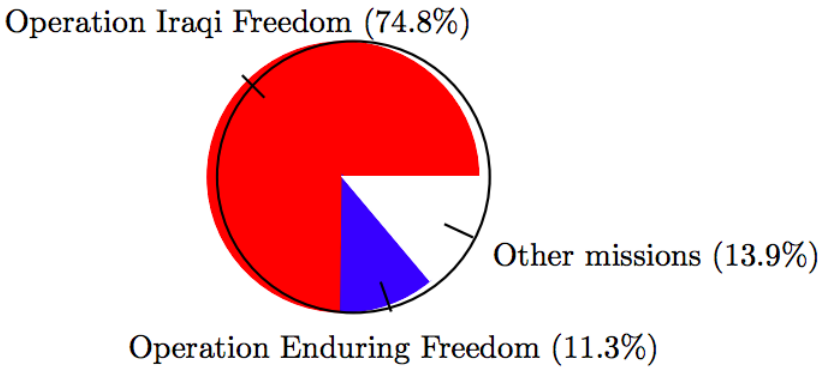
29.

Candidate	Votes	Percent	Degrees
Henry	35	30.7%	110.52°
Bernardo	32	28.1%	101.16°
Estevan	47	41.2%	148.32°



31.

Mission	Troops	Percent	Degrees
Operation Iraqi Freedom	193,598	74.8%	269.28°
Operation Enduring Freedom	29,212	11.3%	40.68°
Other missions	35,849	13.9%	50.04°



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7.7: Solve Proportions and their Applications (Part 1)

Learning Objectives

- Use the definition of proportion
- Solve proportions
- Solve applications using proportions
- Write percent equations as proportions
- Translate and solve percent proportions

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{\frac{1}{3}}{4}$. If you missed this problem, review Example 4.5.8.
2. Solve: $\frac{x}{4} = 20$. If you missed this problem, review Example 4.12.5.
3. Write as a rate: Sale rode his bike 24 miles in 2 hours. If you missed this problem, review Example 5.10.6.

Use the Definition of Proportion

In the section on Ratios and Rates we saw some ways they are used in our daily lives. When two ratios or rates are equal, the equation relating them is called a **proportion**.

Definition: proportion

A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where b ≠ 0, d ≠ 0.

The proportion states two ratios or rates are equal. The proportion is read “a is to b, as c is to d”.

The equation $\frac{1}{2} = \frac{4}{8}$ is a proportion because the two fractions are equal. The proportion $\frac{1}{2} = \frac{4}{8}$ is read “1 is to 2 as 4 is to 8”.

If we compare quantities with units, we have to be sure we are comparing them in the right order. For example, in the proportion $\frac{20\ students}{1\ teacher} = \frac{60\ students}{3\ teachers}$ we compare the number of students to the number of teachers. We put students in the numerators and teachers in the denominators.

Example 7.7.1:

Write each sentence as a proportion: (a) 3 is to 7 as 15 is to 35. (b) 5 hits in 8 at bats is the same as 30 hits in 48 at-bats. (c) $1.50 for 6 ounces is equivalent to $2.25 for 9 ounces.

Solution

(a) 3 is to 7 as 15 is to 35

Write as a proportion.	$\frac{3}{7} = \frac{15}{35}$	(7.7.1)

(b) 5 hits in 8 at bats is the same as 30 hits in 48 at-bats

Write each fraction to compare hits to at-bats.	$\frac{hits}{at-bats} = \frac{hits}{at-bats}$	(7.7.2)

Write as a proportion.	$\frac{5}{8}=\frac{30}{48}$	(7.7.3)

(c) \$1.50 for 6 ounces is equivalent to \$2.25 for 9 ounces

Write each fraction to compare dollars to ounces.	$\frac{\$}{ounces}=\frac{\$}{ounces}$	(7.7.4)
Write as a proportion.	$\frac{1.50}{6}=\frac{2.25}{9}$	(7.7.5)

? Exercise 7.7.1:

Write each sentence as a proportion: (a) 5 is to 9 as 20 is to 36. (b) 7 hits in 11 at-bats is the same as 28 hits in 44 at-bats. (c) \$2.50 for 8 ounces is equivalent to \$3.75 for 12 ounces.

Answer a

$\frac{5}{9}=\frac{20}{36}$

Answer b

$\frac{7}{11}=\frac{28}{44}$

Answer c

$\frac{2.50}{8}=\frac{3.75}{12}$

? Exercise 7.7.2:

Write each sentence as a proportion: (a) 6 is to 7 as 36 is to 42. (b) 8 adults for 36 children is the same as 12 adults for 54 children. (c) \$3.75 for 6 ounces is equivalent to \$2.50 for 4 ounces.

Answer a

$\frac{6}{7}=\frac{36}{42}$

Answer b

$\frac{8}{36}=\frac{12}{54}$

Answer c

$\frac{3.75}{6}=\frac{2.50}{4}$

Look at the proportions $\frac{1}{2}=\frac{4}{8}$ and $\frac{2}{3}=\frac{6}{9}$. From our work with equivalent fractions we know these equations are true. But how do we know if an equation is a proportion with equivalent fractions if it contains fractions with larger numbers? To determine if a proportion is true, we find the **cross products** of each proportion. To find the cross products, we multiply each denominator with the opposite numerator (diagonally across the equal sign). The results are called a cross products because of the cross formed. The cross products of a proportion are equal.

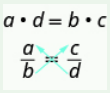
$8 \cdot 1 = 8 \quad 2 \cdot 4 = 8 \qquad 9 \cdot 2 = 18 \quad 3 \cdot 6 = 18$

$\frac{1}{2} \times \frac{4}{8} \qquad \frac{2}{3} \times \frac{6}{9}$

Definition: Cross Products of a Proportion

For any proportion of the form $\frac{a}{b} = \frac{c}{d}$, where b ≠ 0, d ≠ 0, its cross products are equal.

$$a \cdot d = b \cdot c$$



Cross products can be used to test whether a proportion is true. To test whether an equation makes a proportion, we find the cross products. If they are the equal, we have a proportion.

Example 7.7.2:

Determine whether each equation is a proportion: (a) $\frac{4}{9} = \frac{12}{28}$ (b) $\frac{17.5}{37.5} = \frac{7}{15}$

Solution

To determine if the equation is a proportion, we find the cross products. If they are equal, the equation is a proportion.

(a) $\frac{4}{9} = \frac{12}{28}$

Find the cross products.	$\frac{4}{9} \times \frac{12}{28}$ $28 \cdot 4 = 112 \qquad 9 \cdot 12 = 108 \qquad (7.7.6)$

Since the cross products are not equal, 28 · 4 ≠ 9 · 12, the equation is not a proportion.

(b) $\frac{17.5}{37.5} = \frac{7}{15}$

Find the cross products.	$\frac{17.5}{37.5} \times \frac{7}{15}$ $15 \cdot 17.5 = 262.5 \qquad 37.5 \cdot 7 = 262.5 \qquad (7.7.7)$

Since the cross products are equal, 15 • 17.5 = 37.5 • 7, the equation is a proportion.

Exercise 7.7.3:

Determine whether each equation is a proportion: (a) $\frac{7}{9} = \frac{54}{72}$ (b) $\frac{24.5}{45.5} = \frac{7}{13}$

Answer a

no

Answer b

yes

Exercise 7.7.4:

Determine whether each equation is a proportion: (a) $\frac{8}{9} = \frac{56}{73}$ (b) $\frac{28.5}{52.5} = \frac{8}{15}$

Answer a

no

Answer b

no

Solve Proportions

To solve a proportion containing a variable, we remember that the proportion is an equation. All of the techniques we have used so far to solve equations still apply. In the next example, we will solve a proportion by multiplying by the Least Common Denominator (LCD) using the Multiplication Property of Equality.

Example 7.7.3:

Solve: $\frac{x}{63}=\frac{4}{7}$.

Solution

To isolate x, multiply both sides by the LCD, 63.	$63\left(\frac{x}{63}\right)=63\left(\frac{4}{7}\right)$	(7.7.8)
Simplify.	$x=\frac{9\cdot \cancel{7}\cdot 4}{\cancel{7}}$	(7.7.9)
Divide the common factors.	$x=36$	(7.7.10)

Check: To check our answer, we substitute into the original proportion.

Substitute x = 36	$\frac{36}{63}\stackrel{?}{=}\frac{4}{7}$	(7.7.11)
Show common factors.	$\frac{4\cdot 9}{7\cdot 9}\stackrel{?}{=}\frac{4}{7}$	(7.7.12)
Simplify.	$\frac{4}{7}=\frac{4}{7}$ ✓	(7.7.13)

Exercise 7.7.5:

Solve the proportion: $\frac{n}{84}=\frac{11}{12}$.

Answer

77

Exercise 7.7.6:

Solve the proportion: $\frac{y}{96}=\frac{13}{12}$.

Answer

104

When the variable is in a denominator, we'll use the fact that the cross products of a proportion are equal to solve the proportions.

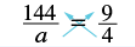
We can find the cross products of the proportion and then set them equal. Then we solve the resulting equation using our familiar techniques.

Example 7.7.4:

Solve: $\frac{144}{a} = \frac{9}{4}$.

Solution

Notice that the variable is in the denominator, so we will solve by finding the cross products and setting them equal.



Find the cross products and set them equal.	4 • 144 = a • 9	
Simplify.	576 = 9a	
Divide both sides by 9.	$\frac{576}{9} = \frac{9a}{9}$	(7.7.14)
Simplify.	$64 = a$	(7.7.15)

Check your answer.

Substitute a = 64	$\frac{144}{64} \stackrel{?}{=} \frac{9}{4}$	(7.7.16)
Show common factors.	$\frac{9 \cdot 16}{4 \cdot 16} \stackrel{?}{=} \frac{9}{4}$	(7.7.17)
Simplify.	$\frac{9}{4} = \frac{9}{4}$ ✓	(7.7.18)

Another method to solve this would be to multiply both sides by the LCD, 4a. Try it and verify that you get the same solution.

Exercise 7.7.7:

Solve the proportion: $\frac{91}{b} = \frac{7}{5}$.

Answer

65

Exercise 7.7.8:

Solve the proportion: $\frac{39}{c} = \frac{13}{8}$.

Answer

24

Example 7.7.5:

Solve: $\frac{52}{91} = \frac{-4}{y}$

Solution

Find the cross products and set them equal.	$\frac{52}{91} \times \frac{-4}{y}$	
	y • 52 = 91(-4)	
Simplify.	52y = -364	
Divide both sides by 52.	$\frac{52y}{52} = \frac{-364}{52}$	(7.7.19)
Simplify.	$y = -7$	(7.7.20)

Check:

Substitute y = -7	$\frac{52}{91} \stackrel{?}{=} \frac{-4}{-7}$	(7.7.21)
Show common factors.	$\frac{13 \cdot 4}{13 \cdot 7} \stackrel{?}{=} \frac{-4}{-7}$	(7.7.22)
Simplify.	$\frac{4}{7} = \frac{4}{7}$ ✓	(7.7.23)

Exercise 7.7.9:

Solve the proportion: $\frac{84}{98} = \frac{-6}{x}$.

Answer

-7

Exercise 7.7.10:

Solve the proportion: $\frac{-7}{y} = \frac{105}{135}$.

Answer

-9

Solve Applications Using Proportions

The strategy for solving applications that we have used earlier in this chapter, also works for proportions, since proportions are equations. When we set up the proportion, we must make sure the units are correct—the units in the numerators match and the units in the denominators match.

Example 7.7.6:

When pediatricians prescribe acetaminophen to children, they prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how many milliliters of acetaminophen will her doctor prescribe?

Solution

Identify what you are asked to find.	How many ml of acetaminophen the doctor will prescribe?
Choose a variable to represent it.	Let a = ml of acetaminophen.
Write a sentence that gives the information to find it.	If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?
Translate into a proportion.	$\frac{ml}{pounds} = \frac{ml}{pounds}$ (6.5.24)
Substitute given values—be careful of the units.	$\frac{5}{25} = \frac{a}{80}$ (6.5.25)
Multiply both sides by 80.	$80 \cdot \frac{5}{25} = 80 \cdot \frac{a}{80}$ (6.5.26)
Multiply and show common factors.	$\frac{16 \cdot 5 \cdot 5}{5 \cdot 5} = \frac{80a}{80}$ (6.5.27)
Simplify.	$16 = a$ (6.5.28)
Check if the answer is reasonable.	Yes. Since 80 is about 3 times 25, the medicine should be about 3 times 5.
Write a complete sentence.	The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

You could also solve this proportion by setting the cross products equal.

Exercise 7.7.11:

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

Answer

12 ml

Exercise 7.7.12:

For every 1 kilogram (kg) of a child's weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

Answer

180 mg

Example 7.7.7:

One brand of microwave popcorn has 120 calories per serving. A whole bag of this popcorn has 3.5 servings. How many calories are in a whole bag of this microwave popcorn?

Solution

Identify what you are asked to find.	How many calories are in a whole bag of microwave popcorn?
Choose a variable to represent it.	Let c = number of calories.
Write a sentence that gives the information to find it.	If there are 120 calories per serving, how many calories are in a whole bag with 3.5 servings?
Translate into a proportion.	$\frac{calories}{serving} = \frac{calories}{serving}$ (6.5.29)
Substitute given values.	$\frac{120}{1} = \frac{c}{3.5}$ (6.5.30)
Multiply both sides by 3.5.	$(3.5)\left(\frac{120}{1}\right) = (3.5)\left(\frac{c}{3.5}\right)$ (6.5.31)
Multiply.	$420 = c$ (6.5.32)
Check if the answer is reasonable.	Yes. Since 3.5 is between 3 and 4, the total calories should be between 360 (3 • 120) and 480 (4 • 120).
Write a complete sentence.	The whole bag of microwave popcorn has 420 calories.

Exercise 7.7.13:

Marissa loves the Caramel Macchiato at the coffee shop. The 16 oz. medium size has 240 calories. How many calories will she get if she drinks the large 20 oz. size?

Answer

300

Exercise 7.7.14:

Yaneli loves Starburst candies, but wants to keep her snacks to 100 calories. If the candies have 160 calories for 8 pieces, how many pieces can she have in her snack?

Answer

5

Example 7.7.8:

Josiah went to Mexico for spring break and changed $325 dollars into Mexican pesos. At that time, the exchange rate had $1 U.S. is equal to 12.54 Mexican pesos. How many Mexican pesos did he get for his trip?

Solution

Identify what you are asked to find.	How many Mexican pesos did Josiah get?

Choose a variable to represent it.	Let p = number of pesos.
Write a sentence that gives the information to find it.	If $1 U.S. is equal to 12.54 Mexican pesos, then $325 is how many pesos?
Translate into a proportion.	$\frac{\$}{pesos} = \frac{\$}{pesos}$ (6.5.33)
Substitute given values.	$\frac{1}{12.54} = \frac{325}{p}$ (6.5.34)
The variable is in the denominator, so find the cross products and set them equal.	$p \cdot 1 = 12.54(325)$ (6.5.35)
Simplify.	$c = 4,075.5$ (6.5.36)
Check if the answer is reasonable.	Yes, $100 would be $1,254 pesos. $325 is a little more than 3 times this amount.
Write a complete sentence.	Josiah has 4075.5 pesos for his spring break trip.

? Exercise 7.7.15:

Yurianna is going to Europe and wants to change $800 dollars into Euros. At the current exchange rate, $1 US is equal to 0.738 Euro. How many Euros will she have for her trip?

Answer

590 Euros

? Exercise 7.7.16:

Corey and Nicole are traveling to Japan and need to exchange $600 into Japanese yen. If each dollar is 94.1 yen, how many yen will they get?

Answer

56,460 yen

Contributors and Attributions

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7.8: Solve Proportions and their Applications (Part 2)

Write Percent Equations As Proportions

Previously, we solved percent equations by applying the properties of equality we have used to solve equations throughout this text. Some people prefer to solve percent equations by using the proportion method. The proportion method for solving percent problems involves a percent proportion. A **percent proportion** is an equation where a percent is equal to an equivalent ratio.

For example, 60% = $\frac{60}{100}$ and we can simplify $\frac{60}{100} = \frac{3}{5}$. Since the equation $\frac{60}{100} = \frac{3}{5}$ shows a percent equal to an equivalent ratio, we call it a **percent proportion**. Using the vocabulary we used earlier:

$$\frac{\text{amount}}{\text{base}} = \frac{\text{percent}}{100}$$
$$\frac{3}{5} = \frac{60}{100}$$

Definition: Percent Proportion

The amount is to the base as the percent is to 100.

$$\frac{amount}{base} = \frac{percent}{100} \tag{6.5.37}$$

If we restate the problem in the words of a proportion, it may be easier to set up the proportion:

The amount is to the base as the percent is to one hundred.

We could also say:

The amount out of the base is the same as the percent out of one hundred.

First we will practice translating into a percent proportion. Later, we'll solve the proportion.

Example 7.8.9:

Translate to a proportion. What number is 75% of 90?

Solution

If you look for the word "of", it may help you identify the base.

Identify the parts of the percent proportion.	$\underbrace{\text{What number}}_{\text{amount}}$ is $\underbrace{75\%}_{\text{percent}}$ of $\underbrace{90?}_{\text{base}}$
Restate as a proportion.	$What\ number\ out\ of\ 90\ is\ the\ same\ as\ 75\ out\ of\ 100?$ (7.8
Set up the proportion. Let n = number.	$\frac{n}{90} = \frac{75}{100}$ (6.5.38)

Exercise 7.8.17

Translate to a proportion: What number is 60% of 105?

Answer

$\frac{n}{105} = \frac{60}{100}$

Exercise 7.8.18

Translate to a proportion: What number is 40% of 85?

Answer

$\frac{n}{85} = \frac{40}{100}$

Example 7.8.10

Translate to a proportion. 19 is 25% of what number?

Solution

Identify the parts of the percent proportion.	$\underbrace{19}_{\text{amount}}$ is $\underbrace{25\%}_{\text{percent}}$ of $\underbrace{\text{what number?}}_{\text{base}}$
Restate as a proportion.	19 *out of* *what number* *is the same as* 25 *out of* 100? (7.8.
Set up the proportion. Let n = number.	$\frac{19}{n} = \frac{25}{100}$ (6.5.39)

Exercise 7.8.19:

Translate to a proportion: 36 is 25% of what number?

Answer

$\frac{36}{n} = \frac{25}{100}$

Exercise 7.8.20:

Translate to a proportion: 27 is 36% of what number?

Answer

$\frac{27}{n} = \frac{36}{100}$

Example 7.8.11:

Translate to a proportion. What percent of 27 is 9?

Solution

Identify the parts of the percent proportion.	$\underbrace{\text{What percent}}_{\text{percent}}$ of $\underbrace{27}_{\text{base}}$ is $\underbrace{9?}_{\text{amount}}$
Restate as a proportion.	9 *out of* 27 *is the same as* *what number* *out of* 100 (7.8.3)
Set up the proportion. Let p = percent.	$\frac{9}{27} = \frac{p}{100}$ (6.5.40)

Exercise 7.8.21:

Translate to a proportion: What percent of 52 is 39?

Answer

$\frac{n}{100} = \frac{39}{52}$

Exercise 7.8.22:

Translate to a proportion: What percent of 92 is 23?

Answer

$\frac{n}{100} = \frac{23}{92}$

Translate and Solve Percent Proportions

Now that we have written percent equations as proportions, we are ready to solve the equations.

Example 7.8.12:

Translate and solve using proportions: What number is 45% of 80?

Solution

Identify the parts of the percent proportion.	$\underbrace{\text{What number}}_{\text{amount}}$ is $\underbrace{45\%}_{\text{percent}}$ of $\underbrace{80?}_{\text{base}}$
Restate as a proportion.	$What\ number\ out\ of\ 80\ is\ the\ same\ as\ 45\ out\ of\ 100?$ (7.8
Set up the proportion. Let n = number.	$\frac{n}{80} = \frac{45}{100}$ (6.5.41)
Find the cross products and set them equal.	$100 \cdot n = 80 \cdot 45$ (6.5.42)
Simplify.	$100n = 3,600$ (6.5.43)
Divide both sides by 100.	$\frac{100n}{100} = \frac{3,600}{100}$ (6.5.44)
Simplify.	$n = 36$ (6.5.45)
Check if the answer is reasonable.	Yes. 45 is a little less than half of 100 and 36 is a little less than half 80.
Write a complete sentence that answers the question.	36 is 45% of 80.

Exercise 7.8.23:

Translate and solve using proportions: What number is 65% of 40?

Answer

26

Exercise 7.8.24:

Translate and solve using proportions: What number is 85% of 40?

Answer

34

In the next example, the percent is more than 100, which is more than one whole. So the unknown number will be more than the base.

Example 7.8.13:

Translate and solve using proportions: 125% of 25 is what number?

Solution

Identify the parts of the percent proportion.	$\underbrace{125\%}_{\text{percent}}$ is $\underbrace{25}_{\text{base}}$ of $\underbrace{\text{what number?}}_{\text{amount}}$

Restate as a proportion.	$What\ number\ out\ of\ 25\ is\ the\ same\ as\ 125\ out\ of\ 100?$	(7.
Set up the proportion. Let n = number.	$\frac{n}{25}=\frac{125}{100}$	(6.5.46)
Find the cross products and set them equal.	$100\cdot n=25\cdot 125$	(6.5.47)
Simplify.	$100n=3,125$	(6.5.48)
Divide both sides by 100.	$\frac{100n}{100}=\frac{3,125}{100}$	(6.5.49)
Simplify.	$n=31.25$	(6.5.50)
Check if the answer is reasonable.	Yes. 125 is more than 100 and 31.25 is more than 25.	
Write a complete sentence that answers the question.	125% of 25 is 31.25.	

Exercise 7.8.25:

Translate and solve using proportions: 125% of 64 is what number?

Answer

80

Exercise 7.8.26:

Translate and solve using proportions: 175% of 84 is what number?

Answer

147

Percents with decimals and money are also used in proportions.

Example 7.8.14:

Translate and solve: 6.5% of what number is $1.56?

Solution

Identify the parts of the percent proportion.	$\underbrace{6.5\%}_{\text{percent}}\ \text{of}\ \underbrace{\text{what number}}_{\text{base}}\ \text{is}\ \underbrace{\$1.56?}_{\text{amount}}$	
Restate as a proportion.	$\$1.56\ out\ of\ what\ number\ is\ the\ same\ as\ 6.5\ out\ of\ 100?$	(
Set up the proportion. Let n = number.	$\frac{1.56}{n}=\frac{6.5}{100}$	(6.5.51)
Find the cross products and set them equal.	$100(1.56)=n\cdot 6.5$	(6.5.52)
Simplify.	$156=6.5n$	(6.5.53)
Divide both sides by 6.5 to isolate the variable.	$\frac{156}{6.5}=\frac{6.5n}{6.5}$	(6.5.54)

Simplify.	$24 = n$	(6.5.55)
Check if the answer is reasonable.	Yes. 6.5% is a small amount and $1.56 is much less than $24.	
Write a complete sentence that answers the question.	6.5% of $24 is $1.56.	

Exercise 7.8.27:

Translate and solve using proportions: 8.5% of what number is $3.23?

Answer

38

Exercise 7.8.28:

Translate and solve using proportions: 7.25% of what number is $4.64?

Answer

64

Example 7.8.15:

Translate and solve using proportions: What percent of 72 is 9?

Solution

Identify the parts of the percent proportion.	$\underbrace{\text{What percent}}_{\text{percent}}\ \text{of}\ \underbrace{72}_{\text{base}}\ \text{is}\ \underbrace{9?}_{\text{amount}}$	
Restate as a proportion.	$9\ out\ of\ 72\ is\ the\ same\ as\ what\ number\ out\ of\ 100?$	(7.8.7
Set up the proportion. Let n = number.	$\frac{9}{72} = \frac{n}{100}$	(6.5.56)
Find the cross products and set them equal.	$72 \cdot n = 100 \cdot 9$	(6.5.57)
Simplify.	$72n = 900$	(6.5.58)
Divide both sides by 72.	$\frac{72n}{72} = \frac{900}{72}$	(6.5.59)
Simplify.	$n = 12.5$	(6.5.60)
Check if the answer is reasonable.	Yes. 9 is $\frac{1}{8}$ of 72 and $\frac{1}{8}$ is 12.5%.	
Write a complete sentence that answers the question.	12.5% of 72 is 9.	

Exercise 7.8.29:

Translate and solve using proportions: What percent of 72 is 27?

Answer

37.5%

Exercise 7.8.30:

Translate and solve using proportions: What percent of 92 is 23?

Answer

25%

Practice Makes Perfect

Use the Definition of Proportion

In the following exercises, write each sentence as a proportion.

244. 4 is to 15 as 36 is to 135.
245. 7 is to 9 as 35 is to 45.
246. 12 is to 5 as 96 is to 40.
247. 15 is to 8 as 75 is to 40.
248. 5 wins in 7 games is the same as 115 wins in 161 games.
249. 4 wins in 9 games is the same as 36 wins in 81 games.
250. 8 campers to 1 counselor is the same as 48 campers to 6 counselors.
251. 6 campers to 1 counselor is the same as 48 campers to 8 counselors.
252. \$9.36 for 18 ounces is the same as \$2.60 for 5 ounces.
253. \$3.92 for 8 ounces is the same as \$1.47 for 3 ounces.
254. \$18.04 for 11 pounds is the same as \$4.92 for 3 pounds.
255. \$12.42 for 27 pounds is the same as \$5.52 for 12 pounds.

In the following exercises, determine whether each equation is a proportion.

256. $\frac{7}{15}=\frac{56}{120}$
257. $\frac{5}{12}=\frac{45}{108}$
258. $\frac{11}{6}=\frac{21}{16}$
259. $\frac{9}{4}=\frac{39}{34}$
260. $\frac{12}{18}=\frac{4.99}{7.56}$
261. $\frac{9}{16}=\frac{2.16}{3.89}$
262. $\frac{13.5}{8.5}=\frac{31.05}{19.55}$
263. $\frac{10.1}{8.4}=\frac{3.03}{2.52}$

Solve Proportions

In the following exercises, solve each proportion.

264. $\frac{x}{56}=\frac{7}{8}$
265. $\frac{n}{91}=\frac{8}{13}$
266. $\frac{49}{63}=\frac{z}{9}$
267. $\frac{56}{72}=\frac{y}{9}$
268. $\frac{5}{a}=\frac{65}{117}$
269. $\frac{4}{b}=\frac{64}{144}$
270. $\frac{98}{154}=\frac{-7}{p}$
271. $\frac{72}{156}=\frac{-6}{q}$

272. $\frac{a}{-8}=\frac{-42}{48}$
273. $\frac{b}{-7}=\frac{-30}{42}$
274. $\frac{2.6}{3.9}=\frac{c}{3}$
275. $\frac{2.7}{3.6}=\frac{d}{4}$
276. $\frac{2.7}{j}=\frac{0.9}{0.2}$
277. $\frac{2.8}{k}=\frac{2.1}{1.5}$
278. $\frac{\frac{1}{2}}{1}=\frac{m}{8}$
279. $\frac{\frac{1}{3}}{3}=\frac{9}{n}$

Solve Applications Using Proportions

In the following exercises, solve the proportion problem.

280. Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?
281. Brianna, who weighs 6 kg, just received her shots and needs a pain killer. The pain killer is prescribed for children at 15 milligrams (mg) for every 1 kilogram (kg) of the child's weight. How many milligrams will the doctor prescribe?
282. At the gym, Carol takes her pulse for 10 sec and counts 19 beats. How many beats per minute is this? Has Carol met her target heart rate of 140 beats per minute?
283. Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds. How many beats per minute is this? Has Kevin met his target heart rate?
284. A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?
285. One 12 ounce can of soda has 150 calories. If Josiah drinks the big 32 ounce size from the local mini-mart, how many calories does he get?
286. Karen eats \(\dfrac{1}{2}) cup of oatmeal that counts for 2 points on her weight loss program. Her husband, Joe, can have 3 points of oatmeal for breakfast. How much oatmeal can he have?
287. An oatmeal cookie recipe calls for $\frac{1}{2}$ cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?
288. Janice is traveling to Canada and will change $250 US dollars into Canadian dollars. At the current exchange rate, $1 US is equal to $1.01 Canadian. How many Canadian dollars will she get for her trip?
289. Todd is traveling to Mexico and needs to exchange $450 into Mexican pesos. If each dollar is worth 12.29 pesos, how many pesos will he get for his trip?
290. Steve changed $600 into 480 Euros. How many Euros did he receive per US dollar?
291. Martha changed $350 US into 385 Australian dollars. How many Australian dollars did she receive per US dollar?
292. At the laundromat, Lucy changed $12.00 into quarters. How many quarters did she get?
293. When she arrived at a casino, Gerty changed $20 into nickels. How many nickels did she get?
294. Jesse's car gets 30 miles per gallon of gas. If Las Vegas is 285 miles away, how many gallons of gas are needed to get there and then home? If gas is $3.09 per gallon, what is the total cost of the gas for the trip?
295. Danny wants to drive to Phoenix to see his grandfather. Phoenix is 370 miles from Danny's home and his car gets 18.5 miles per gallon. How many gallons of gas will Danny need to get to and from Phoenix? If gas is $3.19 per gallon, what is the total cost for the gas to drive to see his grandfather?
296. Hugh leaves early one morning to drive from his home in Chicago to go to Mount Rushmore, 812 miles away. After 3 hours, he has gone 190 miles. At that rate, how long will the whole drive take?
297. Kelly leaves her home in Seattle to drive to Spokane, a distance of 280 miles. After 2 hours, she has gone 152 miles. At that rate, how long will the whole drive take?
298. Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil's lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?
299. April wants to paint the exterior of her house. One gallon of paint covers about 350 square feet, and the exterior of the house measures approximately 2000 square feet. How many gallons of paint will she have to buy?

Write Percent Equations as Proportions

In the following exercises, translate to a proportion.

300. What number is 35% of 250?
301. What number is 75% of 920?
302. What number is 110% of 47?
303. What number is 150% of 64?
304. 45 is 30% of what number?
305. 25 is 80% of what number?
306. 90 is 150% of what number?
307. 77 is 110% of what number?
308. What percent of 85 is 17?
309. What percent of 92 is 46?
310. What percent of 260 is 340?
311. What percent of 180 is 220?

Translate and Solve Percent Proportions

In the following exercises, translate and solve using proportions.

312. What number is 65% of 180?
313. What number is 55% of 300?
314. 18% of 92 is what number?
315. 22% of 74 is what number?
316. 175% of 26 is what number?
317. 250% of 61 is what number?
318. What is 300% of 488?
319. What is 500% of 315?
320. 17% of what number is $7.65?
321. 19% of what number is $6.46?
322. $13.53 is 8.25% of what number?
323. $18.12 is 7.55% of what number?
324. What percent of 56 is 14?
325. What percent of 80 is 28?
326. What percent of 96 is 12?
327. What percent of 120 is 27?

Everyday Math

328. **Mixing a concentrate** Sam bought a large bottle of concentrated cleaning solution at the warehouse store. He must mix the concentrate with water to make a solution for washing his windows. The directions tell him to mix 3 ounces of concentrate with 5 ounces of water. If he puts 12 ounces of concentrate in a bucket, how many ounces of water should he add? How many ounces of the solution will he have altogether?
329. **Mixing a concentrate** Travis is going to wash his car. The directions on the bottle of car wash concentrate say to mix 2 ounces of concentrate with 15 ounces of water. If Travis puts 6 ounces of concentrate in a bucket, how much water must he mix with the concentrate?

Writing Exercises

330. To solve “what number is 45% of 350” do you prefer to use an equation like you did in the section on Decimal Operations or a proportion like you did in this section? Explain your reason.
331. To solve “what percent of 125 is 25” do you prefer to use an equation like you did in the section on Decimal Operations or a proportion like you did in this section? Explain your reason.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use definition of proportion.			
solve proportions.			
solve applications using proportions.			
write percent equations as proportions.			
translate and solve percent proportions.			

(b) Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

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7.E: Percents (Exercises)

6.1 - Understand Percent

In the following exercises, write each percent as a ratio.

332. 32% admission rate for the university
333. 53.3% rate of college students with student loans

In the following exercises, write as a ratio and then as a percent.

334. 13 out of 100 architects are women.
335. 9 out of every 100 nurses are men.

In the following exercises, convert each percent to a fraction.

336. 48%
337. 175%
338. 64.1%
339. $8\frac{1}{4}\%$

In the following exercises, convert each percent to a decimal.

340. 6%
341. 23%
342. 128%
343. 4.9%

In the following exercises, convert each percent to (a) a simplified fraction and (b) a decimal.

344. In 2012, 13.5% of the United States population was age 65 or over. (Source: www.census.gov)
345. In 2012, 6.5% of the United States population was under 5 years old. (Source: www.census.gov)
346. When a die is tossed, the probability it will land with an even number of dots on the top side is 50%.
347. A couple plans to have three children. The probability they will all be girls is 12.5%.

In the following exercises, convert each decimal to a percent.

348. 0.04
349. 0.15
350. 2.82
351. 3
352. 0.003
353. 1.395

In the following exercises, convert each fraction to a percent.

354. 3 4
355. 11 5
356. 3 5 8
357. 2 9
358. According to the Centers for Disease Control, $\frac{2}{5}$ of adults do not take a vitamin or supplement.
359. According to the Centers for Disease Control, among adults who do take a vitamin or supplement, $\frac{3}{4}$ take a multivitamin.

In the following exercises, translate and solve.

360. What number is 46% of 350?
361. 120% of 55 is what number?
362. 84 is 35% of what number?
363. 15 is 8% of what number?
364. 200% of what number is 50?

365. 7.9% of what number is $4.74?
366. What percent of 120 is 81.6?
367. What percent of 340 is 595?

6.2 - Solve General Applications of Percents

In the following exercises, solve.

368. When Aurelio and his family ate dinner at a restaurant, the bill was $83.50. Aurelio wants to leave 20% of the total bill as a tip. How much should the tip be?
369. One granola bar has 2 grams of fiber, which is 8% of the recommended daily amount. What is the total recommended daily amount of fiber?
370. The nutrition label on a package of granola bars says that each granola bar has 190 calories, and 54 calories are from fat. What percent of the total calories is from fat?
371. Elsa gets paid $4,600 per month. Her car payment is $253. What percent of her monthly pay goes to her car payment?

In the following exercises, solve.

372. Jorge got a raise in his hourly pay, from $19.00 to $19.76. Find the percent increase.
373. Last year Bernard bought a new car for $30,000. This year the car is worth $24,000. Find the percent decrease.

6.3 - Solve Sales Tax, Commission, and Discount Applications

In the following exercises, find (a) the sales tax (b) the total cost.

374. The cost of a lawn mower was $750. The sales tax rate is 6% of the purchase price.
375. The cost of a water heater is $577. The sales tax rate is 8.75% of the purchase price.

In the following exercises, find the sales tax rate.

376. Andy bought a piano for $4,600. The sales tax on the purchase was $333.50.
377. Nahomi bought a purse for $200. The sales tax on the purchase was $16.75.

In the following exercises, find the commission.

378. Ginny is a realtor. She receives 3% commission when she sells a house. How much commission will she receive for selling a house for $380,000?
379. Jackson receives 16.5% commission when he sells a dinette set. How much commission will he receive for selling a dinette set for $895?

In the following exercises, find the rate of commission.

380. Ruben received $675 commission when he sold a $4,500 painting at the art gallery where he works. What was the rate of commission?
381. Tori received $80.75 for selling a $950 membership at her gym. What was her rate of commission?

In the following exercises, find the sale price.

382. Aya bought a pair of shoes that was on sale for $30 off. The original price of the shoes was $75.
383. Takwanna saw a cookware set she liked on sale for $145 off. The original price of the cookware was $312.

In the following exercises, find (a) the amount of discount and (b) the sale price.

384. Nga bought a microwave for her office. The microwave was discounted 30% from an original price of $84.90.
385. Jarrett bought a tie that was discounted 65% from an original price of $45.

In the following exercises, find (a) the amount of discount (b) the discount rate. (Round to the nearest tenth of a percent if needed.)

386. Hilda bought a bedspread on sale for $37. The original price of the bedspread was $50.
387. Tyler bought a phone on sale for $49.99. The original price of the phone was $79.99.

In the following exercises, find (a) the amount of the mark-up (b) the list price.

388. Manny paid $0.80 a pound for apples. He added 60% markup before selling them at his produce stand. What price did he charge for the apples?

389. It cost Noelle $17.40 for the materials she used to make a purse. She added a 325% markup before selling it at her friend's store. What price did she ask for the purse?

6.4 - Solve Simple Interest Applications

In the following exercises, solve the simple interest problem.

390. Find the simple interest earned after 4 years on $2,250 invested at an interest rate of 5%.
391. Find the simple interest earned after 7 years on $12,000 invested at an interest rate of 8.5%.
392. Find the principal invested if $660 interest was earned in 5 years at an interest rate of 3%.
393. Find the interest rate if $2,898 interest was earned from a principal of $23,000 invested for 3 years.
394. Kazuo deposited $10,000 in a bank account with interest rate 4.5%. How much interest was earned in 2 years?
395. Brent invested $23,000 in a friend's business. In 5 years the friend paid him the $23,000 plus $9,200 interest. What was the rate of interest?
396. Fresia lent her son $5,000 for college expenses. Three years later he repaid her the $5,000 plus $375 interest. What was the rate of interest?
397. In 6 years, a bond that paid 5.5% earned $594 interest. What was the principal of the bond?

6.5 - Solve Proportions and their Applications

In the following exercises, write each sentence as a proportion.

398. 3 is to 8 as 12 is to 32.
399. 95 miles to 3 gallons is the same as 475 miles to 15 gallons.
400. 1 teacher to 18 students is the same as 23 teachers to 414 students.
401. $7.35 for 15 ounces is the same as $2.94 for 6 ounces.

In the following exercises, determine whether each equation is a proportion.

402. $\frac{5}{13}=\frac{30}{78}$
403. $\frac{16}{7}=\frac{48}{23}$
404. $\frac{12}{18}=\frac{6.99}{10.99}$
405. $\frac{11.6}{9.2}=\frac{37.12}{29.44}$

In the following exercises, solve each proportion.

406. $\frac{x}{36}=\frac{5}{9}$
407. $\frac{7}{a}=\frac{-6}{84}$
408. $\frac{1.2}{1.8}=\frac{d}{6}$
409. $\frac{\frac{1}{2}}{2}=\frac{m}{20}$

In the following exercises, solve the proportion problem.

410. The children's dosage of acetaminophen is 5 milliliters (ml) for every 25 pounds of a child's weight. How many milliliters of acetaminophen will be prescribed for a 60 pound child?
411. After a workout, Dennis takes his pulse for 10 sec and counts 21 beats. How many beats per minute is this?
412. An 8 ounce serving of ice cream has 272 calories. If Lavonne eats 10 ounces of ice cream, how many calories does she get?
413. Alma is going to Europe and wants to exchange $1,200 into Euros. If each dollar is 0.75 Euros, how many Euros will Alma get?
414. Zack wants to drive from Omaha to Denver, a distance of 494 miles. If his car gets 38 miles to the gallon, how many gallons of gas will Zack need to get to Denver?
415. Teresa is planning a party for 100 people. Each gallon of punch will serve 18 people. How many gallons of punch will she need?

In the following exercises, translate to a proportion.

416. What number is 62% of 395?
417. 42 is 70% of what number?
418. What percent of 1,000 is 15?
419. What percent of 140 is 210?

In the following exercises, translate and solve using proportions.

420. What number is 85% of 900?
421. 6% of what number is $24?
422. $3.51 is 4.5% of what number?
423. What percent of 3,100 is 930?

PRACTICE TEST

In the following exercises, convert each percent to (a) a decimal (b) a simplified fraction.

424. 24%
425. 5%
426. 350%

In the following exercises, convert each fraction to a percent. (Round to 3 decimal places if needed.)

427. $\frac{7}{8}$
428. $\frac{1}{3}$
429. $\frac{11}{12}$

In the following exercises, solve the percent problem.

430. 65 is what percent of 260?
431. What number is 27% of 3,000?
432. 150% of what number is 60?
433. Yuki's monthly paycheck is $3,825. She pays $918 for rent. What percent of her paycheck goes to rent?
434. The total number of vehicles on one freeway dropped from 84,000 to 74,000. Find the percent decrease (round to the nearest tenth of a percent).
435. Kyle bought a bicycle in Denver where the sales tax was 7.72% of the purchase price. The purchase price of the bicycle was $600. What was the total cost?
436. Mara received $31.80 commission when she sold a $795 suit. What was her rate of commission?
437. Kiyoshi bought a television set on sale for $899. The original price was $1,200. Find: (a) the amount of discount (b) the discount rate (round to the nearest tenth of a percent)
438. Oxana bought a dresser at a garage sale for $20. She refinished it, then added a 250% markup before advertising it for sale. What price did she ask for the dresser?
439. Find the simple interest earned after 5 years on $3000 invested at an interest rate of 4.2%.
440. Brenda borrowed $400 from her brother. Two years later, she repaid the $400 plus $50 interest. What was the rate of interest?
441. Write as a proportion: 4 gallons to 144 miles is the same as 10 gallons to 360 miles.
442. Solve for a: $\frac{12}{a} = \frac{-15}{65}$
443. Vin read 10 pages of a book in 12 minutes. At that rate, how long will it take him to read 35 pages?

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7.S: Percents (Summary)

Key Terms

commission	A percentage of total sales as determined by the rate of commission
discount	A percent off the original price, determined by the discount rate
mark-up	The amount added to the wholesale price, determined by the mark-up rate
percent	A ratio whose denominator is 100
percent decrease	The percent the amount of decrease is of the original amount
percent increase	The percent the amount of increase is of the original amount
proportion	An equation of the form $\frac{a}{b} = \frac{c}{d}$, where b ≠ 0, d ≠ 0.The proportion states two ratios or rates are equal. The proportion is read " a is to b, as c is to d".
sales tax	A percent of the purchase price
simple interest	If an amount of money, P, the principal, is invested for a period of t years at an annual interest rate r, the amount of interest, I, earned is I = Prt. Interest earned according to this formula is called simple interest.

Key Concepts

6.1 - Understand Percent

- **Convert a percent to a fraction.**
  1. Write the percent as a ratio with the denominator 100.
  2. Simplify the fraction if possible.
- **Convert a percent to a decimal.**
  1. Write the percent as a ratio with the denominator 100.
  2. Convert the fraction to a decimal by dividing the numerator by the denominator.
- **Convert a decimal to a percent.**
  1. Write the decimal as a fraction.
  2. If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.
  3. Write this ratio as a percent.
- **Convert a fraction to a percent.**
  1. Convert the fraction to a decimal.
  2. Convert the decimal to a percent.

6.2 - Solve General Applications of Percent

- **Solve an application.**
  1. Identify what you are asked to find and choose a variable to represent it.
  2. Write a sentence that gives the information to find it.
  3. Translate the sentence into an equation.
  4. Solve the equation using good algebra techniques.
  5. Write a complete sentence that answers the question.
  6. Check the answer in the problem and make sure it makes sense.
- **Find percent increase.**

1. Find the amount of increase: increase = new amount − original amount
2. Find the percent increase as a percent of the original amount.

- **Find percent decrease.**
  1. Find the amount of decrease. decrease = original amount − new amount
  2. Find the percent decrease as a percent of the original amount.

6.3 - Solve Sales Tax, Commission, and Discount Applications

- **Sales Tax**: The sales tax is a percent of the purchase price.
  - sales tax = tax rate • purchase price
  - total cost = purchase price + sales tax
- **Commission**: A commission is a percentage of total sales as determined by the rate of commission.
  - commission = rate of commission • original price
- **Discount**: An amount of discount is a percent off the original price, determined by the discount rate.
  - amount of discount = discount rate • original price
  - sale price = original price – discount
- **Mark-up**: The mark-up is the amount added to the wholesale price, determined by the mark-up rate.
  - amount of mark-up = mark-up rate wholesale price
  - list price = wholesale price + mark up

6.4 - Solve Simple Interest Applications

- **Simple interest**
  - If an amount of money, P, the principal, is invested for a period of t years at an annual interest rate r, the amount of interest, I, earned is I = Prt
  - Interest earned according to this formula is called simple interest.

6.5 - Solve Proportions and their Applications

- **Proportion**
  - A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where b ≠ 0, d ≠ 0.The proportion states two ratios or rates are equal. The proportion is read “ a is to b, as c is to d”.
- **Cross Products of a Proportion**
  - For any proportion of the form $\frac{a}{b} = \frac{c}{d}$, where b ≠ 0, its cross products are equal: a • d = b • c.
- **Percent Proportion**
  - The amount is to the base as the percent is to 100. $\frac{amount}{base} = \frac{percent}{100}$

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CHAPTER OVERVIEW

8: Geometry and Graphing

Which cyclist will win the race? What will the winning time be? How many seconds will separate the winner from the runner-up? One way to summarize the information from the race is by creating a graph. In this chapter, we will discuss the basic concepts of graphing. The applications of graphing go far beyond races. They are used to present information in almost every field, including healthcare, business, and entertainment.

8.1: Solve Geometry Applications- Circles and Irregular Figures
8.2: Use the Rectangular Coordinate System (Part 1)
8.3: Solve Geometry Applications- Volume and Surface Area (Part 1)
8.4: Use the Rectangular Coordinate System (Part 2)
8.5: Graphing Linear Equations (Part 1)
8.6: Solve Geometry Applications- Volume and Surface Area (Part 2)
8.7: Graphing Linear Equations (Part 2)
8.8: Use the Rectangular Coordinate System
8.8E: Exercises
8.9: Graphing with Intercepts (Part 1)
8.10: Graph Linear Equations in Two Variables
8.10E: Exercises
8.11: Graphing with Intercepts (Part 2)
8.12: Graph with Intercepts
8.12E: Exercises
8.13: Understand Slope of a Line (Part 1)
8.14: Understand Slope of a Line (Part 2)
8.E: Graphs (Exercises)
8.E: Review Exercises
8.S: Graphs (Summary)

Figure 11.1 - Cyclists speed toward the finish line. (credit: ewan traveler, Flickr)

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8.1: Solve Geometry Applications- Circles and Irregular Figures

Learning Objectives

- Use the properties of circles
- Find the area of irregular figures

be prepared!

Before you get started, take this readiness quiz.

1. Evaluate x^2 when $x = 5$. If you missed this problem, review Example 2.3.3.
2. Using 3.14 for π, approximate the (a) circumference and (b) the area of a circle with radius 8 inches. If you missed this problem, review Example 5.6.12.
3. Simplify $\frac{22}{7}(0.25)^2$ and round to the nearest thousandth. If you missed this problem, review Example 5.5.9.

In this section, we'll continue working with geometry applications. We will add several new formulas to our collection of formulas. To help you as you do the examples and exercises in this section, we will show the Problem Solving Strategy for Geometry Applications here.

Problem Solving Strategy for Geometry Applications

Step 1. **Read** the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.

Step 2. **Identify** what you are looking for.

Step 3. **Name** what you are looking for. Choose a variable to represent that quantity

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

Step 5. **Solve** the equation using good algebra techniques.

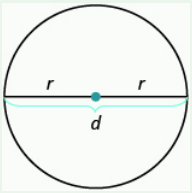
Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

Use the Properties of Circles

Do you remember the properties of circles from Decimals and Fractions? We'll show them here again to refer to as we use them to solve applications.

Definition: Properties of Circles



- r is the length of the radius
- d is the length of the diameter
- $d = 2r$
- Circumference is the perimeter of a circle. The formula for circumference is $C = 2\pi r$
- The formula for area of a circle is $A = \pi r^2$

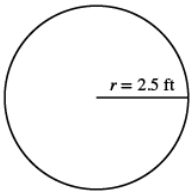
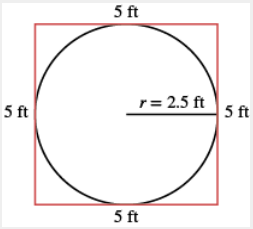
Remember, that we approximate π with 3.14 or $\frac{22}{7}$ depending on whether the radius of the circle is given as a decimal or a fraction. If you use the π key on your calculator to do the calculations in this section, your answers will be slightly different from the answers shown. That is because the π key uses more than two decimal places.

Example 8.1.1:

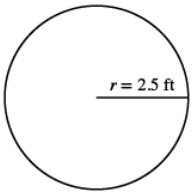
A circular sandbox has a radius of 2.5 feet. Find the (a) circumference and (b) area of the sandbox.

Solution

(a)

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the circumference of the circle
Step 3. **Name**. Choose a variable to represent it.	Let c = circumference of the circle
Step 4. **Translate**. Write the appropriate formula. Substitute.	$C = 2\pi r$ $C = 2\pi(2.5)$
Step 5. **Solve** the equation.	$C \approx 2(3.14)(2.5)$ $C \approx 15\ ft$
Step 6. **Check**. Does this answer make sense?	Yes. If we draw a square around the circle, its sides would be 5 ft (twice the radius), so its perimeter would be 20 ft. This is slightly more than the circle's circumference, 15.7 ft. 
Step 7. **Answer** the question.	The circumference of the sandbox is 15.7 feet.

(b)

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of the circle
Step 3. **Name**. Choose a variable to represent it.	Let A = the area of the circle
Step 4. **Translate**. Write the appropriate formula. Substitute.	$A = \pi r^2$ $A = \pi(2.5)^2$

Step 5. **Solve** the equation.	$A \approx (3.14)(2.5)^2$ $A \approx 19.625\ sq.\ ft$
Step 6. **Check**. Does this answer make sense?	Yes. If we draw a square around the circle, its sides would be 5 ft, as shown in part (a). So the area of the square would be 25 sq. ft. This is slightly more than the circle's area, 19.625 sq. ft.
Step 7. **Answer** the question.	The area of the circle is 19.625 square feet.

? Exercise 8.1.1:

A circular mirror has radius of 5 inches. Find the (a) circumference and (b) area of the mirror.

Answer a

31.4 in.

Answer b

78.5 sq. in.

? Exercise 8.1.2:

A circular spa has radius of 4.5 feet. Find the (a) circumference and (b) area of the spa.

Answer a

28.26 ft

Answer b

63.585 sq. ft

We usually see the formula for circumference in terms of the radius r of the circle:

$$C = 2\pi r \tag{8.1.1}$$

But since the diameter of a circle is two times the radius, we could write the formula for the circumference in terms of d.

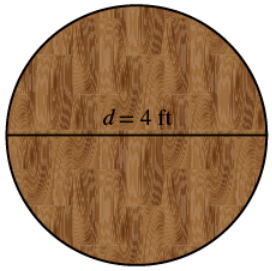
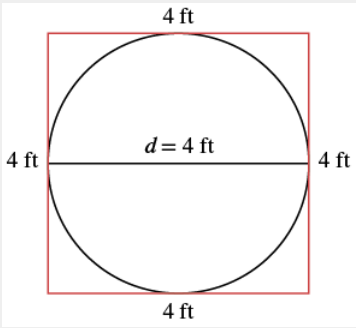
$$\begin{aligned} & & C &= 2\pi r \\ \textit{Using the commutative property, we get} & & C &= \pi \cdot 2r \\ \textit{Then substituting } d = 2r & & C &= \pi \cdot d \\ \textit{So} & & C &= \pi d \end{aligned}$$

We will use this form of the circumference when we're given the length of the diameter instead of the radius.

✓ Example 8.1.2:

A circular table has a diameter of four feet. What is the circumference of the table?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the circumference of the table
Step 3. **Name**. Choose a variable to represent it.	Let c = the circumference of the table
Step 4. **Translate**. Write the appropriate formula for the situation. Substitute.	$C = \pi d$ $C = \pi(4)$
Step 5. **Solve** the equation, using 3.14 for π.	$C \approx (3.14)(4)$ $C \approx 12.56\ ft$
Step 6. **Check**.	If we put a square around the circle, its side would be 4. The perimeter would be 16. It makes sense that the circumference of the circle, 12.56, is a little less than 16. 
Step 7. **Answer** the question.	The diameter of the table is 12.56 square feet.

? Exercise 8.1.3:

Find the circumference of a circular fire pit whose diameter is 5.5 feet.

Answer

17.27 ft

? Exercise 8.1.4:

If the diameter of a circular trampoline is 12 feet, what is its circumference?

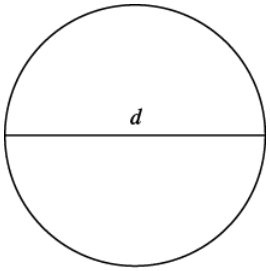
Answer

37.68 ft.

Example 8.1.3:

Find the diameter of a circle with a circumference of 47.1 centimeters.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	 $C = 47.1$ cm
Step 2. **Identify** what you are looking for.	the diameter of the circle
Step 3. **Name**. Choose a variable to represent it.	Let d = the diameter of the circle
Step 4. **Translate**. Write the formula. Substitute, using 3.14 to approximate π.	$C = \pi d$ $47.1 \approx 3.14d$
Step 5. **Solve**.	$\frac{47.11}{3.14} \approx \frac{3.14d}{3.14}$ $15 \approx d$
Step 6. **Check**.	$47.1 \stackrel{?}{=} (3.14)(15)$ $47.1 = 47.1$ ✓
Step 7. **Answer** the question.	The diameter of the circle is approximately 15 centimeters.

Exercise 8.1.5:

Find the diameter of a circle with circumference of 94.2 centimeters.

Answer

30 cm

Exercise 8.1.6:

Find the diameter of a circle with circumference of 345.4 feet.

Answer

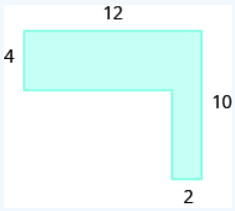
110 ft

Find the Area of Irregular Figures

So far, we have found area for rectangles, triangles, trapezoids, and circles. An **irregular figure** is a figure that is not a standard geometric shape. Its area cannot be calculated using any of the standard area formulas. But some irregular figures are made up of two or more standard geometric shapes. To find the area of one of these irregular figures, we can split it into figures whose formulas we know and then add the areas of the figures.

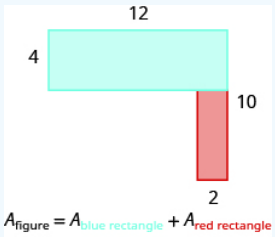
Example 8.1.4:

Find the area of the shaded region.

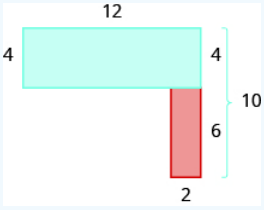


Solution

The given figure is irregular, but we can break it into two rectangles. The area of the shaded region will be the sum of the areas of both rectangles.



The blue rectangle has a width of 12 and a length of 4. The red rectangle has a width of 2, but its length is not labeled. The right side of the figure is the length of the red rectangle plus the length of the blue rectangle. Since the right side of the blue rectangle is 4 units long, the length of the red rectangle must be 6 units.



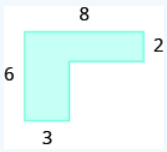
$$
\begin{aligned}
A_{figure} &= A_{rectangle} + A_{rectangle} \\
A_{figure} &= bh + bh \\
A_{figure} &= 12 \cdot 4 + 2 \cdot 6 \\
A_{figure} &= 48 + 12 \\
A_{figure} &= 60
\end{aligned}
$$

The area of the figure is 60 square units.

Is there another way to split this figure into two rectangles? Try it, and make sure you get the same area.

Exercise 8.1.7:

Find the area of each shaded region:

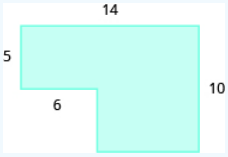


Answer

28 sq. units

Exercise 8.1.8:

Find the area of each shaded region:

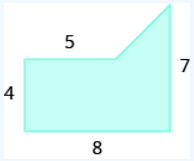


Answer

110 sq. units

Example 8.1.5:

Find the area of the shaded region.



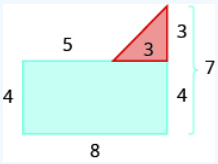
Solution

We can break this irregular figure into a triangle and rectangle. The area of the figure will be the sum of the areas of triangle and rectangle.

The rectangle has a length of 8 units and a width of 4 units.

We need to find the base and height of the triangle.

Since both sides of the rectangle are 4, the vertical side of the triangle is 3, which is 7 − 4. The length of the rectangle is 8, so the base of the triangle will be 3, which is 8 − 4.



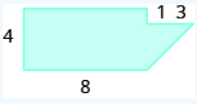
Now we can add the areas to find the area of the irregular figure.

$$\begin{aligned} A_{figure} &= A_{rectangle} + A_{triangle} \\ A_{figure} &= lw + \frac{1}{2}bh \\ A_{figure} &= 8 \cdot 4 + \frac{1}{2} \cdot 3 \cdot 3 \\ A_{figure} &= 32 + 4.5 \\ A_{figure} &= 3.65 \ sq. \ units \end{aligned}$$

The area of the figure is 36.5 square units.

Exercise 8.1.9:

Find the area of each shaded region.

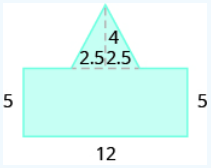


Answer

36.5 sq. units

Exercise 8.1.10:

Find the area of each shaded region.



Answer

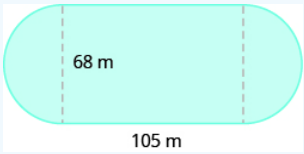
70 sq. units

Example 8.1.6:

A high school track is shaped like a rectangle with a semi-circle (half a circle) on each end. The rectangle has length 105 meters and width 68 meters. Find the area enclosed by the track. Round your answer to the nearest hundredth.

Solution

We will break the figure into a rectangle and two semi-circles. The area of the figure will be the sum of the areas of the rectangle and the semicircles.

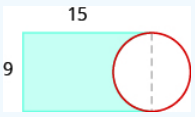


The rectangle has a length of 105 m and a width of 68 m. The semi-circles have a diameter of 68 m, so each has a radius of 34 m.

$$\begin{aligned} A_{figure} &= A_{rectangle} + A_{semicircles} \\ A_{figure} &= bh + 2\left(\frac{1}{2}\pi \cdot r^2\right) \\ A_{figure} &\approx 105 \cdot 68 + 2\left(\frac{1}{2} \cdot 3.14 \cdot 34^2\right) \\ A_{figure} &\approx 7140 + 3629.84 \\ A_{figure} &\approx 10,769.84 \ square \ meters \end{aligned}$$

Exercise 8.1.11:

Find the area:

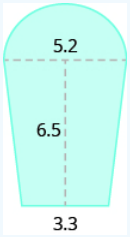


Answer

103.2 sq. units

Exercise 8.1.12:

Find the area:



Answer

38.24 sq. units

ACCESS ADDITIONAL ONLINE RESOURCES

Circumference of a Circle

Area of a Circle

Area of an L-shaped polygon

Area of an L-shaped polygon with Decimals

Perimeter Involving a Rectangle and Circle

Area Involving a Rectangle and Circle

Practice Makes Perfect

Use the Properties of Circles

In the following exercises, solve using the properties of circles.

217. The lid of a paint bucket is a circle with radius 7 inches. Find the (a) circumference and (b) area of the lid.
218. An extra-large pizza is a circle with radius 8 inches. Find the (a) circumference and (b) area of the pizza.
219. A farm sprinkler spreads water in a circle with radius of 8.5 feet. Find the (a) circumference and (b) area of the watered circle.
220. A circular rug has radius of 3.5 feet. Find the (a) circumference and (b) area of the rug.
221. A reflecting pool is in the shape of a circle with diameter of 20 feet. What is the circumference of the pool?
222. A turntable is a circle with diameter of 10 inches. What is the circumference of the turntable?
223. A circular saw has a diameter of 12 inches. What is the circumference of the saw?
224. A round coin has a diameter of 3 centimeters. What is the circumference of the coin?
225. A barbecue grill is a circle with a diameter of 2.2 feet. What is the circumference of the grill?
226. The top of a pie tin is a circle with a diameter of 9.5 inches. What is the circumference of the top?
227. A circle has a circumference of 163.28 inches. Find the diameter.
228. A circle has a circumference of 59.66 feet. Find the diameter.
229. A circle has a circumference of 17.27 meters. Find the diameter.
230. A circle has a circumference of 80.07 centimeters. Find the diameter.

In the following exercises, find the radius of the circle with given circumference.

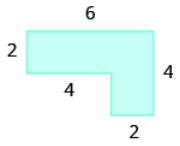
231. A circle has a circumference of 150.72 feet.
232. A circle has a circumference of 251.2 centimeters.
233. A circle has a circumference of 40.82 miles.

234. A circle has a circumference of 78.5 inches.

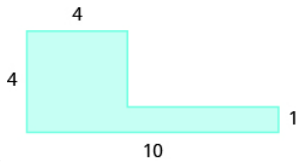
Find the Area of Irregular Figures

In the following exercises, find the area of the irregular figure. Round your answers to the nearest hundredth.

235.



236.



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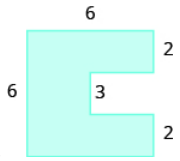
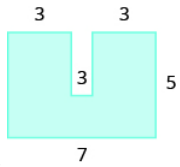
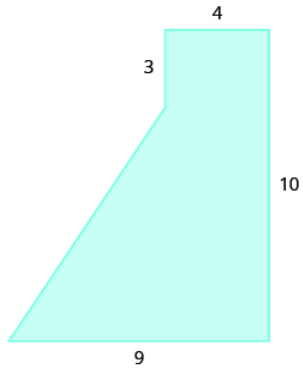


238.



239.



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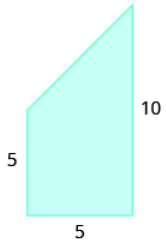

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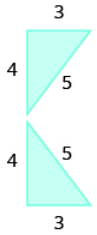

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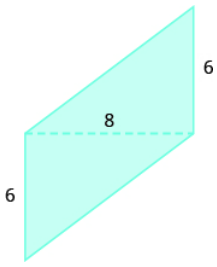

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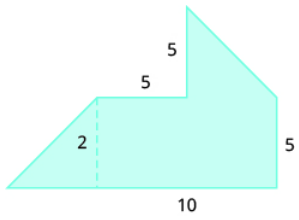

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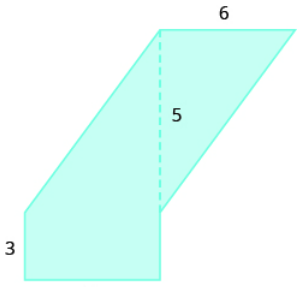

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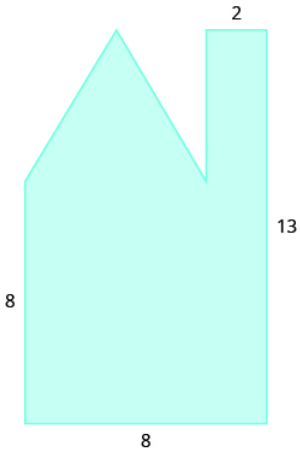

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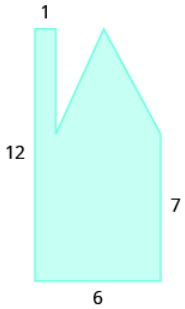
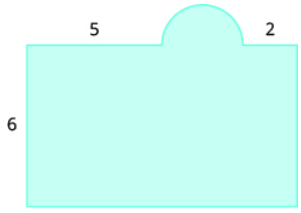


247.

248.



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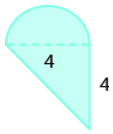
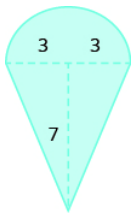
250.



251.



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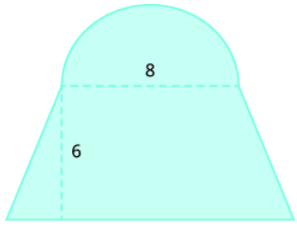

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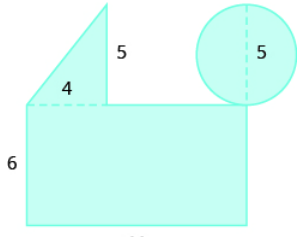
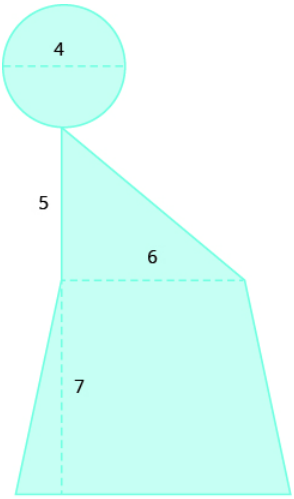




254.

In the following exercises, solve.

255. A city park covers one block plus parts of four more blocks, as shown. The block is a square with sides 250 feet long, and the triangles are isosceles right triangles. Find the area of the park.

256. A gift box will be made from a rectangular piece of cardboard measuring 12 inches by 20 inches, with squares cut out of the corners of the sides, as shown. The sides of the squares are 3 inches. Find the area of the cardboard after the corners are cut out.

257. Perry needs to put in a new lawn. His lot is a rectangle with a length of 120 feet and a width of 100 feet. The house is rectangular and measures 50 feet by 40 feet. His driveway is rectangular and measures 20 feet by 30 feet, as shown. Find the area of Perry's lawn.

258. Denise is planning to put a deck in her back yard. The deck will be a 20-ft by 12-ft rectangle with a semicircle of diameter 6 feet, as shown below. Find the area of the deck.

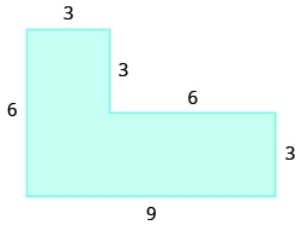
Everyday Math

259. **Area of a Tabletop** Yuki bought a drop-leaf kitchen table. The rectangular part of the table is a 1-ft by 3-ft rectangle with a semicircle at each end, as shown.(a) Find the area of the table with one leaf up. (b) Find the area of the table with both leaves up.

260. **Painting** Leora wants to paint the nursery in her house. The nursery is an 8-ft by 10-ft rectangle, and the ceiling is 8 feet tall. There is a 3-ft by 6.5-ft door on one wall, a 3-ft by 6.5-ft closet door on another wall, and one 4-ft by 3.5-ft window on the third wall. The fourth wall has no doors or windows. If she will only paint the four walls, and not the ceiling or doors, how many square feet will she need to paint?

Writing Exercises

261. Describe two different ways to find the area of this figure, and then show your work to make sure both ways give the same area.



262. A circle has a diameter of 14 feet. Find the area of the circle (a) using 3.14 for π (b) using $\frac{22}{7}$ for π. (c) Which calculation to do prefer? Why?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the properties of circles.			
find the area of irregular figures.			

(b) After looking at the checklist, do you think you are well prepared for the next section? Why or why not?

Contributors and Attributions

- Lynn Marecek (Santa Ana College) and MaryAnne Anthony-Smith (Formerly of Santa Ana College). This content is licensed under Creative Commons Attribution License v4.0 "Download for free at http://cnx.org/contents/fd53eae1-fa2...49835c3c@5.191."

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8.2: Use the Rectangular Coordinate System (Part 1)

Learning Objectives

- Plot points on a rectangular coordinate system
- Identify points on a graph
- Verify solutions to an equation in two variables
- Complete a table of solutions to a linear equation
- Find solutions to linear equations in two variables

be prepared!

Before you get started, take this readiness quiz.

1. Evaluate: $x + 3$ when $x = -1$. If you missed this problem, review Example 3.4.10.
2. Evaluate: $2x - 5y$ when $x = 3$, $y = -2$. If you missed this problem, review Example 3.8.106.
3. Solve for y: $40 - 4y = 20$. If you missed this problem, review Example 8.4.1.

Plot Points on a Rectangular Coordinate System

Many maps, such as the Campus Map shown in Figure 8.2.1, use a grid system to identify locations. Do you see the numbers 1, 2, 3, and 4 across the top and bottom of the map and the letters A, B, C, and D along the sides? Every location on the map can be identified by a number and a letter.

For example, the Student Center is in section 2B. It is located in the grid section above the number 2 and next to the letter B. In which grid section is the Stadium? The Stadium is in section 4D.

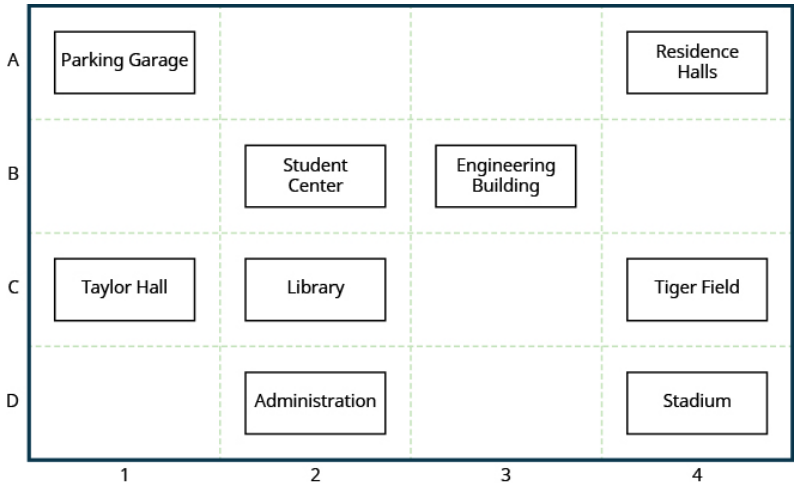


Figure 8.2.1

Example 8.2.1:

Use the map in Figure 8.2.1. (a) Find the grid section of the Residence Halls. (b) What is located in grid section 4C?

Solution

(a) Read the number below the Residence Halls, 4, and the letter to the side, A. So the Residence Halls are in grid section 4A.

(b) Find 4 across the bottom of the map and C along the side. Look below the 4 and next to the C. Tiger Field is in grid section 4C.

Exercise 8.2.1:

Use the map in Figure 8.2.1. (a) Find the grid section of Taylor Hall. (b) What is located in section 3B?

Answer a

1C

Answer b

Engineering Building

Exercise 8.2.1:

Use the map in Figure 8.2.1. (a) Find the grid section of the Parking Garage. (b) What is located in section 2C?

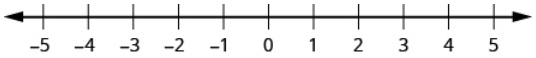
Answer a

1A

Answer b

Library

Just as maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a rectangular coordinate system. To create a rectangular coordinate system, start with a horizontal number line. Show both positive and negative numbers as you did before, using a convenient scale unit. This horizontal number line is called the **x-axis**.



Now, make a vertical number line passing through the x-axis at 0. Put the positive numbers above 0 and the negative numbers below 0. See Figure 8.2.2. This vertical line is called the **y-axis**.

Vertical grid lines pass through the integers marked on the x-axis. Horizontal grid lines pass through the integers marked on the y-axis. The resulting grid is the rectangular coordinate system.

The rectangular coordinate system is also called the x-y plane, the coordinate plane, or the Cartesian coordinate system (since it was developed by a mathematician named René Descartes.)

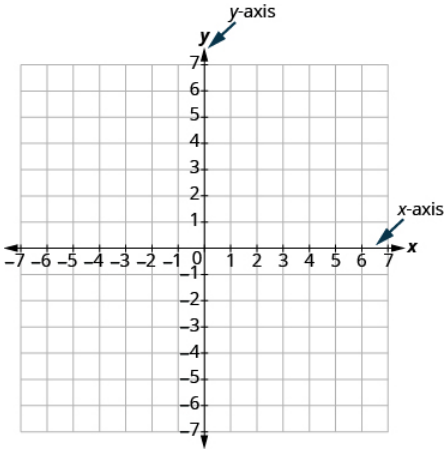


Figure 8.2.2 - The rectangular coordinate system.

The x-axis and the y-axis form the rectangular coordinate system. These axes divide a plane into four areas, called **quadrants**. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See Figure 8.2.3.

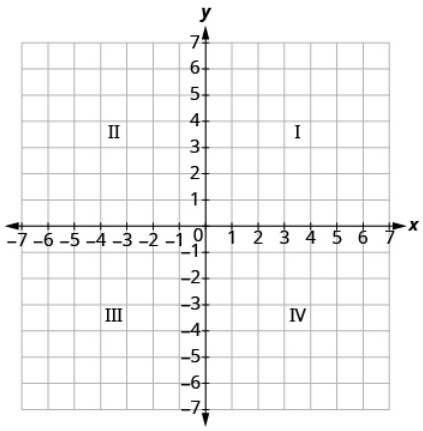
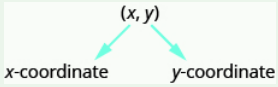


Figure 8.2.3 - The four quadrants of the rectangular coordinate system.

In the rectangular coordinate system, every point is represented by an **ordered pair**. The first number in the ordered pair is the x-coordinate of the point, and the second number is the y-coordinate of the point.

Definition: Ordered pair

An ordered pair, (x, y) gives the coordinates of a point in a rectangular coordinate system. The first number is the x-coordinate. The second number is the y-coordinate.



So how do the coordinates of a point help you locate a point on the x-y plane?

Let's try locating the point (2, 5) . In this ordered pair, the x -coordinate is 2 and the y -coordinate is 5 .

We start by locating the x value, 2, on the x-axis. Then we lightly sketch a vertical line through x = 2, as shown in Figure 8.2.4

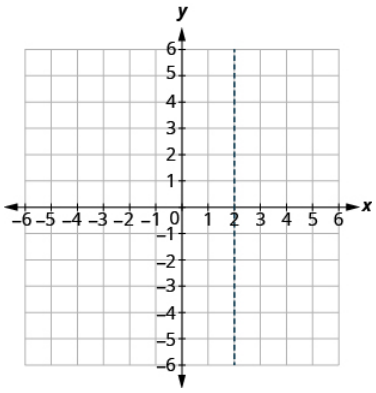


Figure 8.2.4

Now we locate the y value, 5, on the y -axis and sketch a horizontal line through y = 5. The point where these two lines meet is the point with coordinates (2, 5). We plot the point there, as shown in Figure 8.2.5.

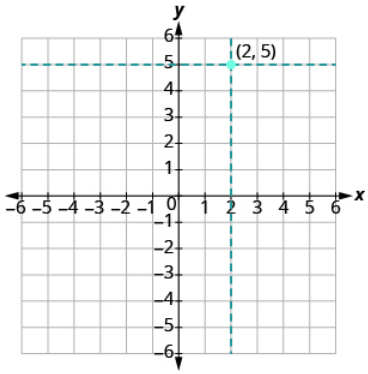


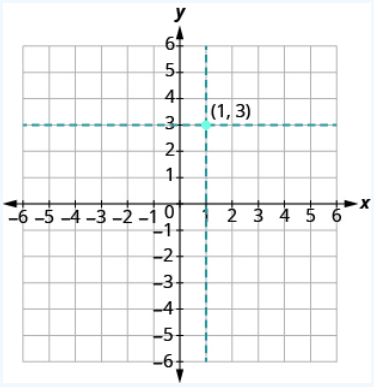
Figure 8.2.5

Example 8.2.2:

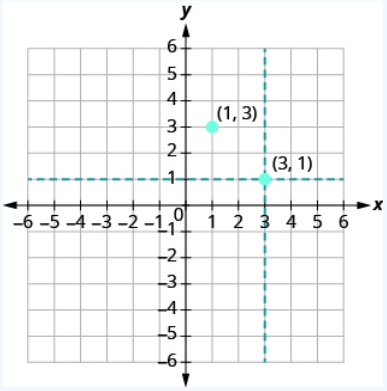
Plot (1, 3) and (3, 1) in the same rectangular coordinate system.

Solution

The coordinate values are the same for both points, but the x and y values are reversed. Let's begin with point (1, 3). The x-coordinate is 1 so find 1 on the x-axis and sketch a vertical line through x = 1. The y-coordinate is 3 so we find 3 on the y-axis and sketch a horizontal line through y = 3. Where the two lines meet, we plot the point (1, 3).



To plot the point (3, 1), we start by locating 3 on the x-axis and sketch a vertical line through x = 3. Then we find 1 on the y-axis and sketch a horizontal line through y = 1. Where the two lines meet, we plot the point (3, 1).

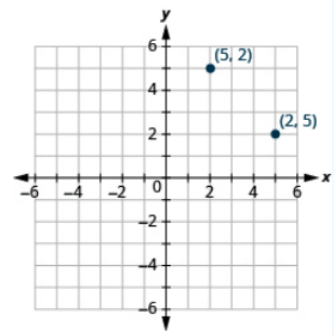


Notice that the order of the coordinates does matter, so, (1, 3) is not the same point as (3, 1).

Exercise 8.2.3:

Plot each point on the same rectangular coordinate system: (2, 5), (5, 2).

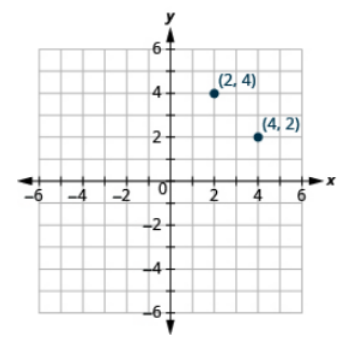
Answer



Exercise 8.2.4:

Plot each point on the same rectangular coordinate system: (4, 2), (2, 4).

Answer



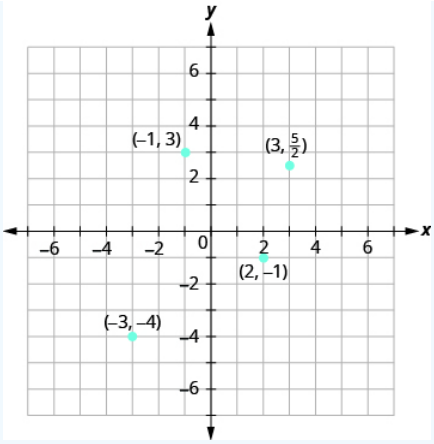
Example 8.2.3:

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located: (a) (−1, 3) (b) (−3, −4) (c) (2, −3) (d) $\left(3, \frac{5}{2}\right)$

Solution

The first number of the coordinate pair is the x-coordinate, and the second number is the y-coordinate.

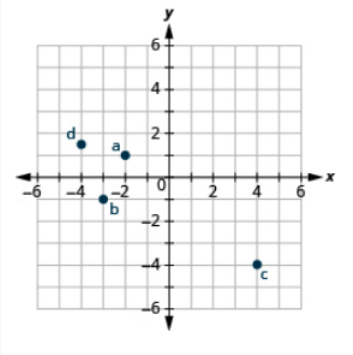
a. Since x = −1, y = 3, the point (−1, 3) is in Quadrant II.
b. Since x = −3, y = −4, the point (−3, −4) is in Quadrant III.
c. Since x = 2, y = −1, the point (2, −1) is in Quadrant lV.
d. Since x = 3, y = $\frac{5}{2}$, the point $\left(3, \frac{5}{2}\right)$ is in Quadrant I. It may be helpful to write $\frac{5}{2}$ as the mixed number, $2\frac{1}{2}$, or decimal, 2.5. Then we know that the point is halfway between 2 and 3 on the y-axis.



Exercise 8.2.5:

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located: (a) (−2, 1) (b) (−3, −1) (c) (4, −4) (d) $\left(-4, \frac{3}{2}\right)$

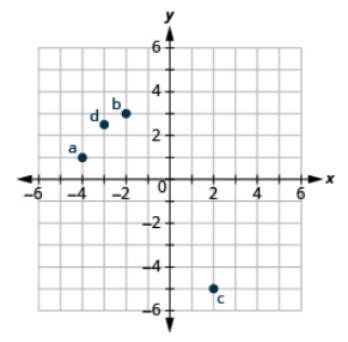
Answer



Exercise 8.2.6:

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located: (a) (−4, 1) (b) (−2, 3) (c) (2, −5) (d) $\left(-3, \frac{5}{2}\right)$

Answer



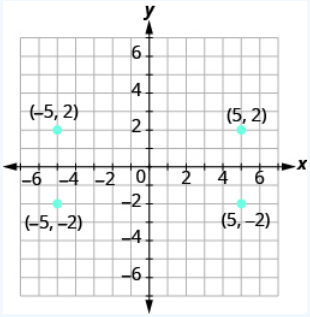
How do the signs affect the location of the points?

Example 8.2.4:

Plot each point: (a) (−5, 2) (b) (−5, −2) (c) (5, 2) (d) (5, −2)

Solution

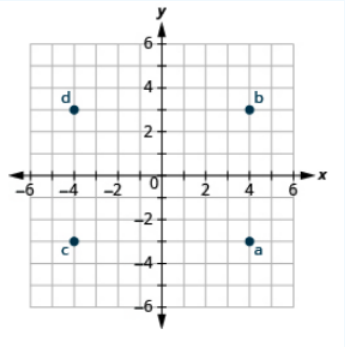
As we locate the x-coordinate and the y-coordinate, we must be careful with the signs.



Exercise 8.2.7:

Plot each point: (a) (4, −3) (b) (4, 3) (c) (−4, −3) (d) (−4, 3)

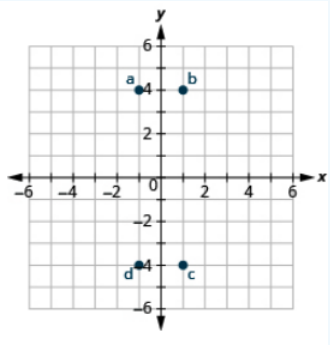
Answer



Exercise 8.2.8:

Plot each point: (a) (−1, 4) (b) (1, 4) (c) (1, −4) (d) (−1, −4)

Answer

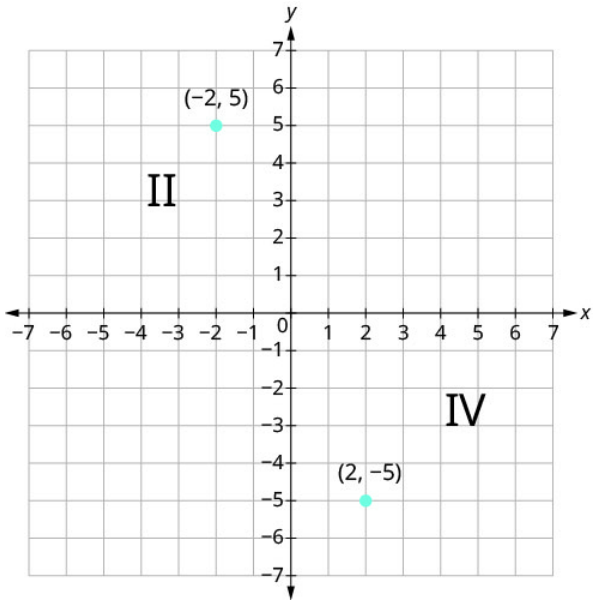


You may have noticed some patterns as you graphed the points in the two previous examples.

For each point in Quadrant IV, what do you notice about the signs of the coordinates?

What about the signs of the coordinates of the points in the third quadrant? The second quadrant? The first quadrant?

Can you tell just by looking at the coordinates in which quadrant the point (−2, 5) is located? In which quadrant is (2, −5) located?



We can summarize sign patterns of the quadrants as follows. Also see Figure 8.2.6.

Table 8.2.1

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x,y)	(x,y)	(x,y)	(x,y)
(+,+)	(-,+)	(-,-)	(+,-)

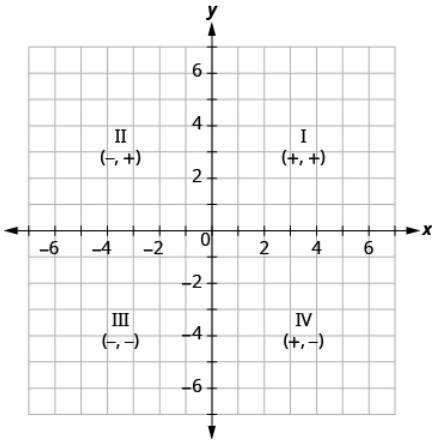


Figure 8.2.6

What if one coordinate is zero? Where is the point (0, 4) located? Where is the point (−2, 0) located? The point (0, 4) is on the y-axis and the point (− 2, 0) is on the x-axis.

Definition: Points on the Axes

Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates (a, 0).

Points with an x-coordinate equal to 0 are on the y-axis, and have coordinates (0, b).

What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is (0, 0). The point has a special name. It is called the *origin*.

Definition: the origin

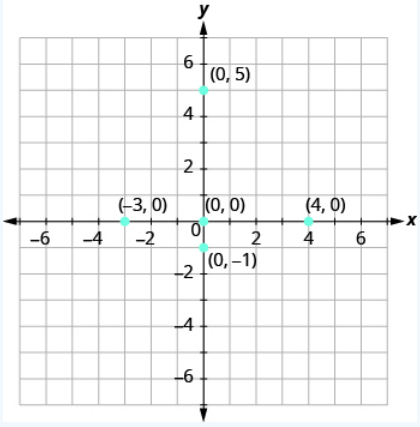
The point (0, 0) is called the origin. It is the point where the x-axis and y-axis intersect.

Example 8.2.5:

Plot each point on a coordinate grid: (a) (0, 5) (b) (4, 0) (c) (−3, 0) (d) (0, 0) (e) (0, −1)

Solution

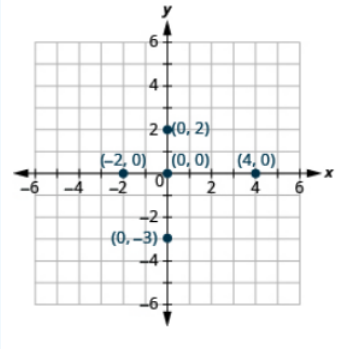
a. Since x = 0, the point whose coordinates are (0, 5) is on the y-axis.
b. Since y = 0, the point whose coordinates are (4, 0) is on the x-axis.
c. Since y = 0, the point whose coordinates are (−3, 0) is on the x-axis.
d. Since x = 0 and y = 0, the point whose coordinates are (0, 0) is the origin.
e. Since x = 0, the point whose coordinates are (0, −1) is on the y-axis.



Exercise 8.2.9:

Plot each point on a coordinate grid: (a) (4, 0) (b) (−2, 0) (c) (0, 0) (d) (0, 2) (e) (0, −3)

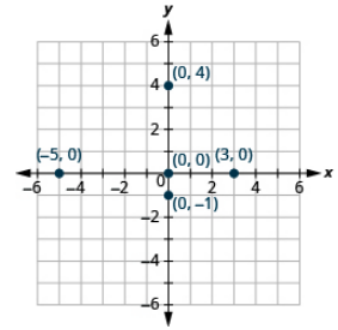
Answer



Exercise 8.2.10:

Plot each point on a coordinate grid: (a) (−5, 0) (b) (3, 0) (c) (0, 0) (d) (0, −1) (e) (0, 4)

Answer

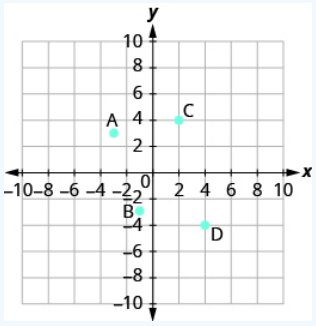


Identify Points on a Graph

In algebra, being able to identify the coordinates of a point shown on a graph is just as important as being able to plot points. To identify the x-coordinate of a point on a graph, read the number on the x-axis directly above or below the point. To identify the y-coordinate of a point, read the number on the y-axis directly to the left or right of the point. Remember, to write the ordered pair using the correct order (x, y).

Example 8.2.6:

Name the ordered pair of each point shown:



Solution

Point A is above −3 on the x-axis, so the x-coordinate of the point is −3. The point is to the left of 3 on the y-axis, so the y-coordinate of the point is 3. The coordinates of the point are (−3, 3).

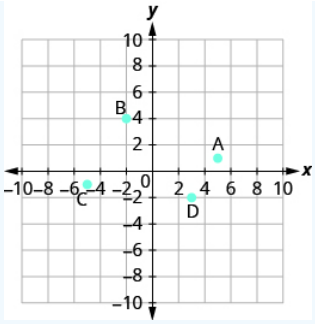
Point B is below −1 on the x-axis, so the x-coordinate of the point is −1. The point is to the left of −3 on the y-axis, so the y-coordinate of the point is −3. The coordinates of the point are (−1, −3).

Point C is above 2 on the x-axis, so the x-coordinate of the point is 2. The point is to the right of 4 on the y-axis, so the y-coordinate of the point is 4. The coordinates of the point are (2, 4).

Point D is below 4 on the x - axis, so the x-coordinate of the point is 4. The point is to the right of −4 on the y-axis, so the y-coordinate of the point is −4. The coordinates of the point are (4, −4)

Exercise 8.2.11:

Name the ordered pair of each point shown:

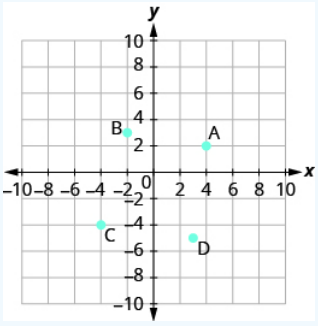


Answer

A: (5,1), B: (−2,4), C: (−5,−1), D: (3,−2)

Exercise 8.2.12:

Name the ordered pair of each point shown:

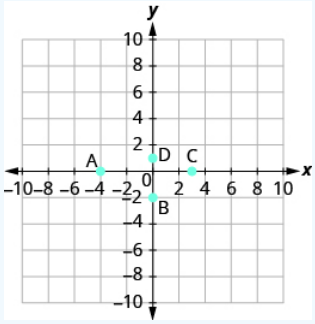


Answer

A: (4,2), B: (−2,3), C: (−4,−4), D: (3,−5)

Example 8.2.7:

Name the ordered pair of each point shown:

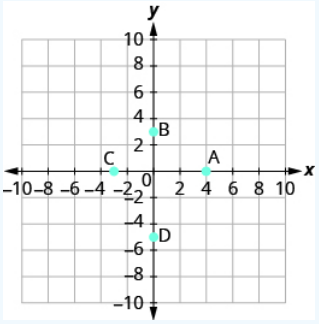


Solution

Point A is on the x-axis at x = − 4.	The coordinates of point A are (− 4, 0).
Point B is on the y-axis at y = − 2.	The coordinates of point B are (0, − 2).
Point C is on the x-axis at x = 3.	The coordinates of point C are (3, 0).
Point D is on the y-axis at y = 1.	The coordinates of point D are (0, 1).

Exercise 8.2.13:

Name the ordered pair of each point shown:

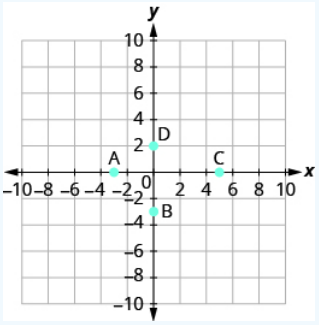


Answer

A: (4,0), B: (0,3), C: (−3,0), D: (0,−5)

Exercise 8.2.14:

Name the ordered pair of each point shown:



Answer

A: (−3,0), B: (0,−3), C: (5,0), D: (0,2)

Contributors and Attributions

- Lynn Marecek (Santa Ana College) and MaryAnne Anthony-Smith (Formerly of Santa Ana College). This content is licensed under Creative Commons Attribution License v4.0 "Download for free at http://cnx.org/contents/fd53eae1-fa2...49835c3c@5.191."

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8.3: Solve Geometry Applications- Volume and Surface Area (Part 1)

Learning Objectives

- Find volume and surface area of rectangular solids
- Find volume and surface area of spheres
- Find volume and surface area of cylinders
- Find volume of cones

be prepared!

Before you get started, take this readiness quiz.

1. Evaluate x^3 when $x = 5$. If you missed this problem, review Example 2.3.3.
2. Evaluate 2^x when $x = 5$. If you missed this problem, review Example 2.3.4.
3. Find the area of a circle with radius $\frac{7}{2}$. If you missed this problem, review Example 5.6.12.

In this section, we will finish our study of geometry applications. We find the volume and surface area of some three-dimensional figures. Since we will be solving applications, we will once again show our Problem-Solving Strategy for Geometry Applications.

Problem Solving Strategy for Geometry Applications

- Step 1. **Read** the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
- Step 2. **Identify** what you are looking for.
- Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.
- Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
- Step 5. **Solve** the equation using good algebra techniques.
- Step 6. **Check** the answer in the problem and make sure it makes sense.
- Step 7. **Answer** the question with a complete sentence.

Find Volume and Surface Area of Rectangular Solids

A cheerleading coach is having the squad paint wooden crates with the school colors to stand on at the games. (See Figure 8.3.1). The amount of paint needed to cover the outside of each box is the **surface area**, a square measure of the total area of all the sides. The amount of space inside the crate is the volume, a cubic measure.

Figure 8.3.1 - This wooden crate is in the shape of a rectangular solid.

Each crate is in the shape of a **rectangular solid**. Its dimensions are the length, width, and height. The rectangular solid shown in Figure 8.3.2 has length 4 units, width 2 units, and height 3 units. Can you tell how many cubic units there are altogether? Let's look layer by layer.

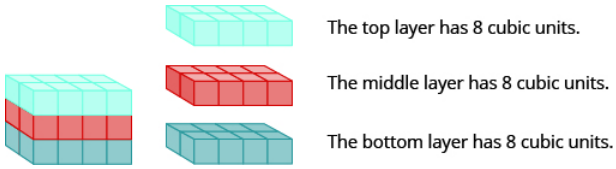


Figure 8.3.2 - Breaking a rectangular solid into layers makes it easier to visualize the number of cubic units it contains. This 4 by 2 by 3 rectangular solid has 24 cubic units.

Altogether there are 24 cubic units. Notice that 24 is the length × width × height.

$$\underset{24}{V} = \underset{4}{L} \cdot \underset{2}{W} \cdot \underset{3}{H}$$

The volume, V, of any rectangular solid is the product of the length, width, and height.

$$V = LWH \tag{8.3.1}$$

We could also write the formula for volume of a rectangular solid in terms of the area of the base. The area of the base, B, is equal to length × width.

$$B = L \cdot W \tag{8.3.2}$$

We can substitute B for L • W in the volume formula to get another form of the volume formula.

$$\begin{aligned} V &= L \cdot W \cdot H \\ V &= (L \cdot W) \cdot H \\ V &= Bh \end{aligned}$$

We now have another version of the volume formula for rectangular solids. Let's see how this works with the 4 × 2 × 3 rectangular solid we started with. See Figure 8.3.3.

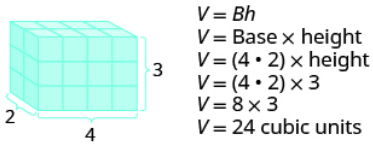


Figure 8.3.3

To find the *surface area* of a rectangular solid, think about finding the area of each of its faces. How many faces does the rectangular solid above have? You can see three of them.

$$\begin{aligned} A_{front} &= L \times W & A_{side} &= L \times W & A_{top} &= L \times W \\ A_{front} &= 4 \cdot 3 & A_{side} &= 2 \cdot 3 & A_{top} &= 4 \cdot 2 \\ A_{front} &= 12 & A_{side} &= 6 & A_{top} &= 8 \end{aligned}$$

Notice for each of the three faces you see, there is an identical opposite face that does not show.

$$\begin{aligned} S &= (front + back) + (left\ side + right\ side) + (top + bottom) \\ S &= (2 \cdot front) + (2 \cdot left\ side) + (2 \cdot top) \\ S &= 2 \cdot 12 + 2 \cdot 6 + 2 \cdot 8 \\ S &= 24 + 12 + 16 \\ S &= 52\ sq.\ units \end{aligned}$$

The surface area S of the rectangular solid shown in Figure 8.3.3 is 52 square units.

In general, to find the surface area of a rectangular solid, remember that each face is a rectangle, so its area is the product of its length and its width (see Figure 8.3.4). Find the area of each face that you see and then multiply each area by two to account for the face on the opposite side.

$$S = 2LH + 2LW + 2WH \tag{8.3.3}$$

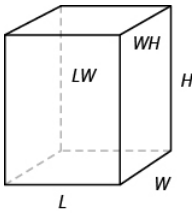
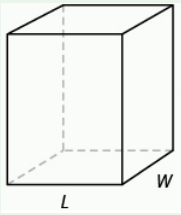


Figure 8.3.4 - For each face of the rectangular solid facing you, there is another face on the opposite side. There are 6 faces in all.

Definition: Volume and Surface Area of a Rectangular Solid

For a rectangular solid with length L, width W, and height H:



Volume: $V = LWH$

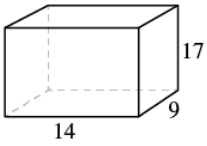
Surface Area: $S = 2LH + 2LW + 2WH$

Example 8.3.1:

For a rectangular solid with length 14 cm, height 17 cm, and width 9 cm, find the (a) volume and (b) surface area.

Solution

Step 1 is the same for both (a) and (b), so we will show it just once.

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

(a)

Step 2. **Identify** what you are looking for.	the volume of the rectangular solid
Step 3. **Name**. Choose a variable to represent it.	Let V = volume
Step 4. **Translate**. Write the appropriate formula. Substitute.	$V = LWH$ $V = 14 \cdot 9 \cdot 9 \cdot 17$
Step 5. **Solve** the equation.	$V = 2,142$ (8.3.4)
Step 6. **Check.**	We leave it to you to check your calculations.
Step 7. **Answer** the question.	The volume is 2,142 cubic centimeters.

(b)

Step 2. **Identify** what you are looking for.	the surface area of the solid
Step 3. **Name**. Choose a variable to represent it.	Let S = surface area
Step 4. **Translate**. Write the appropriate formula. Substitute.	$S = 2LH + 2LW + 2WH$ $S = 2(14 \cdot 17) + 2(14 \cdot 9) + 2(9 \cdot 17)$

Step 5. **Solve** the equation.	$S = 1,034$	(8.3.5)
Step 6. **Check.**	Double-check with a calculator.	
Step 7. **Answer** the question.	The surface area is 1,034 square centimeters.	

? Exercise 8.3.1:

Find the (a) volume and (b) surface area of rectangular solid with the: length 8 feet, width 9 feet, and height 11 feet.

Answer a

792 cu. ft

Answer b

518 sq. ft

? Exercise 8.3.2:

Find the (a) volume and (b) surface area of rectangular solid with the: length 15 feet, width 12 feet, and height 8 feet.

Answer a

1,440 cu. ft

Answer b

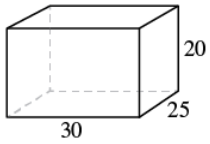
792 sq. ft

✔ Example 8.3.2:

A rectangular crate has a length of 30 inches, width of 25 inches, and height of 20 inches. Find its (a) volume and (b) surface area.

Solution

Step 1 is the same for both (a) and (b), so we will show it just once.

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

(a)

Step 2. **Identify** what you are looking for.	the volume of the crate	
Step 3. **Name**. Choose a variable to represent it.	let V = volume	
Step 4. **Translate**. Write the appropriate formula. Substitute.	$V = LWH$ $V = 30 \cdot 25 \cdot 20$	
Step 5. **Solve** the equation.	$V = 15,000$	(8.3.6)
Step 6. **Check.**	Double check your math.	

Step 7. **Answer** the question.	The volume is 15,000 cubic inches.

(b)

Step 2. **Identify** what you are looking for.	the surface area of the crate
Step 3. **Name**. Choose a variable to represent it.	the surface area of the crate
Step 4. **Translate**. Write the appropriate formula. Substitute.	$S = 2LH + 2LW + 2WH$ $S = 2(30 \cdot 20) + 2(30 \cdot 25) + 2(25 \cdot 20)$
Step 5. **Solve** the equation.	$S = 3,700$ (8.3.7)
Step 6. **Check.**	Check it yourself!
Step 7. **Answer** the question.	The surface area is 3,700 square inches.

Exercise 8.3.3:

A rectangular box has length 9 feet, width 4 feet, and height 6 feet. Find its (a) volume and (b) surface area.

Answer a

216 cu. ft

Answer b

228 sq. ft

Exercise 8.3.4:

A rectangular suitcase has length 22 inches, width 14 inches, and height 9 inches. Find its (a) volume and (b) surface area.

Answer a

2,772 cu. in

Answer b

1,264 sq. in.

Volume and Surface Area of a Cube

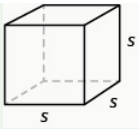
A **cube** is a rectangular solid whose length, width, and height are equal. See Volume and Surface Area of a Cube, below. Substituting, s for the length, width and height into the formulas for volume and surface area of a rectangular solid, we get:

$$\begin{aligned} V &= LWH & S &= 2LH + 2LW + 2WH \\ V &= s \cdot s \cdot s & S &= 2s \cdot s + 2s \cdot s + 2s \cdot s \\ V &= s^3 & S &= 2s^2 + 2s^2 + 2s^2 \\ & & S &= 6s^2 \end{aligned}$$

So for a cube, the formulas for volume and surface area are $V = s^3$ and $S = 6s^2$.

Definition: Volume and Surface Area of a Cube

For any cube with sides of length s,



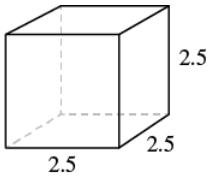
Volume: $V = s^3$
Surface Area: $S = 6s^2$

Example 8.3.3:

A cube is 2.5 inches on each side. Find its (a) volume and (b) surface area.

Solution

Step 1 is the same for both (a) and (b), so we will show it just once.

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

(a)

Step 2. **Identify** what you are looking for.	the volume of the crate	
Step 3. **Name**. Choose a variable to represent it.	let V = volume	
Step 4. **Translate**. Write the appropriate formula.	$V = s^3$	(8.3.8)
Step 5. **Solve.** Substitute and solve.	$V = (2.5)^3$ $V = 15.625$	
Step 6. **Check.**	Check your work.	
Step 7. **Answer** the question.	The volume is 15.625 cubic inches.	

(b)

Step 2. **Identify** what you are looking for.	the surface area of the crate	
Step 3. **Name**. Choose a variable to represent it.	the surface area of the crate	
Step 4. **Translate**. Write the appropriate formula.	$S = 6s^2$	(8.3.9)
Step 5. **Solve.** Substitute and solve.	$S = 6 \cdot (2.5)^2$ $S = 37.5$	
Step 6. **Check.**	The check is left to you.	
Step 7. **Answer** the question.	The surface area is 37.5 square inches.	

Exercise 8.3.5:

For a cube with side 4.5 meters, find the (a) volume and (b) surface area of the cube.

Answer a

91.125 cu. m

Answer b

121.5 sq. m

Exercise 8.3.6:

For a cube with side 7.3 yards, find the (a) volume and (b) surface area of the cube.

Answer a

389.017 cu. yd.

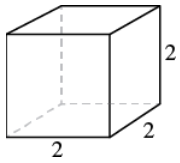
Answer b

319.74 sq. yd.

Example 8.3.4:

A notepad cube measures 2 inches on each side. Find its (a) volume and (b) surface area.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

(a)

Step 2. **Identify** what you are looking for.	the volume of the crate	
Step 3. **Name**. Choose a variable to represent it.	let V = volume	
Step 4. **Translate**. Write the appropriate formula.	$V = s^3$	(8.3.10)
Step 5. **Solve** the equation.	$V = 2^3$ $V = 8$	
Step 6. **Check.**	Check that you did the calculations correctly.	
Step 7. **Answer** the question.	The volume is 8 cubic inches	

(b)

Step 2. **Identify** what you are looking for.	the surface area of the crate	
Step 3. **Name**. Choose a variable to represent it.	the surface area of the crate	
Step 4. **Translate**. Write the appropriate formula.	$S = 6s^2$	(8.3.11)
Step 5. **Solve** the equation.	$S = 6 \cdot 2^2$ $S = 24$	
Step 6. **Check.**	The check is left to you.	
Step 7. **Answer** the question.	The surface area is 24 square inches.	

Exercise 8.3.7:

A packing box is a cube measuring 4 feet on each side. Find its (a) volume and (b) surface area.

Answer a

64 cu. ft

Answer b

96 sq. ft

Exercise 8.3.8:

A wall is made up of cube-shaped bricks. Each cube is 16 inches on each side. Find the (a) volume and (b) surface area of each cube.

Answer a

4,096 cu. in.

Answer b

1,536

Contributors and Attributions

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8.4: Use the Rectangular Coordinate System (Part 2)

Verify Solutions to an Equation in Two Variables

All the equations we solved so far have been equations with one variable. In almost every case, when we solved the equation we got exactly one solution. The process of solving an equation ended with a statement such as x = 4. Then we checked the solution by substituting back into the equation.

Here's an example of a linear equation in one variable, and its one solution.

$$\begin{aligned} 3x+5 &= 17 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

But equations can have more than one variable. Equations with two variables can be written in the general form Ax + By = C. An equation of this form is called a linear equation in two variables.

Definition: Linear Equation

An equation of the form Ax + By = C, where A and B are not both zero, is called a linear equation in two variables.

Notice that the word "line" is in linear.

Here is an example of a linear equation in two variables, x and y:

$$\begin{aligned} Ax+By &= C \\ x+4y &= 8 \\ A=1,\ B=4,\ C &= 8 \end{aligned}$$

Is y = −5x + 1 a linear equation? It does not appear to be in the form Ax + By = C. But we could rewrite it in this form.

	$y=-5x+1$	(8.4.1)
Add 5x to both sides.	$y+5x=-5x+1+5x$	(8.4.2)
Simplify.	$y+5x=1$	(8.4.3)
Use the Commutative Property to put it in Ax + By = C.	$\begin{aligned} Ax+By &= C \\ 5+\ \ y &= 1 \end{aligned}$	

By rewriting y = −5x + 1 as 5x + y = 1, we can see that it is a linear equation in two variables because it can be written in the form Ax + By = C.

Linear equations in two variables have infinitely many solutions. For every number that is substituted for x, there is a corresponding y value. This pair of values is a **solution to the linear equation** and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement because the value on the left side is equal to the value on the right side.

Definition: Solution to a Linear Equation in Two Variables

An ordered pair (x, y) is a solution to the linear equation Ax + By = C, if the equation is a true statement when the x- and y-values of the ordered pair are substituted into the equation.

Example 8.4.8:

Determine which ordered pairs are solutions of the equation x + 4y = 8: (a) (0, 2) (b) (2, −4) (c) (−4, 3)

Solution

Substitute the x- and y-values from each ordered pair into the equation and determine if the result is a true statement.

(a) (0, 2)	(b) (2, −4)	(c) (−4, 3)
$x = 0,\ y = 2$ $x + 4y = 8$ $0 + 4 \cdot 2 \stackrel{?}{=} 8$ $0 + 8 \stackrel{?}{=} 8$ $8 = 8$ ✓	$x = 2,\ y = -4$ $x + 4y = 8$ $2 + 4(-4) \stackrel{?}{=} 8$ $2 + (-16) \stackrel{?}{=} 8$ $-14 \neq 8$	$x = -4,\ y = 3$ $x + 4y = 8$ $-4 + 4 \cdot 3 \stackrel{?}{=} 8$ $-4 + 12 \stackrel{?}{=} 8$ $8 = 8$ ✓
(0, 2) is a solution.	(2, −4) is not a solution.	(−4, 3) is a solution.

Exercise 8.4.15:

Determine which ordered pairs are solutions to the given equation: 2x + 3y = 6

(a) (3, 0) (b) (2, 0) (c) (6, −2)

Answer

(a), (c)

Exercise 8.4.16:

Determine which ordered pairs are solutions to the given equation: 4x − y = 8

(a)(0, 8) (b) (2, 0) (c) (1, −4)

Answer

(b), (c)

Example 8.4.9 :

Determine which ordered pairs are solutions of the equation. y = 5x − 1: (a) (0, −1) (b) (1, 4) (c) (−2, −7)

Solution

Substitute the x- and y-values from each ordered pair into the equation and determine if it results in a true statement.

(a) (0, -1)	(b) (1, 4)	(c) (−2, -7)
$x = 0,\ y = -1$ $y = 5x - 1$ $-1 \stackrel{?}{=} 5(0) - 1$ $-1 \stackrel{?}{=} 0 - 1$ $-1 = -1$ ✓	$x = 1,\ y = 4$ $y = 5x - 1$ $4 \stackrel{?}{=} 5(1) - 1$ $4 \stackrel{?}{=} 5 - 1$ $4 = 4$ ✓	$x = -2,\ y = -7$ $y = 5x - 1$ $-7 \stackrel{?}{=} 5(-2) - 1$ $-7 \stackrel{?}{=} -10 - 1$ $-7 \neq -11$
(0, -1) is a solution.	(1, 4) is a solution.	(−2, -7) is not a solution.

Exercise 8.4.17:

Determine which ordered pairs are solutions of the given equation: y = 4x − 3

(a) (0, 3) (b) (1, 1) (c) (1, 1)

Answer

(b)

Exercise 8.4.18:

Determine which ordered pairs are solutions of the given equation: y = −2x + 6

(a)(0, 6) (b) (1, 4) (c) (−2, −2)

Answer

(a), (b)

Complete a Table of Solutions to a Linear Equation

In the previous examples, we substituted the x- and y-values of a given ordered pair to determine whether or not it was a solution to a linear equation. But how do we find the ordered pairs if they are not given? One way is to choose a value for x and then solve the equation for y. Or, choose a value for y and then solve for x.

We'll start by looking at the solutions to the equation y = 5x − 1 we found in Example 8.4.1. We can summarize this information in a table of solutions.

y = 5x − 1		
x	**y**	**(x, y)**
0	-1	(0, -1)
1	4	(1, 4)

To find a third solution, we'll let x = 2 and solve for y.

	$y = 5x - 1$	(8.4.4)
Substitute x = 2.	$y = 5(2) - 1$	(8.4.5)
Multiply.	$y = 10 - 1$	(8.4.6)
Simplify.	$y = 9$	(8.4.7)

The ordered pair is a solution to y = 5x - 1. We will add it to the table.

y = 5x − 1		
x	**y**	**(x, y)**
0	-1	(0, -1)
1	4	(1, 4)
2	9	(2, 9)

We can find more solutions to the equation by substituting any value of x or any value of y and solving the resulting equation to get another ordered pair that is a solution. There are an infinite number of solutions for this equation.

Example 8.4.10:

Complete the table to find three solutions to the equation $y = 4x - 2$:

y = 4x - 2		
x	**y**	**(x, y)**
0		
-1		
2		

Solution

Substitute $x = 0$, $x = -1$, and $x = 2$ into $y = 4x - 2$.

$x = 0$	$x = -1$	$x = 2$
$y = 4x - 2$	$y = 4x - 2$	$y = 4x - 2$
$y = 4 \cdot 0 - 2$	$y = 4(-1) - 2$	$y = 4 \cdot 2 - 2$
$y = 0 - 2$	$y = -4 - 2$	$y = 8 - 2$
$y = -2$	$y = -6$	$y = 6$
$(0, -2)$	$(-1, -6)$	$(2, 6)$

The results are summarized in the table.

y = 4x - 2		
x	**y**	**(x, y)**
0	-2	(0, -2)
-1	-6	(-1, -6)
2	6	(2, 6)

Exercise 8.4.19:

Complete the table to find three solutions to the equation: $y = 3x - 1$.

y = 3x − 1		
x	**y**	**(x, y)**
0		
-1		
2		

Answer

y = 3x − 1

y = 3x − 1		
x	y	(x, y)
0	-1	(0, -1)
-1	-4	(-1, -4)
2	5	(2, 5)

Exercise 8.4.20:

Complete the table to find three solutions to the equation: y = 6x + 1.

y = 6x + 1		
x	y	(x, y)
0		
1		
-2		

Answer

y = 6x + 1		
x	y	(x, y)
0	1	(0, 1)
-1	7	(1, 7)
2	-11	(-2, -11)

Example 8.4.11:

Complete the table to find three solutions to the equation 5x − 4y = 20:

5x − 4y = 20		
x	y	(x, y)
0		
	0	
	5	

Solution

$x = 0$ $5x - 4y = 20$ $5 \cdot 0 - 4y = 20$ $0 - 4y = 20$ $-4y = 20$ $y = -5$ $(0, -5)$	$y = 0$ $5x - 4y = 20$ $5x - 4 \cdot 0 = 20$ $5x - 0 = 20$ $5x = 20$ $x = 4$ $(4, 0)$	$y = 5$ $5x - 4y = 20$ $5x - 4 \cdot 5 = 20$ $5x - 20 = 20$ $5x = 40$ $x = 8$ $(8, 5)$

The results are summarized in the table.

5x − 4y = 20		
x	**y**	**(x, y)**
0	-5	(0, -5)
4	0	(-4, 0)
8	5	(8, 5)

Exercise 8.4.21:

Complete the table to find three solutions to the equation: 2x − 5y = 20.

2x - 5y = 20		
x	**y**	**(x, y)**
0		
	0	
-5		

Answer

2x - 5y = 20		
x	**y**	**(x, y)**
0	-4	(0, -4)
10	0	(10, 0)
-5	-6	(-5, -6)

Exercise 8.4.22:

Complete the table to find three solutions to the equation: 3x − 4y = 12.

3x - 4y = 12		
x	**y**	**(x, y)**
0		
	0	
-4		

Answer

3x - 4y = 12		
x	**y**	**(x, y)**
0	-3	(0, -3)
4	0	(4, 0)
-4	-6	(-4, -6)

Find Solutions to Linear Equations in Two Variables

To find a solution to a linear equation, we can choose any number we want to substitute into the equation for either x or y. We could choose 1, 100, 1,000, or any other value we want. But it's a good idea to choose a number that's easy to work with. We'll usually choose 0 as one of our values.

Example 8.4.12:

Find a solution to the equation 3x + 2y = 6.

Solution

Step 1: Choose any value for one of the variables in the equation.		We can substitute any value we want for x or any value for y. Let's pick x = 0. What is the value of y if x = 0?
Step 2: Substitute that value into the equation. Solve for the other variable.	Substitute 0 for x. Simplify. Divide both sides by 2.	$3x+2y=6$ $3\cdot 0+2y=6$ $0+2y=6$ $2y=6$ $y=3$
Step 3: Write the solution as an ordered pair.	So, when x = 0, y = 3.	This solution is represented by the ordered pair (0, 3).
Step 4: Check.	Substitute x = 0, y = 3 into the equation 3x + 2y = 6. Is the result a true equation? Yes!	$3x+2y=6$ $3\cdot 0+2\cdot 3 \stackrel{?}{=} 6$ $0+6 \stackrel{?}{=} 6$ $6=6$ ✓

Exercise 8.4.23:

Find a solution to the equation: 4x + 3y = 12.

Answer

Answers will vary

Exercise 8.4.24:

Find a solution to the equation: 2x + 4y = 8.

Answer

Answers will vary

We said that linear equations in two variables have infinitely many solutions, and we've just found one of them. Let's find some other solutions to the equation 3x + 2y = 6.

Example 8.4.13:

Find three more solutions to the equation 3x + 2y = 6.

Solution

To find solutions to 3x + 2y = 6, choose a value for x or y. Remember, we can choose any value we want for x or y. Here we chose 1 for x, and 0 and −3 for y.

Substitute it into the equation.	$y = 0$ $3x + 2y = 6$ $3x + 2(0) = 6$	$x = 1$ $3x + 2y = 6$ $3(1) + 2y = 6$	$y = -3$ $3x + 2y = 6$ $3x + 2(-3) = 6$
Simplify. Solve.	$3x + 0 = 6$ $3x = 6$	$3 + 2y = 6$ $2y = 3$	$3x - 6 = 6$ $3x = 12$
	x = 2	y = $\frac{3}{2}$	x = 4
Write the ordered pair.	(2, 0)	(1, $\frac{3}{2}$)	(4, −3)

Check your answers.

(2, 0)	(1, $\frac{3}{2}$)	(4, −3)
$3x + 2y = 6$ $3 \cdot 2 + 2 \cdot 0 \stackrel{?}{=} 6$ $6 + 0 \stackrel{?}{=} 6$ $6 = 6$ ✓	$3x + 2y = 6$ $3 \cdot 1 + 2 \cdot \frac{3}{2} \stackrel{?}{=} 6$ $3 + 3 \stackrel{?}{=} 6$ $6 = 6$ ✓	$3x + 2y = 6$ $3 \cdot 4 + 2 \cdot (-3) \stackrel{?}{=} 6$ $12 + (-6) \stackrel{?}{=} 6$ $6 = 6$ ✓

So (2, 0), (1, $\frac{3}{2}$) and (4, −3) are all solutions to the equation 3x + 2y = 6. In the previous example, we found that (0, 3) is a solution, too. We can list these solutions in a table.

3x + 2y = 6		
x	**y**	**(x, y)**
0	3	(0, 3)
2	0	(2, 0)
1	$\frac{3}{2}$	(1, $\frac{3}{2}$)
4	-3	(4, -3)

? Exercise 8.4.25:

Find three solutions to the equation: 2x + 3y = 6.

Answer

Answers will vary

? Exercise 8.4.26:

Find three solutions to the equation: 4x + 2y = 8.

Answer

Answers will vary

Let’s find some solutions to another equation now.

Example 8.4.14:

Find three solutions to the equation x − 4y = 8.

Solution

Choose a value for x or y.	x = 0	y = 0	y = 3
Substitute it into the equation.	$0-4y=8 \quad (8.4.8)$	$x-4\cdot 0=8 \quad (8.4.9)$	$x-4\cdot 3=8 \quad (8.4.10)$
Solve.	$-4y=8$ $y=-2$	$x-0=8$ $x=8$	$x-12=8$ $x=20$
Write the ordered pair.	(0, −2)	(8, 0)	(20, 3)

So (0, −2), (8, 0), and (20, 3) are three solutions to the equation x − 4y = 8.

x − 4y = 8		
x	**y**	**(x, y)**
0	-2	(0, -2)
8	0	(8, 0)
20	3	(20, 3)

Remember, there are an infinite number of solutions to each linear equation. Any point you find is a solution if it makes the equation true.

Exercise 8.4.27:

Find three solutions to the equation: 4x + y = 8.

Answer

Answers will vary

Exercise 8.4.28:

Find three solutions to the equation: x + 5y = 10.

Answer

Answers will vary

ACCESS ADDITIONAL ONLINE RESOURCES

Plotting Points

Identifying Quadrants

Verifying Solution to Linear Equation

Practice Makes Perfect

Plot Points on a Rectangular Coordinate System

In the following exercises, plot each point on a coordinate grid.

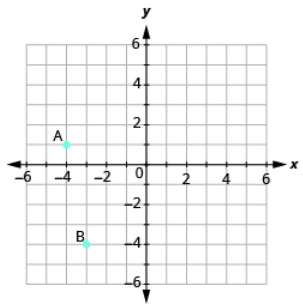
1. (3, 2)
2. (4, 1)
3. (1, 5)
4. (3, 4)
5. (4, 1), (1, 4)
6. (3, 2), (2, 3)
7. (3, 4), (4, 3)

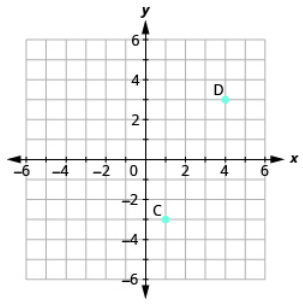
In the following exercises, plot each point on a coordinate grid and identify the quadrant in which the point is located.

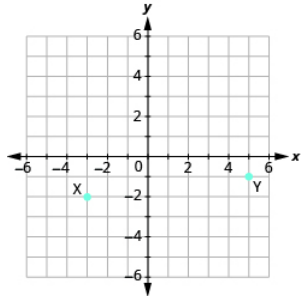
8. (a) (−4, 2) (b) (−1, −2) (c) (3, −5) (d) $\left(2, \frac{5}{2}\right)$
9. (a) (−2, −3) (b) (3, −3) (c) (−4, 1) (d) $\left(1, \frac{3}{2}\right)$
10. (a) (−1, 1) (b) (−2, −1) (c) (1, −4) (d) $\left(3, \frac{7}{2}\right)$
11. (a) (3, -2) (b) (−3, 2) (c) (-3, −2) (d) (3, 2)
12. (a) (4, -1) (b) (−4, 1) (c) (-4, -1) (d) (4, 1)
13. (a) (−2, 0) (b) (−3, 0) (c) (0, 4) (d) (0, 2)

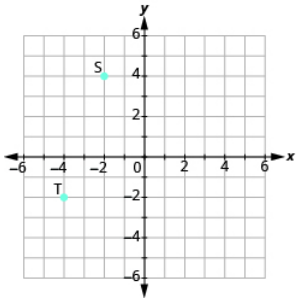
Identify Points on a Graph

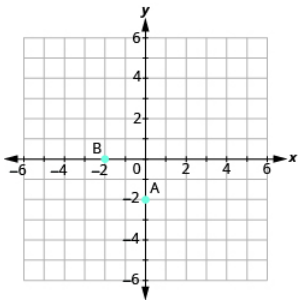
In the following exercises, name the ordered pair of each point shown.

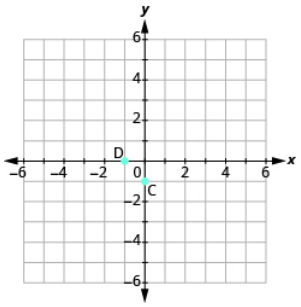
14. 

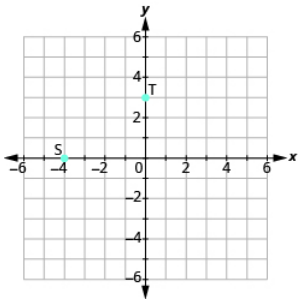
15. 

16. 

17. 

18. 

19. 

20. 

Verify Solutions to an Equation in Two Variables

In the following exercises, determine which ordered pairs are solutions to the given equation.

21. $2x + y = 6$
    a. (1, 4)
    b. (3, 0)
    c. (2, 3)

22. $x + 3y = 9$
    a. (0, 3)
    b. (6, 1)
    c. (-3, -3)

23. $4x - 2y = 8$
    a. (3, 2)
    b. (1, 4)
    c. (0, -4)

24. $3x - 2y = 12$
    a. (4, 0)
    b. (2, -3)
    c. (1, 6)
25. $y = 4x + 3$
    a. (4, 3)
    b. (-1, -1)
    c. $(\frac{1}{2}, 5)$
26. $y = 2x - 5$
    a. (0, -5)
    b. (2, 1)
    c. $(\frac{1}{2}, -4)$
27. $y = \frac{1}{2}x - 1$
    a. (2, 0)
    b. (-6, -4)
    c. (-4, -1)
28. $y = \frac{1}{3}x + 1$
    a. (-3, 0)
    b. (9, 4)
    c. (-6, -1)

Find Solutions to Linear Equations in Two Variables

In the following exercises, complete the table to find solutions to each linear equation.

$y = 2x - 4$

x	y	(x, y)
-1		
0		
2		

30. $y = 3x - 1$

x	y	(x, y)
-1		
0		
2		

31. $y = -x + 5$

x	y	(x, y)
-2		
0		
3		

32. $y = \frac{1}{3}x + 1$

x	y	(x, y)
0		
3		
6		

33. $y = -\frac{3}{2}x - 2$

x	y	(x, y)
-2		
0		
2		

34. $x + 2y = 8$

x	y	(x, y)
0		
4		
	0	

Everyday Math

35. **Weight of a baby** Mackenzie recorded her baby's weight every two months. The baby's age, in months, and weight, in pounds, are listed in the table, and shown as an ordered pair in the third column.
    a. Plot the points on a coordinate grid.
    b. Why is only Quadrant I needed?

Age	Weight	(x, y)
0	7	(0, 7)
2	11	(2, 11)
4	15	(4, 15)
6	16	(6, 16)
8	19	(8, 19)
10	20	(10, 20)
12	21	(12, 21)

36. Weight of a child Latresha recorded her son's height and weight every year. His height, in inches, and weight, in pounds, are listed in the table, and shown as an ordered pair in the third column.
    a. Plot the points on a coordinate grid.
    b. Why is only Quadrant I needed?

Height	Weight	(x, y)
28	22	(28, 22)

31	27	(31, 27)
33	33	(33, 33)
37	35	(37, 35)
40	41	(40, 41)
42	45	(42, 45)

Writing Exercises

37. Have you ever used a map with a rectangular coordinate system? Describe the map and how you used it.
38. How do you determine if an ordered pair is a solution to a given equation?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
plot points on a rectangular coordinate system.			
identify points on a graph.			
verify solutions to an equation in two variables.			
complete a table of solutions to a linear equation.			
find solutions to linear equations in two variables.			

(b) If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Contributors and Attributions

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8.5: Graphing Linear Equations (Part 1)

Learning Objectives

- Recognize the relation between the solutions of an equation and its graph
- Graph a linear equation by plotting points
- Graph vertical and horizontal lines

be prepared!

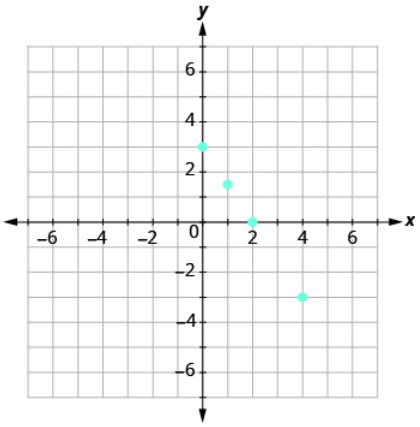
Before you get started, take this readiness quiz.

1. Evaluate: 3x + 2 when x = −1. If you missed this problem, review Example 3.8.10.
2. Solve the formula: 5x + 2y = 20 for y. If you missed this problem, review Example 9.11.6.
3. Simplify: $\frac{3}{8}(-24)$. If you missed this problem, review Example 4.3.10.

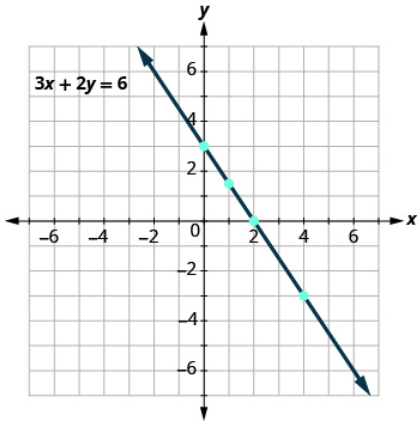
Recognize the Relation Between the Solutions of an Equation and its Graph

In Use the Rectangular Coordinate System, we found a few solutions to the equation 3x + 2y = 6. They are listed in the table below. So, the ordered pairs (0, 3), (2, 0), $\left(1, \frac{3}{2}\right)$, (4, − 3), are some solutions to the equation 3x + 2y = 6. We can plot these solutions in the rectangular coordinate system as shown on the graph at right.

$3x + 2y = 6$		
x	y	(x, y)
0	3	(0, 3)
2	0	(2, 0)
1	$\frac{3}{2}$	$\left(1, \frac{3}{2}\right)$
4	–3	(4, –3)



Notice how the points line up perfectly? We connect the points with a straight line to get the graph of the equation 3x + 2y = 6. Notice the arrows on the ends of each side of the line. These arrows indicate the line continues.



Every point on the line is a solution of the equation. Also, every solution of this equation is a point on this line. Points not on the line are *not* solutions!

Notice that the point whose coordinates are (− 2, 6) is on the line shown in Figure 8.5.1. If you substitute x = − 2 and y = 6 into the equation, you find that it is a solution to the equation.

Test (–2, 6):

$$3x + 2y = 6$$
$$3(-2) + 2(6) \stackrel{?}{=} 6$$
$$-6 + 12 \stackrel{?}{=} 6$$
$$6 = 6 \checkmark$$

So (–2, 6) is a solution to the equation.

What about (4, 1)?

$$3x + 2y = 6$$
$$3 \cdot 4 + 2 \cdot 1 \stackrel{?}{=} 6$$
$$12 + 2 \stackrel{?}{=} 6$$
$$14 \neq 6$$

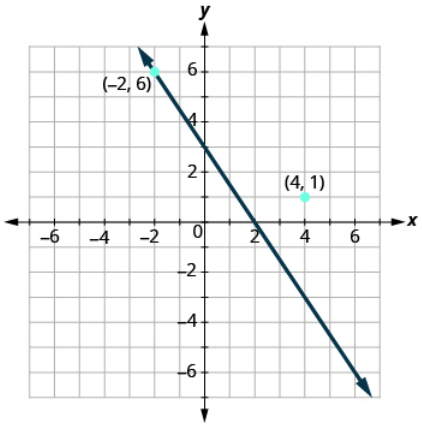


Figure 8.5.1

So (4, 1) is not a solution to the equation 3x + 2y = 6. Therefore the point (4, 1) is not on the line. This is an example of the saying," A picture is worth a thousand words." The line shows you all the solutions to the equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the *graph* of the equation 3x + 2y = 6.

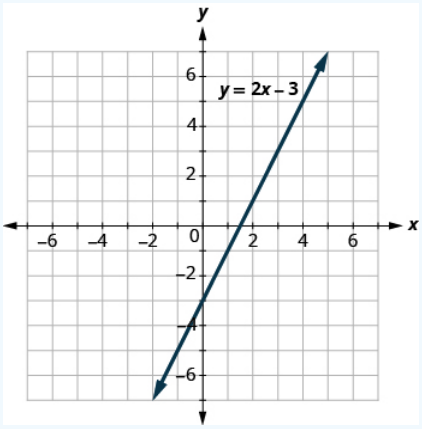
Definition: Graph of a Linear Equation

The graph of a linear equation Ax + By = C is a straight line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

Example 8.5.1:

The graph of y = 2x − 3 is shown below.



For each ordered pair decide (a) Is the ordered pair a solution to the equation? (b) Is the point on the line?

(a) (0, 3) (b) (3, − 3) (c) (2, − 3) (d) (− 1, − 5)

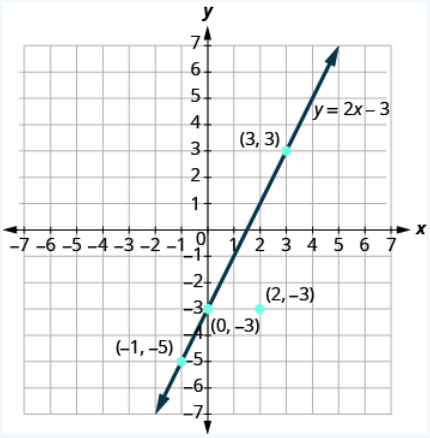
Solution

Substitute the x - and y -values into the equation to check if the ordered pair is a solution to the equation.

(a)

$(a)\ (0,-3)$	$(b)\ (3,3)$	$(c)\ (2,-3)$	$(d)\ (-1,-5)$
$y = 2x-3$	$y = 2x-3$	$y = 2x-3$	$y = 2x-3$
$-3 \stackrel{?}{=} 2(0)-3$	$3 \stackrel{?}{=} 2(3)-3$	$-3 \stackrel{?}{=} 2(2)-3$	$-5 \stackrel{?}{=} 2(-1)-3$
$-3 = -3$ ✓	$3 = 3$ ✓	$-3 \neq 1$	$-5 = -5$ ✓
$(0,-3)$ *is a solution.*	$(3,3)$ *is a solution.*	$(2,-3)$ *is not a solution.*	$(-1,-5)$ *is a solution.*

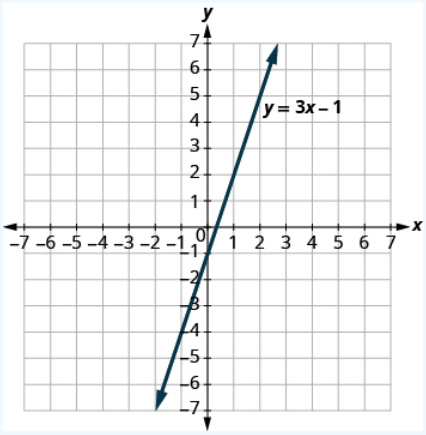
(b) Plot the points A: (0, − 3) B: (3, 3) C: (2, − 3) and D: (− 1, − 5). The points (0, − 3), (3, 3), and (− 1, − 5) are on the line y = 2x − 3, and the point (2, − 3) is not on the line.



The points which are solutions to y = 2x − 3 are on the line, but the point which is not a solution is not on the line.

? Exercise 8.5.1:

The graph of y = 3x − 1 is shown. For each ordered pair, decide (a) is the ordered pair a solution to the equation? (b) is the point on the line?



1. (0, − 1)
2. (2, 2)
3. (3, − 1)
4. (− 1, − 4)

Answer 1.

a. yes, b. no

Answer 2.

a. no, b. no

Answer 3.

a. no, b. no

Answer 4.

a. yes, b. yes

Graph a Linear Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The method we used at the start of this section to graph is called plotting points, or the Point-Plotting Method.

Let's graph the equation y = 2x + 1 by plotting points. We start by finding three points that are solutions to the equation. We can choose any value for x or y, and then solve for the other variable.

Since y is isolated on the left side of the equation, it is easier to choose values for x. We will use 0, 1, and -2 for x for this example. We substitute each value of x into the equation and solve for y.

$x = -2$	$x = 0$	$x = 1$
$y = 2x + 1$	$y = 2x + 1$	$y = 2x + 1$
$y = 2(-2) + 1$	$y = 2(0) + 1$	$y = 2(1) + 1$
$y = -4 + 1$	$y = 0 + 1$	$y = 2 + 1$
$y = -3$	$y = 1$	$y = 3$
$(-2, -3)$	$(0, 1)$	$(1, 3)$

We can organize the solutions in a table. See Table 8.5.1.

Table 8.5.1

y = 2x + 1		
x	**y**	**(x, y)**
0	1	(0, 1)
1	3	(1, 3)
-2	-3	(-2, -3)

Now we plot the points on a rectangular coordinate system. Check that the points line up. If they did not line up, it would mean we made a mistake and should double-check all our work. See Figure 8.5.2.

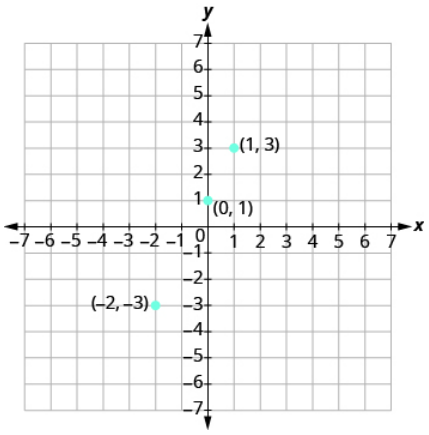


Figure 8.5.2

Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line. The line is the graph of y = 2x + 1.

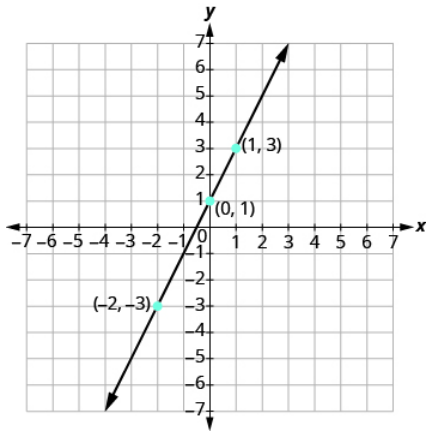


Figure 8.5.3

HOW TO: GRAPH A LINEAR EQUATION BY PLOTTING POINTS

Step 1. Find three points whose coordinates are solutions to the equation. Organize them in a table.

Step 2. Plot the points on a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.

Step 3. Draw the line through the points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you plot only two points and one of them is incorrect, you can still draw a line but it will not represent the solutions to the equation. It will be the wrong line. If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong and you need to check your work. See Figure 8.5.4.

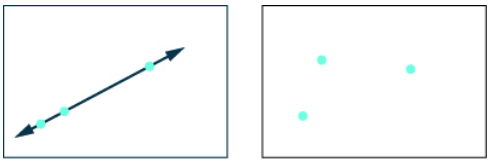


Figure 8.5.4 - Look at the difference between (a) and (b). All three points in (a) line up so we can draw one line through them. The three points in (b) do not line up. We cannot draw a single straight line through all three points.

Example 8.5.2:

Graph the equation y = −3x.

Solution

Find three points that are solutions to the equation. It's easier to choose values for x, and solve for y. Do you see why?

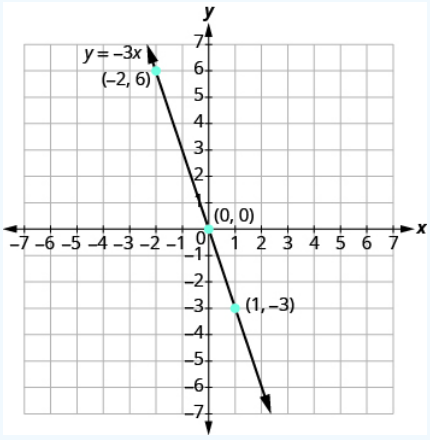
$x = 0$	$x = 1$	$x = -2$
$y = -3x$	$y = -3x$	$y = -3x$
$y = -3(0)$	$y = -3(1)$	$y = -3(-2)$
$y = 0$	$y = -3$	$y = 6$
(0, 0)	(1, −3)	(−2, 6)

List the points in a table.

y = −3x		
x	**y**	**(x, y)**

y = −3x		
0	0	(0, 0)
1	-3	(1, -3)
-2	6	(-2, 6)

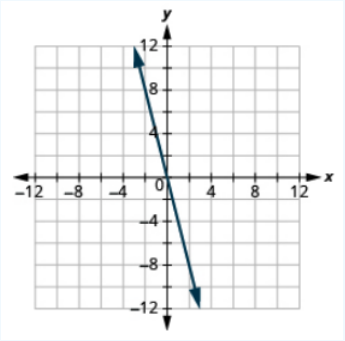
Plot the points, check that they line up, and draw the line as shown.



Exercise 8.5.2:

Graph the equation by plotting points: y = −4x.

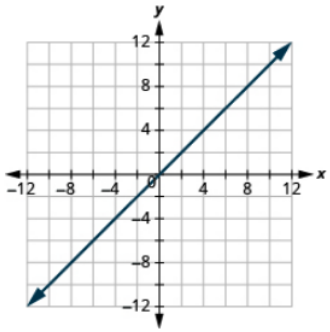
Answer



Exercise 8.5.3:

Graph the equation by plotting points: y = x.

Answer



When an equation includes a fraction as the coefficient of x, we can substitute any numbers for x. But the math is easier if we make 'good' choices for the values of x. This way we will avoid fraction answers, which are hard to graph precisely.

Example 8.5.3:

Graph the equation $y = \frac{1}{2}x + 3$.

Solution

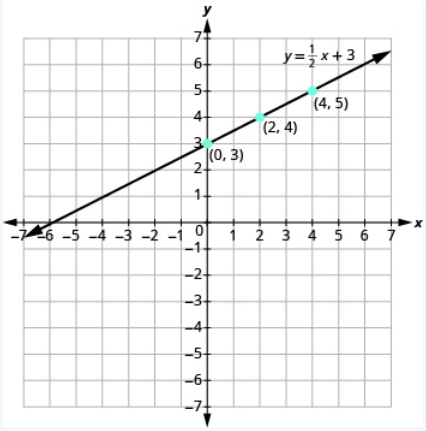
Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices.

$x = 0$	$x = 2$	$x = 4$
$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$
$y = \frac{1}{2}(0) + 3$	$y = \frac{1}{2}(2) + 3$	$y = \frac{1}{2}(4) + 3$
$y = 3$	$y = 4$	$y = 5$
(0, 3)	(2, 4)	(4, 5)

The points are shown in the table.

$y = \frac{1}{2}x + 3$		
x	y	(x, y)
0	3	(0, 3)
2	4	(2, 4)
4	5	(4, 5)

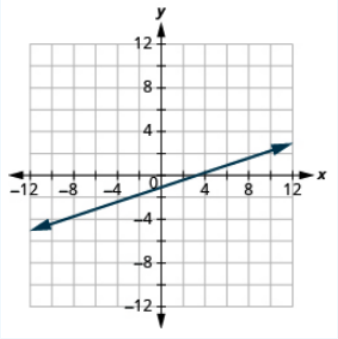
Plot the points, check that they line up, and draw the line as shown.



Exercise 8.5.4:

Graph the equation: $y = \frac{1}{3}x - 1$

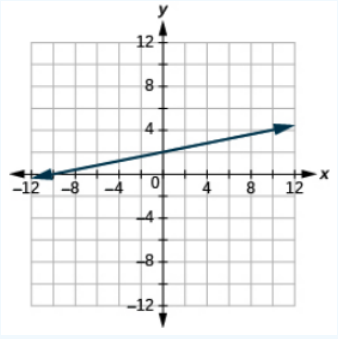
Answer



Exercise 8.5.5:

Graph the equation: $y = \frac{1}{4}x + 2$.

Answer



So far, all the equations we graphed had y given in terms of x. Now we'll graph an equation with x and y on the same side.

Example 8.5.4:

Graph the equation x + y = 5.

Solution

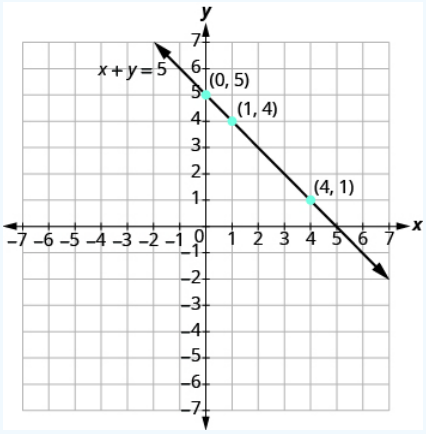
Find three points that are solutions to the equation. Remember, you can start with any value of x or y.

$x = 0$	$x = 1$	$x = 4$
$x + y = 5$	$x + y = 5$	$x + y = 5$
$0 + y = 5$	$1 + y = 5$	$4 + y = 5$
$y = 5$	$y = 4$	$y = 1$
(0, 5)	(1, 4)	(4, 1)

We list the points in a table.

x + y = 5		
x	y	(x, y)
0	5	(0, 5)
1	4	(1, 4)
4	1	(4, 1)

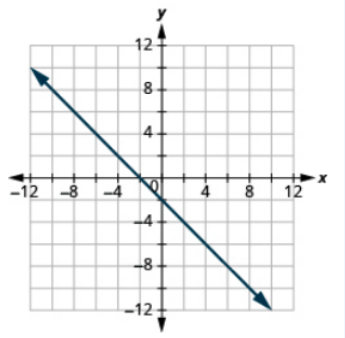
Then plot the points, check that they line up, and draw the line.



Exercise 8.5.6:

Graph the equation: x + y = −2.

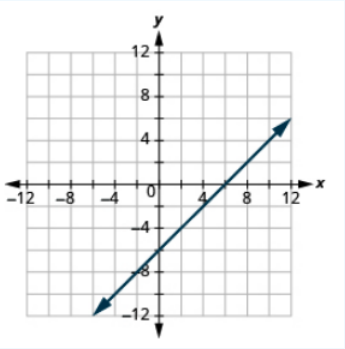
Answer



Exercise 8.5.7:

Graph the equation: x − y = 6.

Answer



In the previous example, the three points we found were easy to graph. But this is not always the case. Let's see what happens in the equation 2x + y = 3. If y is 0, what is the value of x?

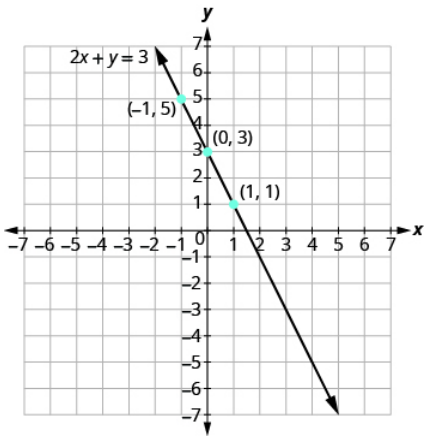
$$\begin{aligned} 2x+y &= 3 \\ 2x+0 &= 3 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

The solution is the point $\left(\frac{3}{2}, 0\right)$. This point has a fraction for the x -coordinate. While we could graph this point, it is hard to be precise graphing fractions. Remember in the example y = $\frac{1}{2}$x + 3, we carefully chose values for x so as not to graph fractions at all. If we solve the equation 2x + y = 3 for y, it will be easier to find three solutions to the equation.

$$\begin{aligned} 2x+y &= 3 \\ y &= -2x+3 \end{aligned}$$

Now we can choose values for x that will give coordinates that are integers. The solutions for x = 0, x = 1, and x = −1 are shown.

y = −2x + 3		
x	y	(x, y)
0	3	(0, 3)
1	1	(1, 1)
-1	5	(-1, 5)



Example 8.5.5:

Graph the equation 3x + y = −1.

Solution

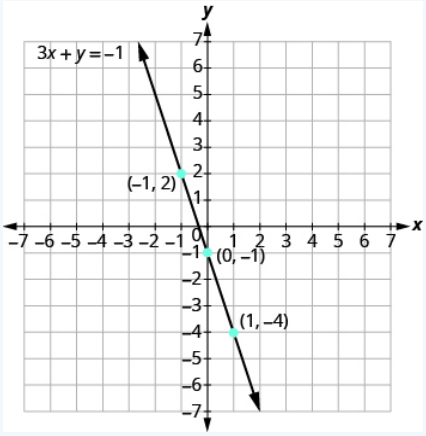
Find three points that are solutions to the equation.

First, solve the equation for y.

$$3x + y = -1$$
$$y = -3x - 1$$

We'll let x be 0, 1, and −1 to find three points. The ordered pairs are shown in the table. Plot the points, check that they line up, and draw the line.

y = −3x − 1		
x	**y**	**(x, y)**
0	-1	(0, -1)
1	-4	(1, -4)
-1	2	(-1, 2)

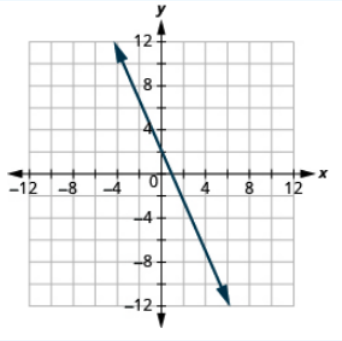


If you can choose any three points to graph a line, how will you know if your graph matches the one shown in the answers in the book? If the points where the graphs cross the x- and y -axes are the same, the graphs match.

Exercise 8.5.8:

Graph each equation: 2x + y = 2.

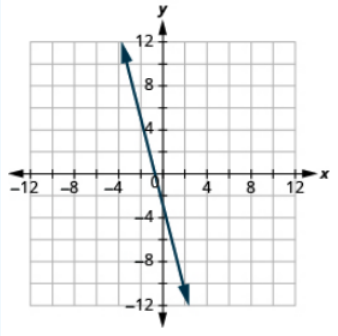
Answer



Exercise 8.5.9:

Graph each equation: 4x + y = −3.

Answer



Contributors and Attributions

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8.6: Solve Geometry Applications- Volume and Surface Area (Part 2)

Find the Volume and Surface Area of Spheres

A **sphere** is the shape of a basketball, like a three-dimensional circle. Just like a circle, the size of a sphere is determined by its radius, which is the distance from the center of the sphere to any point on its surface. The formulas for the volume and surface area of a sphere are given below.

Showing where these formulas come from, like we did for a rectangular solid, is beyond the scope of this course. We will approximate π with 3.14.

Definition: Volume and Surface Area of a Sphere

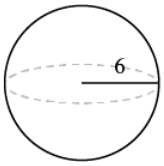
For a sphere with radius r:

Example 8.6.5:

A sphere has a radius 6 inches. Find its (a) volume and (b) surface area.

Solution

Step 1 is the same for both (a) and (b), so we will show it just once.

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

(a)

Step 2. **Identify** what you are looking for.	the volume of the sphere	
Step 3. **Name**. Choose a variable to represent it.	let V = volume	
Step 4. **Translate**. Write the appropriate formula.	$V = \frac{4}{3}\pi r^3$	(9.6.12)
Step 5. **Solve**.	$V \approx \frac{4}{3}(3.14)6^3$ $V \approx 904.32\ cubic\ inches$	
Step 6. **Check**.	Double-check your math on a calculator.	
Step 7. **Answer** the question.	The volume is approximately 904.32 cubic inches.	

(b)

Step 2. **Identify** what you are looking for.	the surface area of the cube	
Step 3. **Name**. Choose a variable to represent it.	let S = surface area	
Step 4. **Translate**. Write the appropriate formula.	$S = 4\pi r^2$	(9.6.13)

Step 5. **Solve**.	$S \approx 4(3.14)6^2$ $S \approx 452.16\ sq.\ inches$
Step 6. **Check**.	Double-check your math on a calculator.
Step 7. **Answer** the question.	The surface area is approximately 452.16 square inches.

? Exercise 8.6.9:

Find the (a) volume and (b) surface area of a sphere with radius 3 centimeters.

Answer a

113.04 cu. cm

Answer b

113.04 sq. cm

? Exercise 8.6.10:

Find the (a) volume and (b) surface area of each sphere with a radius of 1 foot

Answer a

4.19 cu. ft

Answer b

12.56 sq. ft

✓ Example 8.6.6:

A globe of Earth is in the shape of a sphere with radius 14 centimeters. Find its (a) volume and (b) surface area. Round the answer to the nearest hundredth.

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

(a)

Step 2. **Identify** what you are looking for.	the volume of the sphere
Step 3. **Name**. Choose a variable to represent it.	let V = volume
Step 4. **Translate**. Write the appropriate formula. Substitute. (Use 3.14 for π)	$V = \frac{4}{3}\pi r^3$ $V \approx \frac{4}{3}(3.14)14^3$
Step 5. **Solve**.	$V \approx 11,488.21$ $(9.6.14)$
Step 6. **Check**.	We leave it to you to check your calculations.

Step 7. **Answer** the question.	The volume is approximately 11,488.21 cubic inches.

(b)

Step 2. **Identify** what you are looking for.	the surface area of the sphere
Step 3. **Name**. Choose a variable to represent it.	let S = surface area
Step 4. **Translate**. Write the appropriate formula.	$S = 4\pi r^2$ $S \approx 4(3.14)14^2$
Step 5. **Solve**.	$S \approx 2461.76$ (9.6.15)
Step 6. **Check**.	We leave it to you to check your calculations.
Step 7. **Answer** the question.	The surface area is approximately 2461.76 square inches.

? Exercise 8.6.11:

A beach ball is in the shape of a sphere with radius of 9 inches. Find its (a) volume and (b) surface area.

Answer a

3052.08 cu. in

Answer b

1017.36 sq. in.

? Exercise 8.6.12:

A Roman statue depicts Atlas holding a globe with radius of 1.5 feet. Find the (a) volume and (b) surface area of the globe.

Answer a

14.13 cu. ft

Answer b

28.26 sq. ft

Find the Volume and Surface Area of a Cylinder

If you have ever seen a can of soda, you know what a cylinder looks like. A **cylinder** is a solid figure with two parallel circles of the same size at the top and bottom. The top and bottom of a cylinder are called the bases. The height h of a cylinder is the distance between the two bases. For all the cylinders we will work with here, the sides and the height, h, will be perpendicular to the bases.

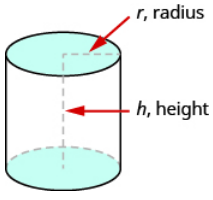


Figure 8.6.5 - A cylinder has two circular bases of equal size. The height is the distance between the bases.

Rectangular solids and cylinders are somewhat similar because they both have two bases and a height. The formula for the volume of a rectangular solid, V = Bh, can also be used to find the volume of a cylinder.

For the rectangular solid, the area of the base, B, is the area of the rectangular base, length × width. For a cylinder, the area of the base, B, is the area of its circular base, πr^2. Figure 8.6.6 compares how the formula V = Bh is used for rectangular solids and cylinders.

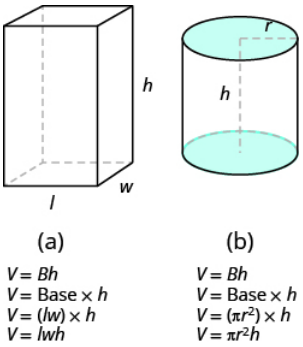


Figure 8.6.6 - Seeing how a cylinder is similar to a rectangular solid may make it easier to understand the formula for the volume of a cylinder.

To understand the formula for the surface area of a cylinder, think of a can of vegetables. It has three surfaces: the top, the bottom, and the piece that forms the sides of the can. If you carefully cut the label off the side of the can and unroll it, you will see that it is a rectangle. See Figure 8.6.7.

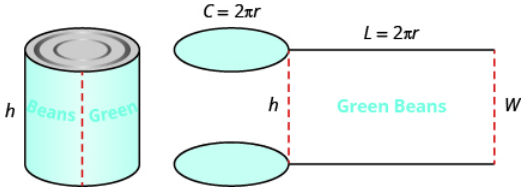
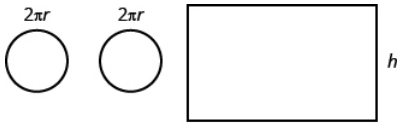


Figure 8.6.7 - By cutting and unrolling the label of a can of vegetables, we can see that the surface of a cylinder is a rectangle. The length of the rectangle is the circumference of the cylinder's base, and the width is the height of the cylinder.

The distance around the edge of the can is the **circumference** of the cylinder's base it is also the length L of the rectangular label. The height of the cylinder is the width W of the rectangular label. So the area of the label can be represented as

$$A = L \cdot W$$
$$A = 2\pi r \cdot h$$

To find the total surface area of the cylinder, we add the areas of the two circles to the area of the rectangle.



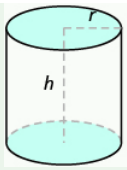
$$S = A_{\text{top circle}} + A_{\text{bottom circle}} + A_{\text{rectangle}}$$
$$S = \underbrace{\pi r^2 + \pi r^2} + 2\pi r \cdot h$$
$$S = 2 \cdot \pi r^2 + 2\pi rh$$
$$S = 2\pi r^2 + 2\pi rh$$

The surface area of a cylinder with radius r and height h, is

$$S = 2\pi r^2 + 2\pi rh \tag{9.6.16}$$

Definition: Volume and Surface Area of a Cylinder

For a cylinder with radius r and height h:

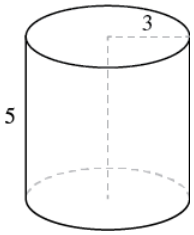


Volume: $V = \pi r^2 h$ or $V = Bh$
Surface Area: $S = 2\pi r^2 + 2\pi rh$

Example 8.6.7:

A cylinder has height 5 centimeters and radius 3 centimeters. Find the (a) volume and (b) surface area.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

(a)

Step 2. **Identify** what you are looking for.	the volume of the cylinder	
Step 3. **Name**. Choose a variable to represent it.	let V = volume	
Step 4. **Translate**. Write the appropriate formula. Substitute. (Use 3.14 for π)	$V = \pi r^2 h$ $V \approx (3.14)3^2 \cdot 5$	
Step 5. **Solve**.	$V \approx 141.3$	$(9.6.17)$
Step 6. **Check**.	We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The volume is approximately 141.3 cubic inches.	

(b)

Step 2. **Identify** what you are looking for.	the surface area of the cylinder	
Step 3. **Name**. Choose a variable to represent it.	let S = surface area	
Step 4. **Translate**. Write the appropriate formula.	$S = 2\pi r^2 + 2\pi rh$ $S \approx 2(3.14)3^2 + 2(3.14)(3)5$	
Step 5. **Solve**.	$S \approx 150.72$	$(9.6.18)$
Step 6. **Check**.	We leave it to you to check your calculations.	
Step 7. **Answer** the question.	The surface area is approximately 150.72 square inches.	

Exercise 8.6.13:

Find the (a) volume and (b) surface area of the cylinder with radius 4 cm and height 7 cm.

Answer a

351.68 cu. cm

Answer b

276.32 sq. ft

Exercise 8.6.14:

Find the (a) volume and (b) surface area of the cylinder with given radius 2 ft and height 8 ft.

Answer a

100.48 cu. ft

Answer b

125.6 sq. ft

Example 8.6.8:

Find the (a) volume and (b) surface area of a can of soda. The radius of the base is 4 centimeters and the height is 13 centimeters. Assume the can is shaped exactly like a cylinder.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

(a)

Step 2. **Identify** what you are looking for.	the volume of the cylinder	
Step 3. **Name**. Choose a variable to represent it.	let V = volume	
Step 4. **Translate**. Write the appropriate formula. Substitute. (Use 3.14 for π)	$V = \pi r^2 h$ $V \approx (3.14)4^2 \cdot 13$	
Step 5. **Solve**.	$V \approx 653.12$	(9.6.19)
Step 6. **Check**.	We leave it to you to check.	
Step 7. **Answer** the question.	The volume is approximately 653.12 cubic centimeters.	

(b)

Step 2. **Identify** what you are looking for.	the surface area of the cylinder	
Step 3. **Name**. Choose a variable to represent it.	let S = surface area	
Step 4. **Translate**. Write the appropriate formula.	$S = 2\pi r^2 + 2\pi r h$ $S \approx 2(3.14)4^2 + 2(3.14)(4)13$	
Step 5. **Solve**.	$S \approx 427.04$	(9.6.20)

Step 6. **Check**.	We leave it to you to check your calculations.
Step 7. **Answer** the question.	The surface area is approximately 427.04 square centimeters.

? Exercise 8.6.15:

Find the (a) volume and (b) surface area of a can of paint with radius 8 centimeters and height 19 centimeters. Assume the can is shaped exactly like a cylinder.

Answer a

3,818.24 cu. cm

Answer b

1,356.48 sq. cm

? Exercise 8.6.16:

Find the (a) volume and (b) surface area of a cylindrical drum with radius 2.7 feet and height 4 feet. Assume the drum is shaped exactly like a cylinder.

Answer a

91.5624 cu. ft

Answer b

113.6052 sq. ft

Find the Volume of Cones

The first image that many of us have when we hear the word ‘cone’ is an ice cream cone. There are many other applications of cones (but most are not as tasty as ice cream cones). In this section, we will see how to find the volume of a cone.

In geometry, a **cone** is a solid figure with one circular base and a vertex. The height of a cone is the distance between its base and the vertex.The cones that we will look at in this section will always have the height perpendicular to the base. See Figure 8.6.8.

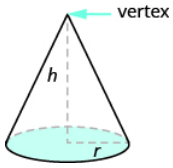


Figure 8.6.8 - The height of a cone is the distance between its base and the vertex.

Earlier in this section, we saw that the volume of a cylinder is $V = \pi r^2 h$. We can think of a cone as part of a cylinder. Figure 8.6.9 shows a cone placed inside a cylinder with the same height and same base. If we compare the volume of the cone and the cylinder, we can see that the volume of the cone is less than that of the cylinder.

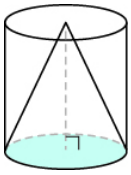


Figure 8.6.9 - The volume of a cone is less than the volume of a cylinder with the same base and height.

In fact, the volume of a cone is exactly one-third of the volume of a cylinder with the same base and height. The volume of a cone is

$$V = \frac{1}{3}Bh \tag{9.6.21}$$

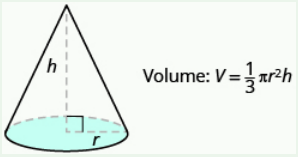
Since the base of a cone is a circle, we can substitute the formula of area of a circle, πr^2, for B to get the formula for volume of a cone.

$$V = \frac{1}{3}\pi r^2 h \tag{9.6.22}$$

In this book, we will only find the volume of a cone, and not its surface area.

Definition: Volume of a Cone

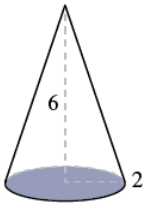
For a cone with radius r and height h .



Example 8.6.9:

Find the volume of a cone with height 6 inches and radius of its base 2 inches.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the volume of the cone
Step 3. **Name**. Choose a variable to represent it.	let V = volume
Step 4. **Translate**. Write the appropriate formula. Substitute. (Use 3.14 for π)	$V = \frac{1}{3}\pi r^2 h$ $V \approx \frac{1}{3}(3.14)(2)^2(6)$
Step 5. **Solve**.	$V \approx 25.12$ (9.6.23)
Step 6. **Check**.	We leave it to you to check your calculations.
Step 7. **Answer** the question.	The volume is approximately 25.12 cubic inches.

Exercise 8.6.17:

Find the volume of a cone with height 7 inches and radius 3 inches

Answer

65.94 cu. in.

Exercise 8.6.18:

Find the volume of a cone with height 9 centimeters and radius 5 centimeters

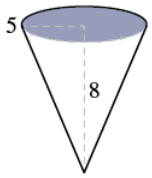
Answer

235.5 cu. cm

Example 8.6.10:

Marty's favorite gastro pub serves French fries in a paper wrap shaped like a cone. What is the volume of a conic wrap that is 8 inches tall and 5 inches in diameter? Round the answer to the nearest hundredth.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the volume of the cone
Step 3. **Name**. Choose a variable to represent it.	let V = volume
Step 4. **Translate**. Write the appropriate formula. Substitute. (Use 3.14 for π, and notice that we were given the distance across the circle, which is its diameter. The radius is 2.5 inches.)	$V = \frac{1}{3}\pi r^2 h$ $V \approx \frac{1}{3}(3.14)(2.5)^2(8)$
Step 5. **Solve**.	$V \approx 52.33$ $(9.6.24)$
Step 6. **Check**.	We leave it to you to check your calculations.
Step 7. **Answer** the question.	The volume of the wrap is approximately 52.33 cubic inches.

Exercise 8.6.19:

How many cubic inches of candy will fit in a cone-shaped piñata that is 18 inches long and 12 inches across its base? Round the answer to the nearest hundredth.

Answer

678.24 cu. in.

Exercise 8.6.20:

What is the volume of a cone-shaped party hat that is 10 inches tall and 7 inches across at the base? Round the answer to the nearest hundredth.

Answer

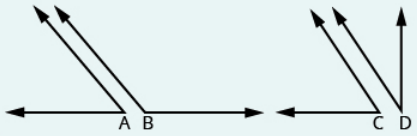
128.2 cu. in.

Summary of Geometry Formulas

The following charts summarize all of the formulas covered in this chapter.

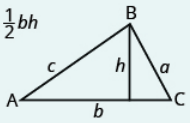
Supplementary and Complementary Angles

$m\angle A + m\angle B = 180°$ for supplementary angles A and B
$m\angle C + m\angle D = 90°$ for complementary angles C and D



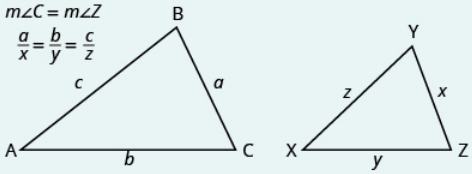
Triangle

For ΔABC, angle measures.
$m\angle A + m\angle B + m\angle C = 180°$
Perimeter. $P = a + b + c$
$A = \frac{1}{2}bh$



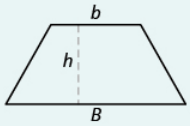
Similar Triangles

$m\angle A = m\angle X$
$m\angle B = m\angle Y$
$m\angle C = m\angle Z$
$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$



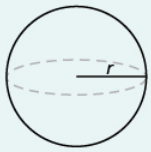
Trapezoid

Area. $A = \frac{1}{2}h(b + B)$



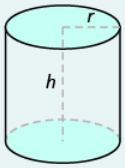
Sphere

Volume: $V = \frac{4}{3}\pi r^3$
Surface Area: $S = 4\pi r^2$



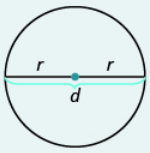
Cylinder

Volume: $V = \pi r^2 h$ or $V = Bh$
Surface Area: $S = 2\pi r^2 + 2\pi rh$

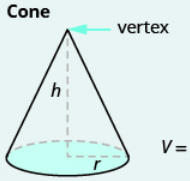


Circle

Circumference: $C = 2\pi r$ or $C = \pi d$
Area: $A = \pi r^2$



Cone



$V = \frac{1}{3}\pi r^2 h$

ACCESS ADDITIONAL ONLINE RESOURCES

Volume of a Cone

Practice Makes Perfect

Find Volume and Surface Area of Rectangular Solids

In the following exercises, find (a) the volume and (b) the surface area of the rectangular solid with the given dimensions.

263. length 2 meters, width 1.5 meters, height 3 meters
264. length 5 feet, width 8 feet, height 2.5 feet
265. length 3.5 yards, width 2.1 yards, height 2.4 yards
266. length 8.8 centimeters, width 6.5 centimeters, height 4.2 centimeters

In the following exercises, solve.

267. **Moving van** A rectangular moving van has length 16 feet, width 8 feet, and height 8 feet. Find its (a) volume and (b) surface area.
268. **Gift box** A rectangular gift box has length 26 inches, width 16 inches, and height 4 inches. Find its (a) volume and (b) surface area.
269. **Carton** A rectangular carton has length 21.3 cm, width 24.2 cm, and height 6.5 cm. Find its (a) volume and (b) surface area.
270. **Shipping container** A rectangular shipping container has length 22.8 feet, width 8.5 feet, and height 8.2 feet. Find its (a) volume and (b) surface area.

In the following exercises, find (a) the volume and (b) the surface area of the cube with the given side length.

271. 5 centimeters
272. 6 inches
273. 10.4 feet
274. 12.5 meters

In the following exercises, solve.

275. **Science center** Each side of the cube at the Discovery Science Center in Santa Ana is 64 feet long. Find its (a) volume and (b) surface area.
276. **Museum** A cube-shaped museum has sides 45 meters long. Find its (a) volume and (b) surface area.
277. **Base of statue** The base of a statue is a cube with sides 2.8 meters long. Find its (a) volume and (b) surface area.
278. **Tissue box** A box of tissues is a cube with sides 4.5 inches long. Find its (a) volume and (b) surface area.

Find the Volume and Surface Area of Spheres

In the following exercises, find (a) the volume and (b) the surface area of the sphere with the given radius. Round answers to the nearest hundredth.

279. 3 centimeters
280. 9 inches
281. 7.5 feet
282. 2.1 yards

In the following exercises, solve. Round answers to the nearest hundredth.

283. **Exercise ball** An exercise ball has a radius of 15 inches. Find its (a) volume and (b) surface area.
284. **Balloon ride** The Great Park Balloon is a big orange sphere with a radius of 36 feet . Find its (a) volume and (b) surface area.
285. **Golf ball** A golf ball has a radius of 4.5 centimeters. Find its (a) volume and (b) surface area.
286. **Baseball** A baseball has a radius of 2.9 inches. Find its (a) volume and (b) surface area.

Find the Volume and Surface Area of a Cylinder

In the following exercises, find (a) the volume and (b) the surface area of the cylinder with the given radius and height. Round answers to the nearest hundredth.

287. radius 3 feet, height 9 feet
288. radius 5 centimeters, height 15 centimeters
289. radius 1.5 meters, height 4.2 meters
290. radius 1.3 yards, height 2.8 yards

In the following exercises, solve. Round answers to the nearest hundredth.

291. **Coffee can** A can of coffee has a radius of 5 cm and a height of 13 cm. Find its (a) volume and (b) surface area.
292. **Snack pack** A snack pack of cookies is shaped like a cylinder with radius 4 cm and height 3 cm. Find its (a) volume and (b) surface area.
293. **Barber shop pole** A cylindrical barber shop pole has a diameter of 6 inches and height of 24 inches. Find its (a) volume and (b) surface area.
294. **Architecture** A cylindrical column has a diameter of 8 feet and a height of 28 feet. Find its (a) volume and (b) surface area.

Find the Volume of Cones

In the following exercises, find the volume of the cone with the given dimensions. Round answers to the nearest hundredth.

295. height 9 feet and radius 2 feet
296. height 8 inches and radius 6 inches
297. height 12.4 centimeters and radius 5 cm
298. height 15.2 meters and radius 4 meters

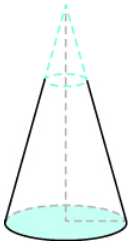
In the following exercises, solve. Round answers to the nearest hundredth.

299. **Teepee** What is the volume of a cone-shaped teepee tent that is 10 feet tall and 10 feet across at the base?

300. **Popcorn cup** What is the volume of a cone-shaped popcorn cup that is 8 inches tall and 6 inches across at the base?
301. **Silo** What is the volume of a cone-shaped silo that is 50 feet tall and 70 feet across at the base?
302. **Sand pile** What is the volume of a cone-shaped pile of sand that is 12 meters tall and 30 meters across at the base?

Everyday Math

303. **Street light post** The post of a street light is shaped like a truncated cone, as shown in the picture below. It is a large cone minus a smaller top cone. The large cone is 30 feet tall with base radius 1 foot. The smaller cone is 10 feet tall with base radius of 0.5 feet. To the nearest tenth, (a) find the volume of the large cone. (b) find the volume of the small cone. (c) find the volume of the post by subtracting the volume of the small cone from the volume of the large cone.



304. **Ice cream cones** A regular ice cream cone is 4 inches tall and has a diameter of 2.5 inches. A waffle cone is 7 inches tall and has a diameter of 3.25 inches. To the nearest hundredth, (a) find the volume of the regular ice cream cone. (b) find the volume of the waffle cone. (c) how much more ice cream fits in the waffle cone compared to the regular cone?

Writing Exercises

305. The formulas for the volume of a cylinder and a cone are similar. Explain how you can remember which formula goes with which shape.
306. Which has a larger volume, a cube of sides of 8 feet or a sphere with a diameter of 8 feet? Explain your reasoning.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find volume and surface area of rectangular solids.			
find volume and surface area of spheres.			
find volume and surface area of cylinders.			
find volume of cones.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

Contributors and Attributions

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8.7: Graphing Linear Equations (Part 2)

Graph Vertical and Horizontal Lines

Can we graph an equation with only one variable? Just x and no y, or just y without an x? How will we make a table of values to get the points to plot?

Let's consider the equation x = −3. The equation says that x is always equal to −3, so its value does not depend on y. No matter what y is, the value of x is always −3.

To make a table of solutions, we write −3 for all the x values. Then choose any values for y. Since x does not depend on y, you can chose any numbers you like. But to fit the size of our coordinate graph, we'll use 1, 2, and 3 for the y-coordinates as shown in the table.

x = -3		
x	**y**	**(x, y)**
-3	1	(-3, 1)
-3	2	(-3, 2)
-3	3	(-3, 3)

Then plot the points and connect them with a straight line. Notice in Figure 8.7.5 that the graph is a **vertical line**.

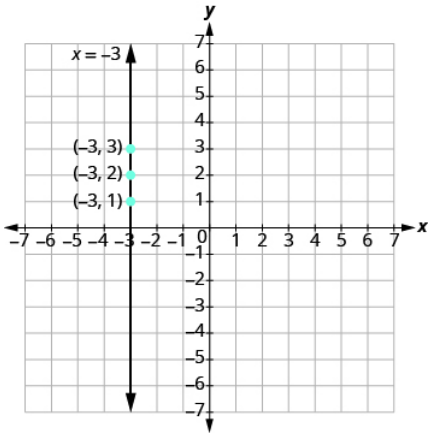


Figure 8.7.5

Definition: Vertical line

A vertical line is the graph of an equation that can be written in the form x = a. The line passes through the x -axis at (a, 0).

Example 8.7.6:

Graph the equation x = 2. What type of line does it form?

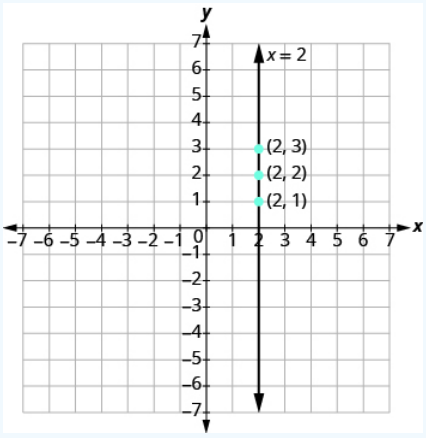
Solution

The equation has only variable, x, and x is always equal to 2. We make a table where x is always 2 and we put in any values for y.

x = -3		
x	**y**	**(x, y)**
2	1	(2, 1)

x = -3		
2	2	(2, 2)
2	3	(2, 3)

Plot the points and connect them as shown.

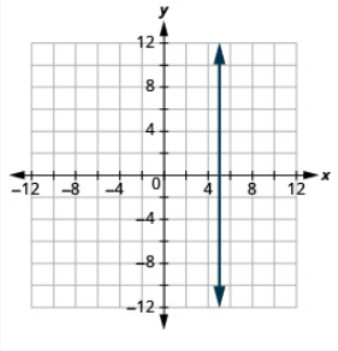


The graph is a vertical line passing through the x -axis at 2.

? Exercise 8.7.10:

Graph the equation: x = 5.

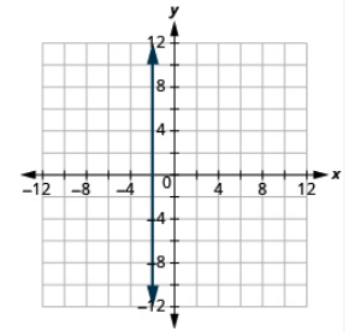
Answer



? Exercise 8.7.11:

Graph the equation: x = −2.

Answer



What if the equation has y but no x ? Let's graph the equation y = 4. This time the y -value is a constant, so in this equation y does not depend on x.

To make a table of solutions, write 4 for all the y values and then choose any values for x.

We'll use 0, 2, and 4 for the x -values.

x = -3		
x	**y**	**(x, y)**
0	4	(0, 4)
2	4	(2, 4)
4	4	(4, 4)

Plot the points and connect them, as shown in Figure 8.7.6. This graph is a horizontal line passing through the y-axis at 4.

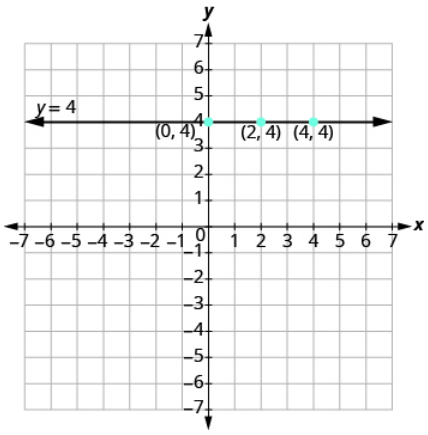


Figure 8.7.6

Definition: Horizontal Line

A horizontal line is the graph of an equation that can be written in the form y = b. The line passes through the y-axis at (0, b).

Example 8.7.7:

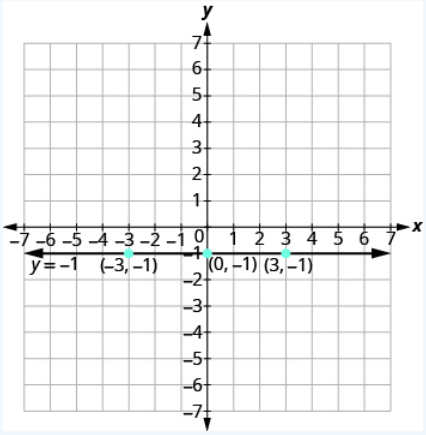
Graph the equation y = −1.

Solution

The equation y = −1 has only variable, y. The value of y is constant. All the ordered pairs in the table have the same y -coordinate, −1 . We choose 0, 3, and −3 as values for x.

x ≡ -3		
x	**y**	**(x, y)**
-3	-1	(-3, -1)
0	-1	(0, -1)
3	-1	(3, -1)

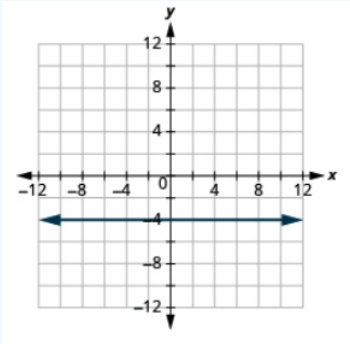
The graph is a horizontal line passing through the y -axis at −1 as shown.



? Exercise 8.7.12:

Graph the equation: y = −4.

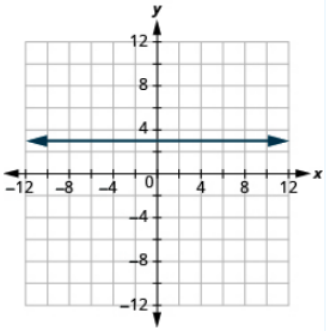
Answer



? Exercise 8.7.13:

Graph the equation: y = 3.

Answer



The equations for vertical and horizontal lines look very similar to equations like y = 4x. What is the difference between the equations y = 4x and y = 4?

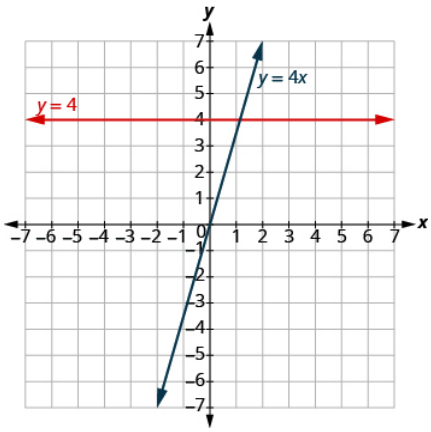
The equation y = 4x has both x and y. The value of y depends on the value of x. The y-coordinate changes according to the value of x.

The equation y = 4 has only one variable. The value of y is constant. The y-coordinate is always 4. It does not depend on the value of x.

y = 4x		
x	**y**	**(x, y)**
0	0	(0, 0)
1	4	(1, 4)
2	8	(2, 8)

$y = 4$		
x	**y**	**(x, y)**
0	4	(0, 4)
1	4	(1, 4)
2	4	(2, 4)

The graph shows both equations.



Notice that the equation y = 4x gives a slanted line whereas y = 4 gives a horizontal line.

Example 8.7.8:

Graph y = −3x and y = −3 in the same rectangular coordinate system.

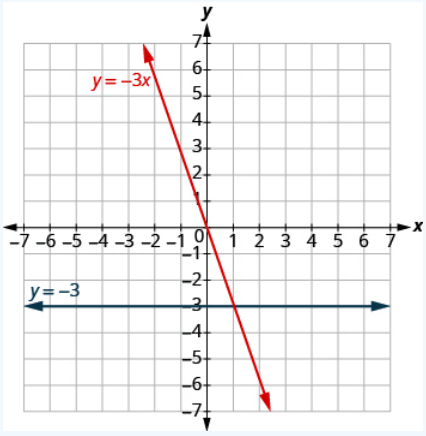
Solution

Find three solutions for each equation. Notice that the first equation has the variable x, while the second does not. Solutions for both equations are listed.

$y = -3x$		
x	**y**	**(x, y)**
0	0	(0, 0)
1	-3	(1, -3)
2	-6	(2, -6)

y = -3		
x	**y**	**(x, y)**
0	-3	(0, -3)
1	-3	(1, -3)
2	-3	(2, -3)

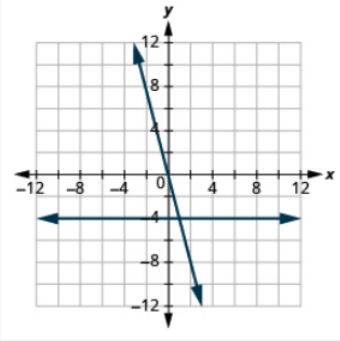
The graph shows both equations.



Exercise 8.7.14:

Graph the equations in the same rectangular coordinate system: y = −4x and y = −4.

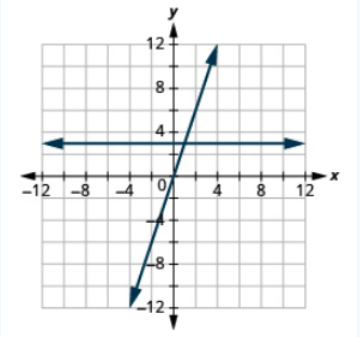
Answer



Exercise 8.7.15:

Graph the equations in the same rectangular coordinate system: y = 3 and y = 3x.

Answer



ACCESS ADDITIONAL ONLINE RESOURCES

Use a Table of Values

Graph a Linear Equation Involving Fractions

Graph Horizontal and Vertical Lines

Practice Makes Perfect

Recognize the Relation Between the Solutions of an Equation and its Graph

For each ordered pair, decide (a) is the ordered pair a solution to the equation? (b) is the point on the line?

39. y = x + 2
    1. (0, 2)
    2. (1, 2)
    3. (− 1, 1)
    4. (− 3, 1)
40. y = x − 4
    1. (0, − 4)
    2. (3, − 1)
    3. (2, 2)
    4. (1, − 5)
41. $y = \frac{1}{2}x - 3$

1. (0, − 3)
2. (2, − 2)
3. (− 2, − 4)
4. (4, 1)

42. $y = \frac{1}{3}x + 2$
    1. (0, 2)
    2. (3, 3)
    3. (− 3, 2)
    4. (− 6, 0)

Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

43. $y = 3x - 1$
44. $y = 2x + 3$
45. $y = -2x + 2$
46. $y = -3x + 1$
47. $y = x + 2$
48. $y = x - 3$
49. $y = - x - 3$
50. $y = - x - 2$
51. $y = 2x$
52. $y = 3x$
53. $y = -4x$
54. $y = -2x$
55. $y = \frac{1}{2}x + 2$
56. $y = \frac{1}{3}x - 1$
57. $y = \frac{4}{3}x - 5$
58. $y = \frac{3}{2}x - 3$
59. $y = -\frac{2}{5}x + 1$
60. $y = -\frac{4}{5}x - 1$
61. $y = -\frac{3}{2}x + 2$
62. $y = -\frac{5}{3}x + 4$
63. $x + y = 6$
64. $x + y = 4$
65. $x + y = -3$
66. $x + y = -2$
67. $x - y = 2$
68. $x - y = 1$
69. $x - y = -1$
70. $x - y = -3$
71. $-x + y = 4$
72. $-x + y = 3$
73. $-x - y = 5$
74. $-x - y = 1$
75. $3x + y = 7$
76. $5x + y = 6$

77. $2x + y = -3$
78. $4x + y = -5$
79. $2x + 3y = 12$
80. $3x - 4y = 12$
81. $\frac{1}{3}x + y = 2$
82. $\frac{1}{2}x + y = 3$

Graph Vertical and Horizontal lines

In the following exercises, graph the vertical and horizontal lines.

83. $x = 4$
84. $x = 3$
85. $x = -2$
86. $x = -5$
87. $y = 3$
88. $y = 1$
89. $y = -5$
90. $y = -2$
91. $x = \frac{7}{3}$
92. $x = \frac{5}{4}$

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

93. $y = -\frac{1}{2}x$ and $y = -\frac{1}{2}$
94. $y = -\frac{1}{3}x$ and $y = -\frac{1}{3}$
95. $y = 2x$ and $y = 2$
96. $y = 5x$ and $y = 5$

Mixed Practice

In the following exercises, graph each equation.

97. $y = 4x$
98. $y = 2x$
99. $y = -\frac{1}{2}x + 3$
100. $y = \frac{1}{4}x - 2$
101. $y = -x$
102. $y = x$
103. $x - y = 3$
104. $x + y = -5$
105. $4x + y = 2$
106. $2x + y = 6$
107. $y = -1$
108. $y = 5$
109. $2x + 6y = 12$
110. $5x + 2y = 10$
111. $x = 3$
112. $x = -4$

Everyday Math

113. **Motor home cost** The Robinsons rented a motor home for one week to go on vacation. It cost them $594 plus $0.32 per mile to rent the motor home, so the linear equation y = 594 + 0.32x gives the cost, y, for driving x miles. Calculate the rental cost for driving 400, 800, and 1,200 miles, and then graph the line.
114. **Weekly earning** At the art gallery where he works, Salvador gets paid $200 per week plus 15% of the sales he makes, so the equation y = 200 + 0.15x gives the amount y he earns for selling x dollars of artwork. Calculate the amount Salvador earns for selling $900, $1,600, and $2,000, and then graph the line.

Writing Exercises

115. Explain how you would choose three x-values to make a table to graph the line $y = \frac{1}{5}x - 2$.
116. What is the difference between the equations of a vertical and a horizontal line?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph a linear equation by plotting points.			
graph vertical and horizontal lines.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

Contributors and Attributions

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8.8: Use the Rectangular Coordinate System

Learning Objectives

By the end of this section, you will be able to:

- Plot points in a rectangular coordinate system
- Verify solutions to an equation in two variables
- Complete a table of solutions to a linear equation
- Find solutions to a linear equation in two variables

Note

Before you get started, take this readiness quiz.

1. Evaluate $x+3$ when $x=-1$.
   If you missed this problem, review Example 1.5.25.
2. Evaluate $2x-5y$ when $x=3$ and y=−2.
   If you missed this problem, review Example 1.5.28.
3. Solve for y: $40-4y=20$
   If you missed this problem, review Example 2.3.1.

Plot Points on a Rectangular Coordinate System

Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a **rectangular coordinate system**. The rectangular coordinate system is also called the *xy*-plane or the 'coordinate plane'.

The horizontal number line is called the *x-axis*. The vertical number line is called the *y-axis*. The *x*-axis and the *y*-axis together form the rectangular coordinate system. These axes divide a plane into four regions, called **quadrants**. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See Figure \(\PageIndex{1}\).

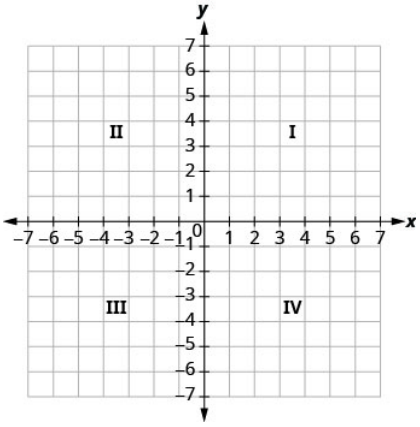


Figure 8.8.1: 'Quadrant' has the root 'quad,' which means 'four.'

In the **rectangular coordinate system**, every point is represented by an *ordered pair*. The first number in the ordered pair is the ***x*-coordinate** of the point, and the second number is the ***y*-coordinate** of the point.

ORDERED PAIR

An **ordered pair**, (x,y)(x,y), gives the coordinates of a point in a rectangular coordinate system.

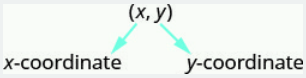


Figure 8.8.2

The first number is the *x*-coordinate.

The second number is the *y*-coordinate.

The phrase 'ordered pair' means the order is important. What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name. It is called the **origin**.

THE ORIGIN

The point $(0, 0)$ is called the **origin**. It is the point where the *x*-axis and *y*-axis intersect.

We use the coordinates to locate a point on the *xy*-plane. Let's plot the point $(1, 3)$ as an example. First, locate 1 on the *x*-axis and lightly sketch a vertical line through $x = 1$. Then, locate 3 on the *y*-axis and sketch a horizontal line through $y = 3$. Now, find the point where these two lines meet — that is the point with coordinates $(1, 3)$.

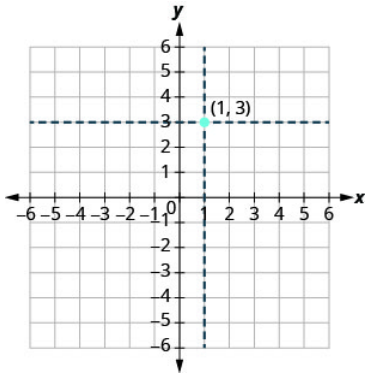


Figure 8.8.3

Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. We just used them to help us locate the point $(1, 3)$.

Example 8.8.1

Plot each point in the rectangular coordinate system and identify the **quadrant** in which the point is located:

a. (−5,4)
b. (−3,−4)
c. (2,−3)
d. (−2,3)
e. $(3, \frac{5}{2})$

Solution

The first number of the coordinate pair is the *x*-coordinate, and the second number is the *y*-coordinate.

a. Since x=−5, the point is to the left of the *y*-axis. Also, since y=4, the point is above the *x*-axis. The point (−5,4) is in Quadrant II.
b. Since x=−3, the point is to the left of the *y*-axis. Also, since y=−4, the point is below the *x*-axis. The point (−3,−4) is in Quadrant III.
c. Since x=2, the point is to the right of the *y*-axis. Since y=−3, the point is below the *x*-axis. The point (2,−3) is in Quadrant IV.
d. Since x=−2, the point is to the left of the *y*-axis. Since y=3, the point is above the *x*-axis. The point (−2,3) is in Quadrant II.
e. Since x=3, the point is to the right of the *y*-axis. Since $y = \frac{5}{2}$, the point is above the *x*-axis. (It may be helpful to write $\frac{5}{2}$ as a mixed number or decimal.) The point $(3, \frac{5}{2})$ is in Quadrant I.

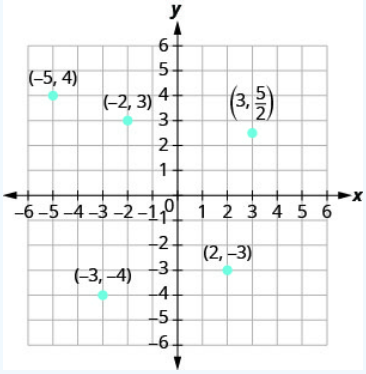


Figure 8.8.4

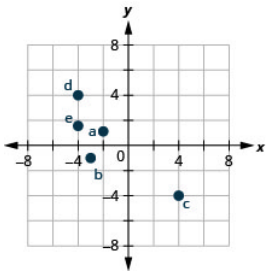
Try It 8.8.2

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

a. (−2,1)
b. (−3,−1)
c. (4,−4)
d. (−4,4)
e. $(-4, \frac{3}{2})$

Answer

The point a (-2,1) is in Quadrant II.
The point b (-3,-1) is in Quadrant III.
The point c (4,-4) is in Quadrant IV.
The point d (-4,4) is in Quadrant II.
The point e $(-4, \frac{3}{2})$ is in Quadrant II.



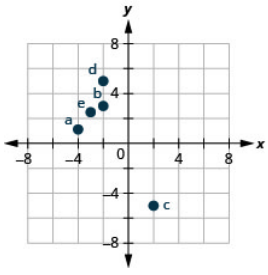
Try It 8.8.3

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

a. (−4,1)
b. (−2,3)
c. (2,−5)
d. (−2,5)
e. $(-3, \frac{5}{2})$

Answer

The point a (-4, 1) is in Quadrant II.
The point b (-2, 3) is in Quadrant II.
The point c (2, -5) is in Quadrant IV.
The point d (-2, 5) is in Quadrant II.
The point e $(-3, \frac{5}{2})$ is in Quadrant II.



How do the signs affect the location of the points? You may have noticed some patterns as you graphed the points in the previous example.

For the point in Figure 8.8.4 in Quadrant IV, what do you notice about the signs of the coordinates? What about the signs of the coordinates of points in the third quadrant? The second quadrant? The first quadrant?

Can you tell just by looking at the coordinates in which quadrant the point (−2,5) is located? In which quadrant is (2,−5) located?

QUADRANTS

We can summarize sign patterns of the quadrants in this way.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV	
(x, y)	(x, y)	(x, y)	(x, y)	(8.8.1)
$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$	

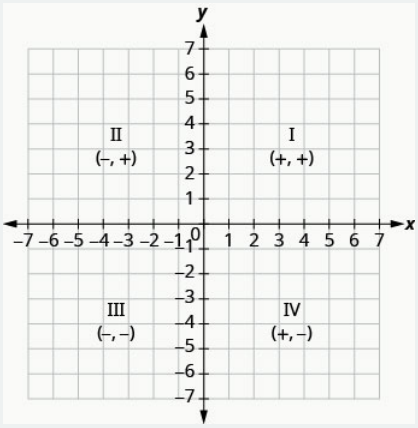


Figure 8.8.5

What if one coordinate is zero as shown in Figure 8.8.6? Where is the point (0,4) located? Where is the point (−2,0) located?

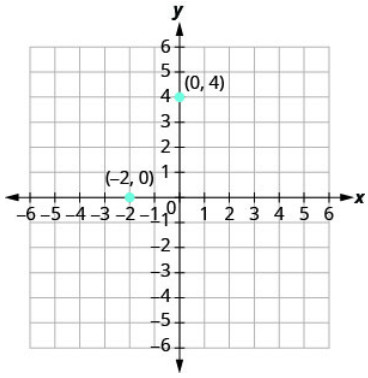


Figure 8.8.6

The point (0,4) is on the *y*-axis and the point (−2,0) is on the *x*-axis.

POINTS ON THE AXES

Points with a ***y*-coordinate** equal to 0 are on the *x*-axis, and have coordinates (a,0).

Points with an ***x*-coordinate** equal to 0 are on the *y*-axis, and have coordinates (0,b).

Example 8.8.4

Plot each point:

1. (0,5)
2. (4,0)
3. (−3,0)
4. (0,0)
5. (0,−1)

Solution

1. Since x=0, the point whose coordinates are (0,5) is on the *y*-axis.
2. Since y=0, the point whose coordinates are (4,0) is on the *x*-axis.
3. Since y=0, the point whose coordinates are (−3,0) is on the *x*-axis.
4. Since x=0 and y=0, the point whose coordinates are (0,0) is the origin.
5. Since x=0, the point whose coordinates are (0,−1) is on the *y*-axis.

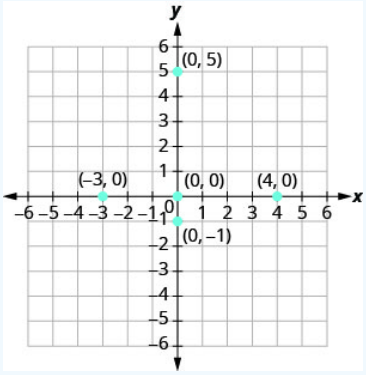


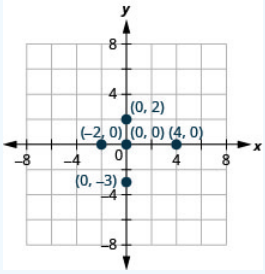
Figure 8.8.7

Try It 8.8.5

Plot each point:

1. (4,0)
2. (−2,0)
3. (0,0)
4. (0,2)
5. (0,−3).

Answer



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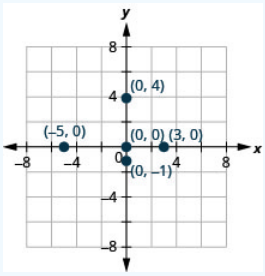
Try It 8.8.6

Plot each point:

1. (−5,0)
2. (3,0)
3. (0,0)
4. (0,−1)
5. (0,4).

Answer



In algebra, being able to identify the coordinates of a point shown on a graph is just as important as being able to plot points. To identify the *x*-coordinate of a point on a graph, read the number on the *x*-axis directly above or below the point. To identify the *y*-coordinate of a point, read the number on the *y*-axis directly to the left or right of the point. Remember, when you write the **ordered pair** use the correct order, (x,y).

Example 8.8.7

Name the ordered pair of each point shown in the rectangular coordinate system.

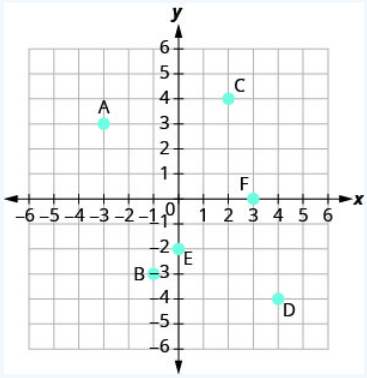


Figure 8.8.8

Solution

Point A is above −3 on the *x*-axis, so the *x*-coordinate of the point is −3.

The point is to the left of 3 on the *y*-axis, so the *y*-coordinate of the point is 3.

The coordinates of the point are (−3,3).

Point B is below −1 on the *x*-axis, so the *x*-coordinate of the point is −1.

The point is to the left of −3 on the *y*-axis, so the *y*-coordinate of the point is −3.

The coordinates of the point are (−1,−3).

Point C is above 2 on the *x*-axis, so the *x*-coordinate of the point is 2.

The point is to the right of 4 on the *y*-axis, so the *y*-coordinate of the point is 4.

The coordinates of the point are (2,4).

Point D is below 4 on the *x*-axis, so the *x*-coordinate of the point is 4.

The point is to the right of −4 on the *y*-axis, so the *y*-coordinate of the point is −4.

The coordinates of the point are (4,−4).

Point E is on the *y*-axis at y=−2. The coordinates of point E are (0,−2).

Point F is on the *x*-axis at x=3. The coordinates of point F are (3,0).

Try It 8.8.8

Name the ordered pair of each point shown in the rectangular coordinate system.

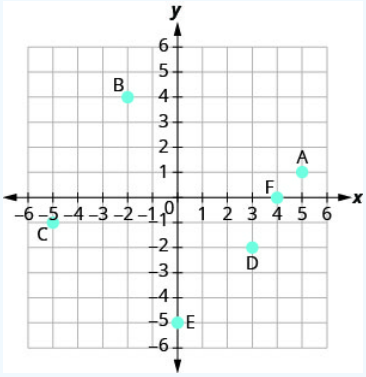


Figure 8.8.9

Answer

A: (5,1) B: (−2,4) C: (−5,−1) D: (3,−2) E: (0,−5) F: (4,0)

Try It 8.8.9

Name the ordered pair of each point shown in the rectangular coordinate system.

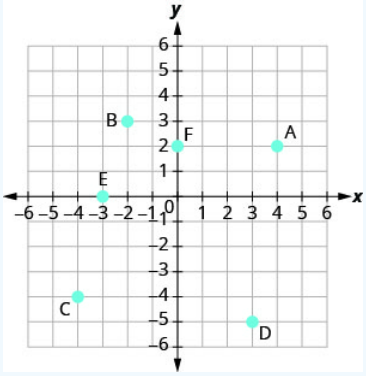


Figure 8.8.10

Answer

A: (4,2) B: (−2,3) C: (−4,−4) D: (3,−5) E: (−3,0) F: (0,2)

Verify Solutions to an Equation in Two Variables

Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation you got exactly one solution. The process of solving an equation ended with a statement like x=4. (Then, you checked the solution by substituting back into the equation.)

Here's an example of an equation in one variable, and its one solution.

$$\begin{aligned} 3x+5 &= 17 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

But equations can have more than one variable. Equations with two variables may be of the form Ax+By=C. Equations of this form are called **linear equations in two variables**.

LINEAR EQUATION

An equation of the form Ax+By=C, where A and B are not both zero, is called a **linear equation in two variables**.

Notice the word *line* in **linear**. Here is an example of a linear equation in two variables, x and y.

$$Ax + By = C$$
$$x + 4y = 8$$
$$A = 1,\ B = 4,\ C = 8$$

Figure 8.8.11

The equation y=−3x+5 is also a **linear equation**. But it does not appear to be in the form Ax+By=C. We can use the Addition Property of Equality and rewrite it in Ax+By=C form.

	y	$=$	$-3x+5$
Add to both sides.	$y+3x$	$=$	$-3x+5+3x$
Simplify.	$y+3x$	$=$	5
Use the Commutative Property to put it in $Ax+By=C$ form.	$3x+y$	$=$	5

By rewriting y=−3x+5 as 3x+y=5, we can easily see that it is a linear equation in two variables because it is of the form Ax+By=C. When an equation is in the form Ax+By=C, we say it is in *standard form*.

STANDARD FORM OF LINEAR EQUATION

A linear equation is in standard form when it is written Ax+By=C.

Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for x there is a corresponding y value. This pair of values is a *solution* to the linear equation and is represented by the ordered pair (x,y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Solution OF A LINEAR EQUATION IN TWO VARIABLES

An **ordered pair** (x,y) is a **solution** of the linear equation Ax+By=C, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

Example 8.8.10

Determine which ordered pairs are solutions to the equation x+4y=8.

(a) (0,2)

(b) (2,−4)

(c) (−4,3)

Solution

Substitute the *x*- and *y*-values from each ordered pair into the equation and determine if the result is a true statement.

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(a)	(b)	(c)
$(0, 2)$	$(2, -4)$	$(-4, 3)$
$x = 0, y = 2$	$x = 2, y = -4$	$x = -4, y = 3$
$x + 4y = 8$	$x + 4y = 8$	$x + 4y = 8$
$0 + 4 \cdot 2 \stackrel{?}{=} 8$	$2 + 4(-4) \stackrel{?}{=} 8$	$-4 + 4 \cdot 3 \stackrel{?}{=} 8$
$0 + 8 \stackrel{?}{=} 8$	$2 + (-16) \stackrel{?}{=} 8$	$-4 + 12 \stackrel{?}{=} 8$
$8 = 8$ ✓	$-14 \neq 8$	$8 = 8$ ✓
$(0, 2)$ is a solution.	$(2, -4)$ is not a solution.	$(-4, 3)$ is a solution.

Try It 8.8.11

Which of the following ordered pairs are solutions to 2x+3y=6?

1. (3,0)
2. (2,0)
3. (6,−2)

Answer

1, 3

Try It 8.8.12

Which of the following ordered pairs are solutions to the equation 4x−y=8?

1. (0,8)
2. (2,0)
3. (1,−4)

Answer

2, 3

Example 8.8.13

Which of the following ordered pairs are solutions to the equation y=5x−1?

(a) (0,−1)

(b) (1,4)

(c) (−2,−7)

Solution

Substitute the *x*- and *y*-values from each **ordered pair** into the equation and determine if it results in a true statement.

(a)	(b)	(c)
$(0, -1)$	$(1, 4)$	$(-2, -7)$
$x = 0, y = -1$	$x = 1, y = 4$	$x = -2, y = -7$
$y = 5x - 1$	$y = 5x - 1$	$y = 5x - 1$
$-1 \stackrel{?}{=} 5(0) - 1$	$4 \stackrel{?}{=} 5(1) - 1$	$-7 \stackrel{?}{=} 5(-2) - 1$
$-1 \stackrel{?}{=} 0 - 1$	$4 \stackrel{?}{=} 5 - 1$	$-7 \stackrel{?}{=} -10 - 1$
$-1 = -1$ ✓	$4 = 4$ ✓	$-7 \neq -11$
$(0, -1)$ is a solution.	$(1, 4)$ is a solution.	$(-2, -7)$ is not a solution.

Try It 8.8.14

Which of the following ordered pairs are solutions to the equation y=4x−3?

1. (0,3)
2. (1,1)
3. (−1,−1)

Answer

2

Try It 8.8.15

Which of the following ordered pairs are solutions to the equation y=−2x+6?

1. (0,6)
2. (1,4)
3. (−2,−2)

Answer

1, 2

Complete a Table of Solutions to a Linear Equation in Two Variables

In the examples above, we substituted the x- and y-values of a given ordered pair to determine whether or not it was a solution to a linear equation. But how do you find the ordered pairs if they are not given? It's easier than you might think—you can just pick a value for x and then solve the equation for y. Or, pick a value for y and then solve for x.

We'll start by looking at the solutions to the equation y = 5x−1 that we found in Example 8.8.13. We can summarize this information in a table of solutions, as shown in Table 8.8.1.

Table 8.8.1

y=5x−1		
x	**y**	**(x,y)**
0	−1	(0,−1)
1	4	(1,4)

To find a third solution, we'll let x=2 and solve for y.

	$y = 5x - 1$
Substitute $x = 2$.	$y = 5(2) - 1$
Multiply.	$y = 10 - 1$
Simplify.	$y = 9$

Figure 8.8.12

The ordered pair (2,9) is a solution to y=5x−1. We will add it to Table 8.8.2.

Table 8.8.2

y=5x−1		
x	**y**	**(x,y)**
0	−1	(0,−1)
1	4	(1,4)
2	9	(2,9)

We can find more solutions to the equation by substituting in any value of x or any value of y and solving the resulting equation to get another ordered pair that is a solution. There are infinitely many solutions of this equation.

Example 8.8.16

Complete the table to find three solutions to the equation $y=4x-2$.

Table 8.8.3

y=4x−2		
x	**y**	**(x,y)**
0		
−1		
2		

Solution

Substitute $x=0$, $x=-1$, and $x=2$ into $y=4x-2$.

$x=0$	$x=-1$	$x=2$
$y=4x-2$	$y=4x-2$	$y=4x-2$
$y=4\cdot 0-2$	$y=4(-1)-2$	$y=4\cdot 2-2$
$y=0-2$	$y=-4-2$	$y=8-2$
$y=-2$	$y=-6$	$y=6$
$(0,-2)$	$(-1,-6)$	$(2,6)$

The results are summarized in Table 8.8.4

Table 8.8.4

y=4x−2		
x	**y**	**(x,y)**
0	−2	(0,−2)
−1	−6	(−1,−6)
2	6	(2,6)

Try It 8.8.17

Complete the table to find three solutions to this equation: $y=3x-1$.

Table 8.8.5

y=3x−1		
x	**y**	**(x,y)**
0		
−1		
2		

Answer

Table 8.8.6

y=3x−1

x	y	(x,y)
0	-1	(0, -1)
−1	-4	(-1, -4)
2	5	(2, 5)

Try It 8.8.18

Complete the table to find three solutions to this equation: y=6x+1.

Table 8.8.7

y=6x+1		
x	**y**	**(x,y)**
-2		

Answer

Table 8.8.8

y=6x+1		
x	**y**	**(x,y)**
0	1	(0,1)
1	7	(1,7)
−2	−11	(−2,−11)

Example 8.8.19

Complete Table 8.8.9 to find three solutions to the equation 5x−4y=20.

Table 8.8.9

5x−4y=20		
x	**y**	**(x,y)**
	0	
	5	

Solution

Substitute the given value into the equation 5x−4y=20 and solve for the other variable. Then, fill in the values in the table.

$x = 0$	$y = 0$	$y = 5$
$5x - 4y = 20$	$5x - 4y = 20$	$5x - 4y = 20$
$5 \cdot 0 - 4y = 20$	$5x - 4 \cdot 0 = 20$	$5x - 4 \cdot 5 = 20$
$0 - 4y = 20$	$5x - 0 = 20$	$5x - 20 = 20$
$-4y = 20$	$5x = 20$	$5x = 40$
$y = -5$	$x = 4$	$x = 8$
$(0, -5)$	$(4, 0)$	$(8, 5)$

The results are summarized in Table 8.8.10

Table 8.8.10

5x−4y=20		
x	**y**	**(x,y)**
0	−5	(0,−5)
4	0	(4,0)
8	5	(8,5)

Try It 8.8.20

Complete the table to find three solutions to this equation: 2x−5y=20.

Table 8.8.11

2x−5y=20		
x	**y**	**(x,y)**
-5		

Answer

Table 8.8.12

2x−5y=20		
x	**y**	**(x,y)**
0	−4	(0,−4)
10	0	(10,0)
−5	−6	(−5,−6)

Try It 8.8.21

Complete the table to find three solutions to this equation: 3x−4y=12.

Table 8.8.13

3x−4y=12		
x	**y**	**(x,y)**

-4		

Answer

Table 8.8.14

3x−4y=12		
x	**y**	**(x,y)**
0	−3	(0,−3)
4	0	(4,0)
−4	−6	(−4,−6)

Find Solutions to a Linear Equation

To find a solution to a linear equation, you really can pick *any* number you want to substitute into the equation for x or y. But since you'll need to use that number to solve for the other variable it's a good idea to choose a number that's easy to work with.

When the equation is in *y*-form, with the *y* by itself on one side of the equation, it is usually easier to choose values of x and then solve for y.

Example 8.8.22

Find three solutions to the equation y=−3x+2.

Solution

We can substitute any value we want for x or any value for y. Since the equation is in *y*-form, it will be easier to substitute in values of x. Let's pick x=0, x=1, and x=−1.

	$x=0$	$x=1$	$x=-1$
	$y=-3x+2$	$y=-3x+2$	$y=-3x+2$
Substitute the value into the equation.	$y=-3\cdot 0+2$	$y=-3\cdot 1+2$	$y=-3(-1)+2$
Simplify.	$y=0+2$	$y=-3+2$	$y=3+2$
Simplify.	$y=2$	$y=-1$	$y=5$
Write the ordered pair.	(0, 2)	(1, -1)	(-1, 5)
Check.			
y=−3x+2	y=−3x+2	y=−3x+2	
$2\stackrel{?}{=}-3\cdot 0+2$	$-1\stackrel{?}{=}-3\cdot 1+2$	$5\stackrel{?}{=}-3(-1)+2$	
$2\stackrel{?}{=}0+2$	$-1\stackrel{?}{=}-3+2$	$5\stackrel{?}{=}-3+2$	
$2=2\checkmark$	$-1=-1\checkmark$	$5=5\checkmark$	

Table 8.8.15

So, (0,2), (1,−1) and (−1,5) are all solutions to y=−3x+2. We show them in Table 8.8.16

Table 8.8.16

y=−3x+2

x	y	(x,y)
0	2	(0,2)
1	−1	(1,−1)
−1	5	(−1,5)

Try It 8.8.23

Find three solutions to this equation: y=−2x+3.

Answer

Answers will vary. Answers might include (0,3), (1,1) and (−1,5).

Try It 8.8.24

Find three solutions to this equation: y=−4x+1.

Answer

Answers will vary. Answers might include (0,1), (1,−3) and (−1,5).

We have seen how using zero as one value of x makes finding the value of y easy. When an equation is in standard form, with both the x and y on the same side of the equation, it is usually easier to first find one solution when x=0 find a second solution when y=0, and then find a third solution.

Example 8.8.25

Find three solutions to the equation 3x+2y=6.

Solution

We can substitute any value we want for x or any value for y. Since the equation is in standard form, let's pick first x=0, then y=0, and then find a third point.

Table 8.8.17

$x=0$	$y=0$	$x=1$			
			$3x+2y=6$	$3x+2y=6$	$3x+2y=6$
Substitute the value into the equation.			$3(0)+2y=6$	$3x+2(0)=6$	$3(1)+2y=6$
Simplify.			$0+2y=6$	$3x+0=6$	$3+2y=6$
Solve.			$2y=6$	$3x=6$	$2y=3$
			$y=3$	$x=2$	$y=\frac{3}{2}$
Write the ordered pair.			(0, 3)	(2, 0)	$(1, \frac{3}{2})$
Check.					
3x+2y=6	3x+2y=6	3x+2y=6			
$3\cdot 0+2\cdot 3\stackrel{?}{=}6$	$3\cdot 2+2\cdot 0\stackrel{?}{=}6$	$3\cdot 1+2\cdot\frac{3}{2}\stackrel{?}{=}6$			
$0+6\stackrel{?}{=}6$	$6+0\stackrel{?}{=}6$	$3+3\stackrel{?}{=}6$			
$6=6$✓	$6=6$✓	$6=6$✓			

So (0,3), (2,0), and $(1, \frac{3}{2})$ are all solutions to the equation 3x+2y=6. We can list these three solutions in Table 8.8.18

Table 8.8.18

3x+2y=6		
x	**y**	**(x,y)**
0	3	(0,3)
2	0	(2,0)
1	$\frac{3}{2}$	$(1, \frac{3}{2})$

? Try It 8.8.26

Find three solutions to the equation 2x+3y=6.

Answer

Answers will vary. Answers might include (0,2), (3,0) and (6, −2).

? Try It 8.8.27

Find three solutions to the equation 4x+2y=8.

Answer

Answers will vary. Answers might include (0,4), (2,0) and (−1,6).

Key Concepts

- **Sign Patterns of the Quadrants**

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$

- **Points on the Axes**
  - On the *x*-axis, y=0. Points with a *y*-coordinate equal to 0 are on the *x*-axis, and have coordinates (a,0).
  - On the *y*-axis, x=0. Points with an *x*-coordinate equal to 0 are on the *y*-axis, and have coordinates (0,b).
- **Solution of a Linear Equation**
  - An ordered pair (x,y) is a solution of the linear equation Ax+By=C, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

Glossary

linear equation

A linear equation is of the form Ax+By=C, where A and B are not both zero, is called a linear equation in two variables.

ordered pair

An ordered pair (x,y) gives the coordinates of a point in a rectangular coordinate system.

origin

The point (0,0) is called the origin. It is the point where the *x*-axis and *y*-axis intersect.

quadrant

The *x*-axis and the *y*-axis divide a plane into four regions, called quadrants.

rectangular coordinate system

A grid system is used in algebra to show a relationship between two variables; also called the *xy*-plane or the ‘coordinate plane’.

***x*-coordinate**

The first number in an ordered pair (x,y).

***y*-coordinate**

The second number in an ordered pair (x,y).

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8.8E: Exercises

Practice Makes Perfect

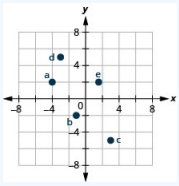
Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

Exercise 8.8E.1

1. $(-4,2)$
2. $(-1,-2)$
3. $(3,-5)$
4. $(-3,5)$
5. $\left(\frac{5}{3},2\right)$

Answer



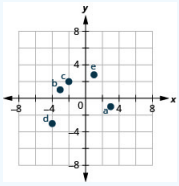
Exercise 8.8E.2

1. $(-2,-3)$
2. $(3,-3)$
3. $(-4,1)$
4. $(4,-1)$
5. $\left(\frac{3}{2},1\right)$

Exercise 8.8E.3

1. $(3,-1)$
2. $(-3,1)$
3. $(-2,2)$
4. $(-4,-3)$
5. $\left(1,\frac{14}{5}\right)$

Answer



Exercise 8.8E.4

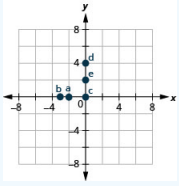
1. $(-1,1)$
2. $(-2,-1)$
3. $(2,1)$
4. $(1,-4)$
5. $\left(3,\frac{7}{2}\right)$

In the following exercises, plot each point in a rectangular coordinate system.

Exercise 8.8E.5

1. $(-2,0)$
2. $(-3,0)$
3. $(0,0)$
4. $(0,4)$
5. $(0,2)$

Answer



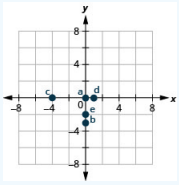
Exercise 8.8E.6

1. (0,1)
2. (0,−4)
3. (−1,0)
4. (0,0)
5. (5,0)

Exercise 8.8E.7

1. (0,0)
2. (0,−3)
3. (−4,0)
4. (1,0)
5. (0,−2)

Answer

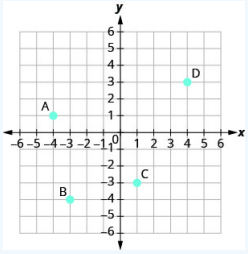


Exercise 8.8E.8

1. (−3,0)
2. (0,5)
3. (0,−2)
4. (2,0)
5. (0,0)

In the following exercises, name the ordered pair of each point shown in the rectangular coordinate system.

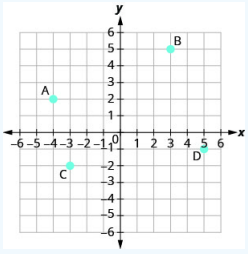
Exercise 8.8E.9



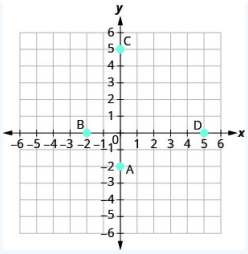
Answer

$A:(-4,1)\quad B:(-3,-4)\quad C:(1,-3)\quad D:(4,3)$

Exercise 8.8E.10



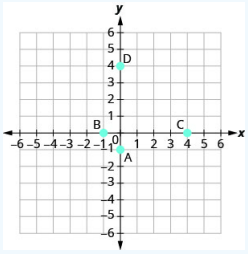
Exercise 8.8E.11



Answer

$A:(0,-2)\quad B:(-2,0)\quad C:(0,5)\quad D:(5,0)$

Exercise 8.8E.12



Verify Solutions to an Equation in Two Variables

In the following exercises, which ordered pairs are solutions to the given equations?

Exercise 8.8E.13

2x+y=6

1. (1,4)
2. (3,0)
3. (2,3)

Answer

1, 2

Exercise 8.8E.14

1. x+3y=9
2. (0,3)
3. (6,1)
4. (−3,−3)

Exercise 8.8E.15

4x−2y=8

1. (3,2)
2. (1,4)
3. (0,−4)

Answer

1, 3

Exercise 8.8E.16

3x−2y=12

1. (4,0)
2. (2,−3)
3. (1,6)

Exercise 8.8E.17

y=4x+3

1. (4,3)
2. (−1,−1)
3. $\left(\frac{1}{2},5\right)$

Answer

2, 3

Exercise 8.8E.18

y=2x−5

1. (0,−5)
2. (2,1)
3. $\left(\frac{1}{2}, -4\right)$

Exercise 8.8E.19

$y = \frac{1}{2}x - 1$

1. (2,0)
2. (−6,−4)
3. (−4,−1)

Answer

1, 2

Exercise 8.8E.20

$y = \frac{1}{3}x + 1$

1. (−3,0)
2. (9,4)
3. (−6,−1)

Complete a Table of Solutions to a Linear Equation

In the following exercises, complete the table to find solutions to each linear equation.

Exercise 8.8E.21

$y = 2x - 4$

x	y	(x,y)
0		
2		
-1		

Answer

x	y	(x,y)
0	−4	(0,−4)
2	0	(2,0)
−1	−6	(−1,−6)

Exercise 8.8E.22

$y = 3x - 1$

x	y	(x,y)
0		
2		
-1		

Exercise 8.8E.23

$y = -x + 5$

x	y	(x,y)
0		
3		
-2		

Answer

x	y	(x,y)
0	5	(0,5)
3	2	(3,2)
−2	7	(−2,7)

Exercise 8.8E.24

$y = -x + 2$

x	y	(x,y)
0		
3		
-2		

Exercise 8.8E.25

$y = \frac{1}{3}x + 1$

x	y	(x,y)
0		
3		
6		

Answer

x	y	(x,y)
0	1	(0,1)
3	2	(3,2)
6	3	(6,3)

Exercise 8.8E.26

$y = \frac{1}{2}x + 4$

x	y	(x,
0		
2		
4		

Exercise 8.8E.27

$y = -\frac{3}{2}x - 2$

x	y	(x,y)
0		
2		
-2		

Answer

x	y	(x,y)
0	−2	(0,−2)
2	−5	(2,−5)
−2	1	(−2,1)

Exercise 8.8E.28

$y = -\frac{2}{3}x - 1$

x	y	(x,y)
0		
3		
-3		

Exercise 8.8E.29

$x + 3y = 6$

x	y	(x,y)
0		
3		

	0	

Answer

x	y	(x,y)
0	2	(0,2)
3	4	(3,1)
6	0	(6,0)

Exercise 8.8E.30

x+2y=8

x	y	(x,y)
0		
4		
	0	

Exercise 8.8E.31

$2x-5y=10$

x	y	(x,y)
0		
10		
	0	

Answer

x	y	(x,y)
0	−2	(0,−2)
10	2	(10,2)
5	0	(5,0)

Exercise 8.8E.32

x	y	(x,y)
0		
8		
	0	

Find Solutions to a Linear Equation

In the following exercises, find three solutions to each linear equation.

Exercise 8.8E.33

$y=5x-8$

Answer

Answers will vary.

Exercise 8.8E.34

$y=3x-9$

Exercise 8.8E.35

$y=-4x+5$

Answer

Answers will vary.

Exercise 8.8E.36

$y=-2x+7$

Exercise 8.8E.37

$x+y=8$

Answer

Answers will vary.

Exercise 8.8E.38

$x+y=6$

Exercise 8.8E.39

$x+y=-2$

Answer

Answers will vary.

Exercise 8.8E.40

$x+y=-1$

Exercise 8.8E.41

$3x+y=5$

Answer

Answers will vary.

Exercise 8.8E.42

$2x+y=3$

Exercise 8.8E.43

$4x-y=8$

Answer

Answers will vary.

Exercise 8.8E.44

$5x-y=10$

Exercise 8.8E.45

$2x+4y=8$

Answer

Answers will vary.

Exercise 8.8E.46

$3x+2y=6$

Exercise 8.8E.47

$5x-2y=10$

Answer

Answers will vary.

Exercise 8.8E.48

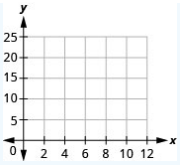
$4x-3y=12$

Everyday Math

Exercise 8.8E.49

Weight of a baby. Mackenzie recorded her baby's weight every two months. The baby's age, in months, and weight, in pounds, are listed in the table below, and shown as an ordered pair in the third column.

1. Plot the points on a coordinate plane.

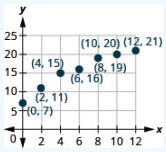


2. Why is only Quadrant I needed?

Age x	Weight y	(x,y)
0	7	(0, 7)
2	11	(2, 11)
4	15	(4, 15)
6	16	(6, 16)
8	19	(8, 19)
10	20	(10, 20)
12	21	(12, 21)

Answer

1.

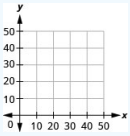


2. Age and weight are only positive.

Exercise 8.8E.50

Weight of a child. Latresha recorded her son's height and weight every year. His height, in inches, and weight, in pounds, are listed in the table below, and shown as an ordered pair in the third column.

1. Plot the points on a coordinate plane.



2. Why is only Quadrant I needed?

Height x	Weight y	(x,y)
28	22	(28, 22)
31	27	(31, 27)
33	33	(33, 33)
37	35	(37, 35)
40	41	(40, 41)
42	45	(42, 45)

Writing Exercises

Exercise 8.8E.51

Explain in words how you plot the point (4,−2) in a rectangular coordinate system?

Answer

Answers will vary.

Exercise 8.8E.52

How do you determine if an ordered pair is a solution to a given equation?

Exercise 8.8E.53

Is the point (−3,0) on the *x*-axis or *y*-axis? How do you know?

Answer

Answers will vary.

? Exercise $8.8E.54$

Is the point (0,8) on the *x*-axis or *y*-axis? How do you know?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
plot points in a rectangular coordinate system.			
identify points on a graph.			
verify solutions to an equation in two variables.			
complete a table of solutions to a linear equation.			
find solutions to a linear equation.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no, I don't get it. This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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8.9: Graphing with Intercepts (Part 1)

Learning Objectives

- Identify the intercepts on a graph
- Find the intercepts from an equation of a line
- Graph a line using the intercepts
- Choose the most convenient method to graph a line

be prepared!

Before you get started, take this readiness quiz.

1. Solve: $3x + 4y = -12$ for x when $y = 0$. If you missed this problem, review Example 9.11.6.
2. Is the point $(0, -5)$ on the x-axis or y-axis? If you missed this problem, review Example 11.1.5.
3. Which ordered pairs are solutions to the equation $2x - y = 6$? (a) (6, 0) (b) (0, −6) (c) (4, −2). If you missed this problem, review Example 11.2.8.

Identify the Intercepts on a Graph

Every linear equation has a unique line that represents all the solutions of the equation. When graphing a line by plotting points, each person who graphs the line can choose any three points, so two people graphing the line might use different sets of points.

At first glance, their two lines might appear different since they would have different points labeled. But if all the work was done correctly, the lines will be exactly the same line. One way to recognize that they are indeed the same line is to focus on where the line crosses the axes. Each of these points is called an **intercept of the line**.

Definition: Intercepts of a Line

Each of the points at which a line crosses the x-axis and the y-axis is called an intercept of the line.

Let's look at the graph of the lines shown in Figure 8.9.1.

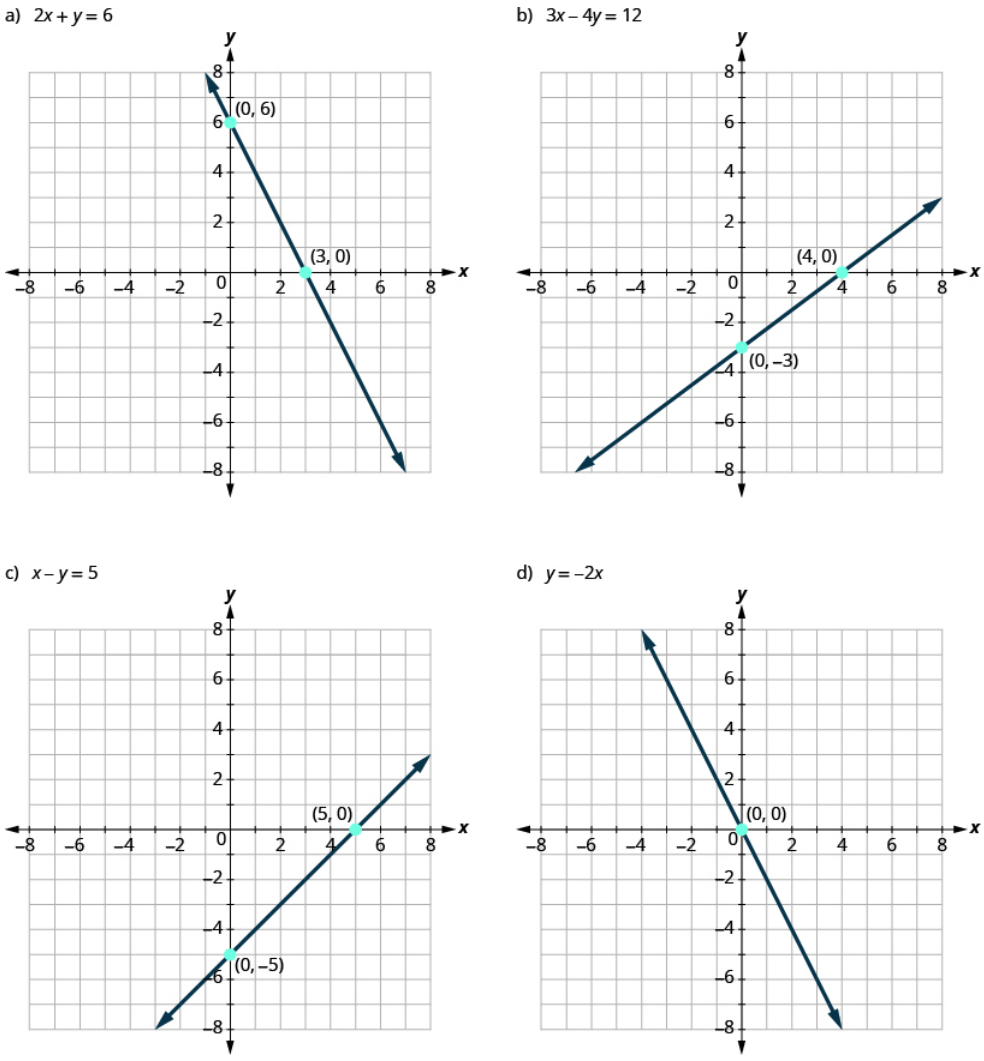


Figure 8.9.1

First, notice where each of these lines crosses the x- axis:

Table 8.9.1

Figure:	The line crosses the x-axis at:	Ordered pair of this point
Figure 8.9.1a	3	(3,0)
Figure 8.9.1b	4	(4,0)
Figure 8.9.1c	5	(5,0)
Figure 8.9.1d	0	(0,0)

Do you see a pattern?

For each row, the y-coordinate of the point where the line crosses the x-axis is zero. The point where the line crosses the x-axis has the form (a, 0); and is called the *x-intercept* of the line. The x-intercept occurs when y is zero.

Now, let's look at the points where these lines cross the y-axis.

Table 8.9.2

Figure:	The line crosses the x-axis at:	Ordered pair of this point
Figure 8.9.1a	6	(0, 6)

Figure:	The line crosses the x-axis at:	Ordered pair of this point
Figure 8.9.1*b*	-3	(0, -3)
Figure 8.9.1*c*	-5	(0, -5)
Figure 8.9.1*d*	0	(0, 0)

Definition: x-intercept and y-intercept of a line

The x-intercept is the point, (a, 0), where the graph crosses the x-axis.

The x-intercept occurs when y is zero.

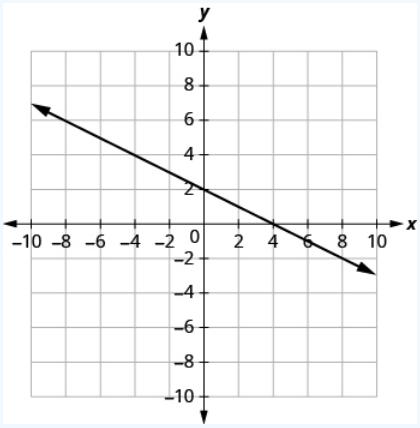
The y-intercept is the point, (0, b), where the graph crosses the y-axis.

The y-intercept occurs when x is zero.

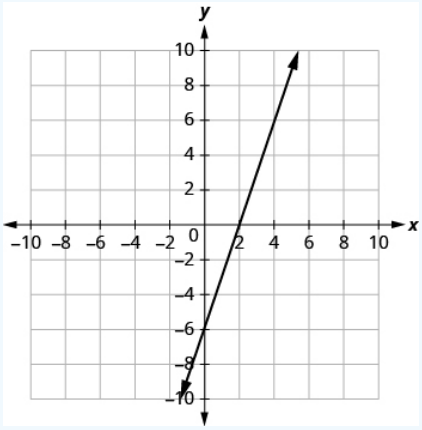
Example 8.9.1

Find the x- and y-intercepts of each line:

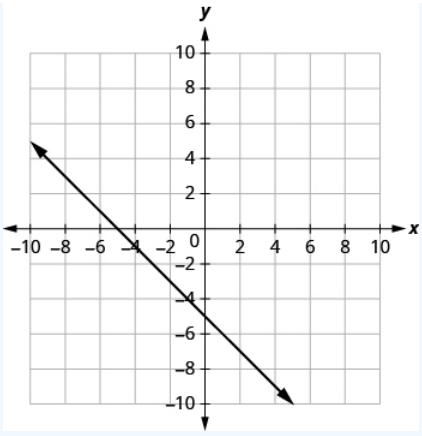
(a) x + 2y = 4



(b) 3x - y = 6



(c) x + y = -5



Solution

(a)

The graph crosses the x-axis at the point (4, 0).	The x-intercept is (4, 0).
The graph crosses the y-axis at the point (0, 2).	The x-intercept is (0, 2).

(b)

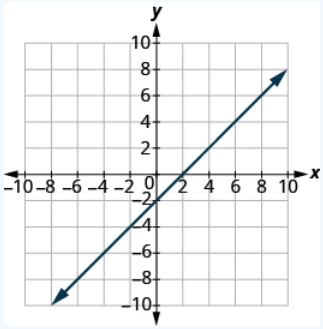
The graph crosses the x-axis at the point (2, 0).	The x-intercept is (2, 0).
The graph crosses the y-axis at the point (0, -6).	The x-intercept is (0, -6).

(c)

The graph crosses the x-axis at the point (-5, 0).	The x-intercept is (-5, 0).
The graph crosses the y-axis at the point (0, -5).	The x-intercept is (0, -5).

Exercise 8.9.1A

Find the x- and y-intercepts of the graph: x − y = 2.

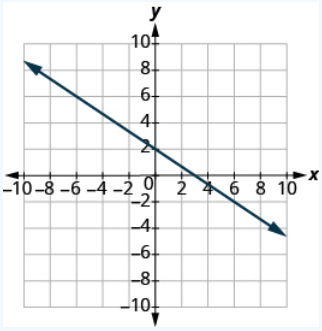


Answer

x-intercept (2,0); y-intercept (0,-2)

Exercise 8.9.1B

Find the x- and y-intercepts of the graph: 2x + 3y = 6.



Answer

x-intercept (3,0); y-intercept (0,2)

Find the Intercepts from an Equation of a Line

Recognizing that the x-intercept occurs when y is zero and that the y-intercept occurs when x is zero gives us a method to find the intercepts of a line from its equation. To find the x-intercept, let y = 0 and solve for x. To find the y-intercept, let x = 0 and solve for y.

Definition: Find the x and y from the Equation of a Line

Use the equation to find:

- the x-intercept of the line, let y = 0 and solve for x.
- the y-intercept of the line, let x = 0 and solve for y

Table 8.9.3

x	y
	0
0	

Example 8.9.2

Find the intercepts of 2x + y = 6

Solution

We'll fill in Figure 8.9.2.

2x + y = 6		
x	y	
	0	x-intercept
0		y-intercept

Figure 8.9.2

To find the x- intercept, let y = 0:

Substitute 0 for y.	$2x + 0 = 6$
Add.	2x = 6

Divide by 2.	x = 3

The x-intercept is (3, 0).

To find the y- intercept, let x = 0:

Substitute 0 for x.	$2 \cdot 0 + y = 6$
Multiply.	0 + y = 6
Add.	y = 6

The y-intercept is (0, 6).

2x + y = 6	
x	**y**
3	0
0	6

Figure 8.9.3

The intercepts are the points (3, 0) and (0, 6).

? Exercise 8.9.2A

Find the intercepts: 3x + y = 12.

Answer

x-intercept (4,0); y-intercept (0,12)

? Exercise 8.9.2B

Find the intercepts: x + 4y = 8.

Answer

x-intercept (8,0); y-intercept (0,2)

✓ Example 8.9.3

Find the intercepts of 4x−3y = 12.

Solution

To find the x-intercept, let y = 0.

Substitute 0 for y.	4x − 3 • 0 = 12
Multiply.	4x − 0 = 12
Subtract.	4x = 12
Divide by 4.	x = 3

The y-intercept is (0, −4). The intercepts are the points (−3, 0) and (0, −4).

4x - 3y = 12	
x	**y**
3	0
0	-4

? Exercise 8.9.3A

Find the intercepts of the line: 3x−4y = 12.

Answer

x-intercept (4,0); y-intercept (0,-3)

? Exercise 8.9.3B

Find the intercepts of the line: 2x−4y = 8.

Answer

x-intercept (4,0); y-intercept (0,-2)

Graph a Line Using the Intercepts

To graph a linear equation by plotting points, you can use the intercepts as two of your three points. Find the two intercepts, and then a third point to ensure accuracy, and draw the line. This method is often the quickest way to graph a line.

✔ Example 8.9.4

Graph −x + 2y = 6 using intercepts.

Solution

First, find the x-intercept. Let y = 0,

$$\begin{aligned} -x+2y &= 6 \\ -x+2(0) &= 6 \\ -x &= 6 \\ x &= -6 \end{aligned}$$

The x-intercept is (–6, 0).

Now find the y-intercept. Let x = 0.

$$\begin{aligned} -x+2y &= 6 \\ -0+2y &= 6 \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

The y-intercept is (0, 3).

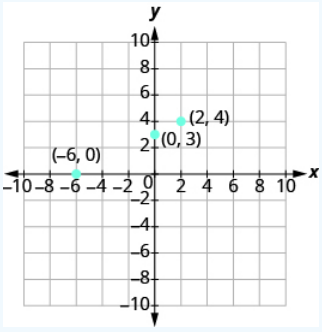
Find a third point. We'll use x = 2,

$$\begin{aligned} -x+2y &= 6 \\ -2+2y &= 6 \\ 2y &= 8 \\ y &= 4 \end{aligned}$$

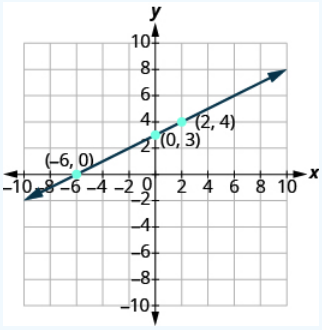
A third solution to the equation is (2, 4).

Summarize the three points in a table and then plot them on a graph.

-x + 2y = 6		
x	**y**	**(x,y)**
-6	0	(−6, 0)
0	3	(0, 3)
2	4	(2, 4)



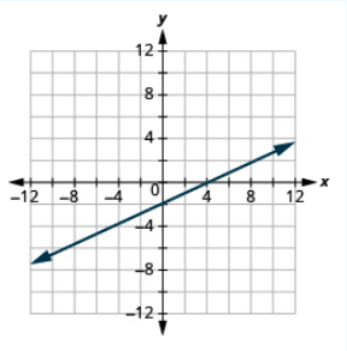
Do the points line up? Yes, so draw line through the points.



Exercise 8.9.4A

Graph the line using the intercepts: x−2y = 4.

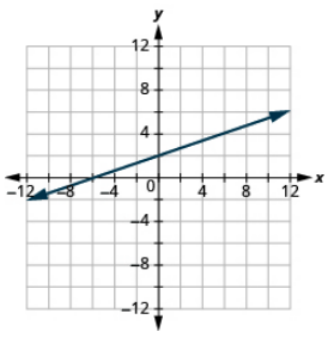
Answer



Exercise 8.9.4B

Graph the line using the intercepts: −x + 3y = 6.

Answer



HOW TO: GRAPH A LINE USING THE INTERCEPTS

Step 1. Find the x - and y-intercepts of the line.

- Let y = 0 and solve for x.
- Let x = 0 and solve for y.

Step 2. Find a third solution to the equation.

Step 3. Plot the three points and then check that they line up.

Step 4. Draw the line.

Example 8.9.5

Graph 4x−3y = 12 using intercepts.

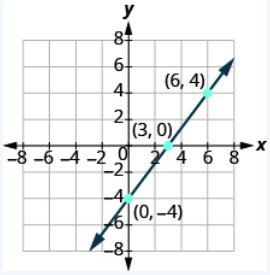
Solution

Find the intercepts and a third point.

$x-intercept,\ let\ y=0$	$y-intercept,\ let\ x=0$	$third\ point,\ let\ y=4$
$4x-3y=12$	$4x-3y=12$	$4x-3y=12$
$4x-3(0)=12$	$4(0)-3y=12$	$4x-12=12$
$4x=12$	$4x-3(4)=12$	$4x=24$
$x=3$	$-3y=12$	$x=6$
	$y=-4$	

We list the points and show the graph.

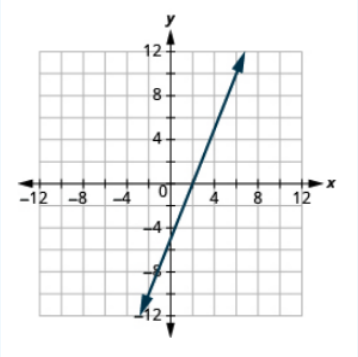
4x - 3y = 12		
x	**y**	**(x. y)**
3	0	(3, 0)
0	-4	(0, −4)
6	4	(6, 4)



Exercise $8.9.5A$

Graph the line using the intercepts: 5x−2y = 10.

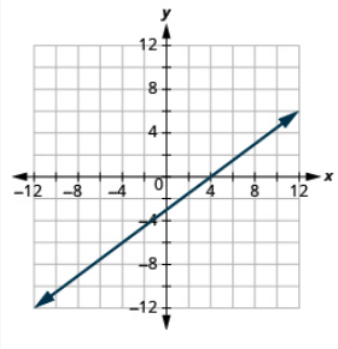
Answer



Exercise $8.9.5B$

Graph the line using the intercepts: 3x−4y = 12.

Answer



Example $8.9.6$

Graph $y = 5x$ using the intercepts.

Solution

$x-intercept;\ Let\ y=0.$	$y-intercept;\ Let\ x=0.$
$y=5x$ $0=5x$ $0=x$ $x=0$ $The\ x-intercept\ is\ (0,0).$	$y=5x$ $y=5(0)$ $y=0$ $The\ y-intercept\ is\ (0,0).$

This line has only one intercept! It is the point (0, 0).

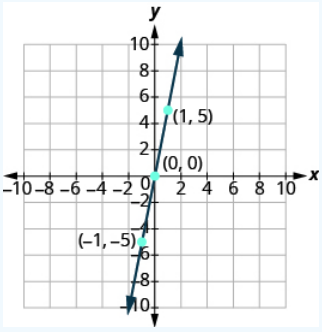
To ensure accuracy, we need to plot three points. Since the intercepts are the same point, we need two more points to graph the line. As always, we can choose any values for x, so we'll let x be 1 and −1.

$x=1$ $y=5x$ $y=5(1)$ $y=5$ $(1,-5)$	$x=-1$ $y=5x$ $y=5(-1)$ $y=-5$ $(-1,-5)$

Organize the points in a table.

y = 5x		
x	**y**	**(x, y)**
0	0	(0, 0)
1	5	(1, 5)
-1	-5	(−1, −5)

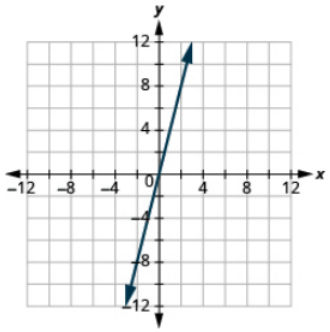
Plot the three points, check that they line up, and draw the line.



Exercise 8.9.6A

Graph using the intercepts: $y=3x$.

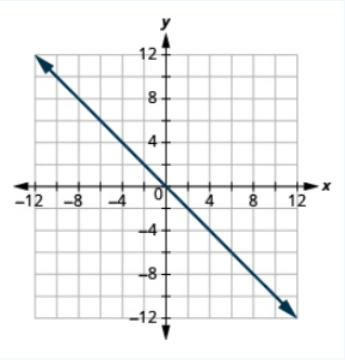
Answer



? Exercise $8.9.6B$

Graph using the intercepts: $y = -x$.

Answer



Contributors and Attributions

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8.10: Graph Linear Equations in Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Recognize the relationship between the solutions of an equation and its graph.
- Graph a linear equation by plotting points.
- Graph vertical and horizontal lines.

Note

Before you get started, take this readiness quiz.

1. Evaluate $3x+2$ when $x=-1$.
   If you missed this problem, review Exercise 1.5.34.
2. Solve $3x+2y=12$ for y in general.
   If you missed this problem, review Exercise 2.6.16.

Recognize the Relationship Between the Solutions of an Equation and its Graph

In the previous section, we found several solutions to the equation $3x+2y=6$. They are listed in Table 8.10.1. So, the ordered pairs (0,3), (2,0), and $(1, \frac{3}{2})$ are some solutions to the equation $3x+2y=6$. We can plot these solutions in the rectangular coordinate system as shown in Figure 8.10.1.

Table 8.10.1

3x+2y=6		
x	**y**	**(x,y)**
0	3	(0,3)
2	0	(2,0)
1	$\frac{3}{2}$	$(1, \frac{3}{2})$

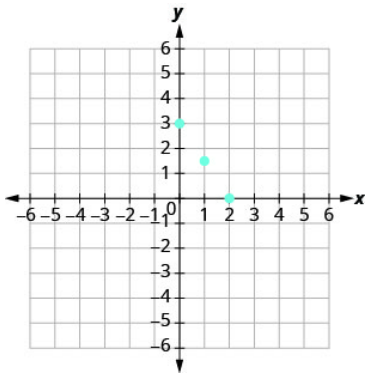


Figure 8.10.1

Notice how the points line up perfectly? We connect the points with a line to get the graph of the equation 3x+2y=6. See Figure 8.10.2 Notice the arrows on the ends of each side of the line. These arrows indicate the line continues.

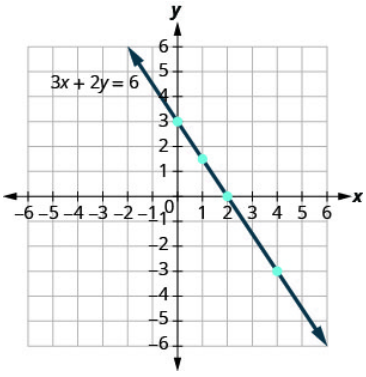


Figure 8.10.2

Every point on the line is a solution of the equation. Also, every solution of this equation is a point on this line. Points *not* on the line are not solutions.

Notice that the point whose coordinates are (−2,6) is on the line shown in Figure 8.10.3. If you substitute x=−2 and y=6 into the equation, you find that it is a solution to the equation.

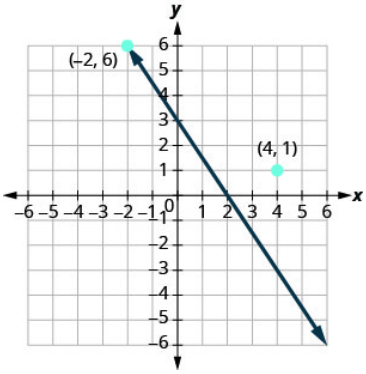


Figure 8.10.3

Test (–2, 6)

$$3x + 2y = 6$$

$$3(-2) + 2(6) = 6$$

$$-6 + 12 = 6$$

$$6 = 6 \checkmark$$

So the point (−2,6) is a solution to the equation $3x + 2y = 6$. (The phrase "the point whose coordinates are (−2,6)" is often shortened to "the point (−2,6).")

What about (4, 1) ?

$$3x + 2y = 6$$

$$3 \cdot 4 + 2 \cdot 1 = 6$$

$$12 + 2 \stackrel{?}{=} 6$$

$$14 \neq 6$$

Figure 8.10.3. This is an example of the saying, "A picture is worth a thousand words." The line shows you *all* the solutions to the equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the *graph* of the equation $3x + 2y = 6$.

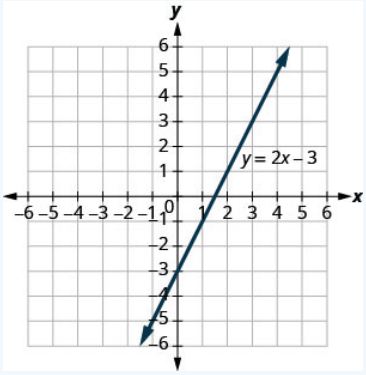
GRAPH OF A LINEAR EQUATION

The **graph of a linear equation** Ax+By=C is a line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

Example 8.10.1

The graph of y=2x−3 is shown.



For each ordered pair, decide:

1. Is the ordered pair a solution to the equation?
2. Is the point on the line?

A (0,−3) B (3,3) C (2,−3) D (−1,−5)

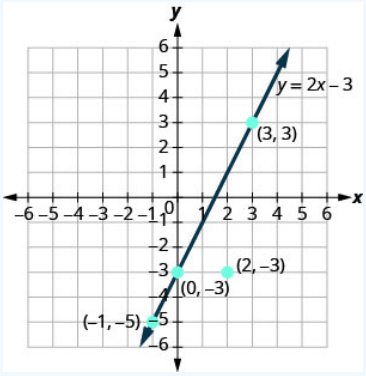
Solution

Substitute the *x*- and *y*- values into the equation to check if the ordered pair is a solution to the equation.

1.

A: (0, –3)	B: (3, 3)	C: (2, –3)	D: (–1, –5)
$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$
$-3 \stackrel{?}{=} 2(0) - 3$	$3 \stackrel{?}{=} 2(3) - 3$	$-3 \stackrel{?}{=} 2(2) - 3$	$-5 \stackrel{?}{=} 2(-1) - 3$
$-3 = -3$ ✓	$3 = 3$ ✓	$-3 \neq 1$	$-5 = -5$ ✓
(0, –3) is a solution.	(3, 3) is a solution.	(2, –3) is not a solution.	(–1, –5) is a solution.

2. Plot the points A (0,−3), B (3,3), C (2,−3), and D (−1,−5).



The points that are solutions to y=2x−3 are on the line, but the point that is not a solution is not on the line.

The points (0,−3), (3,3), and (−1,−5) are on the line y=2x−3, and the point (2,−3) is not on the line.

Try It 8.10.2

Use the graph of y=3x−1 to decide whether each ordered pair is:

- a solution to the equation.

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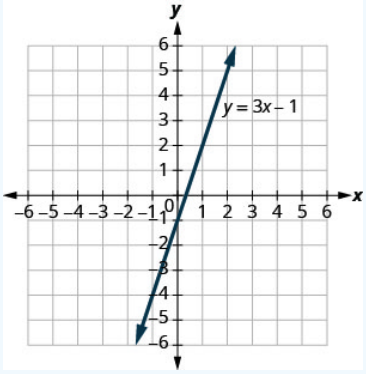
- on the line.

1. (0,−1)
2. (2,5)



Answer

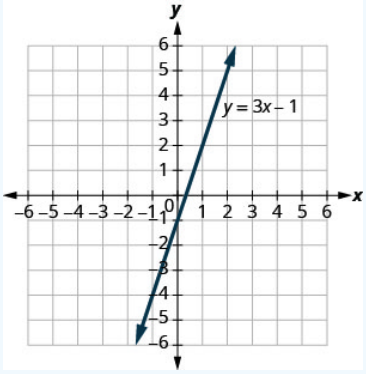
1. yes, yes
2. yes, yes

? Try It 8.10.3

Use graph of y=3x−1 to decide whether each ordered pair is:

- a solution to the equation
- on the line

1. (3,−1)
2. (−1,−4)



Answer

1. no, no
2. yes, yes

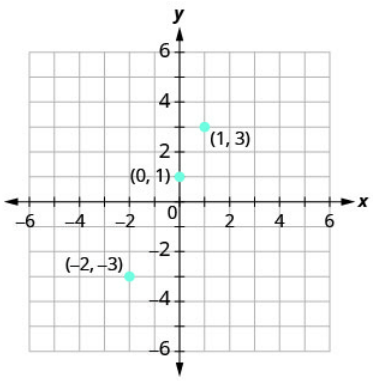
Graph a Linear Equation by Plotting Points

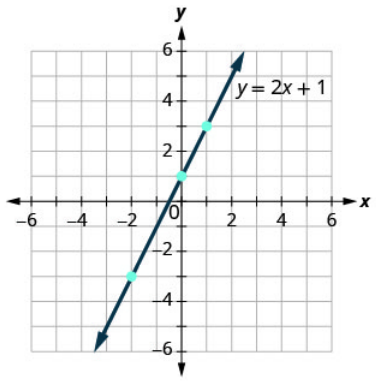
There are several methods that can be used to graph a linear equation. The method we used to graph 3x+2y=6 is called plotting points, or the Point–Plotting Method.

Example 8.10.4: How To Graph an Equation By Plotting Points

Graph the equation y=2x+1 by plotting points.

Solution

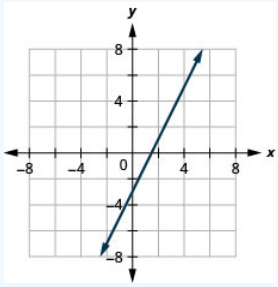
Step 1. Find three points whose coordinates are solutions to the equation.	You can choose any values for x or y. In this case, since y is isolated on the left side of the equation, it is easier to choose values for x.	$y = 2x + 1$ $x = 0$ $y = 2x + 1$ $y = 2 \cdot 0 + 1$ $y = 0 + 1$ $y = 1$ $x = 1$ $y = 2x + 1$ $y = 2 \cdot 1 + 1$ $y = 2 + 1$ $y = 3$ $x = -2$ $y = 2x + 1$ $y = 2(-2) + 1$ $y = -4 + 1$ $y = -3$
Organize the solutions in a table.	Put the three solutions in a table.	$y = 2x + 1$ x | y | (x, y) 0 | 1 | (0, 1) 1 | 3 | (1, 3) –2 | –3 | (–2, –3)
Step 2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work!	Plot: (0, 1), (1, 3), (–2, –3). Do the points line up? Yes, the points line up.	

Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.	This line is the graph of $y = 2x + 1$.	

Try It 8.10.5

Graph the equation by plotting points: y=2x−3.

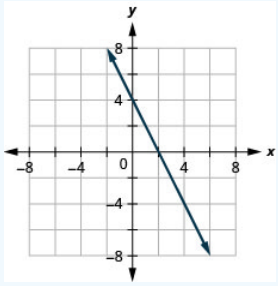
Answer



Try It 8.10.6

Graph the equation by plotting points: y=−2x+4.

Answer



The steps to take when graphing a linear equation by plotting points are summarized below.

GRAPH A LINEAR EQUATION BY PLOTTING POINTS

1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong and you need to check your work. Look at the difference between part (a) and part (b) in Figure 8.10.4

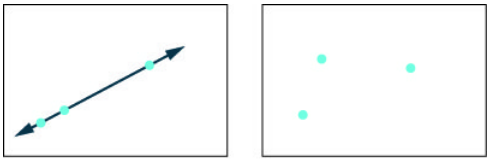


Figure 8.10.4

Let's do another example. This time, we'll show the last two steps all on one grid.

Example 8.10.7

Graph the equation y=−3x.

Solution

Find three points that are solutions to the equation. Here, again, it's easier to choose values for x. Do you see why?

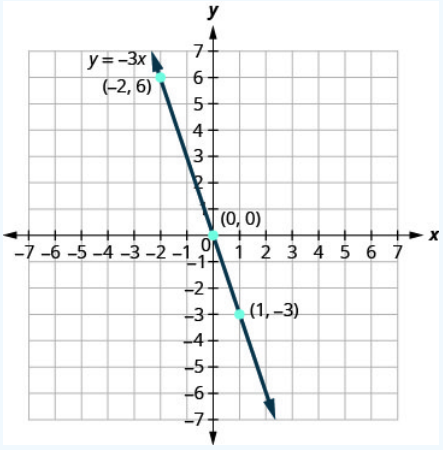
$x = 0$	$x = 1$	$x = -2$
$y = -3x$	$y = -3x$	$y = -3x$
$y = -3 \cdot 0$	$y = -3 \cdot 1$	$y = -3(-2)$
$y = 0$	$y = -3$	$y = 6$

We list the points in Table 8.10.2

Table 8.10.2

y=−3x		
x	**y**	**(x,y)**
0	0	(0,0)
1	−3	(1,−3)
−2	6	(−2,6)

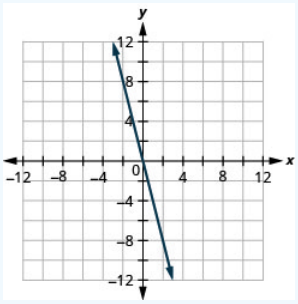
Plot the points, check that they line up, and draw the line.



Try It 8.10.8

Graph the equation by plotting points: y=−4x.

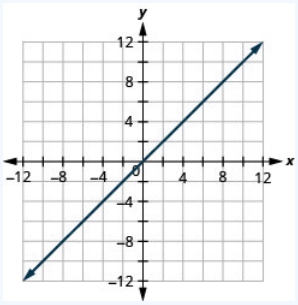
Answer



Try It 8.10.9

Graph the equation by plotting points: y=x.

Answer



When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for x. But the math is easier if we make ‘good’ choices for the values of x. This way we will avoid fraction answers, which are hard to graph precisely.

Example 8.10.10

Graph the equation $y = \frac{1}{2}x + 3$.

Solution

Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of 2 a good choice for values of x?

$x = 0$	$x = 2$	$x = 4$
$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$
$y = \frac{1}{2}(0) + 3$	$y = \frac{1}{2}(2) + 3$	$y = \frac{1}{2}(4) + 3$
$y = 0 + 3$	$y = 1 + 3$	$y = 2 + 3$
$y = 3$	$y = 4$	$y = 5$

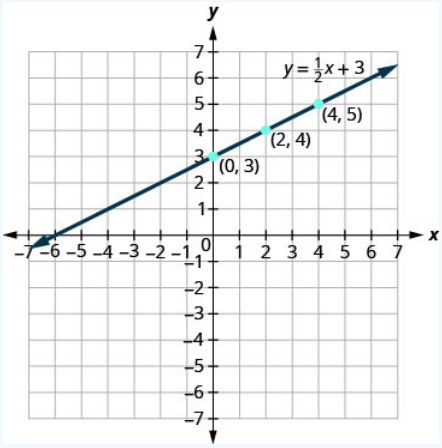
The points are shown in Table 8.10.3

Table 8.10.3

y=12x+3

x	y	(x,y)
0	3	(0,3)
2	4	(2,4)
4	5	(4,5)

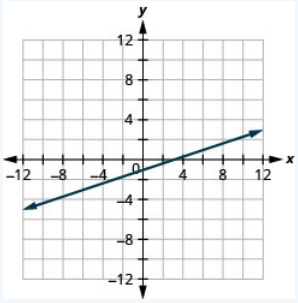
Plot the points, check that they line up, and draw the line.



Try It 8.10.11

Graph the equation $y = \frac{1}{3}x - 1$.

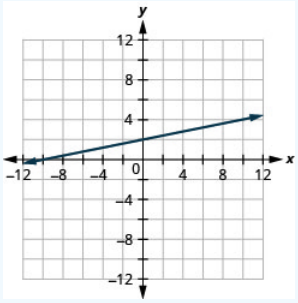
Answer



Try It 8.10.12

Graph the equation $y = \frac{1}{5}x + 2$.

Answer



So far, all the equations we graphed had y given in terms of x. Now we'll graph an equation with x and y on the same side. Let's see what happens in the equation 2x+y=3. If y=0 what is the value of x?

$$\begin{aligned} y &= 0 \\ 2x + y &= 3 \\ 2x + 0 &= 3 \\ 2x &= 3 \\ x &= \frac{3}{2} \\ \left(\frac{3}{2}, 0\right) \end{aligned}$$

This point has a fraction for the *x*- coordinate and, while we could graph this point, it is hard to be precise graphing fractions. Remember in the example y=12x+3, we carefully chose values for x so as not to graph fractions at all. If we solve the equation 2x+y=3 for y, it will be easier to find three solutions to the equation.

$$\begin{aligned} 2x + y &= 3 \\ y &= -2x + 3 \end{aligned}$$

The solutions for x=0, x=1, and x=−1 are shown in the Table 8.10.4 The graph is shown in Figure 8.10.5

Table 8.10.4

2x+y=3		
x	**y**	**(x,y)**
0	3	(0,3)
1	1	(1,1)
−1	5	(−1,5)

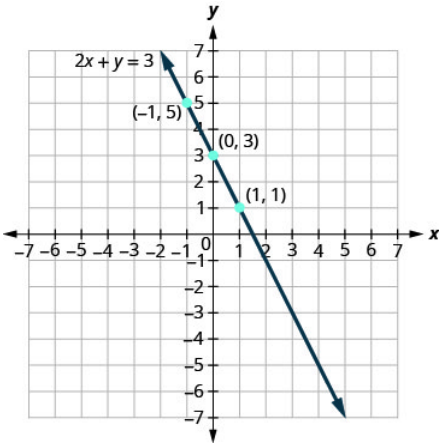


Figure 8.10.5

Can you locate the point $\left(\frac{3}{2}, 0\right)$ which we found by letting y=0, on the line?

Example 8.10.13

Graph the equation 3x+y=−1.

Solution

Find three points that are solutions to the equation. $3x + y = -1$

First solve the equation for y. $y = -3x - 1$

We'll let x be 0, 1, and −1 to find 3 points. The ordered pairs are shown in Table 8.10.5 Plot the points, check that they line up, and draw the line. See Figure 8.10.6

Table 8.10.5

3x+y=−1		
x	**y**	**(x,y)**
0	−1	(0,−1)
1	−4	(1,−4)
−1	2	(−1,2)

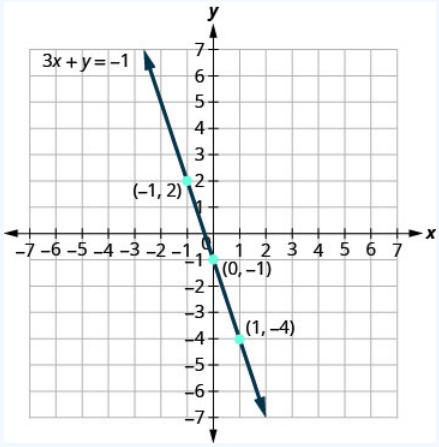
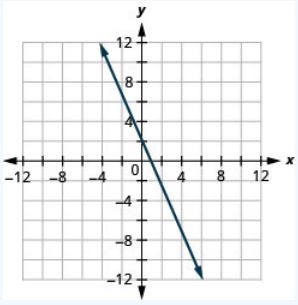


Figure 8.10.6

Try It 8.10.14

Graph the equation 2x+y=2.

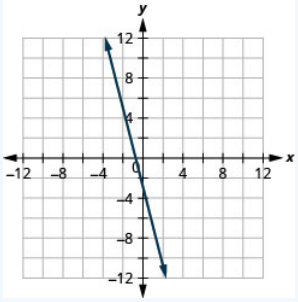
Answer



Try It 8.10.15

Graph the equation 4x+y=−3.

Answer



If you can choose any three points to graph a line, how will you know if your graph matches the one shown in the answers in the book? If the points where the graphs cross the *x*- and *y*-axis are the same, the graphs match!

The equation in Example 8.10.13was written in standard form, with both x and y on the same side. We solved that equation for y in just one step. But for other equations in standard form it is not that easy to solve for y, so we will leave them in standard form. We can still find a first point to plot by letting x=0 and solving for y. We can plot a second point by letting y=0 and then solving for x. Then we will plot a third point by using some other value for x or y.

Example 8.10.16

Graph the equation $2x-3y=6$.

Solution

Find three points that are solutions to the equation.	$2x-3y = 6$
	$2x-3y = 6$
First let $x=0$.	$2(0)-3y = 6$
Solve for y.	$-3y = 6$
	$y = -2$
Now let $y=0$.	$2x-3(0) = 6$
Solve for x.	$2x = 6$
	$x = 3$
We need a third point. Remember, we can choose any value for x or y. We'll let x = 6.	$2(6)-3y = 6$
	$12-3y = 6$
Solve for y.	$-3y = -6$
	$y = 2$

We list the ordered pairs in Table 8.10.6. Plot the points, check that they line up, and draw the line. See Figure 8.10.7.

Table 8.10.6

2x−3y=6		
x	**yT**	**(x,y)**
0	−2	(0,−2)
3	0	(3,0)
6	2	(6,2)

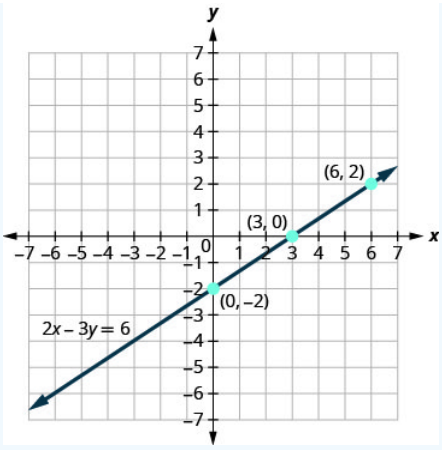
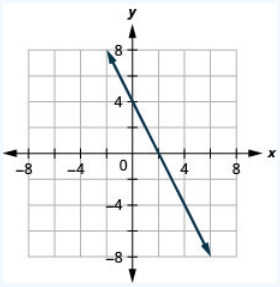


Figure 8.10.7

Try It 8.10.17

Graph the equation $4x + 2y = 8$.

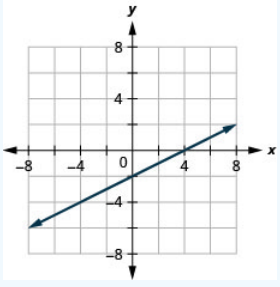
Answer



Try It 8.10.18

Graph the equation $2x - 4y = 8$.

Answer



Graph Vertical and Horizontal Lines

Can we graph an equation with only one variable? Just x and no y, or just y without an x? How will we make a table of values to get the points to plot?

Let's consider the equation x=−3. This equation has only one variable, x. The equation says that x is *always* equal to −3, so its value does not depend on y. No matter what y is, the value of x is always −3.

So to make a table of values, write −3 in for all the x values. Then choose any values for y. Since x does not depend on y, you can choose any numbers you like. But to fit the points on our coordinate graph, we'll use 1, 2, and 3 for the *y*-coordinates. See Table 8.10.7

Table 8.10.7

x=−3		
x	**y**	**(x,y)**
−3	1	(−3,1)
−3	2	(−3,2)
−3	3	(−3,3)

Plot the points from Table 8.10.7 and connect them with a straight line. Notice in Figure 8.10.8 that we have graphed a *vertical line*.

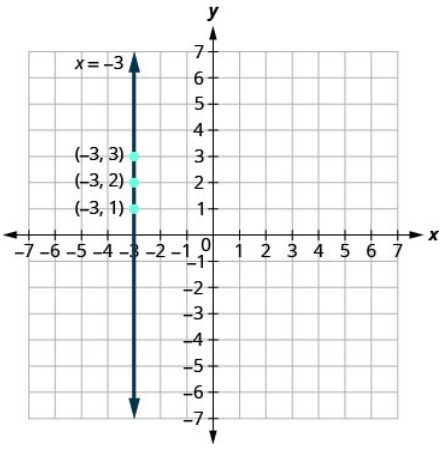


Figure 8.10.8

VERTICAL LINE

A **vertical line** is the graph of an equation of the form x=a.

The line passes through the *x*-axis at (a,0).

Example 8.10.19

Graph the equation x=2.

Solution

The equation has only one variable, x, and x is always equal to 2. We create Table 8.10.8 where x is always 2 and then put in any values for y. The graph is a vertical line passing through the *x*-axis at 2. See Figure 8.10.9

Table 8.10.8

x=2		
x	**y**	**(x,y)**
2	1	(2,1)
2	2	(2,2)
2	3	(2,3)

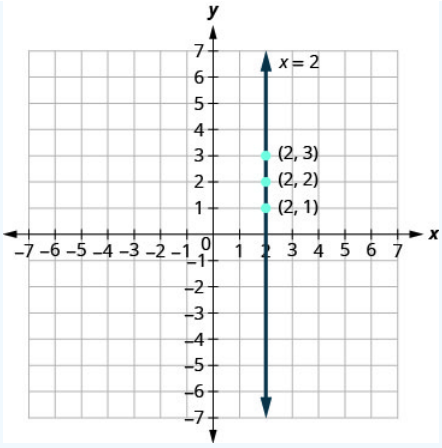
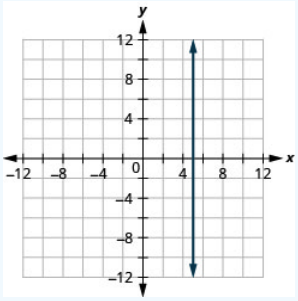


Figure 8.10.9

Try It 8.10.20

Graph the equation x=5.

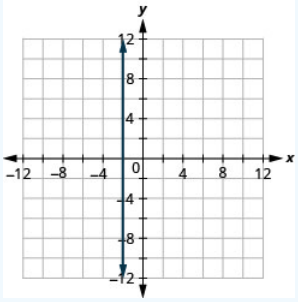
Answer



Try It 8.10.21

Graph the equation x=−2.

Answer



What if the equation has y but no x? Let's graph the equation y=4. This time the *y*- value is a constant, so in this equation, y does not depend on x. Fill in 4 for all the y's in Table 8.10.9 and then choose any values for x. We'll use 0, 2, and 4 for the *x*-coordinates.

Table 8.10.9

y=4		
x	**y**	**(x,y)**

0	4	(0,4)
2	4	(2,4)
4	4	(4,4)

The graph is a horizontal line passing through the *y*-axis at 4. See Figure 8.10.10

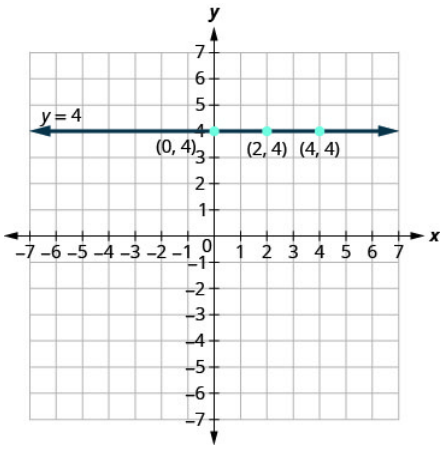


Figure 8.10.10

HORIZONTAL LINE

A **horizontal line** is the graph of an equation of the form y=b.

The line passes through the *y*-axis at (0,b).

Example 8.10.22

Graph the equation y=−1.

Solution

The equation y=−1 has only one variable, y. The value of y is constant. All the ordered pairs in Table 8.10.10have the same *y*-coordinate. The graph is a horizontal line passing through the *y*-axis at −1, as shown in Figure 8.10.11.

Table 8.10.10

y=−1		
x	**y**	**(x,y)**
	−1	(0,−1)
	−1	(3,−1)
−3	−1	(−3,−1)

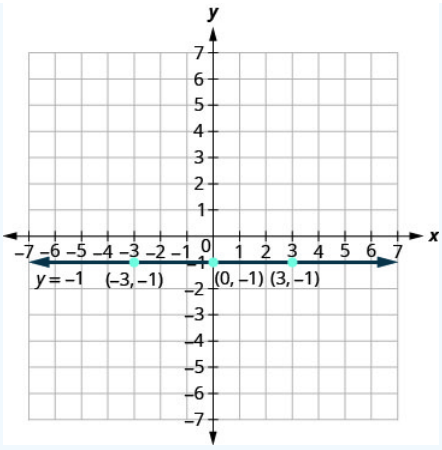
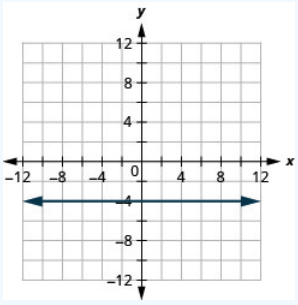


Figure 8.10.11

Try It 8.10.23

Graph the equation y=−4.

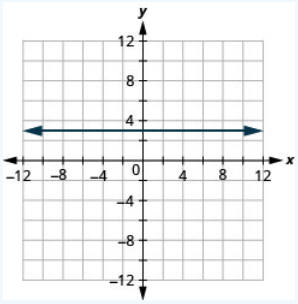
Answer



Try It 8.10.24

Graph the equation y=3.

Answer



The equations for vertical and horizontal lines look very similar to equations like y=4x. What is the difference between the equations y=4x and y=4?

The equation y=4x has both x and y. The value of y depends on the value of x. The *y*-coordinate changes according to the value of x. The equation y=4 has only one variable. The value of y is constant. The *y*-coordinate is always 4. It does not depend on the value of x. See Table 8.10.11.

Table 8.10.11

y=4x				y=4		
x	**y**	**(x,y)**		**x**	**y**	**(x,y)**
0	0	(0,0)		0	4	(0,4)
1	4	(1,4)		1	4	(1,4)
2	8	(2,8)		2	4	(2,4)

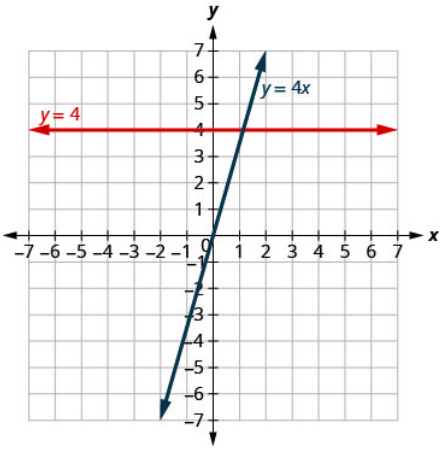


Figure 8.10.12

Notice, in Figure 8.10.12 the equation y=4x gives a slanted line, while y=4 gives a horizontal line.

Example 8.10.25

Graph y=−3x and y=−3 in the same rectangular coordinate system.

Solution

Notice that the first equation has the variable x, while the second does not. See Table 8.10.12 The two graphs are shown in Figure 8.10.13

Table 8.10.12

y=−3x				y=−3		
x	**y**	**(x,y)**		**x**	**y**	**(x,y)**
		(0,0)			−3	(0,−3)
	−3	(1,−3)			−3	(1,−3)
	−6	(2,−6)			−3	(2,−3)

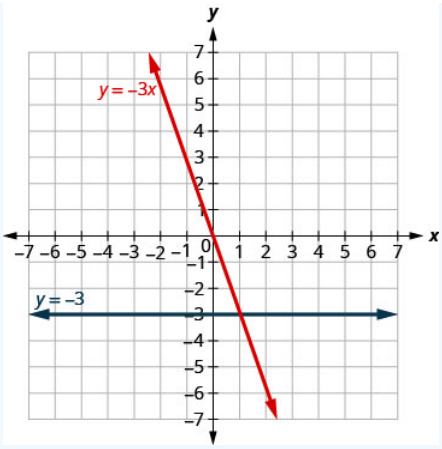
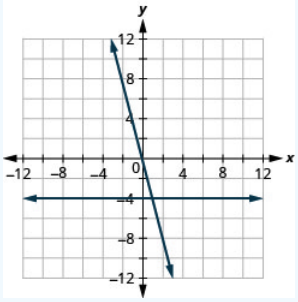


Figure 8.10.13

Try It 8.10.26

Graph y=−4x and y=−4 in the same rectangular coordinate system.

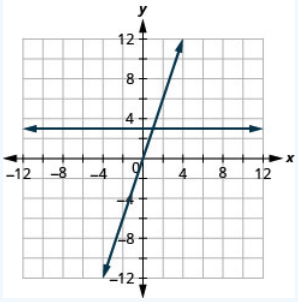
Answer



Try It 8.10.27

Graph y=3 and y=3x in the same rectangular coordinate system.

Answer



Key Concepts

- **Graph a Linear Equation by Plotting Points**
  1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
  2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work!
  3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

Glossary

graph of a linear equation

The graph of a linear equation Ax+By=C is a straight line. Every point on the line is a solution of the equation. Every solution of this equation is a point on this line.

horizontal line

A horizontal line is the graph of an equation of the form y=b. The line passes through the *y*-axis at (0,b).

vertical line

A vertical line is the graph of an equation of the form x=a. The line passes through the *x*-axis at (a,0).

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8.10E: Exercises

Practice Makes Perfect

Recognize the Relationship Between the Solutions of an Equation and its Graph

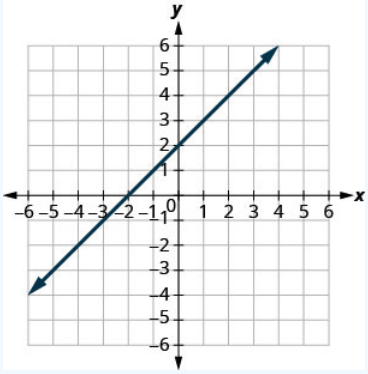
In the following exercises, for each ordered pair, decide:

1. Is the ordered pair a solution to the equation?
2. Is the point on the line?

Exercise 8.10.1

$y=x+2$

1. (0,2)
2. (1,2)
3. $(-1,1)$
4. $(-3,-1)$



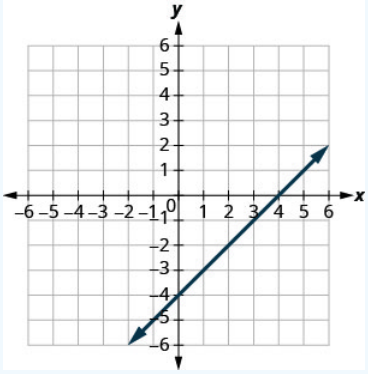
Answer

1. yes; no
2. no; no
3. yes; yes
4. yes; yes

Exercise 8.10.2

$y=x-4$

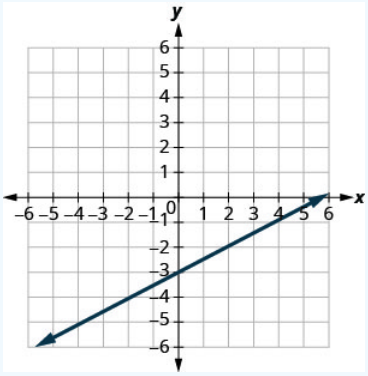
1. $(0,-4)$
2. $(3,-1)$
3. (2,2)
4. $(1,-5)$



Exercise 8.10.3

$y = \frac{1}{2}x - 3$

1. (0,−3)
2. (2,−2)
3. (−2,−4)
4. (4,1)



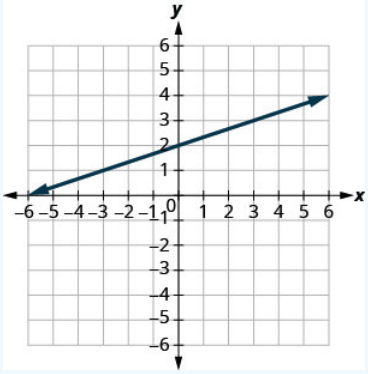
Answer

1. yes; yes
2. yes; yes
3. yes; yes
4. no; no

Exercise 8.10.4

$y = \frac{1}{3}x + 2$

1. (0,2)
2. (3,3)
3. (−3,2)
4. (−6,0)



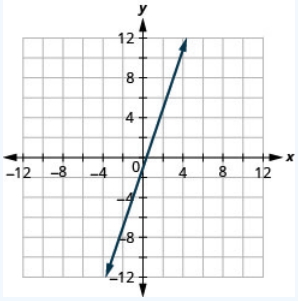
Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

Exercise 8.10.5

$y = 3x - 1$

Answer



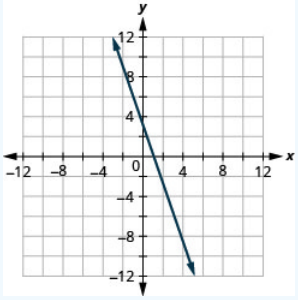
Exercise 8.10.6

$y = 2x + 3$

Exercise 8.10.7

$y = -2x + 2$

Answer



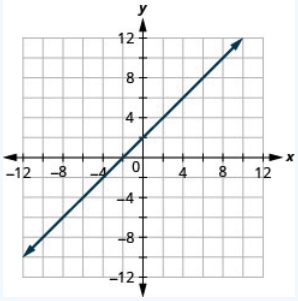
Exercise 8.10.8

$y = -3x + 1$

Exercise 8.10.9

$y = x + 2$

Answer



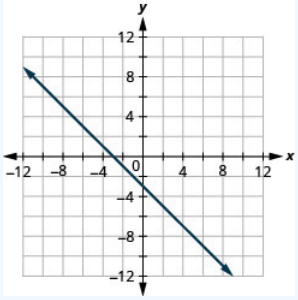
Exercise 8.10.10

$y = x - 3$

Exercise 8.10.11

$y = -x - 3$

Answer



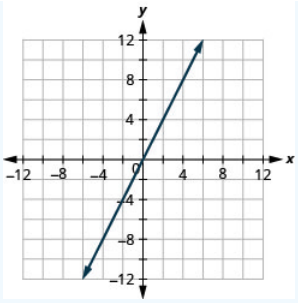
Exercise 8.10.12

$y = -x - 2$

Exercise 8.10.13

$y = 2x$

Answer



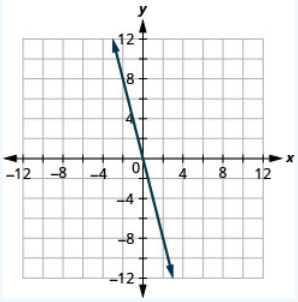
Exercise 8.10.14

$y = 3x$

Exercise 8.10.15

$y = -4x$

Answer



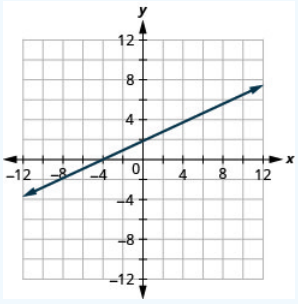
Exercise 8.10.16

$y = -2x$

Exercise 8.10.17

$y = \frac{1}{2}x + 2$

Answer



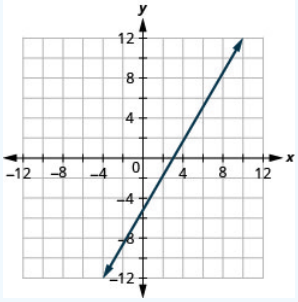
Exercise 8.10.18

$y = \frac{1}{3}x - 1$

Exercise 8.10.19

$y = \frac{4}{3}x - 5$

Answer



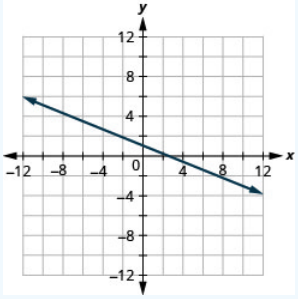
Exercise 8.10.20

$y = \frac{3}{2}x - 3$

Exercise 8.10.21

$y = -\frac{2}{5}x + 1$

Answer



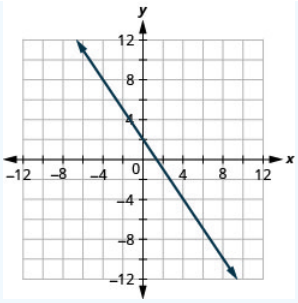
Exercise 8.10.22

$y = -\frac{4}{5}x - 1$

Exercise 8.10.23

$y = -\frac{3}{2}x + 2$

Answer



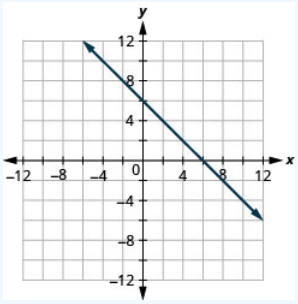
Exercise 8.10.24

$y = -\frac{5}{3}x + 4$

Exercise 8.10.25

$x + y = 6$

Answer



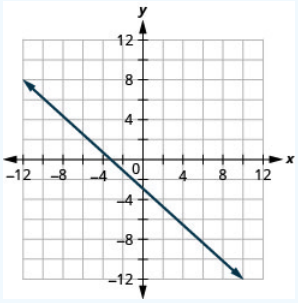
Exercise 8.10.26

$x + y = 4$

Exercise 8.10.27

$x + y = -3$

Answer



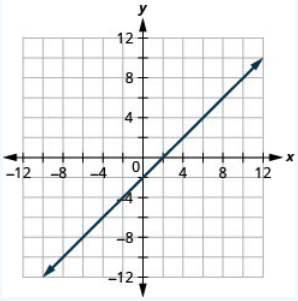
Exercise 8.10.28

$x + y = -2$

Exercise 8.10.29

$x - y = 2$

Answer



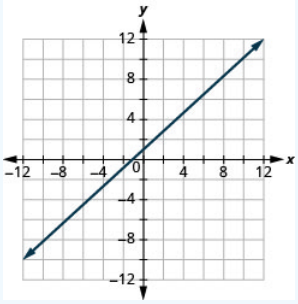
Exercise 8.10.30

$x - y = 1$

Exercise 8.10.31

$x - y = -1$

Answer



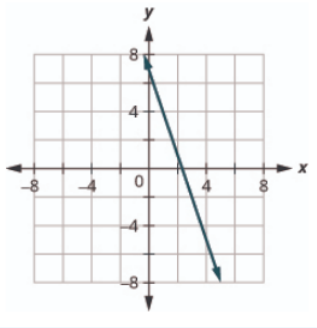
Exercise 8.10.32

$x - y = -3$

Exercise 8.10.33

$3x + y = 7$

Answer



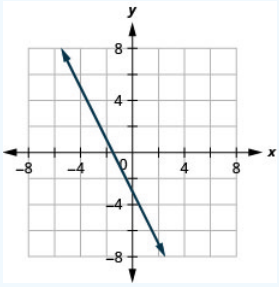
Exercise 8.10.34

$5x + y = 6$

Exercise 8.10.35

2x+y=−3

Answer



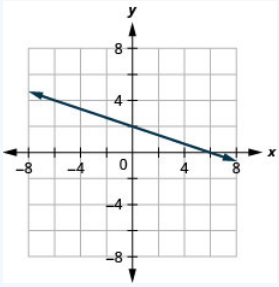
Exercise 8.10.36

$4x + y = -5$

Exercise 8.10.37

$\frac{1}{3}x + y = 2$

Answer



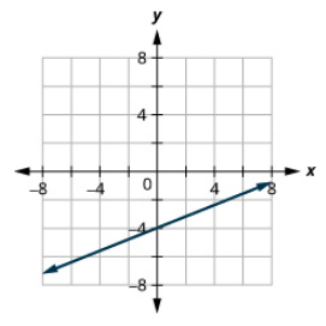
Exercise 8.10.38

$\frac{1}{2}x + y = 3$

Exercise 8.10.39

$\frac{2}{5}x - y = 4$

Answer



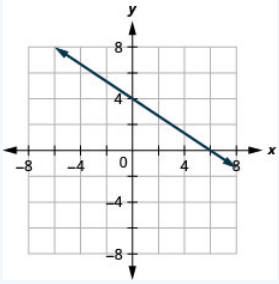
Exercise 8.10.40

$\frac{3}{4}x - y = 6$

Exercise 8.10.41

$2x + 3y = 12$

Answer



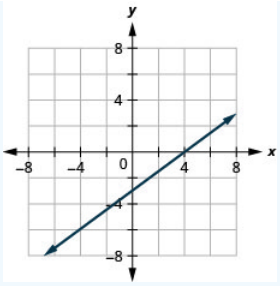
Exercise 8.10.42

4x+2y=12

Exercise 8.10.43

3x−4y=12

Answer



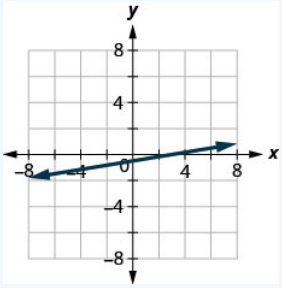
Exercise 8.10.44

$2x-5y=10$

Exercise 8.10.45

$x-6y=3$

Answer



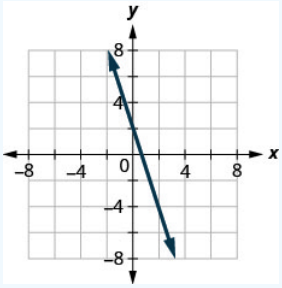
Exercise 8.10.46

$x-4y=2$

Exercise 8.10.47

$5x+2y=4$

Answer



Exercise 8.10.48

$3x+5y=5$

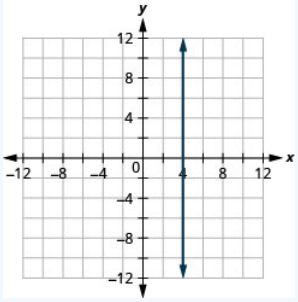
Graph Vertical and Horizontal Lines

In the following exercises, graph each equation.

Exercise 8.10.49

x=4

Answer



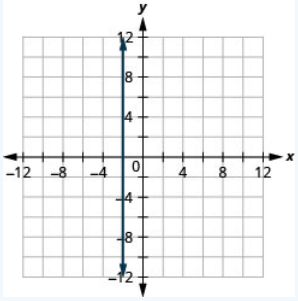
Exercise 8.10.50

x=3

Exercise 8.10.51

x=−2

Answer



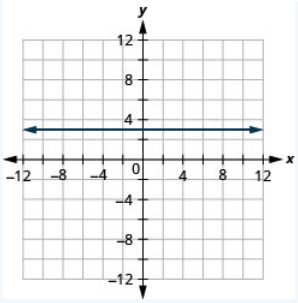
Exercise 8.10.52

x=−5

Exercise 8.10.53

y=3

Answer



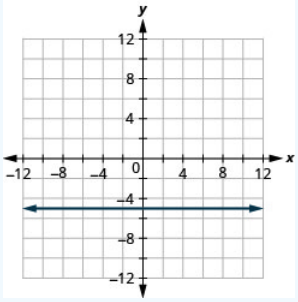
Exercise 8.10.54

y=1

Exercise 8.10.55

y=−5

Answer



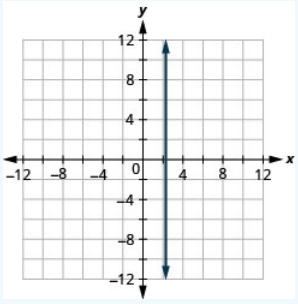
Exercise 8.10.56

y=−2

Exercise 8.10.57

$x = \frac{7}{3}$

Answer



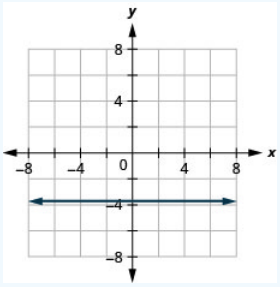
Exercise 8.10.58

$x = \frac{5}{4}$

Exercise 8.10.59

$y = -\frac{15}{4}$

Answer



Exercise 8.10.60

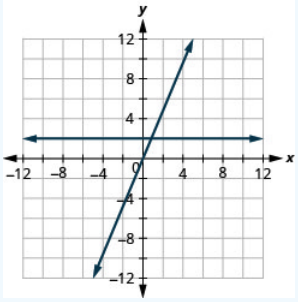
$y = -\frac{5}{3}$

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

Exercise 8.10.61

y=2x and y=2

Answer



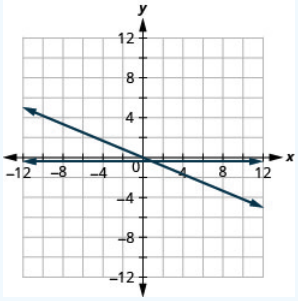
Exercise 8.10.62

y=5x and y=5

Exercise 8.10.63

$y = -\frac{1}{2}x$ and $y = -\frac{1}{2}$

Answer



Exercise 8.10.64

$y = -\frac{1}{3}x$ and $y = -\frac{1}{3}$

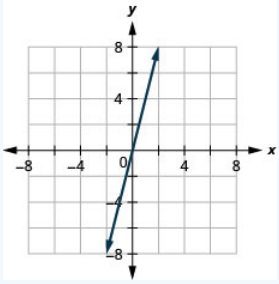
Mixed Practice

In the following exercises, graph each equation.

Exercise 8.10.65

y=4x

Answer



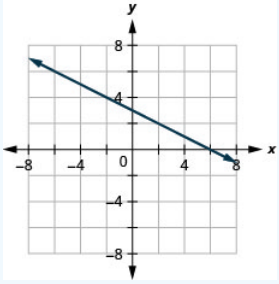
Exercise 8.10.66

y=2x

Exercise 8.10.67

$y = -\frac{1}{2}x + 3$

Answer



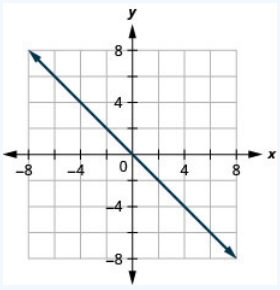
Exercise 8.10.68

$y = \frac{1}{4}x - 2$

Exercise 8.10.69

y=−x

Answer



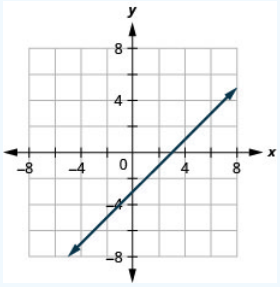
Exercise 8.10.70

y=x

Exercise 8.10.71

x−y=3

Answer



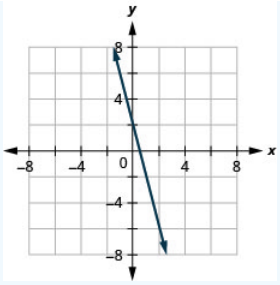
Exercise 8.10.72

x+y=−5

Exercise 8.10.73

4x+y=2

Answer



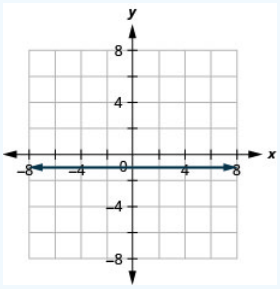
Exercise 8.10.74

2x+y=6

Exercise 8.10.75

y=−1

Answer



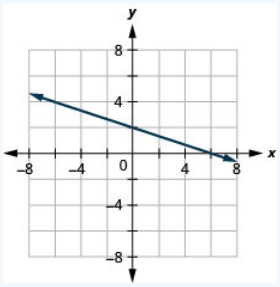
Exercise 8.10.76

y=5

Exercise 8.10.77

2x+6y=12

Answer



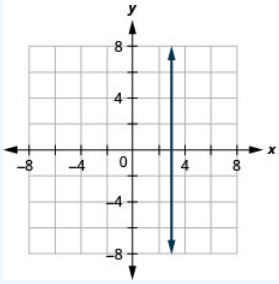
Exercise 8.10.78

5x+2y=10

Exercise 8.10.79

x=3

Answer



Exercise 8.10.80

x=−4

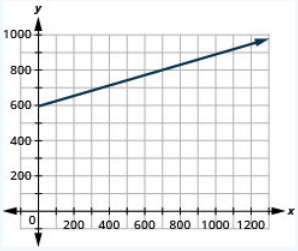
Everyday Math

Exercise 8.10.81

Motor home cost. The Robinsons rented a motor home for one week to go on vacation. It cost them $594 plus $0.32 per mile to rent the motor home, so the linear equation y=594+0.32x gives the cost, yy, for driving xx miles. Calculate the rental cost for driving 400, 800, and 1200 miles, and then graph the line.

Answer

$722, $850, $978



Exercise 8.10.82

Weekly earnings. At the art gallery where he works, Salvador gets paid $200 per week plus 15% of the sales he makes, so the equation y=200+0.15x gives the amount, yy, he earns for selling x dollars of artwork. Calculate the amount Salvador earns for selling $900, $1600, and $2000, and then graph the line.

Writing Exercises

Exercise 8.10.83

Explain how you would choose three x - values to make a table to graph the line $y = \frac{1}{5}x - 2$

Answer

Answers will vary.

Exercise 8.10.84

What is the difference between the equations of a vertical and a horizontal line?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize the relation between the solutions of an equation and its graph.			
graph a linear equation by plotting points.			
graph vertical and horizontal lines.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

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8.11: Graphing with Intercepts (Part 2)

Choose the Most Convenient Method to Graph a Line

While we could graph any linear equation by plotting points, it may not always be the most convenient method. This table shows six of equations we've graphed in this chapter, and the methods we used to graph them.

	Equation	Method
#1	$y = 2x + 1$	Plotting points
#2	$y = \frac{1}{2}x + 3$	Plotting points
#3	$x = -7$	Vertical line
#4	$y = 4$	Horizontal line
#5	$2x + y = 6$	Intercepts
#6	$4x - 3y = 12$	Intercepts

What is it about the form of equation that can help us choose the most convenient method to graph its line?

Notice that in equations #1 and #2, y is isolated on one side of the equation, and its coefficient is 1. We found points by substituting values for x on the right side of the equation and then simplifying to get the corresponding y- values.

Equations #3 and #4 each have just one variable. Remember, in this kind of equation the value of that one variable is constant; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In equations #5 and #6, both x and y are on the same side of the equation. These two equations are of the form $Ax + By = C$. We substituted $y = 0$ and $x = 0$ to find the x- and y- intercepts, and then found a third point by choosing a value for x or y.

This leads to the following strategy for choosing the most convenient method to graph a line.

HOW TO: CHOOSE THE MOST CONVENIENT METHOD TO GRAPH A LINE

Step 1. If the equation has only one variable. It is a vertical or horizontal line.

- $x = a$ is a vertical line passing through the x-axis at a.
- $y = b$ is a horizontal line passing through the y-axis at b.

Step 2. If y is isolated on one side of the equation. Graph by plotting points.

- Choose any three values for x and then solve for the corresponding y- values.

Step 3. If the equation is of the form $Ax + By = C$, find the intercepts.

- Find the x- and y- intercepts and then a third point.

Example 8.11.7:

Identify the most convenient method to graph each line: (a) $y = -3$ (b) $4x-6y = 12$ (c) $x = 2$ (d) $y = 2\ 5\ x-1$

Solution

(a) $y = -3$

This equation has only one variable, y. Its graph is a horizontal line crossing the y-axis at -3.

(b) $4x-6y = 12$

This equation is of the form $Ax + By = C$. Find the intercepts and one more point.

(c) $x = 2$

There is only one variable, x. The graph is a vertical line crossing the x-axis at 2.

(d) $y = \frac{2}{5}x - 1$

Since y is isolated on the left side of the equation, it will be easiest to graph this line by plotting three points.

Exercise 8.11.7A:

Identify the most convenient method to graph each line: (a) $3x + 2y = 12$ (b) $y = 4$ (c) $y = \frac{1}{5}x - 4$ (d) $x = -7$

Answer a

intercepts

Answer b

horizontal line

Answer c

plotting points

Answer d

vertical line

Exercise 8.11.7B:

Identify the most convenient method to graph each line: (a) $x = 6$ (b) $y = -\frac{3}{4}x + 1$ (c) $y = -8$ (d) $4x - 3y = -1$

Answer a

vertical line

Answer b

plotting points

Answer c

horizontal line

Answer d

intercepts

ACCESS ADDITIONAL ONLINE RESOURCES

Graph by Finding Intercepts

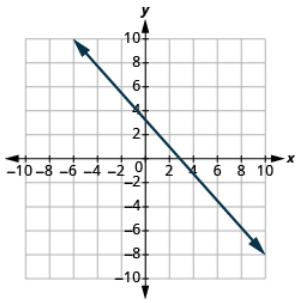
Use Intercepts to Graph

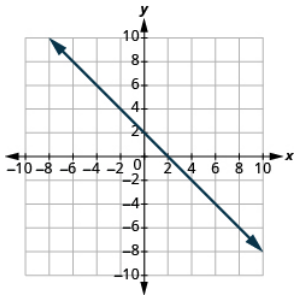
State the Intercepts from a Graph

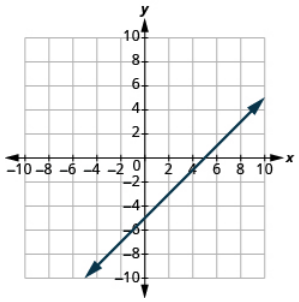
Practice Makes Perfect

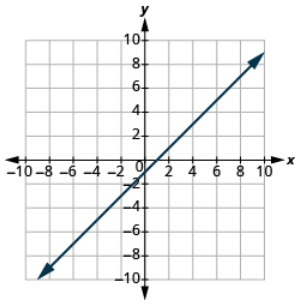
Identify the Intercepts on a Graph

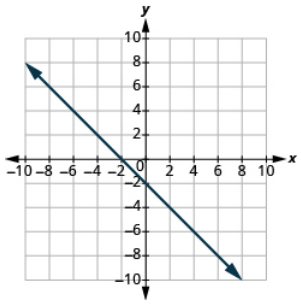
In the following exercises, find the x- and y- intercepts.

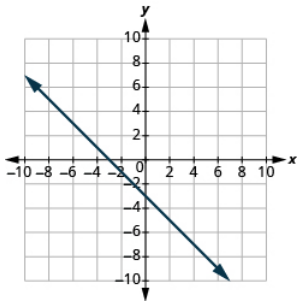
117. 

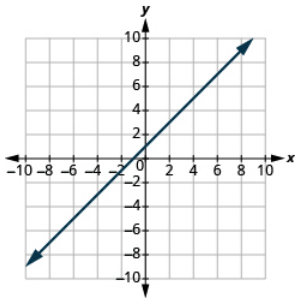
118. 

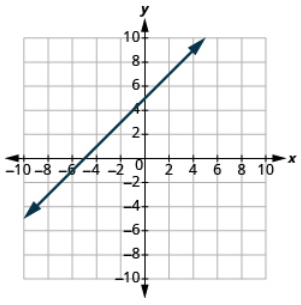
119. 

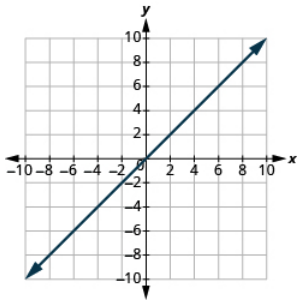
120. 

121. 

122. 

123. 

124. 

125. 

Find the x and y Intercepts from an Equation of a Line

In the following exercises, find the intercepts.

127. x + y = 4
128. x + y = 3
129. x + y = −2
130. x + y = −5
131. x − y = 5
132. x − y = 1
133. x − y = −3
134. x − y = −4
135. x + 2y = 8
136. x + 2y = 10
137. 3x + y = 6
138. 3x + y = 9
139. x−3y = 12

140. $x-2y = 8$
141. $4x - y = 8$
142. $5x - y = 5$
143. $2x + 5y = 10$
144. $2x + 3y = 6$
145. $3x-2y = 12$
146. $3x-5y = 30$
147. $y = \frac{1}{3}x-1$
148. $y = \frac{1}{4}x-1$
149. $y = \frac{1}{5}x + 2$
150. $y = \frac{1}{3}x + 4$
151. $y = 3x$
152. $y = -2x$
153. $y = -4x$
154. $y = 5x$

Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

155. $-x + 5y = 10$
156. $-x + 4y = 8$
157. $x + 2y = 4$
158. $x + 2y = 6$
159. $x + y = 2$
160. $x + y = 5$
161. $x + y = -3$
162. $x + y = -1$
163. $x - y = 1$
164. $x - y = 2$
165. $x - y = -4$
166. $x - y = -3$
167. $4x + y = 4$
168. $3x + y = 3$
169. $3x - y = -6$
170. $2x - y = -8$
171. $2x + 4y = 12$
172. $3x + 2y = 12$
173. $3x-2y = 6$
174. $5x-2y = 10$
175. $2x-5y = -20$
176. $3x-4y = -12$
177. $y = -2x$
178. $y = -4x$
179. $y = x$
180. $y = 3x$

Choose the Most Convenient Method to Graph a Line

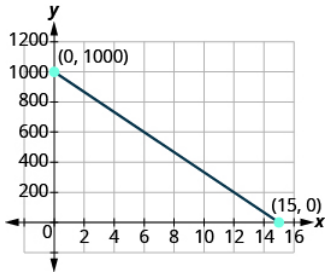
In the following exercises, identify the most convenient method to graph each line.

181. $x = 2$
182. $y = 4$

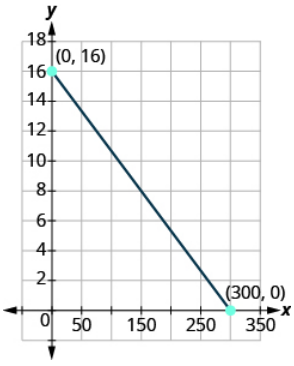
183. $y = 5$
184. $x = -3$
185. $y = -3x + 4$
186. $y = -5x + 2$
187. $x - y = 5$
188. $x - y = 1$
189. $y = \frac{2}{3}x - 1$
190. $y = \frac{4}{5}x - 3$
191. $y = -3$
192. $y = -1$
193. $3x - 2y = -12$
194. $2x - 5y = -10$
195. $y = -\frac{1}{4}x + 3$
196. $y = -\frac{1}{3}x + 5$

Everyday Math

197. **Road trip** Damien is driving from Chicago to Denver, a distance of 1,000 miles. The x-axis on the graph below shows the time in hours since Damien left Chicago. The y-axis represents the distance he has left to drive.
    a. Find the x- and y- intercepts.
    b. Explain what the x- and y- intercepts mean for Damien.



198. **Road trip** Ozzie filled up the gas tank of his truck and went on a road trip. The x-axis on the graph shows the number of miles Ozzie drove since filling up. The y-axis represents the number of gallons of gas in the truck's gas tank.
    a. Find the x- and y- intercepts.
    b. Explain what the x- and y- intercepts mean for Ozzie.



Writing Exercises

199. How do you find the x-intercept of the graph of $3x - 2y = 6$?
200. How do you find the y-intercept of the graph of $5x - y = 10$?
201. Do you prefer to graph the equation $4x + y = -4$ by plotting points or intercepts? Why?

202. Do you prefer to graph the equation $y = \frac{2}{3}x - 2$ by plotting points or intercepts? Why?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify the intercepts on a graph.			
find the intercepts from an equation of a line.			
graph a line using the intercepts.			
choose the most convenient method to graph a line.			

(b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Contributors and Attributions

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8.12: Graph with Intercepts

Learning Objectives

By the end of this section, you will be able to:

- Identify the x- and y- intercepts on a graph
- Find the x- and y- intercepts from an equation of a line
- Graph a line using the intercepts

Note

Before you get started, take this readiness quiz.

1. Solve: $3 \cdot 0 + 4y = -2$.
   If you missed this problem, review Exercise 2.2.13.

Identify the *x*- and *y*- Intercepts on a Graph

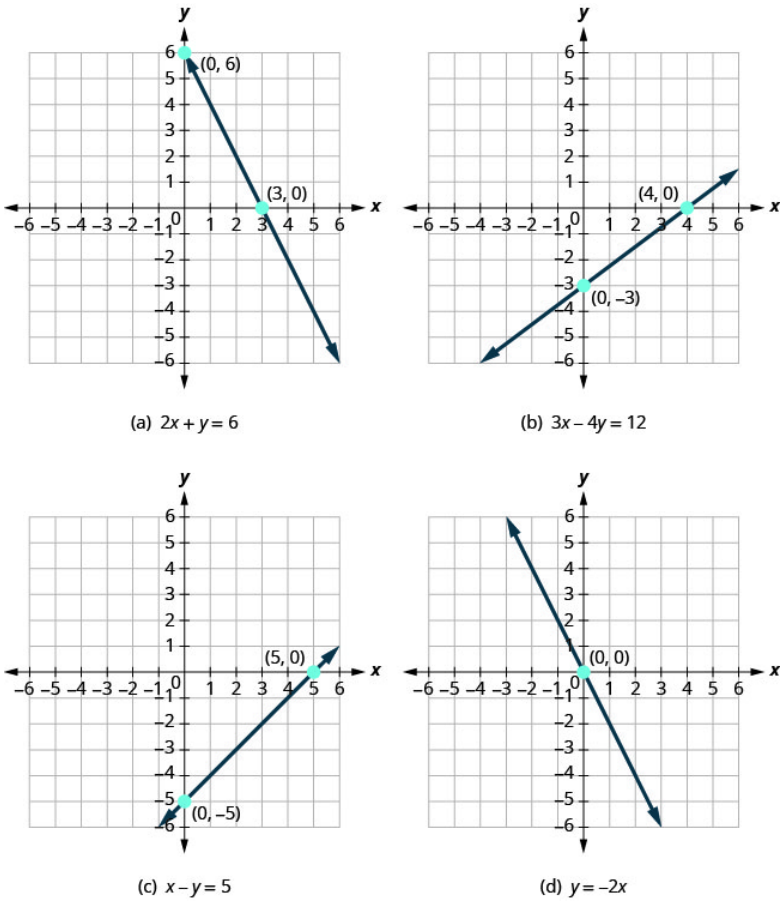
Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line crosses the *x*- axis and the *y*- axis. These points are called the *intercepts* of the line.

INTERCEPTS OF A LINE

The points where a line crosses the *x*- axis and the *y*- axis are called the **intercepts of a line**.

Let's look at the graphs of the lines in Figure 8.12.1.



(a) $2x + y = 6$ (b) $3x - 4y = 12$

(c) $x - y = 5$ (d) $y = -2x$

Figure 8.12.1: Examples of graphs crossing the x-negative axis.

First, notice where each of these lines crosses the x axis. See Figure 8.12.1.

Table 8.12.1

Figure	The line crosses the x-axis at:	Ordered pair of this point
Figure (a)	3	(3,0)
Figure (b)	4	(4,0)
Figure (c)	5	(5,0)
Figure (d)	0	(0,0)

Do you see a pattern?

For each row, the *y*- coordinate of the point where the line crosses the *x*- axis is zero. The point where the line crosses the *x*- axis has the form (a,0) and is called the ***x*- intercept of a line**. The *x*- intercept occurs when y is zero. Now, let's look at the points where these lines cross the *y*- axis. See Table 8.12.2

Table 8.12.2

Figure	The line crosses the x-axis at:	Ordered pair of this point
Figure (a)	6	(0,6)
Figure (b)	−3	(0,−3)
Figure (c)	−5	(0,5)
Figure (d)	0	(0,0)

What is the pattern here?

In each row, the *x*- coordinate of the point where the line crosses the *y*- axis is zero. The point where the line crosses the *y*- axis has the form (0,b) and is called the *y*- *intercept* of the line. The *y*- intercept occurs when x is zero.

X- INTERCEPT AND Y- INTERCEPT OF A LINE

The *x*- intercept is the point (a,0) where the line crosses the *x*- axis.

The *y*- intercept is the point (0,b) where the line crosses the *y*- axis.

	x	*y*
• The *x*-intercept occurs when *y* is zero.	*a*	0
• The *y*-intercept occurs when *x* is zero.	0	*b*

Figure 8.12.2

Example 8.12.1

Find the *x*- and *y*- intercepts on each graph.

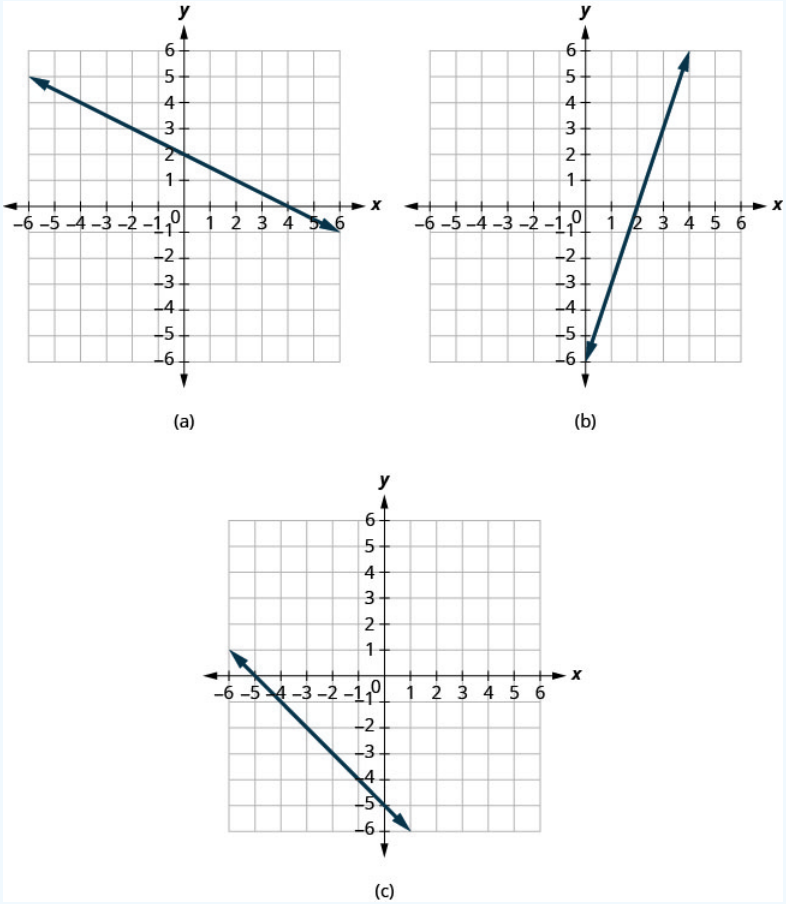


Figure 8.12.3

Solution

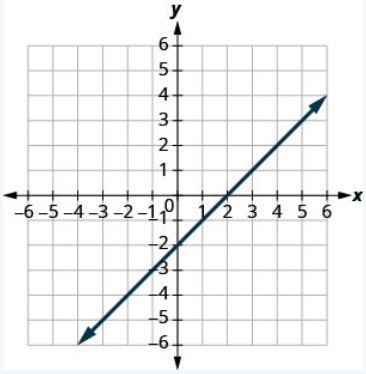
(a) The graph crosses the *x*- axis at the point (4,0). The *x*- intercept is (4,0).
The graph crosses the *y*- axis at the point (0,2). The *y*- intercept is (0,2).

(b) The graph crosses the x- axis at the point (2,0). The x- intercept is (2,0)
The graph crosses the y- axis at the point (0,−6). The y- intercept is (0,−6).

(c) The graph crosses the x- axis at the point (−5,0). The x- intercept is (−5,0).
The graph crosses the y- axis at the point (0,−5). The y- intercept is (0,−5).

Try It 8.12.2

Find the x- and y- intercepts on the graph.

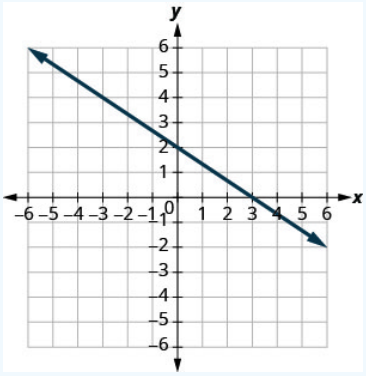


Answer

x- intercept: (2,0); y- intercept: (0,−2)

Try It 8.12.3

Find the x- and y- intercepts on the graph.



Answer

x- intercept: (3,0), y- intercept: (0,2)

Find the x- and y- Intercepts from an Equation of a Line

Recognizing that the ***x*- intercept** occurs when y is zero and that the y- intercept occurs when x is zero, gives us a method to find the intercepts of a line from its equation. To find the x- intercept, let y=0 and solve for x. To find the ***y*- intercept**, let x=0 and solve for y.

X- AND Y- INTERCEPTS FROM THE EQUATION OF A LINE

Use the equation of the line. To find:

- the *x*- intercept of the line, let y=0 and solve for x.
- the *y*- intercept of the line, let x=0 and solve for y.

Example 8.12.4

Find the intercepts of 2x+y=6.

Solution

We will let y=0 to find the *x*- intercept, and let x=0 to find the *y*- intercept. We will fill in the table, which reminds us of what we need to find.

$2x+y=6$	
x	y
	0
0	

x-intercept

y-intercept

To find the *x*- intercept, let y=0.

Table 8.12.3

	$2x+y=6$
Let $y = 0$.	$2x+0=6$
Simplify.	$2x=6$
	$x=3$
The *x*-intercept is	(3, 0)
To find the *y*-intercept, let $x = 0$.	
	$2x+y=6$
Let $x = 0$.	$2\cdot 0+y=6$
Simplify.	$0+y=6$
	$y=6$
The *y*-intercept is	(0, 6)

The intercepts are the points (3,0) and (0,6) as shown in Table 8.12.4

Table 8.12.4

2x+y=6	
x	**y**
3	0
0	6

Try It 8.12.5

Find the intercepts of 3x+y=12.

Answer

x- intercept: (4,0), *y*- intercept: (0,12)

Try It 8.12.6

Find the intercepts of x+4y=8.

Answer

x- intercept: (8,0), *y*- intercept: (0,2)

Example 8.12.7

Find the intercepts of 4x–3y=12.

Solution

To find the *x*-intercept, let $y = 0$.	
	$4x - 3y = 12$
Let $y = 0$.	$4x - 3 \cdot 0 = 12$
Simplify.	$4x - 0 = 12$
	$4x = 12$
	$x = 3$
The *x*-intercept is	(3, 0)
To find the *y*-intercept, let $x = 0$.	
	$4x - 3y = 12$
Let $x = 0$.	$4 \cdot 0 - 3y = 12$
Simplify.	$0 - 3y = 12$
	$-3y = 12$
	$y = -4$
The *y*-intercept is	(0, −4)

Table 8.12.5

The intercepts are the points (3, 0) and (0, −4) as shown in the following table.

Table 8.12.6

4x−3y=12	
x	**y**
3	0
0	−4

Try It 8.12.8

Find the intercepts of 3x–4y=12.

Answer

x- intercept: (4,0), *y*- intercept: (0,−3)

Try It 8.12.9

Find the intercepts of 2x–4y=8.

Answer

x- intercept: (4,0), *y*- intercept: (0,−2)

Graph a Line Using the Intercepts

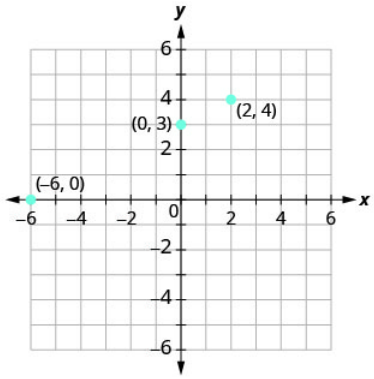
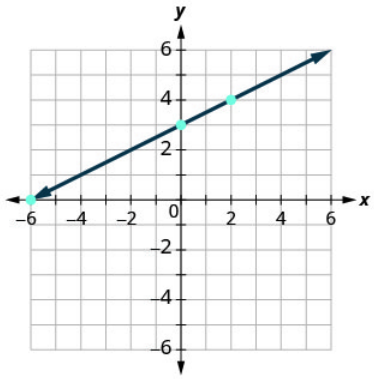
To graph a linear equation by plotting points, you need to find three points whose coordinates are solutions to the equation. You can use the *x*- and *y*- intercepts as two of your three points. Find the intercepts, and then find a third point to ensure accuracy. Make sure the points line up—then draw the line. This method is often the quickest way to graph a line.

Example 8.12.10: How to Graph a Line Using Intercepts

Graph –x+2y=6 using the intercepts.

Solution

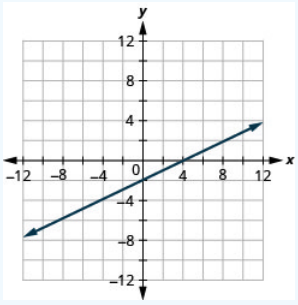
Step 1. Find the *x*- and *y*-intercepts of the line. Let $y = 0$ and solve for x. Let $x = 0$ and solve for y.	Find the *x*-intercept.	Let $y = 0$ $-x + 2y = 6$ $-x + 2(0) = 6$ $-x = 6$ $x = -6$ The *x*-intercept is (–6, 0).
	Find the *y*-intercept.	Let $x = 0$ $-x + 2y = 6$ $-0 + 2y = 6$ $2y = 6$ $y = 3$ The *y*-intercept is (0, 3).
Step 2. Find another solution to the equation.	We'll use $x = 2$.	Let $x = 2$ $-x + 2y = 6$ $-2 + 2y = 6$ $2y = 8$ $y = 4$ A third point is (2, 4).

Step 3. Plot the three points. Check that the points line up.	x | y | (x, y) -6 | 0 | $(-6, 0)$ 0 | 3 | $(0, 3)$ 2 | 4 | $(2, 4)$	
Step 4. Draw the line.	See the graph.	

Try It 8.12.11

Graph x–2y=4 using the intercepts.

Answer

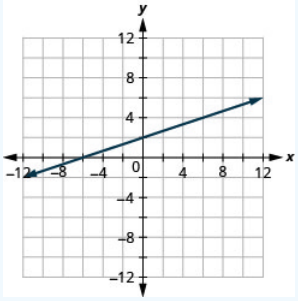


Try It 8.12.12

Graph –x+3y=6 using the intercepts.

Answer

https://math.libretexts.org/@go/page/46281



The steps to graph a linear equation using the intercepts are summarized below.

GRAPH A LINEAR EQUATION USING THE INTERCEPTS.

1. Find the *x*- and *y*- intercepts of the line.
   - Let y=0 and solve for x
   - Let x=0 and solve for y.
2. Find a third solution to the equation.
3. Plot the three points and check that they line up.
4. Draw the line.

Example 8.12.13

Graph 4x–3y=12 using the intercepts.

Solution

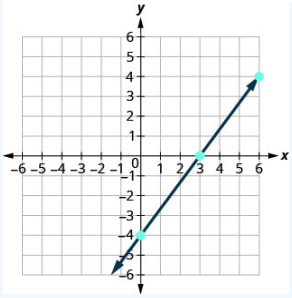
Find the intercepts and a third point.

x-intercept, let $y = 0$	*y*-intercept, let $x = 0$	third point, let $y = 4$
$4x - 3y = 12$	$4x - 3y = 12$	$4x - 3y = 12$
$4x - 3(0) = 12$	$4(0) - 3y = 12$	$4x - 3(4) = 12$
$4x = 12$	$-3y = 12$	$4x - 12 = 12$
$x = 3$	$y = -4$	$4x = 24$
		$x = 6$

We list the points in Table 8.12.7and show the graph below.

4x−3y=12		
x	**y**	**(x,y)**
3	0	(3,0)
0	−4	(0,−4)
6	4	(6,4)

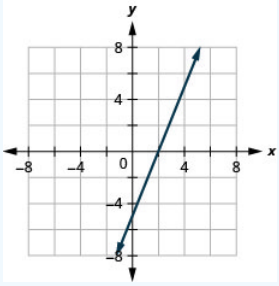
Table 8.12.7



Try It 8.12.14

Graph 5x–2y=10 using the intercepts.

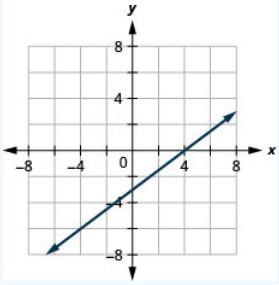
Answer



Try It 8.12.15

Graph 3x–4y=12 using the intercepts.

Answer



Example 8.12.16

Graph y=5x using the intercepts.

Solution

x-intercept	*y*-intercept
Let $y = 0$.	Let $x = 0$.
$y = 5x$	$y = 5x$
$0 = 5x$	$y = 5 \cdot 0$
$0 = x$	$y = 0$
(0, 0)	(0, 0)

This line has only one intercept. It is the point (0,0).

To ensure accuracy we need to plot three points. Since the x- and y- intercepts are the same point, we need *two* more points to graph the line.

Let $x = 1$.
$y = 5x$
$y = 5 \cdot 1$
$y = 5$

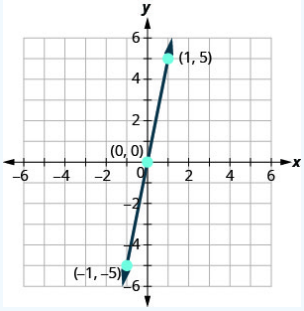
Let $x = -1$.
$y = 5x$
$y = 5(-1)$
$y = -5$

See Table 8.12.8

y=5x		
x	**y**	**(x,y)**
0	0	(0,0)
1	5	(1,5)
−1	−5	(−1,−5)

Table 8.12.8

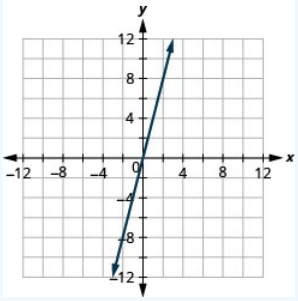
Plot the three points, check that they line up, and draw the line.



Try It 8.12.17

Graph y=4x using the intercepts.

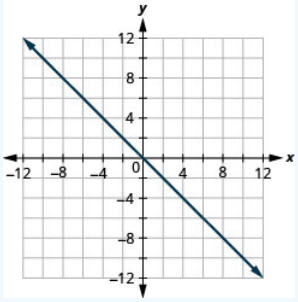
Answer



Try It 8.12.18

Graph y=−x the intercepts.

Answer



Key Concepts

- **Find the *x*- and *y*- Intercepts from the Equation of a Line**
  - Use the equation of the line to find the *x*- intercept of the line, let y=0 and solve for *x*.
  - Use the equation of the line to find the *y*- intercept of the line, let x=0 and solve for *y*.
- **Graph a Linear Equation using the Intercepts**
  1. Find the *x*- and *y*- intercepts of the line.
     Let y=0 and solve for *x*.
     Let x=0 and solve for *y*.
  2. Find a third solution to the equation.
  3. Plot the three points and then check that they line up.
  4. Draw the line.
- **Strategy for Choosing the Most Convenient Method to Graph a Line:**
  - Consider the form of the equation.
  - If it only has one variable, it is a vertical or horizontal line.
    x=a is a vertical line passing through the *x*- axis at a
    y=b is a horizontal line passing through the *y*- axis at b.
  - If *y* is isolated on one side of the equation, graph by plotting points.
  - Choose any three values for *x* and then solve for the corresponding *y*- values.
  - If the equation is of the form ax+by=c, find the intercepts. Find the *x*- and *y*- intercepts and then a third point.

Glossary

intercepts of a line

The points where a line crosses the *x*- axis and the *y*- axis are called the intercepts of the line.

***x*- intercept**

The point (a,0) where the line crosses the *x*- axis; the *x*- intercept occurs when y is zero.

***y*-intercept**

The point (0,b) where the line crosses the *y*- axis; the *y*- intercept occurs when x is zero.

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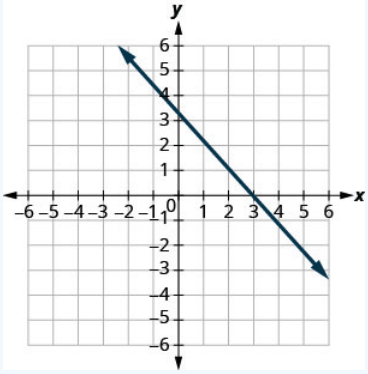
8.12E: Exercises

Practice Makes Perfect

Identify the *x*- and *y*- Intercepts on a Graph

In the following exercises, find the *x*- and *y*- intercepts on each graph.

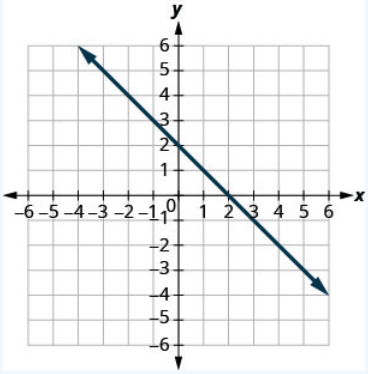
Exercise 8.12E.1



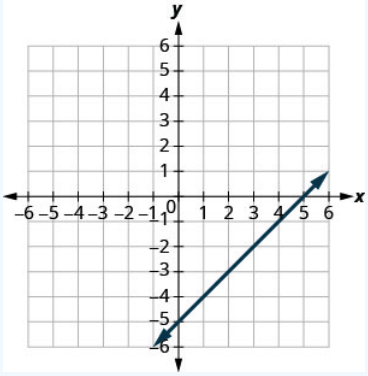
Answer

(3,0),(0,3)

Exercise 8.12E.2



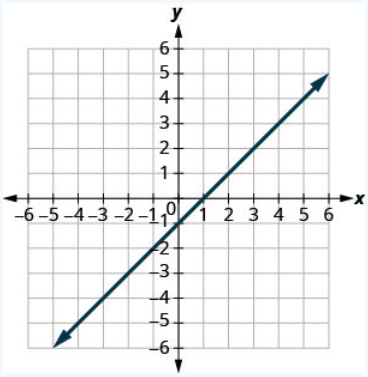
Exercise 8.12E.3



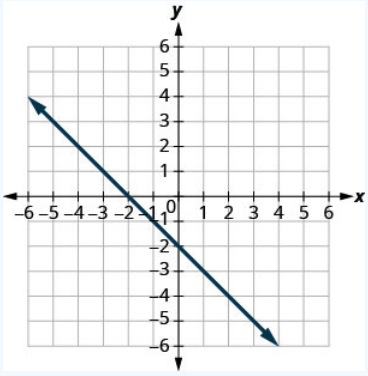
Answer

(5,0),(0,−5)

Exercise 8.12E.4



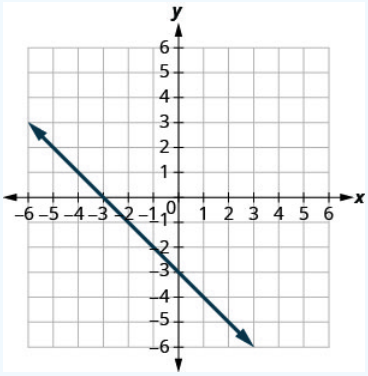
Exercise 8.12E.5



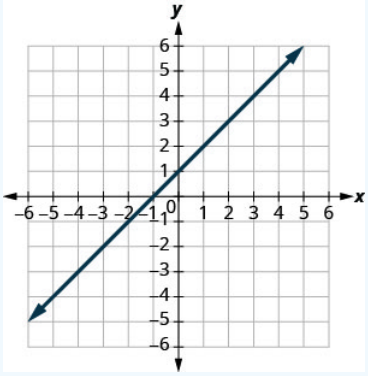
Answer

(−2,0),(0,−2)

Exercise 8.12E.6



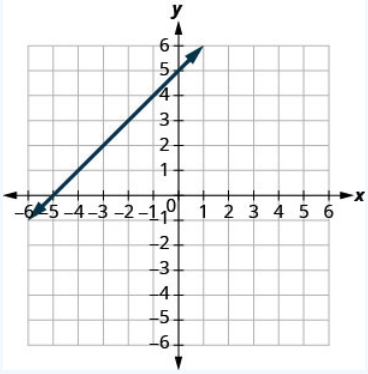
Exercise 8.12E.7



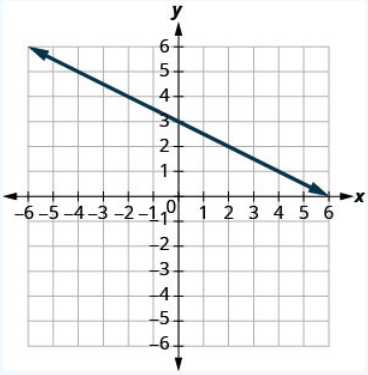
Answer

(−1,0),(0,1)

Exercise 8.12E.8



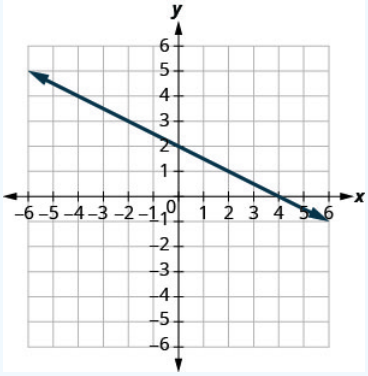
Exercise 8.12E.9



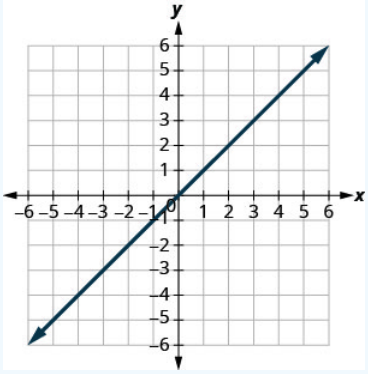
Answer

(6,0),(0,3)

Exercise 8.12E.10



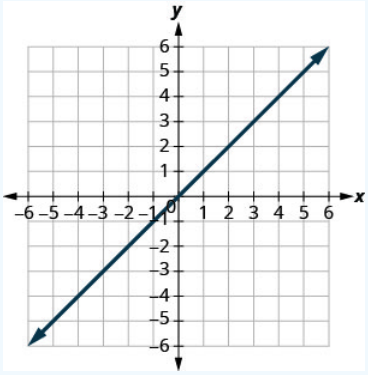
Exercise 8.12E.11



Answer

(0,0)

Exercise 8.12E.12



Find the *x*- and *y*- Intercepts from an Equation of a Line

In the following exercises, find the intercepts for each equation.

Exercise 8.12E.13

x+y=4

Answer

(4,0),(0,4)

Exercise 8.12E.14

x+y=3

Exercise 8.12E.15

x+y=−2

Answer

(−2,0),(0,−2)

Exercise 8.12E.16

x+y=−5

Exercise 8.12E.17

x–y=5

Answer

(5,0),(0,−5)

Exercise 8.12E.18

x–y=1

Exercise 8.12E.19

x–y=−3

Answer

(−3,0),(0,3)

Exercise 8.12E.20

x–y=−4

Exercise 8.12E.21

x+2y=8

Answer

(8,0),(0,4)

Exercise 8.12E.22

x+2y=10

Exercise 8.12E.23

3x+y=6

Answer

(2,0),(0,6)

Exercise 8.12E.24

3x+y=9

Exercise 8.12E.25

x–3y=12

Answer

(12,0),(0,−4)

Exercise 8.12E.26

x–2y=8

Exercise 8.12E.27

4x–y=8

Answer

(2,0),(0,−8)

Exercise 8.12E.28

5x–y=5

Exercise 8.12E.29

2x+5y=10

Answer

(5,0),(0,2)

Exercise 8.12E.30

2x+3y=6

Exercise 8.12E.31

3x–2y=12

Answer

(4,0),(0,−6)

Exercise 8.12E.32

3x–5y=30

Exercise 8.12E.33

$y = \frac{1}{3}x + 1$

Answer

(-3,0),(0,1)

Exercise 8.12E.34

$y = \frac{1}{4}x - 1$

Exercise 8.12E.35

$y = \frac{1}{5}x + 2$

Answer

(−10,0),(0,2)

Exercise 8.12E.36

$y = \frac{1}{3}x + 4$

Exercise 8.12E.37

y=3x

Answer

(0,0)

Exercise 8.12E.38

y=-2x

Exercise 8.12E.39

y=-4x

Answer

(0,0)

Exercise 8.12E.40

y=5x

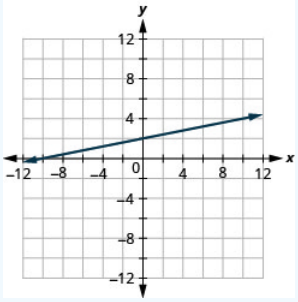
Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

Exercise 8.12E.41

$-x+5y=10$

Answer



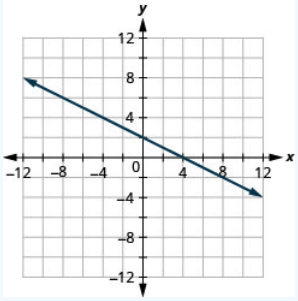
Exercise 8.12E.42

$-x+4y=8$

Exercise 8.12E.43

$x+2y=4$

Answer



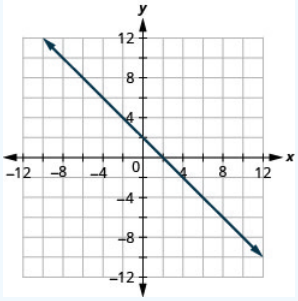
Exercise 8.12E.44

$x+2y=6$

Exercise 8.12E.45

$x+y=2$

Answer



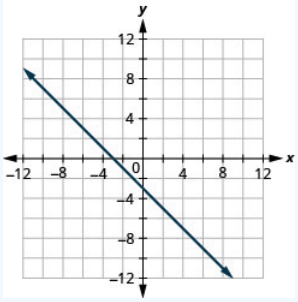
Exercise 8.12E.46

$x + y = 5$

Exercise 8.12E.47

$x + y = -3$

Answer



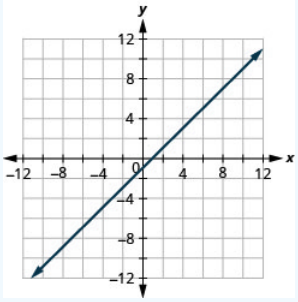
Exercise 8.12E.48

$x + y = -1$

Exercise 8.12E.49

$x - y = 1$

Answer



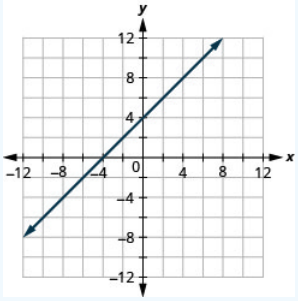
Exercise 8.12E.50

$x - y = 2$

Exercise 8.12E.51

$x - y = -4$

Answer



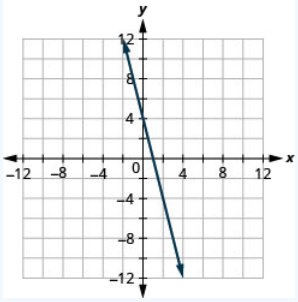
Exercise 8.12E.52

$x - y = -3$

Exercise 8.12E.53

$4x + y = 4$

Answer



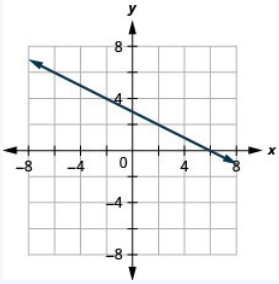
Exercise 8.12E.54

$3x + y = 3$

Exercise 8.12E.55

$2x + 4y = 12$

Answer



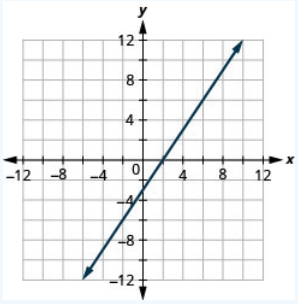
Exercise 8.12E.56

$3x + 2y = 12$

Exercise 8.12E.57

$3x - 2y = 6$

Answer



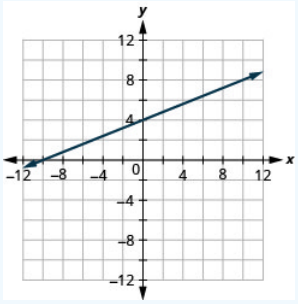
Exercise 8.12E.58

$5x - 2y = 10$

Exercise 8.12E.59

$2x - 5y = -20$

Answer



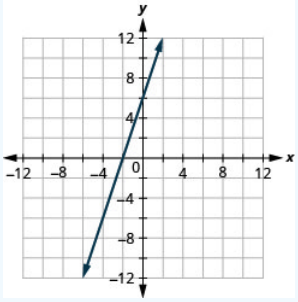
Exercise 8.12E.60

$3x - 4y = -12$

Exercise 8.12E.61

$3x - y = -6$

Answer



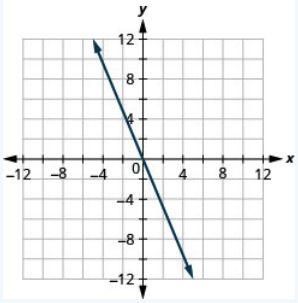
Exercise 8.12E.62

$2x - y = -8$

Exercise 8.12E.63

$y = -2x$

Answer



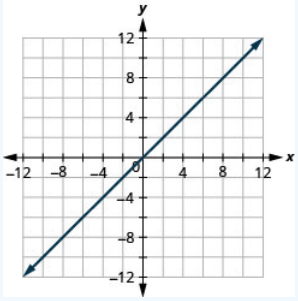
Exercise 8.12E.64

$y = -4x$

Exercise 8.12E.65

$y = x$

Answer



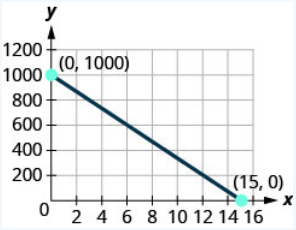
Exercise 8.12E.66

y=3x

Everyday Math

Exercise 8.12E.67

Road trip. Damien is driving from Chicago to Denver, a distance of 1000 miles. The *x*- axis on the graph below shows the time in hours since Damien left Chicago. The *y*- axis represents the distance he has left to drive.



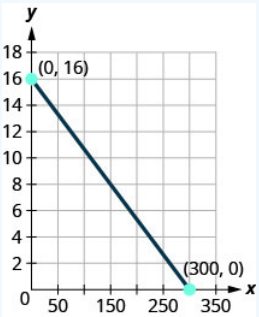
1. Find the *x*- and *y*- intercepts.
2. Explain what the *x*- and *y*- intercepts mean for Damien.

Answer

1. (0,1000),(15,0)
2. At (0,1000), he has been gone 0 hours and has 1000 miles left. At (15,0), he has been gone 15 hours and has 0 miles left to go.

Exercise 8.12E.68

Road trip. Ozzie filled up the gas tank of his truck and headed out on a road trip. The *x*- axis on the graph below shows the number of miles Ozzie drove since filling up. The *y*- axis represents the number of gallons of gas in the truck's gas tank.



1. Find the *x*- and *y*- intercepts.
2. Explain what the *x*- and *y*- intercepts mean for Ozzie.

Writing Exercises

Exercise 8.12E.69

How do you find the x -intercept of the graph of $3x - 2y = 6$?

Answer

Answers will vary.

Exercise 8.12E.70

Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation 4x+y=−4? Why?

Exercise 8.12E.71

Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $y = \frac{2}{3}x - 2$? Why?

Answer

Answers will vary.

Exercise 8.12E.72

Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation y=6? Why?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify the *x* and *y* intercepts on a graph.			
find the *x* and *y* intercepts from an equation of a line.			
graph a line using the intercepts.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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8.13: Understand Slope of a Line (Part 1)

Learning Objectives

- Use geoboards to model slope
- Find the slope of a line from its graph
- Find the slope of horizontal and vertical lines
- Use the slope formula to find the slope of a line between two points
- Graph a line given a point and the slope
- Solve slope applications

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{1-4}{8-2}$. If you missed this problem, review Example 4.6.12.
2. Divide: $\frac{0}{4}, \frac{4}{0}$. If you missed this problem, review Example 7.5.5.
3. Simplify: $\frac{15}{-3}, \frac{-15}{3}, \frac{-15}{-3}$. If you missed this problem, review Example 4.6.11.

As we've been graphing linear equations, we've seen that some lines slant up as they go from left to right and some lines slant down. Some lines are very steep and some lines are flatter. What determines whether a line slants up or down, and if its slant is steep or flat?

The steepness of the slant of a line is called the **slope of the line**. The concept of slope has many applications in the real world. The pitch of a roof and the grade of a highway or wheelchair ramp are just some examples in which you literally see slopes. And when you ride a bicycle, you feel the slope as you pump uphill or coast downhill.

Use Geoboards to Model Slope

In this section, we will explore the concepts of slope.

Using rubber bands on a geoboard gives a concrete way to model lines on a coordinate grid. By stretching a rubber band between two pegs on a geoboard, we can discover how to find the slope of a line. And when you ride a bicycle, you feel the slope as you pump uphill or coast downhill.

We'll start by stretching a rubber band between two pegs to make a line as shown in Figure 8.13.1.

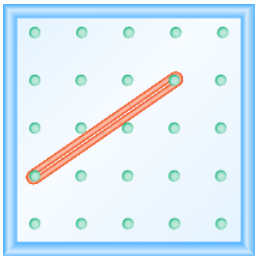


Figure 8.13.1

Does it look like a line?

Now we stretch one part of the rubber band straight up from the left peg and around a third peg to make the sides of a right triangle as shown in Figure 8.13.2. We carefully make a 90° angle around the third peg, so that one side is vertical and the other is horizontal.

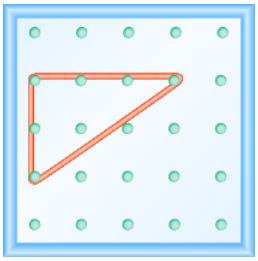


Figure 8.13.2

To find the slope of the line, we measure the distance along the vertical and horizontal legs of the triangle. The vertical distance is called the **rise** and the horizontal distance is called the **run**, as shown in Figure 8.13.3

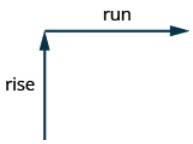


Figure 8.13.3

To help remember the terms, it may help to think of the images shown in Figure 8.13.4

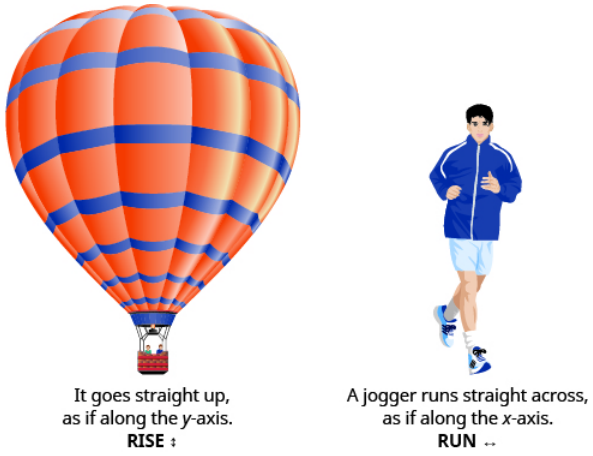


Figure 8.13.4

On our geoboard, the rise is 2 units because the rubber band goes up 2 spaces on the vertical leg. See Figure 8.13.5

What is the run? Be sure to count the spaces between the pegs rather than the pegs themselves! The rubber band goes across 3 spaces on the horizontal leg, so the run is 3 units.

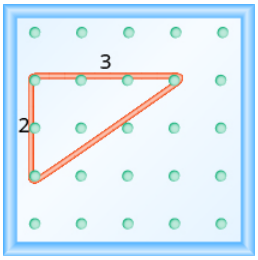


Figure 8.13.5

The slope of a line is the ratio of the rise to the run. So the slope of our line is $\frac{2}{3}$. In mathematics, the slope is always represented by the letter m.

Definition: Slope of a line

The slope of a line is m = $\frac{rise}{run}$.

The rise measures the vertical change and the run measures the horizontal change.

What is the slope of the line on the geoboard in Figure 8.13.5?

$$m = \frac{rise}{run}$$
$$m = \frac{2}{3}$$
$$The\ line\ has\ slope\ \frac{2}{3}.$$

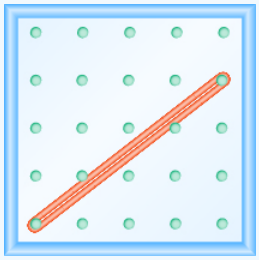
When we work with geoboards, it is a good idea to get in the habit of starting at a peg on the left and connecting to a peg to the right. Then we stretch the rubber band to form a right triangle.

If we start by going up the rise is positive, and if we stretch it down the rise is negative. We will count the run from left to right, just like you read this paragraph, so the run will be positive.

Since the slope formula has rise over run, it may be easier to always count out the rise first and then the run.

Example 8.13.1:

What is the slope of the line on the geoboard shown?

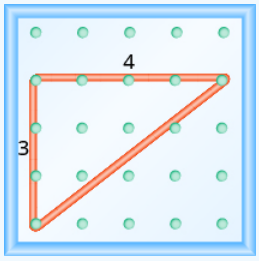


Solution

Use the definition of slope.

$$m = \frac{rise}{run} \tag{8.13.1}$$

Start at the left peg and make a right triangle by stretching the rubber band up and to the right to reach the second peg. Count the rise and the run as shown.



The rise is 3 units.

$$m = \frac{3}{run} \tag{8.13.2}$$

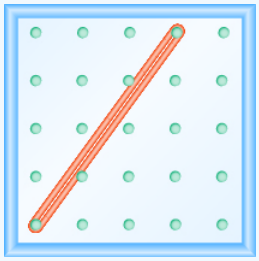
The run is 4 units.

$$m = \frac{3}{4} \tag{8.13.3}$$

The slope is $\frac{3}{4}$.

Exercise 8.13.1:

What is the slope of the line on the geoboard shown?

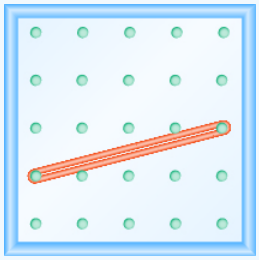


Answer

$\frac{4}{3}$

Exercise 8.13.2:

What is the slope of the line on the geoboard shown?

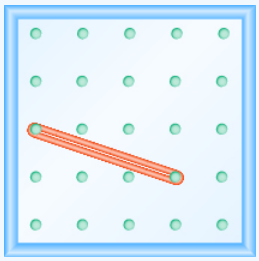


Answer

$\frac{1}{4}$

Example 8.13.2:

What is the slope of the line on the geoboard shown?

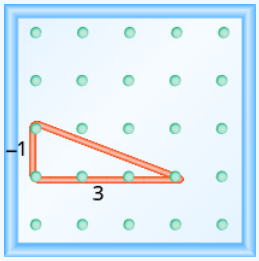


Solution

Use the definition of slope.

$$m = \frac{rise}{run} \tag{8.13.4}$$

Start at the left peg and make a right triangle by stretching the rubber band to the peg on the right. This time we need to stretch the rubber band down to make the vertical leg, so the rise is negative.



The rise is −1.

$$m = \frac{-1}{run} \tag{8.13.5}$$

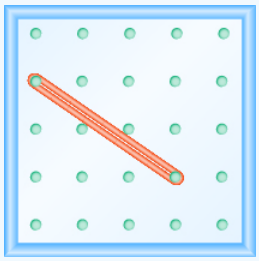
The run is 3.

$$m = \frac{-1}{3}$$
$$m = -\frac{1}{3}$$

The slope is $-\frac{1}{3}$.

? Exercise 8.13.3:

What is the slope of the line on the geoboard?

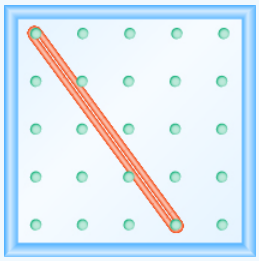


Answer

$-\frac{2}{3}$

? Exercise 8.13.4:

What is the slope of the line on the geoboard?



Answer

$-\frac{4}{3}$

Notice that in the first example, the slope is positive and in the second example the slope is negative. Do you notice any difference in the two lines shown in Figure 8.13.6

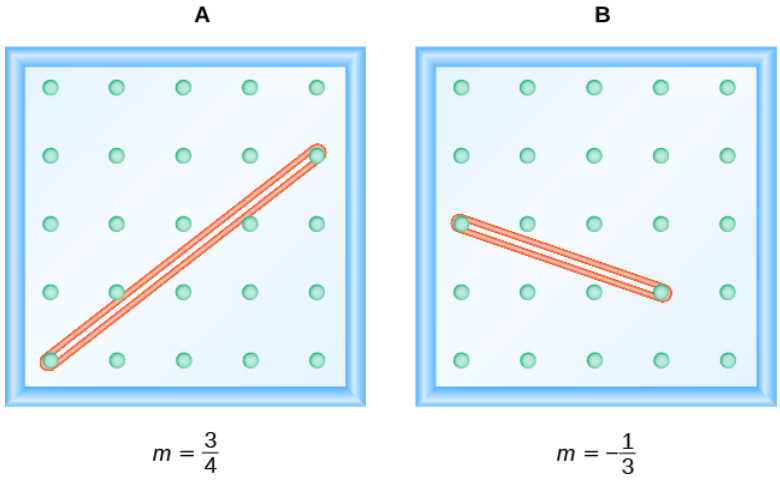


Figure 8.13.6

As you read from left to right, the line in Figure A, is going up; it has positive slope. The line Figure B is going down; it has negative slope.

Figure 8.13.7

Example 8.13.3:

Use a geoboard to model a line with slope $\frac{1}{2}$.

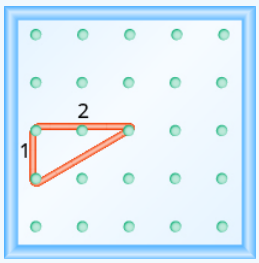
Solution

To model a line with a specific slope on a geoboard, we need to know the rise and the run.

Use the slope formula.	$m = \frac{rise}{run}$	(8.13.6)
Replace m with $\frac{1}{2}$.	$\frac{1}{2} = \frac{rise}{run}$	(8.13.7)

So, the rise is 1 unit and the run is 2 units.

Start at a peg in the lower left of the geoboard. Stretch the rubber band up 1 unit, and then right 2 units.

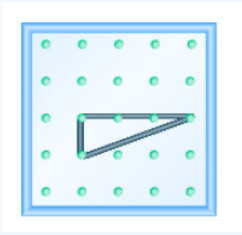


The hypotenuse of the right triangle formed by the rubber band represents a line with a slope of $\frac{1}{2}$.

Exercise 8.13.5:

Use a geoboard to model a line with the given slope: m = $\frac{1}{3}$.

Answer



Exercise 8.13.6:

Use a geoboard to model a line with the given slope: m = $\frac{3}{2}$.

Answer

Example 8.13.4:

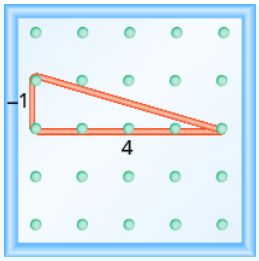
Use a geoboard to model a line with slope $\frac{-1}{4}$.

Solution

Use the slope formula.	$m = \frac{rise}{run}$	(8.13.8)
Replace m with $-\frac{1}{4}$.	$-\frac{1}{4} = \frac{rise}{run}$	(8.13.9)

So, the rise is −1 and the run is 4.

Since the rise is negative, we choose a starting peg on the upper left that will give us room to count down. We stretch the rubber band down 1 unit, then to the right 4 units.

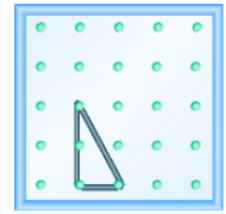


The hypotenuse of the right triangle formed by the rubber band represents a line whose slope is $-\frac{1}{4}$.

Exercise 8.13.7:

Use a geoboard to model a line with the given slope: m = $\frac{-3}{2}$.

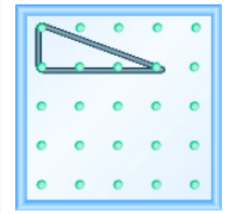
Answer



Exercise 8.13.8:

Use a geoboard to model a line with the given slope: m = $\frac{-1}{3}$.

Answer



Find the Slope of a Line from its Graph

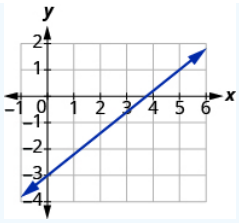
Now we'll look at some graphs on a coordinate grid to find their slopes. The method will be very similar to what we just modeled on our geoboards.

To find the slope, we must count out the rise and the run. But where do we start?

We locate any two points on the line. We try to choose points with coordinates that are integers to make our calculations easier. We then start with the point on the left and sketch a right triangle, so we can count the rise and run.

Example 8.13.5:

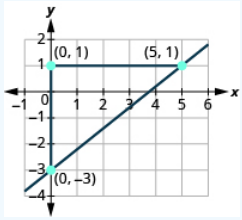
Find the slope of the line shown:



Solution

Locate two points on the graph, choosing points whose coordinates are integers. We will use (0, −3) and (5, 1).

Starting with the point on the left, (0, −3), sketch a right triangle, going from the first point to the second point, (5, 1).



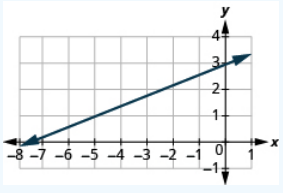
Count the rise on the vertical leg of the triangle.	The rise is 4 units.	
Count the run on the horizontal leg.	The run is 5 units.	
Use the slope formula.	$m = \frac{rise}{run}$	(8.13.10)
Substitute the values of the rise and run.	$m = \frac{4}{5}$	(8.13.11)

The slope of the line is $\frac{4}{5}$.

Notice that the slope is positive since the line slants upward from left to right.

? Exercise 8.13.9:

Find the slope of the line:

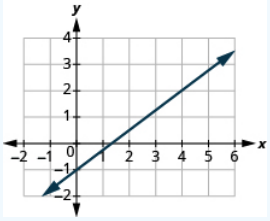


Answer

$\frac{2}{5}$

? Exercise 8.13.10:

Find the slope of the line:



Answer

$\frac{3}{4}$

HOW TO: FIND THE SLOPE FROM A GRAPH

Step 1. Locate two points on the line whose coordinates are integers.

Step 2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.

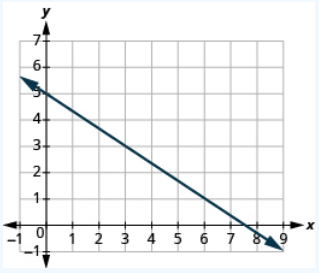
Step 3. Count the rise and the run on the legs of the triangle.

Step 4. Take the ratio of rise to run to find the slope.

$$m = \frac{rise}{run} \tag{8.13.12}$$

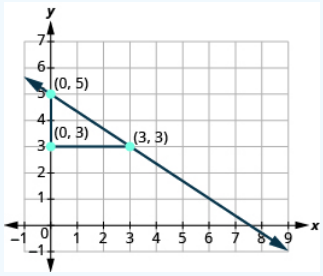
Example 8.13.6:

Find the slope of the line shown:



Solution

Locate two points on the graph. Look for points with coordinates that are integers. We can choose any points, but we will use (0, 5) and (3, 3). Starting with the point on the left, sketch a right triangle, going from the first point to the second point.



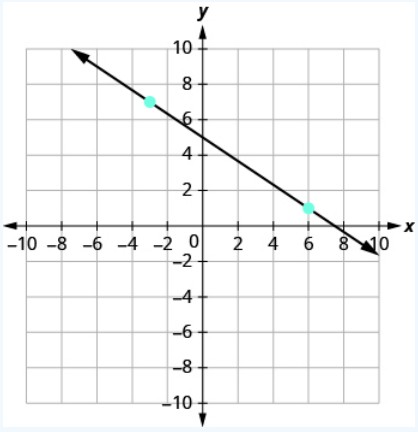
Count the rise – it is negative.	The rise is −2.
Count the run.	The run is 3.
Use the slope formula.	$m = \frac{rise}{run}$ (8.13.13)

Substitute the values of the rise and run.	$m = \frac{-2}{3}$	(8.13.14)
Simplify.	$m = -\frac{2}{3}$	(8.13.15)

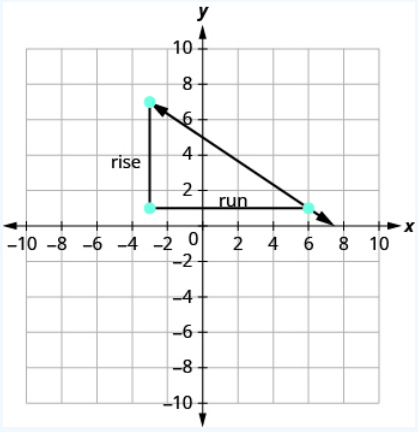
The slope of the line is $-\frac{2}{3}$.

Notice that the slope is negative since the line slants downward from left to right.

What if we had chosen different points? Let's find the slope of the line again, this time using different points. We will use the points (−3, 7) and (6, 1).



Starting at (−3, 7), sketch a right triangle to (6, 1).



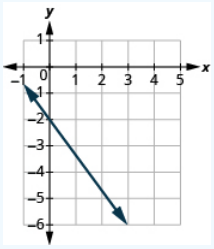
Count the rise.	The rise is −6.	
Count the run.	The run is 9.	
Use the slope formula.	$m = \frac{rise}{run}$	(8.13.16)
Substitute the values of the rise and run.	$m = \frac{-6}{9}$	(8.13.17)
Simplify the fraction.	$m = -\frac{2}{3}$	(8.13.18)

The slope of the line is $-\frac{2}{3}$.

It does not matter which points you use—the slope of the line is always the same. The slope of a line is constant!

Exercise 8.13.11:

Find the slope of the line:

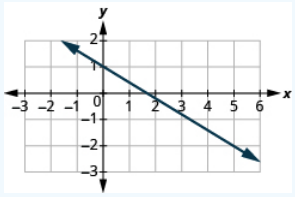


Answer

$-\frac{4}{3}$

Exercise 8.13.12:

Find the slope of the line:



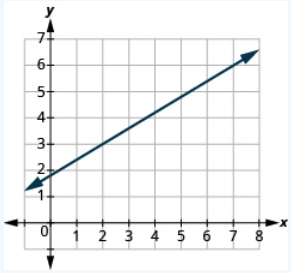
Answer

$-\frac{3}{5}$

The lines in the previous examples had y-intercepts with integer values, so it was convenient to use the y-intercept as one of the points we used to find the slope. In the next example, the y-intercept is a fraction. The calculations are easier if we use two points with integer coordinates.

Example 8.13.7:

Find the slope of the line shown:

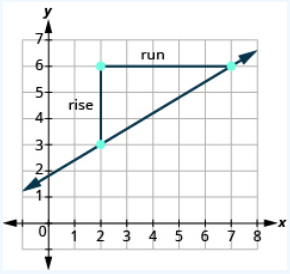


Solution

Locate two points on the graph whose coordinates are integers.	(2, 3) and (7, 6)

Which point is on the left?	(2, 3)

Starting at (2, 3), sketch a right angle to (7, 6) as shown below.

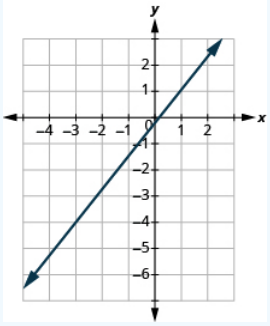


Count the rise.	The rise is 3.	
Count the run.	The run is 5.	
Use the slope formula.	$m = \frac{rise}{run}$	(8.13.19)
Substitute the values of the rise and run.	$m = \frac{3}{5}$	(8.13.20)

The slope of the line is $\frac{3}{5}$.

Exercise 8.13.13:

Find the slope of the line:

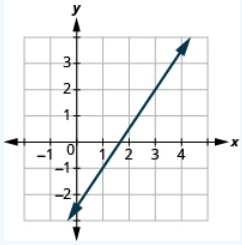


Answer

$\frac{5}{4}$

Exercise 8.13.14:

Find the slope of the line:



Answer

$\frac{3}{2}$

Find the Slope of Horizontal and Vertical Lines

Do you remember what was special about horizontal and vertical lines? Their equations had just one variable.

- horizontal line y = b; all the y -coordinates are the same.
- vertical line x = a; all the x -coordinates are the same.

So how do we find the slope of the horizontal line y = 4? One approach would be to graph the horizontal line, find two points on it, and count the rise and the run. Let's see what happens in Figure 8.13.8. We'll use the two points (0, 4) and (3, 4) to count the rise and run.

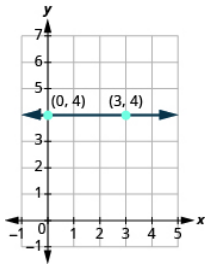


Figure 8.13.8

What is the rise?	The rise is 0.	
What is the run?	The run is 3.	
What is the slope?	$m = \frac{rise}{run}$	(11.4.21)
	$m = \frac{0}{3}$	(11.4.22)
	$m = 0$	(11.4.23)

The slope of the horizontal line y = 4 is 0.

All horizontal lines have slope 0. When the y-coordinates are the same, the rise is 0.

Definition: Slope of a Horizontal Line

The slope of a horizontal line, y = b, is 0.

Now we'll consider a vertical line, such as the line x = 3, shown in Figure 8.13.9. We'll use the two points (3, 0) and (3, 2) to count the rise and run.

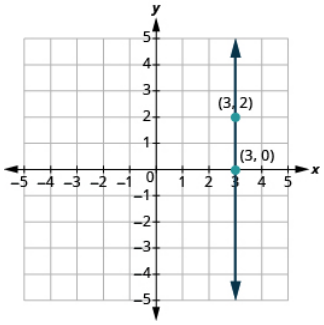


Figure 8.13.9

What is the rise?	The rise is 2.	
What is the run?	The run is 0.	
What is the slope?	$m = \frac{rise}{run}$	(11.4.24)
	$m = \frac{2}{0}$	(11.4.25)

But we can't divide by 0. Division by 0 is undefined. So we say that the slope of the vertical line x = 3 is undefined. The slope of all vertical lines is undefined, because the run is 0.

Definition: Slope of a Vertical Line

The slope of a vertical line, x = a, is undefined.

Example 8.13.8:

Find the slope of each line: (a) x = 8 (b) y = −5

Solution

(a) x = 8

This is a vertical line, so its slope is undefined.

(b) y = −5

This is a horizontal line, so its slope is 0.

Exercise 8.13.15:

Find the slope of the line: x = −4.

Answer

undefined

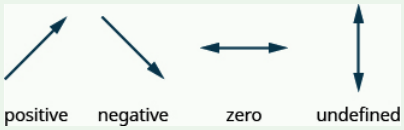
Exercise 8.13.16:

Find the slope of the line: y = 7.

Answer

0

Quick Guide to the Slopes of Lines



Use the Slope Formula to find the Slope of a Line between Two Points

Sometimes we need to find the slope of a line between two points and we might not have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but there is a way to find the slope without graphing.

Before we get to it, we need to introduce some new algebraic notation. We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points?

Mathematicians use subscripts to distinguish between the points. A subscript is a small number written to the right of, and a little lower than, a variable.

- (x_1, y_1) read x sub 1, y sub 1
- (x_2, y_2) read x sub 2, y sub 2

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point. If we had more than two points, we could use (x_3, y_3), (x_4, y_4), and so on.

To see how the rise and run relate to the coordinates of the two points, let's take another look at the slope of the line between the points (2, 3) and (7, 6) in Figure 8.13.10

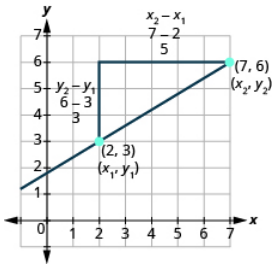


Figure 8.13.10

Since we have two points, we will use subscript notation.

$$\begin{array}{cc} x_1, y_1 & x_2, y_2 \\ (2, 3) & (7, 6) \end{array}$$

On the graph, we counted the rise of 3. The rise can also be found by subtracting the y-coordinates of the points.

$$\begin{array}{c} y_2 - y_1 \\ 6 - 3 \\ 3 \end{array}$$

We counted a run of 5. The run can also be found by subtracting the x-coordinates.

$$\begin{array}{c} x_2 - x_1 \\ 7 - 2 \\ 5 \end{array}$$

We know	$m = \frac{rise}{run}$	(11.4.26)

So	$m = \frac{3}{5}$	(11.4.27)
We rewrite the rise and run by putting in the coordinates.	$m = \frac{6-3}{7-2}$	(11.4.28)
But 6 is the y-coordinate of the second point, y_2 and 3 is the y-coordinate of the first point y_1. So we can rewrite the rise using subscript notation.	$m = \frac{y_2 - y_1}{7-2}$	(11.4.29)
Also 7 is the x-coordinate of the second point, x_2 and 2 is the x-coordinate of the first point x_2. So we rewrite the run using subscript notation.	$m = \frac{y_2 - y_1}{x_2 - x_1}$	(11.4.30)

We've shown that m = $\frac{y_2 - y_1}{x_2 - x_1}$ is really another version of m = $\frac{rise}{run}$. We can use this formula to find the slope of a line when we have two points on the line.

Definition: Slope Formula

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{11.4.31}$$

Say the formula to yourself to help you remember it:

Slope is y of the second point minus y of the first point

over

x of the second point minus x of the first point.

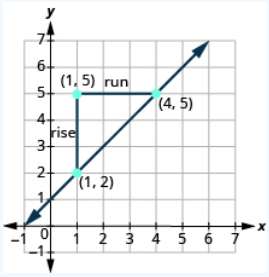
Example 8.13.9:

Find the slope of the line between the points (1, 2) and (4, 5).

Solution

We'll call (1, 2) point #1 and (4, 5) point #2.	$\begin{matrix} x_1, y_1 & x_2, y_2 \\ (1, 2) & (4, 5) \end{matrix}$	
Use the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1}$	(11.4.32)
Substitute the values in the slope formula:		
y of the second point minus y of the first point	$m = \frac{5-2}{x_2 - x_1}$	(11.4.33)
x of the second point minus x of the first point	$m = \frac{5-2}{4-1}$	(11.4.34)
Simplify the numerator and the denominator.	$m = \frac{3}{3}$	(11.4.35)
	m = 1	

Let's confirm this by counting out the slope on the graph.



The rise is 3 and the run is 3, so

$$m = \frac{rise}{run}$$
$$m = \frac{3}{3}$$
$$m = 1$$

Exercise 8.13.17:

Find the slope of the line through the given points: (8, 5) and (6, 3).

Answer

1

Exercise 8.13.18:

Find the slope of the line through the given points: (1, 5) and (5, 9).

Answer

1

How do we know which point to call #1 and which to call #2? Let's find the slope again, this time switching the names of the points to see what happens. Since we will now be counting the run from right to left, it will be negative.

We'll call (4, 5) point #1 and (1, 2) point #2.	$\begin{matrix} x_1, y_1 & x_2, y_2 \\ (4, 5) & (1, 2) \end{matrix}$	
Use the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1}$	(11.4.36)
Substitute the values in the slope formula:		
y of the second point minus y of the first point	$m = \frac{2 - 5}{x_2 - x_1}$	(11.4.37)
x of the second point minus x of the first point	$m = \frac{2 - 5}{1 - 4}$	(11.4.38)
Simplify the numerator and the denominator.	$m = \frac{-3}{-3}$	(11.4.39)
	m = 1	

The slope is the same no matter which order we use the points.

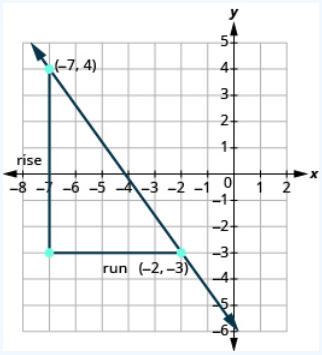
Example 8.13.10:

Find the slope of the line through the points (−2, −3) and (−7, 4).

Solution

We'll call (−2, −3) point #1 and (−7, 4) point #2.	$\begin{matrix} x_1, y_1 & x_2, y_2 \\ (-2,-3) & (-7,4) \end{matrix}$	
Use the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1}$	(11.4.40)
Substitute the values.		
y of the second point minus y of the first point	$m = \frac{4-(-3)}{x_2 - x_1}$	(11.4.41)
x of the second point minus x of the first point	$m = \frac{4-(-3)}{-7-(-2)}$	(11.4.42)
Simplify.	$m = \frac{7}{-5}$	(11.4.43)
	$m = -\frac{7}{5}$	

Let's confirm this on the graph shown.



$$m = \frac{rise}{run}$$
$$m = \frac{-7}{5}$$
$$m = -\frac{7}{5}$$

Exercise 8.13.19:

Find the slope of the line through the pair of points: (−3, 4) and (2, −1).

Answer

-1

Exercise 8.13.20:

Find the slope of the line through the pair of points: (−2, 6) and (−3, −4).

Answer

10

Contributors and Attributions

- Lynn Marecek (Santa Ana College) and MaryAnne Anthony-Smith (Formerly of Santa Ana College). This content is licensed under Creative Commons Attribution License v4.0 "Download for free at http://cnx.org/contents/fd53eae1-fa2...49835c3c@5.191."

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8.14: Understand Slope of a Line (Part 2)

Graph a Line Given a Point and the Slope

In this chapter, we graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

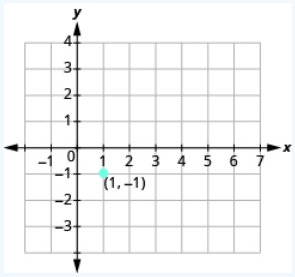
Another method we can use to graph lines is the point-slope method. Sometimes, we will be given one point and the slope of the line, instead of its equation. When this happens, we use the definition of slope to draw the graph of the line.

Example 8.14.11:

Graph the line passing through the point (1, −1) whose slope is m = $\frac{3}{4}$.

Solution

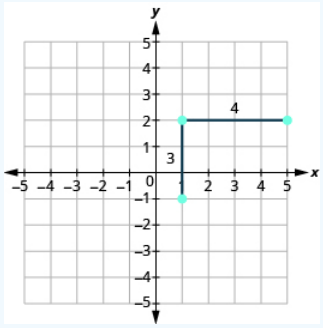
Plot the given point, (1, −1).



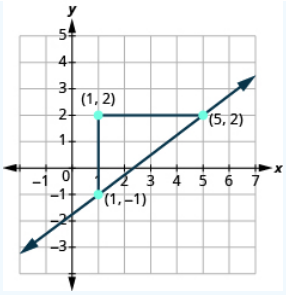
Use the slope formula m = $\frac{rise}{run}$ to identify the rise and the run.

$$m = \frac{3}{4}$$
$$\frac{rise}{run} = \frac{3}{4}$$
$$rise = 3$$
$$run = 4$$

Starting at the point we plotted, count out the rise and run to mark the second point. We count 3 units up and 4 units right.



Then we connect the points with a line and draw arrows at the ends to show it continues.



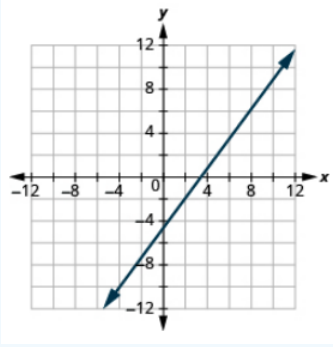
We can check our line by starting at any point and counting up 3 and to the right 4. We should get to another point on the line.

Exercise 8.14.21:

Graph the line passing through the point with the given slope:

$(2, -2)$, $m = \frac{4}{3}$

Answer

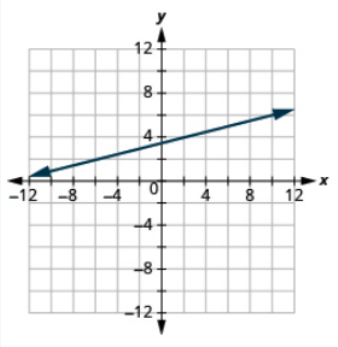


Exercise 8.14.22:

Graph the line passing through the point with the given slope:

$(-2, 3)$, $m = \frac{1}{4}$

Answer



HOW TO: GRAPH A LINE GIVEN A POINT AND A SLOPE

Step 1. Plot the given point.

Step 2. Use the slope formula to identify the rise and the run.

Step 3. Starting at the given point, count out the rise and run to mark the second point.

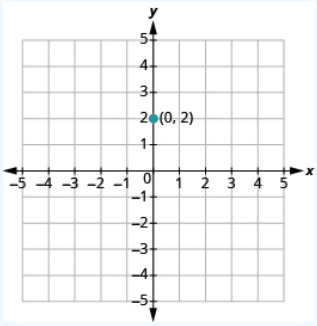
Step 4. Connect the points with a line.

Example 8.14.12:

Graph the line with y -intercept (0, 2) and slope m = $-\frac{2}{3}$.

Solution

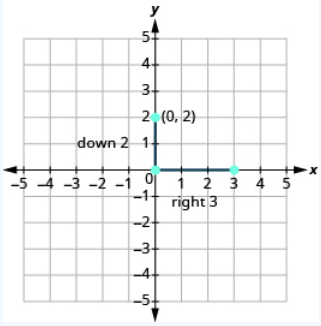
Plot the given point, the y -intercept (0, 2).



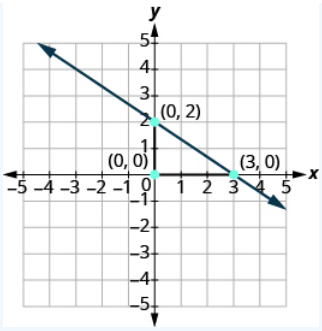
Use the slope formula m = rise run to identify the rise and the run.

$$m = -\frac{2}{3}$$
$$\frac{rise}{run} = \frac{-2}{3}$$
$$rise = -2$$
$$run = 3$$

Starting at (0, 2), count the rise and the run and mark the second point.



Connect the points with a line.

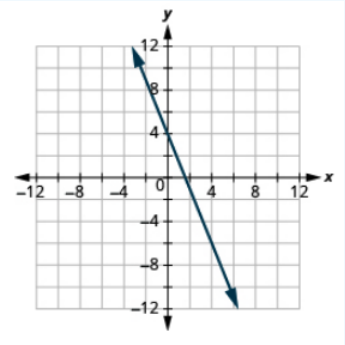


Exercise 8.14.23:

Graph the line with the given intercept and slope:

y-intercept 4, m = $-\frac{5}{2}$

Answer

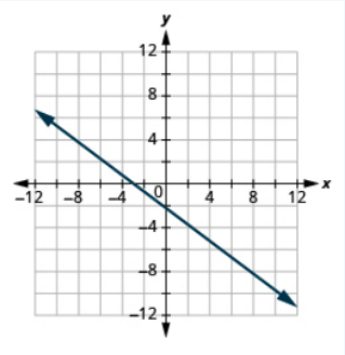


Exercise 8.14.24:

Graph the line with the given intercept and slope:

x-intercept −3, m = $-\frac{3}{4}$

Answer

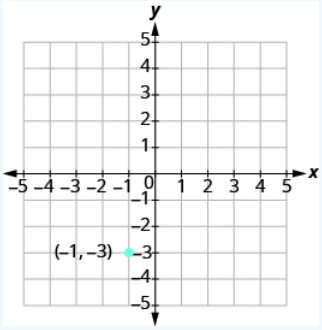


Example 8.14.13:

Graph the line passing through the point (−1, −3) whose slope is m = 4.

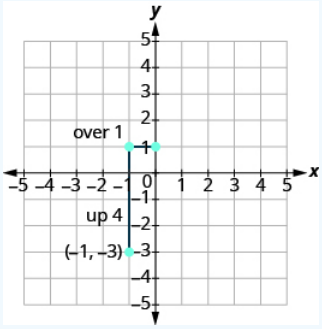
Solution

Plot the given point.

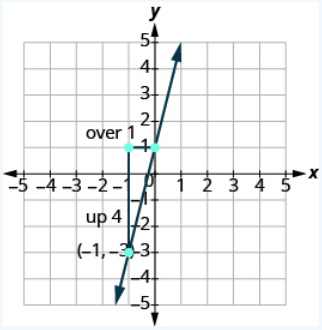


Identify the rise and the run.	$m = 4$	(11.4.44)
Write 4 as a fraction.	$\frac{rise}{run} = \frac{4}{1}$	(11.4.45)
	rise = 4, run = 1	

Count the rise and run.



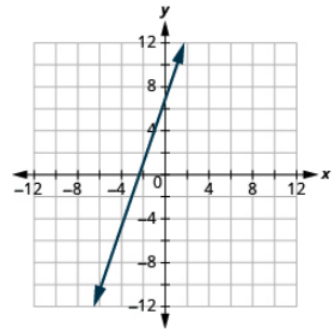
Mark the second point. Connect the two points with a line.



Exercise 8.14.25:

Graph the line with the given intercept and slope: (−2, 1), m = 3.

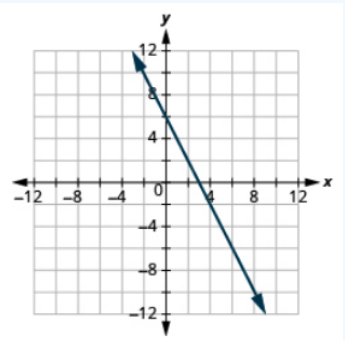
Answer



Exercise 8.14.26:

Graph the line with the given intercept and slope: (4, −2), m = −2.

Answer

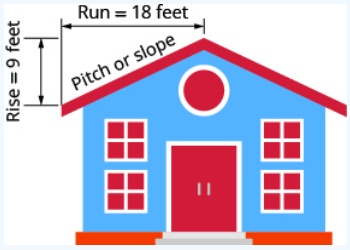


Solve Slope Applications

At the beginning of this section, we said there are many applications of slope in the real world. Let's look at a few now.

Example 8.14.14:

The pitch of a building's roof is the slope of the roof. Knowing the pitch is important in climates where there is heavy snowfall. If the roof is too flat, the weight of the snow may cause it to collapse. What is the slope of the roof shown?



Solution

Use the slope formula.	$m = \frac{rise}{run}$	(11.4.46)
Substitute the values for rise and run.	$m = \frac{9\ ft}{18\ ft}$	(11.4.47)

Simplify.	$m = \frac{1}{2}$	(11.4.48)
	The slope of the roof is $\frac{1}{2}$.	

? Exercise 8.14.27:

Find the slope given rise and run: A roof with a rise = 14 and run = 24.

Answer

$\frac{7}{12}$

? Exercise 8.14.28:

Find the slope given rise and run: A roof with a rise = 15 and run = 36.

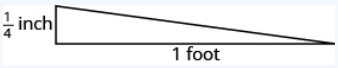
Answer

$\frac{5}{12}$

Have you ever thought about the sewage pipes going from your house to the street? Their slope is an important factor in how they take waste away from your house.

✓ Example 8.14.15:

Sewage pipes must slope down $\frac{1}{4}$ inch per foot in order to drain properly. What is the required slope?



Solution

Use the slope formula.	$m = \frac{rise}{run} = \frac{-\frac{1}{4}\ in.}{1\ ft}$	(11.4.49)
Convert 1 foot to 12 inches.	$m = \frac{-\frac{1}{4}\ in.}{12\ in}$	(11.4.50)
Simplify.	$m = -\frac{1}{48}$	(11.4.51)
	The slope of the pipe is $-\frac{1}{48}$.	

? Exercise 8.14.29:

Find the slope of the pipe: The pipe slopes down $\frac{1}{3}$ inch per foot.

Answer

$-\frac{1}{36}$

Exercise 8.14.30:

Find the slope of the pipe: The pipe slopes down $\frac{3}{4}$ inch per yard.

Answer

$-\frac{1}{48}$

ACCESS ADDITIONAL ONLINE RESOURCES

Determine Positive slope from a Graph

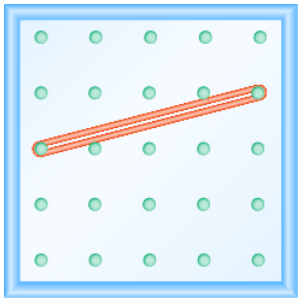
Determine Negative slope from a Graph

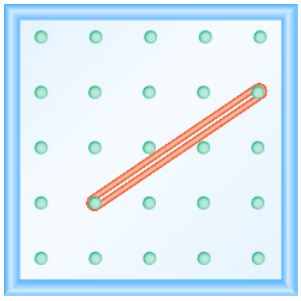
Determine Slope from Two Points

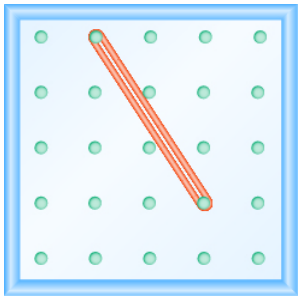
Practice Makes Perfect

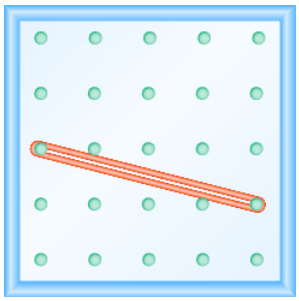
Use Geoboards to Model Slope

In the following exercises, find the slope modeled on each geoboard.

203. 

204. 

205. 

206. 

In the following exercises, model each slope. Draw a picture to show your results.

207. $\frac{2}{3}$

208. $\frac{3}{4}$

209. $\frac{1}{4}$

210. $\frac{4}{3}$

211. $-\frac{1}{2}$

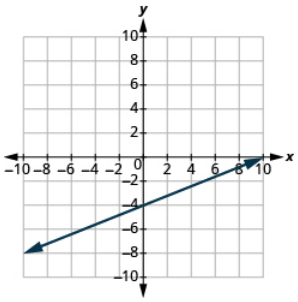
212. $-\frac{3}{4}$

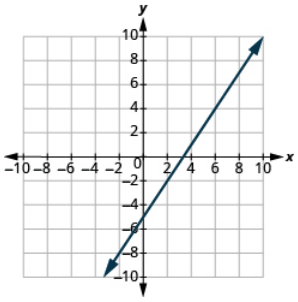
213. $-\frac{2}{3}$

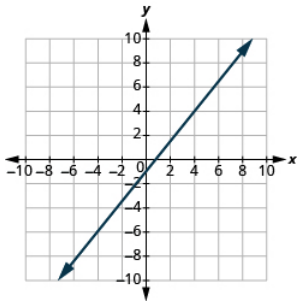
214. $-\frac{3}{2}$

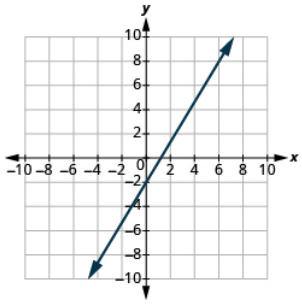
Find the Slope of a Line from its Graph

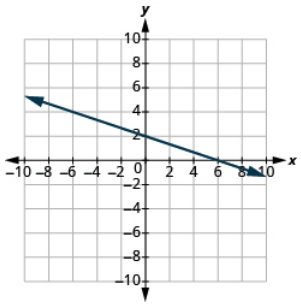
In the following exercises, find the slope of each line shown.

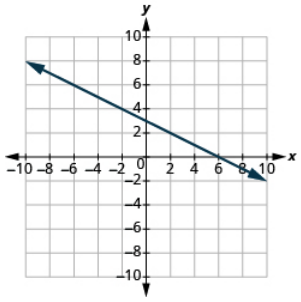
215. 

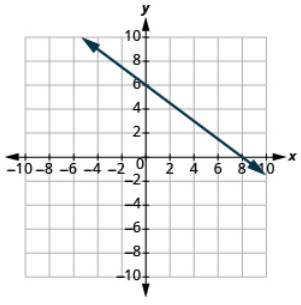
216. 

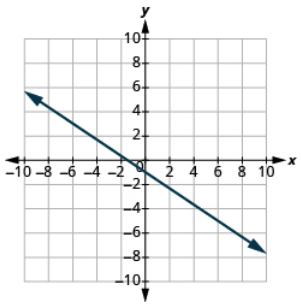
217. 

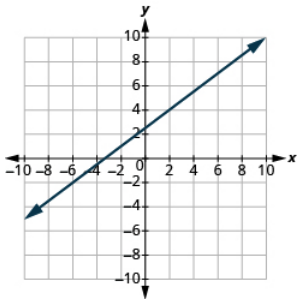
218. 

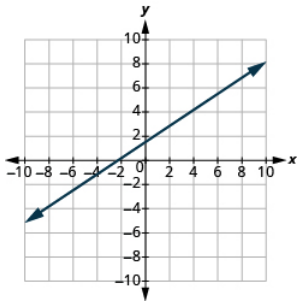
219. 

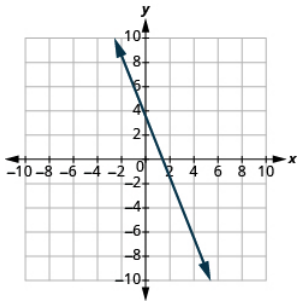
220. 

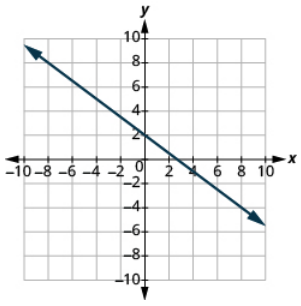
221. 

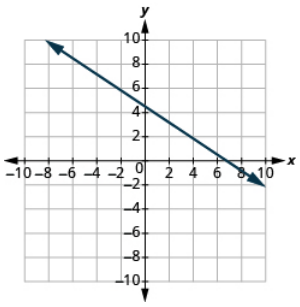
222. 

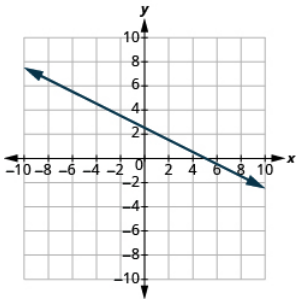
223. 

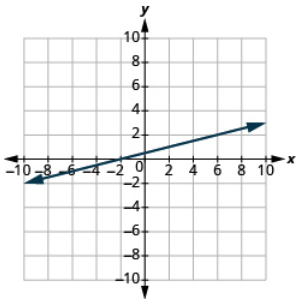
224. 

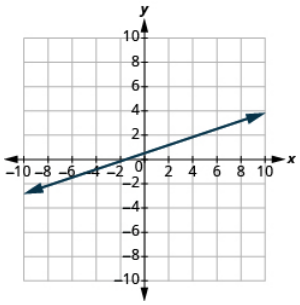
225. 

226. 

227. 

228. 

229. 

230. 

Find the Slope of Horizontal and Vertical Lines

In the following exercises, find the slope of each line.

231. y = 3
232. y = 1
233. x = 4
234. x = 2
235. y = −2
236. y = −3
237. x = −5
238. x = −4

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

239. (1, 4), (3, 9)
240. (2, 3), (5, 7)

241. (0, 3), (4, 6)
242. (0, 1), (5, 4)
243. (2, 5), (4, 0)
244. (3, 6), (8, 0)
245. (−3, 3), (2, −5)
246. (−2, 4), (3, −1)
247. (−1, −2), (2, 5)
248. (−2, −1), (6, 5)
249. (4, −5), (1, −2)
250. (3, −6), (2, −2)

Graph a Line Given a Point and the Slope

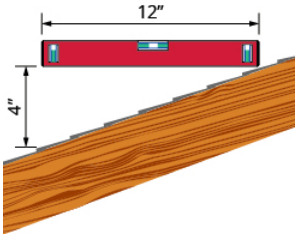
In the following exercises, graph the line given a point and the slope.

251. (1, −2); m = $\frac{3}{4}$
252. (1, −1); m = $\frac{1}{2}$
253. (2, 5); m = $-\frac{1}{3}$
254. (1, 4); m = $-\frac{1}{2}$
255. (−3, 4); m = $-\frac{3}{2}$
256. (−2, 5); m = $-\frac{5}{4}$
257. . (−1, −4); m = $\frac{4}{3}$
258. (−3, −5); m = $\frac{3}{2}$
259. (0, 3); m = $-\frac{2}{5}$
260. (0, 5); m = $-\frac{4}{3}$
261. (−2, 0); m = $-\frac{3}{4}$
262. (−1, 0); m = $\frac{1}{5}$
263. (−3, 3); m = 2
264. (−4, 2); m = 4
265. (1, 5); m = −3
266. (2, 3); m = −1

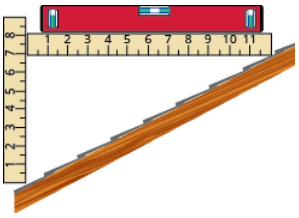
Solve Slope Applications

In the following exercises, solve these slope applications.

267. **Slope of a roof** A fairly easy way to determine the slope is to take a 12-inch level and set it on one end on the roof surface. Then take a tape measure or ruler, and measure from the other end of the level down to the roof surface. You can use these measurements to calculate the slope of the roof. What is the slope of the roof in this picture?



268. What is the slope of the roof shown?



269. **Road grade** A local road has a grade of 6%. The grade of a road is its slope expressed as a percent.
    a. Find the slope of the road as a fraction and then simplify the fraction.
    b. What rise and run would reflect this slope or grade?
270. **Highway grade** A local road rises 2 feet for every 50 feet of highway.
    a. What is the slope of the highway?
    b. The grade of a highway is its slope expressed as a percent. What is the grade of this highway?

Everyday Math

271. **Wheelchair ramp** The rules for wheelchair ramps require a maximum 1 inch rise for a 12 inch run.
    a. How long must the ramp be to accommodate a 24-inch rise to the door?
    b. Draw a model of this ramp.
272. **Wheelchair ramp** A 1-inch rise for a 16-inch run makes it easier for the wheelchair rider to ascend the ramp.
    a. How long must the ramp be to easily accommodate a 24-inch rise to the door?
    b. Draw a model of this ramp.

Writing Exercises

273. What does the sign of the slope tell you about a line?
274. How does the graph of a line with slope m = $\frac{1}{2}$ differ from the graph of a line with slope m = 2?
275. Why is the slope of a vertical line undefined?
276. Explain how you can graph a line given a point and its slope.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use geoboards to model slope.			
find the slope of a line from its graph.			
find the slope of horizontal and vertical lines.			
use the slope formula to find the slope of a line between two points.			
graph a line given a point and the slope.			
solve slope applications.			

(b) On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Contributors and Attributions

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8.E: Graphs (Exercises)

11.1 - Use the Rectangular Coordinate System

Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system.

277. (1, 3), (3, 1)
278. (2, 5), (5, 2)

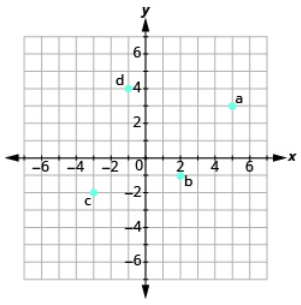
In the following exercises, plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

279. (a) $(-1, -5)$ (b) $(-3, 4)$ (c) $(2, -3)$ (d) $\left(1, \frac{5}{2}\right)$
280. (a) $(3, -2)$ (b) $(-4, -1)$ (c) $(-5, 4)$ (d) $\left(2, \frac{10}{3}\right)$

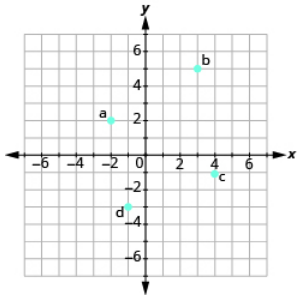
Identify Points on a Graph

In the following exercises, name the ordered pair of each point shown in the rectangular coordinate system.

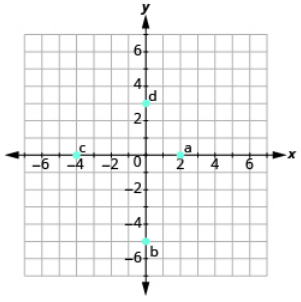
281.

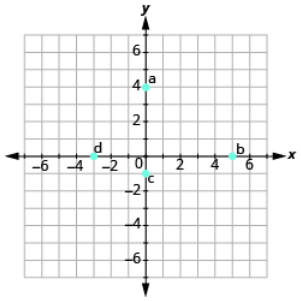


282.



283.



284. 

Verify Solutions to an Equation in Two Variables

In the following exercises, find the ordered pairs that are solutions to the given equation.

285. $5x + y = 10$

a. (5, 1)
b. (2, 0)
c. (4, −10)

286. $y = 6x - 2$

a. (1, 4)
b. $\left(\frac{1}{3}, 0\right)$
c. (6, −2)

Complete a Table of Solutions to a Linear Equation in Two Variables

In the following exercises, complete the table to find solutions to each linear equation.

287. $y = 4x - 1$

x	y	(x, y)
0		
1		
-2		

288. $y = -\frac{1}{2}x + 3$

x	y	(x, y)
0		
1		
-2		

289. $x + 2y = 5$

x	y	(x, y)
	0	
1		
-1		

290. $3x - 2y = 6$

x	y	(x, y)
0		
	0	
-2		

Find Solutions to a Linear Equation in Two Variables

In the following exercises, find three solutions to each linear equation.

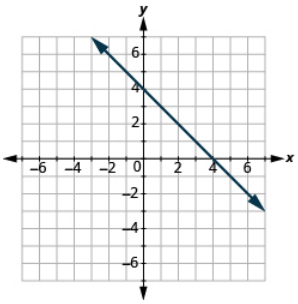
291. x + y = 3
292. x + y = −4
293. y = 3x + 1
294. y = − x − 1

11.2 - Graphing Linear Equations

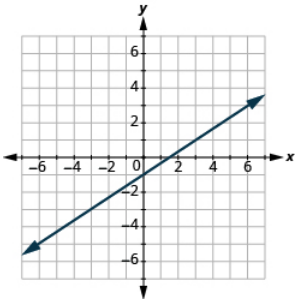
Recognize the Relation Between the Solutions of an Equation and its Graph

In the following exercises, for each ordered pair, decide (a) if the ordered pair is a solution to the equation. (b) if the point is on the line.

295. y = − x + 4
    a. (0, 4)
    b. (−1, 3)
    c. (2, 2)
    d. (−2, 6)



296. y = $\frac{2}{3}$x − 1
    a. (0, −1)
    b. (3, 1)
    c. (−3, −3)
    d. (6, 4)



Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

297. $y = 4x - 3$
298. $y = -3x$
299. $2x + y = 7$

Graph Vertical and Horizontal lines

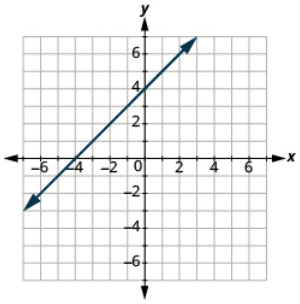
In the following exercises, graph the vertical or horizontal lines.

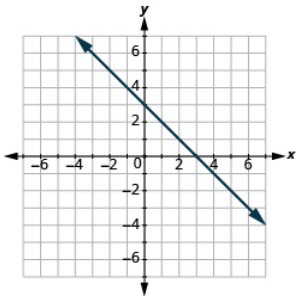
300. $y = -2$
301. $x = 3$

11.3 - Graphing with Intercepts

Identify the Intercepts on a Graph

In the following exercises, find the x- and y-intercepts.

302. 

303. 

Find the Intercepts from an Equation of a Line

In the following exercises, find the intercepts.

304. $x + y = 5$
305. $x - y = -1$
306. $y = \frac{3}{4}x - 12$
307. $y = 3x$

Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

308. $-x + 3y = 3$
309. $x + y = -2$

Choose the Most Convenient Method to Graph a Line

In the following exercises, identify the most convenient method to graph each line.

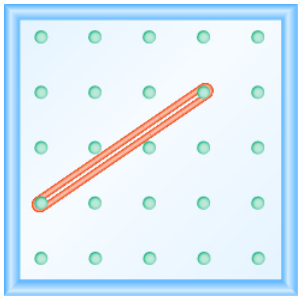
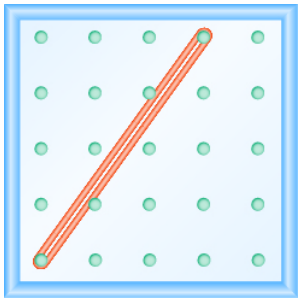
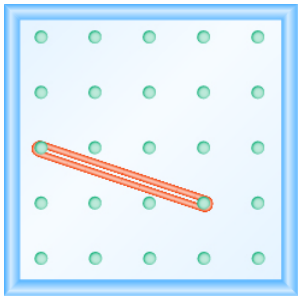
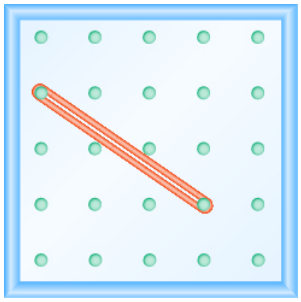
310. $x = 5$
311. $y = -3$

312. $2x + y = 5$
313. $x - y = 2$
314. $y = \frac{1}{2}x + 2$
315. $y = \frac{3}{4}x - 1$

11.4 - Understand Slope of a Line

Use Geoboards to Model Slope

In the following exercises, find the slope modeled on each geoboard.

316. 
317. 
318. 
319. 

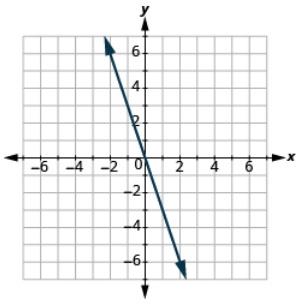
In the following exercises, model each slope. Draw a picture to show your results.

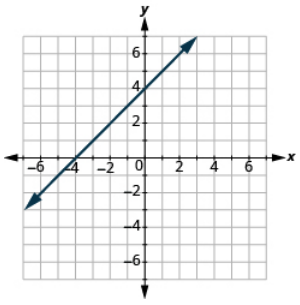
320. $\frac{1}{3}$
321. $\frac{3}{2}$
322. $-\frac{2}{3}$

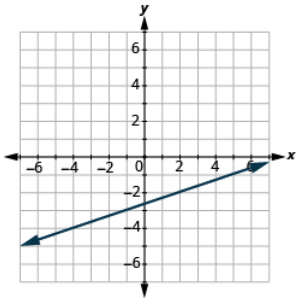
323. $-\frac{1}{2}$

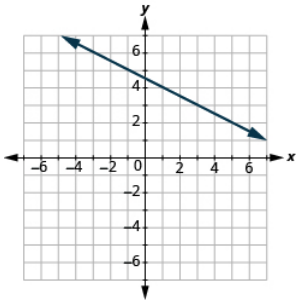
Find the Slope of a Line from its Graph

In the following exercises, find the slope of each line shown.

324. 

325. 

326. 

327. 

Find the Slope of Horizontal and Vertical Lines

In the following exercises, find the slope of each line.

328. y = 2
329. x = 5
330. x = −3
331. y = −1

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

332. (2, 1), (4, 5)

333. (−1, −1), (0, −5)
334. (3, 5), (4, −1)
335. (−5, −2), (3, 2)

Graph a Line Given a Point and the Slope

In the following exercises, graph the line given a point and the slope.

336. (2, −2); m = $\frac{5}{2}$
337. (−3, 4); m = $-\frac{1}{3}$

Solve Slope Applications

In the following exercise, solve the slope application.

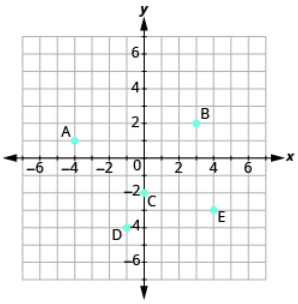
338. A roof has rise 10 feet and run 15 feet. What is its slope?

PRACTICE TEST

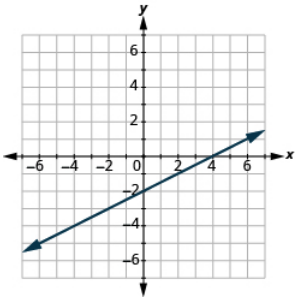
339. Plot and label these points:

a. (2, 5)
b. (−1, −3)
c. (−4, 0)
d. (3, −5)
e. (−2, 1)

340. Name the ordered pair for each point shown.



341. Find the x-intercept and y-intercept on the line shown.



342. Find the x-intercept and y-intercept of the equation 3x − y = 6.
343. Is (1, 3) a solution to the equation x + 4y = 12? How do you know?
344. Complete the table to find four solutions to the equation y = − x + 1.

x	y	(x, y)
0		
1		

x	y	(x, y)
3		
-2		

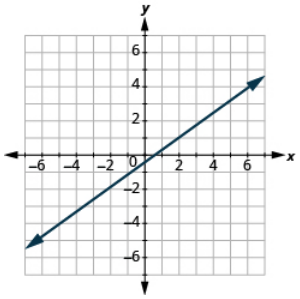
345. Complete the table to find three solutions to the equation 4x + y = 8.

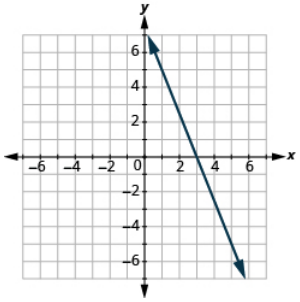
x	y	(x, y)
0		
	0	
3		

In the following exercises, find three solutions to each equation and then graph each line.

346. y = −3x
347. 2x + 3y = −6

In the following exercises, find the slope of each line.

348. 

349. 

350. Use the slope formula to find the slope of the line between (0, −4) and (5, 2).
351. Find the slope of the line y = 2.
352. Graph the line passing through (1, 1) with slope m = $\frac{3}{2}$.
353. A bicycle route climbs 20 feet for 1,000 feet of horizontal distance. What is the slope of the route?

Contributors and Attributions

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8.E: Review Exercises

Chapter 4 Review Exercises

Rectangular Coordinate System

Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system.

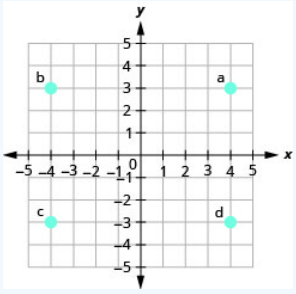
Exercise 8.E.1

1. $(-1,-5)$
2. $(-3,4)$
3. $(2,-3)$
4. $\left(1,\frac{5}{2}\right)$

Exercise 8.E.2

1. $(4,3)$
2. $(-4,3)$
3. $(-4,-3)$
4. $(4,-3)$

Answer



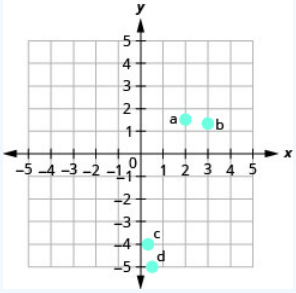
Exercise 8.E.3

1. $(-2,0)$
2. $(0,-4)$
3. $(0,5)$
4. $(3,0)$

Exercise 8.E.4

1. $\left(2,\frac{3}{2}\right)$
2. $\left(3,\frac{4}{3}\right)$
3. $\left(\frac{1}{3},-4\right)$
4. $\left(\frac{1}{2},-5\right)$

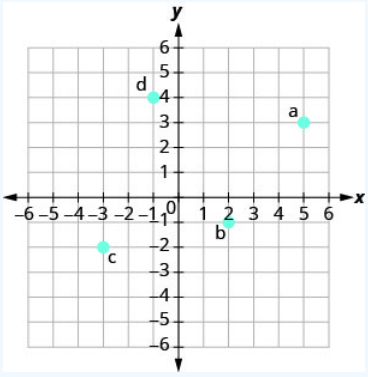
Answer



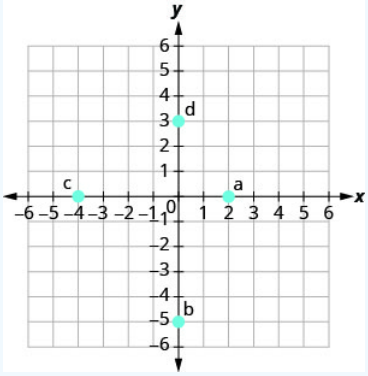
Identify Points on a Graph

In the following exercises, name the ordered pair of each point shown in the rectangular coordinate system.

Exercise 8.E.5



Exercise 8.E.6



Answer

a. (2,0)

b (0,−5)

c (−4.0)

d (0,3)

Verify Solutions to an Equation in Two Variables

In the following exercises, which ordered pairs are solutions to the given equations?

Exercise 8.E.7

$5x + y = 10$

1. (5,1)
2. (2,0)
3. (4,−10)

Exercise 8.E.8

$y = 6x - 2$

1. (1,4)
2. $\left(\frac{1}{3}, 0\right)$
3. (6,−2)

Answer

1, 2

Complete a Table of Solutions to a Linear Equation in Two Variables

In the following exercises, complete the table to find solutions to each linear equation.

Exercise 8.E.9

$y = 4x - 1$

x	y	(x,y)
0		
1		
-2		

Exercise 8.E.10

$y = -\frac{1}{2}x + 3$

x	y	(x,y)
0		
4		
-2		

Answer

x	y	(x,y)
0	3	(0,3)
4	1	(4, 1)
−2	4	(−2,4)

Exercise 8.E.11

$x + 2y = 5$

x	y	(x,y)
	0	
1		
-1		

Exercise 8.E.12

$3x + 2y = 6$

x	y	(x,y)
0		
	0	
-2		

Answer

x	y	(x,y)
0	−3	(0,−3)
2	0	(2,0)
−2	−6	(−2,−6)

Find Solutions to a Linear Equation in Two Variables

In the following exercises, find three solutions to each linear equation.

Exercise 8.E.13

$x + y = 3$

Exercise 8.E.14

$x + y = -4$

Answer

Answers will vary.

Exercise 8.E.15

$y = 3x + 1$

Exercise 8.E.16

$y = -x - 1$

Answer

Answers will vary.

Graphing Linear Equations

Recognize the Relation Between the Solutions of an Equation and its Graph

In the following exercises, for each ordered pair, decide:

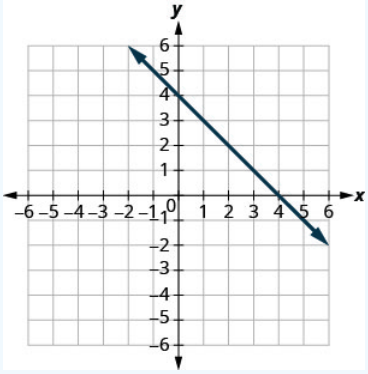
1. Is the ordered pair a solution to the equation?
2. Is the point on the line?

Exercise $8.E.17$

$y = -x + 4$

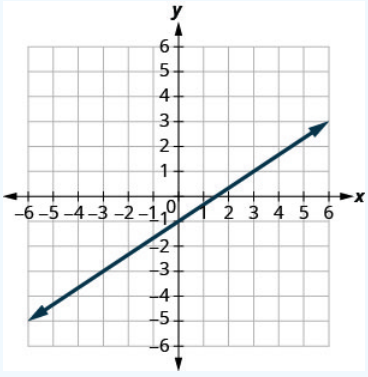
(0,4) (−1,3)

(2,2) (−2,6)



Exercise $8.E.18$

$y = \frac{2}{3}x - 1$
$(0, -1)(3, 1)$
$(-3, -3)(6, 4)$



Answer

1. yes; yes
2. yes; no

Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

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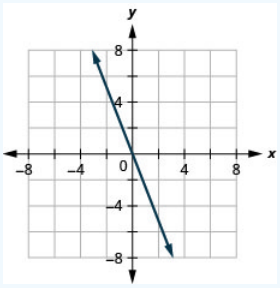
Exercise 8.E.19

$y = 4x - 3$

Exercise 8.E.20

$y = -3x$

Answer



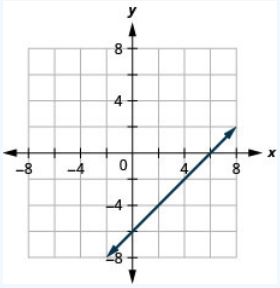
Exercise 8.E.21

$y = \frac{1}{2}x + 3$

Exercise 8.E.22

$x - y = 6$

Answer



Exercise 8.E.23

$2x + y = 7$

Exercise 8.E.24

$3x - 2y = 6$

Answer

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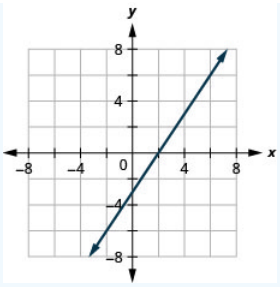


Graph Vertical and Horizontal lines

In the following exercises, graph each equation.

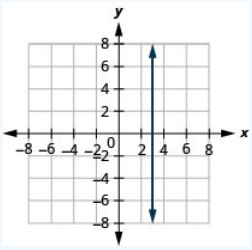
Exercise 8.E.25

$y = -2$

Exercise 8.E.26

$x = 3$

Answer



In the following exercises, graph each pair of equations in the same rectangular coordinate system.

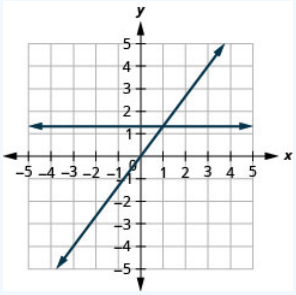
Exercise 8.E.27

$y = -2x$ and $y = -2$

Exercise 8.E.28

$y = \frac{4}{3}x$ and $y = \frac{4}{3}$

Answer



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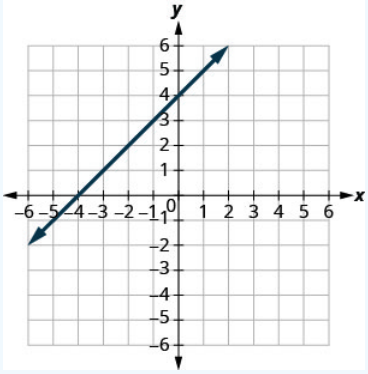
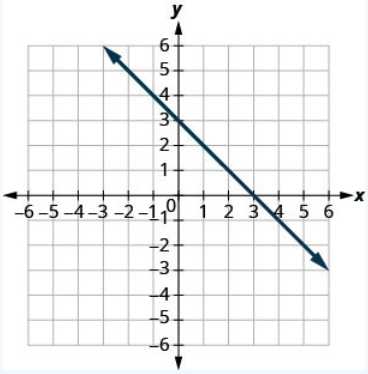
Identify the x- and y-Intercepts on a Graph

In the following exercises, find the x- and y-intercepts.

Exercise $8.E.29$



Exercise $8.E.30$



Answer

$(3, 0)$ and $(0, 3)$

Find the$x$- and y-Intercepts from an Equation of a Line

In the following exercises, find the intercepts of each equation.

Exercise $8.E.31$

$x + y = 5$

Exercise $8.E.32$

$x - y = -1$

Answer

$(-1, 0), (0, 1)$

Exercise 8.E.33

$x+2y=6$

Exercise 8.E.34

$2x+3y=12$

Answer

$(6,0),(0,4)$

Exercise 8.E.35

$y=\frac{3}{4}x-12$

Exercise 8.E.36

$y=3x$

Answer

$(0,0)$

Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

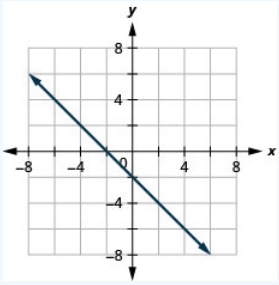
Exercise 8.E.37

$-x+3y=3$

Exercise 8.E.38

$x+y=-2$

Answer



Exercise 8.E.39

$x-y=4$

Exercise 8.E.40

$2x-y=5$

Answer

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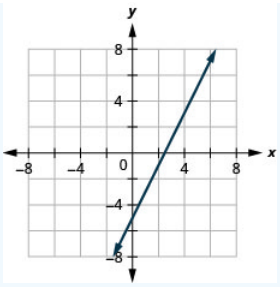


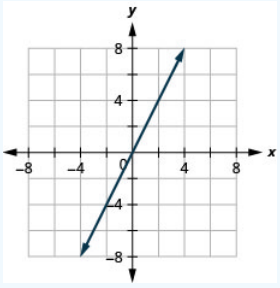
Exercise 8.E.41

$2x - 4y = 8$

Exercise 8.E.42

$y = 2x$

Answer

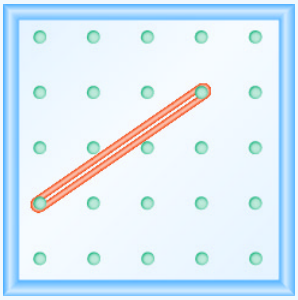


Slope of a Line

Use Geoboards to Model Slope

In the following exercises, find the slope modeled on each geoboard.

Exercise 8.E.43



Exercise 8.E.44

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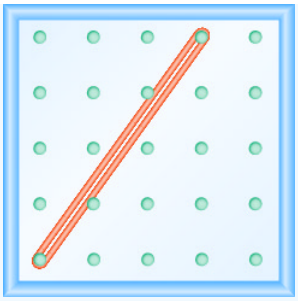
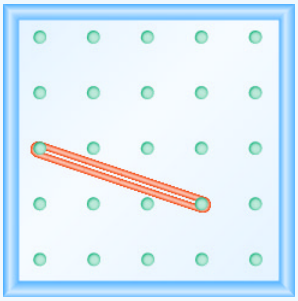


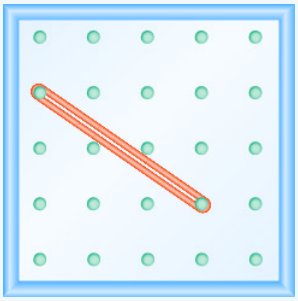
Answer

$\frac{4}{3}$

Exercise 8.E.45



Exercise 8.E.46



Answer

$-\frac{2}{3}$

Exercise 8.E.47

$\frac{1}{3}$

Exercise 8.E.48

$\frac{3}{2}$

Answer

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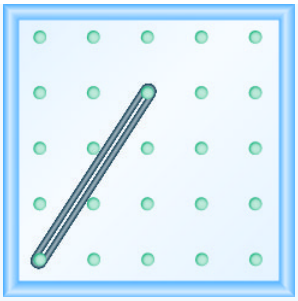


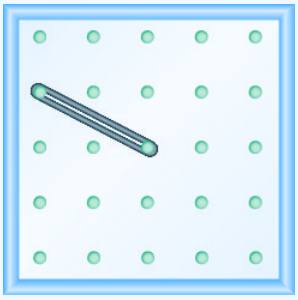
Exercise 8.E.49

$-\frac{2}{3}$

Exercise 8.E.50

$-\frac{1}{2}$

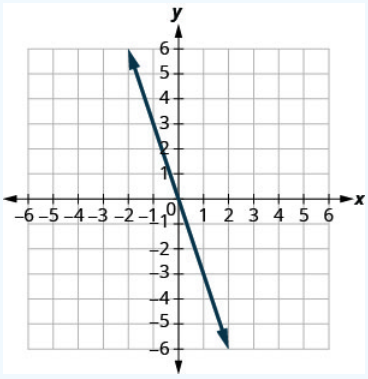
Answer



Use $m = \frac{\text{rise}}{\text{run}}$ to find the Slope of a Line from its Graph

In the following exercises, find the slope of each line shown.

Exercise 8.E.51



Exercise 8.E.52

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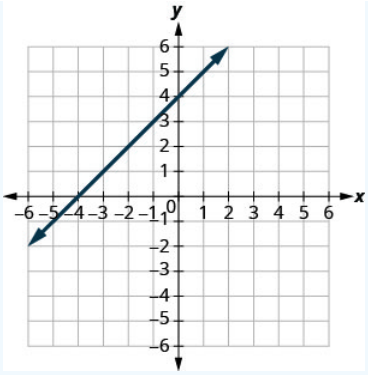
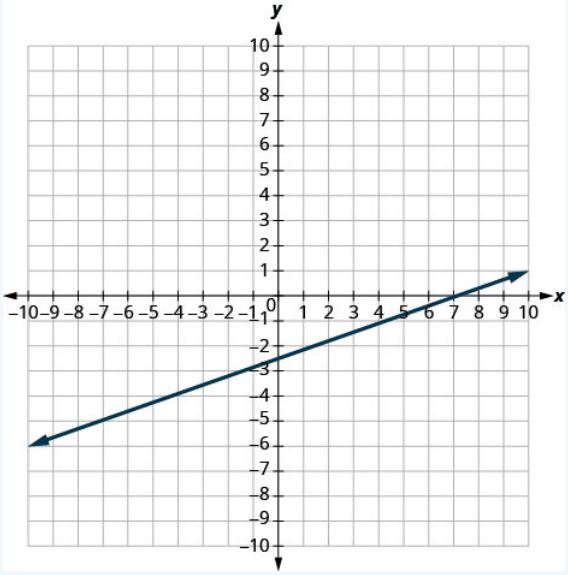


Answer

1

Exercise $8.E.53$



Exercise $8.E.54$

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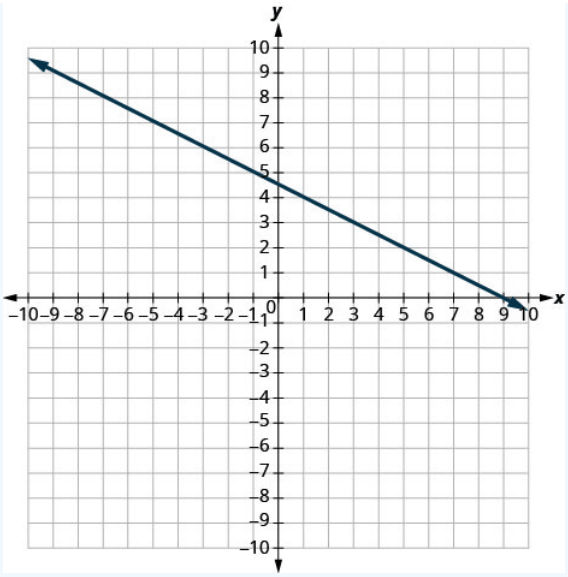


Answer

$-\frac{1}{2}$

Find the Slope of Horizontal and Vertical Lines

In the following exercises, find the slope of each line.

Exercise 8.E.55

$y = 2$

Exercise 8.E.56

$x = 5$

Answer

undefined

Exercise 8.E.57

$x = -3$

Exercise 8.E.58

$y = -1$

Answer

0

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

Exercise 8.E.59

$(-1,-1),(0,5)$

Exercise 8.E.60

$(3,5),(4,-1)$

Answer

-6

Exercise 8.E.61

$(-5,-2),(3,2)$

Exercise 8.E.62

$(2,1),(4,6)$

Answer

$\frac{5}{2}$

Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

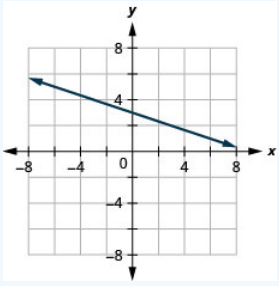
Exercise 8.E.63

$(2,-2);\quad m=\frac{5}{2}$

Exercise 8.E.64

$(-3,4);\quad m=-\frac{1}{3}$

Answer



Exercise 8.E.65

x -intercept $-4;\quad m=3$

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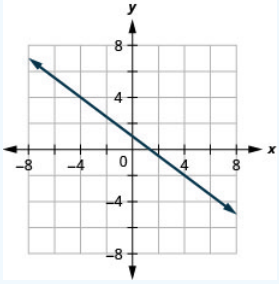
Exercise 8.E.66

y -intercept 1; $m = -\frac{3}{4}$

Answer

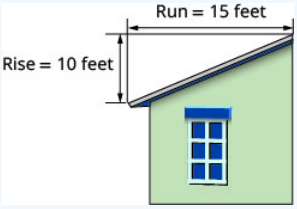


Solve Slope Applications

In the following exercises, solve these slope applications.

Exercise 8.E.67

The roof pictured below has a rise of 10 feet and a run of 15 feet. What is its slope?



Exercise 8.E.68

A mountain road rises 50 feet for a 500-foot run. What is its slope?

Answer

$\frac{1}{10}$

Intercept Form of an Equation of a Line

Recognize the Relation Between the Graph and the Slope–Intercept Form of an Equation of a Line

In the following exercises, use the graph to find the slope and y-intercept of each line. Compare the values to the equation $y = mx + b$.

Exercise 8.E.69

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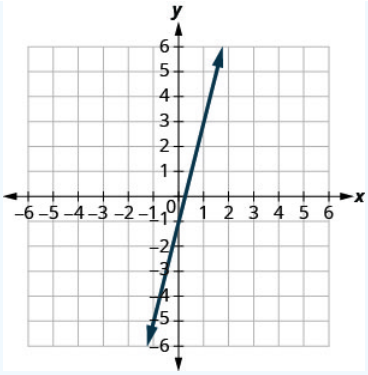
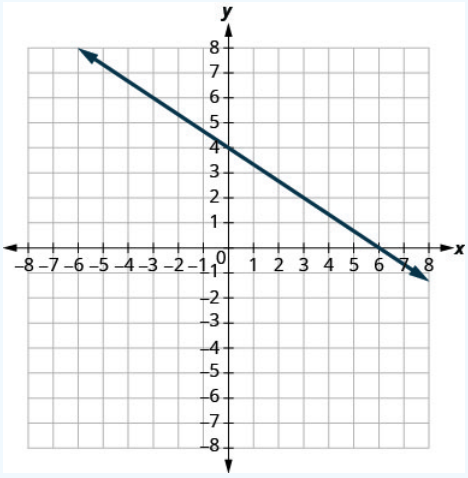


$y = 4x - 1$

Exercise 8.E.70



$y = -\frac{2}{3}x + 4$

Answer

slope $m = -\frac{2}{3}$ and y-intercept $(0, 4)$

Identify the Slope and y-Intercept from an Equation of a Line

In the following exercises, identify the slope and y-intercept of each line.

Exercise 8.E.71

$y = -4x + 9$

Exercise 8.E.72

$y = \frac{5}{3}x - 6$

Answer

$\frac{5}{3}; (0, -6)$

Exercise 8.E.73

$5x + y = 10$

Exercise 8.E.74

$4x - 5y = 8$

Answer

$\frac{4}{5}; \quad \left(0, -\frac{8}{5}\right)$

Graph a Line Using Its Slope and Intercept

In the following exercises, graph the line of each equation using its slope and y-intercept.

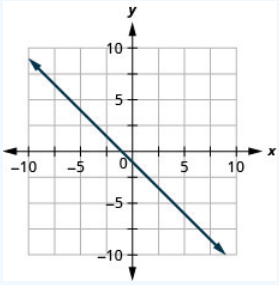
Exercise 8.E.75

$y = 2x + 3$

Exercise 8.E.76

$y = -x - 1$

Answer



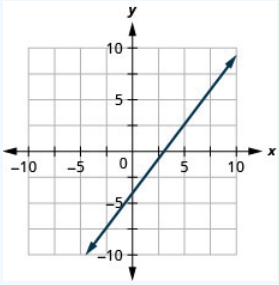
Exercise 8.E.77

$y = -\frac{2}{5}x + 3$

Exercise 8.E.78

$4x - 3y = 12$

Answer



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In the following exercises, determine the most convenient method to graph each line.

Exercise 8.E.79

$x = 5$

Exercise 8.E.80

$y = -3$

Answer

horizontal line

Exercise 8.E.81

$2x + y = 5$

Exercise 8.E.82

$x - y = 2$

Answer

intercepts

Exercise 8.E.83

$y = x + 2$

Exercise 8.E.84

$y = \frac{3}{4}x - 1$

Answer

plotting points

Graph and Interpret Applications of Slope–Intercept

Exercise 8.E.85

Katherine is a private chef. The equation $C = 6.5m + 42$ models the relation between her weekly cost, C, in dollars and the number of meals, m, that she serves.

1. Find Katherine's cost for a week when she serves no meals.
2. Find the cost for a week when she serves 14 meals.
3. Interpret the slope and C-intercept of the equation.
4. Graph the equation.

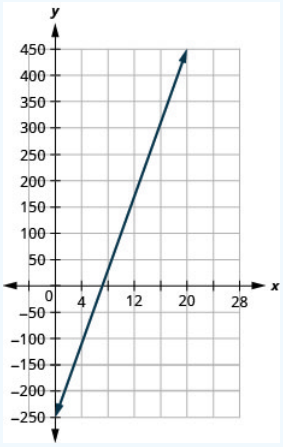
Exercise 8.E.86

Marjorie teaches piano. The equation $P = 35h - 250$ models the relation between her weekly profit, P, in dollars and the number of student lessons, s, that she teaches.

1. Find Marjorie's profit for a week when she teaches no student lessons.
2. Find the profit for a week when she teaches 20 student lessons.

3. Interpret the slope and P-intercept of the equation.
4. Graph the equation.

Answer

1. $-\$250$
2. $\$450$
3. The slope, 35, means that Marjorie's weekly profit, P, increases by $\$35$ for each additional student lesson she teaches. The P-intercept means that when the number of lessons is 0, Marjorie loses $\$250$
4. 

Use Slopes to Identify Parallel Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel.

? Exercise $8.E.87$

$4x-3y=-1; \quad y=\frac{4}{3}x-3$

? Exercise $8.E.88$

$2x-y=8; \quad x-2y=4$

Answer

not parallel

Use Slopes to Identify Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are perpendicular.

? Exercise $8.E.89$

$y=5x-1; \quad 10x+2y=0$

? Exercise $8.E.90$

$3x-2y=5; \quad 2x+3y=6$

Answer

perpendicular

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

Exercise $8.E.91$

slope $\frac{1}{3}$ and y-intercept $(0, -6)$

Exercise $8.E.92$

slope -5 and y-intercept $(0, -3)$

Answer

$y = -5x - 3$

Exercise $8.E.93$

slope 0 and y-intercept $(0, 4)$

Exercise $8.E.94$

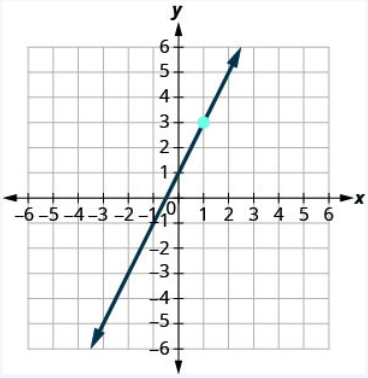
slope -2 and y-intercept $(0, 0)$

Answer

$y = -2x$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.

Exercise $8.E.95$



Exercise $8.E.96$

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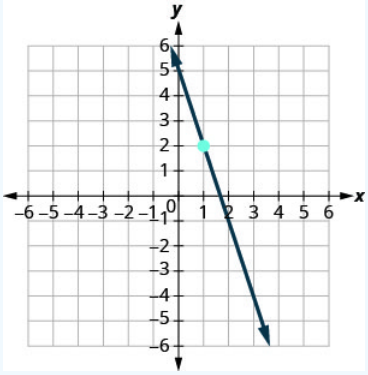
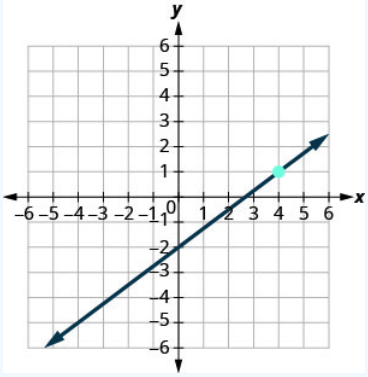


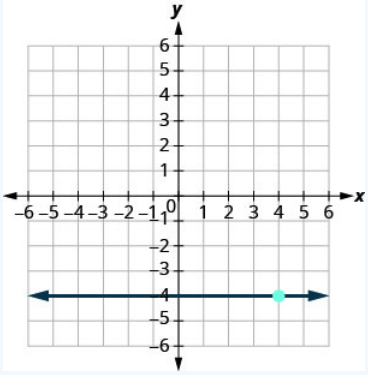
Answer

$y = -3x + 5$

Exercise 8.E.97



Exercise 8.E.98



Answer

$y = -4$

Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

Exercise 8.E.99

$m = -\frac{1}{4}$, point $(-8, 3)$

Exercise 8.E.100

$m = \frac{3}{5}$, point $(10, 6)$

Answer

$y = \frac{3}{5}x$

Exercise 8.E.101

Horizontal line containing $(-2, 7)$

Exercise 8.E.102

$m = -2$, point $(-1, -3)$

Answer

$y = -2x - 5$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

Exercise 8.E.103

$(2, 10)$ and $(-2, -2)$

Exercise 8.E.104

$(7, 1)$ and $(5, 0)$

Answer

$y = \frac{1}{2}x - \frac{5}{2}$

Exercise 8.E.105

$(3, 8)$ and $(3, -4)$

Exercise 8.E.106

$(5, 2)$ and $(-1, 2)$

Answer

$y = 2$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

Exercise 8.E.107

line $y = -3x + 6$, point $(1, -5)$

Exercise 8.E.108

line $2x + 5y = -10$, point $(10, 4)$

Answer

$y = -\frac{2}{5}x + 8$

Exercise 8.E.109

line $x = 4$, point $(-2, -1)$

Exercise 8.E.110

line $y = -5$, point $(-4, 3)$

Answer

$y = 3$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–intercept form.

Exercise 8.E.111

line $y = -\frac{4}{5}x + 2$, point $(8, 9)$

Exercise 8.E.112

line $2x - 3y = 9$, point $(-4, 0)$

Answer

$y = -\frac{3}{2}x - 6$

Exercise 8.E.113

line $y = 3$, point $(-1, -3)$

Exercise 8.E.114

line $x = -5$ point $(2, 1)$

Answer

$y = 1$

Graph Linear Inequalities

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.

Exercise 8.E.115

Determine whether each ordered pair is a solution to the inequality $y < x - 3$:

1. $(0, 1)$
2. $(-2, -4)$
3. $(5, 2)$
4. $(3, -1)$
5. $(-1, -5)$

Exercise 8.E.116

Determine whether each ordered pair is a solution to the inequality $x + y > 4$:

1. $(6, 1)$
2. $(-3, 6)$
3. $(3, 2)$
4. $(-5, 10)$
5. $(0, 0)$

Answer

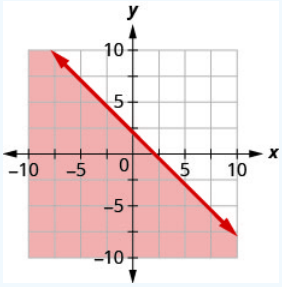
1. yes
2. no
3. yes
4. yes
5. no

Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

Exercise 8.E.117

Write the inequality shown by the graph with the boundary line $y = -x + 2$.



Exercise 8.E.118

Write the inequality shown by the graph with the boundary line $y = \frac{2}{3}x - 3$

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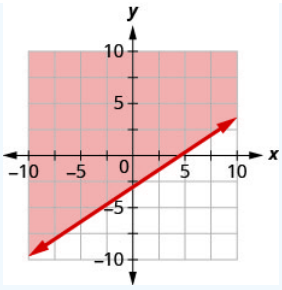
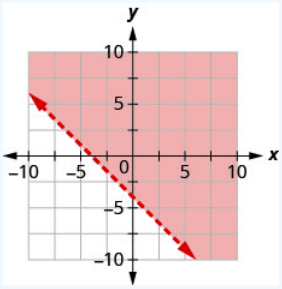


Answer

$y > \frac{2}{3}x - 3$

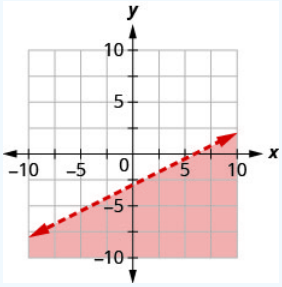
Exercise 8.E.119

Write the inequality shown by the shaded region in the graph with the boundary line $x + y = -4$.



Exercise 8.E.120

Write the inequality shown by the shaded region in the graph with the boundary line $x - 2y = 6$.



Answer

$x - 2y \geq 6$

Graph Linear Inequalities

In the following exercises, graph each linear inequality.

Exercise 8.E.121

Graph the linear inequality $y > \frac{2}{5}x - 4$

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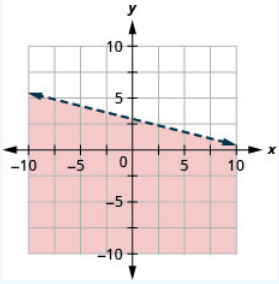
Exercise 8.E.122

Graph the linear inequality $y \leq -\frac{1}{4}x + 3$

Answer



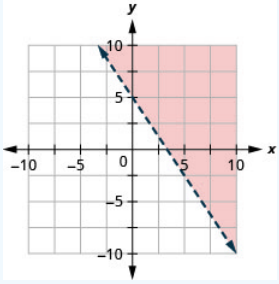
Exercise 8.E.123

Graph the linear inequality $x - y \leq 5$

Exercise 8.E.124

Graph the linear inequality $3x + 2y > 10$

Answer



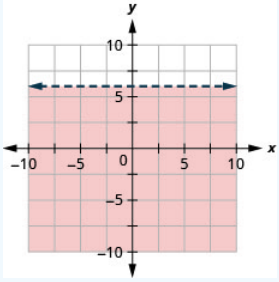
Exercise 8.E.125

Graph the linear inequality $y \leq -3x$

Exercise 8.E.126

Graph the linear inequality $y < 6$

Answer



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Practice Test

Exercise $8.E.1$

Plot each point in a rectangular coordinate system.

1. $(2, 5)$
2. $(-1, -3)$
3. $(0, 2)$
4. $\left(-4, \frac{3}{2}\right)$
5. $(5, 0)$

Exercise $8.E.2$

Which of the given ordered pairs are solutions to the equation $3x - y = 6$?

1. $(3, 3)$
2. $(2, 0)$
3. $(4, -6)$

Answer

1. yes
2. yes
3. no

Exercise $8.E.3$

Find three solutions to the linear equation $y = -2x - 4$

Exercise $8.E.4$

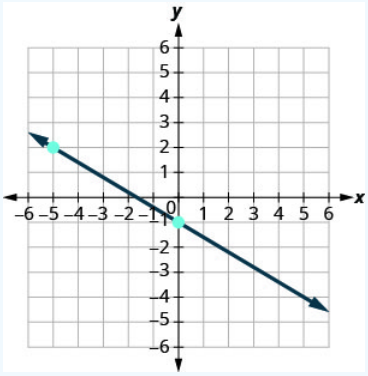
Find the x- and y-intercepts of the equation $4x - 3y = 12$

Answer

$(3, 0), (0, -4)$

Find the slope of each line shown.

Exercise $8.E.5$



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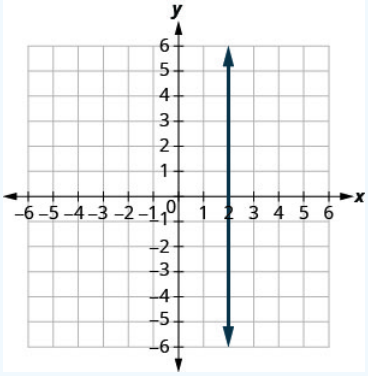
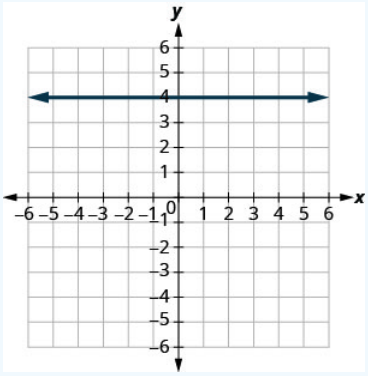
Exercise 8.E.6



Answer

undefined

Exercise 8.E.7



Exercise 8.E.8

Find the slope of the line between the points $(5, 2)$ and $(-1, -4)$

Answer

1

Exercise 8.E.9

Graph the line with slope $\frac{1}{2}$ containing the point $(-3, -4)$

Graph the line for each of the following equations.

Exercise 8.E.10

$y = \frac{5}{3}x - 1$

Answer

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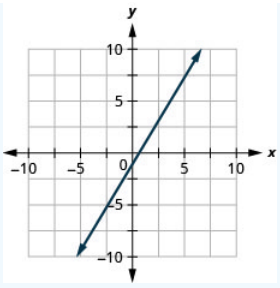


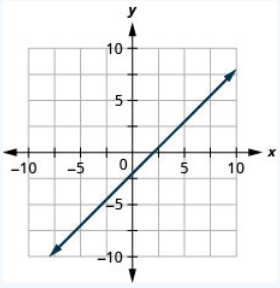
Exercise 8.E.11

$y = -x$

Exercise 8.E.12

$x - y = 2$

Answer



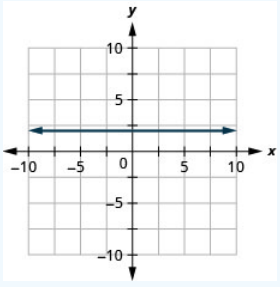
Exercise 8.E.13

$4x + 2y = -8$

Exercise 8.E.14

$y = 2$

Answer



Exercise 8.E.15

$x = -3$

Find the equation of each line. Write the equation in slope–intercept form.

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Exercise 8.E.16

slope $-\frac{3}{4}$ and y-intercept $(0, -2)$

Answer

$y = -\frac{3}{4}x - 2$

Exercise 8.E.17

$m = 2$, point $(-3, -1)$

Exercise 8.E.18

containing $(10, 1)$ and $(6, -1)$

Answer

$y = \frac{1}{2}x - 4$

Exercise 8.E.19

parallel to the line $y = -\frac{2}{3}x - 1$, containing the point $(-3, 8)$

Exercise 8.E.20

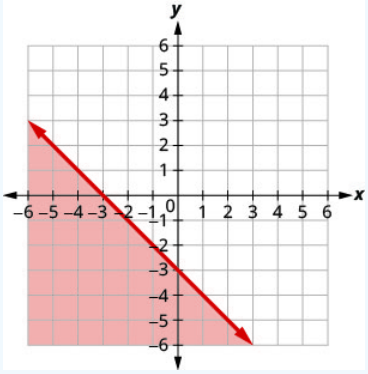
perpendicular to the line $y = \frac{5}{4}x + 2$, containing the point $(-10, 3)$

Answer

$y = -\frac{4}{5}x - 5$

Exercise 8.E.21

Write the inequality shown by the graph with the boundary line $y = -x - 3$.



Graph each linear inequality.

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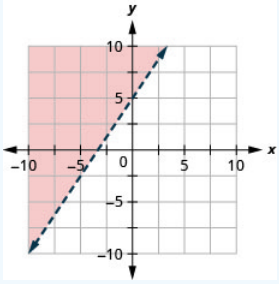
Exercise $8.E.22$

$y > \frac{3}{2}x + 5$

Answer



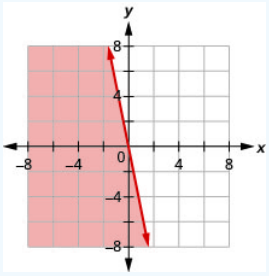
Exercise $8.E.23$

$x - y \geq -4$

Exercise $8.E.24$

$y \leq -5x$

Answer



Exercise $8.E.1$

$y < 3$

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8.S: Graphs (Summary)

KEY TERMS

horizontal line	The graph of an equation that can be written in the form y = b whose line passes through the y-axis at (0, b).
intercepts of a line	Each of the points at which a line crosses the x-axis and the y-axis is called an intercept of the line.
linear equation	An equation of the form Ax + By = C, where A and B are not both zero, is called a linear equation in two variables
ordered pair	An ordered pair (x, y) gives the coordinates of a point in a rectangular coordinate system. The first number is the x-coordinate. The second number is the y-coordinate.
origin	The point (0, 0) is called the origin. It is the point where the point where the x-axis and y-axis intersect.
quadrants	The four areas of a rectangular coordinate system that has been divided by the x-axis and y-axis.
slope of a line	The slope of a line is m = $\frac{rise}{run}$. The rise measures the vertical change and the run measures the horizontal change.
solution to a linear equation in two variables	An ordered pair (x, y) is a solution to the linear equation Ax + By = C, if the equation is a true statement when the x- and y-values of the ordered pair are substituted into the equation.
vertical line	A vertical line is the graph of an equation that can be written in the form x = a. The line passes through the x-axis at (a, 0).
x-axis	The horizontal axis in a rectangular coordinate system.
y-axis	The vertical axis on a rectangular coordinate system.

Key Concepts

11.1 Use the Rectangular Coordinate System

- **Sign Patterns of the Quadrants**

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x,y)	(x,y)	(x,y)	(x,y)
(+,+)	(−,+)	(−,−)	(+,−)

- **Coordinates of Zero**
  - Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates (a, 0).
  - Points with a x-coordinate equal to 0 are on the y-axis, and have coordinates (0, b).
  - The point (0, 0) is called the origin. It is the point where the x-axis and y-axis intersect.

11.2 Graphing Linear Equations

- **Graph a linear equation by plotting points**.
  1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
  2. Plot the points on a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
  3. Draw the line through the points. Extend the line to fill the grid and put arrows on both ends of the line.
- **Graph of a Linear Equation**: The graph of a linear equation ax + by = c is a straight line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

11.3 Graphing with Intercepts

- **Intercepts**
  - The x-intercept is the point, (a, 0), where the graph crosses the x-axis. The x-intercept occurs when y is zero.
  - The y-intercept is the point, (0, b), where the graph crosses the y-axis. The y-intercept occurs when x is zero.
  - The x-intercept occurs when y is zero.
  - The y-intercept occurs when x is zero.
- **Find the x and y intercepts from the equation of a line**
  - To find the x-intercept of the line, let y = 0 and solve for x.
  - To find the y-intercept of the line, let x = 0 and solve for y.

x	y
	0
0	

- **Graph a line using the intercepts**
  1. Find the x- and y-intercepts of the line.
     - Let y = 0 and solve for x.
     - Let x = 0 and solve for y.
  2. Find a third solution to the equation.
  3. Plot the three points and then check that they line up.
  4. Draw the line.
- **Choose the most convenient method to graph a line**
  1. Determine if the equation has only one variable. Then it is a vertical or horizontal line.
     - x = a is a vertical line passing through the x-axis at a.
     - y = b is a horizontal line passing through the y-axis at b.
  2. Determine if y is isolated on one side of the equation. The graph by plotting points. Choose any three values for x and then solve for the corresponding y-values.
  3. Determine if the equation is of the form Ax + By = C, find the intercepts. Find the x- and y-intercepts and then a third point.

11.4 Understand Slope of a Line

- **Find the slope from a graph**
  1. Locate two points on the line whose coordinates are integers.
  2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.
  3. Count the rise and the run on the legs of the triangle.
  4. Take the ratio of rise to run to find the slope, m = $\frac{rise}{run}$.
- **Slope of a Horizontal Line**
  - The slope of a horizontal line, y = b, is 0.
- **Slope of a Vertical Line**
  - The slope of a vertical line, x = a, is undefined.
- **Slope Formula**
  - The slope of the line between two points (x_1, y_1) and (x_2, y_2) is m = $\frac{y_2 - y_1}{x_2 - x_1}$.
- **Graph a line given a point and a slope.**
  1. Plot the given point.
  2. Use the slope formula to identify the rise and the run.

3. Starting at the given point, count out the rise and run to mark the second point.
4. Connect the points with a line.

Contributors and Attributions

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CHAPTER OVERVIEW

9: Solving Linear Equations and Inequalities

9.1: Solve Equations Using the Subtraction and Addition Properties of Equality

9.1E: Exercises

9.2: Solve Equations using the Division and Multiplication Properties of Equality

9.2E: Exercises

9.3: Solve Equations with Variables and Constants on Both Sides

9.3E: Exercises

9.4: Use a General Strategy to Solve Linear Equations

9.4E: Exercises

9.5: Solve Equations with Fractions or Decimals

9.5E: Exercises

9.6: Solve a Formula for a Specific Variable

9.6E: Exercises

9.7: Solve Linear Inequalities

9.7E: Exercises

9.8: Chapter 2 Review Exercises

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9.1: Solve Equations Using the Subtraction and Addition Properties of Equality

Learning Objectives

By the end of this section, you will be able to:

- Verify a solution of an equation
- Solve equations using the Subtraction and Addition Properties of Equality
- Solve equations that require simplification
- Translate to an equation and solve
- Translate and solve applications

Quiz

Before you get started, take this readiness quiz.

1. Evaluate $x+4$ when $x=-3$.
   If you missed this problem, review Exercise 1.5.25.
2. Evaluate $15-y$ when $y=-5$.
   If you missed this problem, review Exercise 1.5.31.
3. Simplify $4(4n+1)-15n$.
   If you missed this problem, review Exercise 1.10.49.
4. Translate into algebra "5 is less than x."
   If you missed this problem, review Exercise 1.3.43.

Verify a Solution of an Equation

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same – so that we end up with a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle!

Definition: Solution of an Equation

A **solution of an equation** is a value of a variable that makes a true statement when substituted into the equation.

TO DETERMINE WHETHER A NUMBER IS A Solution TO AN EQUATION

1. Substitute the number in for the variable in the equation.
2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true (the left side is equal to the right side)
   - If it is true, the number is a solution.
   - If it is not true, the number is not a solution.

Example 9.1.1

Determine whether $x=\frac{3}{2}$ is a solution of $4x-2=2x+1$.

Solution

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

	$4x-2=2x+1$
Substitute $\frac{3}{2}$ for x.	$4\left(\frac{3}{2}\right)-2\stackrel{?}{=}2\left(\frac{3}{2}\right)+1$
Multiply.	$6-2\stackrel{?}{=}3+1$
Subtract.	$4=4\checkmark$

Since $x=\frac{3}{2}$ results in a true equation (4 is in fact equal to 4), $\frac{3}{2}$ is a solution to the equation $4x-2=2x+1$.

Try It 9.1.2

Is $y = \frac{4}{3}$ a solution of $9y + 2 = 6y + 3$?

Answer

no

Try It 9.1.3

Is $y = \frac{7}{5}$ a solution of $5y + 3 = 10y - 4$?

Answer

yes

Solve Equations Using the Subtraction and Addition Properties of Equality

We are going to use a model to clarify the process of solving an equation. An envelope represents the variable – since its contents are unknown – and each counter represents one. We will set out one envelope and some counters on our workspace, as shown in Figure 9.1.1. Both sides of the workspace have the same number of counters, but some counters are "hidden" in the envelope. Can you tell how many counters are in the envelope?

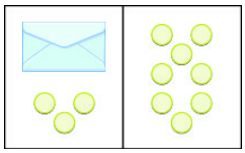

Figure 9.1.1: The illustration shows a model of an equation with one variable. On the left side of the workspace is an unknown (envelope) and three counters, while on the right side of the workspace are eight counters.

What are you thinking? What steps are you taking in your mind to figure out how many counters are in the envelope?

Perhaps you are thinking: "I need to remove the 3 counters at the bottom left to get the envelope by itself. The 3 counters on the left can be matched with 3 on the right and so I can take them away from both sides. That leaves five on the right—so there must be 5 counters in the envelope." See Figure 9.1.2 for an illustration of this process.

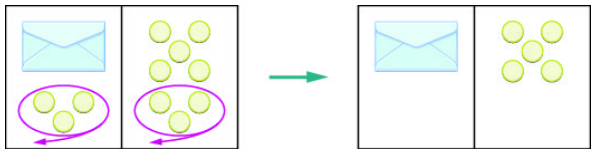

Figure 9.1.2: The illustration shows a model for solving an equation with one variable. On both sides of the workspace remove three counters, leaving only the unknown (envelope) and five counters on the right side. The unknown is equal to five counters.

What algebraic equation would match this situation? In Figure 9.1.3 each side of the workspace represents an expression and the center line takes the place of the equal sign. We will call the contents of the envelope x.

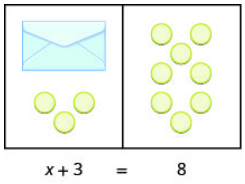

Figure 9.1.3: The illustration shows a model for the equation $x + 3 = 8$.

Let's write algebraically the steps we took to discover how many counters were in the envelope:

Table 9.1.1

	$x + 3 = 8$
First, we took away three from each side.	$x + 3 - 3 = 8 - 3$
Then we were left with five.	$x = 5$

Check:

Five in the envelope plus three more does equal eight!

$$5+3=8$$

Our model has given us an idea of what we need to do to solve one kind of equation. The goal is to isolate the variable by itself on one side of the equation. To solve equations such as these mathematically, we use the **Subtraction Property of Equality**.

SUBTRACTION PROPERTY OF EQUALITY

For any numbers a, b, and c,

$$\begin{aligned} &\text{If} \quad a=b \\ &\text{then} \quad a-c=b-c \end{aligned}$$

When you subtract the same quantity from both sides of an equation, you still have equality.

Note

Doing the Manipulative Mathematics activity "Subtraction Property of Equality" will help you develop a better understanding of how to solve equations by using the **Subtraction Property of Equality**.

Let's see how to use this property to solve an equation. Remember, the goal is to isolate the variable on one side of the equation. And we check our solutions by substituting the value into the equation to make sure we have a true statement.

Example 9.1.4

Solve: $y+37=-13$.

Solution

To get y by itself, we will undo the addition of 37 by using the Subtraction Property of Equality.

	$y+37=-13$
Subtract 37 from each side to 'undo' the addition.	$y+37=-13$ $-37 \quad -37$
Simplify.	$y=-50$

Check:	$y+37=-13$
Substitute $y=-50$	$-50+37=-13$
	$-13 \stackrel{?}{=} -13$ ✓

Since y=−50 makes y+37=−13 a true statement, we have the solution to this equation.

Try It 9.1.5

Solve: $x+19=-27$.

Answer

$x=-46$

Try It 9.1.6

Solve: $x+16=-34$.

Answer

$x=-50$

What happens when an equation has a number subtracted from the variable, as in the equation $x-5=8$? We use another property of equations to solve equations where a number is subtracted from the variable. We want to isolate the variable, so to 'undo' the subtraction we will add the number to both sides. We use the **Addition Property of Equality**.

ADDITION PROPERTY OF EQUALITY

For any numbers a, b, and c,

$$\begin{aligned} &\text{If} \quad a=b \\ &\text{then} \quad a+c=b+c \end{aligned}$$

When you add the same quantity from both sides of an equation, you still have equality.

In Exercise 9.1.4, 37 was added to the y and so we subtracted 37 to 'undo' the addition. In Exercise 9.1.7, we will need to 'undo' subtraction by using the **Addition Property of Equality**.

Example 9.1.7

Solve: $a-28=-37$.

Solution

	$a-28=-37$
Add 28 to each side to 'undo' the subtraction.	$a-28+28=-37+28$
Simplify.	$a=-9$
Check:	$a-28=-37$
Substitute $a=-9$	$-9-28=-37$
	$-37\stackrel{?}{=}-37$ ✓
	The solution to $a-28=-37$ is $a=-9$.

Try It 9.1.8

Solve: $n-61=-75$.

Answer

$n=-14$

Try It 9.1.9

Solve: $p-41=-73$.

Answer

$p=-32$

Example 9.1.10

Solve: $x-\frac{5}{8}=\frac{3}{4}$

Solution

	$x-\frac{5}{8}=\frac{3}{4}$
Use the Addition Property of Equality.	$x-\frac{5}{8}+\frac{5}{8}=\frac{3}{4}+\frac{5}{8}$
Find the LCD to add the fractions on the right.	$x=\frac{6}{8}+\frac{5}{8}$
Simplify.	$x=\frac{11}{8}$

Check:	$x-\frac{5}{8}=\frac{3}{4}$	
Substitute $x=\frac{11}{8}$	$\frac{11}{8}-\frac{5}{8}\stackrel{?}{=}\frac{3}{4}$	
Subtract.	$\frac{6}{8}\stackrel{?}{=}\frac{3}{4}$	
Simplify.	$\frac{3}{4}=\frac{3}{4}\checkmark$	
		The solution to $x-\frac{5}{8}=\frac{3}{4}$ is $x=\frac{11}{8}$.

Try It 9.1.11

Solve: $p-\frac{2}{3}=\frac{5}{6}$.

Answer

$p=\frac{9}{6}p=\frac{3}{2}$

Try It 9.1.12

Solve: $q-\frac{1}{2}=\frac{5}{6}$.

Answer

$q=\frac{4}{3}$

The next example will be an equation with decimals.

Example 9.1.13

Solve: $n-0.63=-4.2$.

Solution

		$n-0.63=-4.2$
Use the Addition Property of Equality.		$n-0.63+0.63=-4.2+0.63$
Add.		$n=-3.57$
Check:	$n=-3.57$	
Let $n=-3.57$.	$-3.57-0.63\stackrel{?}{=}-4.2$	
	$-4.2=-4.2\checkmark$	

Try It 9.1.14

Solve: $b-0.47=-2.1$.

Answer

$b=-1.63$

Try It 9.1.15

Solve: $c-0.93=-4.6$.

Answer

$c=-3.67$

Solve Equations That Require Simplification

In the previous examples, we were able to isolate the variable with just one operation. Most of the equations we encounter in algebra will take more steps to solve. Usually, we will need to simplify one or both sides of an equation before using the Subtraction or Addition Properties of Equality.

You should always simplify as much as possible before you try to isolate the variable. Remember that to simplify an expression means to do all the operations in the expression. Simplify one side of the equation at a time. Note that **simplification** is different from the process used to solve an equation in which we apply an operation to both sides.

Example 9.1.16: How to Solve Equations That Require Simplification

Solve: $9x-5-8x-6=7$.

Solution

Step 1. Simplify the expressions on each side as much as possible.	Rearrange the terms, using the Commutative Property of Addition. Combine like terms. Notice that each side is now simplified as much as possible.	$9x-5-8x-6=7$ $9x-8x-5-6=7$ $x-11=7$
Step 2. Isolate the variable.	Now isolate x. Undo subtraction by adding 11 to both sides.	$x-11+11=7+11$
Step 3. Simplify the expressions on both sides of the equation.		$x=18$
Step 4. Check the solution.		**Check:** Substitute $x=18$. $9x-5-8x-6=7$ $9(18)-5-8(18)-6\stackrel{?}{=}7$ $162-5-144-6\stackrel{?}{=}7$ $157-144-6\stackrel{?}{=}7$ $13-6\stackrel{?}{=}7$ $7=7$ ✓ The solution to $9x-5-8x-6=7$ is $x=18$.

Try It 9.1.17

Solve: $8y-4-7y-7=4$.

Answer

$y=15$

Try It 9.1.18

Solve: $6z+5-5z-4=3$.

Answer

$z=2$

Example 9.1.19

Solve: $5(n-4)-4n=-8$.

Solution

We simplify both sides of the equation as much as possible before we try to isolate the variable.

	$5(n-4)-4n=-8$
Distribute on the left.	$5n-20-4n=-8$
Use the Commutative Property to rearrange terms.	$5n-4n-20=-8$
Combine like terms.	$n-20=-8$
Each side is as simplified as possible. Next, isolate n.	
Undo subtraction by using the Addition Property of Equality.	$n-20+20=-8+20$
Add.	$n=12$
Check. Substitute $n=12$. $5(n-4)-4n=-8$ $5(12-4)-4(12)\stackrel{?}{=}-8z$ $5(8)-48\stackrel{?}{=}-8$ $40-48\stackrel{?}{=}-8$ $-8=-8\checkmark$	
	The solution to $5(n-4)-4n=-8$ is $n=12$.

Try It 9.1.20

Solve: $5(p-3)-4p=-10$.

Answer

$p=5$

Try It 9.1.21

Solve: $4(q+2)-3q=-8$.

Answer

$q=-16$

Example 9.1.22

Solve: $3(2y-1)-5y=2(y+1)-2(y+3)$.

Solution

We simplify both sides of the equation before we isolate the variable.

	$3(2y-1)-5y=2(y+1)-2(y+3)$
Distribute on both sides.	$6y-3-5y=2y+2-2y-6$
Use the Commutative Property of Addition.	$6y-5y-3=2y-2y+2-6$
Combine like terms.	$y-3=-4$
Each side is as simplified as possible. Next, isolate y.	
Undo subtraction by using the Addition Property of Equality.	$y-3+3=-4+3$

Add.	$y = -1$
Check. Let y=−1. $3(2y-1)-5y=2(y+1)-2(y+3)$ $3(2(-1)-1)-5(-1) \stackrel{?}{=} 2(-1+1)-2(-1+3)$ $3(-2-1)+5 \stackrel{?}{=} 2(0)-2(2)$ $3(-3)+5 \stackrel{?}{=} -4$ $-9+5 \stackrel{?}{=} -4$ $-4=-4$ ✓	
	The solution to $3(2y-1)-5y=2(y+1)-2(y+3)3(2y-1)-5y=2(y+1)-2(y+$ is $y=-1$.

Try It 9.1.23

Solve: $4(2h-3)-7h=6(h-2)-6(h-1)$.

Answer

$h=6$

Try It 9.1.24

Solve: $2(5x+2)-9x=3(x-2)-3(x-4)$.

Answer

$x=2$

Translate to an Equation and Solve

To solve applications algebraically, we will begin by translating from English sentences into equations. Our first step is to look for the word (or words) that would translate to the **equals sign**. Here are some of the words that are commonly used.

Equals =

- is
- is equal to
- is the same as
- the result is
- gives
- was
- will be

The steps we use to translate a sentence into an equation are listed below.

TRANSLATE AN ENGLISH SENTENCE TO AN ALGEBRAIC EQUATION

1. Locate the “equals” word(s). Translate to an equals sign (=).
2. Translate the words to the left of the “equals” word(s) into an algebraic expression.
3. Translate the words to the right of the “equals” word(s) into an algebraic expression.

Example 9.1.25

Translate and solve: Eleven more than x is equal to 54.

Solution

Translate.	Eleven more than x is equal to 54 $x+11 = 54$
Subtract 11 from both sides.	$x+11-11 = 54-11$
Simplify.	$x = 43$
Check: Is 54 eleven more than 43? $43+11 = 54\checkmark$	

Try It 9.1.26

Translate and solve: Ten more than x is equal to 41.

Answer

$x+10=41; \quad x=31$

Try It 9.1.27

Translate and solve: Twelve less than x is equal to 51.

Answer

$y-12=51; \quad y=63$

Example 9.1.28

Translate and solve: The difference of $12t$ and $11t$ is -14.

Solution

Translate.	The difference of $12t$ and $11t$ is -14 $12t-11t = -14$
Simplify.	$t = -14$
Check: $12(-14)-11(-14) \stackrel{?}{=} -14$ $-168+154 \stackrel{?}{=} -14$ $-14 = -14\checkmark$	

Try It 9.1.29

Translate and solve: The difference of $4x$ and $3x$ is 14.

Answer

$4x-3x=14; \quad x=14$

Try It 9.1.30

Translate and solve: The difference of $7a$ and $6a$ is -8.

Answer

$7a-6a=-8; \quad a=-8$

Translate and Solve Applications

Most of the time a question that requires an algebraic solution comes out of a real life question. To begin with that question is asked in English (or the language of the person asking) and not in math symbols. Because of this, it is an important skill to be able to translate an everyday situation into algebraic language.

We will start by restating the problem in just one sentence, assign a variable, and then translate the sentence into an equation to solve. When assigning a variable, choose a letter that reminds you of what you are looking for. For example, you might use q for the number of quarters if you were solving a problem about coins.

Example 9.1.31: How to Solve Translate and Solve Applications

The MacIntyre family recycled newspapers for two months. The two months of newspapers weighed a total of 57 pounds. The second month, the newspapers weighed 28 pounds. How much did the newspapers weigh the first month?

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.	The problem is about the weight of newspapers.	
Step 2. Identify what we are asked to find.	What are we asked to find?	"How much did the newspapers weigh the 2nd month?"
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Choose a variable.	Let w = weight of the newspapers the 1st month
Step 4. Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.	Restate the problem. We know the weight of the newspapers the second month is 28 pounds. Translate into an equation, using the variable w.	Weight of newspapers the 1st month plus the weight of the newspapers the 2nd month equals 57 pounds. Weight from 1st month plus 28 equals 57. $w + 28 = 57$
Step 5. Solve the equation using good algebra techniques.	Solve.	$w + 28 - 28 = 57 - 28$ $w = 29$
Step 6. Check the answer in the problem and make sure it makes sense.	Does 1st month's weight plus 2nd month's weight equal 57 pounds?	**Check:** Does 1st month's weight plus 2nd month's weight equal 57 pounds? $29 + 28 \stackrel{?}{=} 57$ $57 = 57$ ✓
Step 7. Answer the question with a complete sentence.	Write a sentence to answer "How much did the newspapers weigh the 2nd month?"	The 2nd month the newspapers weighed 29 pounds.

Try It 9.1.32

Translate into an algebraic equation and solve:

The Pappas family has two cats, Zeus and Athena. Together, they weigh 23 pounds. Zeus weighs 16 pounds. How much does Athena weigh?

Answer

7 pounds

Try It 9.1.33

Translate into an algebraic equation and solve:

Sam and Henry are roommates. Together, they have 68 books. Sam has 26 books. How many books does Henry have?

Answer

42 books

SOLVE AN APPLICATION.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with the important information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Example 9.1.34

Randell paid $28,675 for his new car. This was $875 less than the sticker price. What was the sticker price of the car?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	"What was the sticker price of the car?"
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let s = the sticker price of the car.
Step 4. Translate into an equation. Restate the problem in one sentence.	$28675 is $875 less than the sticker price
	$28675 is $875 less than s
	$28675 = s - 875$
Step 5. Solve the equation.	$28675 + 875 = s - 875 + 875$
	$29550 = s$
Step 6. Check the answer. Is $875 less than $29550 equal to $28675? $29550 - 875 \stackrel{?}{=} 28675$ $28675 = 28675$✓	
Step 7. Answer the question with a complete sentence.	The sticker price of the car was $29550.

Try It 9.1.35

Translate into an algebraic equation and solve:

Eddie paid $19875 for his new car. This was $1025 less than the sticker price. What was the sticker price of the car?

Answer

$20900

Try It 9.1.36

Translate into an algebraic equation and solve:

The admission price for the movies during the day is $7.75. This is $3.25 less the price at night. How much does the movie cost at night?

Answer

$11.00

Key Concepts

- **To Determine Whether a Number is a Solution to an Equation**
  1. **Substitute the number in for the variable in the equation.**
  2. **Simplify the expressions on both sides of the equation.**
  3. **Determine whether the resulting statement is true.**
     - If it is true, the number is a solution.
     - If it is not true, the number is not a solution.
- **Addition Property of Equality**
  - For any numbers a, b and c, if $a = b$, then $a + c = b + c$.
- **Subtraction Property of Equality**
  - For any numbers a, b and c, if $a = b$, then $a - c = b - c$.
- **To Translate a Sentence to an Equation**
  1. Locate the “equals” word(s). Translate to an equal sign (=).
  2. Translate the words to the left of the “equals” word(s) into an algebraic expression.
  3. Translate the words to the right of the “equals” word(s) into an algebraic expression.
- **To Solve an Application**
  1. Read the problem. Make sure all the words and ideas are understood.
  2. Identify what we are looking for.
  3. Name what we are looking for. Choose a variable to represent that quantity.
  4. Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.

Glossary

solution of an equation

A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

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9.1E: Exercises

Practice Makes Perfect

Verify a Solution of an Equation

In the following exercises, determine whether the given value is a solution to the equation.

Exercise 9.1E.1

Is $y = \frac{5}{3}$ a solution of
$6y + 10 = 12y$?

Answer

Yes

Exercise 9.1E.2

Is $x = \frac{9}{4}$ a solution of
$4x + 9 = 8x$?

Exercise 9.1E.3

Is $u = -\frac{1}{2}$ a solution of
$8u - 1 = 6u$?

Answer

No

Exercise 9.1E.4

Is $v = -\frac{1}{3}$ a solution of
$9v - 2 = 3v$?

Solve Equations using the Subtraction and Addition Properties of Equality

In the following exercises, solve each equation using the Subtraction and Addition Properties of Equality.

Exercise 9.1E.5

$x + 24 = 35$

Answer

x = 11

Exercise 9.1E.6

$x + 17 = 22$

Exercise 9.1E.7

$y + 45 = -66$

Answer

y = -111

Exercise 9.1E.8

$y + 39 = -83$

Exercise 9.1E.9

$b + \frac{1}{4} = \frac{3}{4}$

Answer

$b = \frac{1}{2}$

Exercise 9.1E.10

$a + \frac{2}{5} = \frac{4}{5}$

Exercise 9.1E.11

$p + 2.4 = -9.3$

Answer

p = -11.7

Exercise 9.1E.12

$m + 7.9 = 11.6$

Exercise 9.1E.13

$a - 45 = 76$

Answer

a = 121

Exercise 9.1E.14

$a - 30 = 57$

Exercise 9.1E.15

$m - 18 = -200$

Answer

m = -182

Exercise 9.1E.16

$m - 12 = -12$

Exercise 9.1E.17

$x - \frac{1}{3} = 2$

Answer

$x = \frac{7}{3}$

Exercise 9.1E.18

$x - \frac{1}{5} = 4$

Exercise 9.1E.19

$y - 3.8 = 10$

Answer

y = 10.8

Exercise 9.1E.20

$y - 7.2 = 5$

Exercise 9.1E.21

$x - 165 = -420$

Answer

$x = -255$

Exercise 9.1E.22

$z - 101 = -314$

Exercise 9.1E.23

$z + 0.52 = -8.5$

Answer

$z = -9.02$

Exercise 9.1E.24

$x + 0.93 = -4.1$

Exercise 9.1E.25

$q + \frac{3}{4} = \frac{1}{2}$

Answer

$q = -\frac{1}{4}$

Exercise 9.1E.26

$p+\frac{1}{3}=\frac{5}{6}$

Exercise 9.1E.27

$p-\frac{2}{5}=\frac{2}{3}$

Answer

$p=\frac{16}{15}$

Exercise 9.1E.28

$y-\frac{3}{4}=\frac{3}{5}$

Solve Equations that Require Simplification

In the following exercises, solve each equation.

Exercise 9.1E.29

$c+31-10=46$

Answer

c = 25

Exercise 9.1E.30

$m+16-28=5$

Exercise 9.1E.31

$9x+5-8x+14=20$

Answer

x = 1

Exercise 9.1E.32

$6x+8-5x+16=32$

Exercise 9.1E.33

$-6x-11+7x-5=-16$

Answer

x = 0

Exercise 9.1E.34

$-8n-17+9n-4=-41$

Exercise 9.1E.35

$5(y-6)-4y=-6$

Answer

$y=24$

Exercise 9.1E.36

$9(y-2)-8y=-16$

Exercise 9.1E.37

$8(u+1.5)-7u=4.9$

Answer

$u=-7.1$

Exercise 9.1E.38

$5(w+2.2)-4w=9.3$

Exercise 9.1E.39

$6a-5(a-2)+9=-11$

Answer

$a=-30$

Exercise 9.1E.40

$8c-7(c-3)+4=-16$

Exercise 9.1E.41

$6(y-2)-5y=4(y+3)-4(y-1)$

Answer

y =28

Exercise 9.1E.42

$9(x-1)-8x=-3(x+5)+3(x-5)$

Exercise 9.1E.43

$3(5n-1)-14n+9=10(n-4)-6n-4(n+1)$

Answer

n = -50

Exercise 9.1E.44

$2(8m+3)-15m-4=9(m+6)-2(m-1)-7m$

Exercise 9.1E.45

$-(j+2)+2j-1=5$

Answer

j = 8

Exercise 9.1E.46

$-(k+7)+2k+8=7$

Exercise 9.1E.47

$-\left(\frac{1}{4}a-\frac{3}{4}\right)+\frac{5}{4}a=-2$

Answer

$a=-\frac{11}{4}$

Exercise 9.1E.48

$-\left(\frac{2}{3}d-\frac{1}{3}\right)+\frac{5}{3}d=-4$

Exercise 9.1E.49

$8(4x+5)-5(6x)-x$
$=53-6(x+1)+3(2x+2)$

Answer

x=13

Exercise 9.1E.50

$6(9y-1)-10(5y)-3y$
$=22-4(2y-12)+8(y-6)$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve it.

Exercise 9.1E.51

Nine more than x is equal to 52.

Answer

$x+9=52; x=43$

Exercise 9.1E.52

The sum of x and -15 is 23.

Exercise 9.1E.53

Ten less than m is -14.

Answer

$m-10=-14; m=-4$

Exercise 9.1E.54

Three less than y is -19.

Exercise 9.1E.55

The sum of y and -30 is 40.

Answer

$y+(-30)=40; y=70$

Exercise 9.1E.56

Twelve more than p is equal to 67.

Exercise 9.1E.57

The difference of $9x$ and $8x$ is 107.

Answer

$9x-8x=107; 107$

Exercise 9.1E.58

The difference of $5c$ and $4c$ is 602.

Exercise 9.1E.59

The difference of $5c$ and $4c$ is 602

Answer

$n-\frac{1}{6}=\frac{1}{2}; \frac{2}{3}$

Exercise 9.1E.60

The difference of f and $\frac{1}{3}$ is $\frac{1}{12}$.

Exercise 9.1E.61

The sum of $-4n$ and $5n$ is -82

Answer

$-4n+5n=-82;-82$

Exercise 9.1E.62

The sum of $-9m$ and $10m$ is -95

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

Exercise 9.1E.63

Distance Avril rode her bike a total of 18 miles, from home to the library and then to the beach. The distance from Avril's house to the library is 7 miles. What is the distance from the library to the beach?

Answer

11 miles

Exercise 9.1E.64

Reading Jeff read a total of 54 pages in his History and Sociology textbooks. He read 41 pages in his History textbook. How many pages did he read in his Sociology textbook?

Exercise 9.1E.65

Age Eva's daughter is 15 years younger than her son. Eva's son is 22 years old. How old is her daughter?

Answer

7 years old

Exercise 9.1E.66

Age Pablo's father is 3 years older than his mother. Pablo's mother is 42 years old. How old is his father?

Exercise 9.1E.67

Groceries For a family birthday dinner, Celeste bought a turkey that weighed 5 pounds less than the one she bought for Thanksgiving. The birthday turkey weighed 16 pounds. How much did the Thanksgiving turkey weigh?

Answer

21 pounds

Exercise 9.1E.68

Weight Allie weighs 8 pounds less than her twin sister Lorrie. Allie weighs 124 pounds. How much does Lorrie weigh?

Exercise 9.1E.69

Health Connor's temperature was 0.7 degrees higher this morning than it had been last night. His temperature this morning was 101.2 degrees. What was his temperature last night?

Answer

100.5 degrees

Exercise 9.1E.70

Health The nurse reported that Tricia's daughter had gained 4.2 pounds since her last checkup and now weighs 31.6 pounds. How much did Tricia's daughter weigh at her last checkup?

Exercise 9.1E.71

Salary Ron's paycheck this week was $17.43 less than his paycheck last week. His paycheck this week was $103.76. How much was Ron's paycheck last week?

Answer

$121.19

Exercise 9.1E.72

Textbooks Melissa's math book cost $22.85 less than her art book cost. Her math book cost $93.75. How much did her art book cost?

Everyday Math

Exercise 9.1E.73

Construction Miguel wants to drill a hole for a $\frac{5}{8}$ inch screw. The hole should be $\frac{1}{12}$ inch smaller than the screw. Let d equal the size of the hole he should drill. Solve the equation $d = \frac{5}{8} - \frac{1}{12}$ to see what size the hole should be.

Answer

$d = \frac{13}{24}$ inch

Exercise 9.1E.74

Baking Kelsey needs $\frac{2}{3}$ cup of sugar for the cookie recipe she wants to make. She only has $\frac{3}{8}$ cup of sugar and will borrow the rest from her ner neighbor. Let s equal the amount of sugar she will borrow. Solve the equation $\frac{3}{8} + s = \frac{2}{3}$ to find the amount of sugar she should ask to borrow.

Writing Exercises

Exercise 9.1E.75

Is -8 a solution to the equation $3x = 16 - 5x$? How do you know?

Answer

No. Justifications will vary.

? Exercise $9.1E.76$

What is the first step in your solution to the equation $10x + 2 = 4x + 26$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
verify a solution of an equation.			
solve equations using the subtraction and addition properties of equality.			
solve equations that require simplification.			
translate to an equation and solve.			
translate and solve applications.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

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9.2: Solve Equations using the Division and Multiplication Properties of Equality

Learning Objectives

By the end of this section, you will be able to:

- Solve equations using the Division and Multiplication Properties of Equality
- Solve equations that require simplification
- Translate to an equation and solve
- Translate and solve applications

Note

Before you get started, take this readiness quiz.

1. Simplify: $-7\left(\frac{1}{-7}\right)$.
   If you missed this problem, review Exercise 1.6.13.
2. Evaluate $9x+2$ when $x=-3$.
   If you missed this problem, review Exercise 1.5.34.

Solve Equations Using the Division and Multiplication Properties of Equality

You may have noticed that all of the equations we have solved so far have been of the form $x+a=b$ or $x-a=b$. We were able to isolate the variable by adding or subtracting the constant term on the side of the equation with the variable. Now we will see how to solve equations that have a variable multiplied by a constant and so will require division to isolate the variable.

Let's look at our puzzle again with the envelopes and counters in Figure 9.2.1.

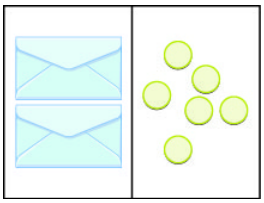

Figure 9.2.1: The illustration shows a model of an equation with one variable multiplied by a constant. On the left side of the workspace are two instances of the unknown (envelope), while on the right side of the workspace are six counters.

In the illustration there are two identical envelopes that contain the same number of counters. Remember, the left side of the workspace must equal the right side, but the counters on the left side are "hidden" in the envelopes. So how many counters are in each envelope?

How do we determine the number? We have to separate the counters on the right side into two groups of the same size to correspond with the two envelopes on the left side. The 6 counters divided into 2 equal groups gives 3 counters in each group (since $6\div 2=3$).

What equation models the situation shown in Figure 9.2.2? There are two envelopes, and each contains xx counters. Together, the two envelopes must contain a total of 6 counters.

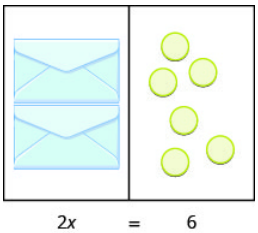

Figure 9.2.2: The illustration shows a model of the equation $2x=6$.

$2x=6$

If we divide both sides of the equation by 2, as we did with the envelopes and counters,	$\frac{2x}{2}=\frac{6}{2}$
we get:	$x=3$

We found that each envelope contains 3 counters. Does this check? We know $2 \cdot 3 = 6$, so it works! Three counters in each of two envelopes does equal six!

This example leads to the **Division Property of Equality**.

THE DIVISION PROPERTY OF EQUALITY

For any numbers a, b, and c, and $c \neq 0$,

$$\begin{array}{rcl} \text{If} & a & = & b \\ \text{then} & \frac{a}{c} & = & \frac{b}{c} \end{array} \qquad (9.2.1)$$

When you divide both sides of an equation by any non-zero number, you still have equality.

Note

Doing the Manipulative Mathematics activity "**Division Property of Equality**" will help you develop a better understanding of how to solve equations by using the Division Property of Equality.

The goal in solving an equation is to 'undo' the operation on the variable. In the next example, the variable is multiplied by 5, so we will divide both sides by 5 to 'undo' the multiplication.

Example 9.2.1

Solve: $5x = -27$.

Solution

To isolate x, "undo" the multiplication by 5.		$5x=-27$
Divide to 'undo' the multiplication.		$\frac{5x}{5}=\frac{-27}{5}$
Simplify.		$x=-\frac{27}{5}$
Check:	$5x=-27$	
Substitute $-\frac{27}{5}$ for x.	$5\left(-\frac{27}{5}\right)\stackrel{?}{=}-27$	
	$-27=-27$ ✓	
		Since this is a true statement, $x=-\frac{27}{5}$ is the solution to $5x=-27$.

Try It 9.2.2

Solve: $3y = -41$.

Answer

$y=-\frac{41}{3}$

Try It 9.2.3

Solve: $4z = -55$.

Answer

$y = -\frac{55}{4}$

Consider the equation $\frac{x}{4} = 3$. We want to know what number divided by 4 gives 3. So to "undo" the division, we will need to multiply by 4. The **Multiplication Property of Equality** will allow us to do this. This property says that if we start with two equal quantities and multiply both by the same number, the results are equal.

THE MULTIPLICATION PROPERTY OF EQUALITY

For any numbers *a*, *b*, and *c*,

$$\begin{array}{lllll} \text{If} & a & = & b \\ \text{then} & ac & = & bc \end{array} \qquad (9.2.2)$$

If you multiply both sides of an equation by the same number, you still have equality.

Example 9.2.4

Solve: $\frac{y}{-7} = -14$

Solution

Here y is divided by −7. We must multiply by −7 to isolate y.

	$\frac{y}{-7} = -14$
Multiply both sides by −7.	$-7\left(\frac{y}{-7}\right) = -7(-14)$
Multiply.	$\frac{-7y}{7} = 98$
Simplify.	$y = 98$
Check: $\frac{y}{-7} = -14$	
Substitute y=98.	$\frac{98}{-7} \stackrel{?}{=} -14$
Divide.	$-14 = -14$ ✓

Try It 9.2.5

Solve: $\frac{a}{-7} = -42$

Answer

$a = 294$

Try It 9.2.6

Solve: $\frac{b}{-6} = -24$

Answer

$b = 144$

Example 9.2.7

Solve: $-n = 9$

Solution

	$-n = 9$
Remember −n is equivalent to −1n.	$-1n = 9$
Divide both sides by −1.	$\frac{-1n}{-1} = \frac{9}{-1}$
Divide.	$n = -9$

Notice that there are two other ways to solve −n=9. We can also solve this equation by multiplying both sides by −1 and also by taking the opposite of both sides.

Check:	$-n = 9$	
Substitute n=−9.	$-(-9) \stackrel{?}{=} 9$	
Simplify.	$9 = 9$ ✓	

Try It 9.2.8

Solve: $-k = 8$.

Answer

$k = -8$

Try It 9.2.9

Solve: $-g = 3$.

Answer

$g = -3$

Example 9.2.10

Solve: $\frac{3}{4}x = 12$

Solution

Since the product of a number and its reciprocal is 1, our strategy will be to isolate x by multiplying by the reciprocal of $\frac{3}{4}$.

	$\frac{3}{4}x = 12$
Multiply by the reciprocal of $\frac{3}{4}$.	$\frac{4}{3} \cdot \frac{3}{4}x = \frac{4}{3} \cdot 12$
Reciprocals multiply to 1.	$1x = \frac{4}{3} \cdot \frac{12}{1}$
Multiply.	$x = 16$

Notice that we could have divided both sides of the equation $\frac{3}{4}x = 12$ by $\frac{3}{4}$ to isolate x. While this would work, most people would find multiplying by the reciprocal easier.

Check:	$\frac{3}{4}x = 12$	

Substitute $x = 16$.	$\frac{3}{4} \cdot 16 \stackrel{?}{=} 12$	
	$12 = 12$ ✓	

Try It 9.2.11

Solve: $\frac{2}{5}n = 14$.

Answer

$n = 35$

Try It 9.2.12

Solve: $\frac{5}{6}y = 15$.

Answer

$y = 18$

In the next example, all the variable terms are on the right side of the equation. As always, our goal in solving the equation is to isolate the variable.

Example 9.2.13

Solve: $\frac{8}{15} = -\frac{4}{5}x$

Solution

	$\frac{8}{15} = -\frac{4}{5}x$
Multiply by the reciprocal of $-\frac{4}{5}$.	$\left(-\frac{5}{4}\right)\left(\frac{8}{15}\right) = \left(-\frac{5}{4}\right)\left(-\frac{4}{5}x\right)$
Reciprocals multiply to 1.	$-\frac{\cancel{5} \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 3 \cdot \cancel{5}} = 1x$
Multiply.	$-\frac{2}{3} = x$

Check:	$\frac{8}{15} = -\frac{4}{5}x$
Let $x = -\frac{2}{3}$.	$\frac{8}{15} = -\frac{4}{5}\left(-\frac{2}{3}\right)$
	$\frac{8}{15} = \frac{8}{15}$ ✓

Try It 9.2.14

Solve: $\frac{9}{25} = -\frac{4}{5}z$

Answer

$z = -\frac{9}{5}$

Try It 9.2.15

$\frac{5}{6} = -\frac{8}{3}r$

Answer

$r = -\frac{5}{16}$

Solve Equations That Require Simplification

Many equations start out more complicated than the ones we have been working with.

With these more complicated equations the first step is to simplify both sides of the equation as much as possible. This usually involves combining like terms or using the distributive property.

Example 9.2.16

Solve: $14 - 23 = 12y - 4y - 5y$.

Solution

Begin by simplifying each side of the equation.

	$14 - 23 \stackrel{?}{=} -36 + 12 + 15$
Simplify each side.	$-9 = -9$ ✓
Divide both sides by 3 to isolate y.	$14 - 23 = 12y - 4y - 5y$
Divide.	$-9 = 3y$
Check:	$14 - 23 = 12y - 4y - 5y$
Substitute $y = -3$.	$14 - 23 \stackrel{?}{=} 12(-3) - 4(-3) - 5(-3)$
	$14 - 23 \stackrel{?}{=} -36 + 12 + 15$
	$-9 = -9$ ✓

Try It 9.2.17

Solve: $18 - 27 = 15c - 9c - 3c$.

Answer

$c = -3$

Try It 9.2.18

Solve: $18 - 22 = 12x - x - 4x$.

Answer

$x = -\frac{4}{7}$

Example 9.2.19

Solve: $-4(a - 3) - 7 = 25$.

Solution

Here we will simplify each side of the equation by using the distributive property first.

	$-4(a-3)-7=25$
Distribute.	$-4a+12-7=25$
Simplify.	$-4a+5=25$
Simplify.	$-4a=20$
Divide both sides by -4 to isolate a.	$\frac{-4a}{-4}=\frac{20}{-4}$
Divide.	$a=-5$

Check:	$-4(a-3)-7=25$
Substitute $a=-5$	$-4(-5-3)-7\stackrel{?}{=}25$
	$-4(-8)-7\stackrel{?}{=}25$
	$32-7\stackrel{?}{=}25$
	$25=25$ ✓

Try It 9.2.20

Solve: $-4(q-2)-8=24$.

Answer

$q=-6$

Try It 9.2.21

Solve: $-6(r-2)-12=30$.

Answer

$r=-5$

Now we have covered all four properties of equality—subtraction, addition, division, and multiplication. We'll list them all together here for easy reference.

PROPERTIES OF EQUALITY

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

$$\begin{array}{ll} \textbf{Subtraction Property of Equality} & \textbf{Addition Property of Equality} \\ \text{For any real numbers a, b, and c,} & \text{For any real numbers a, b, and c,} \\ \text{if } a=b, & \text{if } a=b, \\ \text{then } a-c=b-c & \text{then } a+c=b+c \\ \textbf{Divition Property of Equality} & \textbf{Multiplication Property of Equality} \\ \text{For any real numbers a, b, and c,} & \text{For any real numbers a, b, and c,} \\ \text{if } a=b, & \text{if } a=b, \\ \text{then } a-c=b-c & \text{then } a+c=b+c \end{array} \tag{9.2.3}$$

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

Translate to an Equation and Solve

In the next few examples, we will translate sentences into equations and then solve the equations. You might want to review the translation table in the previous chapter.

Example 9.2.22

Translate and solve: The number 143 is the product of −11 and *y*.

Solution

Translate.	The number 143 is the product of −11 and y. $143 = -11y$
Divide by −11.	$\frac{143}{-11} = \frac{-11y}{-11}$
Simplify.	$-13 = y$
Check: $143 = -11y$ $143 \stackrel{?}{=} -11(-13)$ (9.2.4) $143 = 143\checkmark$	

Try It 9.2.23

Translate and solve: The number 132 is the product of *−12* and *y*.

Answer

132=−12y;y=−11

Try It 9.2.24

Translate and solve: The number 117 is the product of *−13* and *z*.

Answer

117=−13z;z=−9

Example 9.2.25

Translate and solve: *n* divided by 8 is −32.

Solution

Begin by translating the sentence into an equation. Translate.		*n* divided by 8 is −32. $\frac{n}{8} = -32$
Multiple both sides by 8.		$8 \cdot \frac{n}{8} = 8(-32)$
Simplify.		$n = -256$
Check:	Is nn divided by 8 equal to −32?	
Let $n = -256$.	Is −256 divided by 88 equal to −32?	
Translate.	$\frac{-256}{8} \stackrel{?}{=} -32$	

Simplify.	$-32 = -32$✓

Try It 9.2.26

Translate and solve: nn divided by 7 is equal to −21.

Answer

$\frac{n}{7} = -21; n = -147$

Try It 9.2.27

Translate and solve: n divided by 8 is equal to −56.

Answer

$\frac{n}{8} = -56; n = -448$

Example 9.2.28

Translate and solve: The quotient of yy and −4 is 68.

Solution

Begin by translating the sentence into an equation.

Translate.	The quotient of y and -4 is 68. $\frac{y}{-4} = 68$
Multiply both sides by -4.	$-4\left(\frac{y}{-4}\right) = -4(68)$
Simplify.	$y = -272$
Check:	Is the quotient of y and −4 equal to 68?
Let y=−272.	Is the quotient of −272 and −4 equal to 68?
Translate.	$\frac{-272}{-4} \stackrel{?}{=} 68$
Simplify.	$68 = 68$✓

Try It 9.2.29

Translate and solve: The quotient of q and −8 is 72.

Answer

$\frac{q}{-8} = 72; q = -576$

Try It 9.2.30

Translate and solve: The quotient of p and −9 is 81.

Answer

$\frac{p}{-9} = 81; p = -729$

Example 9.2.31

Translate and solve: Three-fourths of p is 18.

Solution

Begin by translating the sentence into an equation. Remember, “of” translates into multiplication.

Translate.	Three-fourths of p is 18. $\frac{3}{4}p = 18$
Multiply both sides by $\frac{4}{3}$.	$\frac{4}{3} \cdot \frac{3}{4}p = \frac{4}{3} \cdot 18$
Simplify.	$p = 24$
Check:	Is three-fourths of p equal to 18?
Let p = 24.	Is three-fourths of 24 equal to 18?
Translate.	$\frac{3}{4} \cdot 24 \stackrel{?}{=} 18$
Simplify.	$18 = 18$✓

Try It 9.2.32

Translate and solve: Two-fifths of f is 16.

Answer

$\frac{2}{5}f = 16; f = 40$

Try It 9.2.33

Translate and solve: Three-fourths of f is 21.

Answer

$\frac{3}{4}f = 21; f = 28$

Example 9.2.34

Translate and solve: The sum of three-eighths and x is one-half.

Solution

Begin by translating the sentence into an equation.

Translate.	The sum of three – eighths and x is $\frac{1}{2}$ $\frac{3}{8} + x = \frac{1}{2}$
Subtract $\frac{3}{8}$ from each side.	$\frac{3}{8} - \frac{3}{8} + x = \frac{1}{2} - \frac{3}{8}$
Simplify and rewrite fractions with common denominators.	$x = \frac{4}{8} - \frac{3}{8}$
Simplify.	$x = \frac{1}{8}$

Check:	Is the sum of three-eighths and x equal to one-half?
Let $x = \frac{1}{8}$.	Is the sum of three-eighths and one-eighth equal to one-half?
Translate.	$\frac{3}{8} + \frac{1}{8} \stackrel{?}{=} \frac{1}{2}$
Simplify.	$\frac{4}{8} \stackrel{?}{=} \frac{1}{2}$
Simplify.	$\frac{1}{2} = \frac{1}{2}$ ✓

Try It 9.2.35

Translate and solve: The sum of five-eighths and *x* is one-fourth.

Answer

$\frac{5}{8} + x = \frac{1}{4}; x = -\frac{3}{8}$

Try It 9.2.36

Translate and solve: The sum of three-fourths and *x* is five-sixths.

Answer

$\frac{3}{4} + x = \frac{5}{6}; x = \frac{1}{12}$

Translate and Solve Applications

To solve applications using the Division and Multiplication Properties of Equality, we will follow the same steps we used in the last section. We will restate the problem in just one sentence, assign a variable, and then translate the sentence into an equation to solve.

Example 9.2.37

Denae bought 6 pounds of grapes for $10.74. What was the cost of one pound of grapes?

Solution

What are you asked to find?	The cost of 1 pound of grapes	
Assign a variable.	Let c = the cost of one pound.	
Write a sentence that gives the information to find it.	The cost of 6 pounds is $10.74	
Translate into an equation.	$6c = 10.74$	
Solve.	$\frac{6c}{c} = \frac{10.74}{6}$ $c = 1.79$	(9.2.5)
	The grapes cost $1.79 per pound.	
Check: If one pound costs $1.79, *do* 6 pounds cost $10.74?		
$6(1.79) \stackrel{?}{=} 10.74$ $10.74 = 10.74$✓		

? Try It 9.2.38

Translate and solve:

Arianna bought a 24-pack of water bottles for $9.36. What was the cost of one water bottle?

Answer

$0.39

? Try It 9.2.39

Translate and solve:

At JB's Bowling Alley, 6 people can play on one lane for $34.98. What is the cost for each person?

Answer

$5.83

✓ Example 9.2.40

Andreas bought a used car for $12,000. Because the car was 4-years old, its price was $\frac{3}{4}$ of the original price, when the car was new. What was the original price of the car?

Solution

What are you asked to find?	The original price of the car
Assign a variable.	Let p = the original price.
Write a sentence that gives the information to find it.	\$12000 is $\frac{3}{4}$ of the original price.
Translate into an equation.	$12000 = \frac{3}{4}p$
Solve.	$\frac{3}{4}(12000) = \frac{4}{3} \cdot \frac{3}{4}p$ $16000 = p$
	The original cost of the car was \$16000.

(9.2.6)

Check: Is $\frac{3}{4}$ of \$16000 equal to \$12000

$\frac{3}{4} \cdot 16000 \stackrel{?}{=} 12000$
$12000 = 12000$✓

? Try It 9.2.41

Translate and solve:

The annual property tax on the Mehta's house is \$1,800, calculated as $\frac{15}{1000}$ of the assessed value of the house. What is the assessed value of the Mehta's house?

Answer

\$120000

? Try It 9.2.42

Translate and solve:

Stella planted 14 flats of flowers in $\frac{2}{3}$ of her garden. How many flats of flowers would she need to fill the whole garden?

Answer

21 flats

Key Concepts

- **The Division Property of Equality**—For any numbers a, b, and c, and $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.
  When you divide both sides of an equation by any non-zero number, you still have equality.
- **The Multiplication Property of Equality**—For any numbers a, b, and c, if $a = b$, then $ac = bc$.
  If you multiply both sides of an equation by the same number, you still have equality.

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9.2E: Exercises

Practice Makes Perfect

Solve Equations Using the Division and Multiplication Properties of Equality

In the following exercises, solve each equation using the Division and Multiplication Properties of Equality and check the solution.

Exercise 9.2E.1

$8x = 56$

Answer

$x = 7$

Exercise 9.2E.2

$7p = 63$

Exercise 9.2E.3

$-5c = 55$

Answer

$c = -11$

Exercise 9.2E.4

$-9x = -27$

Exercise 9.2E.5

$-809 = 15y$

Answer

$y = -\frac{809}{15}$

Exercise 9.2E.6

$-731 = 19y$

Exercise 9.2E.7

$-37p = -541$

Answer

$p = -\frac{541}{37}$

Exercise 9.2E.8

$-19m = -586$

Exercise 9.2E.9

$0.25z = 3.25$

Answer

z= 13

Exercise 9.2E.10

$0.75a = 11.25$

Exercise 9.2E.11

$-13x = 0$

Answer

$x = 0$

Exercise 9.2E.12

$24x = 0$

Exercise 9.2E.13

$\frac{x}{4} = 35$

Answer

$x = 140$

Exercise 9.2E.14

$\frac{z}{2} = 54$

Exercise 9.2E.15

$-20 = \frac{q}{-5}$

Answer

$q = 100$

Exercise 9.2E.16

$\frac{c}{-3} = -12$

Exercise 9.2E.17

$\frac{y}{9} = -16$

Answer

$y = -144$

Exercise 9.2E.18

$\frac{q}{6} = -38$

Exercise 9.2E.19

$\frac{m}{-12} = 45$

Answer

$m = -540$

Exercise 9.2E.20

$-24 = \frac{p}{-20}$

Exercise 9.2E.21

$-y = 6$

Answer

$y = -6$

Exercise 9.2E.22

$-u = 15$

Exercise 9.2E.23

$-v = -72$

Answer

$v = 72$

Exercise 9.2E.24

$-x = -39$

Exercise 9.2E.25

$\frac{2}{3}y = 48$

Answer

$y = 72$

Exercise 9.2E.26

$\frac{3}{5}r = 75$

Exercise 9.2E.27

$-\frac{5}{8}w = 40$

Answer

$w = -64$

Exercise 9.2E.28

$24 = -\frac{3}{4}x$

Exercise 9.2E.29

$-\frac{2}{5} = \frac{1}{10}a$

Answer

$a = -4$

Exercise 9.2E.30

$-\frac{1}{3}q = -\frac{5}{6}$

Exercise 9.2E.31

$-\frac{7}{10}x = -\frac{14}{3}$

Answer

$x = \frac{20}{3}$

Exercise 9.2E.32

$\frac{3}{8}y = -\frac{1}{4}$

Exercise 9.2E.33

$\frac{7}{12} = -\frac{3}{4}p$

Answer

$p = -\frac{7}{9}$

Exercise 9.2E.34

$\frac{11}{18} = -\frac{5}{6}q$

Exercise 9.2E.35

$-\frac{5}{18} = -\frac{10}{9}u$

Answer

$u = \frac{1}{4}$

Exercise 9.2E.36

$-\frac{7}{20}=-\frac{7}{4}v$

Solve Equations That Require Simplification

In the following exercises, solve each equation requiring simplification.

Exercise 9.2E.37

$100-16=4p-10p-p$

Answer

$p=-12$

Exercise 9.2E.38

$-18-7=5t-9t-6t$

Exercise 9.2E.39

$\frac{7}{8}n-\frac{3}{4}n=9+2$

Answer

$n=88$

Exercise 9.2E.40

$\frac{5}{12}q+\frac{1}{2}q=25-3$

Exercise 9.2E.41

$0.25d+0.10d=6-0.75$

Answer

d=15

Exercise 9.2E.42

$0.05p-0.01p=2+0.24$

Exercise 9.2E.43

$-10(q-4)-57=93$

Answer

$q=-11$

Exercise 9.2E.44

$-12(d-5)-29=43$

Exercise 9.2E.45

$-10(x+4)-19=85$

Answer

$x=-\frac{72}{5}$

Exercise 9.2E.46

$-15(z+9)-11=75$

Mixed Practice

In the following exercises, solve each equation.

Exercise 9.2E.47

$\frac{9}{10}x=90$

Answer

$x=100$

Exercise 9.2E.48

$\frac{5}{12}y=60$

Exercise 9.2E.49

$y+46=55$

Answer

$y=9$

Exercise 9.2E.50

$x+33=41$

Exercise 9.2E.51

$\frac{w}{-2}=99$

Answer

$w=-198$

Exercise 9.2E.52

$\frac{s}{-3}=-60$

Exercise 9.2E.53

$27 = 6a$

Answer

$a = \frac{9}{2}$

Exercise 9.2E.54

$-a = 7$

Exercise 9.2E.55

$-x = 2$

Answer

$x = -2$

Exercise 9.2E.56

$z - 16 = -59$

Exercise 9.2E.57

$m - 41 = -14$

Answer

$m = 27$

Exercise 9.2E.58

$0.04r = 52.60$

Exercise 9.2E.59

$63.90 = 0.03p$

Answer

$p = 2130$

Exercise 9.2E.60

$-15x = -120$

Exercise 9.2E.61

$84 = -12z$

Answer

$y = -7$

Exercise 9.2E.62

$19.36 = x - 0.2x$

Exercise 9.2E.63

$c - 0.3c = 35.70$

Answer

$c = 51$

Exercise 9.2E.64

$-y = -9$

Exercise 9.2E.65

$-x = -8$

Answer

$x = 8$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve.

Exercise 9.2E.66

187 is the product of -17 and m

Exercise 9.2E.67

133 is the product of -19 and n

Answer

$133 = -19n; n = -7$

Exercise 9.2E.68

-184 is the product of 23 and p

Exercise 9.2E.69

-152 is the product of 8 and q

Answer

$-152 = 8q; q = -19$

Exercise 9.2E.70

u divided by 7 is equal to -49

Exercise 9.2E.71

r divided by 12 is equal to -48

Answer

$\frac{r}{12} = -48; r = -576$

Exercise 9.2E.72

h divided by -13 is equal to -65

Exercise 9.2E.73

j divided by -20 is equal to -80

Answer

$\frac{j}{-20} = -80; j = 1,600$

Exercise 9.2E.74

The quotient c and -19 is 38.

Exercise 9.2E.75

The quotient of b and -6 is 18

Answer

$\frac{b}{-6} = 18; b = -108$

Exercise 9.2E.76

The quotient of h and 26 is -52

Exercise 9.2E.77

The quotient k and 22 is -66

Answer

$\frac{k}{22} = -66; k = -1,452$

Exercise 9.2E.78

Five-sixths of y is 15

Exercise 9.2E.79

Three-tenths of x is 15

Answer

$\frac{3}{10}x = 15; x = 50$

Exercise 9.2E.80

Four-thirds of w is 36

Exercise 9.2E.81

Five-halves of v is 50

Answer

$\frac{5}{2}v = 50; v = 20$

Exercise 9.2E.82

The sum of nine-tenths and g is two-thirds.

Exercise 9.2E.83

The sum of two-fifths and f is one-half.

Answer

$\frac{2}{5} + f = \frac{1}{2}; f = \frac{1}{10}$

Exercise 9.2E.84

The difference of p and one-sixth is two-thirds.

Exercise 9.2E.85

The difference of q and one-eighth is three-fourths.

Answer

$q - \frac{1}{8} = \frac{3}{4}; q = \frac{7}{8}$

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

Exercise 9.2E.86

Kindergarten Connie's kindergarten class has 24 children. She wants them to get into 4 equal groups. How many children will she put in each group?

Exercise 9.2E.87

Balloons Ramona bought 18 balloons for a party. She wants to make 3 equal bunches. How many balloons did she use in each bunch?

Answer

6 balloons

Exercise 9.2E.88

Tickets Mollie paid $36.25 for 5 movie tickets. What was the price of each ticket?

Exercise 9.2E.89

Shopping Serena paid $12.96 for a pack of 12 pairs of sport socks. What was the price of pair of sport socks?

Answer

$1.08

Exercise 9.2E.90

Sewing Nancy used 14 yards of fabric to make flags for one-third of the drill team. How much fabric, would Nancy need to make flags for the whole team?

Exercise 9.2E.91

MPG John's SUV gets 18 miles per gallon (mpg). This is half as many mpg as his wife's hybrid car. How many miles per gallon does the hybrid car get?

Answer

36 mpg

Exercise 9.2E.92

Height Aiden is 27 inches tall. He is $\frac{3}{8}$ as tall as his father. How tall is his father?

Exercise 9.2E.92

Real estate Bea earned $\$11,700$ commission for selling a house, calculated as $\frac{6}{100}$ of the selling price. What was the selling price of the house?

Answer

$195,000

Everyday Math

Exercise 9.2E.93

Commissions Every week Perry gets paid $\$150$ plus 12% of his total sales amount. Solve the equation $840 = 150 + 0.12(a - 1250)$ for a to find the total amount Perry must sell in order to be paid $\$840$ one week.

Exercise 9.2E.94

Stamps Travis bought $9.45 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was 5 less than the number of 49-cent stamps. Solve the equation 0.49s+0.21(s−5)=9.45 for s, to find the number of 49-cent stamps Travis bought.

Answer

15 49-cent stamps

Writing Exercises

Exercise $9.2E.95$

Frida started to solve the equation −3x=36 by adding 3 to both sides. Explain why Frida's method will not solve the equation.

Exercise $9.2E.96$

Emiliano thinks $x = 40$ is the solution to the equation $\frac{1}{2}x = 80$. Explain why he is wrong.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve equations using the Division and Multiplication Properties of equality.			
solve equations that require simplification.			
translate to an equation and solve.			
translate and solve applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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9.3: Solve Equations with Variables and Constants on Both Sides

Learning Objectives

By the end of this section, you will be able to:

- Solve an equation with constants on both sides
- Solve an equation with variables on both sides
- Solve an equation with variables and constants on both sides

Note

Before you get started, take this readiness quiz.

1. Simplify: 4y−9+9.
   If you missed this problem, review Exercise 1.10.20.

Solve Equations with Constants on Both Sides

In all the equations we have solved so far, all the variable terms were on only one side of the equation with the constants on the other side. This does not happen all the time—so now we will learn to solve equations in which the variable terms, or constant terms, or both are on both sides of the equation.

Our strategy will involve choosing one side of the equation to be the "variable side", and the other side of the equation to be the "constant side." Then, we will use the Subtraction and Addition Properties of Equality to get all the variable terms together on one side of the equation and the constant terms together on the other side.

By doing this, we will transform the equation that began with variables and constants on both sides into the form $ax = b$. We already know how to solve equations of this form by using the Division or Multiplication Properties of Equality.

Example 9.3.1

Solve: $7x + 8 = -13$.

Solution

In this equation, the variable is found only on the left side. It makes sense to call the left side the "variable" side. Therefore, the right side will be the "constant" side. We will write the labels above the equation to help us remember what goes where.

$$\overset{\text{variable}}{7x} + 8 = \overset{\text{constant}}{-13}$$

Since the left side is the "xx", or variable side, the 8 is out of place. We must "undo" adding 8 by subtracting 8, and to keep the equality we must subtract 8 from both sides.

	$\overset{\text{variable}}{7x} + 8 = \overset{\text{constant}}{-13}$
Use the Subtraction Property of Equality.	$7x + 8 - 8 = -13 - 8$
Simplify.	$7x = -21$
Now all the variables are on the left and the constant on the right. The equation looks like those you learned to solve earlier.	
Use the Division Property of Equality.	$\frac{7x}{7} = \frac{-21}{7}$
Simplify.	$x = -3$
Check:	$7x + 8 = -13$
Let x=−3.	$7(-3) + 8 \stackrel{?}{=} -13$

	$-21+8\stackrel{?}{=}-13$ $-13=-13\checkmark$	

Try It 9.3.2

Solve: $3x+4=-8$.

Answer

$x=-4$

Try It 9.3.3

Solve: $5a+3=-37$.

Answer

$a=-8$

Example 9.3.4

Solve: $8y-9=31$.

Solution

Notice, the variable is only on the left side of the equation, so we will call this side the "variable" side, and the right side will be the "constant" side. Since the left side is the "variable" side, the 9 is out of place. It is subtracted from the 8y, so to "undo" subtraction, add 9 to both sides. Remember, whatever you do to the left, you must do to the right.

	variable constant $8y-9=31$
Add 9 to both sides.	$8y-9=31$ $+9\quad+9$
Simplify.	$8y=40$
	The variable is now on one side and the constants on the other. We continue from here as we did earlier.
Divide both sides by 8.	$\frac{8y}{8}=\frac{40}{8}$
Simplify.	$y=5$
Check:	$8y-9=31$
Let y=5.	$8\cdot5-9\stackrel{?}{=}31$
	$40-9\stackrel{?}{=}31$
	$31=31\checkmark$

Try It 9.3.5

Solve: $5y-9=16$.

Answer

$y = 5$

Try It 9.3.6

Solve: $3m - 8 = 19$.

Answer

$m = 9$

Solve Equations with Variables and Constants on Both Sides

The next example will be the first to have variables and constants on both sides of the equation. It may take several steps to solve this equation, so we need a clear and organized strategy.

Example 9.3.7

Solve: $9x = 8x - 6$.

Solution

Here the variable is on both sides, but the constants only appear on the right side, so let's make the right side the "constant" side. Then the left side will be the "variable" side.

	$9x = 8x - 6$
We don't want any x's on the right, so subtract the 8x from both sides.	$9x - 8x = 8x - 8x - 6$
Simplify.	$x = -6$
We succeeded in getting the variables on one side and the constants on the other, and have obtained the solution.	
Check:	$9x = 8x - 6$
Let x=−6.	$9(-6) \stackrel{?}{=} 8(-6) - 6$
	$-54 \stackrel{?}{=} -48 - 6$
	$-54 = -54$ ✓

Try It 9.3.8

Solve: $6n = 5n - 10$.

Answer

$n = -10$

Try It 9.3.9

Solve: $-6c = -7c - 1$

Answer

$c = -1$

Example 9.3.10

Solve: $5y - 9 = 8y$

Solution

The only constant is on the left and the y's are on both sides. Let's leave the constant on the left and get the variables to the right.

	$\overset{\text{constant}}{5y} - 9 = \overset{\text{variable}}{8y}$
Subtract 5y from both sides.	$5y - 5y - 9 = 8y - 5y$
Simplify.	$-9 = 3y$
We have the y's on the right and the constants on the left. Divide both sides by 3.	$\frac{-9}{3} = \frac{3y}{3}$
Simplify.	$-3 = y$
Check:	$5y - 9 = 8y$
Let $y = -3$.	$5(-3) - 9 \stackrel{?}{=} 8(-3)$
	$-15 - 9 \stackrel{?}{=} -24$
	$-24 = -24$ ✓

Try It 9.3.11

Solve: $3p - 14 = 5p$.

Answer

$p = -7$

Try It 9.3.12

Solve: $8m + 9 = 5m$

Answer

$m = -3$

Example 9.3.13

Solve: $12x = -x + 26$

Solution

The only constant is on the right, so let the left side be the "variable" side.

	$\overset{\text{variable}}{12x} = \overset{\text{constant}}{-x + 26}$
Remove the $-x$ from the right side by adding x to both sides.	$12x + x = -x + x + 26$
Simplify.	$13x = 26$
All the x's are on the left and the constants are on the right. Divide both sides by 13.	$\frac{13x}{13} = \frac{26}{13}$

Simplify.	$x = 2$

Try It 9.3.14

Solve: $12j = -4j + 32$

Answer

$j = 2$

Try It 9.3.15

Solve: $8h = -4h + 12$

Answer

$h = 1$

Solve Equations with Variables and Constants on Both Sides

The next example will be the first to have variables and constants on both sides of the equation. It may take several steps to solve this equation, so we need a clear and organized strategy.

Example 9.3.16: How to Solve Equations with Variables and Constants on Both Sides

Solve: $7x + 5 = 6x + 2$

Solution

Step 1. Choose which side will be the "variable" side—the other side will be the "constant" side.	The variable terms are $7x$ and $6x$. Since 7 is greater than 6, we will make the left side the "x" side. The right side will be the "constant" side.	variable constant $7x + 5 = 6x + 2$
Step 2. Collect the variable terms to the "variable" side of the equation, using the addition or subtraction property of equality.	With the right side as the "constant" side, the $6x$ is out of place, so subtract $6x$ from both sides. Combine like terms. Now, the variable is only on the left side!	$7x - 6x + 5 = 6x - 6x + 2$ $x + 5 = 2$
Step 3. Collect all the constants to the other side of the equation, using the addition or subtraction property of equality.	The right side is the "constant" side, so the 5 is out of place. Subtract 5 from both sides. Simplify.	$x + 5 - 5 = 2 - 5$ $x = -3$

Step 4. Make the coefficient of the variable equal 1, using the multiplication or division property of equality.	The coefficient of x is one. The equation is solved.	
Step 5. Check.	Let $x = -3$ Simplify. Add.	**Check:** $7x + 6 = 6x + 2$ $(-3) + 5 = 6(-3) + 2$ $-21 + 5 = -18 + 2$ $-16 = -16$ ✓

? Try It 9.3.17

Solve: $12x + 8 = 6x + 2$.

Answer

$x = -1$

? Try It 9.3.18

Solve: $9y + 4 = 7y + 12$.

Answer

$y = 4$

We'll list the steps below so you can easily refer to them. But we'll call this the 'Beginning Strategy' because we'll be adding some steps later in this chapter.

BEGINNING STRATEGY FOR SOLVING EQUATIONS WITH VARIABLES AND CONSTANTS ON BOTH SIDES OF THE EQUATION.

1. Choose which side will be the "variable" side—the other side will be the "constant" side.
2. Collect the variable terms to the "variable" side of the equation, using the Addition or Subtraction Property of Equality.
3. Collect all the constants to the other side of the equation, using the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable equal 1, using the Multiplication or Division Property of Equality.
5. Check the solution by substituting it into the original equation.

In Step 1, a helpful approach is to make the "variable" side the side that has the variable with the larger coefficient. This usually makes the arithmetic easier.

✓ Example 9.3.19

Solve: $8n - 4 = -2n + 6$.

Solution

In the first step, choose the variable side by comparing the coefficients of the variables on each side.

Since $8 > -2$, make the left side the "variable" side.	variable constant $8n - 4 = -2n + 6$
We don't want variable terms on the right side—add 2n to both sides to leave only constants on the right.	$8n + 2n - 4 = -2n + 2n + 6$
Combine like terms.	$10n - 4 = 6$

We don't want any constants on the left side, so add 4 to both sides.	$10n-4+4=6+4$
Simplify.	$10n=10$
The variable term is on the left and the constant term is on the right. To get the coefficient of nn to be one, divide both sides by 10.	$\frac{10n}{10}=\frac{10}{10}$
Simplify.	$n=1$
Check:	$8n-4=-2n+6$
Let n=1.	$8\cdot 1-4\stackrel{?}{=}-2\cdot 1+6$
	$8-4\stackrel{?}{=}-2+6$
	$4=4\checkmark$

Try It 9.3.20

Solve: $8q-5=-4q+7$

Answer

$q=1$

Try It 9.3.21

Solve: $7n-3=n+3$

Answer

$n=1$

Example 9.3.22

Solve: $7a-3=13a+7$

Solution

In the first step, choose the variable side by comparing the coefficients of the variables on each side.

Since 13>7, make the right side the "variable" side and the left side the "constant" side.

	constant variable $7a-3=13a+7$
Subtract 7a from both sides to remove the variable term from the left.	$7a-7a-3=13a-7a+7$
Combine like terms.	$-3=6a+7$
Subtract 7 from both sides to remove the constant from the right.	$-3-7=6a+7-7$
Simplify.	$-10=6a$
Divide both sides by 6 to make 1 the coefficient of a.	$\frac{-10}{6}=\frac{6a}{6}$
Simplify.	$-\frac{5}{3}=a$
Check:	$7a-3=13a+7$
Let $a=-\frac{5}{3}$.	$7\left(-\frac{5}{3}\right)-3\stackrel{?}{=}13\left(-\frac{5}{3}\right)+7$

	$-\frac{35}{3}-\frac{9}{3}\stackrel{?}{=}-\frac{65}{3}+\frac{21}{3}$	
	$-\frac{54}{3}=-\frac{54}{3}\checkmark$	

Try It 9.3.23

Solve: $2a-2=6a+18$

Answer

$a=-5$

Try It 9.3.24

Solve: $4k-1=7k+17$

Answer

$k=-6$

In the last example, we could have made the left side the "variable" side, but it would have led to a negative coefficient on the variable term. (Try it!) While we could work with the negative, there is less chance of errors when working with positives. The strategy outlined above helps avoid the negatives!

To solve an equation with fractions, we just follow the steps of our strategy to get the solution!

Example 9.3.25

Solve: $\frac{4}{5}x+6=\frac{1}{4}x-2$

Solution

Since $\frac{5}{4}>\frac{1}{4}$, make the left side the "variable" side and the right side the "constant" side.

	variable constant $\frac{5}{4}x+6=\frac{1}{4}x-2$
Subtract $\frac{1}{4}x$ from both sides.	$\frac{5}{4}x-\frac{1}{4}x+6=\frac{1}{4}x-\frac{1}{4}x-2$
Combine like terms.	$x+6=-2$
Subtract 6 from both sides.	$x+6-6=-2-6$
Simplify.	$x=-8$
Check: Let $x=-8$	$\frac{5}{4}x+6 = \frac{1}{4}x-2$ $\frac{5}{4}(-8)+6 \stackrel{?}{=} \frac{1}{4}(-8)-2$ $-10+6 \stackrel{?}{=} -2-2$ $-4 = -4\checkmark$

Try It 9.3.26

Solve: $\frac{7}{8}x-12=-\frac{1}{8}x-2$

Answer

$x = 10$

Try It 9.3.27

Solve: $\frac{7}{6}x + 11 = \frac{1}{6}y + 8$

Answer

$y = -3$

We will use the same strategy to find the solution for an equation with decimals.

Example 9.3.28

Solve: $7.8x + 4 = 5.4x - 8$.

Solution

Since $7.8 > 5.4$, make the left side the "variable" side and the right side the "constant" side.

	variable side constant side $7.8x + 4 = 5.4x - 8$
Subtract 5.4x from both sides.	$7.8x - 5.4x + 4 = 5.4x - 5.4x - 8$
Combine like terms.	$2.4x + 4 = -8$
Subtract 4 from both sides.	$2.4x + 4 - 4 = -8 - 4$
Simplify.	$2.4x = -12$
Use the Division Property of Equality.	$\frac{2.4x}{2.4} = \frac{-12}{2.4}$
Simplify.	$x = -5$
Check:	$7.8x + 4 = 5.4x - 5$
Let $x = -5$	$7.8(-5) + 4 = 5.4(-5) - 8$
	$-39 + 4 \stackrel{?}{=} -27 - 8$
	$-35 = -35$ ✓

Try It 9.3.29

Solve: $2.8x + 12 = -1.4x - 9$

Answer

$x = -5$

Try It 9.3.30

Solve: $3.6y + 8 = 1.2y - 4$

Answer

$y = -5$

Key Concepts

- **Beginning Strategy for Solving an Equation with Variables and Constants on Both Sides of the Equation**
  1. Choose which side will be the “variable” side—the other side will be the “constant” side.
  2. Collect the variable terms to the “variable” side of the equation, using the Addition or Subtraction Property of Equality.
  3. Collect all the constants to the other side of the equation, using the Addition or Subtraction Property of Equality.
  4. Make the coefficient of the variable equal 1, using the Multiplication or Division Property of Equality.
  5. Check the solution by substituting it into the original equation.

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9.3E: Exercises

Practice Makes Perfect

Solve Equations with Constants on Both Sides

In the following exercises, solve the following equations with constants on both sides.

Exercise 9.3E.1

$9x - 3 = 60$

Exercise 9.3E.2

$12x - 8 = 64$

Answer

$x = 6$

Exercise 9.3E.3

$14w + 5 = 117$

Exercise 9.3E.4

$15y + 7 = 97$

Answer

$y = 6$

Exercise 9.3E.5

$2a + 8 = -28$

Exercise 9.3E.6

$3m + 9 = -15$

Answer

$m = -8$

Exercise 9.3E.7

$-62 = 8n - 6$

Exercise 9.3E.8

$-77 = 9b - 5$

Answer

$b = -8$

Exercise 9.3E.9

$35 = -13y + 9$

Exercise 9.3E.10

$60 = -21x - 24$

Answer

$x = -4$

Exercise 9.3E.11

$-12p - 9 = 9$

Exercise 9.3E.12

$-14q - 2 = 16$

Answer

$q = -\frac{9}{7}$

Solve Equations with Variables on Both Sides

In the following exercises, solve the following equations with variables on both sides.

Exercise 9.3E.13

$19z = 18z - 7$

Exercise 9.3E.14

$21k = 20k - 11$

Answer

$k = -11$

Exercise 9.3E.15

$9x + 36 = 15x$

Exercise 9.3E.16

$8x + 27 = 11x$

Answer

$x = 9$

Exercise 9.3E.17

$c = -3c - 20$

Exercise 9.3E.18

$b = -4b - 15$

Answer

$b = -3$

Exercise 9.3E.19

$9q = 44 - 2q$

Exercise 9.3E.20

$5z = 39 - 8z$

Answer

$z = 3$

Exercise 9.3E.21

$6y + \frac{1}{2} = 5y$

Exercise 9.3E.22

$4x + \frac{3}{4} = 3x$

Answer

$x = -\frac{3}{4}$

Exercise 9.3E.23

$-18a - 8 = -22a$

Exercise 9.3E.24

$-11r - 8 = -7r$

Answer

$r = -2$

Solve Equations with Variables and Constants on Both Sides

In the following exercises, solve the following equations with variables and constants on both sides.

Exercise 9.3E.25

$8x - 15 = 7x + 3$

Exercise 9.3E.26

$6x - 17 = 5x + 2$

Answer

$x = 19$

Exercise 9.3E.27

$26 + 13d = 14d + 11$

Exercise 9.3E.28

$21 + 18f = 19f + 14$

Answer

$f = 7$

Exercise 9.3E.29

$2p - 1 = 4p - 33$

Exercise 9.3E.30

$12q - 5 = 9q - 20$

Answer

$q = -5$

Exercise 9.3E.31

$4a + 5 = -a - 40$

Exercise 9.3E.32

$8c + 7 = -3c - 37$

Answer

$c = -4$

Exercise 9.3E.33

$5y - 30 = -5y + 30$

Exercise 9.3E.34

$7x - 17 = -8x + 13$

Answer

$x = 2$

Exercise 9.3E.35

$7s + 12 = 5 + 4s$

Exercise 9.3E.36

$9p+14=6+4p$

Answer

$p=-\frac{8}{5}$

Exercise 9.3E.37

$2z-6=23-z$

Exercise 9.3E.38

$3y-4=12-y$

Answer

$y=4$

Exercise 9.3E.39

$\frac{5}{3}c-3=\frac{2}{3}c-16$

Exercise 9.3E.40

$\frac{7}{4}m-7=\frac{3}{4}m-13$

Answer

$m=-6$

Exercise 9.3E.41

$8-\frac{2}{5}q=\frac{3}{5}q+6$

Exercise 9.3E.42

$11-\frac{1}{5}a=\frac{4}{5}a+4$

Answer

$a=7$

Exercise 9.3E.43

$\frac{4}{3}n+9=\frac{1}{3}n-9$

Exercise 9.3E.44

$\frac{5}{4}a+15=\frac{3}{4}a-5$

Answer

$a=-40$

Exercise 9.3E.45

$\frac{1}{4}y+7=\frac{3}{4}y-3$

Exercise 9.3E.46

$\frac{3}{5}p+2=\frac{4}{5}p-1$

Answer

$p=15$

Exercise 9.3E.47

$14n+8.25=9n+19.60$

Exercise 9.3E.48

$13z+6.45=8z+23.75$

Answer

$z=3.46$

Exercise 9.3E.49

$2.4w-100=0.8w+28$

Exercise 9.3E.50

$2.7w-80=1.2w+10$

Answer

$w=60$

Exercise 9.3E.51

$5.6r+13.1=3.5r+57.2$

Exercise 9.3E.52

$6.6x-18.9=3.4x+54.7$

Answer

$x=23$

Everyday Math

Exercise 9.3E.53

Concert tickets At a school concert the total value of tickets sold was $1506. Student tickets sold for $6 and adult tickets sold for $9. The number of adult tickets sold was 5 less than 3 times the number of student tickets. Find the number of student tickets sold, *s*, by solving the equation 6s+27s−45=1506. Add exercises text here.

Exercise $9.3E.54$

Making a fence Jovani has 150 feet of fencing to make a rectangular garden in his backyard. He wants the length to be 15 feet more than the width. Find the width, w, by solving the equation $150 = 2w + 30 + 2w$.

Answer

30 feet

Writing Exercises

Exercise $9.3E.55$

Solve the equation $\frac{6}{5}y - 8 = \frac{1}{5}y + 7$ explaining all the steps of your solution as in the examples in this section.

Exercise $9.3E.56$

Solve the equation $10x + 14 = -2x + 38$ explaining all the steps of your solution as in this section.

Answer

$x = 2$ Justifications will vary.

Exercise $9.3E.57$

When solving an equation with variables on both sides, why is it usually better to choose the side with the larger coefficient of x to be the "variable" side?

Exercise $9.3E.58$

Is $x = -2$ a solution to the equation $5 - 2x = -4x + 1$? How do you know?

Answer

Yes. Justifications will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve an equation with constants on both sides.			
solve an equation with variables on both sides.			
solve an equation with variables and constants on both sides.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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9.4: Use a General Strategy to Solve Linear Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve equations using a general strategy
- Classify equations

Note

Before you get started, take this readiness quiz.

1. Simplify: $-(a-4)$.
   If you missed this problem, review Exercise 1.10.46
2. Multiply: $\frac{3}{2}(12x+20)$
   If you missed this problem, review Exercise 1.10.34.
3. Simplify: $5-2(n+1)$
   If you missed this problem, review Exercise 1.10.49.
4. Multiply: $3(7y+9)$
   If you missed this problem, review Exercise 1.10.34.
5. Multiply: $(2.5)(6.4)$
   If you missed this problem, review Exercise 1.8.19.

Solve Equations Using the General Strategy

Until now we have dealt with solving one specific form of a linear equation. It is time now to lay out one overall strategy that can be used to solve any linear equation. Some equations we solve will not require all these steps to solve, but many will.

Beginning by simplifying each side of the equation makes the remaining steps easier.

Example 9.4.1: How to Solve Linear Equations Using the General Strategy

Solve: $-6(x+3)=24$.

Solution

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is simplified as much as possible.	$-6(x+3)=24$ $-6x-18=24$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do – all x's are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 18 to each side. Simplify.	$-6x-18+18=24+18$ $-6x=42$

Step 4. Make the coefficient of the variable term to equal to 1.	Divide each side by –6.	$\frac{-6x}{-6} = \frac{42}{-6}$
	Simplify.	$x = -7$
Step 5. Check the solution.	Let $x = -7$	**Check:** $-6(x + 3) = 24$
	Simplify.	$-6(-7 + 3) \stackrel{?}{=} 24$
	Multiply.	$-6(-4) \stackrel{?}{=} 24$
		$24 = 24$ ✓

Try It 9.4.2

Solve: $5(x+3) = 35$

Answer

$x = 4$

Try It 9.4.3

Solve: $6(y-4) = -18$

Answer

$y = 1$

GENERAL STRATEGY FOR SOLVING LINEAR EQUATIONS.

1. **Simplify each side of the equation as much as possible.**
   Use the Distributive Property to remove any parentheses.
   Combine like terms.
2. **Collect all the variable terms on one side of the equation.**
   Use the Addition or Subtraction Property of Equality.
3. **Collect all the constant terms on the other side of the equation.**
   Use the Addition or Subtraction Property of Equality.
4. **Make the coefficient of the variable term to equal to 1.**
   Use the Multiplication or Division Property of Equality.
   State the solution to the equation.
5. **Check the solution.** Substitute the solution into the original equation to make sure the result is a true statement.

Example 9.4.4

Solve: $-(y+9) = 8$

Solution

	$-(y + 9) = 8$
Simplify each side of the equation as much as possible by distributing.	$-y - 9 = 8$
The only y term is on the left side, so all variable terms are on the left side of the equation.	
Add 9 to both sides to get all constant terms on the right side of the equation.	$-y - 9 + 9 = 8 + 9$
Simplify.	$-y = 17$

Rewrite $-y$ as $-1y$.		$-1y = 17$
Make the coefficient of the variable term to equal to 1 by dividing both sides by -1.		$\frac{-1y}{-1} = \frac{17}{-1}$
Simplify.		$y = -17$
Check:	$-(y+9) = 8$	
Let y=−17.	$-(-17+9) \stackrel{?}{=} 8$	
	$-(-8) \stackrel{?}{=} 8$	
	$8 = 8$ ✓	

Try It 9.4.5

Solve: $-(y+8) = -2$

Answer

$y = -6$

Try It 9.4.6

Solve: $-(z+4) = -12$

Answer

$z = 8$

Example 9.4.7

Solve: $5(a-3)+5 = -10$

Solution

		$5(a-3)+5 = -10$
Simplify each side of the equation as much as possible.		
Distribute.		$5a-15+5 = -10$
Combine like terms.		$5a-10 = -10$
The only a term is on the left side, so all variable terms are on one side of the equation.		
Add 10 to both sides to get all constant terms on the other side of the equation.		$5a-10+10 = -10+10$
Simplify.		$5a = 0$
Make the coefficient of the variable term to equal to 11 by dividing both sides by 55.		$\frac{5a}{5} = \frac{0}{5}$
Simplify.		$a = 0$
Check:	$5(a-3)+5 = -10$	
Let a=0.	$5(0-3)+5 \stackrel{?}{=} -10$	
	$5(-3)+5 \stackrel{?}{=} -10$	
	$-15+5 \stackrel{?}{=} -10$	

	$-10 = -10$ ✓

Try It 9.4.8

Solve: $2(m-4)+3=-1$

Answer

$m=2$

Try It 9.4.9

Solve: $7(n-3)-8=-15$

Answer

$n=2$

Example 9.4.10

Solve: $\frac{2}{3}(6m-3)=8-m$

Solution

	$\frac{2}{3}(6m-3)=8-m$
Distribute.	$4m-2=8-m$
Add m to get the variables only to the left.	$4m+m-2=8-m+m$
Simplify.	$5m-2=8$
Add 2 to get constants only on the right.	$5m-2+2=8+2$
Simplify.	$5m=10$
Divide by 5.	$\frac{5m}{5}=\frac{10}{5}$
Simplify.	$m=2$
Check:	$\frac{2}{3}(6m-3)=8-m$
Let m=2.	$\frac{2}{3}(6\cdot 2-3)\stackrel{?}{=}8-2$
	$\frac{2}{3}(12-3)\stackrel{?}{=}6$
	$\frac{2}{3}(9)\stackrel{?}{=}6$
	$6=6$ ✓

Try It 9.4.11

Solve: $\frac{1}{3}(6u+3)=7-u$

Answer

$u=2$

Try It 9.4.12

Solve: $\frac{2}{3}(9x-12)=8+2x$

Answer

$x=4$

Example 9.4.13

Solve: $8-2(3y+5)=0$

Solution

	$8-2(3y+5)=0$
Simplify—use the Distributive Property.	$8-6y-10=0$
Combine like terms.	$-6y-2=0$
Add 2 to both sides to collect constants on the right.	$-6y-2+2=0+2$
Simplify.	$-6y=2$
Divide both sides by -6−6.	$\frac{-6y}{-6}=\frac{2}{-6}$
Simplify.	$y=-\frac{1}{3}$
Check: Let y=−13. $8-2(3y+5)=0$ $8-2\left[3\left(-\frac{1}{3}\right)+5\right]=0$ $8-2(-1+5)\stackrel{?}{=}0$ $8-2(4)\stackrel{?}{=}0$ $8-8\stackrel{?}{=}0$ $0=0$ ✓	

Try It 9.4.14

Solve: $12-3(4j+3)=-17$

Answer

$j=\frac{5}{3}$

Try It 9.4.15

Solve: $-6-8(k-2)=-10$

Answer

$k=\frac{5}{2}$

Example 9.4.16

Solve: $4(x-1)-2=5(2x+3)+6$

Solution

	$4(x-1)-2=5(2x+3)+6$
Distribute.	$4x-4-2=10x+15+6$
Combine like terms.	$4x-6=10x+21$
Subtract 4x to get the variables only on the right side since $10>4$.	$4x-4x-6=10x-4x+21$
Simplify.	$-6=6x+21$
Subtract 21 to get the constants on left.	$-6-21=6x+21-21$
Simplify.	$-27=6x$
Divide by 6.	$\frac{-27}{6}=\frac{6x}{6}$
Simplify.	$-\frac{9}{2}=x$
Check:	$4(x-1)-2=5(2x+3)+6$
Let $x=-\frac{9}{2}$.	$4\left(-\frac{9}{2}-1\right)-2\stackrel{?}{=}5\left[2\left(-\frac{9}{2}\right)+3\right]+6$
	$4\left(-\frac{11}{2}\right)-2\stackrel{?}{=}5(-9+3)+6$
	$-22-2\stackrel{?}{=}5(-6)+6$
	$-24\stackrel{?}{=}-30+6$
	$-24=-24$ ✓

Try It 9.4.17

Solve: $6(p-3)-7=5(4p+3)-12$

Answer

$p=-2$

Try It 9.4.18

Solve: $8(q+1)-5=3(2q-4)-1$

Answer

$q=-8$

Example 9.4.19

Solve: $10[3-8(2s-5)]=15(40-5s)$

Solution

	$10[3-8(2s-5)]=15(40-5s)$

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Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$
Combine like terms in the brackets.	$10[43-16s]=15(40-5s)$
Distribute.	$430-160s=600-75s$
Add 160s to get the s's to the right.	$430-160s+160s=600-75s+160s$
Simplify.	$430=600+85s$
Subtract 600 to get the constants to the left.	$430-600=600+85s-600$
Simplify.	$-170=85s$
Divide.	$\frac{-170}{85}=\frac{85s}{85}$
Simplify.	$-2=s$

Check:	$10[3-8(2s-5)]=15(40-5s)$
Substitute s=−2.	$10[3-8(2(-2)-5)] \stackrel{?}{=} 15(40-5(-2))$
	$10[3-8(-4-5)] \stackrel{?}{=} 15(40+10)$
	$10[3-8(-9)] \stackrel{?}{=} 15(50)$
	$10[3+72] \stackrel{?}{=} 750$
	$10[75] \stackrel{?}{=} 750$
	$750=750$ ✓

Try It 9.4.20

Solve: $6[4-2(7y-1)]=8(13-8y)$.

Answer

$y=-\frac{17}{5}$

Try It 9.4.21

Solve: $12[1-5(4z-1)]=3(24+11z)$.

Answer

$z=0$

Example 9.4.22

Solve: $0.36(100n+5)=0.6(30n+15)$.

Solution

	$0.36(100n+5)=0.6(30n+15)$
Distribute.	$36n+1.8=18n+9$
Subtract 18n to get the variables to the left.	$36n-18n+1.8=18n-18n+9$
Simplify.	$18n+1.8=9$

Subtract 1.8 to get the constants to the right.		$18n + 1.8 - 1.8 = 9 - 1.8$
Simplify.		$18n = 7.2$
Divide.		$\frac{18n}{18} = \frac{7.2}{18}$
Simplify.		$n = 0.4$
Check:	$0.36(100n + 5) = 0.6(30n + 15)$	
Let n=0.4.	$0.36(100(0.4) + 5) \stackrel{?}{=} 0.6(30(0.4) + 15)$	
	$0.36(40 + 5) \stackrel{?}{=} 0.6(12 + 15)$	
	$0.36(45) \stackrel{?}{=} 0.6(27)$	
	$16.2 = 16.2$ ✓	

? Try It 9.4.23

Solve: $0.55(100n+8) = 0.6(85n+14)$.

Answer

$n = 1$

? Try It 9.4.24

Solve: $0.15(40m-120) = 0.5(60m+12)$.

Answer

$m = -1$

Classify Equations

Consider the equation we solved at the start of the last section, 7x+8=−13. The solution we found was x=−3. This means the equation 7x+8=−13 is true when we replace the variable, *x*, with the value −3. We showed this when we checked the solution x=−3 and evaluated 7x+8=−13 for x=−3.

$$7(-3) + 8 \stackrel{?}{=} -13$$
$$-21 + 8 \stackrel{?}{=} -13$$
$$-13 = -13 \checkmark$$

If we evaluate 7x+8 for a different value of *x*, the left side will not be −13.

The equation 7x+8=−13 is true when we replace the variable, *x*, with the value −3, but not true when we replace *x* with any other value. Whether or not the equation 7x+8=−13 is true depends on the value of the variable. Equations like this are called conditional equations.

All the equations we have solved so far are conditional equations.

CONDITIONAL EQUATION

An equation that is true for one or more values of the variable and false for all other values of the variable is a **conditional equation**.

Now let's consider the equation 2y+6=2(y+3). Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for *y*.

	$2y+6=2(y+3)$
Distribute.	$2y+6=2y+6$
Subtract 2y to get the y's to one side.	$2y-2y+6=2y-2y+6$
Simplify—the y's are gone!	$6=6$

But 6=6 is true.

This means that the equation 2y+6=2(y+3) is true for any value of *y*. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable like this is called an **identity**.

IDENTITY

An equation that is true for any value of the variable is called an **identity**.

The solution of an identity is all real numbers.

What happens when we solve the equation 5z=5z−1?

	$5z=5z-1$
Subtract 5z to get the constant alone on the right.	$5z-5z=5z-5z-1$
Simplify—the z's are gone!	$0\neq -1$

But $0 \neq -1$.

Solving the equation 5z=5z−1 led to the false statement 0=−1. The equation 5z=5z−1 will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a **contradiction**.

CONTRADICTION

An equation that is false for all values of the variable is called a **contradiction**.

A contradiction has no solution.

Example 9.4.25

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$6(2n-1)+3=2n-8+5(2n+1)$

Solution

$6(2n-1)+3=2n-8+5(2n+1)$	
Distribute.	$12n-6+3=2n-8+10n+5$
Combine like terms.	$12n-3=12n-3$
Subtract 12n to get the nn's to one side.	$12n-12n-3=12n-12n-3$
Simplify.	$-3=-3$
This is a true statement.	The equation is an identity. The solution is all real numbers.

Try It 9.4.26

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$4+9(3x-7)=-42x-13+23(3x-2)$

Answer

identity; all real numbers

Try It 9.4.27

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$8(1-3x)+15(2x+7)=2(x+50)+4(x+3)+1$

Answer

identity; all real numbers

Example 9.4.28

Classify as a conditional equation, an identity, or a contradiction. Then state the solution.

$10+4(p-5)=0$

Solution

	$10+4(p-5)=0$
Distribute.	$10+4p-20=0$
Combine like terms.	$4p-10=0$
Add 10 to both sides.	$4p-10+10=0+10$
Simplify.	$4p=10$
Divide.	$\frac{4p}{4}=\frac{10}{4}$
Simplify.	$p=\frac{5}{2}$
The equation is true when $p=frac52$.	This is a conditional equation. The solution is $p=frac52$.

Try It 9.4.29

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11(q+3)-5=19$

Answer

conditional equation; \(q = \frac{9}{11}\

Try It 9.4.30

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6+14(k-8)=95$

Answer

conditional equation; $k = \frac{193}{14}$

Example 9.4.31

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$5m + 3(9 + 3m) = 2(7m - 11)$

Solution

	$5m + 3(9 + 3m) = 2(7m - 11)$
Distribute.	$5m + 27 + 9m = 14m - 22$
Combine like terms.	$14m + 27 = 14m - 22$
Subtract 14m from both sides.	$14m + 27 - 14m = 14m - 22 - 14m$
Simplify.	$27 \neq -22$
But $27 \neq -22$.	The equation is a contradiction. It has no solution.

Try It 9.4.32

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$12c + 5(5 + 3c) = 3(9c - 4)$

Answer

contradiction; no solution

Try It 9.4.33

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$4(7d + 18) = 13(3d - 2) - 11d$

Answer

contradiction; no solution

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	**True** for any value of the variable	All real numbers
Contradiction	**False** for all values of the variable	No solution

Key Concepts

- **General Strategy for Solving Linear Equations**
  1. Simplify each side of the equation as much as possible.
     Use the Distributive Property to remove any parentheses.
     Combine like terms.

2. Collect all the variable terms on one side of the equation.
   Use the Addition or Subtraction Property of Equality.
3. Collect all the constant terms on the other side of the equation.
   Use the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable term to equal to 1.
   Use the Multiplication or Division Property of Equality.
   State the solution to the equation.
5. Check the solution.
   Substitute the solution into the original equation.

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9.4E: Exercises

Practice Makes Perfect

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, solve each linear equation.

Exercise 9.4E.1

$15(y-9)=-60$

Exercise 9.4E.2

$21(y-5)=-42$

Answer

$y=3$

Exercise 9.4E.3

$-9(2n+1)=36$

Exercise 9.4E.4

$-16(3n+4)=32$

Answer

$n=-2$

Exercise 9.4E.5

$8(22+11r)=0$

Exercise 9.4E.6

$5(8+6p)=0$

Answer

$p=-\frac{4}{3}$

Exercise 9.4E.7

$-(w-12)=30$

Exercise 9.4E.8

$-(t-19)=28$

Answer

$t=-9$

Exercise 9.4E.9

$9(6a+8)+9=81$

Exercise 9.4E.10

$8(9b-4)-12=100$

Answer

$b=2$

Exercise 9.4E.11

$32+3(z+4)=41$

Exercise 9.4E.12

$21+2(m-4)=25$

Answer

$m=6$

Exercise 9.4E.13

$51+5(4-q)=56$

Exercise 9.4E.14

$-6+6(5-k)=15$

Answer

$k=\frac{3}{2}$

Exercise 9.4E.15

$2(9s-6)-62=16$

Exercise 9.4E.16

$8(6t-5)-35=-27$

Answer

$t=1$

Exercise 9.4E.17

$3(10-2x)+54=0$

Exercise 9.4E.18

$-2(11-7x)+54=4$

Answer

$x=-2$

Exercise 9.4E.19

$\frac{2}{3}(9c-3)=22$

Exercise 9.4E.20

$\frac{3}{5}(10x-5)=27$

Answer

$x=5$

Exercise 9.4E.21

$\frac{1}{5}(15c+10)=c+7$

Exercise 9.4E.22

$\frac{1}{4}(20d+12)=d+7$

Answer

$d=1$

Exercise 9.4E.23

$18-(9r+7)=-16$

Exercise 9.4E.24

$15-(3r+8)=28$

Answer

$r=-7$

Exercise 9.4E.25

$5-(n-1)=19$

Exercise 9.4E.26

$-3-(m-1)=13$

Answer

$m=-15$

Exercise 9.4E.27

$11-4(y-8)=43$

Exercise 9.4E.28

$18-2(y-3)=32$

Answer

$y=-4$

Exercise 9.4E.29

$24-8(3v+6)=0$

Exercise 9.4E.30

$35-5(2w+8)=-10$

Answer

$w=\frac{1}{2}$

Exercise 9.4E.31

$4(a-12)=3(a+5)$

Exercise 9.4E.32

$-2(a-6)=4(a-3)$

Answer

$a=4$

Exercise 9.4E.33

$2(5-u)=-3(2u+6)$

Exercise 9.4E.34

$5(8-r)=-2(2r-16)$

Answer

$r=8$

Exercise 9.4E.35

$3(4n-1)-2=8n+3$

Exercise 9.4E.36

$9(2m-3)-8=4m+7$

Answer

$m=3$

Exercise 9.4E.37

$12+2(5-3y)=-9(y-1)-2$

Exercise 9.4E.38

$-15+4(2-5y)=-7(y-4)+4$

Answer

$y=-3$

Exercise 9.4E.39

$8(x-4)-7x=14$

Exercise 9.4E.40

$5(x-4)-4x=14$

Answer

$x=34$

Exercise 9.4E.41

$5+6(3s-5)=-3+2(8s-1)$

Exercise 9.4E.42

$-12+8(x-5)=-4+3(5x-2)$

Answer

$x=-6$

Exercise 9.4E.43

$4(u-1)-8=6(3u-2)-7$

Exercise 9.4E.44

$7(2n-5)=8(4n-1)-9$

Answer

$n=-1$

Exercise 9.4E.45

$4(p-4)-(p+7)=5(p-3)$

Exercise 9.4E.46

$3(a-2)-(a+6)=4(a-1)$

Answer

$a=-4$

Exercise 9.4E.47

$-(9y+5)-(3y-7)$
$=16-(4y-2)$

Exercise 9.4E.48

$-(7m+4)-(2m-5)$
$=14-(5m-3)$

Answer

$m=-4$

Exercise 9.4E.49

$4[5-8(4c-3)]$
$=12(1-13c)-8$

Exercise 9.4E.50

$5[9-2(6d-1)]$
$=11(4-10d)-139$

Answer

$d=-3$

Exercise 9.4E.51

$3[-9+8(4h-3)]$
$=2(5-12h)-19$

Exercise 9.4E.52

$3[-14+2(15k-6)]$
$=8(3-5k)-24$

Answer

$k=\frac{3}{5}$

Exercise 9.4E.53

$5[2(m+4)+8(m-7)]$
$=2[3(5+m)-(21-3m)]$

Exercise 9.4E.54

$10[5(n+1)+4(n-1)]$
$=11[7(5+n)-(25-3n)]$

Answer

$n=-5$

Exercise 9.4E.55

$5(1.2u-4.8)=-12$

Exercise 9.4E.56

$4(2.5v-0.6)=7.6$

Answer

$v=1$

Exercise 9.4E.57

$0.25(q-6)=0.1(q+18)$

Exercise 9.4E.58

$0.2(p-6)=0.4(p+14)$

Answer

$p=-34$

Exercise 9.4E.59

$0.2(30n+50)=28$

Exercise 9.4E.60

$0.5(16m+34)=-15$

Answer

$m=-4$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

Exercise 9.4E.61

$23z+19=3(5z-9)+8z+46$

Exercise 9.4E.62

$15y+32=2(10y-7)-5y+46$

Answer

identity; all real numbers

Exercise 9.4E.63

$5(b-9)+4(3b+9)=6(4b-5)-7b+21$

Exercise 9.4E.64

$9(a-4)+3(2a+5)=7(3a-4)-6a+7$

Answer

identity; all real numbers

Exercise 9.4E.65

$18(5j-1)+29=47$

Exercise 9.4E.66

$24(3d-4)+100=52$

Answer

conditional equation; $d=\frac{2}{3}$

Exercise 9.4E.67

$22(3m-4)=8(2m+9)$

Exercise 9.4E.68

$30(2n-1)=5(10n+8)$

Answer

conditional equation; $n=7$

Exercise 9.4E.69

$7v+42=11(3v+8)-2(13v-1)$

Exercise 9.4E.70

$18u-51=9(4u+5)-6(3u-10)$

Answer

contradiction; no solution

Exercise 9.4E.71

$3(6q-9)+7(q+4)=5(6q+8)-5(q+1)$

Exercise 9.4E.72

$5(p+4)+8(2p-1)=9(3p-5)-6(p-2)$

Answer

contradiction; no solution

Exercise 9.4E.73

$12(6h-1)=8(8h+5)-4$

Exercise 9.4E.74

$9(4k-7)=11(3k+1)+4$

Answer

conditional equation; $k=26$

Exercise 9.4E.75

$45(3y-2)=9(15y-6)$

Exercise 9.4E.76

$60(2x-1)=15(8x+5)$

Answer

contradiction; no solution

Exercise 9.4E.77

$16(6n+15)=48(2n+5)$

Exercise 9.4E.78

$36(4m+5)=12(12m+15)$

Answer

identity; all real numbers

Exercise 9.4E.79

$9(14d+9)+4d=13(10d+6)+3$

Exercise 9.4E.80

$11(8c+5)-8c=2(40c+25)+5$

Answer

identity; all real numbers

Everyday Math

Exercise 9.4E.81

Fencing Micah has 44 feet of fencing to make a dog run in his yard. He wants the length to be 2.5 feet more than the width. Find the length, L, by solving the equation 2L+2(L−2.5)=44.

Exercise 9.4E.82

Coins Rhonda has $\$1.90$ in nickels and dimes. The number of dimes is one less than twice the number of nickels. Find the number of nickels, n, by solving the equation $0.05n+0.10(2n-1)=1.90$.

Answer

8 nickels

Writing Exercises

Exercise 9.4E.83

Using your own words, list the steps in the general strategy for solving linear equations.

Exercise 9.4E.84

Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.

Answer

Answers will vary.

Exercise 9.4E.85

What is the first step you take when solving the equation $3-7(y-4)=38$? Why is this your first step?

Exercise 9.4E.86

Solve the equation $\frac{1}{4}(8x+20)=3x-4$ explaining all the steps of your solution as in the examples in this section.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objective of this section.

I can...	Confidently	With some help	No-I don't get it!
solve equations using the general strategy for solving linear equations.			
classify equations.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

conditional equation

An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction

An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity

An equation that is true for any value of the variable is called an identity. The solution of an identity is all real numbers.

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9.5: Solve Equations with Fractions or Decimals

Learning Objectives

By the end of this section, you will be able to:

- Solve equations with fraction coefficients
- Solve equations with decimal coefficients

Note

Before you get started, take this readiness quiz.

1. Multiply: $8 \cdot 38$.
   If you missed this problem, review Exercise 1.6.16.
2. Find the LCD of $\frac{5}{6}$ and $\frac{1}{4}$.
   If you missed this problem, review Exercise 1.7.16.
3. Multiply 4.78 by 100.
   If you missed this problem, review Exercise 1.8.22.

Solve Equations with Fraction Coefficients

Let's use the general strategy for solving linear equations introduced earlier to solve the equation, $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

Table 9.5.1

	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
To isolate the x term, subtract $\frac{1}{2}$ from both sides.	$\frac{1}{8}x+\frac{1}{2}-\frac{1}{2}=\frac{1}{4}-\frac{1}{2}$
Simplify the left side.	$\frac{1}{8}x=\frac{1}{4}-\frac{1}{2}$
Change the constants to equivalent fractions with the LCD.	$\frac{1}{8}x=\frac{1}{4}-\frac{2}{4}$
Subtract.	$\frac{1}{8}x=-\frac{1}{4}$
Multiply both sides by the reciprocal of $\frac{1}{8}$.	$\frac{8}{1}\cdot\frac{1}{8}x=\frac{8}{1}\left(-\frac{1}{4}\right)$
Simplify.	$x=-2$

This method worked fine, but many students do not feel very confident when they see all those fractions. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called "clearing" the equation of fractions.

Let's solve a similar equation, but this time use the method that eliminates the fractions.

Example 9.5.1: How to Solve Equations with Fraction Coefficients

Solve: $\frac{1}{6}y - \frac{1}{3} = \frac{5}{6}$

Solution

Step 1. Find the least common denominator of *all* the fractions in the equation.	What is the LCD of $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{5}{6}$?	$\frac{1}{6}y-\frac{1}{3}=\frac{5}{6}$ LCD $=6$
Step 2. Multiply both sides of the equation by that LCD. This clears the fractions.	Multiply both sides of the equation by the LCD 6. Use the Distributive Property. Simplify – and notice, no more fractions!	$6\left(\frac{1}{6}y-\frac{1}{3}\right)=6\left(\frac{5}{6}\right)$ $6\cdot\frac{1}{6}y-6\cdot\frac{1}{3}=6\cdot\frac{5}{6}$ $y-2=5$
Step 3. Solve using the General Strategy for Solving Linear Equations.	To isolate the "y" term, add 2. Simplify.	$y-2+2=5+2$ $y=7$

Try It 9.5.2

Solve: $\frac{1}{4}x+\frac{1}{2}=\frac{5}{8}$

Answer

$x=\frac{1}{2}$

Try It 9.5.3

Solve: $\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$

Answer

$x=-2$

Notice in Exercise 9.5.1, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve! We then used the General Strategy for Solving Linear Equations.

STRATEGY TO SOLVE EQUATIONS WITH FRACTION COEFFICIENTS.

1. Find the least common denominator of *all* the fractions in the equation.
2. Multiply both sides of the equation by that LCD. This clears the fractions.
3. Solve using the General Strategy for Solving Linear Equations.

Example 9.5.4

Solve: $6=\frac{1}{2}v+\frac{2}{5}v-\frac{3}{4}v$

Solution

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the equation.	$6=\frac{1}{2}v+\frac{2}{5}v-\frac{3}{4}v$
The LCD is 20.	
Multiply both sides of the equation by 20.	$20(6)=20\cdot\left(\frac{1}{2}v+\frac{2}{5}v-\frac{3}{4}v\right)$
Distribute.	$20(6)=20\cdot\frac{1}{2}v+20\cdot\frac{2}{5}v-20\cdot\frac{3}{4}v$

Simplify—notice, no more fractions!		$120 = 10v + 8v - 15v$
Combine like terms.		$120 = 3v$
Divide by 3.		$\frac{120}{3} = \frac{3v}{3}$
Simplify.		$40 = v$
Check:	$6 = \frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v$	
Let v=40.	$6 \stackrel{?}{=} \frac{1}{2}(40) + \frac{2}{5}(40) - \frac{3}{4}(40)$	
	$6 \stackrel{?}{=} 20 + 16 - 30$	
	$6 = 6$ ✓	

Try It 9.5.5

Solve: $7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$

Answer

$x = 12$

Try It 9.5.6

Solve: $-1 = \frac{1}{2}u + \frac{1}{4}u - \frac{2}{3}u$

Answer

$u = -12$

In the next example, we again have variables on both sides of the equation.

Example 9.5.7

Solve: $a + \frac{3}{4} = \frac{3}{8}a - \frac{1}{2}$

Solution

	$a + \frac{3}{4} = \frac{3}{8}a - \frac{1}{2}$
Find the LCD of all fractions in the equation. The LCD is 8.	
Multiply both sides by the LCD.	$8\left(a + \frac{3}{4}\right) = 8\left(\frac{3}{8}a - \frac{1}{2}\right)$
Distribute.	$8 \cdot a + 8 \cdot \frac{3}{4} = 8 \cdot \frac{3}{8}a - 8 \cdot \frac{1}{2}$
Simplify—no more fractions.	$8a + 6 = 3a - 4$
Subtract 3a3a from both sides.	$8a - 3a + 6 = 3a - 3a - 4$
Simplify.	$5a + 6 = -4$
Subtract 6 from both sides.	$5a + 6 - 6 = -4 - 6$
Simplify.	$5a = -10$

Divide by 5.	$\frac{5a}{5}=\frac{-10}{5}$
Simplify.	$a=-2$
Check:	$a+\frac{3}{4}=\frac{3}{8}a-\frac{1}{2}$
Let a=−2.	$-2+\frac{3}{4}\stackrel{?}{=}\frac{3}{8}(-2)-\frac{1}{2}$
	$-\frac{8}{4}+\frac{3}{4}\stackrel{?}{=}-\frac{16}{8}-\frac{4}{8}$
	$-\frac{5}{4}=-\frac{10}{8}$
	$-\frac{5}{4}=-\frac{5}{4}\checkmark$

Try It 9.5.8

Solve: $x+\frac{1}{3}=\frac{1}{6}x-\frac{1}{2}$

Answer

$x=-1$

Try It 9.5.9

Solve: $c+\frac{3}{4}=\frac{1}{2}c-\frac{1}{4}$

Answer

$c=-2$

In the next example, we start by using the Distributive Property. This step clears the fractions right away.

Example 9.5.10

Solve: $-5=\frac{1}{4}(8x+4)$

Solution

	$-5=\frac{1}{4}(8x+4)$
Distribute.	$-5=\frac{1}{4}\cdot 8x+\frac{1}{4}\cdot 4$
Simplify. Now there are no fractions.	$-5=2x+1$
Subtract 1 from both sides.	$-5-1=2x+1-1$
Simplify.	$-6=2x$
Divide by 2.	$\frac{-6}{2}=\frac{2x}{2}$
Simplify.	$-3=x$
Check:	$-5=\frac{1}{4}(8x+4)$
Let x=−3.	$-5\stackrel{?}{=}\frac{1}{2}(4(-3)+2)$

	$-5 \stackrel{?}{=} \frac{1}{2}(-12+2)$	
	$-5 \stackrel{?}{=} \frac{1}{2}(-10)$	
	$-5 = -5$ ✓	

Try It 9.5.11

Solve: $-11 = \frac{1}{2}(6p+2)$

Answer

$p = -4$

Try It 9.5.12

Solve: $8 = \frac{1}{3}(9q+6)$

Answer

$q = 2$

In the next example, even after distributing, we still have fractions to clear.

Example 9.5.13

Solve: $\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$

Solution

	$\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$
Distribute.	$\frac{1}{2} \cdot y - \frac{1}{2} \cdot 5 = \frac{1}{4} \cdot y - \frac{1}{4} \cdot 1$
Simplify.	$\frac{1}{2}y - \frac{5}{2} = \frac{1}{4}y - \frac{1}{4}$
Multiply by the LCD, 4.	$4\left(\frac{1}{2}y - \frac{5}{2}\right) = 4\left(\frac{1}{4}y - \frac{1}{4}\right)$
Distribute.	$4 \cdot \frac{1}{2}y - 4 \cdot \frac{5}{2} = 4 \cdot \frac{1}{4}y - 4 \cdot \frac{1}{4}$
Simplify.	$2y - 10 = y - 1$
Collect the variables to the left.	$2y - y - 10 = y - y - 1$
Simplify.	$y - 10 = -1$
Collect the constants to the right.	$y - 10 + 10 = -1 + 10$
Simplify.	$y = 9$
Check:	$\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$
Let y=9.	$\frac{1}{2}(9-5) \stackrel{?}{=} \frac{1}{4}(9-1)$
Finish the check on your own.	

Try It 9.5.14

Solve: $\frac{1}{5}(n+3)=\frac{1}{4}(n+2)$

Answer

$n=2$

Try It 9.5.15

Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$

Answer

$m=-1$

Example 9.5.16

Solve: $\frac{5x-3}{4}=\frac{x}{2}$

Solution

	$\frac{5x-3}{4}=\frac{x}{2}$
Multiply by the LCD, 4.	$4\left(\frac{5x-3}{4}\right)=4\left(\frac{x}{2}\right)$
Simplify.	$5x-3=2x$
Collect the variables to the right.	$5x-5x-3=2x-5x$
Simplify.	$-3=-3x$
Divide.	$\frac{-3}{-3}=\frac{-3x}{-3}$
Simplify.	$1=x$
Check:	$\frac{5x-3}{4}=\frac{x}{2}$
Let x=1.	$\frac{5(1)-3}{4}\stackrel{?}{=}\frac{1}{2}$
	$\frac{2}{4}\stackrel{?}{=}\frac{1}{2}$
	$\frac{1}{2}=\frac{1}{2}$ ✓

Try It 9.5.17

Solve: $\frac{4y-7}{3}=\frac{y}{6}$

Answer

$y=2$

Try It 9.5.18

Solve: $\frac{-2z-5}{4}=\frac{z}{8}$

Answer

$z = -2$

Example 9.5.19

Solve: $\frac{a}{6} + 2 = \frac{a}{4} + 3$

Solution

	$\frac{a}{6} + 2 = \frac{a}{4} + 3$
Multiply by the LCD, 12.	$12\left(\frac{a}{6} + 2\right) = 12\left(\frac{a}{4} + 3\right)$
Distribute.	$12 \cdot \frac{a}{6} + 12 \cdot 2 = 12 \cdot \frac{a}{4} + 12 \cdot 3$
Simplify.	$2a + 24 = 3a + 36$
Collect the variables to the right.	$2a - 2a + 24 = 3a - 2a + 36$
Simplify.	$24 = a + 36$
Collect the constants to the left.	$24 - 36 = a + 36 - 36$
Simplify.	$a = -12$
Check:	$\frac{a}{6} + 2 = \frac{a}{4} + 3$
Let a=−12.	$\frac{-12}{6} + 2 \stackrel{?}{=} \frac{-12}{4} + 3$
	$-2 + 2 \stackrel{?}{=} -3 + 3$
	$0 = 0$ ✓

Try It 9.5.20

Solve: $\frac{b}{10} + 2 = \frac{b}{4} + 5$

Answer

$b = -20$

Try It 9.5.21

Solve: $\frac{c}{6} + 3 = \frac{c}{3} + 4$

Answer

$c = -6$

Example 9.5.22

Solve: $\frac{4q+3}{2} + 6 = \frac{3q+5}{4}$

Solution

	$\frac{4q+3}{2} + 6 = \frac{3q+5}{4}$
Multiply by the LCD, 4.	$4\left(\frac{4q+3}{2} + 6\right) = 4\left(\frac{3q+5}{4}\right)$

Distribute.	$4\left(\frac{4q+3}{2}\right)+4\cdot 6=4\cdot\left(\frac{3q+5}{4}\right)$
Simplify.	$2(4q+3)+24=3q+5$ $8q+6+24=3q+5$ $8q+30=3q+5$
Collect the variables to the left.	$8q-3q+30=3q-3q+5$
Simplify.	$5q+30=5$
Collect the constants to the right.	$5q+30-30=5-30$
Simplify.	$5q=-25$
Divide by 5.	$\frac{5q}{5}=\frac{-25}{5}$
Simplify.	$q=-5$
Check:	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$
Let q=−5.	$\frac{4(-5)+3}{2}+6\stackrel{?}{=}\frac{3(-5)+5}{4}$
Finish the check on your own.	

Try It 9.5.23

Solve: $\frac{3r+5}{6}+1=\frac{4r+3}{3}$

Answer

$r=1$

Try It 9.5.24

Solve: $\frac{2s+3}{2}+1=\frac{3s+2}{4}$

Answer

$s=-8$

Solve Equations with Decimal Coefficients

Some equations have decimals in them. This kind of equation will occur when we solve problems dealing with money or percentages. But decimals can also be expressed as fractions. For example, $0.3=\frac{3}{10}$ and $0.17=\frac{17}{100}$. So, with an equation with decimals, we can use the same method we used to clear fractions—multiply both sides of the equation by the least common denominator.

Example 9.5.25

Solve: $0.06x+0.02=0.25x-1.5$

Solution

Look at the decimals and think of the equivalent fractions.

$0.06=\frac{6}{100}\quad 0.02=\frac{2}{100}\quad 0.25=\frac{25}{100}\quad 1.5=1\frac{5}{10}$

Notice, the LCD is 100.

By multiplying by the LCD, we will clear the decimals from the equation.

	$0.06x + 0.02 = 0.25x - 1.5$
Multiply both side by 100.	$100(0.06x + 0.02) = 100(0.25x - 1.5)$
Distribute.	$100(0.06x) + 100(0.02) = 100(0.25x) - 100(1.5)$
Multiply, and now we have no more decimals.	$6x + 2 = 25x - 150$
Collect the variables to the right.	$6x - 6x + 2 = 25x - 6x - 150$
Simplify.	$2 = 19x - 150$
Collect the variables to the right.	$2 + 150 = 19x - 150 + 150$
Simplify.	$152 = 19x$
Divide by 19.	$\frac{152}{19} = \frac{19x}{19}$
Simplify.	$8 = x$
Check: Let x=8 $0.06(8) + 0.02 \stackrel{?}{=} 0.25(8) - 1.5$ $0.48 + 0.02 \stackrel{?}{=} 2.00 - 1.5$ $0.50 = 0.50$ ✓	

Try It 9.5.26

Solve: $0.14h + 0.12 = 0.35h - 2.4$

Answer

$h = 12$

Try It 9.5.27

Solve: $0.65k - 0.1 = 0.4k - 0.35$

Answer

$k = -1$

The next example uses an equation that is typical of the money applications in the next chapter. Notice that we distribute the decimal before we clear all the decimals.

Example 9.5.28

Solve: $0.25x + 0.05(x + 3) = 2.85$

Solution

	$0.25x + 0.05(x + 3) = 2.85$
Distribute first.	$0.25x + 0.05x + 0.15 = 2.85$
Combine like terms.	$0.30x + 0.15 = 2.85$
To clear decimals, multiply by 100.	$100(0.30x + 0.15) = 100(2.85)$
Distribute.	$30x + 15 = 285$
Subtract 15 from both sides.	$30x + 15 - 15 = 285 - 15$

Simplify.	$30x = 270$
Divide by 30.	$\frac{30x}{30} = \frac{270}{30}$
Simplify.	$x = 9$
Check it yourself by substituting x=9 into the original equation.	

Try It 9.5.29

Solve: $0.25n + 0.05(n + 5) = 2.95$

Answer

$n = 9$

Try It 9.5.30

Solve: $0.10d + 0.05(d - 5) = 2.15$

Answer

$d = 16$

Key Concepts

- **Strategy to Solve an Equation with Fraction Coefficients**
  1. Find the least common denominator of all the fractions in the equation.
  2. Multiply both sides of the equation by that LCD. This clears the fractions.
  3. Solve using the General Strategy for Solving Linear Equations.

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9.5E: Exercises

Practice Makes Perfect

Solve Equations with Fraction Coefficients

In the following exercises, solve each equation with fraction coefficients.

Exercise 9.5E.1

$\frac{1}{4}x - \frac{1}{2} = -\frac{3}{4}$

Exercise 9.5E.2

$\frac{3}{4}x - \frac{1}{2} = \frac{1}{4}$

Answer

x=1

Exercise 9.5E.3

$\frac{5}{6}y - \frac{2}{3} = -\frac{3}{2}$

Exercise 9.5E.4

$\frac{5}{6}y - \frac{1}{3} = -\frac{7}{6}$

Answer

$y = -1$

Exercise 9.5E.5

$\frac{1}{2}a + \frac{3}{8} = \frac{3}{4}$

Exercise 9.5E.6

$\frac{5}{8}b + \frac{1}{2} = -\frac{3}{4}$

Answer

$b = -2$

Exercise 9.5E.7

$2 = \frac{1}{3}x - \frac{1}{2}x + \frac{2}{3}x$

Exercise 9.5E.8

$2 = \frac{3}{5}x - \frac{1}{3}x + \frac{2}{5}x$

Answer

$x = 3$

Exercise 9.5E.9

$\frac{1}{4}m - \frac{4}{5}m + \frac{1}{2}m = -1$

Exercise 9.5E.10

$\frac{5}{6}n - \frac{1}{4}n - \frac{1}{2}n = -2$

Answer

$n = -24$

Exercise 9.5E.11

$x + \frac{1}{2} = \frac{2}{3}x - \frac{1}{2}$

Exercise 9.5E.12

$x + \frac{3}{4} = \frac{1}{2}x - \frac{5}{4}$

Answer

$x = -4$

Exercise 9.5E.13

$\frac{1}{3}w + \frac{5}{4} = w - \frac{1}{4}$

Exercise 9.5E.14

$\frac{3}{2}z + \frac{1}{3} = z - \frac{2}{3}$

Answer

$z = -2$

Exercise 9.5E.15

$\frac{1}{2}x - \frac{1}{4} = \frac{1}{12}x + \frac{1}{6}$

Exercise 9.5E.16

$\frac{1}{2}a - \frac{1}{4} = \frac{1}{6}a + \frac{1}{12}$

Answer

$a = 1$

Exercise 9.5E.17

$\frac{1}{3}b + \frac{1}{5} = \frac{2}{5}b - \frac{3}{5}$

Exercise 9.5E.18

$\frac{1}{3}x+\frac{2}{5}=\frac{1}{5}x-\frac{2}{5}$

Answer

$x=-6$

Exercise 9.5E.19

$1=\frac{1}{6}(12x-6)$

Exercise 9.5E.20

$1=\frac{1}{5}(15x-10)$

Answer

$x=1$

Exercise 9.5E.21

$\frac{1}{4}(p-7)=\frac{1}{3}(p+5)$

Exercise 9.5E.22

$\frac{1}{5}(q+3)=\frac{1}{2}(q-3)$

Answer

$q=7$

Exercise 9.5E.23

$\frac{1}{2}(x+4)=\frac{3}{4}$

Exercise 9.5E.24

$\frac{1}{3}(x+5)=\frac{5}{6}$

Answer

$x=-\frac{5}{2}$

Exercise 9.5E.25

$\frac{5q-8}{5}=\frac{2q}{10}$

Exercise 9.5E.26

$\frac{4m+2}{6}=\frac{m}{3}$

Answer

$m=-1$

Exercise 9.5E.27

$\frac{4n+8}{4} = \frac{n}{3}$

Exercise 9.5E.28

$\frac{3p+6}{3} = \frac{p}{2}$

Answer

$p = -4$

Exercise 9.5E.29

$\frac{u}{3} - 4 = \frac{u}{2} - 3$

Exercise 9.5E.30

$\frac{v}{10} + 1 = \frac{v}{4} - 2$

Answer

$v = 20$

Exercise 9.5E.31

$\frac{c}{15} + 1 = \frac{c}{10} - 1$

Exercise 9.5E.32

$\frac{d}{6} + 3 = \frac{d}{8} + 2$

Answer

$d = -24$

Exercise 9.5E.33

$\frac{3x+4}{2} + 1 = \frac{5x+10}{8}$

Exercise 9.5E.34

$\frac{10y-2}{3} + 3 = \frac{10y+1}{9}$

Answer

$y = -1$

Exercise 9.5E.35

$\frac{7u-1}{4} - 1 = \frac{4u+8}{5}$

Exercise 9.5E.36

$\frac{3v-6}{2}+5=\frac{11v-4}{5}$

Answer

$v=4$

Solve Equations with Decimal Coefficients

In the following exercises, solve each equation with decimal coefficients.

Exercise 9.5E.37

$0.6y+3=9$

Exercise 9.5E.38

$0.4y-4=2$

Answer

$y=15$

Exercise 9.5E.39

$3.6j-2=5.2$

Exercise 9.5E.40

$2.1k+3=7.2$

Answer

$k=2$

Exercise 9.5E.41

$0.4x+0.6=0.5x-1.2$

Exercise 9.5E.42

$0.7x+0.4=0.6x+2.4$

Answer

$x=20$

Exercise 9.5E.43

$0.23x+1.47=0.37x-1.05$

Exercise 9.5E.44

$0.48x+1.56=0.58x-0.64$

Answer

$x = 22$

Exercise 9.5E.45

$0.9x - 1.25 = 0.75x + 1.75$

Exercise 9.5E.46

$1.2x - 0.91 = 0.8x + 2.29$

Answer

$x = 8$

Exercise 9.5E.47

$0.05n + 0.10(n + 8) = 2.15$

Exercise 9.5E.48

$0.05n + 0.10(n + 7) = 3.55$

Answer

$n = 19$

Exercise 9.5E.49

$0.10d + 0.25(d + 5) = 4.05$

Exercise 9.5E.50

$0.10d + 0.25(d + 7) = 5.25$

Answer

$d = 10$

Exercise 9.5E.51

$0.05(q - 5) + 0.25q = 3.05$

Exercise 9.5E.52

$0.05(q - 8) + 0.25q = 4.10$

Answer

$q = 15$

Everyday Math

Exercise 9.5E.53

Coins Taylor has $200 in dimes and pennies. The number of pennies is 2 more than the number of dimes. Solve the equation $0.10d+0.01(d+2)=2$ for d, the number of dimes.

Exercise 9.5E.54

Stamps Paula bought $22.82 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was 8 less than the number of 49-cent stamps. Solve the equation $0.49s+0.21(s-8)=22.82$ for s, to find the number of 49-cent stamps Paula bought.

Answer

$s=35$

Writing Exercises

Exercise 9.5E.55

Explain how you find the least common denominator of $\frac{3}{8}$, $\frac{1}{6}$, and $\frac{2}{3}$

Exercise 9.5E.56

If an equation has several fractions, how does multiplying both sides by the LCD make it easier to solve?

Answer

Answers will vary.

Exercise 9.5E.57

If an equation has fractions only on one side, why do you have to multiply both sides of the equation by the LCD?

Exercise 9.5E.58

In the equation $0.35x+2.1=3.85$ what is the LCD? How do you know?

Answer

100. Justifications will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve equations with fraction coefficients.			
solve equations with decimal coefficients.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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9.6: Solve a Formula for a Specific Variable

Learning Objectives

By the end of this section, you will be able to:

- Use the Distance, Rate, and Time formula
- Solve a formula for a specific variable

Quiz

Before you get started, take this readiness quiz.

1. Solve: $15t = 120$.
   If you missed this problem, review Exercise 2.2.1.
2. Solve: $6x + 24 = 96$.
   If you missed this problem, review Exercise 2.3.1.

Use the Distance, Rate, and Time Formula

One formula you will use often in algebra and in everyday life is the formula for distance traveled by an object moving at a constant rate. Rate is an equivalent word for "speed." The basic idea of rate may already familiar to you. Do you know what distance you travel if you drive at a steady rate of 60 miles per hour for 2 hours? (This might happen if you use your car's cruise control while driving on the highway.) If you said 120 miles, you already know how to use this formula!

DISTANCE, RATE, AND TIME

For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula:

$$d = rt \quad \text{where} \quad \begin{aligned} d &= \text{distance} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned} \tag{9.6.1}$$

We will use the Strategy for Solving Applications that we used earlier in this chapter. When our problem requires a formula, we change Step 4. In place of writing a sentence, we write the appropriate formula. We write the revised steps here for reference.

SOLVE AN APPLICATION (WITH A FORMULA).

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

You may want to create a mini-chart to summarize the information in the problem. See the chart in this first example.

Example 9.6.1

Jamal rides his bike at a uniform rate of 12 miles per hour for $3\frac{1}{2}$ hours. What distance has he traveled?

Solution

Step 1. Read the problem.	

Step 2. Identify what you are looking for.	distance traveled
Step 3. Name. Choose a variable to represent it.	Let d = distance.
Step 4. Translate: Write the appropriate formula.	$d = rt$
	$d = ?$ $r = 12$ mph $t = 3\frac{1}{2}$ hours
Substitute in the given information.	$d = 12 \cdot 3\frac{1}{2}$
Step 5. Solve the equation.	$d = 42$ miles
Step 6. Check	
Does 42 miles make sense?	
Jamal rides:	
12 miles in 1 hour, 24 miles in 2 hours, 36 miles in 3 hours, 42 miles in $3\frac{1}{2}$ hours is reasonable 48 miles in 4 hours.	
Step 7. Answer the question with a complete sentence.	Jamal rode 42 miles.

Try It 9.6.2

Lindsay drove for $5\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Answer

330 miles

Try It 9.6.3

Trinh walked for $2\frac{1}{3}$ hours at 3 miles per hour. How far did she walk?

Answer

7 miles

Example 9.6.4

Rey is planning to drive from his house in San Diego to visit his grandmother in Sacramento, a distance of 520 miles. If he can drive at a steady rate of 65 miles per hour, how many hours will the trip take?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	How many hours (time)
Step 3. Name. Choose a variable to represent it.	Let t = time.
	$d = 520$ miles $r = 65$ mph $t = ?$ hours

Step 4. Translate. Write the appropriate formula.	$d = rt$
Substitute in the given information.	$520 = 65t$
Step 5. Solve the equation.	$t = 8$
Step 6. Check. Substitute the numbers into the formula and make sure the result is a true statement.	
$d = rt$ $520 \stackrel{?}{=} 65 \cdot 8$ $520 = 520 \checkmark$	
Step 7. Answer the question with a complete sentence. Rey's trip will take 8 hours.	

Try It 9.6.5

Lee wants to drive from Phoenix to his brother's apartment in San Francisco, a distance of 770 miles. If he drives at a steady rate of 70 miles per hour, how many hours will the trip take?

Answer

11 hours

Try It 9.6.6

Yesenia is 168 miles from Chicago. If she needs to be in Chicago in 3 hours, at what rate does she need to drive?

Answer

56 mph

Solve a Formula for a Specific Variable

You are probably familiar with some geometry formulas. A formula is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be familiar with formulas and be able to manipulate them easily.

In Exercise 9.6.1 and Exercise 9.6.4, we used the formula $d = rt$. This formula gives the value of d, distance, when you substitute in the values of r and t, the rate and time. But in Exercise 9.6.4, we had to find the value of t. We substituted in values of d and r and then used algebra to solve for tt. If you had to do this often, you might wonder why there is not a formula that gives the value of t when you substitute in the values of d and r. We can make a formula like this by solving the formula $d = rt$ for t.

To solve a formula for a specific variable means to isolate that variable on one side of the equals sign with a coefficient of 1. All other variables and constants are on the other side of the equals sign. To see how to solve a formula for a specific variable, we will start with the distance, rate and time formula.

Example 9.6.7

Solve the formula d=rt for t:

1. when d=520 and r=65
2. in general

Solution

We will write the solutions side-by-side to demonstrate that solving a formula in general uses the same steps as when we have numbers to substitute.

1. when d=520 and r=65		2. in general	
Write the formula.	$d = rt$	Write the formula.	$d = rt$
Substitute.	$520 = 65t$		
Divide, to isolate t.	$\frac{520}{65} = \frac{65t}{65}$	Divide, to isolate tt.	$\frac{d}{r} = \frac{rt}{t}$
Simplify.	$8 = t$	Simplify.	$\frac{d}{r} = t$

We say the formula $t = \frac{d}{r}$ is solved for t.

Try It 9.6.8

Solve the formula $d = rt$ for r:

1. when d=180 and t=4
2. in general

Answer

1. $r = 45$
2. $r = \frac{d}{t}$

Try It 9.6.9

Solve the formula $d = rt$ for r:

1. when d=780 and t=12
2. in general

Answer

1. $r = 65$
2. \(r = \frac{d}{rt}\)

Example 9.6.10

Solve the formula $A = \frac{1}{2}bh$ for h:

1. when $A = 90$ and $b = 15$
2. in general

Solution

1. when $A = 90$ and $b = 15$		2. in general	
Write the formula.	$A = \frac{1}{2}bh$	Write the formula.	$A = \frac{1}{2}bh$
Substitute.	$90 = \frac{1}{2} \cdot 15 \cdot h$		
Clear the fractions.	$2 \cdot 90 = 2 \cdot \frac{1}{2}15h$	Clear the fractions.	$2 \cdot A = 2 \cdot \frac{1}{2}bh$
Simplify.	$180 = 15h$	Simplify.	$2A = bh$
Solve for h.	$12 = h$	Solve for hh.	$\frac{2A}{b} = h$

We can now find the height of a triangle, if we know the area and the base, by using the formula $h = \frac{2A}{b}$

Try It 9.6.11

Solve the formula $A = \frac{1}{2}bh$ for h:

1. when $A = 170$ and $b = 17$
2. in general

Answer

1. $h = 20$
2. $h = \frac{2A}{b}$

Try It 9.6.12

Solve the formula $A = \frac{1}{2}bh$ for h:

1. when $A = 62$ and $h = 31$
2. in general

Answer

1. $b = 4$
2. $b = \frac{2A}{h}$

The formula $I = Prt$ is used to calculate simple interest, *I*, for a principal, *P*, invested at rate, *r*, for *t* years.

Example 9.6.13

Solve the formula I=Prt to find the principal, P:

1. when I=$5,600, r=4% ,t=7years
2. in general

Solution

1. I=$5,600, r=4% ,t=7years		2. in general	
Write the formula.	$I = Prt$	Write the formula.	$I = Prt$
Substitute.	$5600 = P(0.04)(7)$		
Simplify.	$5600 = P(0.28)$	Simplify.	$I = P(rt)$
Divide, to isolate *P*.	$\frac{5600}{0.28} = \frac{P(0.28)}{0.28}$	Divide, to isolate *P*.	$\frac{I}{rt} = \frac{P(rt)}{rt}$
Simplify.	$20,000 = P$	Simplify.	$\frac{I}{rt} = P$
The principal is	$20,000		$P = \frac{I}{rt}$

Try It 9.6.14

Solve the formula I=Prt to find the principal, P:

1. when I=$2160, r=6% ,t=3 years
2. in general

Answer

1. $12000
2. $P = \frac{1}{rt}$

Try It 9.6.15

Solve the formula I=Prt to find the principal, P:

1. when I=$5400, r= 12% ,t=5 years
2. in general

Answer

1. $9000
2. $P = \frac{1}{rt}$

Later in this class, and in future algebra classes, you'll encounter equations that relate two variables, usually *x* and *y*. You might be given an equation that is solved for *y* and need to solve it for *x*, or vice versa. In the following example, we're given an equation with both *x* and *y* on the same side and we'll solve it for *y*.

Example 9.6.16

Solve the formula 3x+2y=18 for *y*:

1. when x=4
2. in general

Solution

1. when x=4		2. in general	
	$3x + 2y = 18$		$3x + 2y = 18$
Substitute.	$3(4) + 2y = 18$		
Subtract to isolate the y-term.	$12 - 12 + 2y = 18 - 12$	Subtract to isolate the y-term.	$3x - 3x + 2y = 18 - 3x$
Divide.	$\frac{2y}{2} = \frac{6}{2}$	Divide.	$\frac{2y}{2} = \frac{18}{2} - \frac{3x}{2}$
Simplify.	$y = 3$	Simplify.	$y = -\frac{3x}{2} + 9$

Try It 9.6.17

Solve the formula 3x+4y=10 for *y*:

1. when $x = \frac{14}{3}$
2. in general

Answer

1. $y = -1$
2. $y = \frac{10-3x}{4}$

Try It 9.6.18

Solve the formula 5x+2y=18 for *y*:

1. when $x = 4$

2. in general

Answer

1. $y = -1$
2. $y = \frac{18-5x}{2}$

In Exercise $9.6.7$ through Exercise $9.6.18$ we used the numbers in part 1 as a guide to solving in general in part 2. Now we will solve a formula in general without using numbers as a guide.

Example 9.6.19

Solve the formula P=a+b+c for a.

Solution

We will isolate aa on one side of the equation.	$P = a + b + c$
Both b and c are added to a, so we subtract them from both sides of the equation.	$P - b - c = a + b + c - b - c$
Simplify.	$P - b - c = a$ $a = P - b - c$

Try It 9.6.20

Solve the formula P=a+b+c for *b*.

Answer

b=P−a−c

Try It 9.6.21

Solve the formula P=a+b+c for *c*.

Answer

c=P−a−b

Example 9.6.22

Solve the formula 6x+5y=13 for *y*.

Solution

	$6x + 5y = 13$
Subtract 6x from both sides to isolate the term with y.	$6x - 6x + 5y = 13 - 6x$
Simplify.	$5y = 13 - 6x$
Divide by 5 to make the coefficient 1.	$\frac{5y}{5} = \frac{13-6x}{5}$
Simplify.	$y = \frac{13-6x}{5}$

The fraction is simplified. We cannot divide 13−6x by 5.

? Try It 9.6.23

Solve the formula 4x+7y=9 for *y*.

Answer

$y = \frac{9-4x}{7}$

? Try It 9.6.24

Solve the formula 5x+8y=1 for *y*.

Answer

$y = \frac{1-5x}{8}$

Key Concepts

- **To Solve an Application (with a formula)**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute in the given information.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- **Distance, Rate and Time**
  For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula: d=rt where *d* = distance, *r* = rate, *t* = time.
- **To solve a formula for a specific variable** means to get that variable by itself with a coefficient of 1 on one side of the equation and all other variables and constants on the other side.

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9.6E: Exercises

Practice Makes Perfect

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

Exercise 9.6E.1

Steve drove for $8\frac{1}{2}$ hours at 72 miles per hour. How much distance did he travel?

Exercise 9.6E.2

Socorro drove for $4\frac{5}{6}$ hours at 60 miles per hour. How much distance did she travel?

Answer

290 miles

Exercise 9.6E.3

Yuki walked for $1\frac{3}{4}$ hours at 4 miles per hour. How far did she walk?

Exercise 9.6E.4

Francie rode her bike for $2\frac{1}{2}$ hours at 12 miles per hour. How far did she ride?

Answer

30 miles

Exercise 9.6E.5

Connor wants to drive from Tucson to the Grand Canyon, a distance of 338 miles. If he drives at a steady rate of 52 miles per hour, how many hours will the trip take?

Exercise 9.6E.6

Megan is taking the bus from New York City to Montreal. The distance is 380 miles and the bus travels at a steady rate of 76 miles per hour. How long will the bus ride be?

Answer

5 hours

Exercise 9.6E.7

Aurelia is driving from Miami to Orlando at a rate of 65 miles per hour. The distance is 235 miles. To the nearest tenth of an hour, how long will the trip take?

Exercise 9.6E.8

Kareem wants to ride his bike from St. Louis to Champaign, Illinois. The distance is 180 miles. If he rides at a steady rate of 16 miles per hour, how many hours will the trip take?

Answer

11.25 hours

Exercise 9.6E.9

Javier is driving to Bangor, 240 miles away. If he needs to be in Bangor in 4 hours, at what rate does he need to drive?

Exercise 9.6E.10

Alejandra is driving to Cincinnati, 450 miles away. If she wants to be there in 6 hours, at what rate does she need to drive?

Answer

75 mph

Exercise 9.6E.11

Aisha took the train from Spokane to Seattle. The distance is 280 miles and the trip took 3.5 hours. What was the speed of the train?

Exercise 9.6E.12

Philip got a ride with a friend from Denver to Las Vegas, a distance of 750 miles. If the trip took 10 hours, how fast was the friend driving?

Answer

75 mph

Solve a Formula for a Specific Variable

In the following exercises, use the formula $d = rt$.

Exercise 9.6E.13

Solve for t

1. when $d = 350$ and $r = 70$
2. in general

Exercise 9.6E.14

Solve for t

1. when $d = 240$ and $r = 60$
2. in general

Answer

1. $t = 4$
2. $t = \frac{d}{r}$

Exercise 9.6E.15

Solve for t

1. when $d = 510$ and $r = 60$

2. in general

Exercise 9.6E.16

Solve for t

1. when $d = 175$ and $r = 50$
2. in general

Answer

1. $t = 3.5$
2. $t = \frac{d}{r}$

Exercise 9.6E.17

Solve for r

1. when $d = 204$ and $t = 3$
2. in general

Exercise 9.6E.18

Solve for r

1. when $d = 420$ and $t = 6$
2. in general

Answer

1. $r = 70$
2. $r = \frac{d}{t}$

Exercise 9.6E.19

Solve for r

1. when $d = 160$ and $t = 2.5$
2. in general

Exercise 9.6E.20

Solve for r

1. when $d = 180$ and $t = 4.5$
2. in general

Answer

1. $r = 40$
2. $r = \frac{d}{t}$

In the following exercises, use the formula $A = \frac{1}{2}bh$

Exercise 9.6E.21

Solve for b

1. when $A = 126$ and $h = 18$
2. in general

Exercise 9.6E.22

Solve for h

1. when $A = 176$ and $b = 22$
2. in general

Answer

1. $h = 16$
2. $h = \frac{2A}{b}$

Exercise 9.6E.23

Solve for h

1. when $A = 375$ and $b = 25$
2. in general

Exercise 9.6E.24

Solve for b

1. when $A = 65$ and $h = 13$
2. in general

Answer

1. $b = 10$
2. $b = \frac{2A}{h}$

In the following exercises, use the formula $I = Prt$.

Exercise 9.6E.25

Solve for the principal, P for

1. $I = \$5,480$, $r = 4\%$, $t = 7$ years
2. in general

Exercise 9.6E.26

Solve for the principal, P for

1. $I = \$3,950$, $r = 6\%$, $t = 5$ years
2. in general

Answer

1. $P = \$13,166.67$
2. $P = \frac{I}{rt}$

Exercise 9.6E.27

Solve for the time, t for

1. $I = \$2,376$, $P = \$9,000$, $r = 4.4\%$
2. in general

Exercise 9.6E.28

Solve for the time, t for

1. $I = \$624$, $P = \$6,000$, $r = 5.2\%$
2. in general

Answer

1. $t = 2$ years
2. $t = \frac{I}{Pr}$

In the following exercises, solve.

Exercise 9.6E.29

Solve the formula $2x + 3y = 12$ for y

1. when $x = 3$
2. in general

Exercise 9.6E.30

Solve the formula $5x + 2y = 10$ for y

1. when $x = 4$
2. in general

Answer

1. $y = -5$
2. $y = \frac{10-5x}{2}$

Exercise 9.6E.31

Solve the formula $3x - y = 7$ for y

1. when $x = -2$
2. in general

Exercise 9.6E.32

Solve the formula $4x + y = 5$ for y

1. when $x = -3$
2. in general

Answer

1. $y = 17$
2. $y = 5 - 4x$

Exercise 9.6E.33

Solve $a+b=90$ for b.

Exercise 9.6E.34

Solve $a+b=90$ for a

Answer

$a=90-b$

Exercise 9.6E.35

Solve $180=a+b+c$ for a

Exercise 9.6E.36

Solve $180=a+b+c$ for c

Answer

$c=180-a-b$

Exercise 9.6E.37

Solve the formula $8x+y=15$ for y

Exercise 9.6E.38

Solve the formula $9x+y=13$ for y

Answer

$y=13-9x$

Exercise 9.6E.39

Solve the formula $-4x+y=-6$ for y

Exercise 9.6E.40

Solve the formula $-5x+y=-1$ for y

Answer

$y=-1+5x$

Exercise 9.6E.41

Solve the formula $4x+3y=7$ for y

Exercise 9.6E.42

Solve the formula $3x+2y=11$ for y

Answer

$y=\frac{11-3x}{2}$

Exercise 9.6E.43

Solve the formula $x-y=-4$ for y

Exercise 9.6E.44

Solve the formula $x-y=-3$ for y

Answer

$y=3+x$

Exercise 9.6E.45

Solve the formula $P=2L+2W$ for L

Exercise 9.6E.46

Solve the formula $P=2L+2W$ for W

Answer

$W=\frac{P-2L}{2}$

Exercise 9.6E.47

Solve the formula $C=\pi d$ for d

Exercise 9.6E.48

Solve the formula $C=\pi d$ for π

Answer

$\pi=\frac{C}{d}$

Exercise 9.6E.49

Solve the formula $V=LWH$ for L

Exercise 9.6E.50

Solve the formula $V=LWH$ for H

Answer

$H=\frac{V}{LW}$

Everyday Math

Exercise 9.6E.51

Converting temperature While on a tour in Greece, Tatyana saw that the temperature was 40° Celsius. Solve for F in the formula $C = \frac{5}{9}(F - 32)$ to find the Fahrenheit temperature.

Exercise 9.6E.52

Converting temperature Yon was visiting the United States and he saw that the temperature in Seattle one day was 50ºFahrenheit. Solve for C in the formula $F = \frac{9}{5}C + 32$ to find the Celsius temperature.

Answer

10°C

Writing Exercises

Exercise 9.6E.53

Solve the equation $2x + 3y = 6$ for y

1. when $x = -3$
2. in general
3. Which solution is easier for you, 1 or 2? Why?

Exercise 9.6E.54

Solve the equation $5x - 2y = 10$ for x

1. when $y = 10$
2. in general
3. Which solution is easier for you, 1 or 2? Why?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the distance, rate, and time formula.			
solve a formula for a specific variable.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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9.7: Solve Linear Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Graph inequalities on the number line
- Solve inequalities using the Subtraction and Addition Properties of inequality
- Solve inequalities using the Division and Multiplication Properties of inequality
- Solve inequalities that require simplification
- Translate to an inequality and solve

Note

Before you get started, take this readiness quiz.

1. Translate from algebra to English: $15 > x$.
   If you missed this problem, review Exercise 1.3.1.
2. Solve: $n - 9 = -42$.
   If you missed this problem, review Exercise 2.1.7.
3. Solve: $-5p = -23$.
   If you missed this problem, review Exercise 2.2.1.
4. Solve: $3a - 12 = 7a - 20$.
   If you missed this problem, review Exercise 2.3.22.

Graph Inequalities on the Number Line

Do you remember what it means for a number to be a solution to an equation? A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

What about the solution of an inequality? What number would make the inequality $x > 3$ true? Are you thinking, 'x could be 4'? That's correct, but x could be 5 too, or 20, or even 3.001. Any number greater than 3 is a solution to the inequality $x > 3$.

We show the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of 3, to show that all numbers greater than 3 are solutions. Because the number 3 itself is not a solution, we put an open parenthesis at 3. The graph of $x > 3$ is shown in Figure 9.7.1. Please note that the following convention is used: light blue arrows point in the positive direction and dark blue arrows point in the negative direction.

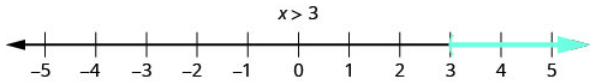


Figure 9.7.1: The inequality $x > 3$ is graphed on this number line.

The graph of the inequality $x \geq 3$ is very much like the graph of $x > 3$, but now we need to show that 3 is a solution, too. We do that by putting a bracket at $x = 3$, as shown in Figure 9.7.2.

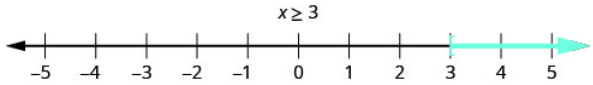


Figure 9.7.2: The inequality $x \geq 3$ is graphed on this number line.

Notice that the open parentheses symbol, (, shows that the endpoint of the inequality is not included. The open bracket symbol, [, shows that the endpoint is included.

Exercise 9.7.1

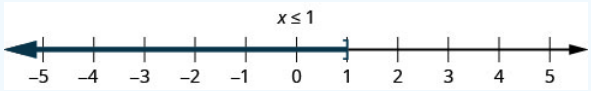
Graph on the number line:

1. $x \leq 1$
2. $x < 5$

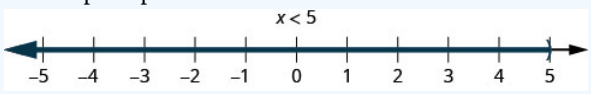
3. $x > -1$

Answer

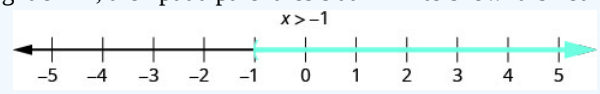
1. $x \leq 1$ This means all numbers less than or equal to 1. We shade in all the numbers on the number line to the left of 1 and put a bracket at x=1 to show that it is included.



2. $x < 5$ This means all numbers less than 5, but not including 5. We shade in all the numbers on the number line to the left of 5 and put a parenthesis at x=5 to show it is not included.



3. $x > -1$ This means all numbers greater than −1, but not including −1. We shade in all the numbers on the number line to the right of −1, then put a parenthesis at x=−1 to show it is not included.

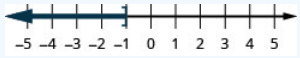


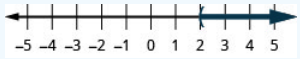
Exercise 9.7.2

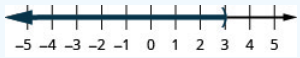
Graph on the number line:

1. $x \leq -1$
2. $x > 2$
3. $x < 3$

Answer

1. 

2. 

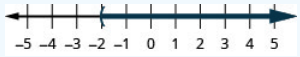
3. 

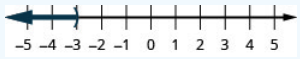
Exercise 9.7.3

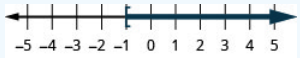
Graph on the number line:

1. $x > -2$
2. $x < -3$
3. $x \geq -1$

Answer

1. 

2. 

3. 

We can also represent inequalities using *interval notation.* As we saw above, the inequality $x > 3$ means all numbers greater than 3. There is no upper end to the solution to this inequality. In **interval notation**, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as 'infinity'. It is not an actual number. Figure $9.7.3$ shows both the number line and the interval notation.

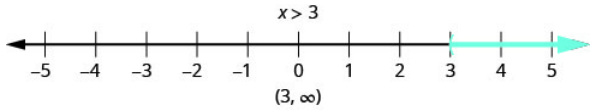


Figure 9.7.3: The inequality $x > 3$ is graphed on this number line and written in interval notation.

The inequality $x \leq 1$ means all numbers less than or equal to 1. There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as 'negative infinity'. Figure 9.7.4 shows both the number line and interval notation.

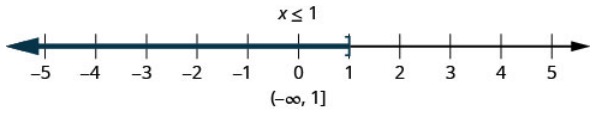


Figure 9.7.4: The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

INEQUALITIES, NUMBER LINES, AND INTERVAL NOTATION

Did you notice how the parenthesis or bracket in the interval notation matches the symbol at the endpoint of the arrow? These relationships are shown in Figure 9.7.5.

Figure 9.7.5: The notation for inequalities on a number line and in interval notation use similar symbols to express the endpoints of intervals.

Exercise 9.7.4

Graph on the number line and write in interval notation.

1. $x \geq -3$
2. $x < 2.5$
3. $x \leq \frac{3}{5}$

Answer

1.

	$x \geq -3$
Shade to the right of −3, and put a bracket at −3.	
Write in interval notation.	[–3, ∞)

2.

	$x < 2.5$
Shade to the left of 2.5, and put a parenthesis at 2.5.	
Write in interval notation.	(–∞, 2.5)

3.

	$x \leq -\frac{3}{5}$

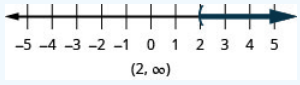
Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	
Write in interval notation.	$\left(-\infty, -\frac{3}{5}\right]$

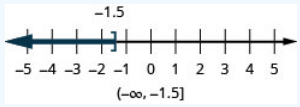
Exercise 9.7.5

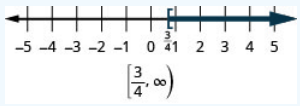
Graph on the number line and write in interval notation:

1. $x > 2$
2. $x \leq -1.5$
3. $x \geq \frac{3}{4}$

Answer

1. 

2. 

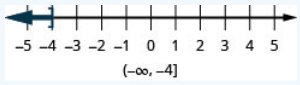
3. 

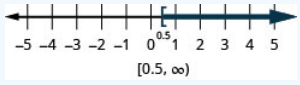
Exercise 9.7.6

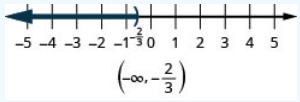
Graph on the number line and write in interval notation:

1. $x \leq -4$
2. $x \geq 0.5$
3. $x < -\frac{2}{3}$

Answer

1. 

2. 

3. 

Solve Inequalities using the Subtraction and Addition Properties of Inequality

The Subtraction and Addition Properties of Equality state that if two quantities are equal, when we add or subtract the same amount from both quantities, the results will be equal.

PROPERTIES OF EQUALITY

Subtraction Property of Equality	**Addition Property of Equality**
For any numbers $a, b,$ and c,	For any numbers $a, b,$ and c
if $a = b$,	if $a = b$
then $a - c = b - c$.	then $a + c = b + c$

(9.7.1)

Similar properties hold true for inequalities.

Table 9.7.1

For example, we know that −4 is less than 2.	$-4 < 2$
If we subtract 5 from both quantities, is the left side still less than the right side?	$-4-5 \stackrel{?}{<} 2-5$
We get −9 on the left and −3 on the right.	$-9 \stackrel{?}{<} -3$
And we know −9 is less than −3.	$-9 < -3$
	The inequality sign stayed the same.

Similarly we could show that the inequality also stays the same for addition.

This leads us to the Subtraction and Addition Properties of Inequality.

PROPERTIES OF INEQUALITY

Subtraction Property of Inequality	**Addition Property of Inequality**
For any numbers $a, b,$ and c,	For any numbers $a, b,$ and c
if $a < b$	if $a < b$
then $a - c < b - c$.	then $a + c < b + c$
if $a > b$	if $a > b$
then $a - c > b - c$.	then $a + c > b + c$

(9.7.2)

We use these properties to solve inequalities, taking the same steps we used to solve equations. Solving the inequality $x + 5 > 9$, the steps would look like this:

$$\begin{array}{lrcl} & x+5 & > & 9 \\ \text{Subtract 5 from both sides to isolate } x. & x+5-5 & > & 9-5 \\ & x & > & 4 \end{array} \tag{9.7.3}$$

Any number greater than 4 is a solution to this inequality.

Exercise 9.7.7

Solve the inequality $n - \frac{1}{2} \leq \frac{5}{8}$, graph the solution on the number line, and write the solution in interval notation.

Answer

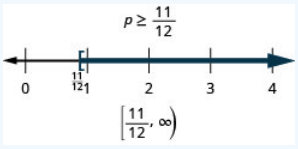
	$n - \frac{1}{2} \leq \frac{5}{8}$
Add $\frac{1}{2}$ to both sides of the inequality.	$n - \frac{1}{2} + \frac{1}{2} \leq \frac{5}{8} + \frac{1}{2}$
Simplify.	$n \leq \frac{9}{8}$
Graph the solution on the number line.	
Write the solution in interval notation.	

Exercise 9.7.8

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$p - \frac{3}{4} \geq \frac{1}{6}$

Answer

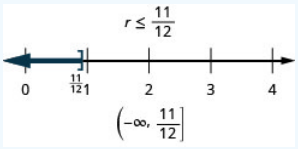


Exercise 9.7.9

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$r - \frac{1}{3} \leq \frac{7}{12}$

Answer



Solve Inequalities using the Division and Multiplication Properties of Inequality

The Division and Multiplication Properties of Equality state that if two quantities are equal, when we divide or multiply both quantities by the same amount, the results will also be equal (provided we don't divide by 0).

PROPERTIES OF EQUALITY

Division Property of Equality	**MUltiplication Property of Equality**
For any numbers a, b, c, and $c \neq 0$	For any numbers a, b, c
if $a = b$	if $a = b$
then $\frac{a}{c} = \frac{b}{c}$	then $ac = bc$

(9.7.4)

Are there similar properties for inequalities? What happens to an inequality when we divide or multiply both sides by a constant?

Consider some numerical examples.

Table 9.7.2

	$10 < 15$		$10 < 15$
Divide both sides by 5.	$\frac{10}{5} \,?\, \frac{15}{5}$	Multiply both sides by 5.	$10(5) \,?\, 15(5)$
Simplify.	$2 \,?\, 3$		$50 \,?\, 75$
Fill in the inequality signs.	$2 < 3$		$50 < 75$

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

Table 9.7.3

	$10 < 15$		$10 < 15$

Divide both sides by -5.	$\frac{10}{-5} \stackrel{?}{} \frac{15}{-5}$	Multiply both sides by -5.	$10(-5) \stackrel{?}{} 15(-5)$
Simplify.	$-2 \stackrel{?}{} -3$		$-50 \stackrel{?}{} -75$
Fill in the inequality signs.	$-2 > -3$		$-50 > -75$

The inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

Here are the Division and Multiplication Properties of Inequality for easy reference.

DIVISION AND MULTIPLICATION PROPERTIES OF INEQUALITY

For any real numbers a,b,c

$$\begin{aligned} &\text{if } a<b \text{ and } c>0, \text{ then } && \tfrac{a}{c}<\tfrac{b}{c} \text{ and } ac<bc \\ &\text{if } a>b \text{ and } c>0, \text{ then } && \tfrac{a}{c}>\tfrac{b}{c} \text{ and } ac>bc \\ &\text{if } a<b \text{ and } c<0, \text{ then } && \tfrac{a}{c}>\tfrac{b}{c} \text{ and } ac>bc \\ &\text{if } a>b \text{ and } c<0, \text{ then } && \tfrac{a}{c}<\tfrac{b}{c} \text{ and } ac<bc \end{aligned} \tag{9.7.5}$$

When we **divide or multiply** an inequality by a:

- **positive** number, the inequality stays the **same**.
- **negative** number, the inequality **reverses**.

Exercise 9.7.10

Solve the inequality $7y < 42$, graph the solution on the number line, and write the solution in interval notation.

Answer

	$7y < 42$
Divide both sides of the inequality by 7. Since $7 > 0$, the inequality stays the same.	$\frac{7y}{7} < \frac{42}{7}$
Simplify.	$y < 6$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, 6)$

Exercise 9.7.11

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$9c > 72$

Answer

$c > 8$

$(8, \infty)$

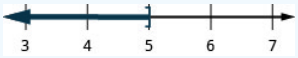
Exercise 9.7.12

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$12d \leq 60$

Answer

$d \leq 5$



$(-\infty, 5]$

Exercise 9.7.13

Solve the inequality $-10a \geq 50$, graph the solution on the number line, and write the solution in interval notation.

Answer

	$-10a \geq 50$
Divide both sides of the inequality by -10. Since $-10 < 0$, the inequality reverses.	$\frac{-10a}{-10} \leq \frac{50}{-10}$
Simplify.	$a \leq -5$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -5]$

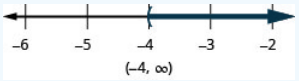
Exercise 9.7.14

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$-8q < 32$

Answer

$q > -4$

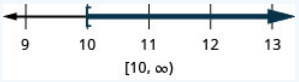


Exercise 9.7.15

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$-7r \leq -70$

Answer



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SOLVING INEQUALITIES

Sometimes when solving an inequality, the variable ends up on the right. We can rewrite the inequality in reverse to get the variable to the left.

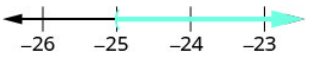
$$x > a \text{ has the same meaning as } a < x \tag{9.7.6}$$

Think about it as "If Xavier is taller than Alex, then Alex is shorter than Xavier."

Exercise 9.7.16

Solve the inequality $-20 < \frac{4}{5}u$, graph the solution on the number line, and write the solution in interval notation.

Answer

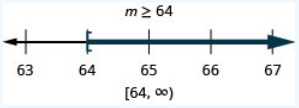
	$-20 < \frac{4}{5}u$
Multiply both sides of the inequality by $\frac{5}{4}$. Since $\frac{5}{4} > 0$, the inequality stays the same.	$\frac{5}{4}(-20) < \frac{5}{4}\left(\frac{4}{5}u\right)$
Simplify.	$-25 < u$
Rewrite the variable on the left.	$u > -25$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-25, \infty)$

Exercise 9.7.17

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$24 \leq \frac{3}{8}m$

Answer

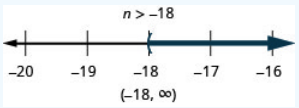


Exercise 9.7.18

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$-24 < \frac{4}{3}n$

Answer



Exercise 9.7.19

Solve the inequality $\frac{t}{-2} \geq 8$, graph the solution on the number line, and write the solution in interval notation.

Answer

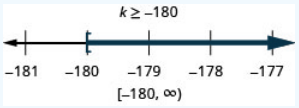
	$\frac{t}{-2} \geq 8$
Multiply both sides of the inequality by -2. Since $-2 < 0$, the inequality reverses.	$-2\left(\frac{t}{-2}\right) \leq -2(8)$
Simplify.	$t \leq -16$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -16]$

Exercise 9.7.20

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$\frac{k}{-12} \leq 15$

Answer

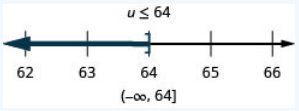


Exercise 9.7.21

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$\frac{u}{-4} \geq -16$

Answer



Solve Inequalities That Require Simplification

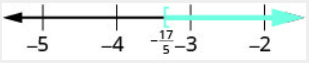
Most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but be sure to pay close attention during multiplication or division.

Exercise 9.7.22

Solve the inequality $4m \leq 9m + 17$, graph the solution on the number line, and write the solution in interval notation.

Answer

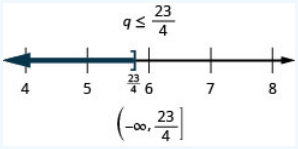
	$4m \leq 9m + 17$
Subtract 9m from both sides to collect the variables on the left.	$4m - 9m \leq 9m - 9m + 17$
Simplify.	$-5m \leq 17$
Divide both sides of the inequality by -5, and reverse the inequality.	$\frac{-5m}{-5} \geq \frac{17}{-5}$

Simplify.	$m \geq -\frac{17}{5}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left[-\frac{17}{5}, \infty\right)$

Exercise 9.7.23

Solve the inequality $3q \geq 7q - 23$, graph the solution on the number line, and write the solution in interval notation.

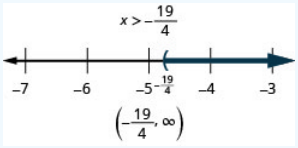
Answer



Exercise 9.7.24

Solve the inequality $6x < 10x + 19$, graph the solution on the number line, and write the solution in interval notation.

Answer



Exercise 9.7.25

Solve the inequality $8p + 3(p - 12) > 7p - 28$ graph the solution on the number line, and write the solution in interval notation.

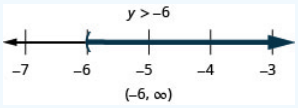
Answer

Simplify each side as much as possible.	8p+3(p−12)>7p−28
Distribute.	8p+3p−36>7p−28
Combine like terms.	11p−36>7p−28
Subtract 7p from both sides to collect the variables on the left.	11p−36−7p>7p−28−7p
Simplify.	4p−36>−28
Add 36 to both sides to collect the constants on the right.	4p−36+36>−28+36
Simplify.	4p>8
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4} > 84$
Simplify.	$p > 2$
Graph the solution on the number line.	
Write the solution in interval notation.	$(2, \infty)$

Exercise 9.7.26

Solve the inequality $9y+2(y+6)>5y-24$, graph the solution on the number line, and write the solution in interval notation.

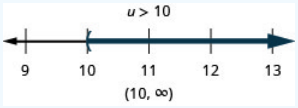
Answer



Exercise 9.7.27

Solve the inequality $6u+8(u-1)>10u+32$, graph the solution on the number line, and write the solution in interval notation.

Answer



Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

Exercise 9.7.28

Solve the inequality $8x-2(5-x)<4(x+9)+6x$, graph the solution on the number line, and write the solution in interval notation.

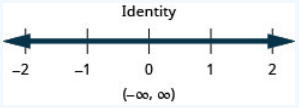
Answer

Simplify each side as much as possible.	8x−2(5−x)<4(x+9)+6x
Distribute.	8x−10+2x<4x+36+6x
Combine like terms.	10x−10<10x+36
Subtract 10x from both sides to collect the variables on the left.	10x−10−10x<10x+36−10x
Simplify.	−10<36
The xx's are gone, and we have a true statement.	The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty,\infty)$

Exercise 9.7.29

Solve the inequality $4b-3(3-b)>5(b-6)+2b$, graph the solution on the number line, and write the solution in interval notation.

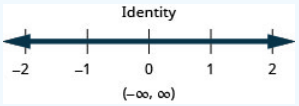
Answer



Exercise 9.7.30

Solve the inequality $9h-7(2-h)<8(h+11)+8h$, graph the solution on the number line, and write the solution in interval notation.

Answer



Exercise 9.7.31

Solve the inequality $\frac{1}{3}a-\frac{1}{8}a>\frac{5}{24}a+\frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

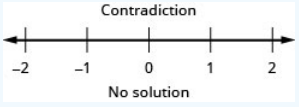
Answer

	$\frac{1}{3}a-\frac{1}{8}a>\frac{5}{24}a+\frac{3}{4}$
Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a-\frac{1}{8}a\right)>24\left(\frac{5}{24}a+\frac{3}{4}\right)$
Simplify.	$8a-3a>5a+18$
Combine like terms.	$5a>5a+18$
Subtract 5a from both sides to collect the variables on the left.	$5a-5a>5a-5a+18$
Simplify.	$0>18$
The statement is false!	The inequality is a contradiction.
	There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

Exercise 9.7.32

Solve the inequality $\frac{1}{4}x-\frac{1}{12}x>\frac{1}{6}x+\frac{7}{8}$, graph the solution on the number line, and write the solution in interval notation.

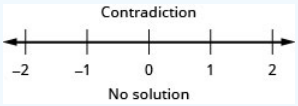
Answer



Exercise 9.7.33

Solve the inequality $\frac{2}{5}z - \frac{1}{3}z < \frac{1}{15}z - \frac{3}{5}$, graph the solution on the number line, and write the solution in interval notation.

Answer



Translate to an Inequality and Solve

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like ‘more than’ and ‘less than’. But others are not as obvious.

Think about the phrase ‘at least’ – what does it mean to be ‘at least 21 years old’? It means 21 or more. The phrase ‘at least’ is the same as ‘greater than or equal to’.

Table 9.7.4 shows some common phrases that indicate inequalities.

Table 9.7.4

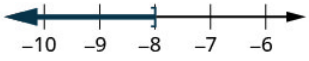
>	≥	<	≤
" data-valign="middle" class="lt-math-15134">is greater than	is greater than or equal to	is less than	is less than or equal to
" data-valign="middle" class="lt-math-15134">is more than	is at least	is smaller than	is at most
" data-valign="middle" class="lt-math-15134">is larger than	is no less than	has fewer than	is no more than
" data-valign="middle" class="lt-math-15134">exceeds	is the minimum	is lower than	is the maximum

Exercise 9.7.34

Translate and solve. Then write the solution in interval notation and graph on the number line.

Twelve times c is no more than 96.

Answer

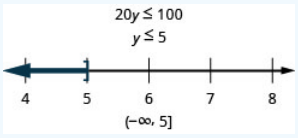
Translate.	Twelve times c is no more than 96 $12c \leq 96$
Solve—divide both sides by 12.	$\frac{12c}{12} \leq \frac{96}{12}$
Simplify.	$c \leq 8$
Write in interval notation.	$(-\infty, -8]$
Graph on the number line.	

Exercise 9.7.35

Translate and solve. Then write the solution in interval notation and graph on the number line.

Twenty times y is at most 100

Answer

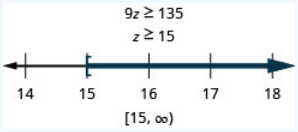


Exercise 9.7.36

Translate and solve. Then write the solution in interval notation and graph on the number line.

Nine times z is no less than 135

Answer



Exercise 9.7.37

Translate and solve. Then write the solution in interval notation and graph on the number line.

Thirty less than x is at least 45.

Answer

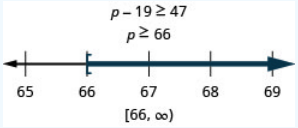
Translate.	Thirty less than x is at least 45. $x - 30 \ge 45$
Solve—add 30 to both sides.	$x - 30 + 30 \ge 45 + 30$
Simplify.	$x \ge 75$
Write in interval notation.	$[75, \infty)$
Graph on the number line.	

Exercise 9.7.38

Translate and solve. Then write the solution in interval notation and graph on the number line.

Nineteen less than p is no less than 47

Answer



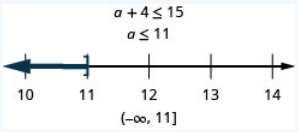
https://math.libretexts.org/@go/page/46259

Exercise 9.7.39

Translate and solve. Then write the solution in interval notation and graph on the number line.

Four more than a is at most 15.

Answer



Key Concepts

- **Subtraction Property of Inequality**
  For any numbers a, b, and c,
  if a<b then a−c<b−c and
  if a>b then a−c>b−c.
- **Addition Property of Inequality**
  For any numbers a, b, and c,
  if a<b then a+c<b+c and
  if a>b then a+c>b+c.
- **Division and Multiplication Properties of Inequality**
  For any numbers a, b, and c,
  if a<b and c>0, then ac<bc and ac>bc.
  if a>b and c>0, then ac>bc and ac>bc.
  if a<b and c<0, then ac>bc and ac>bc.
  if a>b and c<0, then ac<bc and ac<bc.
- When we **divide or multiply** an inequality by a:
  - **positive** number, the inequality stays the **same**.
  - **negative** number, the inequality **reverses**.

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9.7E: Exercises

Practice Makes Perfect

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line

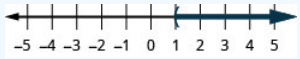
Exercise 9.7E.1

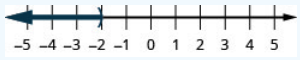
1. $x \leq 2$
2. x>−1
3. x<0

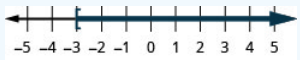
Exercise 9.7E.2

1. x>1
2. x<−2
3. $x \geq -3$

Answer

1. 

2. 

3. 

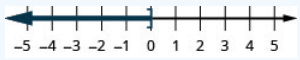
Exercise 9.7E.3

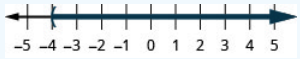
1. $x \geq -3$
2. x<4
3. $x \leq -2$

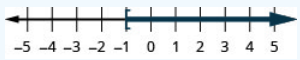
Exercise 9.7E.4

1. $x \leq 0$
2. x>−4
3. $x \geq -1$

Answer

1. 

2. 

3. 

In the following exercises, graph each inequality on the number line and write in interval notation.

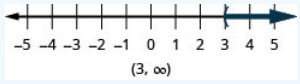
Exercise 9.7E.5

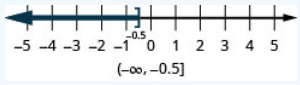
1. x<−2
2. $x \geq -3.5$
3. $x \leq \frac{2}{3}$

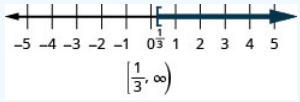
Exercise 9.7E.6

1. $x > 3$
2. $x \leq -0.5$
3. $x \geq \frac{1}{3}$

Answer

1. 

2. 

3. 

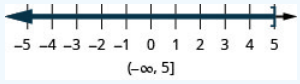
Exercise 9.7E.7

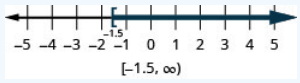
1. $x \geq -4$
2. x<2.5
3. $x > -\frac{3}{2}$

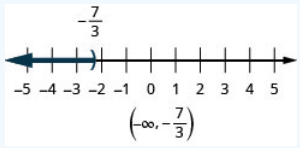
Exercise 9.7E.8

1. $x \leq 5$
2. $x \geq -1.5x$
3. x<−73

Answer

1. 

2. 

3. 

Solve Inequalities using the Subtraction and Addition Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

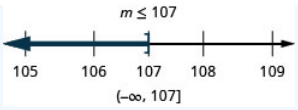
Exercise 9.7E.9

$n - 11 < 33$

Exercise 9.7E.10

$m - 45 \leq 62$

Answer



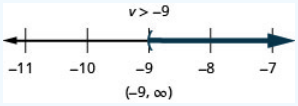
Exercise 9.7E.11

$u+25>21$

Exercise 9.7E.12

$v+12>3$

Answer



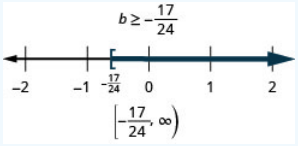
Exercise 9.7E.13

$a+\frac{3}{4}\geq\frac{7}{10}$

Exercise 9.7E.14

$b+\frac{7}{8}\geq\frac{1}{6}$

Answer



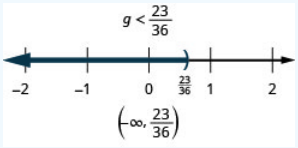
Exercise 9.7E.15

$f-\frac{13}{20}<-\frac{5}{12}$

Exercise 9.7E.16

$g-\frac{11}{12}<-\frac{5}{18}$

Answer



Solve Inequalities using the Division and Multiplication Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

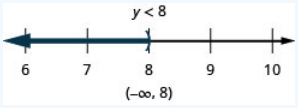
Exercise 9.7E.17

$8x > 72$

Exercise 9.7E.18

$6y < 48$

Answer



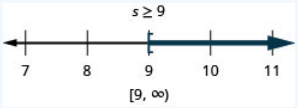
Exercise 9.7E.19

$7r \leq 56$

Exercise 9.7E.20

$9s \geq 81$

Answer



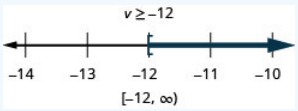
Exercise 9.7E.21

$-5u \geq 65$

Exercise 9.7E.22

$-8v \leq 96$

Answer



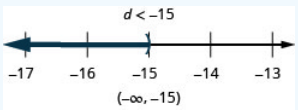
Exercise 9.7E.23

$-9c < 126$

Exercise 9.7E.24

$-7d > 105$

Answer



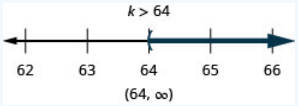
Exercise 9.7E.25

$20 > \frac{2}{5}h$

Exercise 9.7E.26

$40 < \frac{5}{8}k$

Answer



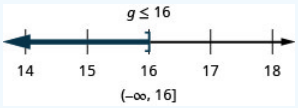
Exercise 9.7E.27

$\frac{7}{6}j \geq 42$

Exercise 9.7E.28

$\frac{9}{4}g \leq 36$

Answer



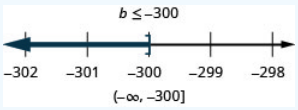
Exercise 9.7E.29

$\frac{a}{-3} \leq 9$

Exercise 9.7E.30

$\frac{b}{-10} \geq 30$

Answer



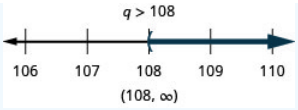
Exercise 9.7E.31

$-25 < \frac{p}{-5}$

Exercise 9.7E.32

$-18 > \frac{q}{-6}$

Answer



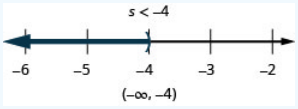
Exercise 9.7E.33

$9t \geq -27$

Exercise 9.7E.34

$7s < -28$

Answer



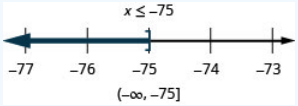
Exercise 9.7E.35

$\frac{2}{3}y > -36$

Exercise 9.7E.36

$\frac{3}{5}x \leq -45$

Answer



Solve Inequalities That Require Simplification

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

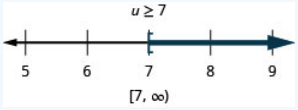
Exercise 9.7E.37

$4v \geq 9v - 40$

Exercise 9.7E.38

$5u \leq 8u - 21$

Answer



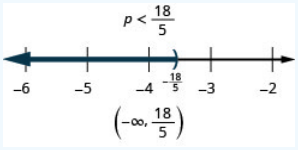
Exercise 9.7E.39

$13q < 7q - 29$

Exercise 9.7E.40

$9p > 14p - 18$

Answer



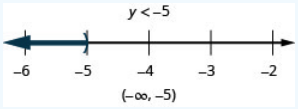
Exercise 9.7E.41

$12x + 3(x + 7) > 10x - 24$

Exercise 9.7E.42

$9y + 5(y + 3) < 4y - 35$

Answer



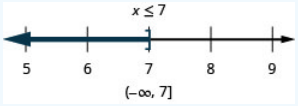
Exercise 9.7E.43

$6h - 4(h - 1) \leq 7h - 11$

Exercise 9.7E.44

$4k - (k - 2) \geq 7k - 26$

Answer



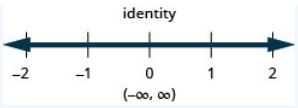
Exercise 9.7E.45

$8m - 2(14 - m) \geq 7(m - 4) + 3m$

Exercise 9.7E.46

$6n - 12(3 - n) \leq 9(n - 4) + 9n$

Answer



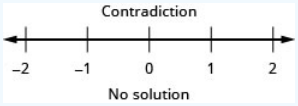
Exercise 9.7E.47

$\frac{3}{4}b-\frac{1}{3}b<\frac{5}{12}b-\frac{1}{2}$

Exercise 9.7E.48

$9u+5(2u-5)\geq 12(u-1)+7u$

Answer



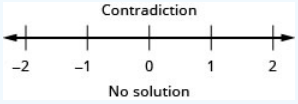
Exercise 9.7E.49

$\frac{2}{3}g-\frac{1}{2}(g-14)\leq \frac{1}{6}(g+42)$

Exercise 9.7E.50

$\frac{5}{6}a-\frac{1}{4}a>\frac{7}{12}a+\frac{2}{3}$

Answer



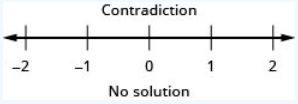
Exercise 9.7E.51

$\frac{4}{5}h-\frac{2}{3}(h-9)\geq \frac{1}{15}(2h+90)$

Exercise 9.7E.52

$12v+3(4v-1)\leq 19(v-2)+5v$

Answer



Mixed practice

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

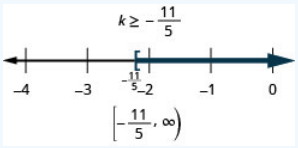
Exercise 9.7E.53

$15k\leq -40$

Exercise 9.7E.54

$35k \geq -77$

Answer



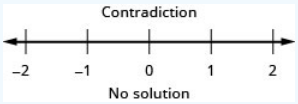
Exercise 9.7E.55

$23p - 2(6 - 5p) > 3(11p - 4)$

Exercise 9.7E.56

$18q - 4(10 - 3q) < 5(6q - 8)$

Answer



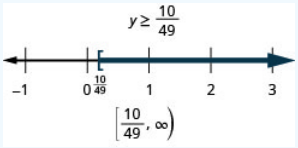
Exercise 9.7E.57

$-\frac{9}{4}x \geq -\frac{5}{12}$

Exercise 9.7E.58

$-\frac{21}{8}y \leq -\frac{15}{28}$

Answer



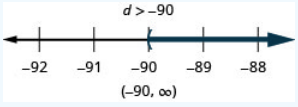
Exercise 9.7E.59

$c + 34 < -99$

Exercise 9.7E.60

$d + 29 > -61$

Answer



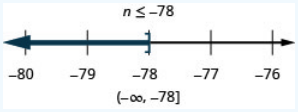
Exercise 9.7E.61

$\frac{m}{18} \geq -4$

Exercise 9.7E.62

$\frac{n}{13} \leq -6$

Answer



Translate to an Inequality and Solve

In the following exercises, translate and solve .Then write the solution in interval notation and graph on the number line.

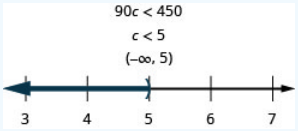
Exercise 9.7E.63

Fourteen times d is greater than 56.

Exercise 9.7E.64

Ninety times c is less than 450.

Answer



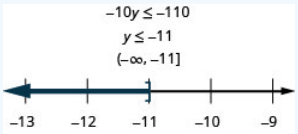
Exercise 9.7E.65

Eight times z is smaller than -40

Exercise 9.7E.66

Ten times y is at most -110

Answer



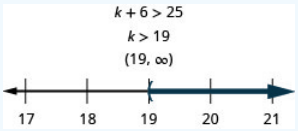
Exercise 9.7E.67

Three more than h is no less than 25

Exercise 9.7E.68

Six more than k exceeds 25

Answer



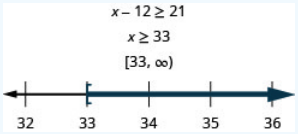
Exercise 9.7E.69

Ten less than w is at least 39.

Exercise 9.7E.70

Twelve less than x is no less than 21

Answer



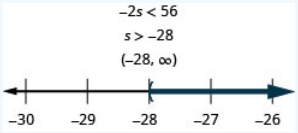
Exercise 9.7E.71

Negative five times r is no more than 95.

Exercise 9.7E.72

Negative two times s is lower than 56

Answer



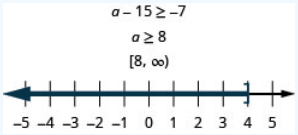
Exercise 9.7E.73

Nineteen less than b is at most -22

Exercise 9.7E.74

Fifteen less than a is at least -7

Answer



Everyday Math

Exercise 9.7E.75

Safety A child's height, *h*, must be at least 57 inches for the child to safely ride in the front seat of a car. Write this as an inequality.

Exercise 9.7E.76

Fighter pilots The maximum height, *h*, of a fighter pilot is 77 inches. Write this as an inequality.

Answer

$h \leq 77$

Exercise 9.7E.77

Elevators The total weight, *w*, of an elevator's passengers can be no more than 1,200 pounds. Write this as an inequality.

Exercise 9.7E.78

Shopping The number of items, *n*, a shopper can have in the express check-out lane is at most 8. Write this as an inequality.

Answer

$n \leq 8$

Writing Exercises

Exercise 9.7E.79

Give an example from your life using the phrase 'at least'.

Exercise 9.7E.80

Give an example from your life using the phrase 'at most'.

Answer

Answers will vary.

Exercise 9.7E.81

Explain why it is necessary to reverse the inequality when solving $-5x > 10$

Exercise 9.7E.82

Explain why it is necessary to reverse the inequality when solving $\frac{n}{-3} < 12$

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph inequalities on the number line.			
solve inequalities using the Subtraction and Addition Properties of Inequality.			
solve inequalities using the Division and Multiplication Properties of Inequality.			
solve inequalities that require simplification.			
translate to an inequality and solve.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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9.8: Chapter 2 Review Exercises

Chapter 2 Review Exercises

Solve Equations using the Subtraction and Addition Properties of Equality

Verify a Solution of an Equation

In the following exercises, determine whether each number is a solution to the equation.

Exercise 9.8.1

$10x-1=5x; x=\frac{1}{5}$

Exercise 9.8.2

$w+2=\frac{5}{8}; w=\frac{3}{8}$

Answer

no

Exercise 9.8.3

$-12n+5=8n; n=-\frac{5}{4}$

Exercise 9.8.4

$6a-3=-7a, a=\frac{3}{13}$

Answer

yes

Solve Equations using the Subtraction and Addition Properties of Equality

In the following exercises, solve each equation using the Subtraction Property of Equality.

Exercise 9.8.5

$x+7=19$

Exercise 9.8.6

$y+2=-6$

Answer

$y=-8$

Exercise 9.8.7

$a+\frac{1}{3}=\frac{5}{3}$

Exercise 9.8.8

$n+3.6=5.1$

Answer

$n=1.5$

In the following exercises, solve each equation using the Addition Property of Equality.

Exercise 9.8.9

$u-7=10$

Exercise 9.8.10

$x-9=-4$

Answer

$x=5$

Exercise 9.8.11

$c-\frac{3}{11}=\frac{9}{11}$

Exercise 9.8.12

$p-4.8=14$

Answer

$p=18.8$

In the following exercises, solve each equation.

Exercise 9.8.13

$n-12=32$

Exercise 9.8.14

$y+16=-9$

Answer

$y=-25$

Exercise 9.8.15

$f+\frac{2}{3}=4$

Exercise 9.8.16

$d-3.9=8.2$

Answer

$d = 12.1$

Solve Equations That Require Simplification

In the following exercises, solve each equation.

Exercise 9.8.17

$y + 8 - 15 = -3$

Exercise 9.8.18

$7x + 10 - 6x + 3 = 5$

Answer

$x = -8$

Exercise 9.8.19

$6(n - 1) - 5n = -14$

Exercise 9.8.20

$8(3p + 5) - 23(p - 1) = 35$

Answer

$p = -28$

Translate to an Equation and Solve

In the following exercises, translate each English sentence into an algebraic equation and then solve it.

Exercise 9.8.21

The sum of -6 and m is 25

Exercise 9.8.22

Four less than n is 13

Answer

$n - 4 = 13; n = 17$

Translate and Solve Applications

In the following exercises, translate into an algebraic equation and solve.

Exercise 9.8.23

Rochelle's daughter is 11 years old. Her son is 3 years younger. How old is her son?

Exercise 9.8.24

Tan weighs 146 pounds. Minh weighs 15 pounds more than Tan. How much does Minh weigh?

Answer

161 pounds

Exercise 9.8.25

Peter paid $9.75 to go to the movies, which was $46.25 less than he paid to go to a concert. How much did he pay for the concert?

Exercise 9.8.26

Elissa earned $\$152.84$this week, which was $\$2. .65$more than she earned last week. How much did she earn last week?

Answer

$\$131.19$

Solve Equations using the Division and Multiplication Properties of Equality

Solve Equations Using the Division and Multiplication Properties of Equality

In the following exercises, solve each equation using the division and multiplication properties of equality and check the solution.

Exercise 9.8.27

$8x = 72$

Exercise 9.8.28

$13a = -65$

Answer

$a = -5$

Exercise 9.8.29

$0.25p = 5.25$

Exercise 9.8.30

$-y = 4$

Answer

$y = -4$

Exercise 9.8.31

$\frac{n}{6} = 18$

Exercise 9.8.32

$\frac{y}{-10} = 30$

Answer

$y = -300$

Exercise 9.8.33

$36 = \frac{3}{4}x$

Exercise 9.8.34

$\frac{5}{8}u = \frac{15}{16}$

Answer

$u = \frac{3}{2}$

Exercise 9.8.35

$-18m = -72$

Exercise 9.8.36

$\frac{c}{9} = 36$

Answer

$c = 324$

Exercise 9.8.37

$0.45x = 6.75$

Exercise 9.8.38

$\frac{11}{12} = \frac{2}{3}y$

Answer

$y = \frac{11}{8}$

Solve Equations That Require Simplification

In the following exercises, solve each equation requiring simplification.

Exercise 9.8.39

$5r - 3r + 9r = 35 - 2$

Exercise 9.8.40

$24x + 8x - 11x = -7 - 14$

Answer

$x = -1$

Exercise 9.8.41

$\frac{11}{12}n - \frac{5}{6}n = 9 - 5$

Exercise 9.8.42

$-9(d-2) - 15 = -24$

Answer

$d = 3$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve.

Exercise 9.8.43

143 is the product of -11 and y

Exercise 9.8.44

The quotient of b and and 9 is -27

Answer

$\frac{b}{9} = -27; b = -243$

Exercise 9.8.45

The sum of q and one-fourth is one.

Exercise 9.8.46

The difference of s and one-twelfth is one fourth.

Answer

$s - \frac{1}{12} = \frac{1}{4}; s = \frac{1}{3}$

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

Exercise 9.8.47

Ray paid $21 for 12 tickets at the county fair. What was the price of each ticket?

Exercise 9.8.48

Janet gets paid $\$24$ per hour. She heard that this is $\frac{3}{4}$ of what Adam is paid. How much is Adam paid per hour?

Answer

$\$32$

Solve Equations with Variables and Constants on Both Sides

Solve an Equation with Constants on Both Sides

In the following exercises, solve the following equations with constants on both sides.

Exercise 9.8.49

$8p+7=47$

Exercise 9.8.50

$10w-5=65$

Answer

$w=7$

Exercise 9.8.51

$3x+19=-47$

Exercise 9.8.52

$32=-4-9n$

Answer

$n=-4$

Solve an Equation with Variables on Both Sides

In the following exercises, solve the following equations with variables on both sides.

Exercise 9.8.53

$7y=6y-13$

Exercise 9.8.54

$5a+21=2a$

Answer

$a=-7$

Exercise 9.8.55

$k=-6k-35$

Exercise 9.8.56

$4x - \frac{3}{8} = 3x$

Answer

$x = \frac{3}{8}$

Solve an Equation with Variables and Constants on Both Sides

In the following exercises, solve the following equations with variables and constants on both sides.

Exercise 9.8.57

$12x - 9 = 3x + 45$

Exercise 9.8.58

$5n - 20 = -7n - 80$

Answer

$n = -5$

Exercise 9.8.59

$4u + 16 = -19 - u$

Exercise 9.8.60

$\frac{5}{8}c - 4 = \frac{3}{8}c + 4$

Answer

$c = 32$

Use a General Strategy for Solving Linear Equations

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, solve each linear equation.

Exercise 9.8.61

$6(x + 6) = 24$

Exercise 9.8.62

$9(2p - 5) = 72$

Answer

$p = \frac{13}{2}$

Exercise 9.8.63

$-(s+4)=18$

Exercise 9.8.64

$8+3(n-9)=17$

Answer

$n=12$

Exercise 9.8.65

$23-3(y-7)=8$

Exercise 9.8.66

$\frac{1}{3}(6m+21)=m-7$

Answer

$m=-14$

Exercise 9.8.67

$4(3.5y+0.25)=365$

Exercise 9.8.68

$0.25(q-8)=0.1(q+7)$

Answer

$q=18$

Exercise 9.8.69

$8(r-2)=6(r+10)$

Exercise 9.8.70

$5+7(2-5x)=2(9x+1)$
$-(13x-57)$

Answer

$x=-1$

Exercise 9.8.71

$(9n+5)-(3n-7)$
$=20-(4n-2)$

Exercise 9.8.72

$2[-16+5(8k-6)]$
$=8(3-4k)-32$

Answer

$k=\frac{3}{4}$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

Exercise 9.8.73

$17y-3(4-2y)=11(y-1)$
$+12y-1$

Exercise 9.8.74

$9u+32=15(u-4)$
$-3(2u+21)$

Answer

contradiction; no solution

Exercise 9.8.75

$-8(7m+4)=-6(8m+9)$

Exercise 9.8.76

$21(c-1)-19(c+1)$
$=2(c-20)$

Answer

identity; all real numbers

Solve Equations with Fractions and Decimals

Solve Equations with Fraction Coefficients

In the following exercises, solve each equation with fraction coefficients.

Exercise 9.8.77

$\frac{2}{5}n-\frac{1}{10}=\frac{7}{10}$

Exercise 9.8.78

$\frac{1}{3}x+\frac{1}{5}x=8$

Answer

$x=15$

Exercise 9.8.79

$\frac{3}{4}a-\frac{1}{3}=\frac{1}{2}a-\frac{5}{6}$

Exercise 9.8.80

$\frac{1}{2}(k-3)=\frac{1}{3}(k+16)$

Answer

$k=41$

Exercise 9.8.81

$\frac{3x-2}{5}=\frac{3x+4}{8}$

Exercise 9.8.82

$\frac{5y-1}{3}+4=\frac{-8y+4}{6}$

Answer

$y=-1$

Solve Equations with Decimal Coefficients

In the following exercises, solve each equation with decimal coefficients.

Exercise 9.8.83

$0.8x-0.3=0.7x+0.2$

Exercise 9.8.84

$0.36u+2.55=0.41u+6.8$

Answer

$u=-85$

Exercise 9.8.85

$0.6p-1.9=0.78p+1.7$

Exercise 9.8.86

$0.6p-1.9=0.78p+1.7$

Answer

$d=-20$

Solve a Formula for a Specific Variable

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

Exercise 9.8.87

Natalie drove for $7\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Exercise 9.8.88

Mallory is taking the bus from St. Louis to Chicago. The distance is 300 miles and the bus travels at a steady rate of 60 miles per hour. How long will the bus ride be?

Answer

5 hours

Exercise 9.8.89

Aaron's friend drove him from Buffalo to Cleveland. The distance is 187 miles and the trip took 2.75 hours. How fast was Aaron's friend driving?

Exercise 9.8.90

Link rode his bike at a steady rate of 15 miles per hour for $2\frac{1}{2}$ hours. How much distance did he travel?

Answer

37.5 miles

Solve a Formula for a Specific Variable

In the following exercises, solve.

Exercise 9.8.91

Use the formula. d=rt to solve for t

1. when d=510 and r=60
2. in general

Exercise 9.8.92

Use the formula. d=rt to solve for r

1. when when d=451 and t=5.5
2. in general

Answer

1. r=82mph
2. $r = \frac{D}{t}$

Exercise 9.8.93

Use the formula $A = \frac{1}{2}bh$ to solve for b

1. when A=390 and h=26
2. in general

Exercise 9.8.94

Use the formula $A = \frac{1}{2}bh$ to solve for b

1. when A=153 and b=18
2. in general

Answer

1. $h = 17$
2. $h = \frac{2A}{b}$

Exercise 9.8.95

Use the formula I=Prt to solve for the principal, *P* for

1. I=$2,501,r=4.1%, t=5 years
2. in general

Exercise 9.8.96

Solve the formula 4x+3y=6 for *y*

1. when x=−2
2. in general

Answer

ⓐ $y = \frac{14}{3}$ ⓑ $y = \frac{6-4x}{3}$

Exercise 9.8.97

Solve $180 = a + b + c$ for c

Exercise 9.8.98

Solve the formula $V = LWH$ for H

Answer

$H = \frac{V}{LW}$

Solve Linear Inequalities

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line.

Exercise 9.8.99

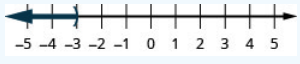
1. $x \leq 4$
2. x>−2
3. x<1

Exercise 9.8.100

1. x>0
2. x<−3
3. $x \geq -1$

Answer

1.

2. 

3.

In the following exercises, graph each inequality on the number line and write in interval notation.

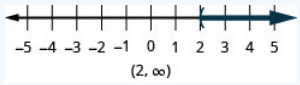
Exercise 9.8.101

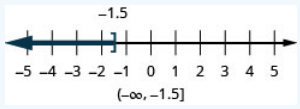
1. $x < -1$
2. $x \geq -2.5$
3. $x \leq \frac{5}{4}$

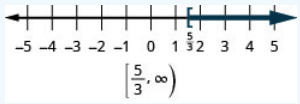
Exercise 9.8.102

1. $x > 2$
2. $x \leq -1.5$
3. $x \geq \frac{5}{3}$

Answer

1. 

2. 

3. 

Solve Inequalities using the Subtraction and Addition Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

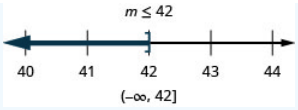
Exercise 9.8.103

$n - 12 \leq 23$

Exercise 9.8.104

$m + 14 \leq 56$

Answer



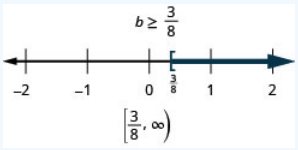
Exercise 9.8.105

$a+\frac{2}{3} \geq \frac{7}{12}$

Exercise 9.8.106

$b-\frac{7}{8} \geq -\frac{1}{2}$

Answer



Solve Inequalities using the Division and Multiplication Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

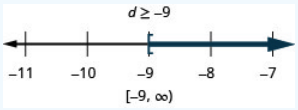
Exercise 9.8.107

$9x > 54$

Exercise 9.8.108

$-12d \leq 108$

Answer



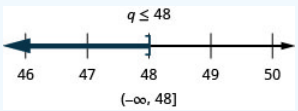
Exercise 9.8.109

$\frac{5}{2}j < -60$

Exercise 9.8.110

$\frac{q}{-2} \geq -24$

Answer



Solve Inequalities That Require Simplification

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

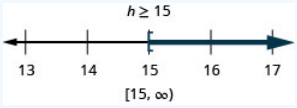
Exercise 9.8.111

$6p > 15p - 30$

Exercise 9.8.112

$9h - 7(h-1) \leq 4h - 23$

Answer



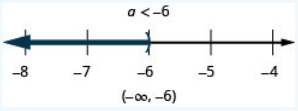
Exercise 9.8.113

$5n - 15(4-n) < 10(n-6) + 10n$

Exercise 9.8.114

$\frac{3}{8}a - \frac{1}{12}a > \frac{5}{12}a + \frac{3}{4}$

Answer



Translate to an Inequality and Solve

In the following exercises, translate and solve. Then write the solution in interval notation and graph on the number line.

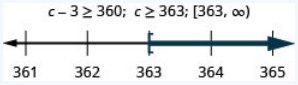
Exercise 9.8.115

Five more than z is at most 19.

Exercise 9.8.116

Three less than c is at least 360.

Answer



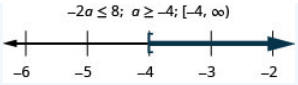
Exercise 9.8.117

Nine times n exceeds 42.

Exercise 9.8.118

Negative two times a is no more than 8.

Answer



Everyday Math

Exercise 9.8.119

Describe how you have used two topics from this chapter in your life outside of your math class during the past month.

Chapter 2 Practice Test

Exercise 9.8.1

Determine whether each number is a solution to the equation $6x - 3 = x + 20$

1. 5
2. $\frac{23}{5}$

Answer

1. no
2. yes

In the following exercises, solve each equation.

Exercise 9.8.2

$n - \frac{2}{3} = \frac{1}{4}$

Exercise 9.8.3

$\frac{9}{2}c = 144$

Answer

c=32

Exercise 9.8.4

$4y - 8 = 16$

Exercise 9.8.5

$-8x - 15 + 9x - 1 = -21$

Answer

$x = -5$

Exercise 9.8.6

$-15a = 120$

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Exercise 9.8.7

$\frac{2}{3}x = 6$

Answer

$x = 9$

Exercise 9.8.8

$x - 3.8 = 8.2$

Exercise 9.8.9

$10y = -5y - 60$

Answer

$y = -4$

Exercise 9.8.10

$8n - 2 = 6n - 12$

Exercise 9.8.11

$9m - 2 - 4m - m = 42 - 8$

Answer

$m = 9$

Exercise 9.8.12

$-5(2x - 1) = 45$

Exercise 9.8.13

$-(d - 9) = 23$

Answer

$d = -14$

Exercise 9.8.14

$\frac{1}{4}(12m - 28) = 6 - 2(3m - 1)$

Exercise 9.8.15

$2(6x - 5) - 8 = -22$

Answer

$x = -\frac{1}{3}$

Exercise 9.8.16

$8(3a-5)-7(4a-3)=20-3a$

Exercise 9.8.17

$\frac{1}{4}p-\frac{1}{3}=\frac{1}{2}$

Answer

$p=\frac{10}{3}$

Exercise 9.8.18

$0.1d+0.25(d+8)=4.1$

Exercise 9.8.19

$14n-3(4n+5)=-9+2(n-8)$

Answer

contradiction; no solution

Exercise 9.8.20

$9(3u-2)-4[6-8(u-1)]=3(u-2)$

Exercise 9.8.21

Solve the formula x−2y=5 for *y*

1. when x=−3
2. in general

Answer

1. y=4
2. $y=\frac{5-x}{2}$

In the following exercises, graph on the number line and write in interval notation.

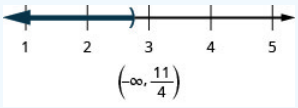
Exercise 9.8.22

$x\geq -3.5$

Exercise 9.8.23

$x<\frac{11}{4}$

Answer



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In the following exercises,, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

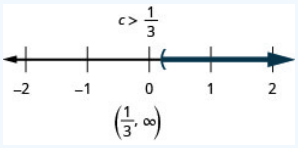
Exercise 9.8.24

$8k \geq 5k - 120$

Exercise 9.8.25

$3c - 10(c-2) < 5c + 16$

Answer



In the following exercises, translate to an equation or inequality and solve.

Exercise 9.8.26

4 less than twice *x* is 16.

Exercise 9.8.27

Fifteen more than *n* is at least 48.

Answer

$n + 15 \geq 48; n \geq 33$

Exercise 9.8.28

Samuel paid $25.82 for gas this week, which was $3.47 less than he paid last week. How much had he paid last week?

Exercise 9.8.29

Jenna bought a coat on sale for $\$120$, which was $\frac{2}{3}$ of the original price. What was the original price of the coat?

Answer

$120 = \frac{2}{3}p$; The original price was $\$180$

Exercise 9.8.30

Sean took the bus from Seattle to Boise, a distance of 506 miles. If the trip took $7\frac{2}{3}$ hours, what was the speed of the bus?

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CHAPTER OVERVIEW

10: Graphs

10.1: Understanding the Slope of a Line

10.1E: Exercises

10.2: Use the Slope–Intercept Form of an Equation of a Line

10.2E: Exercises

10.3: Find the Equation of a Line

10.3E: Exercises

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10.1: Understanding the Slope of a Line

Learning Objectives

- By the end of this section, you will be able to:
- Use geoboards to model slope
- Use $m = \frac{\text{rise}}{\text{run}}$ to find the slope of a line from its graph
- Find the slope of horizontal and vertical lines
- Use the slope formula to find the slope of a line between two points
- Graph a line given a point and the slope
- Solve slope applications

Note

Before you get started, take this readiness quiz.

1. Simplify: $\frac{1-4}{8-2}$.
   If you missed this problem, review Exercise 1.6.31
2. Divide: $\frac{0}{4}, \frac{4}{0}$.
   If you missed this problem, review Exercise 1.10.16.
3. Simplify: $\frac{15}{-3}, \frac{-15}{3}, \frac{-15}{-3}$.
   If you missed this problem, review Exercise 1.6.4.

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter. What determines whether a line tilts up or down or if it is steep or flat?

In mathematics, the 'tilt' of a line is called the *slope* of the line. The concept of slope has many applications in the real world. The pitch of a roof, grade of a highway, and a ramp for a wheelchair are some examples where you literally see slopes. And when you ride a bicycle, you feel the slope as you pump uphill or coast downhill.

In this section, we will explore the concept of slope.

Use Geoboards to Model Slope

A **geoboard** is a board with a grid of pegs on it. Using rubber bands on a geoboard gives us a concrete way to model lines on a coordinate grid. By stretching a rubber band between two pegs on a geoboard, we can discover how to find the slope of a line.

Doing the Manipulative Mathematics activity "Exploring Slope" will help you develop a better understanding of the slope of a line. (Graph paper can be used instead of a geoboard, if needed.)

We'll start by stretching a rubber band between two pegs as shown in Figure 10.1.1.

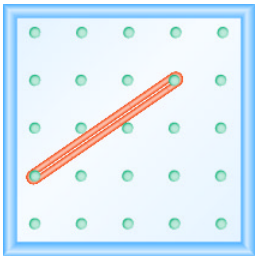


Figure 10.1.1

Doesn't it look like a line?

Now we stretch one part of the rubber band straight up from the left peg and around a third peg to make the sides of a right triangle, as shown in Figure 10.1.2

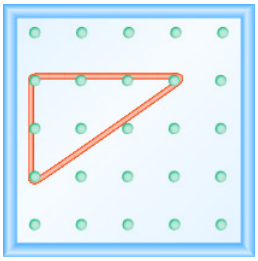


Figure 10.1.2

We carefully make a 90º angle around the third peg, so one of the newly formed lines is vertical and the other is horizontal.

To find the slope of the line, we measure the distance along the vertical and horizontal sides of the triangle. The vertical distance is called the **rise** and the horizontal distance is called the **run**, as shown in Figure 10.1.3.

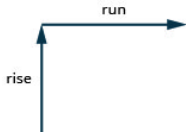


Figure 10.1.3

If our **geoboard** and rubber band look just like the one shown in Figure 10.1.4, the rise is 2. The rubber band goes up 2 units. (Each space is one unit.)

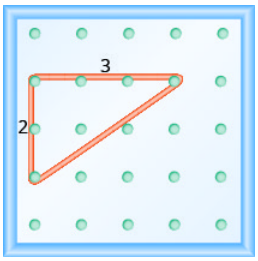


Figure 10.1.4

The rise on this geoboard is 2, as the rubber band goes up two units.

What is the run?

The rubber band goes across 3 units. The run is 3 (see Figure 10.1.4).

The slope of a line is the ratio of the rise to the run. In mathematics, it is always referred to with the letter m.

SLOPE OF A LINE

The **slope of a line** of a line is $m = \frac{\text{rise}}{\text{run}}$.

The **rise** measures the vertical change and the **run** measures the horizontal change between two points on the line.

What is the slope of the line on the geoboard in Figure 10.1.4?

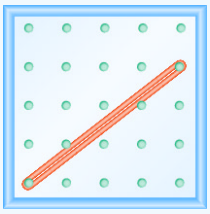
$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{2}{3}$$

The line has slope $\frac{2}{3}$. This means that the line rises 2 units for every 3 units of run.

When we work with geoboards, it is a good idea to get in the habit of starting at a peg on the left and connecting to a peg to the right. If the rise goes up it is positive and if it goes down it is negative. The run will go from left to right and be positive.

Example 10.1.1

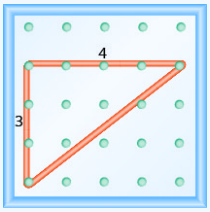
What is the slope of the line on the geoboard shown?



Solution

Use the definition of slope: $m = \frac{\text{rise}}{\text{run}}$.

Start at the left peg and count the spaces up and to the right to reach the second peg.

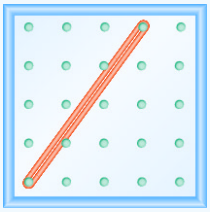


$$\begin{aligned} &\text{The rise is } 3. && m = \tfrac{3}{\text{run}} \\ &\text{The run is } 4. && m = \tfrac{3}{4} \\ & && \text{The slope is } \tfrac{3}{4}. \end{aligned} \tag{10.1.1}$$

This means that the line rises 3 units for every 4 units of run.

Try It 10.1.2

What is the slope of the line on the geoboard shown?

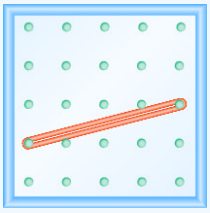


Answer

$\frac{4}{3}$

Try It 10.1.3

What is the slope of the line on the geoboard shown?

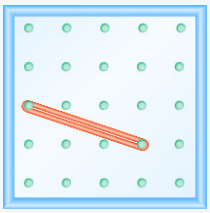


Answer

$\frac{1}{4}$

Example 10.1.4

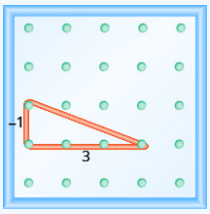
What is the slope of the line on the geoboard shown?



Solution

Use the definition of slope: $m = \frac{\text{rise}}{\text{run}}$.

Start at the left peg and count the units down and to the right to reach the second peg.

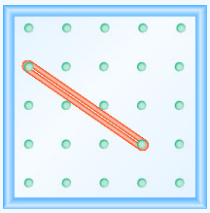


$$\begin{aligned} &\text{The rise is } -1. && m = \frac{-1}{\text{run}} \\ &\text{The run is } 3. && m = \frac{-1}{3} \\ & && m = -\frac{1}{3} \\ & && \text{The slope is } -\frac{1}{3} \end{aligned} \tag{10.1.2}$$

This means that the line drops 1 unit for every 3 units of run.

Try It 10.1.5

What is the slope of the line on the geoboard?

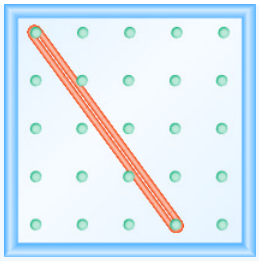


Answer

$-\frac{2}{3}$

Try It 10.1.6

What is the slope of the line on the geoboard?



Answer

$-\frac{4}{3}$

Notice that in Exercise 10.1.1 the slope is positive and in Exercise 10.1.4 the slope is negative. Do you notice any difference in the two lines shown in Figure(a) and Figure(b)?

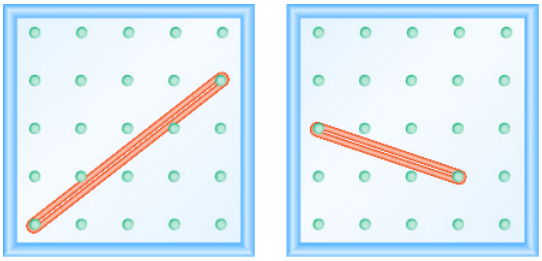


Figure 10.1.5

We 'read' a line from left to right just like we read words in English. As you read from left to right, the line in Figure(a) is going up; it has **positive slope**. The line in Figure(b) is going down; it has **negative slope**.

POSITIVE AND NEGATIVE SLOPES

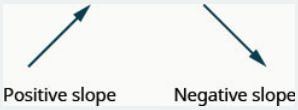


Figure 10.1.6

Example 10.1.7

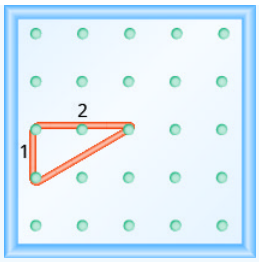
Use a geoboard to model a line with slope $\frac{1}{2}$.

Solution

To model a line on a geoboard, we need the rise and the run.

Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Replace m with $\frac{1}{2}$.	$\frac{1}{2} = \frac{\text{rise}}{\text{run}}$

So, the rise is 1 and the run is 2 .
Start at a peg in the lower left of the geoboard.
Stretch the rubber band up 1 unit, and then right 2 units.

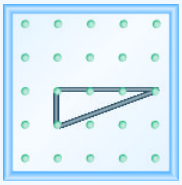


The hypotenuse of the right triangle formed by the rubber band represents a line whose slope is $\frac{1}{2}$.

Try It 10.1.8

Model the slope $m = \frac{1}{3}$. Draw a picture to show your results.

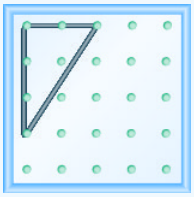
Answer



Try It 10.1.9

Model the slope $m = \frac{3}{2}$. Draw a picture to show your results.

Answer

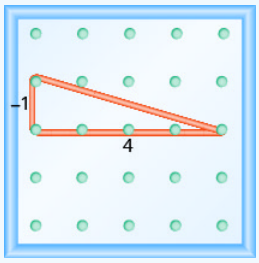


Example 10.1.10

Use a geoboard to model a line with slope $\frac{-1}{4}$.

Solution

Use the slope formula. m = $\frac{\text{rise}}{\text{run}}$.
Replace m with $\frac{-1}{4}$.
So, the rise is -1 and the run is 4.
Since the rise is negative, we choose a starting peg on the upper left that will give us room to count down.
We stretch the rubber band down 1 unit, and then right 4 units.

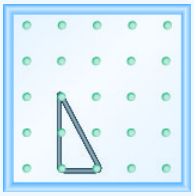


The hypotenuse of the right triangle formed by the rubber band represents a line whose slope is $\frac{-1}{4}$.

Try It 10.1.11

Model the slope $m = -2 = \frac{-2}{1}$. Draw a picture to show your results.

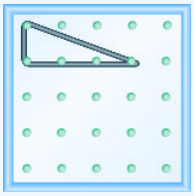
Answer



Try It 10.1.12

Model the slope $m = \frac{-1}{3}$. Draw a picture to show your results.

Answer



Use $m = \frac{\text{rise}}{\text{run}}$ to Find the Slope of a Line from its Graph

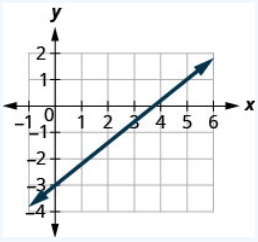
Now, we'll look at some graphs on the xy-coordinate plane and see how to find their slopes. The method will be very similar to what we just modeled on our geoboards.

To find the slope, we must count out the rise and the run. But where do we start?

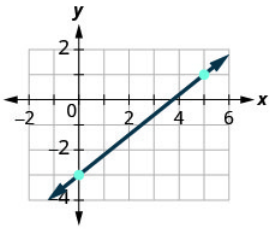
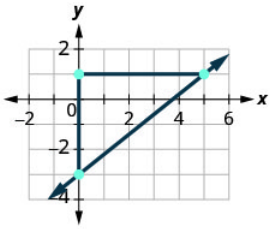
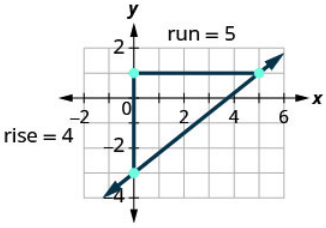
We locate two points on the line whose coordinates are integers. We then start with the point on the left and sketch a right triangle, so we can count the rise and run.

Example 10.1.13:

Find the slope of the line shown.

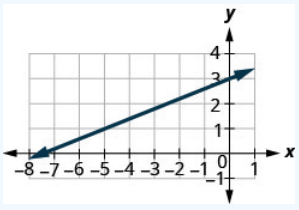


Solution

Step 1. Locate two points on the graph whose coordinates are integers.	Mark (0, –3) and (5, 1).	
Step 2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.	Starting at (0, –3), sketch a right triangle to (5, 1).	
Step 3. Count the rise and the run on the legs of the triangle.	Count the rise. Count the run.	 The rise is 4. The run is 5.
Step 4. Take the ratio of rise to run to find the slope. $m = \frac{\text{rise}}{\text{run}}$	Use the slope formula. Substitute the values of the rise and run.	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{4}{5}$ The slope of the line is $\frac{4}{5}$. This means that y increases 4 units as x increases 5 units.

Try It 10.1.14

Find the slope of the line shown.

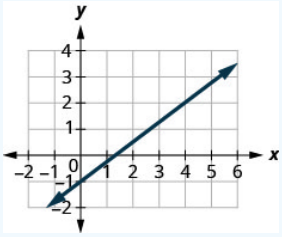


Answer

$\frac{2}{5}$

Try It 10.1.15

Find the slope of the line shown.



Answer

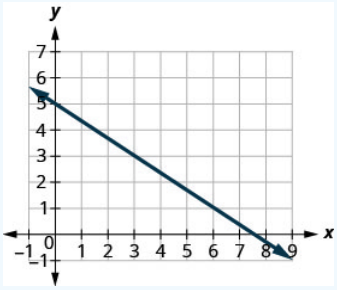
$\frac{3}{4}$

FIND THE SLOPE OF A LINE FROM ITS GRAPH

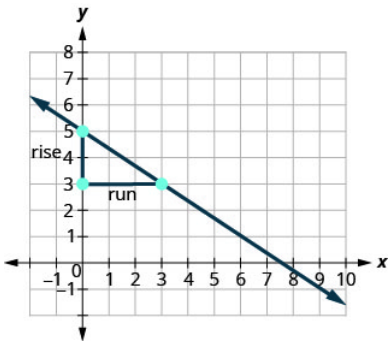
1. Locate two points on the line whose coordinates are integers.
2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.
3. Count the rise and the run on the legs of the triangle.
4. Take the ratio of rise to run to find the slope, $m = \frac{\text{rise}}{\text{run}}$.

Example 10.1.16

Find the slope of the line shown.



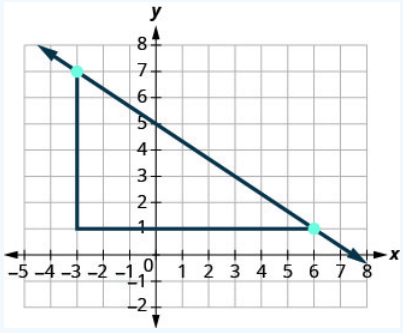
Solution

Locate two points on the graph whose coordinates are integers.	(0,5) and (3,3)
Which point is on the left?	(0,5)
Starting at (0,5), sketch a right triangle to (3,3).	
Count the rise—it is negative.	The rise is −2.
Count the run.	The run is 3.

Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{-2}{3}$
Simplify.	$m = -\frac{2}{3}$
	The slope of the line is $-\frac{2}{3}$.

So y increases by 3 units as x decreases by 2 units.

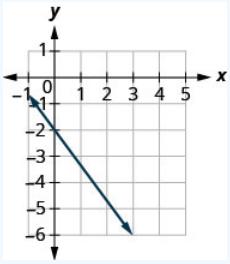
What if we used the points (−3,7) and (6,1) to find the slope of the line?



The rise would be −6 and the run would be 9. Then $m = \frac{-6}{9}$, and that simplifies to $m = -\frac{2}{3}$. Remember, it does not matter which points you use—the slope of the line is always the same.

Try It 10.1.17

Find the slope of the line shown.

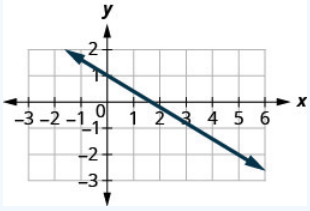


Answer

$-\frac{4}{3}$

Try It 10.1.18

Find the slope of the line shown.



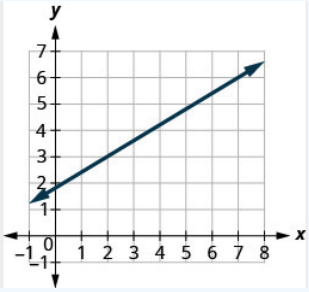
Answer

$-\frac{3}{5}$

In the last two examples, the lines had *y*-intercepts with integer values, so it was convenient to use the *y*-intercept as one of the points to find the slope. In the next example, the *y*-intercept is a fraction. Instead of using that point, we'll look for two other points whose coordinates are integers. This will make the slope calculations easier.

Example 10.1.19

Find the slope of the line shown.



Solution

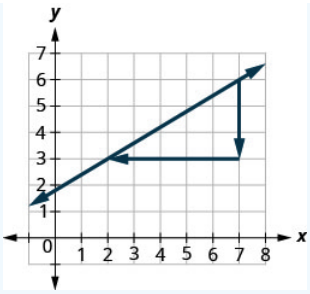
Locate two points on the graph whose coordinates are integers.	(2,3) and (7,6)
Which point is on the left?	(2,3)
Starting at (2,3), sketch a right triangle to (7,6).	
Count the rise.	The rise is 3.
Count the run.	The run is 5.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{3}{5}$
	The slope of the line is $\frac{3}{5}$.

This means that y increases 5 units as x increases 3 units.

When we used geoboards to introduce the concept of slope, we said that we would always start with the point on the left and count the rise and the run to get to the point on the right. That way the run was always positive and the rise determined whether the slope was positive or negative.

What would happen if we started with the point on the right?

Let's use the points (2,3) and (7,6) again, but now we'll start at (7,6).

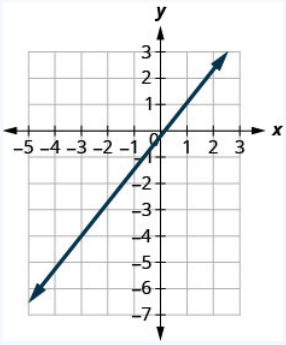


Count the rise.	The rise is -3.
Count the run. It goes from right to left, so it is negative.	The run is -5.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{-3}{-5}$
	The slope of the line is $\frac{3}{5}$

It does not matter where you start—the slope of the line is always the same.

Try It 10.1.20

Find the slope of the line shown.

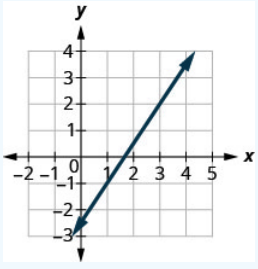


Answer

$\frac{5}{4}$

Try It 10.1.21

Find the slope of the line shown.



Answer

$\frac{3}{2}$

Find the Slope of Horizontal and Vertical Lines

Do you remember what was special about horizontal and vertical lines? Their equations had just one variable.

Horizontal line $y = b$	**Vertical line** $x = a$
y -coordinates are the same.	x -coordinates are the same.

(10.1.3)

So how do we find the slope of the horizontal line y=4? One approach would be to graph the horizontal line, find two points on it, and count the rise and the run. Let's see what happens when we do this.

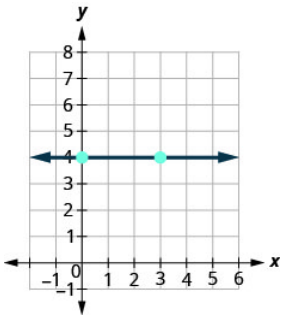


Figure 10.1.7

What is the rise?	The rise is 0.
What is the run?	The run is 3.
	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{0}{3}$
What is the slope?	$m = 0$
	The slope of the horizontal line y = 4 is 0.

All horizontal lines have slope 0. When the *y*-coordinates are the same, the rise is 0.

SLOPE OF A HORIZONTAL LINE

The slope of a horizontal line, y=b, is 0.

The floor of your room is horizontal. Its slope is 0. If you carefully placed a ball on the floor, it would not roll away.

Now, we'll consider a vertical line, the line.

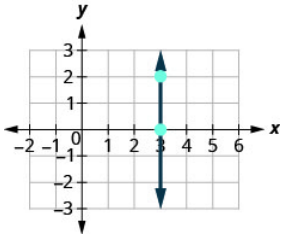


Figure 10.1.8

What is the rise?	The rise is 2.
What is the run?	The run is 0.
	$m = \frac{\text{rise}}{\text{run}}$
What is the slope?	$m = \frac{2}{0}$

But we can't divide by 0. Division by 0 is not defined. So we say that the slope of the vertical line x=3 is undefined.

The slope of any vertical line is undefined. When the *x*-coordinates of a line are all the same, the run is 0.

SLOPE OF A VERTICAL LINE

The slope of a vertical line, x=a, is undefined.

Example 10.1.22

Find the slope of each line:

(a) x=8 (b) y=−5.

Solution

(a) x=8
This is a vertical line.
Its slope is undefined.

(b) y=−5
This is a horizontal line.
It has slope 0.

Try It 10.1.23

Find the slope of the line: x=−4.

Answer

undefined

Try It 10.1.24

Find the slope of the line: y=7.

Answer

0

QUICK GUIDE TO THE SLOPES OF LINES

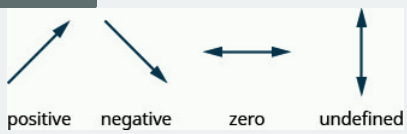


Figure 10.1.9

Remember, we ‘read’ a line from left to right, just like we read written words in English.

Use the Slope Formula to find the Slope of a Line Between Two Points

Doing the Manipulative Mathematics activity “Slope of Lines Between Two Points” will help you develop a better understanding of how to find the slope of a line between two points.

Sometimes we’ll need to find the slope of a line between two points when we don’t have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we’ll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

We have seen that an ordered pair (x,y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x,y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.

$$\begin{array}{ll} (x_1, y_1) & \text{read } `x \text{ sub } 1, y \text{ sub } 1' \\ (x_2, y_2) & \text{read } `x \text{ sub } 2, y \text{ sub } 2' \end{array} \tag{10.1.4}$$

The use of subscripts in math is very much like the use of last name initials in elementary school. Maybe you remember Laura C. and Laura M. in your third grade class?

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point.

If we had more than two points, we could use (x_3, y_3), (x_4, y_4), and so on.

Let's see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points (2,3) and (7,6).

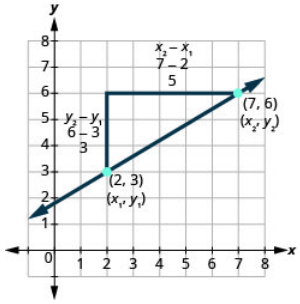


Figure 10.1.10

Since we have two points, we will use subscript notation, $\begin{pmatrix} x_1, y_1 \\ 2, 3 \end{pmatrix} \begin{pmatrix} x_2, y_2 \\ 7, 6 \end{pmatrix}$.

On the graph, we counted the rise of 3 and the run of 5.

Notice that the rise of 3 can be found by subtracting the *y*-coordinates 6 and 3.

$$3 = 6 - 3 \tag{10.1.5}$$

And the run of 5 can be found by subtracting the *x*-coordinates 7 and 2.

$$5 = 7 - 2 \tag{10.1.6}$$

We know $m = \frac{\text{rise}}{\text{run}}$. So $m = \frac{3}{5}$.

We rewrite the rise and run by putting in the coordinates $m = \frac{6-3}{7-2}$

But 6 is y_2, the *y*-coordinate of the second point and 3 is y_1, the *y*-coordinate of the first point.

So we can rewrite the slope using subscript notation. $m = \frac{y2-y1}{7-2}$

Also, 7 is x_2, the *x*-coordinate of the second point and 2 is x_1, the *x*-coordinate of the first point.

So, again, we rewrite the slope using subscript notation. $m = \frac{y2-y1}{x2-x1}$

We've shown that $m = \frac{y2-y1}{x2-x1}$ is really another version of $m = \frac{\text{rise}}{\text{run}}$. We can use this formula to find the slope of a line when we have two points on the line.

SLOPE FORMULA

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{10.1.7}$$

This is the **slope formula**.

The slope is:

$$\begin{array}{c} y \text{ of the second point minus } y \text{ of the first point} \\ \text{over} \\ x \text{ of the second point minus } x \text{ of the first point.} \end{array} \tag{10.1.8}$$

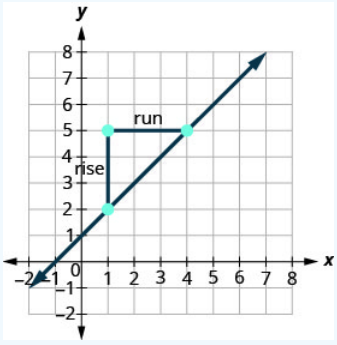
Example 10.1.25

Use the **slope formula** to find the slope of the line between the points (1,2) and (4,5).

Solution

We'll call (1,2) point #1 and (4,5) point #2.	$\begin{pmatrix} x_1, y_1 \\ 1, 2 \end{pmatrix} \begin{pmatrix} x_2, y_2 \\ 4, 5 \end{pmatrix}$
Use the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Substitute the values.	
y of the second point minus y of the first point	$m = \frac{5-2}{x_2 - x_1}$
x of the second point minus x of the first point	$m = \frac{5-2}{4-1}$
Simplify the numerator and the denominator.	$m = \frac{3}{3}$
Simplify.	$m = 1$

Let's confirm this by counting out the slope on a graph using $m = \frac{\text{rise}}{\text{run}}$.



It doesn't matter which point you call point #1 and which one you call point #2. The slope will be the same. Try the calculation yourself.

Try It 10.1.26

Use the slope formula to find the slope of the line through the points: (8,5) and (6,3).

Answer

1

Try It 10.1.27

Use the slope formula to find the slope of the line through the points: (1,5) and (5,9).

Answer

1

Example 10.1.28

Use the slope formula to find the slope of the line through the points (−2,−3) and (−7,4).

Solution

We'll call (-2, -3) point #1 and (-7,4) point #2. $\begin{pmatrix} x_1, y_1 \\ -2, -3 \end{pmatrix} \begin{pmatrix} x_2, y_2 \\ -7, 4 \end{pmatrix}$

Use the slope formula. $m = \frac{y_2 - y_1}{x_2 - x_1}$

Substitute the values.

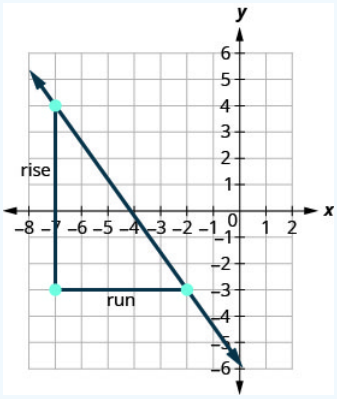
y of the second point minus y of the first point $m = \frac{4-(-3)}{x_2 - x_1}$

x of the second point minus x of the first point $m = \frac{4-(-3)}{-7-(-2)}$

Simplify the numerator and the denominator. $m = \frac{7}{-5}$

Simplify. $m = -\frac{7}{5}$

Let's verify this slope on the graph shown.



$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{-7}{5}$$
$$m = -\frac{7}{5}$$

Try It 10.1.29

Use the slope formula to find the slope of the line through the points: (−3,4) and (2,−1).

Answer

-1

Try It 10.1.30

Use the slope formula to find the slope of the line through the pair of points: (−2,6) and (−3,−4).

Answer

10

Graph a Line Given a Point and the Slope

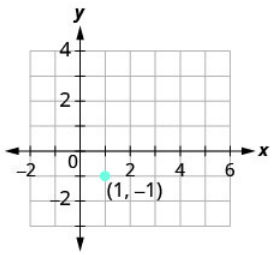
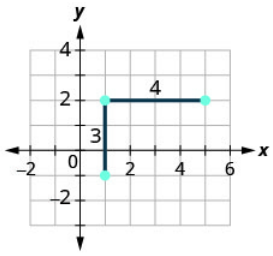
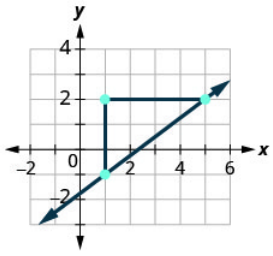
Up to now, in this chapter, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

One other method we can use to graph lines is called the **point–slope method**. We will use this method when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

Example 10.1.31

Graph the line passing through the point $(1,-1)$ whose slope is $m = \frac{3}{4}$.

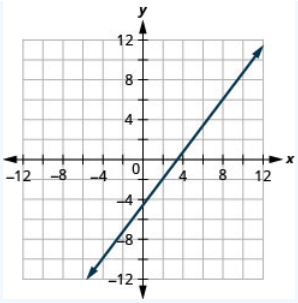
Solution

Step 1. Plot the given point.	Plot (1, –1).	
Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = \frac{3}{4}$ $\frac{\text{rise}}{\text{run}} = \frac{3}{4}$ rise = 3 run = 4
Step 3. Starting at the given point, count out the rise and run to mark the second point.	Start at (1, –1) and count the rise and the run. Up 3 units, right 4 units.	
Step 4. Connect the points with a line.	Connect the two points with a line.	

Try It 10.1.32

Graph the line passing through the point $(2,-2)$ with the slope $m = \frac{4}{3}$.

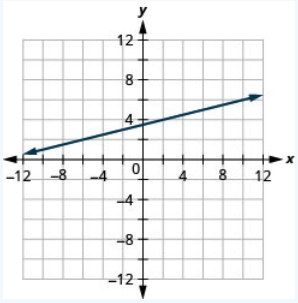
Answer



Try It 10.1.33

Graph the line passing through the point (−2,3) with the slope $m = \frac{1}{4}$.

Answer



GRAPH A LINE GIVEN A POINT AND THE SLOPE.

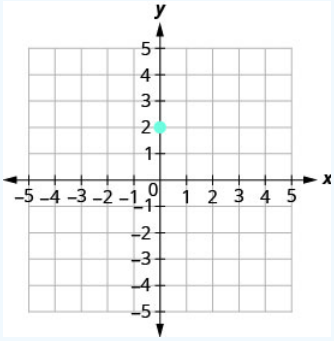
1. Plot the given point.
2. Use the slope formula $m = \frac{\text{rise}}{\text{rise}}$ to identify the rise and the run.
3. Starting at the given point, count out the rise and run to mark the second point.
4. Connect the points with a line.

Example 10.1.34

Graph the line with *y*-intercept 2 whose slope is $m = -\frac{2}{3}$.

Solution

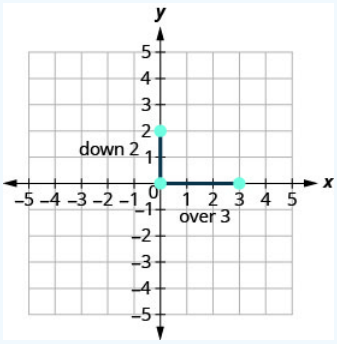
Plot the given point, the *y*-intercept, (0,2).



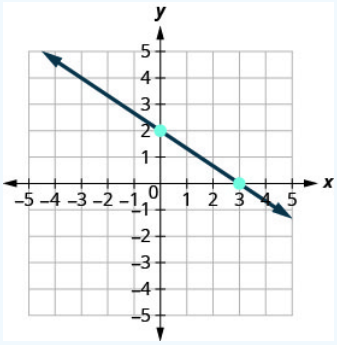
Identify the rise and the run. $m = -\frac{2}{3}$

$$\frac{\text{rise}}{\text{run}} = \frac{-2}{3}$$
$$\text{rise} = -2$$
$$\text{run} = 3$$

Count the rise and the run. Mark the second point.



Connect the two points with a line.

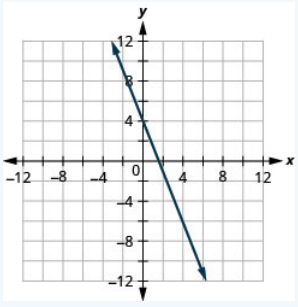


You can check your work by finding a third point. Since the slope is $m = -\frac{2}{3}$, it can be written as $m = \frac{2}{-3}$. Go back to (0,2) and count out the rise, 2, and the run, −3.

? Try It 10.1.35

Graph the line with the *y*-intercept 4 and slope $m = -\frac{5}{2}$.

Answer

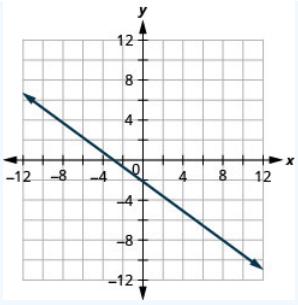


? Try It 10.1.36

Graph the line with the *x*-intercept −3 and slope $m = -\frac{3}{4}$.

Answer

https://math.libretexts.org/@go/page/46283

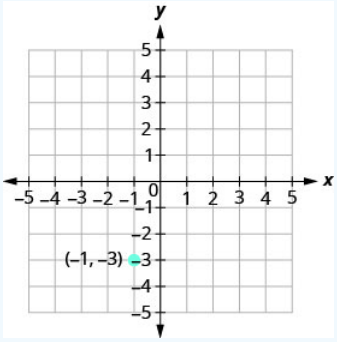


Example 10.1.37

Graph the line passing through the point $(-1,-3)$ whose slope is $m=4$.

Solution

Plot the given point.

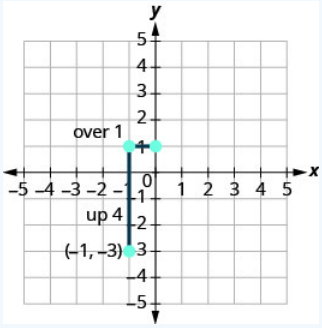


Identify the rise and the run. $m = 4$

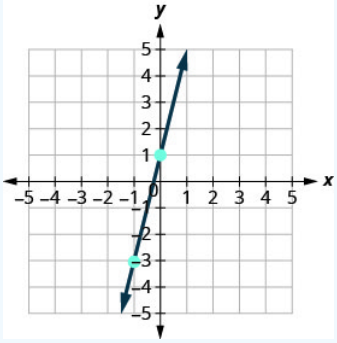
Write 4 as a fraction. $\frac{\text{rise}}{\text{run}} = \frac{4}{1}$

$\text{rise} = 4 \quad \text{run} = 1$

Count the rise and run and mark the second point.



Connect the two points with a line.

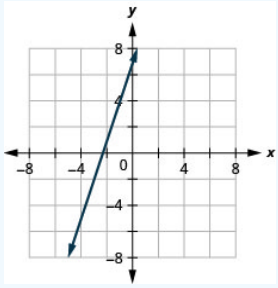


You can check your work by finding a third point. Since the slope is m=4, it can be written as $m = \frac{-4}{-1}$. Go back to (−1,−3) and count out the rise, −4, and the run, −1.

Try It 10.1.38

Graph the line with the point (−2,1) and slope m=3.

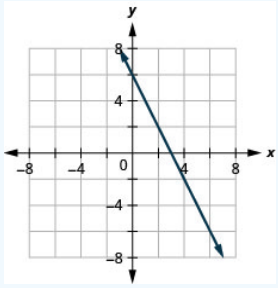
Answer



Try It 10.1.39

Graph the line with the point (4,−2) and slope m=−2.

Answer

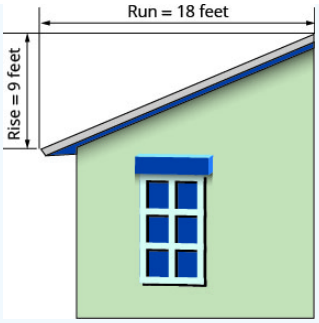


Solve Slope Applications

At the beginning of this section, we said there are many applications of slope in the real world. Let's look at a few now.

Example 10.1.40

The 'pitch' of a building's roof is the slope of the roof. Knowing the pitch is important in climates where there is heavy snowfall. If the roof is too flat, the weight of the snow may cause it to collapse. What is the slope of the roof shown?



Solution

Use the slope formula.	$m = \frac{\text{rise}}{\text{rise}}$
Substitute the values for rise and run.	$m = \frac{9}{18}$
Simplify.	$m = \frac{1}{2}$

The slope of the roof is $\frac{1}{2}$.

The roof rises 1 foot for every 2 feet of horizontal run.

Try It 10.1.41

Use Exercise 10.1.40, substituting the rise = 14 and run = 24.

Answer

$\frac{7}{12}$

Try It 10.1.42

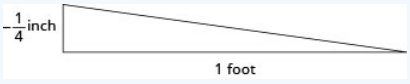
Use Exercise 10.1.40, substituting rise = 15 and run = 36.

Answer

$\frac{5}{12}$

Example 10.1.43

Have you ever thought about the sewage pipes going from your house to the street? They must slope down $\frac{1}{4}$ inch per foot in order to drain properly. What is the required slope?



Solution

Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{-\frac{1}{4}\text{inch}}{1\text{ foot}}$
	$m = \frac{-\frac{1}{4}\text{ inch}}{12\text{ inches}}$
Simplify.	$m = -\frac{1}{48}$

The slope of the pipe is $-\frac{1}{48}$

The pipe drops 1 inch for every 48 inches of horizontal run.

? Try It 10.1.44

Find the slope of a pipe that slopes down $\frac{1}{3}$ inch per foot.

Answer

$-\frac{1}{36}$

? Try It 10.1.45

Find the slope of a pipe that slopes down $\frac{3}{4}$ inch per yard.

Answer

$-\frac{1}{48}$

Access these online resources for additional instruction and practice with understanding slope of a line.

- Practice Slope with a Virtual Geoboard
- Small, Medium, and Large Virtual Geoboards
- Explore Area and Perimeter with a Geoboard

Key Concepts

- **Find the Slope of a Line from its Graph using $m = \frac{\text{rise}}{\text{run}}$**
  1. Locate two points on the line whose coordinates are integers.
  2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.
  3. Count the rise and the run on the legs of the triangle.
  4. Take the ratio of rise to run to find the slope.
- **Graph a Line Given a Point and the Slope**
  1. Plot the given point.
  2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
  3. Starting at the given point, count out the rise and run to mark the second point.
  4. Connect the points with a line.
- **Slope of a Horizontal Line**
  - The slope of a horizontal line, y=b, is 0.
- **Slope of a vertical line**
  - The slope of a vertical line, x=a, is undefined

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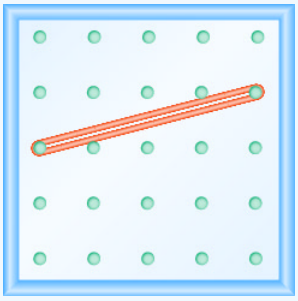
10.1E: Exercises

Practice Makes Perfect

Use Geoboards to Model Slope

In the following exercises, find the slope modeled on each geoboard.

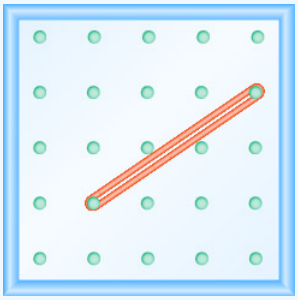
Exercise 10.1E.1



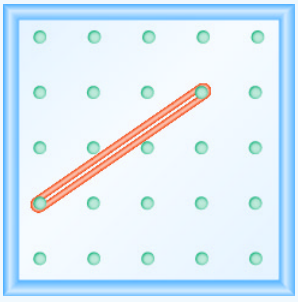
Answer

$\frac{1}{4}$

Exercise 10.1E.2



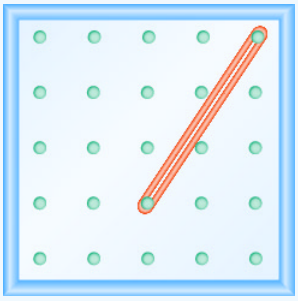
Exercise 10.1E.3



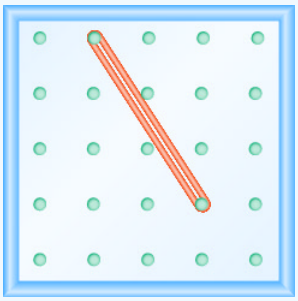
Answer

$\frac{2}{3}$

Exercise 10.1E.4



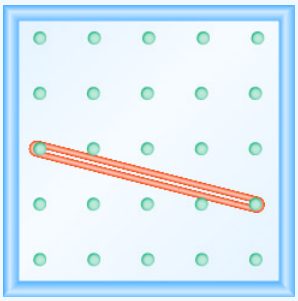
Exercise 10.1E.5



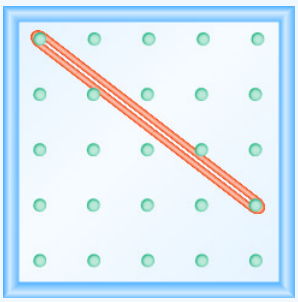
Answer

$\frac{-3}{2} = -\frac{3}{2}$

Exercise 10.1E.6



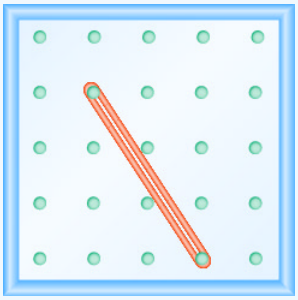
Exercise 10.1E.7



Answer

$-\frac{2}{3}$

Exercise 10.1E.8

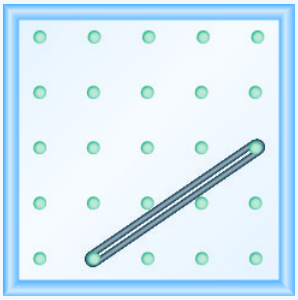


In the following exercises, model each slope. Draw a picture to show your results.

Exercise 10.1E.9

$\frac{2}{3}$

Answer



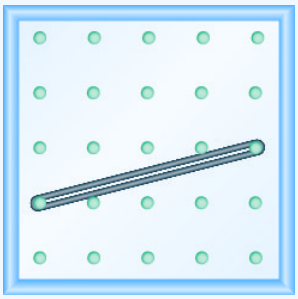
Exercise 10.1E.10

$\frac{3}{4}$

Exercise 10.1E.11

$\frac{1}{4}$

Answer



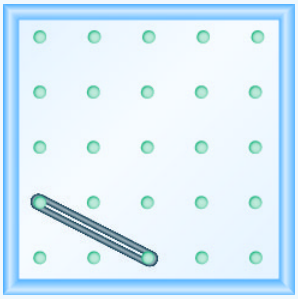
Exercise 10.1E.12

$\frac{4}{3}$

Exercise 10.1E.13

$-\frac{1}{2}$

Answer



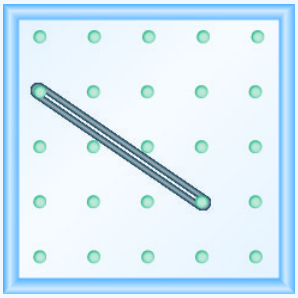
Exercise 10.1E.14

$-\frac{3}{4}$

Exercise 10.1E.15

$-\frac{2}{3}$

Answer



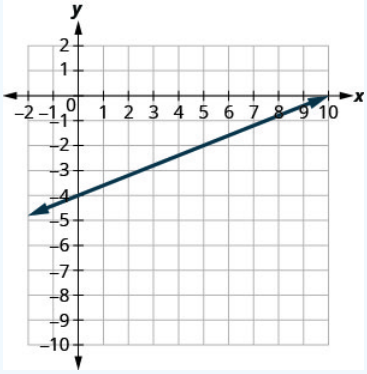
Exercise 10.1E.16

$-\frac{3}{2}$

Use $m = \frac{rise}{run}$ to find the Slope of a Line from its Graph

In the following exercises, find the slope of each line shown.

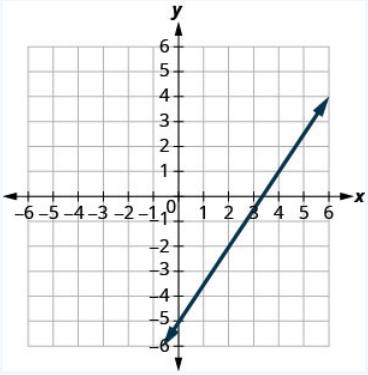
Exercise 10.1E.17



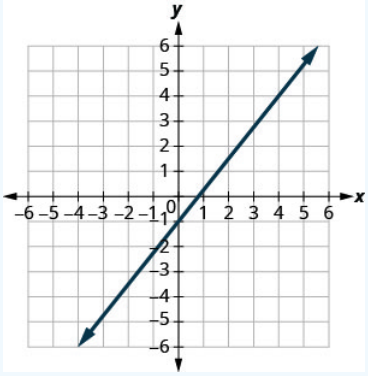
Answer

$\frac{2}{5}$

Exercise $10.1E.18$



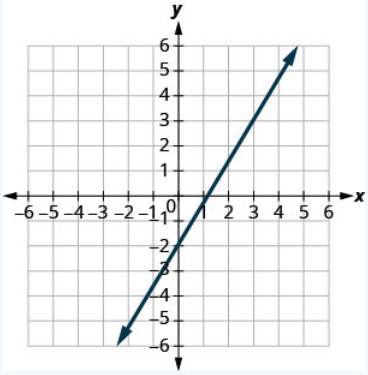
Exercise $10.1E.19$



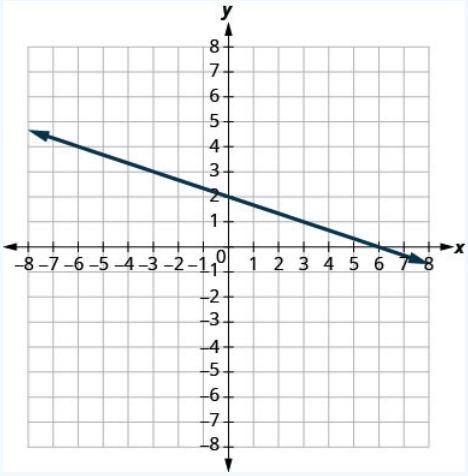
Answer

$\frac{5}{4}$

Exercise $10.1E.20$



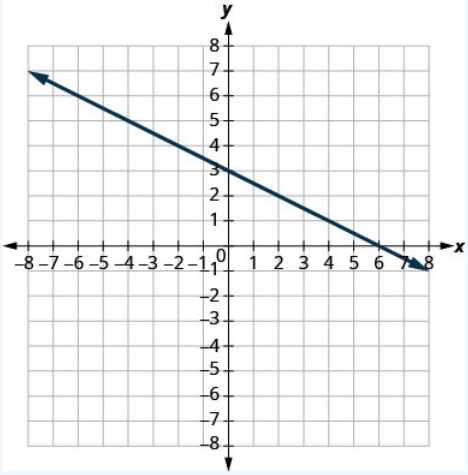
Exercise 10.1E.21



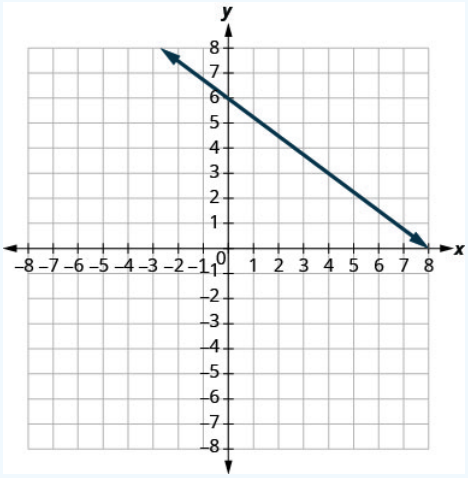
Answer

$-\frac{1}{3}$

Exercise 10.1E.22



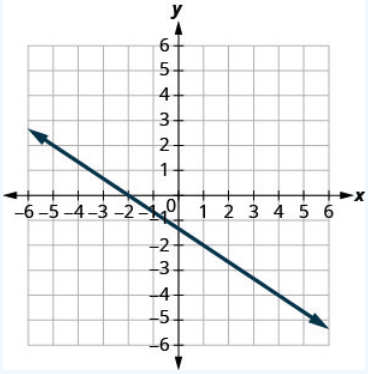
Exercise $10.1E.23$



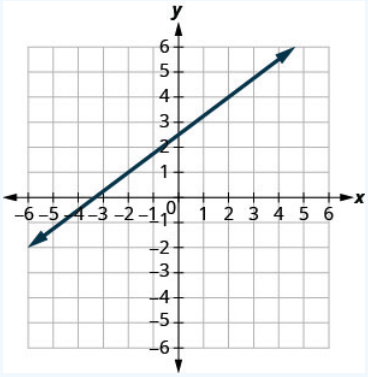
Answer

$-\frac{3}{4}$

Exercise $10.1E.24$



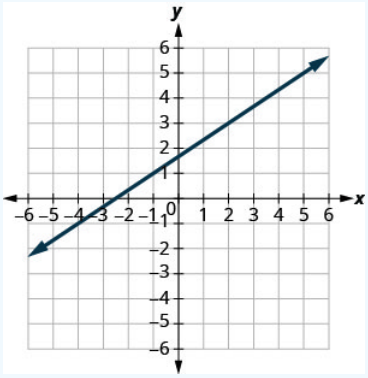
Exercise $10.1E.25$



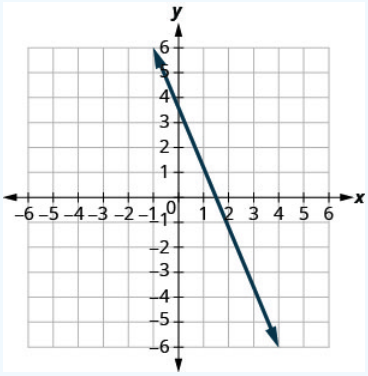
Answer

$\frac{3}{4}$

Exercise $10.1E.26$



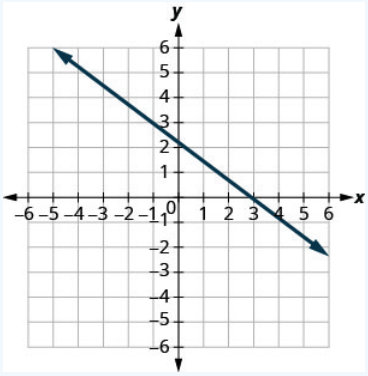
Exercise $10.1E.27$



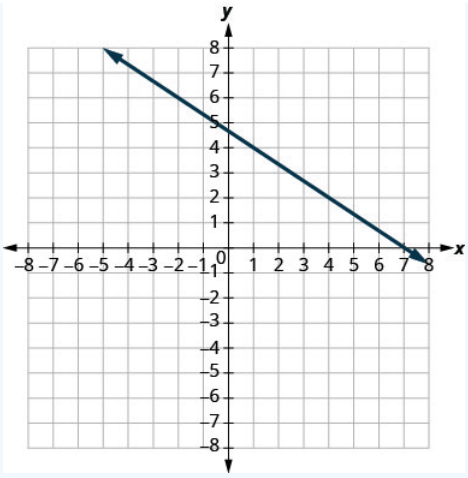
Answer

$-\frac{5}{2}$

Exercise $10.1E.28$



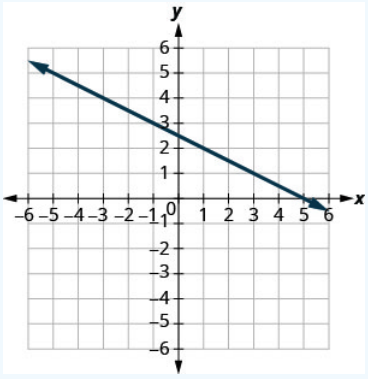
Exercise $10.1E.29$



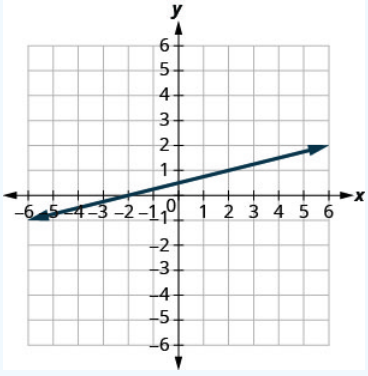
Answer

$-\frac{2}{3}$

Exercise $10.1E.30$



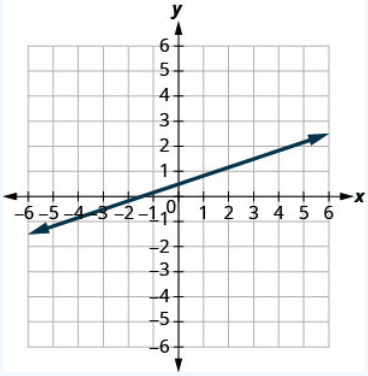
Exercise $10.1E.31$



Answer

$\frac{1}{4}$

Exercise 10.1E.32



Find the Slope of Horizontal and Vertical Lines

In the following exercises, find the slope of each line.

Exercise 10.1E.33

y=3

Answer

0

Exercise 10.1E.34

y=1

Exercise 10.1E.35

x=4

Answer

undefined

Exercise 10.1E.36

x=2

Exercise 10.1E.37

y=−2

Answer

0

Exercise 10.1E.38

y=−3

Exercise 10.1E. 39

x=−5

Answer

undefined

Exercise 10.1E. 40

x=−4

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

Exercise 10.1E. 41

(1,4),(3,9)

Answer

$\frac{5}{2}$

Exercise 10.1E. 42

(2,3),(5,7)

Exercise 10.1E. 43

(0,3),(4,6)

Answer

$\frac{3}{4}$

Exercise 10.1E. 44

(0,1),(5,4)

Exercise 10.1E. 45

(2,5),(4,0)

Answer

$-\frac{5}{2}$

Exercise 10.1E. 46

(3,6),(8,0)

Exercise 10.1E. 47

(−3,3),(4,−5)

Answer

$-\frac{8}{7}$

Exercise 10.1E.48

(−2,4),(3,−1)

Exercise 10.1E.49

(−1,−2),(2,5)

Answer

$\frac{7}{3}$

Exercise 10.1E.50

(−2,−1),(6,5)

Exercise 10.1E.51

(4,−5),(1,−2)

Answer

−1

Exercise 10.1E.52

(3,−6),(2,−2)

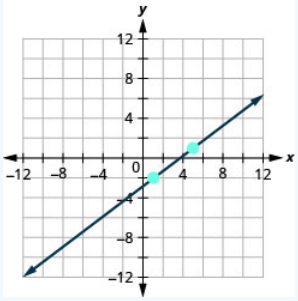
Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

Exercise 10.1E.53

$(1,-2); m=\frac{3}{4}$

Answer



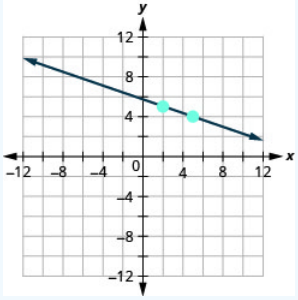
Exercise 10.1E.54

$(1,-1); m=\frac{2}{3}$

Exercise 10.1E.55

$(2, 5); m = -\frac{1}{3}$

Answer



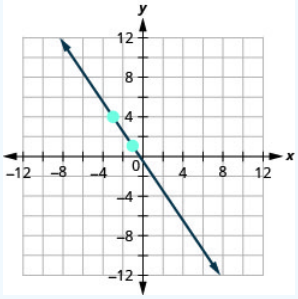
Exercise 10.1E.56

$(1, 4); m = -\frac{1}{2}$

Exercise 10.1E.57

$(-3, 4); m = -\frac{3}{2}$

Answer



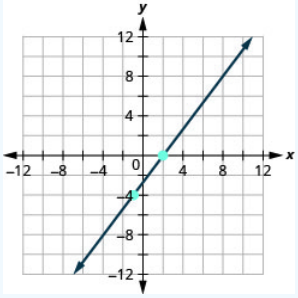
Exercise 10.1E.58

$(-2, 5); m = -\frac{5}{4}$

Exercise 10.1E.59

$(-1, -4); m = \frac{4}{3}$

Answer



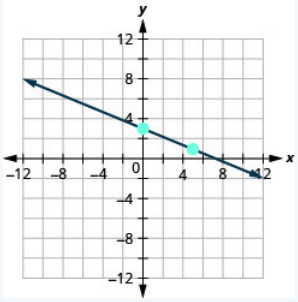
Exercise 10.1E. 60

$(-3,-5); m=\frac{3}{2}$

Exercise 10.1E. 61

y -intercept $3; m=-\frac{2}{5}$

Answer



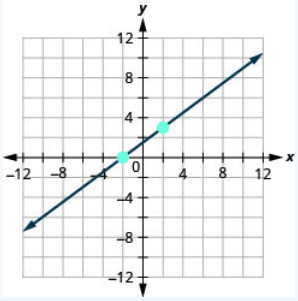
Exercise 10.1E. 62

y -intercept $5; m=-\frac{4}{3}$

Exercise 10.1E. 63

x -intercept $-2; m=\frac{3}{4}$

Answer



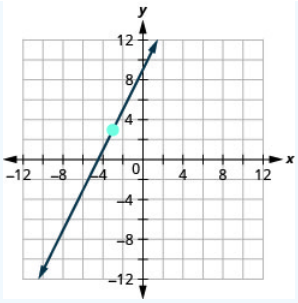
Exercise 10.1E. 64

x -intercept $-1; m=\frac{1}{5}$

Exercise 10.1E. 65

$(-3,3); m=2$

Answer



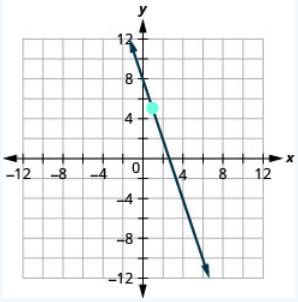
Exercise 10.1E.66

$(-4, 2); m = 4$

Exercise 10.1E.67

$(1, 5); m = -3$

Answer



Exercise 10.1E.68

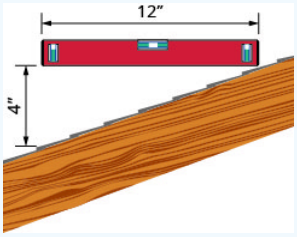
$(2, 3); m = -1$

Everyday Math

Exercise 10.1E.69

Slope of a roof. An easy way to determine the slope of a roof is to set one end of a 12 inch level on the roof surface and hold it level. Then take a tape measure or ruler and measure from the other end of the level down to the roof surface. This will give you the slope of the roof. Builders, sometimes, refer to this as pitch and state it as an "x 12 pitch" meaning $\frac{x}{12}$, where x is the measurement from the roof to the level—the rise. It is also sometimes stated as an "x-in-12 pitch".

1. What is the slope of the roof in this picture?
2. What is the pitch in construction terms?

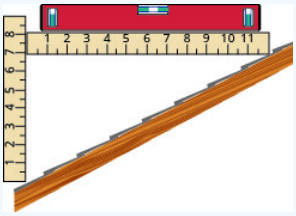


Answer

1. $\frac{1}{3}$
2. 4 12 pitch or 4-in-12 pitch

Exercise 10.1E.70

The slope of the roof shown here is measured with a 12” level and a ruler. What is the slope of this roof?



Exercise 10.1E.71

Road grade. A local road has a grade of 6%. The grade of a road is its slope expressed as a percent. Find the slope of the road as a fraction and then simplify. What rise and run would reflect this slope or grade?

Answer

$\frac{3}{50}$; rise $= 3$, run $= 50$

Exercise 10.1E.72

Highway grade. A local road rises 2 feet for every 50 feet of highway.

1. What is the slope of the highway?
2. The grade of a highway is its slope expressed as a percent. What is the grade of this highway?

Exercise 10.1E.73

Wheelchair ramp. The rules for wheelchair ramps require a maximum 1-inch rise for a 12-inch run.

1. How long must the ramp be to accommodate a 24-inch rise to the door?
2. Create a model of this ramp.

Answer

1. 288 inches (24 feet)
2. Models will vary.

Exercise 10.1E.74

Wheelchair ramp. A 1-inch rise for a 16-inch run makes it easier for the wheelchair rider to ascend a ramp.

1. How long must a ramp be to easily accommodate a 24-inch rise to the door?
2. Create a model of this ramp.

Writing Exercises

Exercise 10.1E.75

What does the sign of the slope tell you about a line?

Answer

When the slope is a positive number the line goes up from left to right. When the slope is a negative number the line goes down from left to right.

Exercise 10.1E.76

How does the graph of a line with slope $m = \frac{1}{2}$ differ from the graph of a line with slope $m = 2$?

Exercise 10.1E.77

Why is the slope of a vertical line “undefined”?

Answer

A vertical line has 0 run and since division by 0 is undefined the slope is undefined.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it
use geoboards to model slope.			
use $m = \frac{rise}{run}$ to find the slope of a line from its graph.			
find the slope of horizontal and vertical lines.			
use the slope formula to find the slope of a line between two points.			
graph a line given a point and the slope.			
solve slope applications.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

geoboard

A geoboard is a board with a grid of pegs on it.

negative slope

A negative slope of a line goes down as you read from left to right.

positive slope

A positive slope of a line goes up as you read from left to right.

rise

The rise of a line is its vertical change.

run

The run of a line is its horizontal change.

slope formula

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

slope of a line

The slope of a line is $m = \frac{\text{rise}}{\text{run}}$. The rise measures the vertical change and the run measures the horizontal change.

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10.2: Use the Slope–Intercept Form of an Equation of a Line

Learning Objectives

By the end of this section, you will be able to:

- Recognize the relation between the graph and the slope–intercept form of an equation of a line
- Identify the slope and y-intercept form of an equation of a line
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope–intercept
- Use slopes to identify parallel lines
- Use slopes to identify perpendicular lines

Note

Before you get started, take this readiness quiz.

1. Add: $\frac{x}{4}+\frac{1}{4}$.
   If you missed this problem, review Exercise 1.7.1.
2. Find the reciprocal of $\frac{3}{7}$.
   If you missed this problem, review Exercise 1.6.19.
3. Solve $2x-3y=12$ for y.
   If you missed this problem, review Exercise 2.6.16.

Recognize the Relation Between the Graph and the Slope–Intercept Form of an Equation of a Line

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using the point–slope method. Once we see how an equation in slope–intercept form and its graph are related, we'll have one more method we can use to graph lines.

In **Graph Linear Equations in Two Variables**, we graphed the line of the equation $y=12x+3$ by plotting points. See Figure 10.2.1. Let's find the slope of this line.

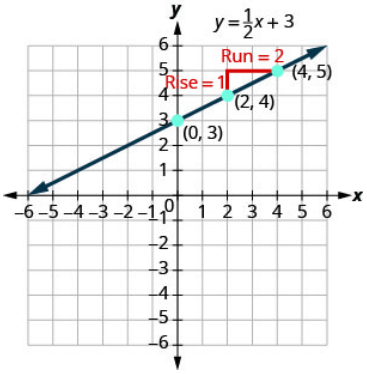


Figure 10.2.1

The red lines show us the rise is 1 and the run is 2. Substituting into the slope formula:

$$m=\frac{\text{rise}}{\text{run}}$$
$$m=\frac{1}{2}$$

What is the y-intercept of the line? The y-intercept is where the line crosses the y-axis, so y-intercept is $(0,3)$. The equation of this line is:

$y = \frac{1}{2}x + 3$

Notice, the line has:

slope $m = \frac{1}{2}$ and y-intercept $(0, 3)$

When a linear equation is solved for y, the coefficient of the x-term is the slope and the constant term is the y-coordinate of the y-intercept. We say that the equation $y = \frac{1}{2}x + 3$ is in slope–intercept form.

$m = \frac{1}{2}$; y-intercept is $(0, 3)$

$y = \frac{1}{2}x + 3$

$y = mx + b$

SLOPE-INTERCEPT FORM OF AN EQUATION OF A LINE

The **slope–intercept form** of an equation of a line with slope m and y-intercept, $(0, b)$ is,

$$y = mx + b \tag{10.2.1}$$

Sometimes the slope–intercept form is called the "y-form."

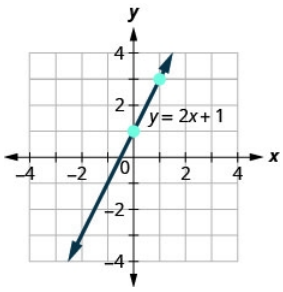
Example 10.2.1

Use the graph to find the slope and y-intercept of the line, $y = 2x + 1$.

Compare these values to the equation $y = mx + b$.

Solution

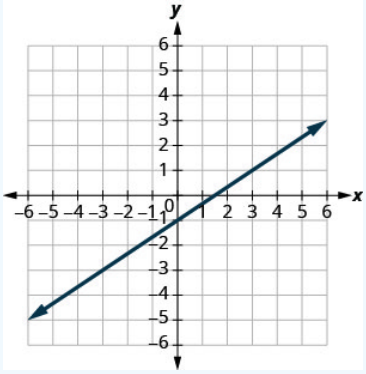
To find the slope of the line, we need to choose two points on the line. We'll use the points $(0, 1)$ and $(1, 3)$.

	
Find the rise and run.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{1}$
	$m = 2$
Find the y-intercept of the line.	The y-intercept is the point $(0, 1)$.
We found slope $m = 2$ and y-intercept $(0, 1)$.	$y = 2x + 1$ $y = mx + b$

The slope is the same as the coefficient of x and the y-coordinate of the y-intercept is the same as the constant term.

Try It 10.2.2

Use the graph to find the slope and y-intercept of the line $y = \frac{2}{3}x - 1$. Compare these values to the equation $y = mx + b$.

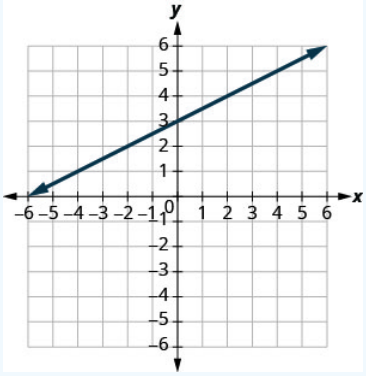


Answer

slope $m = \frac{2}{3}$ and y-intercept $(0, -1)$

Try It 10.2.3

Use the graph to find the slope and y-intercept of the line $y = \frac{1}{2}x + 3$. Compare these values to the equation $y = mx + b$.



Answer

slope $m = \frac{1}{2}$ and y-intercept $(0, 3)$

Identify the Slope and y-Intercept From an Equation of a Line

In **Understand Slope of a Line**, we graphed a line using the slope and a point. When we are given an equation in slope–intercept form, we can use the y-intercept as the point, and then count out the slope from there. Let's practice finding the values of the slope and y-intercept from the equation of a line.

Example 10.2.4

Identify the slope and y-intercept of the line with equation $y = -3x + 5$.

Solution

We compare our equation to the slope–intercept form of the equation.

	$y = mx + b$
Write the equation of the line.	$y = -3x + 5$
Identify the slope.	$m = -3$
Identify the y-intercept.	y-intercept is $(0, 5)$

Try It 10.2.5

Identify the slope and y-intercept of the line $y = \frac{2}{5}x - 1$.

Answer

slope $\frac{2}{5}$; y-intercept $(0,-1)$

Try It 10.2.6

Identify the slope and y-intercept of the line $y = -\frac{4}{3}x + 1$.

Answer

slope $-\frac{4}{3}$; y-intercept $(0,1)$

When an equation of a line is not given in slope–intercept form, our first step will be to solve the equation for y.

Example 10.2.7

Identify the slope and y-intercept of the line with equation $x + 2y = 6$.

Solution

This equation is not in slope–intercept form. In order to compare it to the slope–intercept form we must first solve the equation for y.

Solve for y.	$x + 2y = 6$
Subtract x from each side.	$2y = -x + 6$
Divide both sides by 2.	$\frac{2y}{2} = \frac{-x+6}{2}$
Simplify.	$\frac{2y}{2} = \frac{-x}{2} + \frac{6}{2}$
(Remember: $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$)	
Simplify.	$y = -\frac{1}{2}x + 3$
Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{1}{2}x + 3$
Identify the slope.	$m = -\frac{1}{2}$
Identify the y-intercept.	y-intercept is $(0, 3)$

Try It 10.2.8

Identify the slope and y-intercept of the line $x + 4y = 8$.

Answer

slope $-\frac{1}{4}$; y-intercept $(0,2)$

Try It 10.2.9

Identify the slope and y-intercept of the line $3x + 2y = 12$.

Answer

slope $-\frac{3}{2}$; y-intercept (0,6)

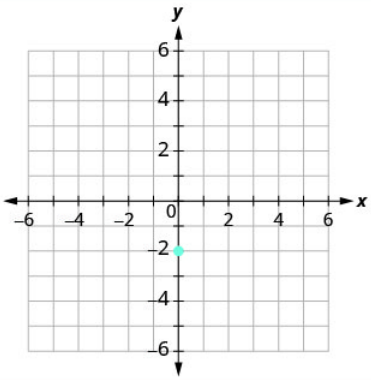
Graph a Line Using its Slope and Intercept

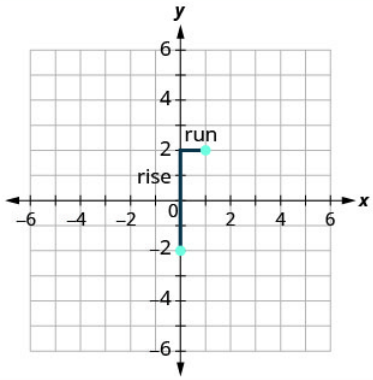
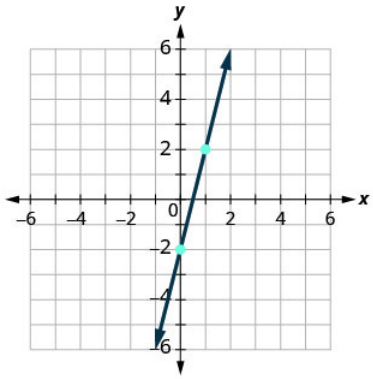
Now that we know how to find the slope and y-intercept of a line from its equation, we can graph the line by plotting the y-intercept and then using the slope to find another point.

Example 10.2.10: How to Graph a Line Using its Slope and Intercept

Graph the line of the equation $y = 4x - 2$ using its slope and y-intercept.

Solution

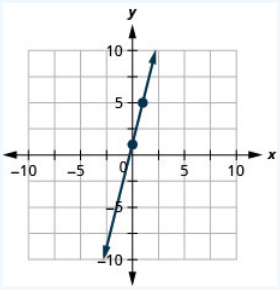
Step 1. Find the slope–intercept form of the equation.	This equation is in slope–intercept form.	$y = 4x - 2$
Step 2. Identify the slope and y-intercept.	Use $y = mx + b$ Find the slope. Find the y-intercept.	$y = mx + b$ $y = 4x + (-2)$ $m = 4$ $b = -2$, $(0, -2)$
Step 3. Plot the y-intercept.	Plot $(0, -2)$.	
Step 4. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = 4$ $\frac{\text{rise}}{\text{run}} = \frac{4}{1}$ rise $= 4$ run $= 1$

Step 5. Starting at the *y*-intercept, count out the rise and run to mark the second point.	Start at (0, –2) and count the rise and the run. Up 4, right 1.	
Step 6. Connect the points with a line.	Connect the two points with a line.	

? Try It 10.2.11

Graph the line of the equation $y = 4x + 1$ using its slope and y-intercept.

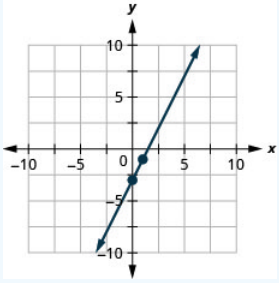
Answer



? Try It 10.2.12

Graph the line of the equation $y = 2x - 3$ using its slope and y-intercept.

Answer



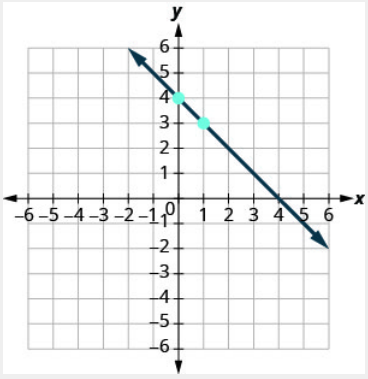
GRAPH A LINE USING ITS SLOPE AND y-INTERCEPT.

1. Find the slope-intercept form of the equation of the line.
2. Identify the slope and y-intercept.
3. Plot the y-intercept.
4. Use the slope formula $\frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
5. Starting at the y-intercept, count out the rise and run to mark the second point.
6. Connect the points with a line.

Example 10.2.13

Graph the line of the equation $y = -x + 4$ using its slope and y-intercept.

Solution

	$y = mx + b$
The equation is in slope–intercept form.	$y = -x + 4$
Identify the slope and y-intercept.	$m = -1$
	y-intercept is $(0, 4)$
Plot the y-intercept.	See graph below.
Identify the rise and the run.	$m = \frac{-1}{1}$
Count out the rise and run to mark the second point.	rise -1, run 1
Draw the line.	
To check your work, you can find another point on the line and make sure it is a solution of the equation. In the graph we see the line goes through $(4, 0)$.	

check.

$y = -x + 4$

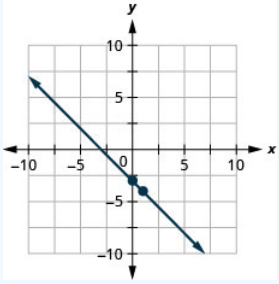
$0 \stackrel{?}{=} -4 + 4$

$0 = 0 \checkmark$

Try It 10.2.14

Graph the line of the equation $y = -x - 3$ using its slope and y-intercept.

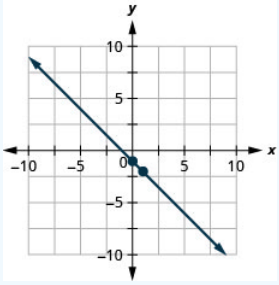
Answer



Try It 10.2.15

Graph the line of the equation $y = -x - 1$ using its slope and y-intercept.

Answer

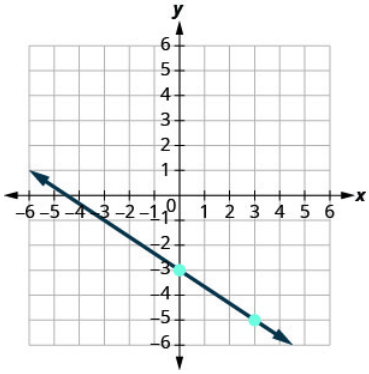


Example 10.2.16

Graph the line of the equation $y = -\frac{2}{3}x - 3$ using its slope and y-intercept.

Solution

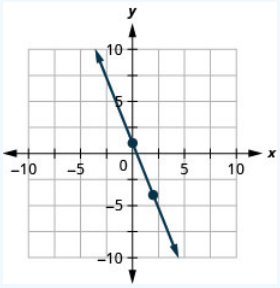
	$y = mx + b$
The equation is in slope–intercept form.	$y = -\frac{2}{3}x - 3$
Identify the slope and y-intercept.	$m = -\frac{2}{3}$; y-intercept is $(0, -3)$
Plot the y-intercept.	See graph below.
Identify the rise and the run.	
Count out the rise and run to mark the second point.	

Draw the line.	

Try It 10.2.17

Graph the line of the equation $y = -\frac{5}{2}x + 1$ using its slope and y-intercept.

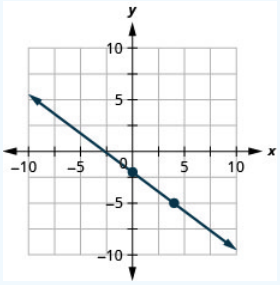
Answer



Try It 10.2.18

Graph the line of the equation $y = -\frac{3}{4}x - 2$ using its slope and y-intercept.

Answer

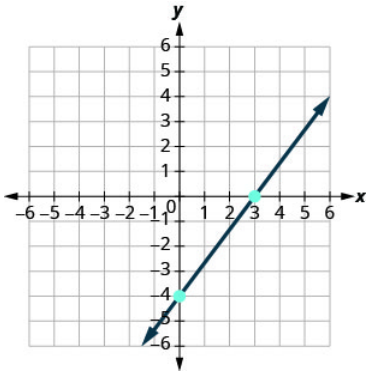


Example 10.2.19

Graph the line of the equation $4x - 3y = 12$ using its slope and y-intercept.

Solution

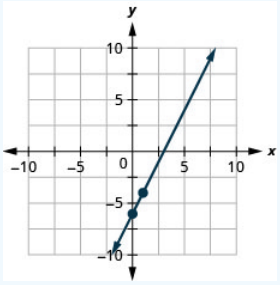
	$4x - 3y = 12$
Find the slope–intercept form of the equation.	$-3y = -4x + 12$
	$-\frac{3y}{3} = \frac{-4x+12}{-3}$

The equation is now in slope–intercept form.	$y = \frac{4}{3}x - 4$
Identify the slope and y-intercept.	$m = \frac{4}{3}$
	y-ntercept is $(0, -4)$
Plot the y-intercept.	See graph below.
Identify the rise and the run; count out the rise and run to mark the second point.	
Draw the line.	

Try It 10.2.20

Graph the line of the equation $2x - y = 6$ using its slope and y-intercept.

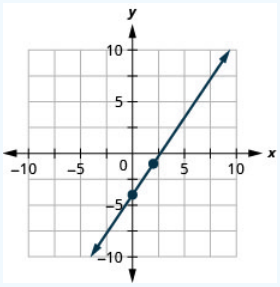
Answer



Try It 10.2.21

Graph the line of the equation $3x - 2y = 8$ using its slope and y-intercept.

Answer



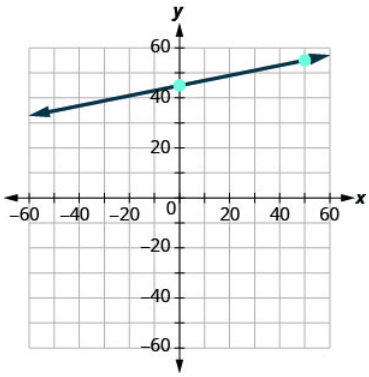
We have used a grid with x and y both going from about -10 to 10 for all the equations we've graphed so far. Not all linear equations can be graphed on this small grid. Often, especially in applications with real-world data, we'll need to extend the axes to bigger positive or smaller negative numbers.

Example 10.2.22

Graph the line of the equation $y = 0.2x + 45$ using its slope and y-intercept.

Solution

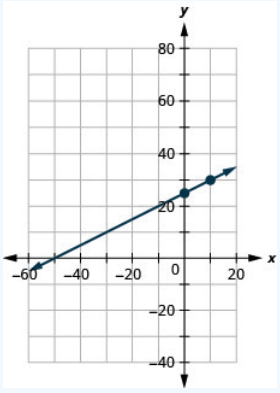
We'll use a grid with the axes going from about -80 to 80.

	$y = mx + b$
The equation is in slope–intercept form.	$y = 0.2x + 45$
Identify the slope and y-intercept.	$m = 0.2$
	The y-intercept is $(0, 45)$
Plot the y-intercept.	See graph below.
Count out the rise and run to mark the second point. The slope is $m = 0.2$; in fraction form this means $m = \frac{2}{10}$. Given the scale of our graph, it would be easier to use the equivalent fraction $m = \frac{10}{50}$.	
Draw the line.	

Try It 10.2.23

Graph the line of the equation $y = 0.5x + 25$ using its slope and y-intercept.

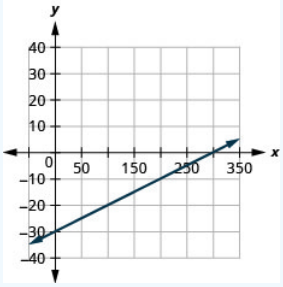
Answer



? Try It 10.2.24

Graph the line of the equation $y = 0.1x - 30$ using its slope and y-intercept.

Answer



Now that we have graphed lines by using the slope and y-intercept, let's summarize all the methods we have used to graph lines. See Figure 10.2.2

Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
x | y	$y = mx + b$	x | y 0 | | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and *y*-intercept. Start at the *y*-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Figure 10.2.2

Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope–intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier. Generally, plotting points is not the most efficient way to graph a line. We saw better methods in sections 4.3, 4.4, and earlier in this section. Let's look for some patterns to help determine the most convenient method to graph a line.

Here are six equations we graphed in this chapter, and the method we used to graph each of them.

$$\begin{array}{lll} \#1 & \text{Equation} & \text{Method} \\ \#2 & x = 2 & \text{Vertical line} \\ \#3 & y = 4 & \text{Horizontal line} \\ \#4 & -x + 2y = 6 & \text{Intercepts} \\ \#5 & 4x - 3y = 12 & \text{Intercepts} \\ \#6 & y = 4x - 2 & \text{Slope-intercept} \\ \#7 & y = -x + 4 & \text{Slope-intercept} \end{array} \tag{10.2.2}$$

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is constant; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In equations #3 and #4, both x and y are on the same side of the equation. These two equations are of the form $Ax + By = C$. We substituted $y = 0$ to find the x-intercept and $x = 0$ to find the y-intercept, and then found a third point by choosing another value for x or y.

Equations #5 and #6 are written in slope–intercept form. After identifying the slope and y-intercept from the equation we used them to graph the line.

This leads to the following strategy.

STRATEGY FOR CHOOSING THE MOST CONVENIENT METHOD TO GRAPH A LINE

Consider the form of the equation.

- If it only has one variable, it is a vertical or horizontal line.
  - $x = a$ is a vertical line passing through the x-axis at a.
  - $y = b$ is a horizontal line passing through the y-axis at b.
- If y is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and y-intercept.
  - Identify the slope and y-intercept and then graph.
- If the equation is of the form $Ax + By = C$, find the intercepts.
  - Find the x- and y-intercepts, a third point, and then graph.

Example 10.2.25

Determine the most convenient method to graph each line.

1. $y = -6$
2. $5x - 3y = 15$
3. $x = 7$
4. $y = \frac{2}{5}x - 1$.

Solution

1. $y = -6$
This equation has only one variable, y. Its graph is a horizontal line crossing the y-axis at -6.

2. $5x - 3y = 15$
This equation is of the form $Ax + By = C$. The easiest way to graph it will be to find the intercepts and one more point.

3. $x = 7$
There is only one variable, x. The graph is a vertical line crossing the x-axis at 7.

4. $y = \frac{2}{5}x - 1$
Since this equation is in $y = mx + b$ form, it will be easiest to graph this line by using the slope and y-intercept.

Try It 10.2.26

Determine the most convenient method to graph each line:

1. $3x + 2y = 12$
2. $y = 4$
3. $y = \frac{1}{5}x - 4$
4. $x = -7$

Answer

1. intercepts
2. horizontal line
3. slope–intercept
4. vertical line

Try It 10.2.27

Determine the most convenient method to graph each line:

1. $x = 6$
2. $y = -\frac{3}{4}x + 1$
3. $y = -8$
4. $4x - 3y = -1$

Answer

1. vertical line
2. slope–intercept
3. horizontal line
4. intercepts

Graph and Interpret Applications of Slope–Intercept

Many real-world applications are modeled by linear equations. We will take a look at a few applications here so you can see how equations written in slope–intercept form relate to real-world situations.

Usually when a linear equation models a real-world situation, different letters are used for the variables, instead of x and y. The variable names remind us of what quantities are being measured.

Example 10.2.28

The equation $F = \frac{9}{5}C + 32$ is used to convert temperatures, C, on the Celsius scale to temperatures, F, on the Fahrenheit scale.

1. Find the Fahrenheit temperature for a Celsius temperature of 0.
2. Find the Fahrenheit temperature for a Celsius temperature of 20.
3. Interpret the slope and F-intercept of the equation.
4. Graph the equation.

Solution

1.

Find the Fahrenheit temperature for a Celsius temperature of 0.	$F = \frac{9}{5}C + 32$
Find F when $C = 0$.	$F = \frac{9}{5}(0) + 32$
Simplify.	$F = 32$

2.

$$\begin{array}{ll} \text{Find the Fahrenheit temperature for a Celsius temperature of } 20. & F = \frac{9}{5}C + 32 \\ \text{Find } F \text{ when } C = 20. & F = \frac{9}{5}(20) + 32 \\ \text{Simplify.} & F = 36 + 32 \\ \text{Simplify.} & F = 68 \end{array} \tag{10.2.3}$$

3. Interpret the slope and F-intercept of the equation.

Even though this equation uses F and C, it is still in slope–intercept form.

$y = mx + b$

$F = mC + b$

$F = \frac{9}{5}C + 32$

The slope, $\frac{9}{5}$, means that the temperature Fahrenheit (F) increases 9 degrees when the temperature Celsius (C) increases 5 degrees.

The F-intercept means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.

4. Graph the equation.

We'll need to use a larger scale than our usual. Start at the F-intercept $(0, 32)$ then count out the rise of 9 and the run of 5 to get a second point. See Figure $10.2.3$

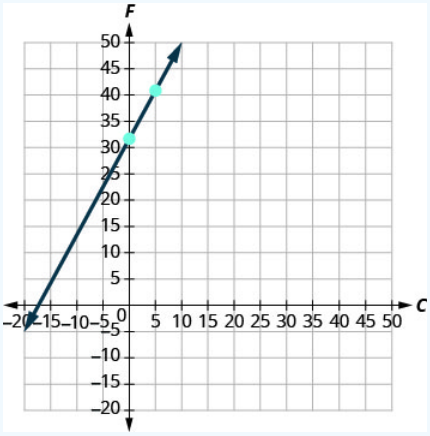


Figure 10.2.3

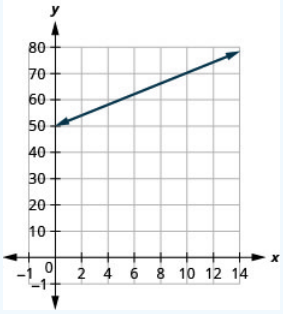
Try It 10.2.29

The equation $h = 2s + 50$ is used to estimate a woman's height in inches, h, based on her shoe size, s.

1. Estimate the height of a child who wears women's shoe size 0.
2. Estimate the height of a woman with shoe size 8.
3. Interpret the slope and h-intercept of the equation.
4. Graph the equation.

Answer

1. 50 inches
2. 66 inches
3. The slope, 2, means that the height, h, increases by 2 inches when the shoe size, s, increases by 1. The h-intercept means that when the shoe size is 0, the height is 50 inches.
4.

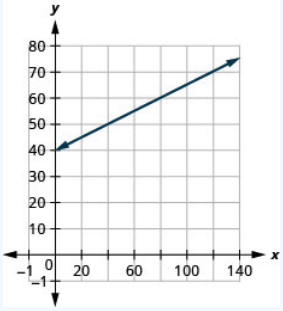


Try It 10.2.30

The equation $T = \frac{1}{4}n + 40$ is used to estimate the temperature in degrees Fahrenheit, T, based on the number of cricket chirps, n, in one minute.

1. Estimate the temperature when there are no chirps.
2. Estimate the temperature when the number of chirps in one minute is 100.
3. Interpret the slope and T-intercept of the equation.
4. Graph the equation.

Answer

1. 40 degrees
2. 65 degrees
3. The slope, $\frac{1}{4}$, means that the temperature Fahrenheit (F) increases 1 degree when the number of chirps, n, increases by 4. The T-intercept means that when the number of chirps is 0, the temperature is 40°.
4. 

The cost of running some types business has two components—a *fixed cost* and a *variable cost*. The fixed cost is always the same regardless of how many units are produced. This is the cost of rent, insurance, equipment, advertising, and other items that must be paid regularly. The variable cost depends on the number of units produced. It is for the material and labor needed to produce each item.

Example 10.2.31

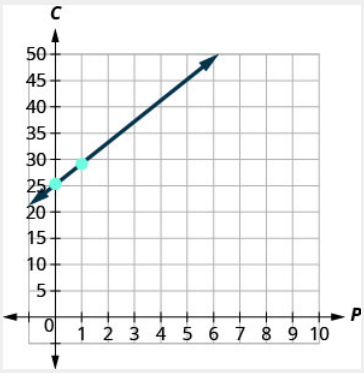
Stella has a home business selling gourmet pizzas. The equation $C = 4p + 25$ models the relation between her weekly cost, C, in dollars and the number of pizzas, p, that she sells.

1. Find Stella's cost for a week when she sells no pizzas.
2. Find the cost for a week when she sells 15 pizzas.
3. Interpret the slope and C-intercept of the equation.
4. Graph the equation.

Solution

1. Find Stella's cost for a week when she sells no pizzas.	$C = 4p + 25$
Find C when $p = 0$.	$C = 4(0) + 25$
Simplify.	$C = 25$
	Stella's fixed cost is \$25 when she sells no pizzas.
2. Find the cost for a week when she sells 15 pizzas.	$C = 4p + 25$
Find C when $p = 15$.	$C = 4(15) + 25$
Simplify.	$C = 60 + 25$
	$C = 85$
	Stella's costs are \$85 when she sells 15 pizzas.
3. Interpret the slope and C-intercept of the equation.	$y = mx + b$ $C = 4p + 25$
	The slope, 4, means that the cost increases by \$4 for each pizza Stella sells. The C-intercept means that even when Stella sells no pizzas, her costs for the week are \$25.

4. Graph the equation. We'll need to use a larger scale than our usual. Start at the C-intercept $(0, 25)$ then count out the rise of 4 and the run of 1 to get a second point.



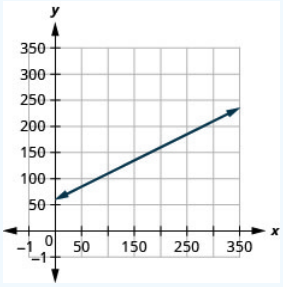
Try It 10.2.32

Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, C, in dollars and the number of miles, m, that he drives.

1. Find Sam's cost for a week when he drives 0 miles.
2. Find the cost for a week when he drives 250 miles.
3. Interpret the slope and C-intercept of the equation.
4. Graph the equation.

Answer

1. $\$60$
2. $\$185$
3. The slope, 0.5, means that the weekly cost, C, increases by $\$0.50$ when the number of miles driven, n, increases by 1. The C-intercept means that when the number of miles driven is 0, the weekly cost is $\$60$.
4.



Try It 10.2.33

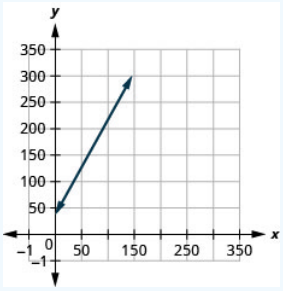
Loreen has a calligraphy business. The equation $C = 1.8n + 35$ models the relation between her weekly cost, C, in dollars and the number of wedding invitations, n, that she writes.

1. Find Loreen's cost for a week when she writes no invitations.
2. Find the cost for a week when she writes 75 invitations.
3. Interpret the slope and C-intercept of the equation.
4. Graph the equation.

Answer

1. $\$35$
2. $\$170$
3. The slope, 1.8, means that the weekly cost, C, increases by $\$1.80$ when the number of invitations, n, increases by 1. The C-intercept means that when the number of invitations is 0, the weekly cost is $\$35$.

4.



Use Slopes to Identify Parallel Lines

The slope of a line indicates how steep the line is and whether it rises or falls as we read it from left to right. Two lines that have the same slope are called parallel lines. Parallel lines never intersect.

Figure 10.2.4.

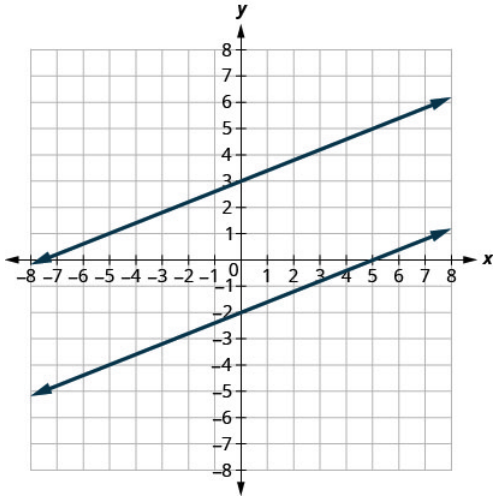


Figure 10.2.4: Verify that both lines have the same slope, $m = \frac{2}{5}$, and different y-intercepts.

What about vertical lines? The slope of a vertical line is undefined, so vertical lines don't fit in the definition above. We say that vertical lines that have different x-intercepts are parallel. See Figure $10.2.5$

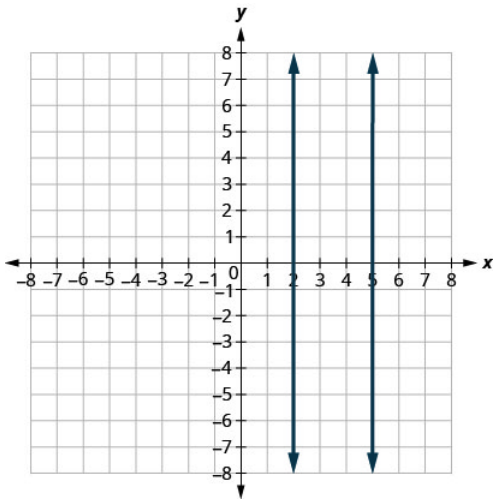


Figure 10.2.5: Vertical lines with different x-intercepts are parallel.

PARALLEL LINES

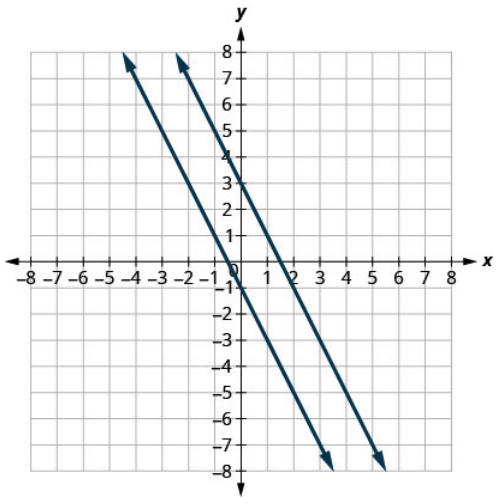
Parallel lines are lines in the same plane that do not intersect.

- Parallel lines have the same slope and different y-intercepts.
- If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
- Parallel vertical lines have different x-intercepts.

Let's graph the equations $y = -2x + 3$ and $2x + y = -1$ on the same grid. The first equation is already in slope–intercept form: $y = -2x + 3$. We solve the second equation for y:

$$\begin{aligned} 2x + y &= -1 \\ y &= -2x - 1 \end{aligned}$$

Graph the lines.



Notice the lines look parallel. What is the slope of each line? What is the y-intercept of each line?

$$\begin{array}{llll} y & = mx + b & y = mx + b \\ y & = -2x + 3 & y = -2x - 1 \\ m & = -2 & m = -2 \\ b & = 3, (0, 3) & b = -1, (0, -1) \end{array} \tag{10.2.4}$$

The slopes of the lines are the same and the y-intercept of each line is different. So we know these lines are parallel.

Since parallel lines have the same slope and different y-intercepts, we can now just look at the slope–intercept form of the equations of lines and decide if the lines are parallel.

Example 10.2.34

Use slopes and y-intercepts to determine if the lines $3x-2y=6$ and $y=\frac{3}{2}x+1$ are parallel.

Solution

Solve the first equation for y.

$$\begin{aligned} 3x-2y &= 6 \\ -2y &= -3x+6 \\ \frac{-2y}{-2} &= \frac{-3x+6}{-2} \\ y &= \frac{3}{2}x-3 \end{aligned}$$

The equation is now in slope–intercept form.

The equation of the second line is already in slope–intercept form.

Identify the slope and y-intercept of both lines.

$y=\frac{3}{2}x+1$	$y=\frac{3}{2}x-3$
$y=mx+b$	$y=mx+b$
$m=\frac{3}{2}$	$m=\frac{3}{2}$
y-intercept is $(0,1)$	y-intercept is $(0,-3)$

The lines have the same slope and different y-intercepts and so they are parallel. You may want to graph the lines to confirm whether they are parallel.

Try It 10.2.35

Use slopes and y-intercepts to determine if the lines $2x+5y=5$ and $y=-\frac{2}{5}x-4$ are parallel.

Answer

parallel

Try It 10.2.36

Use slopes and y-intercepts to determine if the lines $4x-3y=6$ and $y=\frac{4}{3}x-1$ are parallel.

Answer

parallel

Example 10.2.37

Use slopes and y-intercepts to determine if the lines $y=-4$ and $y=3$ are parallel.

Solution

Write each equation in slope-intercept form.	$y=-4$	and	$y=3$
Since there is no x term we write $0x$.	$y=0x-4$		$y=0x+3$
Identify the slope and y-intercept of both lines.	$y=mx+b$		$y=mx+b$
	$m=0$		$m=0$
	y-intercept is $(0,4)$		y-intercept is $(0,3)$

The lines have the same slope and different y-intercepts and so they are parallel.

There is another way you can look at this example. If you recognize right away from the equations that these are horizontal lines, you know their slopes are both 0. Since the horizontal lines cross the y-axis at $y=-4$ and at $y=3$, we know the y-intercepts are $(0,-4)$ and $(0,3)$. The lines have the same slope and different y-intercepts and so they are parallel.

Try It 10.2.38

Use slopes and y-intercepts to determine if the lines $y=8$ and $y=-6$ are parallel.

Answer

parallel

Try It 10.2.39

Use slopes and y-intercepts to determine if the lines $y=1$ and $y=-5$ are parallel.

Answer

parallel

Example 10.2.40

Use slopes and y-intercepts to determine if the lines $x=-2$ and $x=-5$ are parallel.

Solution

$$x=-2 \text{ and } x=-5 \tag{10.2.5}$$

Since there is no y, the equations cannot be put in slope–intercept form. But we recognize them as equations of vertical lines. Their x-intercepts are -2 and -5. Since their x-intercepts are different, the vertical lines are parallel.

Try It 10.2.41

Use slopes and y-intercepts to determine if the lines $x=1$ and $x=-5$ are parallel.

Answer

parallel

Try It 10.2.42

Use slopes and y-intercepts to determine if the lines $x=8$ and $x=-6$ are parallel.

Answer

parallel

Example 10.2.43

Use slopes and y-intercepts to determine if the lines $y=2x-3$ and $-6x+3y=-9$ are parallel. You may want to graph these lines, too, to see what they look like.

Solution

The first equation is already in slope-intercept form. $y = 2x - 3$

Solve the second equation for y

$$\begin{aligned} -6x + 3y &= -9 \\ 3y &= 6x - 9 \\ \tfrac{3y}{3} &= \tfrac{6x-9}{3} \\ y &= 2x - 3 \end{aligned}$$

The second equation is now in slope–intercept form as well.

Identify the slope and y-intercept of both lines.

$y = 2x - 3$	$y = 2x - 3$
$y = mx + b$	$y = mx + b$
$m = 2$	$m = 2$
The y-intercept is $(0, -3)$	The y-intercept is $(0, -3)$

The lines have the same slope, but they also have the same y-intercepts. Their equations represent the same line. They are not parallel; they are the same line.

Try It 10.2.44

Use slopes and y-intercepts to determine if the lines $y = -\frac{1}{2}x + 1$ and $x + 2y = 2$ are parallel.

Answer

not parallel; same line

Try It 10.2.45

Use slopes and y-intercepts to determine if the lines $y = \frac{3}{4}x - 3$ and $3x - 4y = 12$ are parallel.

Answer

not parallel; same line

Use Slopes to Identify Perpendicular Lines

Let's look at the lines whose equations are $y = \frac{1}{4}x - 1$ and $y = -4x + 2$, shown in Figure 10.2.5

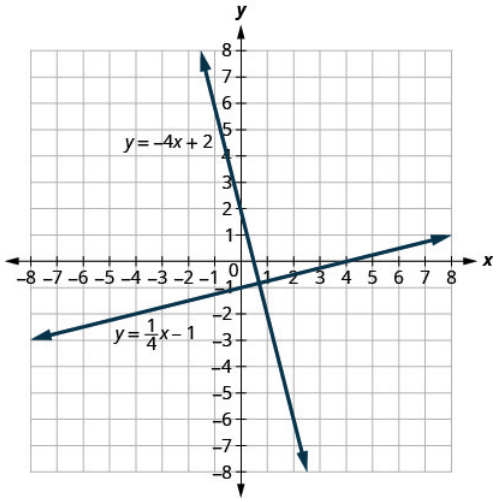


Figure 10.2.5

These lines lie in the same plane and intersect in right angles. We call these lines **perpendicular**.

What do you notice about the slopes of these two lines? As we read from left to right, the line $y = \frac{1}{4}x - 1$ rises, so its slope is positive. The line $y = -4x + 2$ drops from left to right, so it has a negative slope. Does it make sense to you that the slopes of two perpendicular lines will have opposite signs?

If we look at the slope of the first line, $m_1 = \frac{1}{4}$, and the slope of the second line, $m_2 = -4$, we can see that they are *negative reciprocals* of each other. If we multiply them, their product is -1.

$$\begin{array}{c} m_1 \cdot m_2 \\ \frac{1}{4}(-4) \\ -1 \end{array} \tag{10.2.6}$$

This is always true for **perpendicular lines** and leads us to this definition.

PERPENDICULAR LINES

Perpendicular lines are lines in the same plane that form a right angle.

If m_1 and m_2 are the slopes of two perpendicular lines, then:

$$m_1 \cdot m_2 = -1 \text{ and } m_1 = \frac{-1}{m_2} \tag{10.2.7}$$

Vertical lines and horizontal lines are always perpendicular to each other.

We were able to look at the slope–intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope–intercept form of the equation, and then see if the slopes are negative reciprocals. If the product of the slopes is -1, the lines are perpendicular. Perpendicular lines may have the same y-intercepts.

Example 10.2.46

Use slopes to determine if the lines, $y = -5x - 4$ and $x - 5y = 5$ are perpendicular.

Solution

The first equation is already in slope–intercept form: $y = -5x - 4$

Solve the second equation for y.

$$\begin{array}{rcl} x - 5y & = & 5 \\ -5y & = & -x + 5 \\ \frac{-5y}{-5} & = & \frac{-x+5}{-5} \\ y & = & \frac{1}{5}x - 1 \end{array}$$

The second equation is now in slope-intercept form as well.

Identify the slope of each line.

$$\begin{array}{rclrcl} y & = & -5x - 4 & y & = & \frac{1}{5}x - 1 \\ y & = & mx + b & y & = & mx + b \\ m_1 & = & -5 & m_2 & = & \frac{1}{5} \end{array}$$

The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes,

$$\begin{array}{c} m_1 \cdot m_2 \\ -5\left(\frac{1}{5}\right) \\ -1\checkmark \end{array} \tag{10.2.8}$$

Try It 10.2.47

Use slopes to determine if the lines $y = -3x + 2$ and $x - 3y = 4$ are perpendicular.

Answer

perpendicular

Try It 10.2.48

Use slopes to determine if the lines $y = 2x - 5$ and $x + 2y = -6$ are perpendicular.

Answer

perpendicular

Example 10.2.49

Use slopes to determine if the lines, $7x + 2y = 3$ and $2x + 7y = 5$ are perpendicular.

Solution

Solve the equations for y.

$$\begin{aligned} 7x + 2y &= 3 \\ 2y &= -7x + 3 \\ \frac{2y}{2} &= \frac{-7x+3}{2} \\ y &= -\frac{7}{2}x + \frac{3}{2} \end{aligned} \qquad \begin{aligned} 2x + 7y &= 5 \\ 7y &= -2x + 5 \\ \frac{7y}{7} &= \frac{-2x+5}{7} \\ y &= \frac{-2}{7}x + \frac{5}{7} \end{aligned}$$

Identify the slope of each line.

$$\begin{aligned} y &= mx + b \\ m_1 &= -\frac{7}{2} \end{aligned} \qquad \begin{aligned} y &= mx + b \\ m_2 &= -\frac{2}{7} \end{aligned}$$

The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.

Try It 10.2.50

Use slopes to determine if the lines $5x + 4y = 1$ and $4x + 5y = 3$ are perpendicular.

Answer

not perpendicular

Try It 10.2.51

Use slopes to determine if the lines $2x - 9y = 3$ and $9x - 2y = 1$ are perpendicular.

Answer

not perpendicular

Note

Access this online resource for additional instruction and practice with graphs.

- Explore the Relation Between a Graph and the Slope–Intercept Form of an Equation of a Line

Key Concepts

- The slope–intercept form of an equation of a line with slope mm and y-intercept, $(0, b)$ is, $y = mx + b$.
- **Graph a Line Using its Slope and y-Intercept**
  1. Find the slope-intercept form of the equation of the line.
  2. Identify the slope and y-intercept.
  3. Plot the y-intercept.
  4. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
  5. Starting at the y-intercept, count out the rise and run to mark the second point.
  6. Connect the points with a line.
- **Strategy for Choosing the Most Convenient Method to Graph a Line:** Consider the form of the equation.
  - If it only has one variable, it is a vertical or horizontal line.
    $x = a$ is a vertical line passing through the x-axis at a.
    $y = b$ is a horizontal line passing through the y-axis at b.
  - If y is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and y-intercept.
    Identify the slope and y-intercept and then graph.
  - If the equation is of the form $Ax + By = C$, find the intercepts.
    Find the x- and y-intercepts, a third point, and then graph.
- Parallel lines are lines in the same plane that do not intersect.
  - Parallel lines have the same slope and different y-intercepts.
  - If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
  - Parallel vertical lines have different x-intercepts.
- Perpendicular lines are lines in the same plane that form a right angle.
  - If m_1 and m_2 are the slopes of two perpendicular lines, then $m_1 \cdot m_2 = -1$ and $m_1 = \frac{-1}{m_2}$.
  - Vertical lines and horizontal lines are always perpendicular to each other.

Glossary

parallel lines

Lines in the same plane that do not intersect.

perpendicular lines

Lines in the same plane that form a right angle.

slope-intercept form of an equation of a line

The slope–intercept form of an equation of a line with slope mm and y-intercept, $(0, b)$ is, $y = mx + b$.

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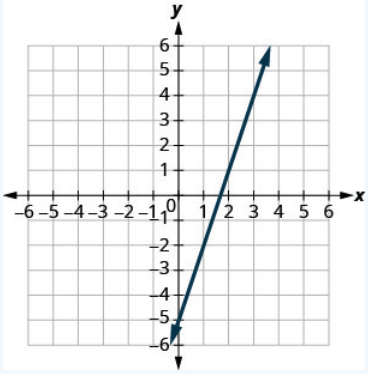
10.2E: Exercises

Practice Makes Perfect

Recognize the Relation Between the Graph and the Slope–Intercept Form of an Equation of a Line

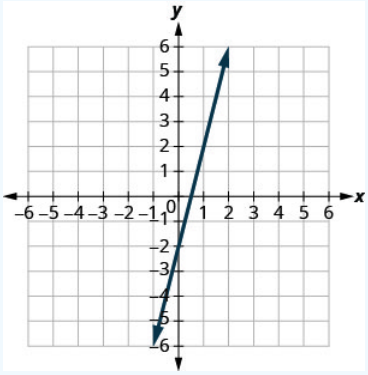
In the following exercises, use the graph to find the slope and y-intercept of each line. Compare the values to the equation $y = mx + b$.

Exercise $10.2E.1$



$y = 3x - 5$

Exercise $10.2E.2$

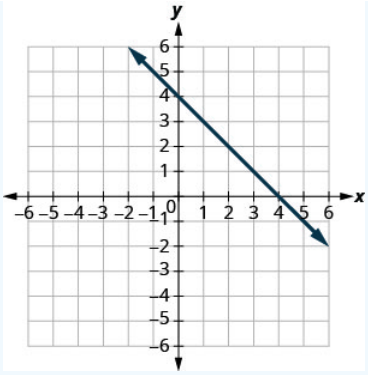


$y = 4x - 2$

Answer

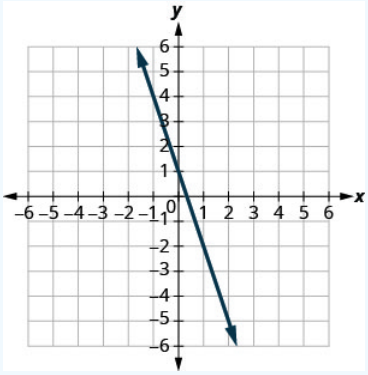
slope $m = 4$ and y-intercept $(0, -2)$

Exercise $10.2E.3$



$y = -x + 4$

Exercise $10.2E.4$

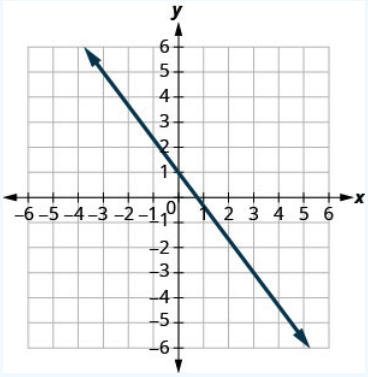


$y = -3x + 1$

Answer

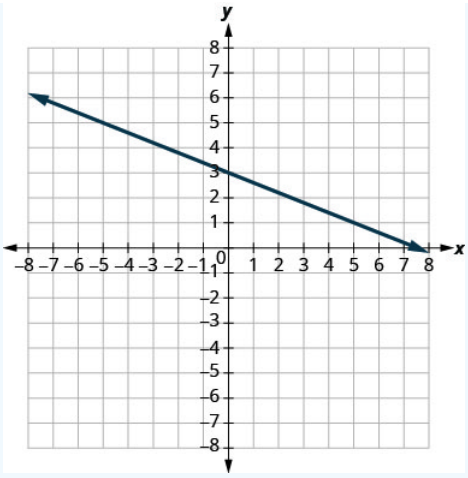
slope $m = -3$ and y-intercept $(0, 1)$

Exercise $10.2E.5$



$y = -\frac{4}{3}x + 1$

Exercise $10.2E.6$



$y = -\frac{2}{5}x + 3$

Answer

slope $m = -\frac{2}{5}$ and y -intercept $(0, 3)$

Identify the Slope and y-Intercept From an Equation of a Line

In the following exercises, identify the slope and y-intercept of each line.

Exercise $10.2E.7$

$y = -7x + 3$

Exercise $10.2E.8$

$y = -9x + 7$

Answer

$m = -9$; y-intercept: $(0, 7)$

Exercise $10.2E.9$

$y = 6x - 8$

Exercise $10.2E.10$

$y = 4x - 10$

Answer

$m = 4$; y-intercept: $(0, -10)$

Exercise $10.2E.11$

$3x + y = 5$

Exercise 10.2E.12

$4x + y = 8$

Answer

$m = -4 \backslash 0$; \(y-intercept: $(0, 8)$

Exercise 10.2E.13

$6x + 4y = 12$

Exercise 10.2E.14

$8x + 3y = 12$

Answer

$m = -\frac{8}{3}$; y-intercept: $(0, 4)$

Exercise 10.2E.15

$5x - 2y = 6$

Exercise 10.2E.16

$7x - 3y = 9$

Answer

$m = \frac{7}{3}$; y-intercept: $(0, -3)$

Graph a Line Using Its Slope and Intercept

In the following exercises, graph the line of each equation using its slope and y-intercept.

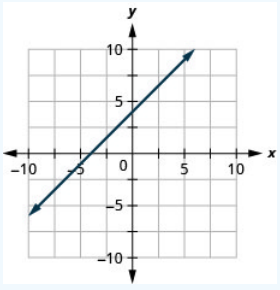
Exercise 10.2E.17

$y = x + 3$

Exercise 10.2E.18

$y = x + 4$

Answer



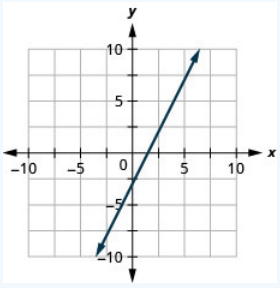
Exercise 10.2E.19

$y = 3x - 1$

Exercise 10.2E.20

$y = 2x - 3$

Answer



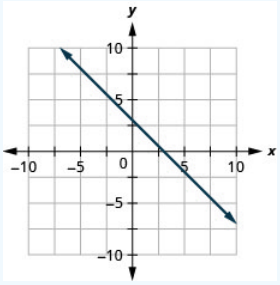
Exercise 10.2E.21

$y = -x + 2$

Exercise 10.2E.22

$y = -x + 3$

Answer



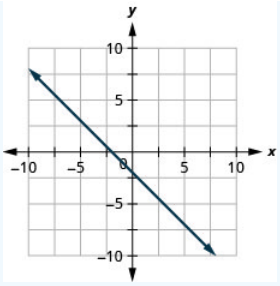
Exercise 10.2E.23

$y = -x - 4$

Exercise 10.2E.24

$y = -x - 2$

Answer



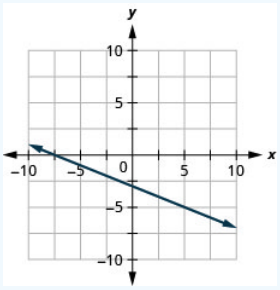
Exercise 10.2E.25

$y = -\frac{3}{4}x - 1$

Exercise 10.2E.26

$y = -\frac{2}{5}x - 3$

Answer



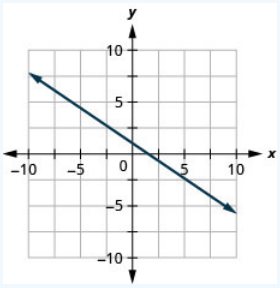
Exercise 10.2E.27

$y = -\frac{3}{5}x + 2$

Exercise 10.2E.28

$y = -\frac{2}{3}x + 1$

Answer



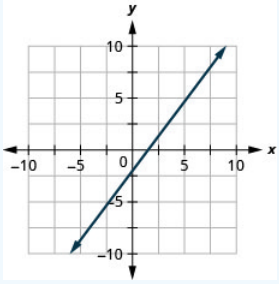
Exercise 10.2E.29

$3x - 4y = 8$

Exercise 10.2E.30

$4x - 3y = 6$

Answer



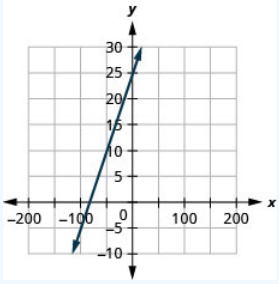
Exercise 10.2E.31

$y = 0.1x + 15$

Exercise 10.2E.32

$y = 0.3x + 25$

Answer



Choose the Most Convenient Method to Graph a Line

In the following exercises, determine the most convenient method to graph each line.

Exercise 10.2E.33

$x = 2$

Exercise 10.2E.34

$y = 4$

Answer

horizontal line

Exercise 10.2E.35

$y = 5$

Exercise 10.2E.36

$x = -3$

Answer

vertical line

Exercise 10.2E.37

$y = -3x + 4$

Exercise 10.2E.38

$y = -5x + 2$

Answer

slope–intercept

Exercise 10.2E.39

$x - y = 5$

Exercise 10.2E.40

$x - y = 1$

Answer

intercepts

Exercise 10.2E.41

$y = \frac{2}{3}x - 1$

Exercise 10.2E.42

$y = \frac{4}{5}x - 3$

Answer

slope–intercept

Exercise 10.2E.43

$y = -3$

Exercise 10.2E.44

$y = -1$

Answer

horizontal line

Exercise 10.2E.45

$3x - 2y = -12$

Exercise 10.2E.46

$2x - 5y = -10$

Answer

intercepts

Exercise 10.2E.47

$y = -\frac{1}{4}x + 3$

Exercise 10.2E.48

$y = -\frac{1}{3}x + 5$

Answer

slope–intercept

Graph and Interpret Applications of Slope–Intercept

Exercise 10.2E.49

The equation $P = 31 + 1.75w$ models the relation between the amount of Tuyet's monthly water bill payment, P, in dollars, and the number of units of water, w, used.

1. Find Tuyet's payment for a month when 0 units of water are used.
2. Find Tuyet's payment for a month when 12 units of water are used.
3. Interpret the slope and P-intercept of the equation.
4. Graph the equation.

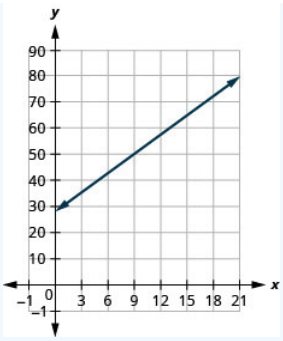
Exercise 10.2E.50

The equation $P = 28 + 2.54w$ models the relation between the amount of Randy's monthly water bill payment, P, in dollars, and the number of units of water, w, used.

1. Find the payment for a month when Randy used 0 units of water.
2. Find the payment for a month when Randy used 15 units of water.
3. Interpret the slope and P-intercept of the equation.
4. Graph the equation.

Answer

1. $28
2. $66.10
3. The slope, 2.54, means that Randy's payment, P, increases by $2.54 when the number of units of water he used, w, ncreases by 1. The P-intercept means that if the number units of water Randy used was 0, the payment would be $28
4.



Exercise $10.2E.51$

Bruce drives his car for his job. The equation $R = 0.575m + 42$ models the relation between the amount in dollars, R, that he is reimbursed and the number of miles, m, he drives in one day.

1. Find the amount Bruce is reimbursed on a day when he drives 0 miles.
2. Find the amount Bruce is reimbursed on a day when he drives 220 miles.
3. Interpret the slope and R-intercept of the equation.
4. Graph the equation.

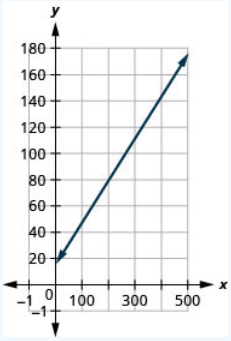
Exercise $10.2E.52$

Janelle is planning to rent a car while on vacation. The equation $C = 0.32m + 15$ models the relation between the cost in dollars, C, per day and the number of miles, m, she drives in one day.

1. Find the cost if Janelle drives the car 0 miles one day.
2. Find the cost on a day when Janelle drives the car 400 miles.
3. Interpret the slope and C-intercept of the equation.
4. Graph the equation.

Answer

1. $\$15$
2. $\$143$
3. The slope, 0.32, means that the cost, C, increases by $\$0.32$ when the number of miles driven, m, increases by 1. The C-intercept means that if Janelle drives 0 miles one day, the cost would be $\$15$.
4.



Exercise $10.2E.53$

Cherie works in retail and her weekly salary includes commission for the amount she sells. The equation $S = 400 + 0.15c$ models the relation between her weekly salary, S, in dollars and the amount of her sales, c, in dollars.

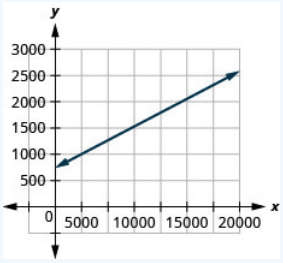
1. Find Cherie's salary for a week when her sales were 0.
2. Find Cherie's salary for a week when her sales were 3600.
3. Interpret the slope and S-intercept of the equation.
4. Graph the equation.

Exercise $10.2E.54$

Patel's weekly salary includes a base pay plus commission on his sales. The equation $S = 750 + 0.09c$ models the relation between his weekly salary, S, in dollars and the amount of his sales, c, in dollars.

1. Find Patel's salary for a week when his sales were 0.
2. Find Patel's salary for a week when his sales were $18,540$.
3. Interpret the slope and S-intercept of the equation.
4. Graph the equation.

Answer

1. $\$750$
2. $\$2418.60$
3. The slope, 0.09, means that Patel's salary, S, increases by $\$0.09$ for every $\$1$ increase in his sales. The S-intercept means that when his sales are $\$0$, his salary is $\$750$
4. 

Exercise $10.2E.55$

Costa is planning a lunch banquet. The equation $C = 450 + 28g$ models the relation between the cost in dollars, C, of the banquet and the number of guests, g.

1. Find the cost if the number of guests is 40.
2. Find the cost if the number of guests is 80.
3. Interpret the slope and C-intercept of the equation.
4. Graph the equation.

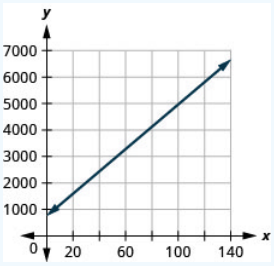
Exercise $10.2E.56$

Margie is planning a dinner banquet. The equation $C = 750 + 42g$ models the relation between the cost in dollars, C, of the banquet and the number of guests, g.

1. Find the cost if the number of guests is 50.
2. Find the cost if the number of guests is 100.
3. Interpret the slope and C-intercept of the equation.
4. Graph the equation.

Answer

1. $\$2850$
2. $\$4950$

3. The slope, 42, means that the cost, C, increases by $\$42$ for when the number of guests increases by 1. The C-intercept means that when the number of guests is 0, the cost would be $\$750$
4. 

Use Slopes to Identify Parallel Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel.

Exercise 10.2E.57

$y = \frac{3}{4}x - 3; \quad 3x - 4y = -2$

Exercise 10.2E.58

$y = \frac{2}{3}x - 1; \quad 2x - 3y = -2$

Answer

parallel

Exercise 10.2E.59

$2x - 5y = -3; \quad y = \frac{2}{5}x + 1$

Exercise 10.2E.60

$3x - 4y = -2; \quad y = \frac{3}{4}x - 3$

Answer

parallel

Exercise 10.2E.61

$2x - 4y = 6; \quad x - 2y = 3$

Exercise 10.2E.62

$6x - 3y = 9; \quad 2x - y = 3$

Answer

not parallel

Exercise 10.2E.63

$4x+2y=6; \quad 6x+3y=3$

Exercise 10.2E.64

$8x+6y=6; \quad 12x+9y=12$

Answer

parallel

Exercise 10.2E.65

$x=5; \quad x=-6$

Exercise 10.2E.66

$x=7; \quad x=-8$

Answer

parallel

Exercise 10.2E.67

$x=-4; \quad x=-1$

Exercise 10.2E.68

$x=-3; \quad x=-2$

Answer

parallel

Exercise 10.2E.69

$y=2; \quad y=6$

Exercise 10.2E.70

$y=5; \quad y=1$

Answer

parallel

Exercise 10.2E.71

$y=-4; \quad y=3$

Exercise 10.2E.72

$y = -1; \quad y = 2$

Answer

parallel

Exercise 10.2E.73

$x - y = 2; \quad 2x - 2y = 4$

Exercise 10.2E.74

$4x + 4y = 8; \quad x + y = 2$

Answer

not parallel

Exercise 10.2E.75

$x - 3y = 6; \quad 2x - 6y = 12$

Exercise 10.2E.76

$5x - 2y = 11; \quad 5x - y = 7$

Answer

not parallel

Exercise 10.2E.77

$3x - 6y = 12; \quad 6x - 3y = 3$

Exercise 10.2E.78

$4x - 8y = 16; \quad x - 2y = 4$

Answer

not parallel

Exercise 10.2E.79

$9x - 3y = 6; \quad 3x - y = 2$

Exercise 10.2E.80

$x - 5y = 10; \quad 5x - y = -10$

Answer

not parallel

Exercise 10.2E.81

$7x-4y=8; \quad 4x+7y=14$

Exercise 10.2E.82

$9x-5y=4; \quad 5x+9y=-1$

Answer

not parallel

Use Slopes to Identify Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are perpendicular.

Exercise 10.2E.83

$3x-2y=8; \quad 2x+3y=6$

Exercise 10.2E.84

$x-4y=8; \quad 4x+y=2$

Answer

perpendicular

Exercise 10.2E.85

$2x+5y=3; \quad 5x-2y=6$

Exercise 10.2E.86

$2x+3y=5; \quad 3x-2y=7$

Answer

perpendicular

Exercise 10.2E.87

$3x-2y=1; \quad 2x-3y=2$

Exercise 10.2E.88

$3x-4y=8; \quad 4x-3y=6$

Answer

not perpendicular

Exercise 10.2E.89

$5x+2y=6; \quad 2x+5y=8$

Exercise 10.2E.90

$2x+4y=3; \quad 6x+3y=2$

Answer

not perpendicular

Exercise 10.2E.91

$4x-2y=5; \quad 3x+6y=8$

Exercise 10.2E.92

$2x-6y=4; \quad 12x+4y=9$

Answer

perpendicular

Exercise 10.2E.93

$6x-4y=5; \quad 8x+12y=3$

Exercise 10.2E.94

$8x-2y=7; \quad 3x+12y=9$

Answer

perpendicular

Everyday Math

Exercise 10.2E.95

The equation $C=\frac{5}{9}F-17.8$ can be used to convert temperatures, F, on the Fahrenheit scale to temperatures, C, on the Celsius scale.

1. Explain what the slope of the equation means.
2. Explain what the C-intercept of the equation means.

Exercise 10.2E.96

The equation $n=4T-160$ is used to estimate the number of cricket chirps, n, in one minute based on the temperature in degrees Fahrenheit, T.

1. Explain what the slope of the equation means.
2. Explain what the n-intercept of the equation means. Is this a realistic situation?

Answer

1. For every increase of one degree Fahrenheit, the number of chirps increases by four.
2. There would be -160 chirps when the Fahrenheit temperature is 0°. (Notice that this does not make sense; this model cannot be used for all possible temperatures.)

Writing Exercises

Exercise $10.2E.97$

Explain in your own words how to decide which method to use to graph a line.

Exercise $10.2E.98$

Why are all horizontal lines parallel?

Answer

Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize the relation between the graph and the slope-intercept form of an equation of a line.			
identify the slope and *y*-intercept from an equation of a line.			
graph a line using its slope and intercept.			
choose the most convenient method to graph a line.			
graph and interpret applications of slope-intercept.			
use slopes to identify parallel lines.			

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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10.3: Find the Equation of a Line

Learning Objectives

By the end of this section, you will be able to:

- Find an equation of the line given the slope and y-intercept
- Find an equation of the line given the slope and a point
- Find an equation of the line given two points
- Find an equation of a line parallel to a given line
- Find an equation of a line perpendicular to a given line

Note

Before you get started, take this readiness quiz.

1. Solve: $\frac{2}{3} = \frac{x}{5}$.
   If you missed this problem, review Exercise 2.2.4.
2. Simplify: $-\frac{2}{5}(x - 15)$.
   If you missed this problem, review Exercise 1.10.34.

How do online retailers know that 'you may also like' a particular item based on something you just ordered? How can economists know how a rise in the minimum wage will affect the unemployment rate? How do medical researchers create drugs to target cancer cells? How can traffic engineers predict the effect on your commuting time of an increase or decrease in gas prices? It's all mathematics.

You are at an exciting point in your mathematical journey as the mathematics you are studying has interesting applications in the real world.

The physical sciences, social sciences, and the business world are full of situations that can be modeled with linear equations relating two variables. Data is collected and graphed. If the data points appear to form a straight line, an equation of that line can be used to predict the value of one variable based on the value of the other variable.

To create a mathematical model of a linear relation between two variables, we must be able to find the equation of the line. In this section we will look at several ways to write the equation of a line. The specific method we use will be determined by what information we are given.

Find an Equation of the Line Given the Slope and *y*-Intercept

We can easily determine the slope and intercept of a line if the equation was written in slope–intercept form, y=mx+b. Now, we will do the reverse—we will start with the slope and *y*-intercept and use them to find the equation of the line.

Example 10.3.1

Find an equation of a line with slope −7 and *y*-intercept (0,−1).

Solution

Since we are given the slope and *y*-intercept of the line, we can substitute the needed values into the slope–intercept form, y=mx+b.

Name the slope.	$m = -7$
Name the *y*-intercept.	*y*-intercept (0, −1)
Substitute the values into y=mx+b.	$y = mx + b$
	$y = -7x + (-1)$
	$y = -7x + -1$

Try It 10.3.2

Find an equation of a line with slope $\frac{2}{5}$ and y-intercept (0,4).

Answer

$y = \frac{2}{5}x + 4$

Try It 10.3.3

Find an equation of a line with slope −1 and y-intercept (0,−3).

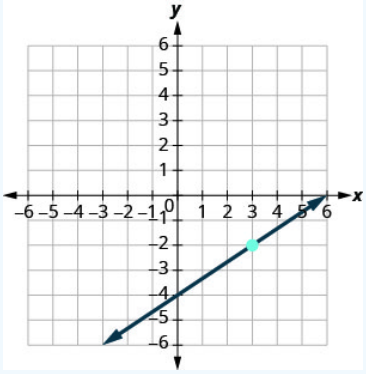
Answer

$y = -x - 3$

Sometimes, the slope and intercept need to be determined from the graph.

Example 10.3.4

Find the equation of the line shown.



Solution

We need to find the slope and y-intercept of the line from the graph so we can substitute the needed values into the slope–intercept form, $y = mx + b$.

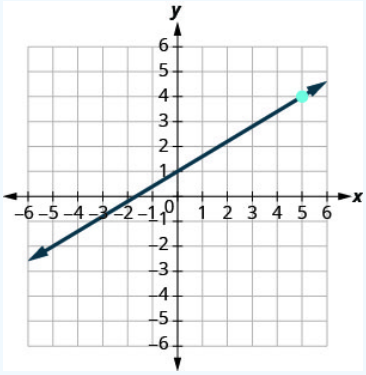
To find the slope, we choose two points on the graph.

The y-intercept is (0,−4) and the graph passes through (3,−2).

Find the slope by counting the rise and run.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{3}$
Find the y-intercept.	y-intercept (0, −4)
Substitute the values into y=mx+b.	$y = mx + b$
	$y = \frac{2}{3}x - 4$

Try It 10.3.5

Find the equation of the line shown in the graph.

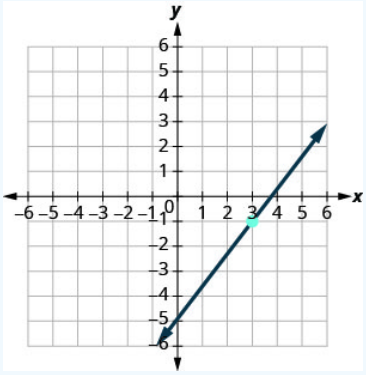


Answer

$y = \frac{3}{5}x + 1$

Try It 10.3.6

Find the equation of the line shown in the graph.



Answer

$y = \frac{4}{3}x - 5$

Find an Equation of the Line Given the Slope and a Point

Finding an equation of a line using the slope–intercept form of the equation works well when you are given the slope and *y*-intercept or when you read them off a graph. But what happens when you have another point instead of the *y*-intercept?

We are going to use the slope formula to derive another form of an equation of the line. Suppose we have a line that has slope mm and that contains some specific point (x_1, y_1) and some other point, which we will just call (x,y). We can write the slope of this line and then change it to a different form.

	m	$= \frac{y-y_1}{x-x_1}$
Multiply both sides of the equation by $x - x_1$.	$m(x-x_1)$	$= \left(\frac{y-y_1}{x-x_1}\right)(x-x_1)$
Simplify.	$m(x-x_1)$	$= y - y_1$
Rewrite the equation with the y terms on the left.	$y - y_1$	$= m(x-x_1)$

This format is called the point–slope form of an equation of a line.

POINT–SLOPE FORM OF AN EQUATION OF A LINE

The **point–slope form** of an equation of a line with slope m and containing the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

We can use the point–slope form of an equation to find an equation of a line when we are given the slope and one point. Then we will rewrite the equation in slope–intercept form. Most applications of linear equations use the the slope–intercept form.

Example 10.3.7: Find an Equation of a Line Given the Slope and a Point

Find an equation of a line with slope $m = \frac{2}{5}$ that contains the point (10,3). Write the equation in slope–intercept form.

Solution

Step 1. Identify the slope.	The slope is given.	$m = \frac{2}{5}$
Step 2. Identify the point.	The point is given.	(x_1, y_1) $(10, 3)$
Step 3. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.		$y - y_1 = m(x - x_1)$
		$y - 3 = \frac{2}{5}(x - 10)$
	Simplify.	$y - 3 = \frac{2}{5}x - 4$
Step 4. Write the equation in slope–intercept form.		$y = \frac{2}{5}x - 1$

Try It 10.3.8

Find an equation of a line with slope $m = \frac{5}{6}$ and containing the point (6,3).

Answer

$y = \frac{5}{6}x - 2$

Try It 10.3.9

Find an equation of a line with slope $m = \frac{2}{3}$ and containing the point (9,2).

Answer

$y = \frac{2}{3}x - 4$

FIND AN EQUATION OF A LINE GIVEN THE SLOPE AND A POINT.

1. Identify the slope.
2. Identify the point.
3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
4. Write the equation in slope–intercept form.

Example 10.3.10

Find an equation of a line with slope $m = -\frac{1}{3}$ that contains the point (6,−4). Write the equation in slope–intercept form.

Solution

Since we are given a point and the slope of the line, we can substitute the needed values into the point–slope form, $y - y_1 = m(x - x_1)$.

Identify the slope.	$m = -\frac{1}{3}$
Identify the point.	$\begin{matrix} x_1, & y_1 \\ 6, & -4 \end{matrix}$
Substitute the values into $y - y_1 = m(x - x_1)$.	$y - y_1 = m(x - x_1)$
	$y - (-4) = -\frac{1}{3}(x - 6)$
Simplify.	$y + 4 = -\frac{1}{3}x + 2$
Write in slope–intercept form.	$y = -\frac{1}{3}x - 2$

Try It 10.3.11

Find an equation of a line with slope $m = -\frac{2}{5}$ and containing the point (10,−5).

Answer

$y = -\frac{2}{5}x - 1$

Try It 10.3.12

Find an equation of a line with slope $m = -\frac{3}{4}$, and containing the point (4,−7).

Answer

$y = -\frac{3}{4}x - 4$

Example 10.3.13

Find an equation of a horizontal line that contains the point (−1,2). Write the equation in slope–intercept form.

Solution

Every horizontal line has slope 0. We can substitute the slope and points into the point–slope form, $y - y_1 = m(x - x_1)$.

Identify the slope.	$m = 0$
Identify the point.	$\begin{matrix} x_1, & y_1 \\ -1, & 2 \end{matrix}$
Substitute the values into $y - y_1 = m(x - x_1)$.	$y - y_1 = m(x - x_1)$
	$y - 2 = 0(x - (-1))$
Simplify.	$y - 2 = 0(x + 1)$
	$y - 2 = 0$
	$y = 2$

Write in slope–intercept form.	It is in *y*-form, but could be written y=0x+2.

Did we end up with the form of a horizontal line, y=a?

? Try It 10.3.14

Find an equation of a horizontal line containing the point (−3,8).

Answer

y = 8

? Try It 10.3.15

Find an equation of a horizontal line containing the point (−1,4).

Answer

y = 4

Find an Equation of the Line Given Two Points

When real-world data is collected, a linear model can be created from two data points. In the next example we'll see how to find an equation of a line when just two points are given.

We have two options so far for finding an equation of a line: slope–intercept or point–slope. Since we will know two points, it will make more sense to use the point–slope form.

But then we need the slope. Can we find the slope with just two points? Yes. Then, once we have the slope, we can use it and one of the given points to find the equation.

✓ Example 10.3.16: Find an Equation of a Line Given Two Points

Find an equation of a line that contains the points (5,4) and (3,6). Write the equation in slope–intercept form.

Solution

Step 1. Find the slope using the given points.	To use the point-slope form, we first find the slope.	$m=\frac{y_2-y_1}{x_2-x_1}$ $m=\frac{6-4}{3-5}$ $m=\frac{2}{-2}$ $m=-1$
Step 2. Choose one point.	Choose either point.	$(\overset{x_1}{5},\ \overset{y_1}{4})$
Step 3. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.	Simplify.	$y-y_1=m(x-x_1)$ $y-4=-1(x-5)$ $y-4=-1x+5$
Step 4. Write the equation in slope–intercept form.		$y=-1x+9$

Use the point (3,6) and see that you get the same equation.

Try It 10.3.17

Find an equation of a line containing the points (3,1) and (5,6).

Answer

$y = \frac{5}{2}x - \frac{13}{2}$

Try It 10.3.18

Find an equation of a line containing the points (1,4) and (6,2).

Answer

$y = -\frac{2}{5}x + \frac{22}{5}$

FIND AN EQUATION OF A LINE GIVEN TWO POINTS.

1. Find the slope using the given points.
2. Choose one point.
3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
4. Write the equation in slope–intercept form.

Example 10.3.19

Find an equation of a line that contains the points (−3,−1) and (2,−2). Write the equation in slope–intercept form.

Solution

Since we have two points, we will find an equation of the line using the point–slope form. The first step will be to find the slope.

Find the slope of the line through (−3, −1) and (2, −2).	$m = \frac{y_2 - y_1}{x_2 - x_1}$
	$m = \frac{-2-(-1)}{2-(-3)}$
	$m = \frac{-1}{5}$
	$m = -\frac{1}{5}$
Choose either point.	$(2, -2)$
Substitute the values into $y - y_1 = m(x - x_1)$.	$y - y_1 = m(x - x_1)$
	$y-(-2) = -\frac{1}{5}(x-2)$
	$y + 2 = -\frac{1}{5}x + \frac{2}{5}$
Write in slope–intercept form.	$y = -\frac{1}{5}x - \frac{8}{5}$

Try It 10.3.20

Find an equation of a line containing the points (−2,−4) and (1,−3).

Answer

$y = \frac{1}{3}x - \frac{10}{3}$

Try It 10.3.21

Find an equation of a line containing the points (−4,−3) and (1,−5).

Answer

$y = -\frac{2}{5}x - \frac{23}{5}$

Example 10.3.22

Find an equation of a line that contains the points (−2,4) and (−2,−3). Write the equation in slope–intercept form.

Solution

Again, the first step will be to find the slope.

Find the slope of the line through $(-2, 4)$ and $(-2, -3)$

$$\begin{aligned} m &= \frac{y_2 - x_1}{x_2 - x_1} \\ m &= \frac{-3-4}{-2-(-2)} \\ m &= \frac{-7}{0} \end{aligned}$$

The slope is undefined.

This tells us it is a vertical line. Both of our points have an *x*-coordinate of −2. So our equation of the line is x=−2. Since there is no yy, we cannot write it in slope–intercept form.

You may want to sketch a graph using the two given points. Does the graph agree with our conclusion that this is a vertical line?

Try It 10.3.23

Find an equation of a line containing the points (5,1) and (5,−4).

Answer

x = 5

Try It 10.3.24

Find an equation of a line containing the points (−4,4) and (−4,3).

Answer

x=−4

We have seen that we can use either the slope–intercept form or the point–slope form to find an equation of a line. Which form we use will depend on the information we are given. This is summarized in Table 10.3.1.

Table 10.3.1

To Write an Equation of a Line		
If given:	Use:	Form:
Slope and *y*-intercept	slope–intercept	y=mx+b

Slope and a point	point–slope	$y - y_1 = m(x - x_1)$
Two points	point–slope	$y - y_1 = m(x - x_1)$

Find an Equation of a Line Parallel to a Given Line

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line. We can use the fact that parallel lines have the same slope. So we will have a point and the slope—just what we need to use the point–slope equation.

First let's look at this graphically.

The graph shows the graph of y=2x−3. We want to graph a line parallel to this line and passing through the point (−2,1).

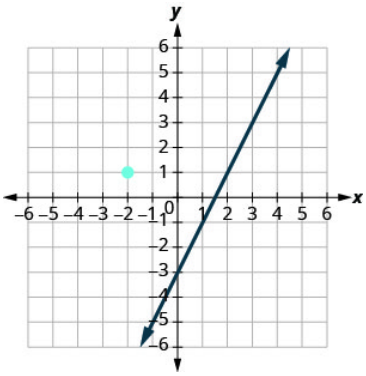


Figure 10.3.1

We know that parallel lines have the same slope. So the second line will have the same slope as y=2x−3. That slope is $m_{\parallel} = 2$. We'll use the notation $m_{\parallel}$ to represent the slope of a line parallel to a line with slope m. (Notice that the subscript ∥ looks like two parallel lines.)

The second line will pass through (−2,1) and have m=2. To graph the line, we start at (−2,1) and count out the rise and run. With m=2 (or $m = \frac{2}{1}$), we count out the rise 2 and the run 1. We draw the line.

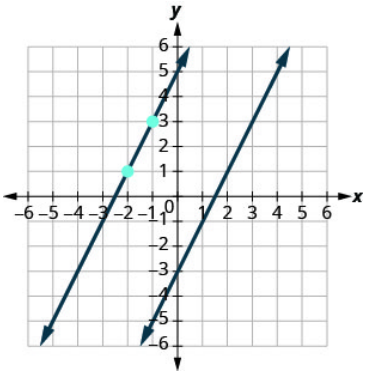


Figure 10.3.2

Do the lines appear parallel? Does the second line pass through (−2,1)?

Now, let's see how to do this algebraically.

We can use either the slope–intercept form or the point–slope form to find an equation of a line. Here we know one point and can find the slope. So we will use the point–slope form.

Example 10.3.25: How to Find an Equation of a Line Parallel to a Given Line

Find an equation of a line parallel to y=2x−3 that contains the point (−2,1). Write the equation in slope–intercept form.

Solution

Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$
Step 2. Find the slope of the parallel line.	Parallel lines have the same slope.	$m_{\parallel} = 2$
Step 3. Identify the point.	The given point is, (–2, 1).	$\begin{pmatrix} x_1 & y_1 \\ -2, & 1 \end{pmatrix}$
Step 4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = 2(x - (-2))$ $y - 1 = 2(x + 2)$ $y - 1 = 2x + 4$
Step 5. Write the equation in slope–intercept form.		$y = 2x + 5$

Does this equation make sense? What is the *y*-intercept of the line? What is the slope?

? Try It 10.3.26

Find an equation of a line parallel to the line y=3x+1 that contains the point (4,2). Write the equation in slope–intercept form.

Answer

y=3x−10

? Try It 10.3.27

Find an equation of a line parallel to the line $y = \frac{1}{2}x - 3$ that contains the point (6,4).

Answer

$y = \frac{1}{2}x + 1$

FIND AN EQUATION OF A LINE PARALLEL TO A GIVEN LINE.

1. Find the slope of the given line.
2. Find the slope of the parallel line.
3. Identify the point.
4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.
5. Write the equation in slope–intercept form.

Find an Equation of a Line Perpendicular to a Given Line

Now, let's consider perpendicular lines. Suppose we need to find a line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point–slope equation, like we did with parallel lines.

The graph shows the graph of y=2x−3. Now, we want to graph a line perpendicular to this line and passing through (−2,1).

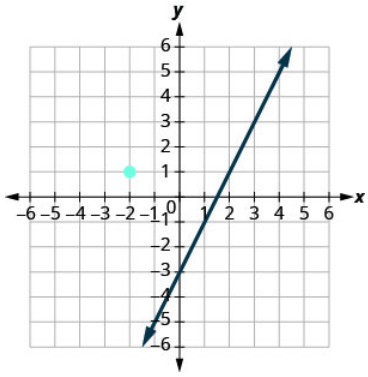


Figure 10.3.3

We know that perpendicular lines have slopes that are negative reciprocals. We'll use the notation $m_{\perp}$ to represent the slope of a line perpendicular to a line with slope m. (Notice that the subscript $_{\perp}$ looks like the right angles made by two perpendicular lines.)

$$\begin{aligned} y &= 2x - 3 \quad \text{perpendicular line} \\ m &= 2 \qquad m_{\perp} = -\tfrac{1}{2} \end{aligned} \tag{10.3.1}$$

We now know the perpendicular line will pass through (−2,1) with $m_{\perp} = -\frac{1}{2}$.

To graph the line, we will start at (−2,1) and count out the rise −1 and the run 2. Then we draw the line.

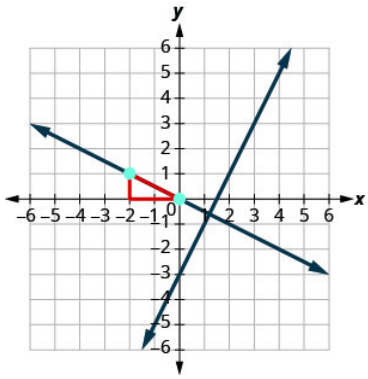


Figure 10.3.4

Do the lines appear perpendicular? Does the second line pass through (−2,1)?

Now, let's see how to do this algebraically. We can use either the slope–intercept form or the point–slope form to find an equation of a line. In this example we know one point, and can find the slope, so we will use the point–slope form.

Example 10.3.28

Find an equation of a line perpendicular to y=2x−3 that contains the point (−2,1). Write the equation in slope–intercept form.

Solution

Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$
Step 2. Find the slope of the perpendicular line.	The slopes of perpendicular lines are negative reciprocals.	$m_{\perp} = -\frac{1}{2}$
Step 3. Identify the point.	The given point is, (–2, 1)	$\begin{pmatrix} x_1 & y_1 \\ -2, & 1 \end{pmatrix}$

Step 4. Substitute the values into the point–slope form, $y-y_1=m(x-x_1)$.	Simplify.	$y-y_1=m(x-x_1)$ $y-1=-\frac{1}{2}(x-(-2))$ $y-1=-\frac{1}{2}(x+2)$ $y-1=-\frac{1}{2}x-1$
Step 5. Write the equation in slope–intercept form.		$y=-\frac{1}{2}x$

Try It 10.3.29

Find an equation of a line perpendicular to the line y=3x+1 that contains the point (4,2). Write the equation in slope–intercept form.

Answer

$y=-\frac{1}{3}x+\frac{10}{3}$

Try It 10.3.30

Find an equation of a line perpendicular to the line $y=\frac{1}{2}x-3$ that contains the point (6,4).

Answer

y=−2x+16

FIND AN EQUATION OF A LINE PERPENDICULAR TO A GIVEN LINE.

1. Find the slope of the given line.
2. Find the slope of the perpendicular line.
3. Identify the point.
4. Substitute the values into the point–slope form, $y-y_1=m(x-x_1)$.
5. Write the equation in slope–intercept form.

Example 10.3.31

Find an equation of a line perpendicular to x=5 that contains the point (3,−2). Write the equation in slope–intercept form.

Solution

Again, since we know one point, the point–slope option seems more promising than the slope–intercept option. We need the slope to use this form, and we know the new line will be perpendicular to x=5. This line is vertical, so its perpendicular will be horizontal. This tells us the $m_\perp=0$.

Identify the point.	$(3$	,	$-2)$
Identify the slope of the perpendicular line.	$m_\perp$	$=$	0
Substitute the values into $y-y_1=m\,(x-x_1)$.	$y-y_1$	$=$	$m\,(x-x_1)$
	$y-(-2)$	$=$	$0(x-3)$
Simplify.	$y+2$	$=$	0
	y	$=$	-2

Sketch the graph of both lines. Do they appear to be perpendicular?

Try It 10.3.32

Find an equation of a line that is perpendicular to the line $x=4$ that contains the point $(4,-5)$. Write the equation in slope–intercept form.

Answer

$y=-5$

Try It 10.3.33

Find an equation of a line that is perpendicular to the line $x=2$ that contains the point $(2,-1)$. Write the equation in slope–intercept form.

Answer

$y=-1$

In Example 10.3.31, we used the point–slope form to find the equation. We could have looked at this in a different way.

We want to find a line that is perpendicular to $x=5$ that contains the point $(3,-2)$. The graph shows us the line $x=5$ and the point $(3,-2)$.

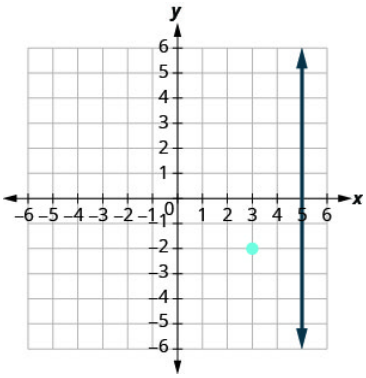


Figure 10.3.5

We know every line perpendicular to a vertical line is horizontal, so we will sketch the horizontal line through $(3,-2)$.

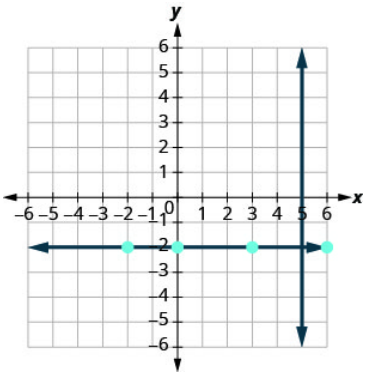


Figure 10.3.6

Do the lines appear perpendicular?

If we look at a few points on this horizontal line, we notice they all have *y*-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x=5$ is $y=-2$.

Example 10.3.34

Find an equation of a line that is perpendicular to y=−4 that contains the point (−4,2). Write the equation in slope–intercept form.

Solution

The line y=−4 is a horizontal line. Any line perpendicular to it must be vertical, in the form x=a. Since the perpendicular line is vertical and passes through (−4,2), every point on it has an *x*-coordinate of −4. The equation of the perpendicular line is x=−4. You may want to sketch the lines. Do they appear perpendicular?

Try It 10.3.35

Find an equation of a line that is perpendicular to the line y=1 that contains the point (−5,1). Write the equation in slope–intercept form.

Answer

x=−5

Try It 10.3.36

Find an equation of a line that is perpendicular to the line y=−5 that contains the point (−4,−5).

Answer

x=−4

Note

Access this online resource for additional instruction and practice with finding the equation of a line.

- Use the Point-Slope Form of an Equation of a Line

Key Concepts

- **To Find an Equation of a Line Given the Slope and a Point**
  1. Identify the slope.
  2. Identify the point.
  3. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.
  4. Write the equation in slope-intercept form.
- **To Find an Equation of a Line Given Two Points**
  1. Find the slope using the given points.
  2. Choose one point.
  3. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.
  4. Write the equation in slope-intercept form.
- **To Write an Equation of a Line**
  - If given slope and y-intercept, use slope–intercept form $y=mx+b$.
  - If given slope and a point, use point–slope form $y-y_1=m(x-x_1)$.
  - If given two points, use point–slope form $y-y_1=m(x-x_1)$.
- **To Find an Equation of a Line Parallel to a Given Line**
  1. Find the slope of the given line.
  2. Find the slope of the parallel line.
  3. Identify the point.
  4. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.

5. Write the equation in slope-intercept form.

- **To Find an Equation of a Line Perpendicular to a Given Line**
  1. Find the slope of the given line.
  2. Find the slope of the perpendicular line.
  3. Identify the point.
  4. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
  5. Write the equation in slope-intercept form.

Glossary

point–slope form

The point–slope form of an equation of a line with slope mm and containing the point (x_1, y_1) is $y - y_1 = m\,(x - x_1)$.

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10.3E: Exercises

Practice Makes Perfect

Find an Equation of the Line Given the Slope and y-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

Exercise 10.3E.1

slope 3 and y-intercept $(0, 5)$

Exercise 10.3E.2

slope 4 and y-intercept $(0, 1)$

Answer

$y = 4x + 1$

Exercise 10.3E.3

slope 6 and y-intercept $(0, -4)$

Exercise 10.3E.4

slope 8 and y-intercept $(0, -6)$

Answer

$y = 8x - 6$

Exercise 10.3E.5

slope -1 and y-intercept $(0, 3)$

Exercise 10.3E.6

slope -1 and y-intercept $(0, 7)$

Answer

$y = -x + 7$

Exercise 10.3E.7

slope -2 and y-intercept $(0, -3)$

Exercise 10.3E.8

slope -3 and y-intercept $(0, -1)$

Answer

$y = -3x - 1$

Exercise 10.3E.9

slope $\frac{3}{5}$ and y-intercept $(0, -1)$

Exercise 10.3E.10

slope $\frac{1}{5}$ and y-intercept $(0, -5)$

Answer

$y = \frac{1}{5}x - 5$

Exercise 10.3E.11

slope $-\frac{3}{4}$ and y-intercept $(0, -2)$

Exercise 10.3E.12

slope $-\frac{2}{3}$ and y-intercept $(0, -3)$

Answer

$y = -\frac{2}{3}x - 3$

Exercise 10.3E.13

slope 0 and y-intercept $(0, -1)$

Exercise 10.3E.14

slope 0 and y-intercept $(0, 2)$

Answer

$y = 2$

Exercise 10.3E.15

slope -3 and y-intercept $(0, 0)$

Exercise 10.3E.16

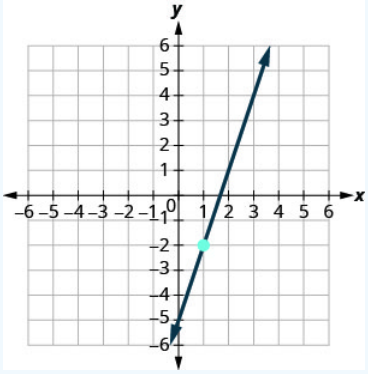
slope -4 and y-intercept $(0, 0)$

Answer

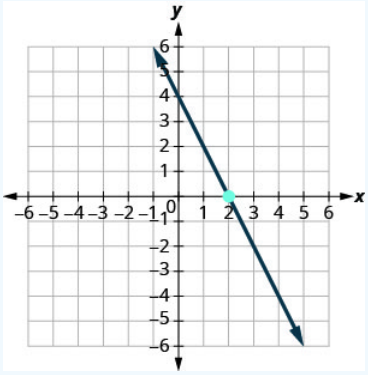
$y = -4x$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.

Exercise 10.3E.17



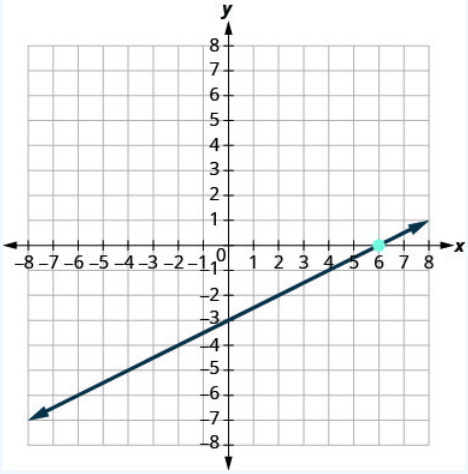
Exercise $10.3E.18$



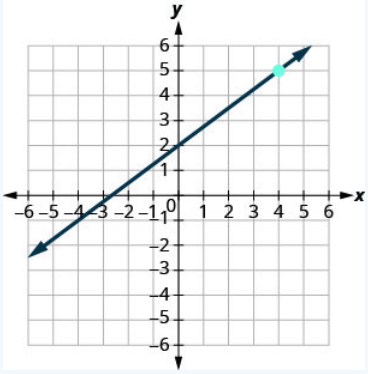
Answer

$y = -2x + 4$

Exercise $10.3E.19$



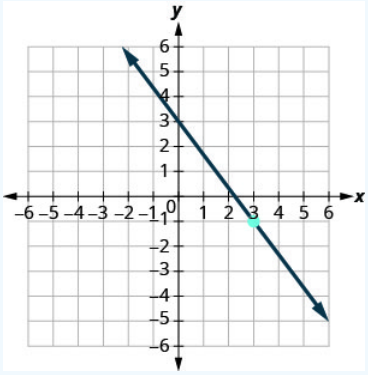
Exercise $10.3E.20$



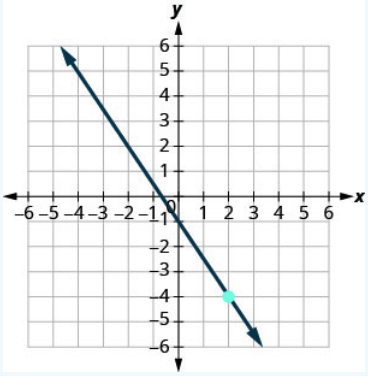
Answer

$y = \frac{3}{4}x + 2$

Exercise $10.3E.21$



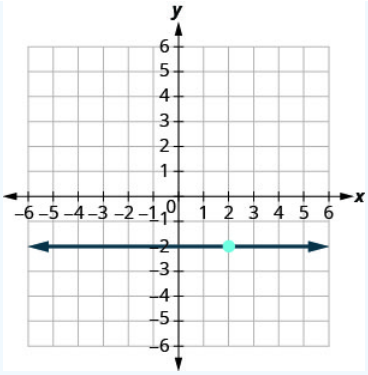
Exercise $10.3E.22$



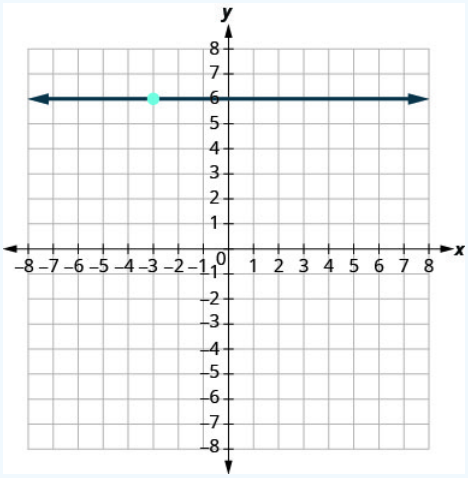
Answer

$y = -\frac{3}{2}x - 1$

Exercise $10.3E.23$



Exercise 10.3E.24



Answer

$y = 6$

Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

Exercise 10.3E.25

$m = \frac{5}{8}$, point $(8, 3)$

Exercise 10.3E.26

$m = \frac{3}{8}$, point $(8, 2)$

Answer

$y = \frac{3}{8}x - 1$

Exercise 10.3E.27

$m = \frac{1}{6}$, point $(6, 1)$

Exercise 10.3E.28

$m = \frac{5}{6}$, point $(6, 7)$

Answer

$y = \frac{5}{6}x + 2$

Exercise 10.3E.29

$m = -\frac{3}{4}$, point $(8, -5)$

Exercise 10.3E.30

$m = -\frac{3}{5}$, point $(10, -5)$

Answer

$y = -\frac{3}{5}x + 1$

Exercise 10.3E.31

$m = -\frac{1}{4}$, point $(-12, -6)$

Exercise 10.3E.32

$m = -\frac{1}{3}$, point $(-9, -8)$

Answer

$y = -\frac{1}{3}x - 11$

Exercise 10.3E.33

Horizontal line containing $(-2, 5)$

Exercise 10.3E.34

Horizontal line containing $(-1, 4)$

Answer

$y = 4$

Exercise 10.3E.35

Horizontal line containing $(-2, -3)$

Exercise 10.3E.36

Horizontal line containing $(-1, -7)$

Answer

$y = -7$

Exercise 10.3E.37

$m = -\frac{3}{2}$, point $(-4, -3)$

Exercise 10.3E.38

$m = -\frac{5}{2}$, point $(-8, -2)$

Answer

$y = -\frac{5}{2}x - 22$

Exercise 10.3E.39

$m = -7$, point $(-1, -3)$

Exercise 10.3E.40

$m = -4$, point $(-2, -3)$

Answer

$y = -4x - 11$

Exercise 10.3E.41

Horizontal line containing $(2, -3)$

Exercise 10.3E.42

Horizontal line containing $(4, -8)$

Answer

$y = -8$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

Exercise 10.3E.43

$(2, 6)$ and $(5, 3)$

Exercise 10.3E.44

$(3, 1)$ and $(2, 5)$

Answer

$y = -4x + 13$

Exercise 10.3E.45

$(4, 3)$ and $(8, 1)$

Exercise 10.3E.46

$(2, 7)$ and $(3, 8)$

Answer

$y = x + 5$

Exercise 10.3E.47

$(-3, -4)$ and $(5, -2)$

Exercise 10.3E.48

$(-5, -3)$ and $(4, -6)$

Answer

$y = -\frac{1}{3}x - \frac{14}{3}$

Exercise 10.3E.49

$(-1, 3)$ and $(-6, -7)$

Exercise 10.3E.50

$(-2, 8)$ and $(-4, -6)$

Answer

$y = 7x + 22$

Exercise 10.3E.51

$(6, -4)$ and $(-2, 5)$

Exercise 10.3E.52

$(3, -2)$ and $(-4, 4)$

Answer

$y = -\frac{6}{7}x + \frac{4}{7}$

Exercise 10.3E.53

$(0, 4)$ and $(2, -3)$

Exercise 10.3E.54

$(0, -2)$ and $(-5, -3)$

Answer

$y = \frac{1}{5}x - 2$

Exercise 10.3E.55

$(7, 2)$ and $(7, -2)$

Exercise 10.3E.56

$(4, 2)$ and $(4, -3)$

Answer

$x = 4$

Exercise 10.3E.57

$(-7, -1)$ and $(-7, -4)$

Exercise 10.3E.58

$(-2, 1)$ and $(-2, -4)$

Answer

$x = -2$

Exercise 10.3E.59

$(6, 1)$ and $(0, 1)$

Exercise 10.3E.60

$(6, 2)$ and $(-3, 2)$

Answer

$y = 2$

Exercise 10.3E.61

$(3, -4)$ and $(5, -4)$

Exercise 10.3E.62

$(-6, -3)$ and $(-1, -3)$

Answer

$y = -3$

Exercise 10.3E.63

$(4, 3)$ and $(8, 0)$

Exercise 10.3E.64

$(0, 0)$ and $(1, 4)$

Answer

$y = 4x$

Exercise 10.3E.65

$(-2, -3)$ and $(-5, -6)$

Exercise 10.3E.66

$(-3, 0)$ and $(-7, -2)$

Answer

$y = \frac{1}{2}x + \frac{3}{2}$

Exercise 10.3E.67

$(8, -1)$ and $(8, -5)$

Exercise 10.3E.68

$(3, 5)$ and $(-7, 5)$

Answer

$y = 5$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

Exercise 10.3E.69

line $y = 4x + 2$, point $(1, 2)$

Exercise 10.3E.70

line $y = 3x + 4$, point $(2, 5)$

Answer

$y = 3x - 1$

Exercise 10.3E.71

line $y = -2x - 3$, point $(-1, 3)$

Exercise 10.3E.72

line $y = -3x - 1$, point $(2, -3)$

Answer

$y = -3x + 3$

Exercise 10.3E.73

line $3x - y = 4$, point $(3, 1)$

Exercise 10.3E.74

line $2x - y = 6$, point $(3, 0)$

Answer

$y = 2x - 6$

Exercise 10.3E.75

line $4x + 3y = 6$, point $(0, -3)$

Exercise 10.3E.76

line $2x + 3y = 6$, point $(0, 5)$

Answer

$y = -\frac{2}{3}x + 5$

Exercise 10.3E.77

line $x = -3$, point $(-2, -1)$

Exercise 10.3E.78

line $x = -4$, point $(-3, -5)$

Answer

$x = -3$

Exercise 10.3E.79

line $x - 2 = 0$, point $(1, -2)$

Exercise 10.3E.80

line $x-6=0$, point $(4,-3)$

Answer

$x=4$

Exercise 10.3E.81

line $y=5$, point $(2,-2)$

Exercise 10.3E.82

line $y=1$, point $(3,-4)$

Answer

$y=-4$

Exercise 10.3E.83

line $y+2=0$, point $(3,-3)$

Exercise 10.3E.84

line $y+7=0$, point $(1,-1)$

Answer

$y=-1$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–intercept form.

Exercise 10.3E.85

line $y=-2x+3$, point $(2,2)$

Exercise 10.3E.86

line $y=-x+5$, point $(3,3)$

Answer

$y=x$

Exercise 10.3E.87

line $y=\frac{3}{4}x-2$, point $(-3,4)$

Exercise 10.3E.88

line $y = \frac{2}{3}x - 4$, point $(2, -4)$

Answer

$y = -\frac{3}{2}x - 1$

Exercise 10.3E.89

line $2x - 3y = 8$, point $(4, -1)$

Exercise 10.3E.90

line $4x - 3y = 5$, point $(-3, 2)$

Answer

$y = -\frac{3}{4}x - \frac{1}{4}$

Exercise 10.3E.91

line $2x + 5y = 6$, point $(0, 0)$

Exercise 10.3E.92

line $4x + 5y = -3$, point $(0, 0)$

Answer

$y = \frac{5}{4}x$

Exercise 10.3E.93

line $y - 3 = 0$, point $(-2, -4)$

Exercise 10.3E.94

line $y - 6 = 0$, point $(-5, -3)$

Answer

$x = -5$

Exercise 10.3E.95

line y-axis, point $(3, 4)$

Exercise 10.3E.96

line y-axis, point $(2, 1)$

Answer

$y = 1$

Mixed Practice

In the following exercises, find the equation of each line. Write the equation in slope–intercept form.

Exercise 10.3E.97

Containing the points $(4, 3)$ and $(8, 1)$

Exercise 10.3E.98

Containing the points $(2, 7)$ and $(3, 8)$

Answer

$y = x + 5$

Exercise 10.3E.99

$m = \frac{1}{6}$, containing point $(6, 1)$

Exercise 10.3E.100

$m = \frac{5}{6}$, containing point $(6, 7)$

Answer

$y = \frac{5}{6}x + 2$

Exercise 10.3E.101

Parallel to the line $4x + 3y = 6$, containing point $(0, -3)$

Exercise 10.3E.102

Parallel to the line $2x + 3y = 6$, containing point $(0, 5)$

Answer

$y = -\frac{2}{3}x + 5$

Exercise 10.3E.103

$m = -\frac{3}{4}$, containing point $(8, -5)$

Exercise 10.3E.104

$m = -\frac{3}{5}$, containing point $(10, -5)$

Answer

$y = -\frac{3}{5}x + 1$

Exercise 10.3E.105

Perpendicular to the line $y-1=0,$ point $(-2,6)$

Exercise 10.3E.106

Perpendicular to the line y-axis, point $(-6,2)$

Answer

$y=2$

Exercise 10.3E.107

Containing the points $(4,3)$ and $(8,1)$

Exercise 10.3E.108

Containing the points $(-2,0)$ and $(-3,-2)$

Answer

$y=x+2$

Exercise 10.3E.109

Parallel to the line $x=-3,$ containing point $(-2,-1)$

Exercise 10.3E.110

Parallel to the line $x=-4,$ containing point $(-3,-5)$

Answer

$x=-3$

Exercise 10.3E.111

Containing the points $(-3,-4)$ and $(2,-5)$

Exercise 10.3E.112

Containing the points $(-5,-3)$ and $(4,-6)$

Answer

$y=-\frac{1}{3}x-\frac{14}{3}$

Exercise 10.3E.113

Perpendicular to the line $x-2y=5,$ containing point $(-2,2)$

Exercise 10.3E.114

Perpendicular to the line $4x+3y=1$, containing point $(0,0)$

Answer

$y=\frac{3}{4}x$

Everyday Math

Exercise 10.3E.115

Cholesterol. The age, x, and LDL cholesterol evel, y, of two men are given by the points $(18,68)$ and $(27,122)$. Find a linear equation that models the relationship between age and LDL cholesterol level.

Exercise 10.3E.116

Fuel consumption. The city mpg, x, and highway mpg, y, of two cars are given by the points $(29,40)$ and $(19,28)$. Find a linear equation that models the relationship between city mpg and highway mpg.

Answer

$y=1.2x+5.2$

Writing Exercises

Exercise 10.3E.117

Why are all horizontal lines parallel?

Exercise 10.3E.118

Explain in your own words why the slopes of two perpendicular lines must have opposite signs.

Answer

Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the equation of the line given the slope and *y*-intercept.			
find an equation of the line given the slope and a point.			
find an equation of the line given two points.			
find an equation of a line parallel to a given line.			
find an equation of a line perpendicular to a given line.			

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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CHAPTER OVERVIEW

11: Systems of Linear Equations

- 11.1: Solve Systems of Linear Equations with Two Variables
  - 11.1E: Exercises
- 11.2: Solve Applications with Systems of Equations
  - 11.2E: Exercises
- 11.3: Solve Mixture Applications with Systems of Equations
  - 11.3E: Exercises
- 11.4: Graphs of Linear Inequalities
  - 11.4E: Exercises

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11.1: Solve Systems of Linear Equations with Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Solve a system of equations by substitution
- Solve a system of equations by elimination
- Choose the most convenient method to solve a system of linear equations

Before you get started, take this readiness quiz.

1. For the equation $y = \frac{2}{3}x - 4$,
   ⓐ Is $(6, 0)$ a solution? ⓑ Is $(-3, -2)$ a solution?
   If you missed this problem, review [link].
2. Find the slope and y-intercept of the line $3x - y = 12$.
   If you missed this problem, review [link].
3. Find the x- and y-intercepts of the line $2x - 3y = 12$.
   If you missed this problem, review [link].

Determine Whether an Ordered Pair is a Solution of a System of Equations

In Solving Linear Equations, we learned how to solve linear equations with one variable. Now we will work with two or more linear equations grouped together, which is known as a **system of linear equations**.

SYSTEM OF LINEAR EQUATIONS

When two or more linear equations are grouped together, they form a **system of linear equations**.

In this section, we will focus our work on systems of two linear equations in two unknowns. We will solve larger systems of equations later in this chapter.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, such as $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to *both* equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the **solutions of a system of equations**.

SolutionS OF A SYSTEM OF EQUATIONS

The **solutions of a system of equations** are the values of the variables that make *all* the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Example 11.1.1

Determine whether the ordered pair is a solution to the system $\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$.

ⓐ $(-2, -1)$ ⓑ $(-4, -3)$

Answer

ⓐ

$$\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$$

We substitute $x=-2$ and $y=-1$ into both equations.

$x-y=-1$	$2x-y=-5$
$-2-(-1)\stackrel{?}{=}-1$	$2\cdot -2-(-1)\stackrel{?}{=}-5$
$-1=-1$ ✓	$5\neq -5$
$(-2,-1)$ does not make both equations true.	$(-2,-1)$ is not a solution.

ⓑ

We substitute $x=-4$ and $y=-3$ into both equations.

$x-y=-1$	$2x-y=-5$
$-4-(-3)\stackrel{?}{=}-1$	$2\cdot(-4)-(-3)\stackrel{?}{=}-5$
$-1=-1$ ✓	$-5=-5$ ✓
$(-4,-3)$ does make both equations true.	$(-4,-3)$ is a solution.

? Example 11.1.2

Determine whether the ordered pair is a solution to the system $\begin{cases} x+y=0 \\ x+2y=-5 \end{cases}$.

ⓐ $(1,-3)$ ⓑ $(0,0)$

Answer

ⓐ yes ⓑ no

? Example 11.1.3

Determine whether the ordered pair is a solution to the system $\begin{cases} -3y=-8 \\ -3x-y=4 \end{cases}$.

ⓐ $(2,-2)$ ⓑ $(-2,2)$

Answer

ⓐ no ⓑ yes

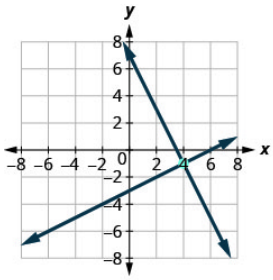
Solve a System of Linear Equations by Graphing

In this section, we will use three methods to solve a system of linear equations. The first method we'll use is graphing.

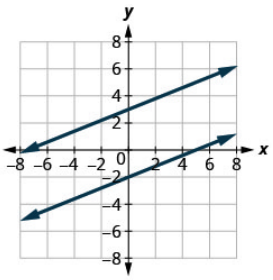
The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions.

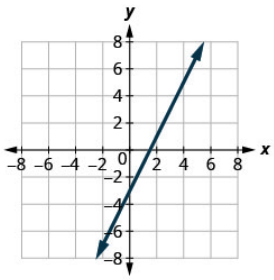
Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.



The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.



The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.



Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

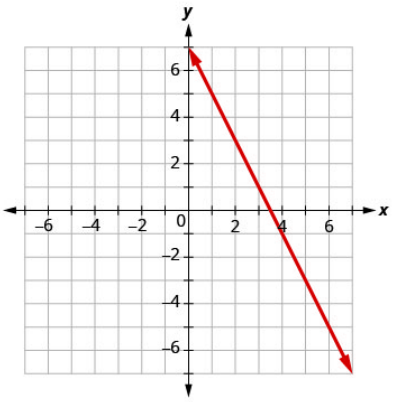
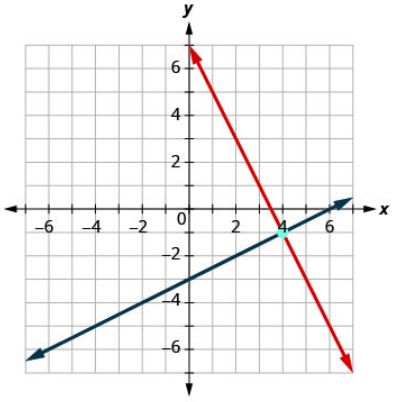
Figure 11.1.1

Each time we demonstrate a new method, we will use it on the same system of linear equations. At the end of the section you'll decide which method was the most convenient way to solve this system.

Example 11.1.4: How to Solve a System of Equations by Graphing

Solve the system by graphing $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$.

Answer

Step 1. Graph the first equation.	To graph the first line, write the equation in slope–intercept form. $2x+y=7$ $y=-2x+7$ $m=-2 \quad b=7$	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$ 
Step 2. Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $x-2y=6$ $(0,-3) \quad (6,0)$	

Step 3. Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Step 4. Identify the solution to the system. • If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. • If the lines are parallel, the system has no solution. • If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at (4, –1). $2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $8 - 1 \stackrel{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6$ ✓ The solution is (4, –1).

Example 11.1.5

Solve the system by graphing: $\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$.

Answer

$(3, 2)$

Example 11.1.6

Solve the system by graphing: $\begin{cases} -x + y = 1 \\ 3x + 2y = 12 \end{cases}$

Answer

$(2, 3)$

The steps to use to solve a system of linear equations by graphing are shown here.

SOLVE A SYSTEM OF LINEAR EQUATIONS BY GRAPHING.

1. Graph the first equation.
2. Graph the second equation on the same rectangular coordinate system.
3. Determine whether the lines intersect, are parallel, or are the same line.
4. Identify the solution to the system.
   - If the lines intersect, identify the point of intersection. This is the solution to the system.
   - If the lines are parallel, the system has no solution.
   - If the lines are the same, the system has an infinite number of solutions.
5. Check the solution in both equations.

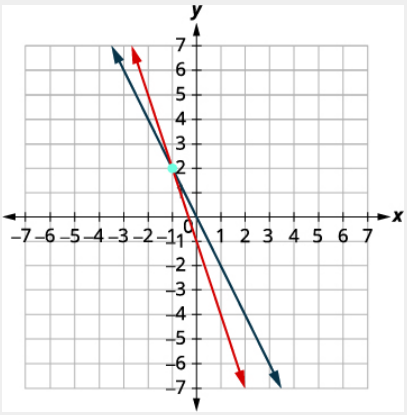
In the next example, we'll first re-write the equations into slope–intercept form as this will make it easy for us to quickly graph the lines.

Example 11.1.7

Solve the system by graphing: $\begin{cases} 3x + y = -1 \\ 2x + y = 0 \end{cases}$

Answer

We'll solve both of these equations for y so that we can easily graph them using their slopes and y-intercepts.

	$\begin{cases} 3x+y=-1 \\ 2x+y=0 \end{cases}$
Solve the first equation for y.	$\begin{aligned} 3x+y&=-1 \\ y&=-3x-1 \end{aligned}$
Find the slope and y-intercept.	$\begin{aligned} m&=-3 \\ b&=-1 \end{aligned}$
Solve the second equation for y.	$\begin{aligned} 2x+y&=0 \\ y&=-2x \end{aligned}$
Find the slope and y-intercept.	$\begin{aligned} m&=-2 \\ b&=0 \end{aligned}$
Graph the lines.	
Determine the point of intersection.	The lines intersect at $(-1, 2)$.
Check the solution in both equations. $\begin{aligned} 3x+y&=-1 \\ 3(-1)+2&\stackrel{?}{=}-1 \\ -1&=-1\ \checkmark \end{aligned}$ $\begin{aligned} 2x+y&=0 \\ 2(-1)+2&\stackrel{?}{=}0 \\ 0&=0\ \checkmark \end{aligned}$	
	The solution is $(-1, 2)$.

? Example 11.1.8

Solve the system by graphing: $\begin{cases} -x+y=1 \\ 2x+y=10 \end{cases}$.

Answer

$(3, 4)$

? Example 11.1.9

Solve the system by graphing: $\begin{cases} 2x+y=6 \\ x+y=1 \end{cases}$.

Answer

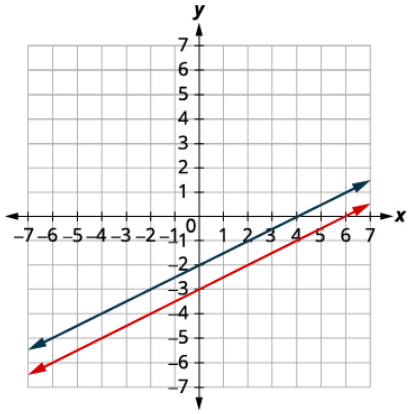
$(5, -4)$

In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

? Example 11.1.10

Solve the system by graphing: $\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$.

Answer

	$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$
To graph the first equation, we will use its slope and y-intercept.	$y = \frac{1}{2}x - 3$ $m = \frac{1}{2}$ $b = -3$
To graph the second equation, we will use the intercepts.	$x - 2y = 4$
	x | y 0 | −2 4 | 0
Graph the lines.	
Determine the points of intersection.	The lines are parallel. Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

? Example 11.1.11

Solve the system by graphing: $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = 4 \end{cases}$.

Answer

no solution

? Example 11.1.12

Solve the system by graphing: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 6 \end{cases}$.

Answer

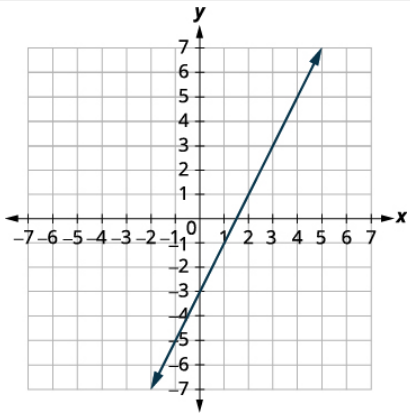
no solution

Sometimes the equations in a system represent the same line. Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to the system.

Example 11.1.13

Solve the system by graphing: $\begin{cases} y = 2x - 3 \\ -6x + 3y = 9 \end{cases}$.

Answer

	$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$
Find the slope and y-intercept of the first equation.	$y = 2x - 3$ $m = 2$ $b = -3$
Find the intercepts of the second equation.	$-6x + 3y = -9$
	x | y 0 | -3 $\frac{3}{2}$ | 0
Graph the lines.	
	The lines are the same! Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to this system.

If you write the second equation in slope-intercept form, you may recognize that the equations have the same slope and same y-intercept.

Example 11.1.14

Solve the system by graphing: $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$.

Answer

infinitely many solutions

Example 11.1.15

Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x - 4 \\ 2x - 4y = 16 \end{cases}$.

Answer

infinitely many solutions

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are **coincident**. Coincident lines have the same slope and same *y*-intercept.

COINCIDENT LINES

Coincident lines have the same slope and same *y*-intercept.

The systems of equations in Example and Example each had two intersecting lines. Each system had one solution.

In Example, the equations gave coincident lines, and so the system had infinitely many solutions.

The systems in those three examples had at least one solution. A system of equations that has at least one solution is called a *consistent* system.

A system with parallel lines, like Example, has no solution. We call a system of equations like this *inconsistent*. It has no solution.

CONSISTENT AND INCONSISTENT SYSTEMS

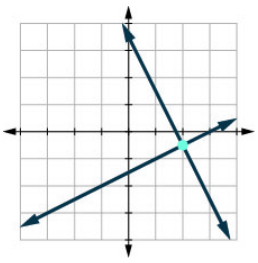
A **consistent system of equations** is a system of equations with at least one solution.

An **inconsistent system of equations** is a system of equations with no solution.

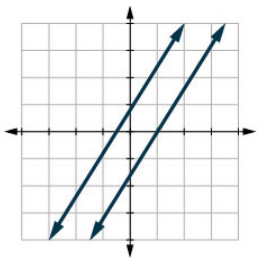
We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are independent, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines.

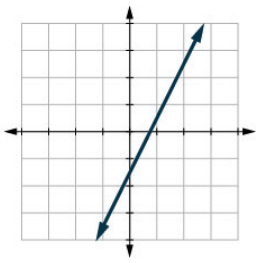
Let's sum this up by looking at the graphs of the three types of systems. See below and Table.



Intersecting



Parallel



Coincident

Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/ independent	Independent	Independent	Dependent

? Example 11.1.16

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$ ⓑ $\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$

Answer

ⓐ We will compare the slopes and intercepts of the two lines.

$$\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$$

$y = 3x - 1$

The first equation is already in slope-intercept form.
Write the second equation in slope-intercept form.

$$\begin{aligned} 6x - 2y &= 12 \\ -2y &= -6x + 12 \\ \tfrac{-2y}{-2} &= \tfrac{-6x+12}{-2} \\ y &= 3x - 6 \end{aligned}$$

$$\begin{aligned} y &= 3x - 1 \\ m &= 3 \\ b &= -1 \end{aligned} \qquad \begin{aligned} y &= 3x - 6 \\ m &= 3 \\ b &= -6 \end{aligned}$$

Find the slope and intercept of each line.

Since the slopes are the same andy-intercepts are different, the lines are parallel.

ⓑ We will compare the slope and intercepts of the two lines.

$$\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$$

Write both equations in slope–intercept form.

$$\begin{aligned} 2x + y &= -3 \\ y &= -2x - 3 \end{aligned} \qquad \begin{aligned} x - 5y &= 5 \\ -5y &= -x + 5 \\ \tfrac{-5y}{-5} &= \tfrac{-x+5}{-5} \\ y &= \tfrac{1}{5} - 1 \end{aligned}$$

Find the slope and intercept of each line.

$$\begin{aligned} y &= -2x - 3 \\ m &= -2 \\ b &= -3 \end{aligned} \qquad \begin{aligned} y &= \tfrac{1}{5} - 1 \\ m &= \tfrac{1}{5} \\ b &= -1 \end{aligned}$$

Since the slopes are different, the lines intersect.

A system of equations whose graphs are intersect has 1 solution and is consistent and independent.

? Example 11.1.17

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y = -2x - 4 \\ 4x + 2y = 9 \end{cases}$ ⓑ $\begin{cases} 3x + 2y = 2 \\ 2x + y = 1 \end{cases}$

Answer

ⓐ no solution, inconsistent, independent ⓑ one solution, consistent, independent

? Example 11.1.18

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y = \frac{1}{3}x - 5 \\ x - 3y = 6 \end{cases}$ ⓑ $\begin{cases} x + 4y = 12 \\ -x + y = 3 \end{cases}$

Answer

ⓐ no solution, inconsistent, independent ⓑ one solution, consistent, independent

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between -10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

Solve a System of Equations by Substitution

We will now solve systems of linear equations by the substitution method.

We will use the same system we used first for graphing.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

? Example 11.1.19: How to Solve a System of Equations by Substitution

Solve the system by substitution: $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

Answer

Step 1. Solve one of the equations for either variable.	We'll solve the first equation for y.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $2x + y = 7$ $y = 7 - 2x$
Step 2. Substitute the expression from Step 1 into the other equation.	We replace y in the second equation with the expression $7 - 2x$.	$x - 2y = 6$ $x - 2(7 - 2x) = 6$

Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x-2(7-2x)=6$ $x-14+4x=6$ $5x=20$ $x=4$
Step 4. Substitute the solution from Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace x with 4.	$2x+y=7$ $2(4)+y=7$ $8+y=7$ $y=-1$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $x=4$, $y=-1$ into both equations and make sure they are both true.	$2x+y=7$ $\quad$ $x-2y=6$ $2(4)+(-1)\stackrel{?}{=}7$ $\quad$ $4-2(-1)\stackrel{?}{=}6$ $7=7$ ✓ $\quad$ $6=6$ ✓ Both equations are true. $(4, -1)$ is the solution to the system.

Example 11.1.20

Solve the system by substitution: $\begin{cases} -2x+y=-11 \\ x+3y=9 \end{cases}$

Answer

$(6, 1)$

Example 11.1.21

Solve the system by substitution: $\begin{cases} 2x+y=-1 \\ 4x+3y=3 \end{cases}$

Answer

$(-3, 5)$

SOLVE A SYSTEM OF EQUATIONS BY SUBSTITUTION.

1. Solve one of the equations for either variable.
2. Substitute the expression from Step 1 into the other equation.
3. Solve the resulting equation.
4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to **both** original equations.

Be very careful with the signs in the next example.

Example 11.1.22

Solve the system by substitution: $\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$

Answer

We need to solve one equation for one variable. We will solve the first equation for y.

	$\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$
Solve the first equation for y. Substitute $-2x+2$ for y in the second equation.	$4x+2y=4$ $2y=-4x+4$ $y=-2x+2$ $6x-y=8$
Replace the y with $-2x+2$.	$6x-(-2x+2)=8$
Solve the equation for x.	$6x+2x-2=8$ $8x-2=8$ $8x=10$ $x=\frac{5}{4}$
Substitute $x=54$ into $4x+2y=4$ to find y.	$4x+2y=4$ $4\left(\frac{5}{4}\right)+2y=4$ $5+2y=4$ $2y=-1$ $y=-\frac{1}{2}$
	The ordered pair is $(54,-12)$.
Check the ordered pair in both equations. $4x+2y=4$ $4\left(\frac{5}{4}\right)+2\left(-\frac{1}{2}\right)\stackrel{?}{=}4$ $5-1\stackrel{?}{=}4$ $4=4$ ✓ $6x-y=8$ $6\left(\frac{5}{4}\right)-\left(-\frac{1}{2}\right)\stackrel{?}{=}8$ $\frac{15}{4}-\left(-\frac{1}{2}\right)\stackrel{?}{=}8$ $\frac{16}{2}\stackrel{?}{=}8$ $8=8$ ✓	
	The solution is $(54,-12)$.

? Example 11.1.23

Solve the system by substitution: $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$

Answer

$(2,32)$

? Example 11.1.24

Solve the system by substitution: $\begin{cases} 4x-y=0 \\ 2x-3y=5 \end{cases}$

Answer

$(-12,-2)$

Solve a System of Equations by Elimination

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Elimination Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination

method, too, but we'll have a different way to get there.

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions *a*, *b*, *c*, and *d*.

$$\begin{aligned} &\text{if} && a=b \\ &\text{and} && c=d \\ &\text{then} && a+c=b+d. \end{aligned}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:

$$\begin{cases} 3x+y=5 \\ 2x-y=0 \end{cases}$$
$$5x=5$$

The *y*'s add to zero and we have one equation with one variable.

Let's try another one:

$$\begin{cases} +4y=2 \\ 2x+5y=-2 \end{cases}$$

This time we don't see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by -2, we will make the coefficients of *x* opposites. We must multiply every term on both sides of the equation by -2.

$$\begin{aligned} -2(x+4y) &= -2(2) \\ 2x+5y &= -2 \end{aligned}$$

Then rewrite the system of equations.

$$\begin{aligned} -2x-8y &= -4 \\ 2x+5y &= -2 \end{aligned}$$

Now we see that the coefficients of the *x* terms are opposites, so *x* will be eliminated when we add these two equations.

$$\begin{array}{r} -2x-8y=-4 \\ 2x+5y=-2 \\ \hline -3y=-6 \end{array}$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we'll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

? Exercise 11.1.25: How to Solve a System of Equations by Elimination

Solve the system by elimination: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Answer

Step 1. Write both equations in standard form. • If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
Step 2. Make the coefficients of one variable opposites. • Decide which variable you will eliminate. • Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $\begin{cases} 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$
Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$\begin{array}{r} 4x + 2y = 14 \\ x - 2y = 6 \\ \hline 5x \quad\quad = 20 \end{array}$
Step 4. Solve for the remaining variable.	Solve for x.	$x = 4$
Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$
Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $x = 4$, $y = -1$ into $2x + y = 7$ and $x - 2y = 6$. Do they make both equations true? Yes!	$2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6$ ✓ The solution is $(4, -1)$.

? Exercise 11.1.26

Solve the system by elimination: $\begin{cases} 3x + y = 5 \\ 2x - 3y = 7 \end{cases}$

Answer

$(2, -1)$

? Exercise 11.1.27

Solve the system by elimination: $\begin{cases} 4x + y = -5 \\ -2x - 2y = -2 \end{cases}$

Answer

$(-2, 3)$

The steps are listed here for easy reference.

SOLVE A SYSTEM OF EQUATIONS BY ELIMINATION.

1. Write both equations in standard form. If any coefficients are fractions, clear them.
2. Make the coefficients of one variable opposites.
   - Decide which variable you will eliminate.

- Multiply one or both equations so that the coefficients of that variable are opposites.

3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to **both** original equations.

Now we'll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Exercise 11.1.28

Solve the system by elimination: $\begin{cases} 4x-3y=9 \\ 7x+2y=-6 \end{cases}$

Answer

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by different constants to get the opposites.

	$\begin{aligned} 4x-3y&=9 \\ 7x+2y&=-6 \end{aligned}$
Both equations are in standard form. To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.	$\begin{aligned} 2(4x-3y)&=2(9) \\ 3(7x+2y)&=3(-6) \end{aligned}$
Simplify.	$\begin{aligned} 8x-6y&=18 \\ 21x+6y&=-18 \end{aligned}$
Add the two equations to eliminate y.	$\begin{aligned} 8x-6y&=18 \\ 21x+6y&=-18 \\ \hline 29x&=0 \end{aligned}$
Solve for x.	$\begin{aligned} x&=0 \\ 7x+2y&=-6 \end{aligned}$
Substitute x=0x=0 into one of the original equations.	$7\cdot 0+2y=-6$
Solve for y.	$\begin{aligned} 2y&=-6 \\ y&=-3 \end{aligned}$
Write the solution as an ordered pair.	The ordered pair is $(0,-3)$.
Check that the ordered pair is a solution to **both** original equations. $\begin{aligned} 4x-3y&=9 \\ 4(0)-3(-3)&\stackrel{?}{=}9 \\ 9&=9\checkmark \end{aligned}$ $\begin{aligned} 7x+2y&=-6 \\ 7(0)+2(-3)&\stackrel{?}{=}-6 \\ -6&=-6\checkmark \end{aligned}$	
	The solution is $(0,-3)$.

Exercise 11.1.29

Solve the system by elimination: $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$

Answer

$(1,3)$

Exercise 11.1.30

Solve each system by elimination: $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$

Answer

$(4,-3)$

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by the LCD of all the fractions in the equation.

Exercise 11.1.31

Solve the system by elimination: $\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$

Answer

In this example, both equations have fractions. Our first step will be to multiply each equation by the LCD of all the fractions in the equation to clear the fractions.

	$\begin{aligned} x+\tfrac{1}{2}y&=6 \\ \tfrac{3}{2}x+\tfrac{2}{3}y&=\tfrac{17}{2} \end{aligned}$
To clear the fractions, multiply each equation by its LCD.	$\begin{aligned} 2\left(x+\tfrac{1}{2}y\right)&=2(6) \\ 6\left(\tfrac{3}{2}x+\tfrac{2}{3}y\right)&=6\left(\tfrac{17}{2}\right) \end{aligned}$
Simplify.	$\begin{aligned} 2x+\ y&=12 \\ 9x+4y&=51 \end{aligned}$
Now we are ready to eliminate one of the variables. Notice that both equations are in standard form.	
We can eliminate y by multiplying the top equation by -4.	$\begin{aligned} -4(2x+\ y)&=-4(12) \\ 9x+4y\ &=51 \end{aligned}$
Simplify and add. Substitute $x=3$ into one of the original equations.	$\begin{aligned} -8x-4y&=-48 \\ 9x+4y\ &=\ 51 \\ \hline x\quad &=\ 3 \end{aligned}$ $x+\frac{1}{2}y=6$
Solve for y.	$\begin{aligned} 3+\tfrac{1}{2}y&=6 \\ \tfrac{1}{2}y&=3 \\ y&=6 \end{aligned}$
Write the solution as an ordered pair.	The ordered pair is $(3,6)$.
Check that the ordered pair is a solution to both original equations.	$\begin{aligned} x+\tfrac{1}{2}y&=6 \\ 3+\tfrac{1}{2}(6)&\stackrel{?}{=}6 \\ 3+3&\stackrel{?}{=}6 \\ 6&=6\ \checkmark \end{aligned}$ $\begin{aligned} \tfrac{3}{2}x+\tfrac{2}{3}y&=\tfrac{17}{2} \\ \tfrac{3}{2}(3)+\tfrac{2}{3}(6)&\stackrel{?}{=}\tfrac{17}{2} \\ \tfrac{9}{2}+4&\stackrel{?}{=}\tfrac{17}{2} \\ \tfrac{9}{2}+\tfrac{8}{2}&\stackrel{?}{=}\tfrac{17}{2} \\ \tfrac{17}{2}&=\tfrac{17}{2}\ \checkmark \end{aligned}$
	The solution is $(3,6)$.

Exercise 11.1.32

Solve each system by elimination: $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 1 \\ \frac{3}{4}x - y = \frac{5}{2} \end{cases}$

Answer

$(6, 2)$

Exercise 11.1.33

Solve each system by elimination: $\begin{cases} x + \frac{3}{5}y = -\frac{1}{5} \\ -\frac{1}{2}x - \frac{2}{3}y = \frac{5}{6} \end{cases}$

Answer

$(1, -2)$

When we solved the system by graphing, we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

The same is true using substitution or elimination. If the equation at the end of substitution or elimination is a true statement, we have a consistent but dependent system and the system of equations has infinitely many solutions. If the equation at the end of substitution or elimination is a false statement, we have an inconsistent system and the system of equations has no solution.

Exercise 11.1.34

Solve the system by elimination: $\begin{cases} 3x + 4y = 12 \\ y = 3 - \frac{3}{4}x \end{cases}$

Answer

$$\begin{cases} 3x + 4y = 12 \\ y = 3 - \frac{3}{4}x \end{cases}$$

Write the second equation in standard form.

$$\begin{cases} 3x + 4y = 12 \\ \frac{3}{4}x + y = 3 \end{cases}$$

Clear the fractions by multiplying the second equation by 4.

$$\begin{cases} 3x + 4y = 12 \\ 4(\frac{3}{4}x + y) = 4(3) \end{cases}$$

Simplify.

$$\begin{cases} 3x + 4y = 12 \\ 3x + 4y = 12 \end{cases}$$

To eliminate a variable, we multiply the second equation by -1. Simplify and add.

$$\begin{cases} 3x + 4y = 12 \\ \underline{-3x - 4y = -12} \end{cases}$$
$$0 = 0$$

This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

? Exercise 11.1.35

Solve the system by elimination: $\begin{cases} 5x-3y=15 \\ 5y=-5+\frac{5}{3}x \end{cases}$

Answer

infinitely many solutions

? Exercise 11.1.36

Solve the system by elimination: $\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$

Answer

infinitely many solutions

Choose the Most Convenient Method to Solve a System of Linear Equations

When you solve a system of linear equations in in an application, you will not be told which method to use. You will need to make that decision yourself. So you'll want to choose the method that is easiest to do and minimizes your chance of making mistakes.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations a rein standard form.

? Example 11.1.37

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$ ⓑ $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Answer

ⓐ

$$\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$$

Since both equations are in standard form, using elimination will be most convenient.

ⓑ

$$\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$$

Since one equation is already solved for y, using substitution will be most convenient.

? Example 11.1.38

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 4x-5y=-32 \\ 3x+2y=-1 \end{cases}$ ⓑ $\begin{cases} x=2y-1 \\ 3x-5y=-7 \end{cases}$

Answer

ⓐ Since both equations are in standard form, using elimination will be most convenient. ⓑ Since one equation is already solved for x, using substitution will be most convenient.

Example 11.1.39

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} y = 2x - 1 \\ 3x - 4y = -6 \end{cases}$ ⓑ $\begin{cases} 6x - 2y = 12 \\ 3x + 7y = -13 \end{cases}$

Answer

ⓐ Since one equation is already solved for y, using substitution will be most convenient. ⓑ Since both equations are in standard form, using elimination will be most convenient.

Key Concepts

- **How to solve a system of linear equations by graphing.**
  1. Graph the first equation.
  2. Graph the second equation on the same rectangular coordinate system.
  3. Determine whether the lines intersect, are parallel, or are the same line.
  4. Identify the solution to the system.
     If the lines intersect, identify the point of intersection. This is the solution to the system.
     If the lines are parallel, the system has no solution.
     If the lines are the same, the system has an infinite number of solutions.
  5. Check the solution in both equations.
- **How to solve a system of equations by substitution.**
  1. Solve one of the equations for either variable.
  2. Substitute the expression from Step 1 into the other equation.
  3. Solve the resulting equation.
  4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
  5. Write the solution as an ordered pair.
  6. Check that the ordered pair is a solution to **both** original equations.
- **How to solve a system of equations by elimination.**
  1. Write both equations in standard form. If any coefficients are fractions, clear them.
  2. Make the coefficients of one variable opposites.
     Decide which variable you will eliminate.
     Multiply one or both equations so that the coefficients of that variable are opposites.
  3. Add the equations resulting from Step 2 to eliminate one variable.
  4. Solve for the remaining variable.
  5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
  6. Write the solution as an ordered pair.
  7. Check that the ordered pair is a solution to **both** original equations.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations a rein standard form.

Glossary

coincident lines

Coincident lines have the same slope and same y-intercept.

consistent and inconsistent systems

Consistent system of equations is a system of equations with at least one solution; inconsistent system of equations is a system of equations with no solution.

solutions of a system of equations

Solutions of a system of equations are the values of the variables that make *all* the equations true; solution is represented by an ordered pair (x,y).(x,y).

system of linear equations

When two or more linear equations are grouped together, they form a system of linear equations.

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11.1E: Exercises

Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Equations

In the following exercises, determine if the following points are solutions to the given system of equations.

1. $\begin{cases} 2x-6y=0 \\ 3x-4y=5 \end{cases}$

ⓐ $(3,1)$
ⓑ $(-3,4)$

Answer

ⓐ yes ⓑ no

2. $\begin{cases} -3x+y=8 \\ -x+2y=-9 \end{cases}$

ⓐ $(-5,-7)$
ⓑ $(-5,7)$

3. $\begin{cases} x+y=2 \\ y=\frac{3}{4}x \end{cases}$

ⓐ $(87,67)$
ⓑ $(1,34)$

Answer

ⓐ yes ⓑ no

4. $\begin{cases} 2x+3y=6 \\ y=\frac{2}{3}x+2 \end{cases}$

ⓐ $(-6,2)$
ⓑ $(-3,4)$

Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.

5. $\begin{cases} 3x+y=-3 \\ 2x+3y=5 \end{cases}$

Answer

$(-3,2)$

6. $\begin{cases} -x+y=2 \\ 2x+y=-4 \end{cases}$

7. $\begin{cases} y=x+2 \\ y=-2x+2 \end{cases}$

Answer

$(0, 2)$

8. $\begin{cases} y = x - 2 \\ y = -3x + 2 \end{cases}$

9. $\begin{cases} y = \frac{3}{2}x + 1 \\ y = -\frac{1}{2}x + 5 \end{cases}$

Answer

$(2, 4)$

10. $\begin{cases} y = \frac{2}{3}x - 2 \\ y = -\frac{1}{3}x - 5 \end{cases}$

11. $\begin{cases} x + y = -4 \\ -x + 2y = -2 \end{cases}$

Answer

$(-2, 2)$

12. $\begin{cases} -x + 3y = 3 \\ x + 3y = 3 \end{cases}$

13. $\begin{cases} -2x + 3y = 3 \\ x + 3y = 12 \end{cases}$

Answer

$(3, 3)$

14. $\begin{cases} 2x - y = 4 \\ 2x + 3y = 12 \end{cases}$

15. $\begin{cases} x + 3y = -6 \\ y = -\frac{4}{3}x + 4 \end{cases}$

Answer

$(6, -4)$

16. $\begin{cases} -x + 2y = -6 \\ y = -\frac{1}{2}x - 1 \end{cases}$

17. $\begin{cases} -2x + 4y = 4 \\ y = \frac{1}{2}x \end{cases}$

Answer

no solution

18. $\begin{cases} 3x+5y=10 \\ y=-\frac{3}{5}x+1 \end{cases}$

19. $\begin{cases} 4x-3y=8 \\ 8x-6y=14 \end{cases}$

Answer

no solution

20. $\begin{cases} x+3y=4 \\ -2x-6y=3 \end{cases}$

21. $\begin{cases} x=-3y+4 \\ 2x+6y=8 \end{cases}$

Answer

infinite solutions with solution set: $\{(x,y)|2x+6y=8\}$

22. $\begin{cases} 4x=3y+7 \\ 8x-6y=14 \end{cases}$

23. $\begin{cases} 2x+y=6 \\ -8x-4y=-24 \end{cases}$

Answer

infinite solutions with solution set: $\{(x,y)|2x+y=6\}$

24. $\begin{cases} 5x+2y=7 \\ -10x-4y=-14 \end{cases}$

Without graphing, determine the number of solutions and then classify the system of equations.

25. $\begin{cases} y=\frac{2}{3}x+1 \\ -2x+3y=5 \end{cases}$

Answer

1 point, consistent and independent

26. $\begin{cases} y=\frac{3}{2}x+1 \\ 2x-3y=7 \end{cases}$

27. $\begin{cases} 5x+3y=4 \\ 2x-3y=5 \end{cases}$

Answer

1 point, consistent and independent

28. $\begin{cases} y=-12x+5 \\ x+2y=10 \end{cases}$

29. $\begin{cases} 5x-2y=10 \\ y=52x-5 \end{cases}$

Answer

infinite solutions, consistent, dependent

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

30. $\begin{cases} 2x+y=-4 \\ 3x-2y=-6 \end{cases}$

31. $\begin{cases} 2x+y=-2 \\ 3x-y=7 \end{cases}$

Answer

$(1,-4)$

32. $\begin{cases} x-2y=-5 \\ 2x-3y=-4 \end{cases}$

33. $\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$

Answer

$(-3,2)$

34. $\begin{cases} 5x-2y=-6 \\ y=3x+3 \end{cases}$

35. $\begin{cases} -2x+2y=6 \\ y=-3x+1 \end{cases}$

Answer

$(-1/2,5/2)$

36. $\begin{cases} 2x+5y=1 \\ y=\frac{1}{3}x-2 \end{cases}$

37. $\begin{cases} 3x+4y=1 \\ y=-\frac{2}{5}x+2 \end{cases}$

Answer

$(-5,4)$

38. $\begin{cases} 2x+y=5 \\ x-2y=-15 \end{cases}$

39. $\begin{cases} 4x+y=10 \\ x-2y=-20 \end{cases}$

Answer

$(0, 10)$

40. $\begin{cases} y=-2x-1 \\ y=-\frac{1}{3}x+4 \end{cases}$

41. $\begin{cases} y=x-6 \\ y=-\frac{3}{2}x+4 \end{cases}$

Answer

$(4, -2)$

42. $\begin{cases} x=2y \\ 4x-8y=0 \end{cases}$

43. $\begin{cases} 2x-16y=8 \\ -x-8y=-4 \end{cases}$

Answer

$(4, 0)$

44. $\begin{cases} y=\frac{7}{8}x+4 \\ -7x+8y=6 \end{cases}$

45. $\begin{cases} y=-\frac{2}{3}x+5 \\ 2x+3y=11 \end{cases}$

Answer

no solution

Solve a System of Equations by Elimination

In the following exercises, solve the systems of equations by elimination.

46. $\begin{cases} 5x+2y=2 \\ -3x-y=0 \end{cases}$

47. $\begin{cases} 6x-5y=-1 \\ 2x+y=13 \end{cases}$

Answer

$(4, 5)$

48. $\begin{cases} 2x-5y=7 \\ 3x-y=17 \end{cases}$

49. $\begin{cases} 5x-3y=-1 \\ 2x-y=2 \end{cases}$

Answer

$(7,12)$

50. $\begin{cases} 3x-5y=-9 \\ 5x+2y=16 \end{cases}$

51. $\begin{cases} 4x-3y=3 \\ 2x+5y=-31 \end{cases}$

Answer

$(-3,-5)$

52. $\begin{cases} 3x+8y=-3 \\ 2x+5y=-3 \end{cases}$

53. $\begin{cases} 11x+9y=-5 \\ 7x+5y=-1 \end{cases}$

Answer

$(2,-3)$

54. $\begin{cases} 3x+8y=67 \\ 5x+3y=60 \end{cases}$

55. $\begin{cases} 2x+9y=-4 \\ 3x+13y=-7 \end{cases}$

Answer

$(-11,2)$

56. $\begin{cases} \frac{1}{3}x-y=-3 \\ x+\frac{5}{2}y=2 \end{cases}$

57. $\begin{cases} x+\frac{1}{2}y=\frac{3}{2} \\ \frac{1}{5}x-\frac{1}{5}y=3 \end{cases}$

Answer

$(6/-9,24/7)$

58. $\begin{cases} x+\frac{1}{3}y=-1 \\ \frac{1}{3}x+\frac{1}{2}y=1 \end{cases}$

59. $\begin{cases} \frac{1}{3}x-y=-3 \\ \frac{2}{3}x+\frac{5}{2}y=3 \end{cases}$

Answer

$(-3, 2)$

60. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$

61. $\begin{cases} x-4y=-1 \\ -3x+12y=3 \end{cases}$

Answer

infinitely many solutions with solution set: $\{(x, y)|x-4y=-1\}$

62. $\begin{cases} -3x-y=8 \\ 6x+2y=-16 \end{cases}$

63. $\begin{cases} 4x+3y=2 \\ 20x+15y=10 \end{cases}$

Answer

infinitely many solutions with solution set: $\{(x, y)|4x+3y=2\}$

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

64.

ⓐ $\begin{cases} 8x-15y=-32 \\ 6x+3y=-5 \end{cases}$

ⓑ $\begin{cases} x=4y-3 \\ 4x-2y=-6 \end{cases}$

65.

ⓐ $\begin{cases} y=7x-5 \\ 3x-2y=16 \end{cases}$

ⓑ $\begin{cases} 12x-5y=-42 \\ 3x+7y=-15 \end{cases}$

Answer

ⓐ substitution ⓑ elimination

66.

ⓐ $\begin{cases} y=4x+95 \\ x-2y=-21 \end{cases}$

ⓑ $\begin{cases} 9x-4y=24 \\ 3x+5y=-14 \end{cases}$

67.

ⓐ $\begin{cases} 14x-15y=-30 \\ 7x+2y=10 \end{cases}$

ⓑ $\begin{cases} x = 9y - 11 \\ 2x - 7y = -27 \end{cases}$

Answer

ⓐ elimination ⓑ substitution

Writing Exercises

68. In a system of linear equations, the two equations have the same intercepts. Describe the possible solutions to the system.

69. Solve the system of equations by substitution and explain all your steps in words: $\begin{cases} 3x + y = 1 \\ 2x = y - 8 \end{cases}$

Answer

Answers will vary.

70. Solve the system of equations by elimination and explain all your steps in words: $\begin{cases} 5x + 4y = 10 \\ 2x = 3y + 27 \end{cases}$

71. Solve the system of equations $\begin{cases} x + y = 10 \\ x - y = 6 \end{cases}$

ⓐ by graphing ⓑ by substitution
ⓒ Which method do you prefer? Why?

Answer

Answers will vary.

Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether an ordered pair is a solution of a system of equations.			
solve a system of linear equations by graphing.			
solve a system of equations by substitution.			
solve a system of equations by elimination.			
choose the most convenient method to solve a system of linear equations.			

If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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11.2: Solve Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve direct translation applications
- Solve geometry applications
- Solve uniform motion applications

Before you get started, take this readiness quiz.

1. The sum of twice a number and nine is 31. Find the number.
   If you missed this problem, review [link].
2. Twins Jon and Ron together earned $96,000 last year. Ron earned $8000 more than three times what Jon earned. How much did each of the twins earn?
   If you missed this problem, review [link].
3. An express train and a local train leave Pittsburgh to travel to Washington, D.C. The express train can make the trip in four hours and the local train takes five hours for the trip. The speed of the express train is 12 miles per hour faster than the speed of the local train. Find the speed of both trains.
   If you missed this problem, review [link].

Solve Direct Translation Applications

Systems of linear equations are very useful for solving applications. Some people find setting up word problems with two variables easier than setting them up with just one variable. To solve an application, we'll first translate the words into a system of linear equations. Then we will decide the most convenient method to use, and then solve the system.

SOLVE APPLICATIONS WITH SYSTEMS OF EQUATIONS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose variables to represent those quantities.
4. **Translate** into a system of equations.
5. **Solve** the system of equations using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We solved number problems with one variable earlier. Let's see how differently it works using two variables.

?

The sum of two numbers is zero. One number is nine less than the other. Find the numbers.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let $n =$ the first number. $m =$ the second number
Step 4. Translate into a system of equations.	The sum of two numbers is zero.
	$n + m = 0$
	One number is nine less than the other.

	$n = m - 9$
The system is:	$n + m = 0$ $n = m - 9$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for n.	
Substitute $m - 9$ for n in the first equation.	$n = (m - 9)$ $n + m = 0$
Solve for m.	$m - 9 + m = 0$
	$2m - 9 = 0$
	$2m = 9$
Substitute $m = \frac{9}{2}$ into the second equation and then solve for n.	$m = \frac{9}{2}$ $n = m - 9$
	$m = \frac{9}{2} - 9$ $m = \frac{9}{2} - \frac{18}{2}$
	$n = -\frac{9}{2}$
Step 6. Check the answer in the problem.	Do these numbers make sense in the problem? We will leave this to you!
Step 7. Answer the question.	The numbers are $\frac{9}{2}$ and $-\frac{9}{2}$.

? Example 11.2.2

The sum of two numbers is 10. One number is 4 less than the other. Find the numbers.

Answer

$3, 7$

? Example 11.2.3

The sum of two numbers is -6. One number is 10 less than the other. Find the numbers.

Answer

$2, -8$

?

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her \$25,000 plus \$15 for each training session. Option B would pay her $\$10,000 + \$40$ for each training session. How many training sessions would make the salary options equal?

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of training sessions that would make the pay equal.
Step 3. Name what we are looking for.	Let s=s= Heather's salary. n=n= the number of training sessions
Step 4. Translate into a system of equations.	Option A would pay her $25,000 plus $15 for each training session.
	$s = 25{,}000 + 15n$
	Option B would pay her $10,000 + $40 for each training session.
	$s = 10{,}000 + 40n$
The system is shown.	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Step 5. Solve the system of equations. We will use substitution.	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Substitute 25,000 +15n for s in the second equation.	$25{,}000 + 15n = 10{,}000 + 40n$
Solve for n.	$25{,}000 = 10{,}000 + 25n$ $15{,}000 = 25n$ $600 = n$
Step 6. Check the answer.	Are 600 training sessions a year reasonable? Are the two options equal when n = 600?
Step 7. Answer the question.	The salary options would be equal for 600 training sessions.

Example 11.2.5

Geraldine has been offered positions by two insurance companies. The first company pays a salary of $12,000 plus a commission of $100 for each policy sold. The second pays a salary of $20,000 plus a commission of $50 for each policy sold. How many policies would need to be sold to make the total pay the same?

Answer

160 policies

Example 11.2.6

Kenneth currently sells suits for company A at a salary of $22,000 plus a $10 commission for each suit sold. Company B offers him a position with a salary of $28,000 plus a $4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Answer

1000 suits

As you solve each application, remember to analyze which method of solving the system of equations would be most convenient.

Example 11.2.7

Translate to a system of equations and then solve:

When Jenna spent 10 minutes on the elliptical trainer and then did circuit training for 20 minutes, her fitness app says she burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes circuit training she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories for each minute of circuit training?

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of calories burned each minute on the elliptical trainer and each minute of circuit training.
Step 3. Name what we are looking for.	Let e=e= number of calories burned per minute on the elliptical trainer. c=c= number of calories burned per minute while circuit training
Step 4. Translate into a system of equations.	10 minutes on the elliptical and circuit training for 20 minutes, burned 278 calories
	$10e + 20c = 278$
	20 minutes on the elliptical and 30 minutes of circuit training burned 473 calories
	$20e + 30c = 473$
The system is:	$\begin{cases} 10e + 20c = 278 \\ 20e + 30c = 473 \end{cases}$
Step 5. Solve the system of equations.	
Multiply the first equation by −2 to get opposite coefficients of e.	$\begin{cases} -2(10e + 20c) = -2(278) \\ 20e + 30c = 473 \end{cases}$
Simplify and add the equations. Solve for c.	$\begin{cases} -20e - 40c = -556 \\ 20e + 30c = 473 \end{cases}$ $-10c = -83$ $c = 8.3$
Substitute $c = 8.3$ into one of the original equations to solve for e.	$10e + 20c = 278$ $10e + 20(8.3) = 278$ $10e + 166 = 278$ $10e = 112$ $e = 11.2$
Step 6. Check the answer in the problem.	Check the math on your own.
	$\begin{cases} 10(11.2) + 20(8.3) \stackrel{?}{=} 278 \\ 20(11.2) + 30(8.3) \stackrel{?}{=} 473 \end{cases}$
Step 7. Answer the question.	Jenna burns 8.3 calories per minute circuit training and 11.2 calories per minute while on the elliptical trainer.

Example 11.2.8

Translate to a system of equations and then solve:

Mark went to the gym and did 40 minutes of Bikram hot yoga and 10 minutes of jumping jacks. He burned 510 calories. The next time he went to the gym, he did 30 minutes of Bikram hot yoga and 20 minutes of jumping jacks burning 470 calories. How many calories were burned for each minute of yoga? How many calories were burned for each minute of jumping jacks?

Answer

Mark burned 11 calories for each minute of yoga and 7 calories for each minute of jumping jacks.

Example 11.2.9

Translate to a system of equations and then solve:

Erin spent 30 minutes on the rowing machine and 20 minutes lifting weights at the gym and burned 430 calories. During her next visit to the gym she spent 50 minutes on the rowing machine and 10 minutes lifting weights and burned 600 calories. How many calories did she burn for each minutes on the rowing machine? How many calories did she burn for each minute of weight lifting?

Answer

Erin burned 11 calories for each minute on the rowing machine and 5 calories for each minute of weight lifting.

Solve Geometry Applications

We will now solve geometry applications using systems of linear equations. We will need to add complementary angles and supplementary angles to our list some properties of angles.

The measures of two **complementary angles** add to 90 degrees. The measures of two **supplementary angles** add to 180 degrees.

COMPLEMENTARY AND SUPPLEMENTARY ANGLES

Two angles are complementary if the sum of the measures of their angles is 90 degrees.

Two angles are supplementary if the sum of the measures of their angles is 180 degrees

If two angles are complementary, we say that *one angle is the complement of the other.*

If two angles are supplementary, we say that *one angle is the supplement of the other.*

Example 11.2.10

Translate to a system of equations and then solve.

The difference of two complementary angles is 26 degrees. Find the measures of the angles.

Answer

Step 1. Read the problem.

Step 2. Identify what we are looking for. We are looking for the measure of each angle.

Step 3. Name what we are looking for. Let $x =$ the measure of the first angle.
$y =$ the measure of the second angle

Step 4. Translate into a system of equations. The angles are complementary.

$$x + y = 90$$

The difference of the two angles is 26 degrees.

$$x - y = 26$$

The system is shown.

$$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$$

Step 5. Solve the system of equations by elimination.

$$\begin{cases} x + y = 90 \\ \underline{x - y = 26} \end{cases}$$
$$2x = 116$$
$$x = 58$$

Substitute $x = 58$ into the first equation.

$$x + y = 90$$
$$58 + y = 90$$
$$y = 32$$

Step 6. Check the answer in the problem.

$58 + 32 = 90$✓
$58 - 32 = 26$✓

Step 7. Answer the question. The angle measures are 58 and 32 degrees.

Example 11.2.11

Translate to a system of equations and then solve:

The difference of two complementary angles is 20 degrees. Find the measures of the angles.

Answer

The angle measures are 55 and 35.

Example 11.2.12

Translate to a system of equations and then solve:

The difference of two complementary angles is 80 degrees. Find the measures of the angles.

Answer

The angle measures are 5 and 85.

In the next example, we remember that the measures of supplementary angles add to 180.

Example 11.2.13

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is twelve degrees less than five times the measure of the smaller angle. Find the measures of both angles.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for measure of each angle.
Step 3. Name what we are looking for.	Let x=x= the measure of the first angle. y=y= the measure of the second angle
Step 4. Translate into a system of equations.	The angles are supplementary.
	$x+y=180$
	The larger angle is twelve less than five times the smaller angle.
	$y=5x-12$
The system is shown: **Step 5. Solve** the system of equations substitution.	$\begin{cases} x+y=180 \\ y=5x-12 \end{cases}$ $x+y=180$
Substitute $5x-12$ for y in the first equation. Solve for x.	$x+5x-12=180$ $6x-12=180$
Substitute 32 for x in the second equation, then solve for y.	$6x=192$ $x=32$ $y=5x-12$ $y=5\cdot 32-12$ $y=160-12$ $y=148$
Step 6. Check the answer in the problem.	$32+148=180$ ✓ $5\cdot 32-12=148$ ✓
Step 7. Answer the question.	The angle measures are 148 and 32 degrees.

Example 11.2.14

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 12 degrees more than three times the smaller angle. Find the measures of the angles.

Answer

The angle measures are 42 and 138.

Example 11.2.15

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 18 less than twice the measure of the smaller angle. Find the measures of the angles.

Answer

The angle measures are 66 and 114.

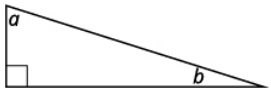
Recall that the angles of a triangle add up to 180 degrees. A right triangle has one angle that is 90 degrees. What does that tell us about the other two angles? In the next example we will be finding the measures of the other two angles.

Example 11.2.16

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle. Find the measures of both angles.

Answer

We will draw and label a figure.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the measures of the angles.
Step 3. Name what we are looking for.	Let a=a= the measure of the first angle. b=b= the measure of the second angle
Step 4. Translate into a system of equations.	The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle.
	$a = 3b + 10$
	The sum of the measures of the angles of a triangle is 180.
	$a + b + 90 = 180$
The system is shown.	$a = 3b + 10$ $a + b + 90 = 180$
Step 5. Solve the system of equations. We will use substitution since the first equation is solved for a.	$a = 3b + 10$ $a + b + 90 = 180$
Substitute 3b+103b+10 for a in the second equation.	$(3b + 10) + b + 90 = 180$
Solve for b.	$4b + 100 = 180$ $4b = 80$ $b = 20$ $a = 3b + 10$
Substitute b=20b=20 into the first equation and then solve for a.	$a = 3 \cdot 20 + 10$ $a = 70$
Step 6. Check the answer in the problem.	We will leave this to you!
Step 7. Answer the question.	The measures of the small angles are 20 and 70 degrees.

Example 11.2.17

The measure of one of the small angles of a right triangle is 2 more than 3 times the measure of the other small angle. Find the measure of both angles.

Answer

$22, 68$

Example 11.2.18

The measure of one of the small angles of a right triangle is 18 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

$36, 54$

Often it is helpful when solving geometry applications to draw a picture to visualize the situation.

Example 11.2.19

Translate to a system of equations and then solve:

Randall has 125 feet of fencing to enclose the part of his backyard adjacent to his house. He will only need to fence around three sides, because the fourth side will be the wall of the house. He wants the length of the fenced yard (parallel to the house wall) to be 5 feet more than four times as long as the width. Find the length and the width.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let L=L= the length of the fenced yard. W=W= the width of the fenced yard
Step 4. Translate into a system of equations.	One lenth and two widths equal 125.
	$L + 2W = 125$
	The length will be 5 feet more than four times the width.
	$L = 4W + 5$
The system is shown. **Step 5. Solve** The system of equations by substitution.	$\begin{cases} L + 2W = 125 \\ L = 4W + 5 \end{cases}$ $L + 2W = 125$
Substitute $L = 4W + 5$ into the first equation, then solve for W.	$4W + 5 + 2W = 125$

	$6W + 5 = 125$ $6W = 120$ $W = 20$
Substitute 20 for W in the second equation, then solve for L.	$L = 4W + 5$ $L = 4 \cdot 20 + 5$ $L = 80 + 5$ $L = 85$
Step 6. Check the answer in the problem.	$20 + 85 + 20 = 125$ $85 = 4 \cdot 20 + 5$ $85 = 85$ ✓
Step 7. Answer the equation.	The length is 85 feet and the width is 20 feet.

? Example 11.2.20

Translate to a system of equations and then solve:

Mario wants to put a fence around the pool in his backyard. Since one side is adjacent to the house, he will only need to fence three sides. There are two long sides and the one shorter side is parallel to the house. He needs 155 feet of fencing to enclose the pool. The length of the long side is 10 feet less than twice the width. Find the length and width of the pool area to be enclosed.

Answer

The length is 60 feet and the width is 35 feet.

? Example 11.2.21

Translate to a system of equations and then solve:

Alexis wants to build a rectangular dog run in her yard adjacent to her neighbor's fence. She will use 136 feet of fencing to completely enclose the rectangular dog run. The length of the dog run along the neighbor's fence will be 16 feet less than twice the width. Find the length and width of the dog run.

Answer

The length is 60 feet and the width is 38 feet.

Solve uniform motion applications

We used a table to organize the information in uniform motion problems when we introduced them earlier. We'll continue using the table here. The basic equation was $D = rt$ where D is the distance traveled, r is the rate, and t is the time.

Our first example of a uniform motion application will be for a situation similar to some we have already seen, but now we can use two variables and two equations.

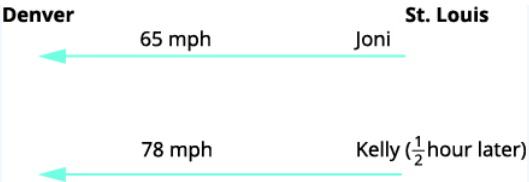
? Example 11.2.22

Translate to a system of equations and then solve:

Joni left St. Louis on the interstate, driving west towards Denver at a speed of 65 miles per hour. Half an hour later, Kelly left St. Louis on the same route as Joni, driving 78 miles per hour. How long will it take Kelly to catch up to Joni?

Answer

A diagram is useful in helping us visualize the situation.



Identify and name what we are looking for. A chart will help us organize the data. We know the rates of both Joni and Kelly, and so we enter them in the chart. We are looking for the length of time Kelly, *k*, and Joni, *j*, will each drive.

	Rate · Time = Distance		
Joni	65	j	$65j$
Kelly	78	k	$78k$

Since $D = r \cdot t$ we can fill in the Distance column.

Translate into a system of equations.

To make the system of equations, we must recognize that Kelly and Joni will drive the same distance. So,

$$65j = 78k$$

Also, since Kelly left later, her time will be $\frac{1}{2}$ hour less than Joni's time. So,

$$k = j - \tfrac{1}{2}$$

Now we have the system.

$$\begin{cases} k = j - \frac{1}{2} \\ 65j = 78k \end{cases}$$

Solve the system of equations by substitution.

Substitute $k = j - 12$ into the second equation, then solve for j.

$$\begin{aligned} 65j &= 78k \\ 65j &= 78(j - \tfrac{1}{2}) \\ 65j &= 78j - 39 \\ -13j &= -39 \\ j &= 3 \end{aligned}$$

To find Kelly's time, substitute $j = 3$ into the first equation, then solve for k.

$$\begin{aligned} k &= j - \tfrac{1}{2} \\ k &= 3 - \tfrac{1}{2} \\ k &= \tfrac{5}{2} \text{ or } k = 2\tfrac{1}{2} \end{aligned}$$

Check the answer in the problem.

Joni 3 hours (65 mph) = 195 miles

Kelly $2\frac{1}{2}$ hours (78 mph) = 195 miles

Yes, they will have traveled the same distance when they meet.

Answer the question.

Kelly will catch up to Joni in $2\frac{1}{2}$ hours. By then, Joni will have traveled 3 hours.

Example 11.2.23

Translate to a system of equations and then solve:

Mitchell left Detroit on the interstate driving south towards Orlando at a speed of 60 miles per hour. Clark left Detroit 1 hour later traveling at a speed of 75 miles per hour, following the same route as Mitchell. How long will it take Clark to catch Mitchell?

Answer

It will take Clark 4 hours to catch Mitchell.

Example 11.2.24

Translate to a system of equations and then solve:

Charlie left his mother's house traveling at an average speed of 36 miles per hour. His sister Sally left 15 minutes ($\frac{1}{4}$ hour) later traveling the same route at an average speed of 42 miles per hour. How long before Sally catches up to Charlie?

Answer

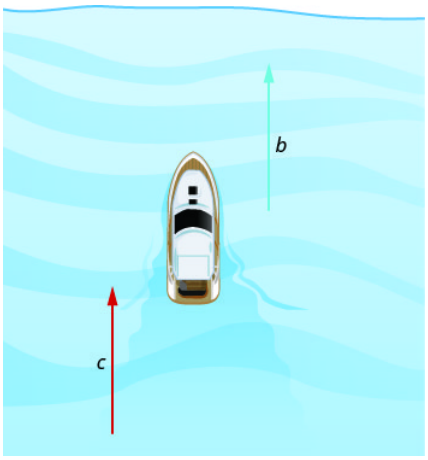
It will take Sally 112 hours to catch up to Charlie.

Many real-world applications of uniform motion arise because of the effects of currents—of water or air—on the actual speed of a vehicle. Cross-country airplane flights in the United States generally take longer going west than going east because of the prevailing wind currents.

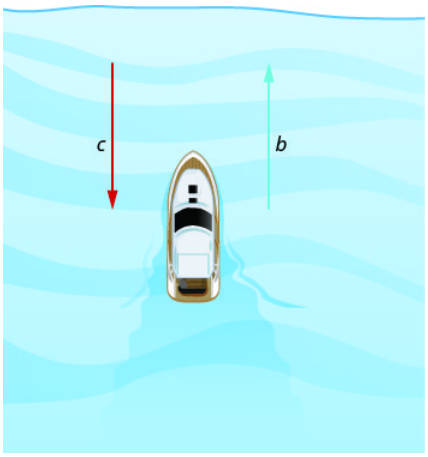
Let's take a look at a boat travelling on a river. Depending on which way the boat is going, the current of the water is either slowing it down or speeding it up.

The images below show how a river current affects the speed at which a boat is actually travelling. We'll call the speed of the boat in still water b and the speed of the river current c.

The boat is going downstream, in the same direction as the river current. The current helps push the boat, so the boat's actual speed is faster than its speed in still water. The actual speed at which the boat is moving is $b+c$.



Now, the boat is going upstream, opposite to the river current. The current is going against the boat, so the boat's actual speed is slower than its speed in still water. The actual speed of the boat is $b-c$.



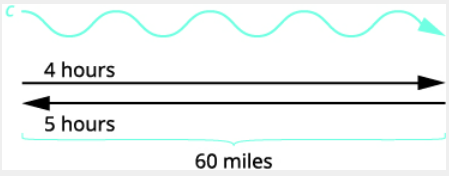
We'll put some numbers to this situation in the next example.

? Example 11.2.25

Translate to a system of equations and then solve.

A river cruise ship sailed 60 miles downstream for 4 hours and then took 5 hours sailing upstream to return to the dock. Find the speed of the ship in still water and the speed of the river current.

Answer

Read the problem.	This is a uniform motion problem and a picture will help us visualize the situation.
	
Identify what we are looking for.	We are looking for the speed of the ship in still water and the speed of the current.
Name what we are looking for.	Let $s =$ the rate of the ship in still water. $c =$ the rate of the current
A chart will help us organize the information. The ship goes downstream and then upstream. Going downstream, the current helps the ship and so the ship's actual rate is $s + c$. Going upstream, the current slows the ship and so the actual rate is $s - c$.	<table><tr><th></th><th colspan="3">Rate • Time = Distance</th></tr><tr><td>downstream</td><td>$s + c$</td><td>4</td><td>60</td></tr><tr><td>upstream</td><td>$s - c$</td><td>5</td><td>60</td></tr></table>
Downstream it takes 4 hours. Upstream it takes 5 hours. Each way the distance is 60 miles.	
Translate into a system of equations. Since rate times time is distance, we can write the system of equations.	$\begin{cases} 4(s + c) = 60 \\ 5(s - c) = 60 \end{cases}$
Solve the system of equations. Distribute to put both equations in standard form, then solve by elimination.	$\begin{cases} 4s + 4c = 60 \\ 5s - 5c = 60 \end{cases}$

Multiply the top equation by 5 and the bottom equation by 4. Add the equations, then solve for s.	$20s + 20c = 300$ $20s - 20c = 240$ $40s = 540$ $s = 13.5$ $4(s + c) = 60$
Substitute $s = 13.5$ into of the original equations.	$4(13.5 + c) = 60$ $54 + 4c = 60$ $4c = 6$ $c = 1.5$
Check the answer in the problem. The downstream rate would be $13.5 + 1.5 = 15$ mph. In 4 hours the ship would travel $15 \cdot 4 = 60$ miles. The upstream rate would be $13.5 - 1.5 = 12$ mph. In 5 hours the ship would travel $12 \cdot 5 = 60$ miles.	
Answer the question.	The rate of the ship is 13.5 mph and the rate of the current is 1.5 mph.

Example 11.2.26

Translate to a system of equations and then solve:

A Mississippi river boat cruise sailed 120 miles upstream for 12 hours and then took 10 hours to return to the dock. Find the speed of the river boat in still water and the speed of the river current.

Answer

The rate of the boat is 11 mph and the rate of the current is 1 mph.

Example 11.2.27

Translate to a system of equations and then solve:

Jason paddled his canoe 24 miles upstream for 4 hours. It took him 3 hours to paddle back. Find the speed of the canoe in still water and the speed of the river current.

Answer

The speed of the canoe is 7 mph and the speed of the current is 1 mph.

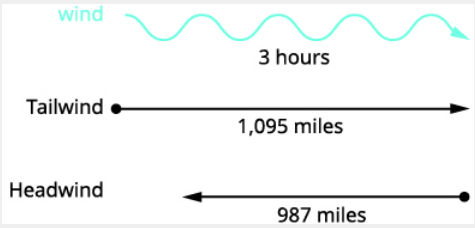
Wind currents affect airplane speeds in the same way as water currents affect boat speeds. We'll see this in the next example. A wind current in the same direction as the plane is flying is called a *tailwind*. A wind current blowing against the direction of the plane is called a *headwind*.

Example 11.2.28

Translate to a system of equations and then solve:

A private jet can fly 1,095 miles in three hours with a tailwind but only 987 miles in three hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

Read the problem.	This is a uniform motion problem and a picture will help us visualize.
	
Identify what we are looking for.	We are looking for the speed of the jet in still air and the speed of the wind.
Name what we are looking for.	Let j=j= the speed of the jet in still air. w=w= the speed of the wind.
A chart will help us organize the information. The jet makes two trips—one in a tailwind and one in a headwind. In a tailwind, the wind helps the jet and so the rate is $j + w$. In a headwind, the wind slows the jet and so the rate is $j - w$.	
Each trip takes 3 hours. In a tailwind the jet flies 1,095 miles. In a headwind the jet flies 987 miles.	
Translate into a system of equations. Since rate times time is distance, we get the system of equations.	$3(j + w) = 1095$ $3(j - w) = 987$
Solve the system of equations. Distribute, then solve by elimination. Add, and solve for j.	$3j + 3w = 1095$ $3j - 3w = 987$ $6j = 2082$ $j = 347$ $4(j + w) = 1095$
Substitute $j = 347$ into one of the original equations, then solve for w.	$3(347 + w) = 1095$ $1041 + 3w = 1095$ $3w = 54$ $w = 18$
Check the answer in the problem. With the tailwind, the actual rate of the jet would be $347 + 18 = 365$ mph. In 3 hours the jet would travel $365 \cdot 3 = 1,095$ miles Going into the headwind, the jet's actual rate would be $347 - 18 = 329$ mph. In 3 hours the jet would travel $329 \cdot 3 = 987$ miles.	
Answer the question.	The rate of the jet is 347 mph and the rate of the wind is 18 mph.

	Rate • Time = Distance		
tailwind	$j + w$	3	1095
headwind	$j - w$	3	987

? Example 11.2.29

Translate to a system of equations and then solve:

A small jet can fly 1,325 miles in 5 hours with a tailwind but only 1,035 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

The speed of the jet is 235 mph and the speed of the wind is 30 mph.

? Example 11.2.30

Translate to a system of equations and then solve:

A commercial jet can fly 1,728 miles in 4 hours with a tailwind but only 1,536 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

The speed of the jet is 408 mph and the speed of the wind is 24 mph.

Access this online resource for additional instruction and practice with systems of equations.

- Systems of Equations

Key Concepts

- **How To Solve Applications with Systems of Equations**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose variables to represent those quantities.
  4. **Translate** into a system of equations.
  5. **Solve** the system of equations using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Glossary

complementary angles

Two angles are complementary if the sum of the measures of their angles is 90 degrees.

supplementary angles

Two angles are supplementary if the sum of the measures of their angles is 180 degrees.

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11.2E: Exercises

Practice Makes Perfect

Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

1. The sum of two number is 15. One number is 3 less than the other. Find the numbers.

2. The sum of two number is 30. One number is 4 less than the other. Find the numbers.

Answer

13 and 17

3. The sum of two number is −16. One number is 20 less than the other. Find the numbers.

4. The sum of two number is -26. One number is 12 less than the other. Find the numbers.

Answer

-7 and -19

5. The sum of two numbers is 65. Their difference is 25. Find the numbers.

6. The sum of two numbers is 37. Their difference is 9. Find the numbers.

Answer

14 and 23

7. The sum of two numbers is -27. Their difference is -59. Find the numbers.

8. The sum of two numbers is -45. Their difference is -89. Find the numbers.

Answer

22 and -67

9. Maxim has been offered positions by two car companies. The first company pays a salary of $10,000 plus a commission of $1000 for each car sold. The second pays a salary of $20,000 plus a commission of $500 for each car sold. How many cars would need to be sold to make the total pay the same?

10. Jackie has been offered positions by two cable companies. The first company pays a salary of $14,000 plus a commission of $100 for each cable package sold. The second pays a salary of $20,000 plus a commission of $25 for each cable package sold. How many cable packages would need to be sold to make the total pay the same?

Answer

Eighty cable packages would need to be sold to make the total pay the same.

11. Amara currently sells televisions for company A at a salary of $17,000 plus a $100 commission for each television she sells. Company B offers her a position with a salary of $29,000 plus a $20 commission for each television she sells. How televisions would Amara need to sell for the options to be equal?

12. Mitchell currently sells stoves for company A at a salary of $12,000 plus a $150 commission for each stove he sells. Company B offers him a position with a salary of $24,000 plus a $50 commission for each stove he sells. How many stoves would Mitchell need to sell for the options to be equal?

Answer

Mitchell would need to sell 120 stoves for the companies to be equal.

13. Two containers of gasoline hold a total of fifty gallons. The big container can hold ten gallons less than twice the small container. How many gallons does each container hold?

14. June needs 48 gallons of punch for a party and has two different coolers to carry it in. The bigger cooler is five times as large as the smaller cooler. How many gallons can each cooler hold?

Answer

8 and 40 gallons

15. Shelly spent 10 minutes jogging and 20 minutes cycling and burned 300 calories. The next day, Shelly swapped times, doing 20 minutes of jogging and 10 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?

16. Drew burned 1800 calories Friday playing one hour of basketball and canoeing for two hours. Saturday he spent two hours playing basketball and three hours canoeing and burned 3200 calories. How many calories did he burn per hour when playing basketball? How many calories did he burn per hour when canoeing?

Answer

1000 calories playing basketball and 400 calories canoeing

17. Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and thumb drive. Troy bought four notebooks and five thumb drives for $116. Lisa bought two notebooks and three thumb dives for $68. Find the cost of each notebook and each thumb drive.

18. Nancy bought seven pounds of oranges and three pounds of bananas for $17. Her husband later bought three pounds of oranges and six pounds of bananas for $12. What was the cost per pound of the oranges and the bananas?

Answer

Oranges cost $2 per pound and bananas cost $1 per pound

19. Andrea is buying some new shirts and sweaters. She is able to buy 3 shirts and 2 sweaters for $114 or she is able to buy 2 shirts and 4 sweaters for $164. How much does a shirt cost? How much does a sweater cost?

20. Peter is buying office supplies. He is able to buy 3 packages of paper and 4 staplers for $40 or he is able to buy 5 packages of paper and 6 staplers for $62. How much does a package of paper cost? How much does a stapler cost?

Answer

Package of paper $4, stapler $7

21. The total amount of sodium in 2 hot dogs and 3 cups of cottage cheese is 4720 mg. The total amount of sodium in 5 hot dogs and 2 cups of cottage cheese is 6300 mg. How much sodium is in a hot dog? How much sodium is in a cup of cottage cheese?

22. The total number of calories in 2 hot dogs and 3 cups of cottage cheese is 960 calories. The total number of calories in 5 hot dogs and 2 cups of cottage cheese is 1190 calories. How many calories are in a hot dog? How many calories are in a cup of cottage cheese?

Answer

Hot dog 150 calories, cup of cottage cheese 220 calories

23. Molly is making strawberry infused water. For each ounce of strawberry juice, she uses three times as many ounces of water as juice. How many ounces of strawberry juice and how many ounces of water does she need to make 64 ounces of strawberry infused water?

24. Owen is making lemonade from concentrate. The number of quarts of water he needs is 4 times the number of quarts of concentrate. How many quarts of water and how many quarts of concentrate does Owen need to make 100 quarts of lemonade?

Answer

Owen will need 80 quarts of water and 20 quarts of concentrate to make 100 quarts of lemonade.

Solve Geometry Applications

In the following exercises, translate to a system of equations and solve.

25. The difference of two complementary angles is 55 degrees. Find the measures of the angles.

26. The difference of two complementary angles is 17 degrees. Find the measures of the angles.

Answer

53.5degrees and 36.5degree.

27. Two angles are complementary. The measure of the larger angle is twelve less than twice the measure of the smaller angle. Find the measures of both angles.

28. Two angles are complementary. The measure of the larger angle is ten more than four times the measure of the smaller angle. Find the measures of both angles.

Answer

16 degrees and 74 degrees

29. The difference of two supplementary angles is 8 degrees. Find the measures of the angles.

30. The difference of two supplementary angles is 88 degrees. Find the measures of the angles.

Answer

134 degrees and 46 degrees

31. Two angles are supplementary. The measure of the larger angle is four more than three times the measure of the smaller angle. Find the measures of both angles.

32. Two angles are supplementary. The measure of the larger angle is five less than four times the measure of the smaller angle. Find the measures of both angles.

Answer

37 degrees and 143 degrees

33. The measure of one of the small angles of a right triangle is 14 more than 3 times the measure of the other small angle. Find the measure of both angles.

34. The measure of one of the small angles of a right triangle is 26 more than 3 times the measure of the other small angle. Find the measure of both angles.

Answer

16° and 74°

35. The measure of one of the small angles of a right triangle is 15 less than twice the measure of the other small angle. Find the measure of both angles.

36. The measure of one of the small angles of a right triangle is 45 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

45° and 45°

37. Wayne is hanging a string of lights 45 feet long around the three sides of his patio, which is adjacent to his house. The length of his patio, the side along the house, is five feet longer than twice its width. Find the length and width of the patio.

38. Darrin is hanging 200 feet of Christmas garland on the three sides of fencing that enclose his front yard. The length is five feet less than three times the width. Find the length and width of the fencing.

Answer

Width is 41 feet and length is 118 feet.

39. A frame around a family portrait has a perimeter of 90 inches. The length is fifteen less than twice the width. Find the length and width of the frame.

40. The perimeter of a toddler play area is 100 feet. The length is ten more than three times the width. Find the length and width of the play area.

Answer

Width is 10 feet and length is 40 feet.

Solve Uniform Motion Applications

In the following exercises, translate to a system of equations and solve.

41. Sarah left Minneapolis heading east on the interstate at a speed of 60 mph. Her sister followed her on the same route, leaving two hours later and driving at a rate of 70 mph. How long will it take for Sarah's sister to catch up to Sarah?

42. College roommates John and David were driving home to the same town for the holidays. John drove 55 mph, and David, who left an hour later, drove 60 mph. How long will it take for David to catch up to John?

Answer

11 hours

43. At the end of spring break, Lucy left the beach and drove back towards home, driving at a rate of 40 mph. Lucy's friend left the beach for home 30 minutes (half an hour) later, and drove 50 mph. How long did it take Lucy's friend to catch up to Lucy?

44. Felecia left her home to visit her daughter driving 45 mph. Her husband waited for the dog sitter to arrive and left home twenty minutes (1/3 hour) later. He drove 55 mph to catch up to Felecia. How long before he reaches her?

Answer

1.5 hour

45. The Jones family took a 12-mile canoe ride down the Indian River in two hours. After lunch, the return trip back up the river took three hours. Find the rate of the canoe in still water and the rate of the current.

46. A motor boat travels 60 miles down a river in three hours but takes five hours to return upstream. Find the rate of the boat in still water and the rate of the current.

Answer

Boat rate is 16 mph and current rate is 4 mph.

47. A motor boat traveled 18 miles down a river in two hours but going back upstream, it took 4.54.5 hours due to the current. Find the rate of the motor boat in still water and the rate of the current. (Round to the nearest hundredth.)

48. A river cruise boat sailed 80 miles down the Mississippi River for four hours. It took five hours to return. Find the rate of the cruise boat in still water and the rate of the current.

Answer

Boat rate is 18 mph and current rate is 2 mph.

49. A small jet can fly 1072 miles in 4 hours with a tailwind but only 848 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

50. A small jet can fly 1435 miles in 5 hours with a tailwind but only 1,215 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

Jet rate is 265 mph and wind speed is 22 mph.

51. A commercial jet can fly 868 miles in 2 hours with a tailwind but only 792 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

52. A commercial jet can fly 1,320 miles in 3 hours with a tailwind but only 1170 miles in 3 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

Jet rate is 415 mph and wind speed is 25 mph.

Writing Exercises

53. Write an application problem similar to Example. Then translate to a system of equations and solve it.\

54. Write a uniform motion problem similar to Example that relates to where you live with your friends or family members. Then translate to a system of equations and solve it.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve direct translation applications.			
solve geometry applications.			
solve uniform motion applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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11.3: Solve Mixture Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve mixture applications
- Solve interest applications
- Solve applications of cost and revenue functions

Before you get started, take this readiness quiz.

1. Multiply: $4.025(1,562)$
   If you missed this problem, review [link].
2. Write 8.2% as a decimal.
   If you missed this problem, review [link].
3. Earl's dinner bill came to $32.50 and he wanted to leave an 18% tip. How much should the tip be?
   If you missed this problem, review [link].

Solve Mixture Applications

Mixture application involve combining two or more quantities. When we solved mixture applications with coins and tickets earlier, we started by creating a table so we could organize the information. For a coin example with nickels and dimes, the table looked like this:

	Number •	Value =	Total Value
nickels			
dimes			

Using one variable meant that we had to relate the number of nickels and the number of dimes. We had to decide if we were going to let n be the number of nickels and then write the number of dimes in terms of n, or if we would let d be the number of dimes and write the number of nickels in terms of d.

Now that we know how to solve systems of equations with two variables, we'll just let n be the number of nickels and d be the number of dimes. We'll write one equation based on the total value column, like we did before, and the other equation will come from the number column.

For the first example, we'll do a ticket problem where the ticket prices are in whole dollars, so we won't need to use decimals just yet.

Example 11.3.1

Translate to a system of equations and solve:

A science center sold 1,363 tickets on a busy weekend. The receipts totaled $12,146. How many $12 adult tickets and how many $7 child tickets were sold?

Answer

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of adult tickets and the number of child tickets sold.
Step 3. Name what we are looking for.	Let $a =$ the number of adult tickets. $c =$ the number of child tickets
A table will help us organize the data. We have two types of tickets, adult and child.	Write in a and c for the number of tickets.

Write the total number of tickets sold at the bottom of the Number column.	Altogether 1,363 were sold.
Write the value of each type of ticket in the Value column.	The value of each adult ticket is \$12. The value of each child tickets is \$7.
The number times the value gives the total value, so the total value of adult tickets is $a \cdot 12 = 12a$, and the total value of child tickets is $c \cdot 7 = 7c$.	Fill in the Total Value column.
Altogether the total value of the tickets was \$12,146.	<table><tr><th>Type</th><th>Number</th><th>•</th><th>Value</th><th>=</th><th>Total Value</th></tr><tr><td>adult</td><td>a</td><td></td><td>12</td><td></td><td>$11a$</td></tr><tr><td>child</td><td>c</td><td></td><td>7</td><td></td><td>$8c$</td></tr><tr><td></td><td>1,363</td><td></td><td></td><td></td><td>12,146</td></tr></table>
Step 4. Translate into a system of equations.	
The Number column and the Total value column give us the system of equations.	$\begin{cases} a + c = 1{,}363 \\ 12a + 7c = 12{,}146 \end{cases}$
We will use the elimination method to solve this system. Multiply the first equation by -7.	$\begin{cases} -7(a + c) = -7(1{,}363) \\ 12a + 7c = 12{,}146 \end{cases}$
Simplify and add, then solve for a.	$\begin{array}{rl} -7a - 7c = & -9{,}541 \\ 12a + 7c = & 12{,}146 \\ \hline 5a = & 2{,}605 \\ a = & 521 \end{array}$
Substitute $a = 521$ into the first equation, then solve for c.	$\begin{aligned} a + c &= 1{,}363 \\ 521 + c &= 1{,}363 \\ c &= 842 \end{aligned}$
Step 6. Check the answer in the problem. 521 adult at \$12 per ticket makes \$ 6,252 842 child at \$7 per ticket makes \$58,994 The total receipts are \$12,146✓	
Step 7. Answer the question.	The science center sold 521 adult tickets and 842 child tickets.

Example 11.3.2

Translate to a system of equations and solve:

The ticket office at the zoo sold 553 tickets one day. The receipts totaled \$3,936. How many \$9 adult tickets and how many \$6 child tickets were sold?

Answer

206 adults, 347 children

Example 11.3.3

Translate to a system of equations and solve:

The box office at a movie theater sold 147 tickets for the evening show, and receipts totaled \$1,302. How many \$11 adult and how many \$8 child tickets were sold?

Answer

42 adults, 105 children

In the next example, we'll solve a coin problem. Now that we know how to work with systems of two variables, naming the variables in the 'number' column will be easy.

Example 11.3.4

Translate to a system of equations and solve:

Juan has a pocketful of nickels and dimes. The total value of the coins is \$8.10. The number of dimes is 9 less than twice the number of nickels. How many nickels and how many dimes does Juan have?

Answer

Step 1. Read the problem. We will create a table to organize the information.	
Step 2. Identify what we are looking for.	We are looking for the number of nickels and the number of dimes.
Step 3. Name what we are looking for.	Let $n =$ the number of nickels. $d =$ the number of dimes
A table will help us organize the data. We have two types of coins, nickels and dimes.	Write *n* and *d* for the number of each type of coin.
Fill in the Value column with the value of each type of coin.	The value of each nickel is \$0.05. The value of each dime is \$0.10.
The number times the value gives the total value, so, the total value of the nickels is $n(0.05) = 0.05n$ and the total value of dimes is $d(0.10) = 0.10d$. Altogether the total value of the coins is \$8.10.	**Type** | **Number** • **Value** = **Total Value** nickels | n | 0.05 | $0.05n$ dimes | d | 0.10 | $0.10d$ | | | 8.10
Step 4. Translate into a system of equations.	
The Total Value column gives one equation.	$0.05n + 0.10d = 8.10$
We also know the number of dimes is 9 less than twice the number of nickels.	
Translate to get the second equation.	$d = 2n - 9$
Now we have the system to solve.	$\begin{cases} 0.05n + 0.10d = 8.10 \\ d = 2n - 9 \end{cases}$
Step 5. Solve the system of equations We will use the substitution method.	
Substitute $d = 2n - 9$ into the first equation.	$0.05n + 0.10d = 8.10$ $0.05n + 0.10(2n - 9) = 8.10$
Simplify and solve for *n*.	$0.05n + 0.2n - 0.90 = 8.10$ $0.25n - 0.90 = 8.10$ $0.25n = 9.00$ $n = 36$
To find the number of dimes, substitute $n = 36$ into the second equation.	$d = 2n - 9$ $d = 2 \cdot 36 - 9$ $d = 63$

Step 6. Check the answer in the problem 63 dimes at $\$0.10 = \$6.30$ 36 nickels at $\$0.05 = \$1.80$ Total $= \$8.10$✓	
Step 7. Answer the question.	Juan has 36 nickels and 63 dimes.

Example 11.3.5

Translate to a system of equations and solve:

Matilda has a handful of quarters and dimes, with a total value of $8.55. The number of quarters is 3 more than twice the number of dimes. How many dimes and how many quarters does she have?

Answer

13 dimes and 29 quarters

Example 11.3.6

Translate to a system of equations and solve:

Priam has a collection of nickels and quarters, with a total value of $7.30. The number of nickels is six less than three times the number of quarters. How many nickels and how many quarters does he have?

Answer

19 quarters and 51 nickels

Some mixture applications involve combining foods or drinks. Example situations might include combining raisins and nuts to make a trail mix or using two types of coffee beans to make a blend.

Example 11.3.7

Translate to a system of equations and solve:

Carson wants to make 20 pounds of trail mix using nuts and chocolate chips. His budget requires that the trail mix costs him $7.60. per pound. Nuts cost $9.00 per pound and chocolate chips cost $2.00 per pound. How many pounds of nuts and how many pounds of chocolate chips should he use?

Answer

Step 1. Read the problem. We will create a table to organize the information.	
Step 2. Identify what we are looking for.	We are looking for the number of pounds of nuts and the number of pounds of chocolate chips.
Step 3. Name what we are looking for.	Let $n =$ the number of pound of nuts. $c =$ the number of pounds of chips
Carson will mix nuts and chocolate chips to get trail mix. Write in n and c for the number of pounds of nuts and chocolate chips.	

Type	Number of pounds	• Value	= Total Value
Nuts	n	9.00	$9n$
Chocolate chips	c	2.00	$2c$
Trail mix	20	7.60	$7.60(20) = 152$

There will be 20 pounds of trail mix. Put the price per pound of each item in the Value column. Fill in the last column using $\text{Number} \bullet \text{Value} = \text{Total Value}$	
Step 4. Translate into a system of equations. We get the equations from the Number and Total Value columns.	$\begin{cases} n + c = 20 \\ 9n + 2c = 152 \end{cases}$
Step 5. Solve the system of equations We will use elimination to solve the system. Multiply the first equation by -2 to eliminate c.	$\begin{cases} -2(n + c) = -2(20) \\ 9n + 2c = 152 \end{cases}$
Simplify and add. Solve for n.	$\begin{cases} -2n - 2c = -40 \\ 9n + 2c = 152 \end{cases}$ $7n = 112$ $n = 16$
To find the number of pounds of chocolate chips, substitute $n = 16$ into the first equation, then solve for c.	$n + c = 20$ $16 + c = 20$ $c = 4$
Step 6. Check the answer in the problem. $16 + 4 = 20\checkmark$ $9 \cdot 16 + 2 \cdot 4 = 152\checkmark$	
Step 7. Answer the question.	Carson should mix 16 pounds of nuts with 4 pounds of chocolate chips to create the trail mix.

Example 11.3.8

Translate to a system of equations and solve:

Greta wants to make 5 pounds of a nut mix using peanuts and cashews. Her budget requires the mixture to cost her \$6 per pound. Peanuts are \$4 per pound and cashews are \$9 per pound. How many pounds of peanuts and how many pounds of cashews should she use?

Answer

3 pounds peanuts and 2 pounds cashews

Example 11.3.9

Translate to a system of equations and solve:

Sammy has most of the ingredients he needs to make a large batch of chili. The only items he lacks are beans and ground beef. He needs a total of 20 pounds combined of beans and ground beef and has a budget of \$3 per pound. The price of beans is \$1 per pound and the price of ground beef is \$5 per pound. How many pounds of beans and how many pounds of ground beef should he purchase?

Answer

10 pounds of beans, 10 pounds of ground beef

Another application of mixture problems relates to concentrated cleaning supplies, other chemicals, and mixed drinks. The concentration is given as a percent. For example, a 20% concentrated household cleanser means that 20% of the total amount is

cleanser, and the rest is water. To make 35 ounces of a 20% concentration, you mix 7 ounces (20% of 35) of the cleanser with 28 ounces of water.

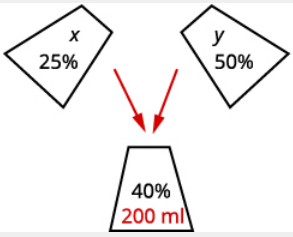
For these kinds of mixture problems, we'll use "percent" instead of "value" for one of the columns in our table.

? Example 11.3.10

Translate to a system of equations and solve:

Sasheena is lab assistant at her community college. She needs to make 200 milliliters of a 40% solution of sulfuric acid for a lab experiment. The lab has only 25% and 50% solutions in the storeroom. How much should she mix of the 25% and the 50% solutions to make the 40% solution?

Answer

Step 1. Read the problem. A figure may help us visualize the situation, then we will create a table to organize the information.	Sasheena must mix some of the 25 solution and some of the 50 solution together to get 200 ml of the 40 solution.
	
Step 2. Identify what we are looking for.	We are looking for how much of each solution she needs.
Step 3. Name what we are looking for.	Let $x =$ number of ml of 25 \(y= \text{number of }ml\text{ of }50%\text{ solution}\)
A table will help us organize the data. She will mix x ml of 25 with y ml of 50 to get 200 ml of 40 solution. We write the percents as decimals in the chart. We multiply the number of units times the concentration to get the total amount of sulfuric acid in each solution.	

Type	Number of units	• Concentration %	= Amount
25%	x	0.25	$0.25x$
50%	y	0.50	$0.50y$
40%	200	0.40	$0.40(200)$

Step 4. Translate into a system of equations. We get the equations from the Number column and the Amount column. Now we have the system.	$\begin{cases} x + y = 200 \\ 0.25x + 0.50y = 0.40(200) \end{cases}$
Step 5. Solve the system of equations We will solve the system by elimination. Multiply the first equation by -0.5 to eliminate y.	$\begin{cases} -0.5(x + y) = -0.5(200) \\ 0.25x + 0.50y = 80 \end{cases}$
Simplify and add to solve for x.	$\begin{array}{rcl} -0.5x - 0.5y &=& -100 \\ 0.25x + 0.5y &=& 80 \\ \hline -0.25x &=& -20 \\ x &=& 80 \end{array}$
To solve for y, substitute $x = 80$ into the first equation.	$\begin{aligned} x + y &= 200 \\ 80 + y &= 200 \\ y &= 120 \end{aligned}$

Step 6. Check the answer in the problem. $80+120 = 200$✓ $0.25(80)+0.50(120) = 200$✓ Yes!	
Step 7. Answer the question.	Sasheena should mix 80 ml of the 25 solution with 120 ml of the 50 solution to get the 200 ml of the 40 solution.

? Example 11.3.11

Translate to a system of equations and solve:

LeBron needs 150 milliliters of a 30% solution of sulfuric acid for a lab experiment but only has access to a 25% and a 50% solution. How much of the 25% and how much of the 50% solution should he mix to make the 30% solution?

Answer

120 ml of 25% solution and 30 ml of 50% solution

? Example 11.3.12

Translate to a system of equations and solve:

Anatole needs to make 250 milliliters of a 25% solution of hydrochloric acid for a lab experiment. The lab only has a 10% solution and a 40% solution in the storeroom. How much of the 10% and how much of the 40% solutions should he mix to make the 25% solution?

Answer

125 ml of 10% solution and 125 ml of 40% solution

Solve Interest Applications

The formula to model simple interest applications is $I = Prt$. Interest, I, is the product of the principal, P, the rate, r, and the time, t. In our work here, we will calculate the interest earned in one year, so t will be 1.

We modify the column titles in the mixture table to show the formula for interest, as you'll see in the next example.

Translate to a system of equations and solve:

Adnan has $40,000 to invest and hopes to earn 7.1 interest per year. He will put some of the money into a stock fund that earns 8% per year and the rest into bonds that earns 3% per year. How much money should he put into each fund?

Answer

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the amount to invest in each fund.
Step 3. Name what we are looking for.	Let $s =$ the amount invested in stocks. $b =$ the amount invested in stocks
Write the interest rate as a decimal for each fund. Multiply: Principal · Rate · Time	

Account	Principal •	Rate •	Time =	Interest
Stock fund	s	0.08	1	$0.08s$
Bonds	b	0.03	1	$0.03b$
Total	40,000	0.071		$0.071(40{,}000)$

Step 4. Translate into a system of equations. We get our system of equations from the Principal column and the Interest column.	$\begin{cases} s+b=40{,}000 \\ 0.08s+0.03b=0.071(40{,}000) \end{cases}$
Step 5. Solve the system of equations by elimination. Multiply the top equation by -0.03.	$\begin{cases} -0.03(s+b)=-0.03(40{,}000) \\ 0.08s+0.03b=2{,}840 \end{cases}$
Simplify and add to solve for s.	$\begin{cases} -0.03s-0.03b=-1{,}200 \\ 0.08s+0.03b=2{,}840 \end{cases}$ $0.05s=1{,}640$ $s=32{,}800$
To find b, substitute $s = 32{,}800$ into the first equation.	$s+b=40{,}000$ $32{,}800+b=40{,}000$ $b=7{,}200$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	Adnan should invest \$32,800 in stock and \$7,200 in bonds.

Did you notice that the Principal column represents the total amount of money invested while the Interest column represents only the interest earned? Likewise, the first equation in our system, $s+b=40,000$, represents the total amount of money invested and the second equation, $0.08s+0.03b=0.071(40,000)$represents the interest earned.

Example 11.3.14

Translate to a system of equations and solve:

Leon had \$50,000 to invest and hopes to earn 6.2 interest per year. He will put some of the money into a stock fund that earns 7% per year and the rest in to a savings account that earns 2% per year. How much money should he put into each fund?

Answer

\$42,000 in the stock fund and \$8000 in the savings account

Example 11.3.15

Translate to a system of equations and solve:

Julius invested \$7000 into two stock investments. One stock paid 11% interest and the other stock paid 13% interest. He earned 12.5 interest on the total investment. How much money did he put in each stock?

Answer

\$1750 at 11% and \$5250 at 13%

The next example requires that we find the principal given the amount of interest earned.

Example 11.3.16

Translate to a system of equations and solve:

Rosie owes \$21,540 on her two student loans. The interest rate on her bank loan is 10.5 and the interest rate on the federal loan is 5.9. The total amount of interest she paid last year was $\$1,669.68$ What was the principal for each loan?

Answer

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the principal of each loan.
Step 3. Name what we are looking for.	Let $b =$ the principal for the bank loan. $f =$ the principal on the federal loan
The total loans are $21,540.	
Record the interest rates as decimals in the chart. Multiply using the formula $I = Prt$ to get the Interest.	

Account	Principal	• Rate	• Time	= Interest
Bank	b	0.105	1	$0.105b$
Federal	f	0.059	1	$0.059f$
Total	21,540			1669.68

Step 4. Translate into a system of equations. The system of equations comes from the Principal column and the Interest column.	$b + f = 21{,}540$ $0.105b + 0.059f = 1669.68$
Step 5. Solve the system of equations We will use substitution to solve. Solve the first equation for b.	$b + f = 21{,}540$ $b = -f + 21{,}540$
	$0.105b + 0.059f = 1669.68$
Substitute $b = -f + 21.540$ into the second equation.	$0.105(-f + 21{,}540) + 0.059f = 1669.68$ $-0.105f + 2261.70 + 0.059f = 1669.68$
Simplify and solve for f.	$-0.046f + 2261.70 = 1669.68$ $-0.046f = -592.02$ $f = 12{,}870$
To find b, substitute $f = 12{,}870$ into the first equation.	$b + f = 21{,}540$ $b + 12{,}870 = 21{,}540$ $b = 8{,}670$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	The principal of the federal loan was $12,870 and the principal for the bank loan was $8,670.

Example 11.3.17

Translate to a system of equations and solve:

Laura owes $18,000 on her student loans. The interest rate on the bank loan is 2.5% and the interest rate on the federal loan is 6.9%. The total amount of interest she paid last year was $1,066. What was the principal for each loan?

Answer

Bank $4,000; Federal $14,000

Example 11.3.18

Translate to a system of equations and solve:

Jill's Sandwich Shoppe owes $65,200 on two business loans, one at 4.5% interest and the other at 7.2% interest. The total amount of interest owed last year was $3,582. What was the principal for each loan?

Answer

$41,200 at 4.5%, $24,000 at 7.2%

Solve applications of cost and revenue functions

Suppose a company makes and sells *x* units of a product. The cost to the company is the total costs to produce *x* units. This is the cost to manufacture for each unit times *x*, the number of units manufactured, plus the fixed costs.

The **revenue** is the money the company brings in as a result of selling *x* units. This is the selling price of each unit times the number of units sold.

When the costs equal the revenue we say the business has reached the **break-even point**.

COST AND REVENUE FUNCTIONS

The **cost function** is the cost to manufacture each unit times *x*, the number of units manufactured, plus the fixed costs.

$$C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$$

The **revenue** function is the selling price of each unit times *x*, the number of units sold.

$$R(x) = (\text{selling price per unit}) \cdot x$$

The **break-even point** is when the revenue equals the costs.

$$C(x) = R(x)$$

Example 11.3.19

The manufacturer of a weight training bench spends $105 to build each bench and sells them for $245. The manufacturer also has fixed costs each month of $7,000.

ⓐ Find the cost function *C* when *x* benches are manufactured.

ⓑ Find the revenue function *R* when *x* benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ The manufacturer has $7,000 of fixed costs no matter how many weight training benches it produces. In addition to the fixed costs, the manufacturer also spends $105 to produce each bench. Suppose *x* benches are sold.

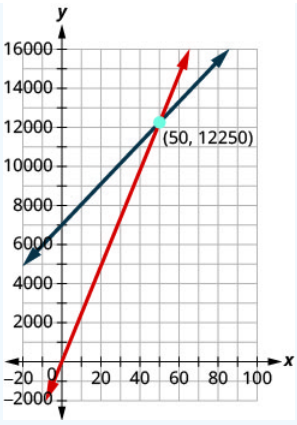
Write the general Cost function formula.	$C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$
Substitute in the cost values.	$C(x) = 105x + 7000$

ⓑ The manufacturer sells each weight training bench for $245. We get the total revenue by multiplying the revenue per unit times the number of units sold.

Write the general Revenue function.	$C(x) = (\text{selling price per unit}) \cdot x$
Substitute in the revenue per unit.	$R(x) = 245x$

ⓒ Essentially we have a system of linear equations. We will show the graph of the system as this helps make the idea of a break-even point more visual.

$$\begin{cases} C(x) = 105x + 7000 \\ R(x) = 245x \end{cases} \quad \text{or} \quad \begin{cases} y = 105x + 7000 \\ y = 245x \end{cases}$$



ⓓ To find the actual value, we remember the break-even point occurs when costs equal revenue.

Write the break-even formula. $C(x) = R(x)$

Solve.
$$105x + 7000 = 245x$$
$$7000 = 140x$$
$$50 = x$$

When 50 benches are sold, the costs equal the revenue.

$C(x) = 105x + 7000$	$R(x) = 245x$
$C(50) = 105(50) + 7000$	$R(50) = 245 \cdot 50$
$C(50) = 12{,}250$	$R(50) = 12{,}250$

When 50 benches are sold, the revenue and costs are both \$12,250. Notice this corresponds to the ordered pair $(50, 12250)$

Example 11.3.20

The manufacturer of a weight training bench spends \$15 to build each bench and sells them for \$32. The manufacturer also has fixed costs each month of \$25,500.

ⓐ Find the cost function C when x benches are manufactured.

ⓑ Find the revenue function R when x benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

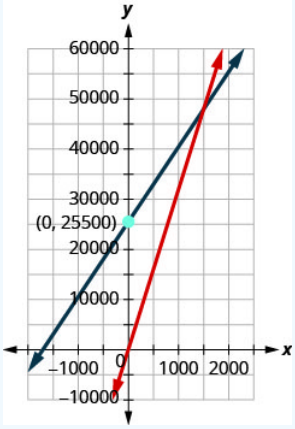
ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ $C(x) = 15x + 25,500$

ⓑ $R(x) = 32x$

ⓒ



ⓓ 1,5001,500; when 1,500 benches are sold, the cost and revenue will be both 48,000

Example 11.3.21

The manufacturer of a weight training bench spends $120 to build each bench and sells them for $170. The manufacturer also has fixed costs each month of $150,000.

ⓐ Find the cost function C when x benches are manufactured.

ⓑ Find the revenue function R when x benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

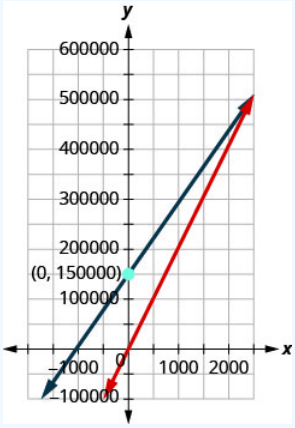
ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ $C(x) = 120x + 150,000$

ⓑ $R(x) = 170x$

ⓒ



ⓓ $3,000$; when 3,000 benches are sold, the revenue and costs are both $510,000

Access this online resource for additional instruction and practice with interest and mixtures.

- Interest and Mixtures

Key Concepts

- **Cost function:** The cost function is the cost to manufacture each unit times *x*, the number of units manufactured, plus the fixed costs.

  $C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$

- **Revenue:** The revenue function is the selling price of each unit times *x*, the number of units sold.

  $R(x) = (\text{selling price per unit}) \cdot x$

- **Break-even point:** The break-even point is when the revenue equals the costs.

  $C(x) = R(x)$

Glossary

cost function

The cost function is the cost to manufacture each unit times xx, the number of units manufactured, plus the fixed costs; $C(x)$ = (cost per unit)x + fixed costs.

revenue

The revenue is the selling price of each unit times *x*, the number of units sold; $R(x)$ = (selling price per unit)x.

break-even point

The point at which the revenue equals the costs is the break-even point; C(x)=R(x).

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11.3E: Exercises

Practice Makes Perfect

Solve Mixture Applications

In the following exercises, translate to a system of equations and solve.

1. Tickets to a Broadway show cost $35 for adults and $15 for children. The total receipts for 1650 tickets at one performance were $47,150. How many adult and how many child tickets were sold?

2. Tickets for the Cirque du Soleil show are $70 for adults and $50 for children. One evening performance had a total of 300 tickets sold and the receipts totaled $17,200. How many adult and how many child tickets were sold?

Answer

110 adult tickets, 190 child tickets

3. Tickets for an Amtrak train cost $10 for children and $22 for adults. Josie paid $1200 for a total of 72 tickets. How many children tickets and how many adult tickets did Josie buy?

4. Tickets for a Minnesota Twins baseball game are $69 for Main Level seats and $39 for Terrace Level seats. A group of sixteen friends went to the game and spent a total of $804 for the tickets. How many of Main Level and how many Terrace Level tickets did they buy?

Answer

6 good seats, 10 cheap seats

5. Tickets for a dance recital cost $15 for adults and $7 dollars for children. The dance company sold 253 tickets and the total receipts were $2771. How many adult tickets and how many child tickets were sold?

6. Tickets for the community fair cost $12 for adults and $5 dollars for children. On the first day of the fair, 312 tickets were sold for a total of $2204. How many adult tickets and how many child tickets were sold?

Answer

92 adult tickets, 220 children tickets

7. Brandon has a cup of quarters and dimes with a total value of $\$3.80$. The number of quarters is four less than twice the number of quarters. How many quarters and how many dimes does Brandon have?

8. Sherri saves nickels and dimes in a coin purse for her daughter. The total value of the coins in the purse is $\$0.95$. The number of nickels is two less than five times the number of dimes. How many nickels and how many dimes are in the coin purse?

Answer

13 nickels, 3 dimes

9. Peter has been saving his loose change for several days. When he counted his quarters and nickels, he found they had a total value $\$13.10$. The number of quarters was fifteen more than three times the number of dimes. How many quarters and how many dimes did Peter have?

10. Lucinda had a pocketful of dimes and quarters with a value of $6.20. The number of dimes is eighteen more than three times the number of quarters. How many dimes and how many quarters does Lucinda have?

Answer

42 dimes, 8 quarters

11. A cashier has 30 bills, all of which are $10 or $20 bills. The total value of the money is $460. How many of each type of bill does the cashier have?

12. A cashier has 54 bills, all of which are $10 or $20 bills. The total value of the money is $910. How many of each type of bill does the cashier have?

Answer

17 $10 bills, 37 $20 bills

13. Marissa wants to blend candy selling for $1.80 per pound with candy costing $1.20 per pound to get a mixture that costs her $1.40 per pound to make. She wants to make 90 pounds of the candy blend. How many pounds of each type of candy should she use?

14. How many pounds of nuts selling for $6 per pound and raisins selling for $3 per pound should Kurt combine to obtain 120 pounds of trail mix that cost him $5 per pound?

Answer

80 pounds nuts and 40 pounds raisins

15. Hannah has to make twenty-five gallons of punch for a potluck. The punch is made of soda and fruit drink. The cost of the soda is $1.79 per gallon and the cost of the fruit drink is $2.49 per gallon. Hannah's budget requires that the punch cost $2.21 per gallon. How many gallons of soda and how many gallons of fruit drink does she need?

16. Joseph would like to make twelve pounds of a coffee blend at a cost of $6 per pound. He blends Ground Chicory at $5 a pound with Jamaican Blue Mountain at $9 per pound. How much of each type of coffee should he use?

Answer

9 pounds of Chicory coffee, 3 pounds of Jamaican Blue Mountain coffee

17. Julia and her husband own a coffee shop. They experimented with mixing a City Roast Columbian coffee that cost $7.80 per pound with French Roast Columbian coffee that cost $8.10 per pound to make a twenty-pound blend. Their blend should cost them $7.92 per pound. How much of each type of coffee should they buy?

18. Twelve-year old Melody wants to sell bags of mixed candy at her lemonade stand. She will mix M&M's that cost $4.89 per bag and Reese's Pieces that cost $3.79 per bag to get a total of twenty-five bags of mixed candy. Melody wants the bags of mixed candy to cost her $4.23 a bag to make. How many bags of M&M's and how many bags of Reese's Pieces should she use?

Answer

10 bags of M&M's, 15 bags of Reese's Pieces

19. Jotham needs 70 liters of a 50% solution of an alcohol solution. He has a 30% and an 80% solution available. How many liters of the 30% and how many liters of the 80% solutions should he mix to make the 50% solution?

20. Joy is preparing 15 liters of a 25% saline solution. She only has 40% and 10% solution in her lab. How many liters of the 40% and how many liters of the 10% should she mix to make the 25% solution?

Answer

7.5 liters of each solution

21. A scientist needs 65 liters of a 15% alcohol solution. She has available a 25% and a 12% solution. How many liters of the 25% and how many liters of the 12% solutions should she mix to make the 15% solution?

22. A scientist needs 120 milliliters of a 20% acid solution for an experiment. The lab has available a 25% and a 10% solution. How many liters of the 25% and how many liters of the 10% solutions should the scientist mix to make the 20% solution?

Answer

80 liters of the 25% solution and 40 liters of the 10% solution

23. A 40% antifreeze solution is to be mixed with a 70% antifreeze solution to get 240 liters of a 50% solution. How many liters of the 40% and how many liters of the 70% solutions will be used?

24. A 90% antifreeze solution is to be mixed with a 75% antifreeze solution to get 360 liters of an 85% solution. How many liters of the 90% and how many liters of the 75% solutions will be used?

Answer

240 liters of the 90% solution and 120 liters of the 75% solution

Solve Interest Applications

In the following exercises, translate to a system of equations and solve.

25. Hattie had $3000 to invest and wants to earn 10.6% interest per year. She will put some of the money into an account that earns 12% per year and the rest into an account that earns 10% per year. How much money should she put into each account?

26. Carol invested $2560 into two accounts. One account paid 8% interest and the other paid 6% interest. She earned 7.25% interest on the total investment. How much money did she put in each account?

Answer

$1600 at 8%, 960 at 6%

27. Sam invested $48,000, some at 6% interest and the rest at 10%. How much did he invest at each rate if he received $4000 in interest in one year?

28. Arnold invested $64,000, some at 5.5% interest and the rest at 9%. How much did he invest at each rate if he received $4500 in interest in one year?

Answer

$28,000 at 9%, $36,000 at 5.5%

29. After four years in college, Josie owes $65, 800 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owes for one year was $2878.50 What is the amount of each loan?

30. Mark wants to invest $10,000 to pay for his daughter's wedding next year. He will invest some of the money in a short term CD that pays 12% interest and the rest in a money market savings account that pays 5% interest. How much should he

invest at each rate if he wants to earn $1095 in interest in one year?

Answer

$8500 CD, $1500 savings account

31. A trust fund worth $25,000 is invested in two different portfolios. This year, one portfolio is expected to earn 5.25% interest and the other is expected to earn 4%. Plans are for the total interest on the fund to be $1150 in one year. How much money should be invested at each rate?

32. A business has two loans totaling $85,000. One loan has a rate of 6% and the other has a rate of 4.5% This year, the business expects to pay $4,650 in interest on the two loans. How much is each loan?

Answer

$55,000 on loan at 6% and $30,000 on loan at 4.5%

Solve Applications of Cost and Revenue Functions

33. The manufacturer of an energy drink spends $1.20 to make each drink and sells them for $2. The manufacturer also has fixed costs each month of $8,000.

ⓐ Find the cost function C when x energy drinks are manufactured.

ⓑ Find the revenue function R when x drinks are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

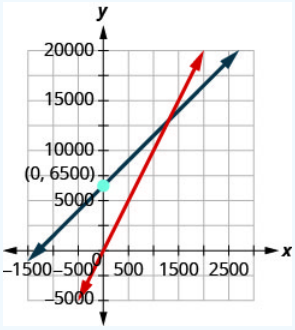
34. The manufacturer of a water bottle spends $5 to build each bottle and sells them for $10. The manufacturer also has fixed costs each month of $6500. ⓐ Find the cost function C when x bottles are manufactured. ⓑ Find the revenue function R when x bottles are sold. ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid. ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ $C(x) = 5x + 6500$

ⓑ $R(x) = 10x$

ⓒ



ⓓ 1,500; when 1,500 water bottles are sold, the cost and the revenue equal $15,000

Writing Exercises

35. Take a handful of two types of coins, and write a problem similar to Example relating the total number of coins and their total value. Set up a system of equations to describe your situation and then solve it.

36. In Example, we used elimination to solve the system of equations
$\begin{cases} s+b=40,000 \\ 0.08s+0.03b=0.071(40,000). \end{cases}$
Could you have used substitution or elimination to solve this system? Why?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve mixture applications.			
solve interest applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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11.4: Graphs of Linear Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Verify solutions to an inequality in two variables
- Recognize the relation between the solutions of an inequality and its graph
- Graph linear inequalities

Note

Before you get started, take this readiness quiz.

1. Solve: $4x+3>23$.
   If you missed this problem, review Exercise 2.7.22.
2. Translate from algebra to English: $x<5$.
   If you missed this problem, review Exercise 1.3.1.
3. Evaluate $3x-2y$ when $x=1,\ y=-2$.
   If you missed this problem, review Exercise 1.5.28.

Verify Solutions to an Inequality in Two Variables

We have learned how to solve inequalities in one variable. Now, we will look at inequalities in two variables. Inequalities in two variables have many applications. If you ran a business, for example, you would want your revenue to be greater than your costs—so that your business would make a profit.

LINEAR INEQUALITY

A **linear inequality** is an inequality that can be written in one of the following forms:

$$Ax+By>C \quad Ax+By\geq C \quad Ax+By<C \quad Ax+By\leq C$$

where A and B are not both zero.

Do you remember that an inequality with one variable had many solutions? The solution to the inequality $x>3$ is any number greater than 3. We showed this on the number line by shading in the number line to the right of 3, and putting an open parenthesis at 3. See Figure 11.4.1.

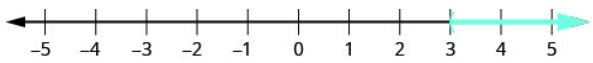


Figure 11.4.1

Similarly, inequalities in two variables have many solutions. Any ordered pair (x,y) that makes the inequality true when we substitute in the values is a solution of the inequality.

Solution OF A LINEAR INEQUALITY

An ordered pair (x,y) is a **solution of a linear inequality** if the inequality is true when we substitute the values of x and y.

Example 11.4.1

Determine whether each ordered pair is a solution to the inequality $y>x+4$:

1. $(0,0)$
2. $(1,6)$
3. $(2,6)$
4. $(-5,-15)$

5. $(-8, 12)$

Solution

1.

$(0, 0)$	$y > x + 4$
Substitute 0 for x and 0 for y.	$0 \stackrel{?}{>} 0 + 4$
Simplify.	$0 \not> 4$ So, $(0, 0)$ is not a solution to $y > x + 4$.

2.

$(1, 6)$	$y > x + 4$
Substitute 1 for x and 6 for y.	$6 \stackrel{?}{>} 1 + 4$
Simplify.	$6 > 5$ So, $(1, 6)$ is a solution to $y > x + 4$.

3.

$(2, 6)$	$y > x + 4$
Substitute 2 for x and 6 for y.	$6 \stackrel{?}{>} 2 + 4$
Simplify.	$6 \not> 6$ So, $(2, 6)$ is not a solution to $y > x + 4$.

4.

$(-5, -15)$	$y > x + 4$
Substitute –5 for x and –15 for y.	$-15 \stackrel{?}{>} -5 + 4$
Simplify.	$-15 \not> -1$ So, $(-5, -15)$ is not a solution to $y > x + 4$.

5.

$(-8,12)$	$y > x + 4$
Substitute –8 for x and 12 for y.	$12 \stackrel{?}{>} -8 + 4$
Simplify.	$12 > -4$ So, $(-8, 12)$ is a solution to $y > x + 4$.

? Try It 11.4.2

Determine whether each ordered pair is a solution to the inequality $y > x - 3$:

1. $(0, 0)$
2. $(4, 9)$
3. $(-2, 1)$
4. $(-5, -3)$
5. $(5, 1)$

Answer

1. yes

2. yes
3. yes
4. yes
5. no

Try It 11.4.3

Determine whether each ordered pair is a solution to the inequality $y < x + 1$:

1. $(0, 0)$
2. $(8, 6)$
3. $(-2, -1)$
4. $(3, 4)$
5. $(-1, -4)$

Answer

1. yes
2. yes
3. no
4. no
5. yes

Recognize the Relation Between the Solutions of an Inequality and its Graph

Now, we will look at how the solutions of an inequality relate to its graph.

Let's think about the number line in Figure 11.4.1 again. The point $x = 3$ separated that number line into two parts. On one side of 3 are all the numbers less than 3. On the other side of 3 all the numbers are greater than 3. See Figure 11.4.2

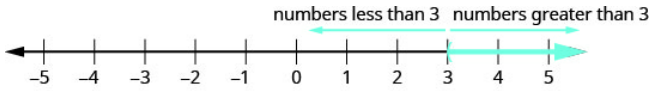


Figure 11.4.2

The solution to $x > 3$ is the shaded part of the number line to the right of $x = 3$.

Similarly, the line $y = x + 4$ separates the plane into two regions. On one side of the line are points with $y < x + 4$. On the other side of the line are the points with $y > x + 4$. We call the line $y = x + 4$ a boundary line.

BOUNDARY LINE

The line with equation $Ax + By = C$ is the **boundary line** that separates the region where $Ax + By > C$ from the region where $Ax + By < C$.

For an inequality in one variable, the endpoint is shown with a parenthesis or a bracket depending on whether or not aa is included in the solution:

Similarly, for an inequality in two variables, the boundary line is shown with a solid or dashed line to indicate whether or not it the line is included in the solution. This is summarized in Table 11.4.1.

Table 11.4.1

$Ax + By < C$	$Ax + By \leq C$
$Ax + By > C$	$Ax + By \geq C$
Boundary line is not included in solution.	Boundary line is included in solution.

Boundary line is dashed.	Boundary line is solid.

Now, let's take a look at what we found in Example 11.4.1. We'll start by graphing the line $y = x + 4$, and then we'll plot the five points we tested. See Figure 11.4.3.

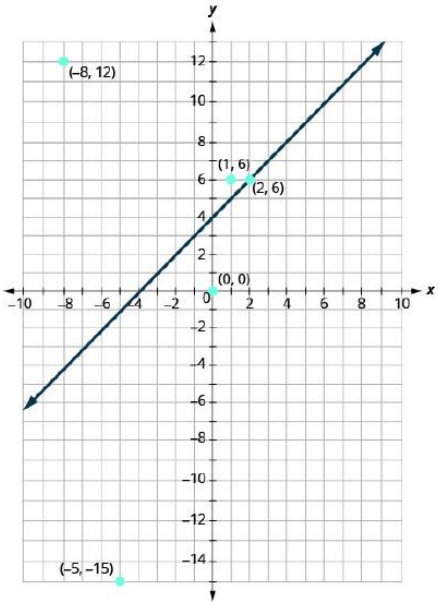


Figure 11.4.3

In Example 11.4.1 we found that some of the points were solutions to the inequality $y > x + 4$ and some were not.

Which of the points we plotted are solutions to the inequality $y > x + 4$? The points $(1, 6)$ and $(-8, 12)$ are solutions to the inequality $y > x + 4$. Notice that they are both on the same side of the boundary line $y = x + 4$.

The two points $(0, 0)$ and $(-5, -15)$ are on the other side of the **boundary line** $y = x + 4$, and they are not solutions to the inequality $y > x + 4$. For those two points, $y < x + 4$.

What about the point $(2, 6)$? Because $6 = 2 + 4$, the point is a solution to the equation $y = x + 4$. So the point $(2, 6)$ is on the boundary line.

Let's take another point on the left side of the boundary line and test whether or not it is a solution to the inequality $y > x + 4$. The point $(0, 10)$ clearly looks to be to the left of the boundary line, doesn't it? Is it a solution to the inequality?

$$\begin{aligned} &y > x + 4 \\ &10 \overset{?}{>} 0 + 4 \\ &10 > 4 \qquad \text{So, } (0, 10) \text{ is a solution to } y > x + 4. \end{aligned} \tag{11.4.1}$$

Any point you choose on the left side of the boundary line is a solution to the inequality $y > x + 4$. All points on the left are solutions.

Similarly, all points on the right side of the boundary line, the side with $(0, 0)$ and $(-5, -15)$, are not solutions to $y > x + 4$. See Figure 11.4.4

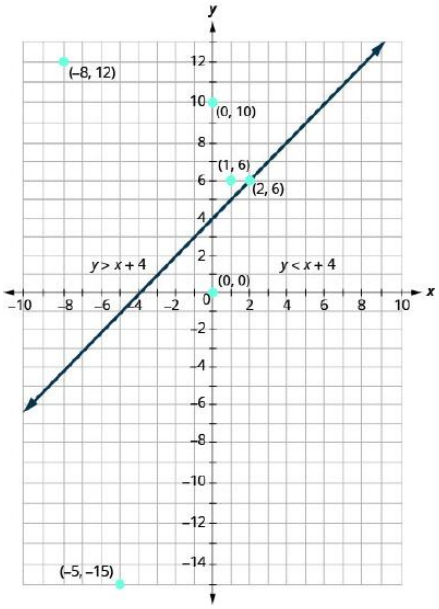


Figure 11.4.4

The graph of the inequality $y > x + 4$ is shown in Figure $11.4.5$ below. The line $y = x + 4$ divides the plane into two regions. The shaded side shows the solutions to the inequality $y > x + 4$.

The points on the boundary line, those where $y = x + 4$, are not solutions to the inequality $y > x + 4$, so the line itself is not part of the solution. We show that by making the line dashed, not solid.

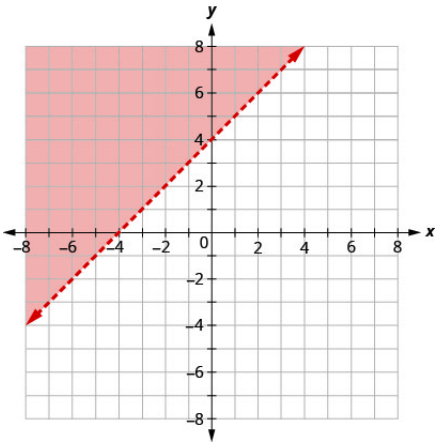
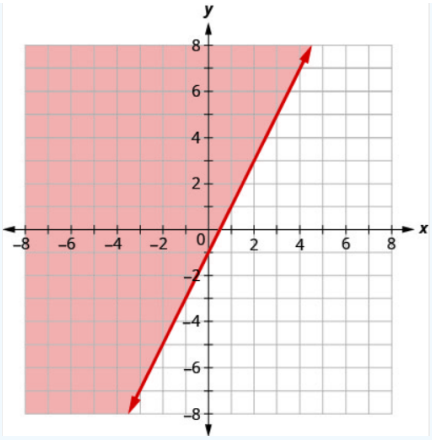


Figure 11.4.5: The graph of the inequality y>x+4.

Example 11.4.4

The boundary line shown is $y = 2x - 1$. Write the inequality shown by the graph.

https://math.libretexts.org/@go/page/46289



Solution

The line $y = 2x - 1$ is the boundary line. On one side of the line are the points with $y > 2x - 1$ and on the other side of the line are the points with $y < 2x - 1$.

Let's test the point $(0, 0)$ and see which inequality describes its side of the boundary line.

At $(0, 0)$, which inequality is true:

$$\begin{array}{lll} y > 2x - 1 & \text{or} & y < 2x - 1? \\ y > 2x - 1 & & y < 2x - 1 \\ 0 > 2 \cdot 0 - 1 & & 0 < 2 \cdot 0 - 1 \\ 0 > -1 \text{ True} & & 0 < -1 \text{ False} \end{array} \tag{11.4.2}$$

Since $y > 2x - 1$ is true, the side of the line with $(0, 0)$, is the solution. The shaded region shows the solution of the inequality $y > 2x - 1$.

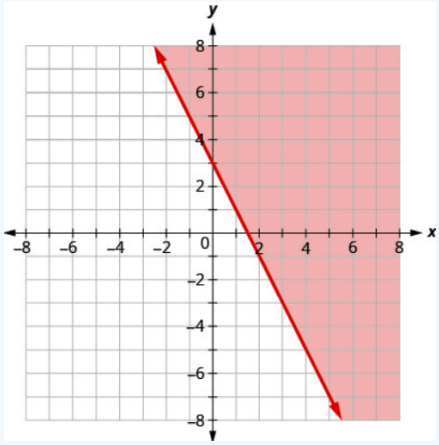
Since the boundary line is graphed with a solid line, the inequality includes the equal sign.

The graph shows the inequality $y \geq 2x - 1$.

We could use any point as a test point, provided it is not on the line. Why did we choose $(0, 0)$? Because it's the easiest to evaluate. You may want to pick a point on the other side of the boundary line and check that $y < 2x - 1$.

? Try It 11.4.5

Write the inequality shown by the graph with the boundary line $y = -2x + 3$.

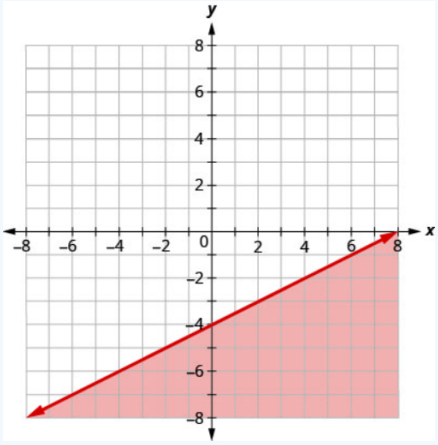


Answer

$y \geq -2x + 3$

Try It 11.4.6

Write the inequality shown by the graph with the boundary line $y = \frac{1}{2}x - 4$.

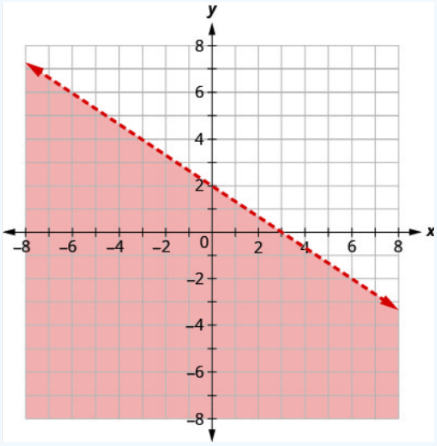


Answer

$y \leq \frac{1}{2}x - 4$

Example 11.4.7

The boundary line shown is $2x + 3y = 6$. Write the inequality shown by the graph.



Solution

The line $2x + 3y = 6$ is the boundary line. On one side of the line are the points with $2x + 3y > 6$ and on the other side of the line are the points with $2x + 3y < 6$.

Let's test the point $(0, 0)$ and see which inequality describes its side of the boundary line.

At $(0, 0)$, which inequality is true:

$$\begin{aligned} 2x + 3y &> 6 \quad \text{or} \quad 2x + 3y < 6? \\ 2x + 3y &> 6 \qquad\quad 2x + 3y < 6 \\ 2(0) + 3(0) &> 6 \qquad\quad 2(0) + 3(0) < 6 \\ 0 &> 6 \quad \text{False} \qquad\quad 0 < 6 \quad \text{True} \end{aligned} \tag{11.4.3}$$

So the side with $(0, 0)$ is the side where $2x + 3y < 6$.

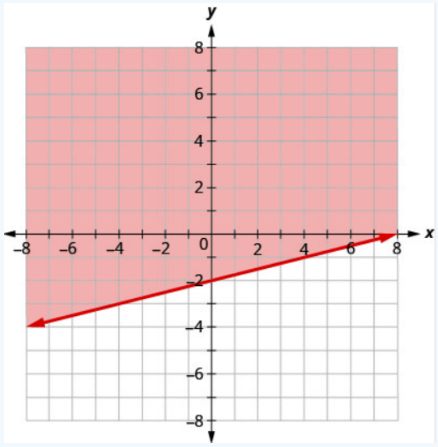
(You may want to pick a point on the other side of the boundary line and check that $2x + 3y > 6$.)

Since the boundary line is graphed as a dashed line, the inequality does not include an equal sign.

The graph shows the solution to the inequality $2x + 3y < 6$.

Try It 11.4.8

Write the inequality shown by the shaded region in the graph with the boundary line $x - 4y = 8$.

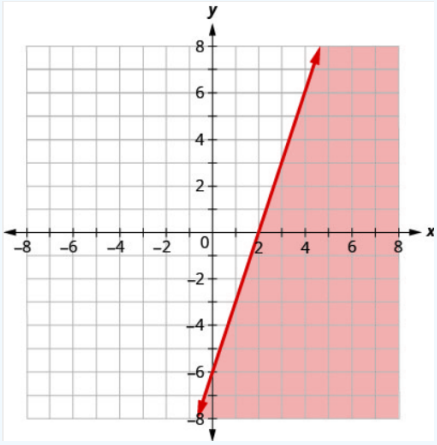


Answer

$x - 4y \leq 8$

Try It 11.4.9

Write the inequality shown by the shaded region in the graph with the boundary line $3x - y = 6$.



Answer

$3x - y \leq 6$

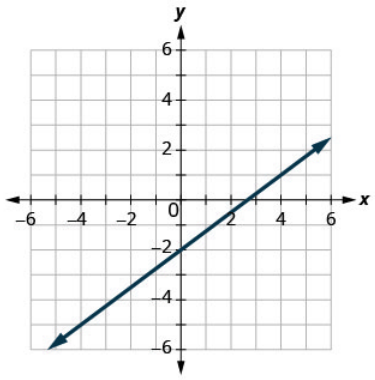
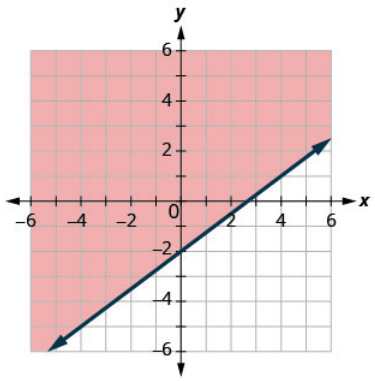
Graph Linear Inequalities

Now, we're ready to put all this together to graph linear inequalities.

Example 11.4.10: How to Graph Linear Inequalities

Graph the **linear inequality** $y \geq \frac{3}{4}x - 2$.

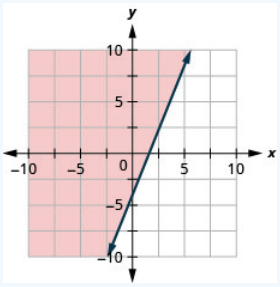
Solution

Step 1. Identify and graph the boundary line. • If the inequality is ≤ or ≥, the boundary line is solid. • If the inequality is < or >, the boundary line is dashed.	Replace the inequality sign with an equal sign to find the boundary line. Graph the boundary line $y = \frac{3}{4}x - 2$. The inequality sign is ≥, so we draw a solid line.	
Step 2. Test a point that is not on the boundary line. Is it a solution of the inequality?	We'll test (0, 0). Is it a solution of the inequality?	At (0, 0), is $y \geq \frac{3}{4}x - 2$? $0 \stackrel{?}{\geq} \frac{3}{4}(0) - 2$ $0 \geq -2$ So, (0, 0) is a solution.
Step 3. Shade in one side of the boundary line. • If the test point is a solution, shade in the side that includes the point. • If the test point is not a solution, shade in the opposite side.	The test point (0, 0), is a solution to $y \geq \frac{3}{4}x - 2$. So we shade in that side.	

Try It 11.4.11

Graph the linear inequality $y \geq \frac{5}{2}x - 4$.

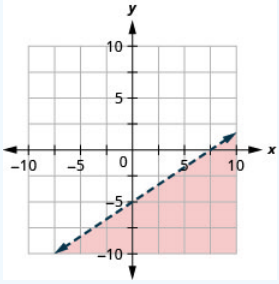
Answer



Try It 11.4.12

Graph the linear inequality $y < \frac{2}{3}x - 5$.

Answer



The steps we take to graph a linear inequality are summarized here.

GRAPH A LINEAR INEQUALITY.

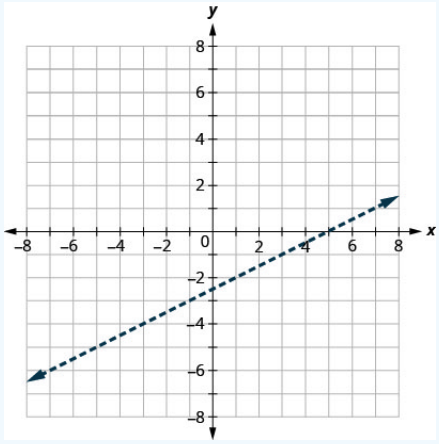
1. Identify and graph the boundary line.
   - If the inequality is $\leq$ or $\geq$, the boundary line is solid.
   - If the inequality is $<$ or $>$, the boundary line is dashed.
2. Test a point that is not on the boundary line. Is it a solution of the inequality?
3. Shade in one side of the boundary line.
   - If the test point is a solution, shade in the side that includes the point.
   - If the test point is not a solution, shade in the opposite side.

Example 11.4.13

Graph the linear inequality $x - 2y < 5$.

Solution

First we graph the boundary line $x - 2y = 5$. The inequality is $<$ so we draw a dashed line.



Then we test a point. We'll use $(0, 0)$ again because it is easy to evaluate and it is not on the boundary line.

Is $(0, 0)$ a solution of $x - 2y < 5$?

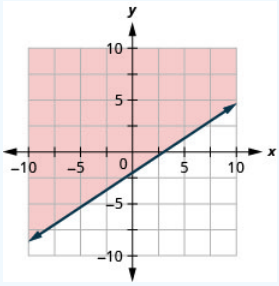
$$0 - 2(0) \stackrel{?}{<} 5$$
$$0 - 0 \stackrel{?}{<} 5$$
$$0 < 5$$

The point $(0, 0)$ is a solution of $x - 2y < 5$, so we shade in that side of the boundary line.

Try It 11.4.14

Graph the linear inequality $2x - 3y \leq 6$.

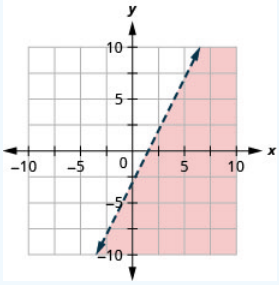
Answer



Try It 11.4.15

Graph the linear inequality $2x - y > 3$.

Answer



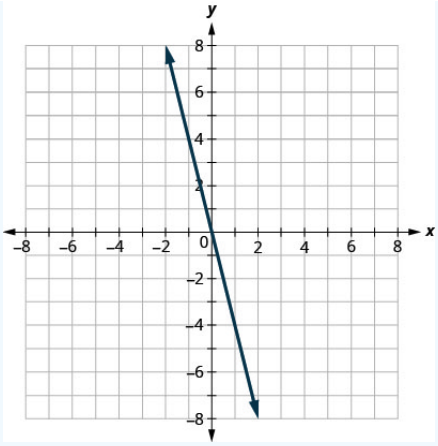
What if the boundary line goes through the origin? Then we won't be able to use $(0, 0)$ as a test point. No problem—we'll just choose some other point that is not on the boundary line.

Example 11.4.16

Graph the linear inequality $y \leq -4x$.

Solution

First we graph the boundary line $y = -4x$. It is in slope–intercept form, with $m = -4$ and $b = 0$. The inequality is $\leq$ so we draw a solid line.



Now, we need a test point. We can see that the point $(1, 0)$ is not on the boundary line.

Is $(1, 0)$ a solution of $y \leq -4x$?

$0 \stackrel{?}{\leq} -4(1)$

$0 \not\leq -4$

The point $(1, 0)$ is not a solution to $y \leq -4x$, so we shade in the opposite side of the boundary line. See Figure 11.4.6

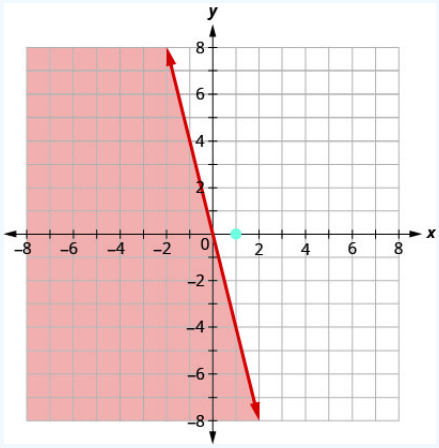
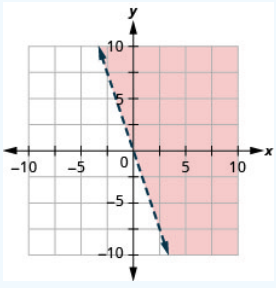


Figure 11.4.6

Try It 11.4.17

Graph the linear inequality $y > -3x$.

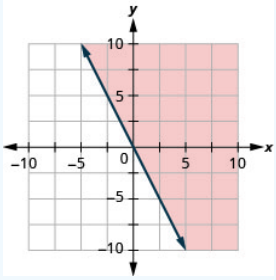
Answer



Try It 11.4.18

Graph the linear inequality $y \geq -2x$.

Answer



Some linear inequalities have only one variable. They may have an x but no y, or a y but no x. In these cases, the boundary line will be either a vertical or a horizontal line. Do you remember?

$x = a$ vertical line
$y = b$ horizontal line

Example 11.4.19

Graph the linear inequality $y > 3$.

Solution

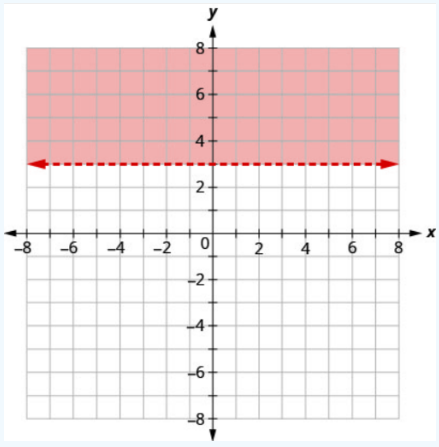
First we graph the boundary line $y = 3$. It is a horizontal line. The inequality is $>$ so we draw a dashed line.

We test the point $(0, 0)$.

$$\begin{aligned} y &> 3 \\ 0 &\not> 3 \end{aligned} \tag{11.4.4}$$

$(0, 0)$ is not a solution to $y > 3$.

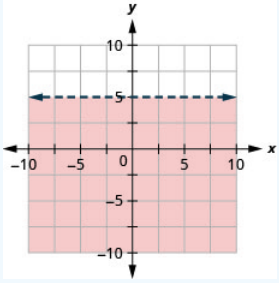
So we shade the side that does not include $(0, 0)$.



Try It 11.4.20

Graph the linear inequality $y < 5$.

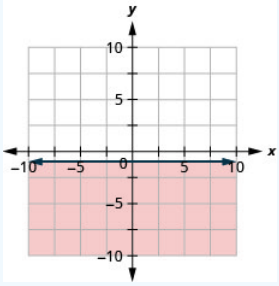
Answer



Try It 11.4.21

Graph the linear inequality $y \leq -1$.

Answer



Key Concepts

- **To Graph a Linear Inequality**
  1. Identify and graph the boundary line.
     If the inequality is $\leq$ or $\geq$, the boundary line is solid.
     If the inequality is $<$ or $>$, the boundary line is dashed.
  2. Test a point that is not on the boundary line. Is it a solution of the inequality?
  3. Shade in one side of the boundary line.
     If the test point is a solution, shade in the side that includes the point.
     If the test point is not a solution, shade in the opposite side.

Glossary

boundary line

The line with equation $Ax + By = C$ that separates the region where $Ax + By > C$ from the region where $Ax + By < C$.

linear inequality

An inequality that can be written in one of the following forms:

$$Ax + By > C \quad Ax + By \geq C \quad Ax + By < C \quad Ax + By \leq C \tag{11.4.5}$$

where A and B are not both zero.

solution of a linear inequality

An ordered pair $(x,\ y)$ is a solution to a linear inequality the inequality is true when we substitute the values of x and y.

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11.4E: Exercises

Practice Makes Perfect

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.

Exercise 11.4E.1

Determine whether each ordered pair is a solution to the inequality $y > x - 1$:

1. $(0, 1)$
2. $(-4, -1)$
3. $(4, 2)$
4. $(3, 0)$
5. $(-2, -3)$

Exercise 11.4E.2

Determine whether each ordered pair is a solution to the inequality $y > x - 3$:

1. $(0, 0)$
2. $(2, 1)$
3. $(-1, -5)$
4. $(-6, -3)$
5. $(1, 0)$

Answer

1. yes
2. no
3. no
4. yes
5. no

Exercise 11.4E.3

Determine whether each ordered pair is a solution to the inequality $y < x + 2$:

1. $(0, 3)$
2. $(-3, -2)$
3. $(-2, 0)$
4. $(0, 0)$
5. $(-1, 4)$

Exercise 11.4E.4

Determine whether each ordered pair is a solution to the inequality $y < x + 5$:

1. $(-3, 0)$
2. $(1, 6)$
3. $(-6, -2)$
4. $(0, 1)$
5. $(5, -4)$

Answer

1. yes
2. no
3. no
4. yes
5. yes

Exercise 11.4E.5

Determine whether each ordered pair is a solution to the inequality $x+y>4$:

1. $(5,1)$
2. $(-2,6)$
3. $(3,2)$
4. $(10,-5)$
5. $(0,0)$

Exercise 11.4E.6

Determine whether each ordered pair is a solution to the inequality $x+y>2$:

1. $(1,1)$
2. $(4,-3)$
3. $(0,0)$
4. $(-8,12)$
5. $(3,0)$

Answer

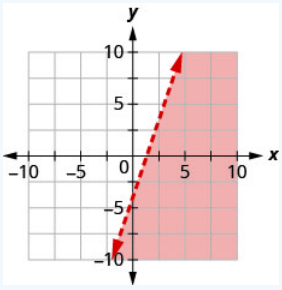
1. no
2. no
3. no
4. yes
5. yes

Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

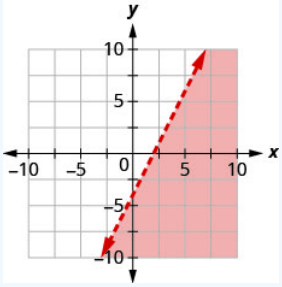
Exercise 11.4E.7

Write the inequality shown by the graph with the boundary line $y=3x-4$.



Exercise 11.4E. 8

Write the inequality shown by the graph with the boundary line $y = 2x - 4$.

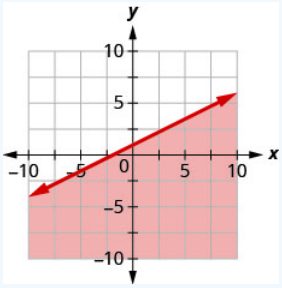


Answer

$y < 2x - 4$

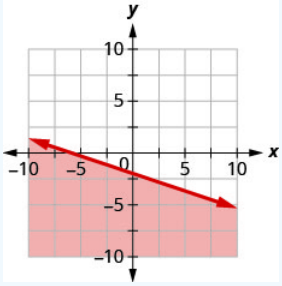
Exercise 11.4E. 9

Write the inequality shown by the graph with the boundary line $y = \frac{1}{2}x + 1$



Exercise 11.4E. 10

Write the inequality shown by the graph with the boundary line $y = -\frac{1}{3}x - 2$



Answer

$y \leq -\frac{1}{3}x - 2$

Exercise 11.4E. 11

Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 5$.

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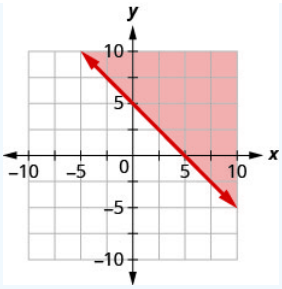
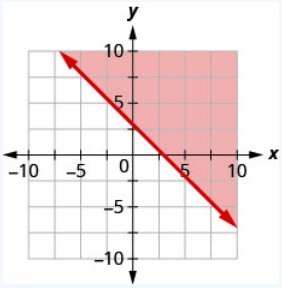


Exercise 11.4E.12

Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 3$.

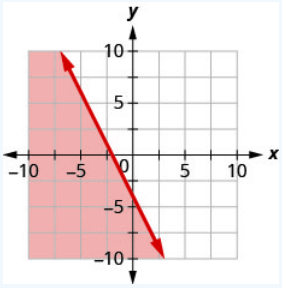


Answer

$x + y \geq 3$

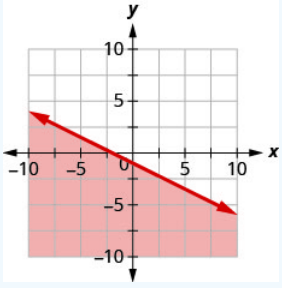
Exercise 11.4E.13

Write the inequality shown by the shaded region in the graph with the boundary line $2x + y = -4$.



Exercise 11.4E.14

Write the inequality shown by the shaded region in the graph with the boundary line $x + 2y = -2$.



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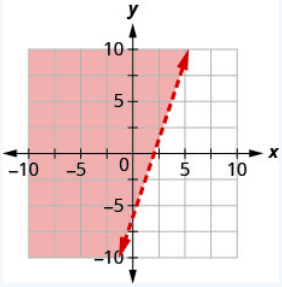
Answer

$x + 2y \geq -2$

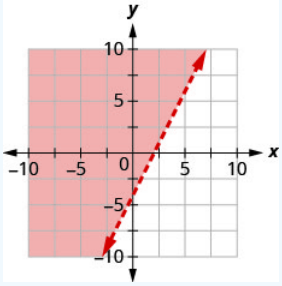
Exercise 11.4E. 15

Write the inequality shown by the shaded region in the graph with the boundary line $3x - y = 6$.



Exercise 11.4E. 16

Write the inequality shown by the shaded region in the graph with the boundary line $2x - y = 4$.

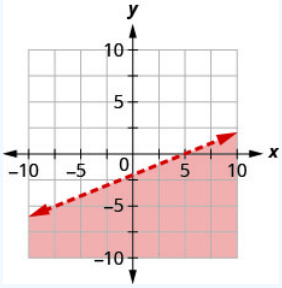


Answer

$2x - y < 4$

Exercise 11.4E. 17

Write the inequality shown by the shaded region in the graph with the boundary line $2x - 5y = 10$.



Exercise 11.4E. 18

Write the inequality shown by the shaded region in the graph with the boundary line $4x - 3y = 12$.

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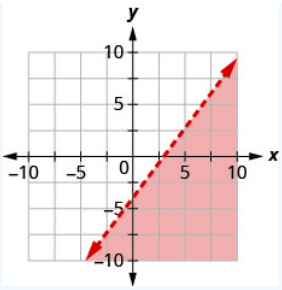


Answer

$4x - 3y > 12$

Graph Linear Inequalities

In the following exercises, graph each linear inequality.

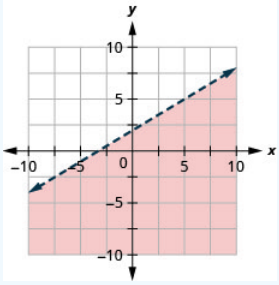
Exercise 11.4E. 19

Graph the linear inequality $y > \frac{2}{3}x - 1$

Exercise 11.4E. 20

Graph the linear inequality $y < \frac{3}{5}x + 2$

Answer



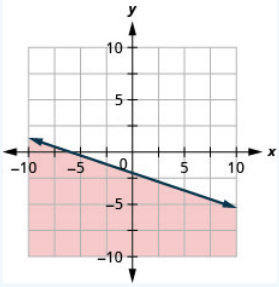
Exercise 11.4E. 21

Graph the linear inequality $y \leq -\frac{1}{2}x + 4$

Exercise 11.4E. 22

Graph the linear inequality $y \geq -\frac{1}{3}x - 2$

Answer



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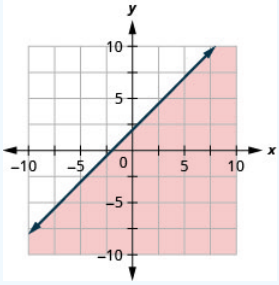
Exercise 11.4E. 23

Graph the linear inequality $x - y \leq 3$

Exercise 11.4E. 24

Graph the linear inequality $x - y \geq -2$

Answer



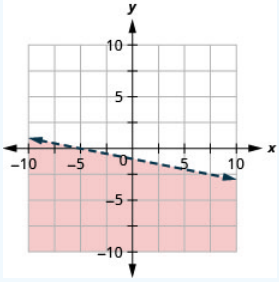
Exercise 11.4E. 25

Graph the linear inequality $4x + y > -4$

Exercise 11.4E. 26

Graph the linear inequality $x + 5y < -5$

Answer



Exercise 11.4E. 27

Graph the linear inequality $3x + 2y \geq -6$

Exercise 11.4E. 28

Graph the linear inequality $4x + 2y \geq -8$

Answer

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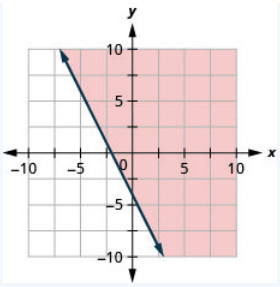


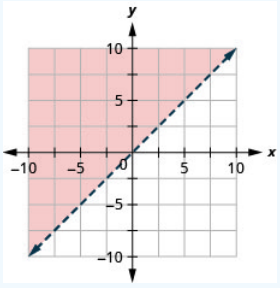
Exercise 11.4E.29

Graph the linear inequality $y > 4x$

Exercise 11.4E.30

Graph the linear inequality $y > x$

Answer



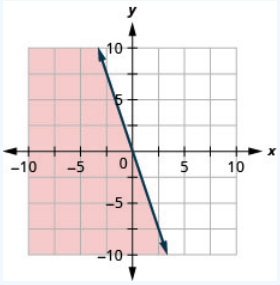
Exercise 11.4E.31

Graph the linear inequality $y \leq -x$

Exercise 11.4E.32

Graph the linear inequality $y \leq -3x$

Answer



Exercise 11.4E.33

Graph the linear inequality $y \geq -2$

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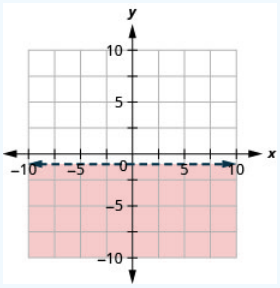
Exercise $11.4E.34$

Graph the linear inequality $y < -1$

Answer



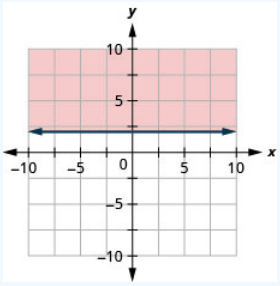
Exercise $11.4E.35$

Graph the linear inequality $y < 4$

Exercise $11.4E.36$

Graph the linear inequality $y \geq 2$

Answer



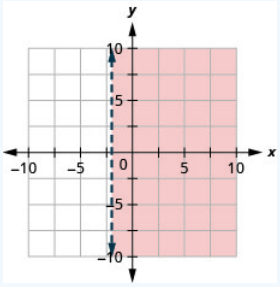
Exercise $11.4E.37$

Graph the linear inequality $x \leq 5$

Exercise $11.4E.38$

Graph the linear inequality $x > -2$

Answer



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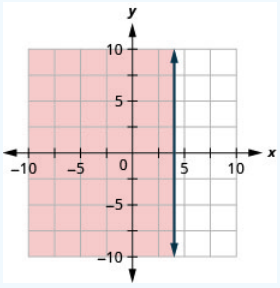
Exercise 11.4E. 39

Graph the linear inequality $x > -3$

Exercise 11.4E. 40

Graph the linear inequality $x \leq 4$

Answer



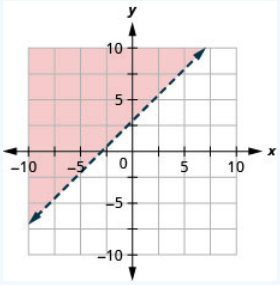
Exercise 11.4E. 41

Graph the linear inequality $x - y < 4$

Exercise 11.4E. 42

Graph the linear inequality $x - y < -3$

Answer



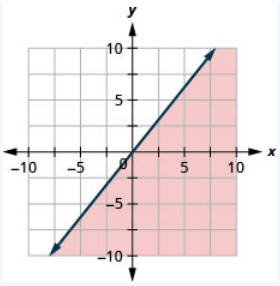
Exercise 11.4E. 43

Graph the linear inequality $y \geq \frac{3}{2}x$

Exercise 11.4E. 44

Graph the linear inequality $y \leq \frac{5}{4}x$

Answer



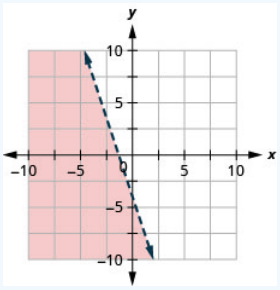
Exercise 11.4E. 45

Graph the linear inequality $y > -2x + 1$

Exercise 11.4E. 46

Graph the linear inequality $y < -3x - 4$

Answer



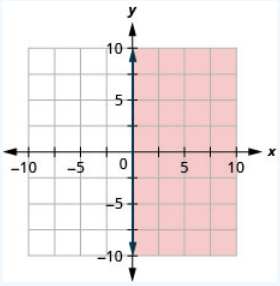
Exercise 11.4E. 47

Graph the linear inequality $x \leq -1$

Exercise 11.4E. 48

Graph the linear inequality $x \geq 0$

Answer



Everyday Math

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Exercise 11.4E.49

Money. Gerry wants to have a maximum of \$100 cash at the ticket booth when his church carnival opens. He will have \$1 bills and \$5 bills. If x is the number of \$1 bills and y is the number of \$5 bills, the inequality $x+5y \leq 100$ models the situation.

1. Graph the inequality.
2. List three solutions to the inequality $x+5y \leq 100$ where both x and y are integers.

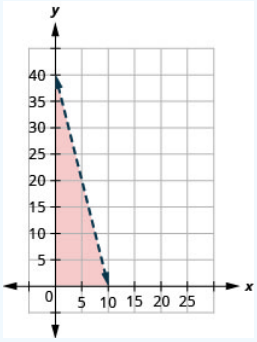
Exercise 11.4E.50

Shopping. Tula has \$20 to spend at the used book sale. Hardcover books cost \$2 each and paperback books cost \$0.50 each. If x is the number of hardcover books Tula can buy and y is the number of paperback books she can buy, the inequality $2x+\frac{1}{2}y \leq 20$ models the situation.

1. Graph the inequality.
2. List three solutions to the inequality $2x+\frac{1}{2}y \leq 20$ where both x and y are whole numbers.

Answer

1.



2. Answers will vary.

Writing Exercises

Exercise 11.4E.51

Lester thinks that the solution of any inequality with a $>$ sign is the region above the line and the solution of any inequality with a $<$ sign is the region below the line. Is Lester correct? Explain why or why not.

Exercise 11.4E.52

Explain why in some graphs of linear inequalities the boundary line is solid but in other graphs it is dashed.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
verify solutions to an inequality in two variables.			
recognize the relation between the solutions of an inequality and its graph.			
graph linear inequalities.			

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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CHAPTER OVERVIEW

12: Polynomials

Expressions known as polynomials are used widely in algebra. Applications of these expressions are essential to many careers, including economists, engineers, and scientists. In this chapter, we will find out what polynomials are and how to manipulate them through basic mathematical operations.

12.1: Add and Subtract Polynomials
12.2: Use Multiplication Properties of Exponents (Part 1)
12.3: Use Multiplication Properties of Exponents (Part 2)
12.4: Multiply Polynomials (Part 1)
12.5: Multiply Polynomials (Part 2)
12.6: Divide Monomials (Part 1)
12.7: Divide Monomials (Part 2)
12.8: Integer Exponents and Scientific Notation (Part 1)
12.9: Integer Exponents and Scientific Notation (Part 2)
12.10: Introduction to Factoring Polynomials
12.E: Polynomials (Exercises)
12.S: Polynomials (Summary)

Figure 10.1 - The paths of rockets are calculated using polynomials. (credit: NASA, Public Domain)

Contributors

- Lynn Marecek (Santa Ana College) and MaryAnne Anthony-Smith (Formerly of Santa Ana College). This content is licensed under Creative Commons Attribution License v4.0 "Download for free at http://cnx.org/contents/fd53eae1-fa2...49835c3c@5.191."

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12.1: Add and Subtract Polynomials

Learning Objectives

- Identify polynomials, monomials, binomials, and trinomials
- Determine the degree of polynomials
- Add and subtract monomials
- Add and subtract polynomials
- Evaluate a polynomial for a given value

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: 8x + 3x. If you missed this problem, review Example 2.3.10.
2. Subtract: (5n + 8) − (2n − 1). If you missed this problem, review Example 7.4.13.
3. Evaluate: $4y^2$ when y = 5 If you missed this problem, review Example 2.3.6.

Identify Polynomials, Monomials, Binomials, and Trinomials

In Evaluate, Simplify, and Translate Expressions, you learned that a term is a constant or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial. A monomial, or a sum and/or difference of monomials, is called a polynomial.

Definition: Polynomials

polynomial — A monomial, or two or more monomials, combined by addition or subtraction

monomial — A polynomial with exactly one term

binomial — A polynomial with exactly two terms

trinomial — A polynomial with exactly three terms

Notice the roots:

- *poly-* means many
- *mono-* means one
- *bi-* means two
- *tri-* means three

Here are some examples of polynomials:

Polynomial	b + 1	$4y^2 - 7y + 2$	$5x^5 - 4x^4 + x^3 + 8x^2 - 9x + 1$
Monomial	5	$4b^2$	$-9x^3$
Binomial	3a - 7	$y^2 - 9$	$17x^3 + 14x^2$
Trinomial	$x^2 - 5x + 6$	$4y^2 - 7y + 2$	$5a^4 - 3a^3 + a$

Notice that every monomial, binomial, and trinomial is also a polynomial. They are special members of the family of polynomials and so they have special names. We use the words ‘monomial’, ‘binomial’, and ‘trinomial’ when referring to these special polynomials and just call all the rest ‘polynomials’.

Example 12.1.1:

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial: (a) $8x^2 - 7x - 9$ (b) $-5a^4$ (c) $x^4 - 7x^3 - 6x^2 + 5x + 2$ (d) $11 - 4y^3$ (e) n

Solution

Polynomial	Number of terms	Type
(a) $8x^2 - 7x - 9$	3	Trinomial
(b) $-5a^4$	1	Monomial
(c) $x^4 - 7x^3 - 6x^2 + 5x + 2$	5	Polynomial
(d) $11 - 4y^3$	2	Binomial
(e) n	1	Monomial

Exercise 12.1.1:

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. (a) z (b) $2x^3 - 4x^2 - x - 8$ (c) $6x^2 - 4x + 1$ (d) $9 - 4y^2$ (e) $3x^7$

Answer a

monomial

Answer b

polynomial

Answer c

trinomial

Answer d

binomial

Answer e

monomial

Exercise 12.1.2:

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. (a) $y^3 - 8$ (b) $9x^3 - 5x^2 - x$ (c) $x^4 - 3x^2 - 4x - 7$ (d) $-y^4$ (e) w

Answer a

binomial

Answer b

trinomial

Answer c

polynomial

Answer d

monomial

Answer e

monomial

Determine the Degree of Polynomials

In this section, we will work with polynomials that have only one variable in each term. The degree of a polynomial and the degree of its terms are determined by the exponents of the variable.

A monomial that has no variable, just a constant, is a special case. The degree of a constant is 0 —it has no variable.

Definition: Degree of a Polynomial

The degree of a term is the exponent of its variable.

The degree of a constant is 0.

The degree of a polynomial is the highest degree of all its terms.

Let's see how this works by looking at several polynomials. We'll take it step by step, starting with monomials, and then progressing to polynomials with more terms.

Remember: Any base written without an exponent has an implied exponent of 1.

Monomials	5	$4b^2$	$-9x^3$	-18
Degree	0	2	3	0
Binomial	$b+1$	$3a-7$	y^2-9	$17x^3+14x^2$
Degree of each term	1 0	1 0	2 0	3 2
Degree of polynomial	1	1	2	3
Trinomial	x^2-5x+6	$4y^2-7y+2$	$5a^4-3a^3+a$	x^4+2x^2-5
Degree of each term	2 1 0	2 1 0	4 3 1	4 2 0
Degree of polynomial	2	2	4	4
Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	1 0	2 1 0	4 3 2 1 0	
Degree of polynomial	1	2	4	

Example 12.1.2:

Find the degree of the following polynomials: (a) 4x (b) $3x^3 - 5x + 7$ (c) −11 (d) $-6x^2 + 9x - 3$ (e) $8x + 2$

Solution

(a) 4x

The exponent of x is one. $x = x^1$	The degree is 1.

(b) $3x^3 - 5x + 7$

The highest degree of all the terms is 3.	The degree is 3.

(c) −11

The degree of a constant is 0.	The degree is 0.

(d) $-6x^2 + 9x - 3$

The highest degree of all the terms is 2.	The degree is 2.

(e) $8x + 2$

The highest degree of all the terms is 1.	The degree is 1.

? Exercise 12.1.3:

Find the degree of the following polynomials: (a) −6y (b) $4x - 1$ (c) $3x^4 + 4x^2 - 8$ (d) $2y^2 + 3y + 9$ (e) −18

Answer a

1

Answer b

1

Answer c

4

Answer d

2

Answer e

0

? Exercise 12.1.4:

Find the degree of the following polynomials: (a) 47 (b) $2x^2 - 8x + 2$ (c) $x^4 - 16$ (d) $y^5 - 5y^3 + y$ (e) $9a^3$

Answer a

0

Answer b

2

Answer c

4

Answer d

5

Answer e

3

Working with polynomials is easier when you list the terms in descending order of degrees. When a polynomial is written this way, it is said to be in **standard form**. Look back at the polynomials in Example 10.2. Notice that they are all written in standard form. Get in the habit of writing the term with the highest degree first.

Add and Subtract Monomials

In The Language of Algebra, you simplified expressions by combining like terms. Adding and subtracting monomials is the same as combining like terms. Like terms must have the same variable with the same exponent. Recall that when combining like terms only the coefficients are combined, never the exponents.

Example 12.1.3:

Add: $17x^2 + 6x^2$.

Solution

Combine like terms.	$23x^2$

Exercise 12.1.5:

Add: $12x^2 + 5x^2$.

Answer

$17x^2$

Exercise 12.1.6:

Add: $-11y^2 + 8y^2$.

Answer

$-3y^2$

Example 12.1.4:

Subtract: $11n - (-8n)$.

Solution

Combine like terms.	$19n$

Exercise 12.1.7:

Subtract: $9n - (-5n)$.

Answer

$14n$

Exercise 12.1.8:

Subtract: $-7a^3 - (-5a^3)$.

Answer

$-2a^3$

Example 12.1.5:

Simplify: $a^2 + 4b^2 - 7a^2$.

Solution

Combine like terms.	$-6a^2 + 4b^2$

Remember, $-6a^2$ and $4b^2$ are not like terms. The variables are not the same.

Exercise 12.1.9:

Add: $3x^2 + 3y^2 - 5x^2$.

Answer

$-2x^2 + 3y^2$

Exercise 12.1.10:

Add: $2a^2 + b^2 - 4a^2$.

Answer

$-2a^2 + b^2$

Add and Subtract Polynomials

Adding and subtracting polynomials can be thought of as just adding and subtracting like terms. Look for like terms—those with the same variables with the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together. It may also be helpful to underline, circle, or box like terms.

Example 12.1.6:

Find the sum: $(4x^2 - 5x + 1) + (3x^2 - 8x - 9)$

Solution

Identify like terms.	$\underline{4x^2} - \underline{5x} + \boxed{1} + \underline{3x^2} - \underline{8x} - \boxed{9}$
Rearrange to get the like terms together.	$\underline{4x^2 + 3x^2} - \underline{5x - 8x} + \boxed{1} - \boxed{9}$
Combine like terms.	$7x^2 - 13x - 8$

Exercise 12.1.11:

Find the sum: $(3x^2 - 2x + 8) + (x^2 - 6x + 2)$.

Answer

$4x^2 - 8x + 10$

Exercise 12.1.12:

Find the sum: $(7y^2 + 4y - 6) + (4y^2 + 5y + 1)$

Answer

$11y^2 + 9y - 5$

Parentheses are grouping symbols. When we add polynomials as we did in Example 10.6, we can rewrite the expression without parentheses and then combine like terms. But when we subtract polynomials, we must be very careful with the signs.

Example 12.1.7:

Find the difference: $(7u^2 - 5u + 3) - (4u^2 - 2)$.

Solution

Distribute and identify like terms.	$7u^2 - 5u + 3 - 4u^2 + 2$
Rearrange the terms.	$7u^2 - 4u^2 - 5u + 3 + 2$
Combine like terms.	$3u^2 - 5u + 5$

Exercise 12.1.13:

Find the difference: $(6y^2 + 3y - 1) - (3y^2 - 4)$.

Answer

$3y^2 + 3y + 3$

Exercise 12.1.14:

Find the difference: $(8u^2 - 7u - 2) - (5u^2 - 6u - 4)$.

Answer

$3u^2 - u + 2$

Example 12.1.8:

Subtract: $(m^2 - 3m + 8)$ from $(9m^2 - 7m + 4)$.

Solution

Distribute and identify like terms.	$9m^2 - 7m + 4 - m^2 + 3m - 8$
Rearrange the terms.	$9m^2 - m^2 - 7m + 3m + 4 - 8$
Combine like terms.	$8m^2 - 4m - 4$

Exercise 12.1.15:

Subtract: $(4n^2 - 7n - 3)$ from $(8n^2 + 5n - 3)$.

Answer

$4n^2 + 12n$

Exercise 12.1.16:

Subtract: $(a^2 - 4a - 9)$ from $(6a^2 + 4a - 1)$.

Answer

$5a^2 + 8a + 8$

Evaluate a Polynomial for a Given Value

In The Language of Algebra we evaluated expressions. Since polynomials are expressions, we'll follow the same procedures to evaluate polynomials—substitute the given value for the variable into the polynomial, and then simplify.

Example 12.1.9:

Evaluate $3x\ 2 - 9x + 7$ when (a) $x = 3$ (b) $x = -1$

Solution

(a) $x = 3$

Substitute 3 for x.	$3(3)^2 - 9(3) + 7$
Simplify the expression with the exponent.	$3 \cdot 9 - 9(3) + 7$
Multiply.	$27 - 27 + 7$
Simplify.	7

(b) $x = -1$

Substitute -1 for x.	$3(-1)^2 - 9(-1) + 7$
Simplify the expression with the exponent.	$3 \cdot 1 - 9(-1) + 7$
Multiply.	$3 + 9 + 7$
Simplify.	19

Exercise 12.1.17:

Evaluate: $2x^2 + 4x - 3$ when (a) $x = 2$ (b) $x = -3$

Answer a

13

Answer b

3

Exercise 12.1.18:

Evaluate: $7y^2 - y - 2$ when (a) $y = -4$ (b) $y = 0$

Answer a

114

Answer b

-2

Example 12.1.10:

The polynomial $-16t\ 2 + 300$ gives the height of an object t seconds after it is dropped from a 300 foot tall bridge. Find the height after t = 3 seconds.

Solution

Substitute 3 for t.	$-16(3)^2 + 300$
Simplify the expression with the exponent.	$-16 \cdot 9 + 300$
Multiply.	$-144 + 300$
Simplify.	156

Exercise 12.1.19:

The polynomial $-8t^2 + 24t + 4$ gives the height, in feet, of a ball t seconds after it is tossed into the air, from an initial height of 4 feet. Find the height after t = 3 seconds.

Answer

4 feet

Exercise 12.1.20:

The polynomial $-8t^2 + 24t + 4$ gives the height, in feet, of a ball x seconds after it is tossed into the air, from an initial height of 4 feet. Find the height after t = 2 seconds.

Answer

20 feet

ACCESS ADDITIONAL ONLINE RESOURCES

Adding Polynomials

Subtracting Polynomials

Practice Makes Perfect

Identify Polynomials, Monomials, Binomials and Trinomials

In the following exercises, determine if each of the polynomials is a monomial, binomial, trinomial, or other polynomial.

1. $5x + 2$
2. $z^2 - 5z - 6$
3. $a^2 + 9a + 18$
4. $-12p^4$
5. $y^3 - 8y^2 + 2y - 16$
6. $10 - 9x$
7. $23y^2$
8. $m^4 + 4m^3 + 6m^2 + 4m + 1$

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

9. $8a^5 - 2a^3 + 1$
10. $5c^3 + 11c^2 - c - 8$

11. $3x - 12$
12. $4y + 17$
13. -13
14. -22

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

15. $6x^2 + 9x^2$
16. $4y^3 + 6y^3$
17. $-12u + 4u$
18. $-3m + 9m$
19. $5a + 7b$
20. $8y + 6z$
21. Add: $4a, -3b, -8a$
22. Add: $4x, 3y, -3x$
23. $18x - 2x$
24. $13a - 3a$
25. Subtract $5x^6$ from $-12x^6$
26. Subtract $2p^4$ from $-7p^4$

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

27. $(4y^2 + 10y + 3) + (8y^2 - 6y + 5)$
28. $(7x^2 - 9x + 2) + (6x^2 - 4x + 3)$
29. $(x^2 + 6x + 8) + (-4x^2 + 11x - 9)$
30. $(y^2 + 9y + 4) + (-2y^2 - 5y - 1)$
31. $(3a^2 + 7) + (a^2 - 7a - 18)$
32. $(p^2 - 5p - 11) + (3p^2 + 9)$
33. $(6m^2 - 9m - 3) - (2m^2 + m - 5)$
34. $(3n^2 - 4n + 1) - (4n^2 - n - 2)$
35. $(z^2 + 8z + 9) - (z^2 - 3z + 1)$
36. $(z^2 - 7z + 5) - (z^2 - 8z + 6)$
37. $(12s^2 - 15s) - (s - 9)$
38. $(10r^2 - 20r) - (r - 8)$
39. Find the sum of $(2p^3 - 8)$ and $(p^2 + 9p + 18)$
40. Find the sum of $(q^2 + 4q + 13)$ and $(7q^3 - 3)$
41. Subtract $(7x^2 - 4x + 2)$ from $(8x^2 - x + 6)$
42. Subtract $(5x^2 - x + 12)$ from $(9x^2 - 6x - 20)$
43. Find the difference of $(w^2 + w - 42)$ and $(w^2 - 10w + 24)$
44. Find the difference of $(z^2 - 3z - 18)$ and $(z^2 + 5z - 20)$

Evaluate a Polynomial for a Given Value

In the following exercises, evaluate each polynomial for the given value.

45. Evaluate $8y^2 - 3y + 2$
    a. $y = 5$
    b. $y = -2$
    c. $y = 0$
46. Evaluate $5y^2 - y - 7$ when:
    a. $y = -4$
    b. $y = 1$
    c. $y = 0$

47. Evaluate $4 - 36x$ when:
    a. $x = 3$
    b. $x = 0$
    c. $x = -1$
48. Evaluate $16 - 36x^2$ when:
    a. $x = -1$
    b. $x = 0$
    c. $x = 2$
49. A window washer drops a squeegee from a platform 275 feet high. The polynomial $-16t^2 + 275$ gives the height of the squeegee t seconds after it was dropped. Find the height after $t = 4$ seconds.
50. A manufacturer of microwave ovens has found that the revenue received from selling microwaves at a cost of p dollars each is given by the polynomial $-5p^2 + 350p$. Find the revenue received when $p = 50$ dollars.

Everyday Math

51. **Fuel Efficiency** The fuel efficiency (in miles per gallon) of a bus going at a speed of x miles per hour is given by the polynomial $-\frac{1}{160}x^2 + \frac{1}{2}x$. Find the fuel efficiency when $x = 40$ mph.
52. **Stopping Distance** The number of feet it takes for a car traveling at x miles per hour to stop on dry, level concrete is given by the polynomial $0.06x^2 + 1.1x$. Find the stopping distance when $x = 60$ mph.

Writing Exercises

53. Using your own words, explain the difference between a monomial, a binomial, and a trinomial.
54. Eloise thinks the sum $5x^2 + 3x^4$ is $8x^6$. What is wrong with her reasoning?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify polynomials, monomials, binomials, and trinomials.			
determine the degree of polynomials.			
add and subtract monomials.			
add and subtract polynomials.			
evaluate a polynomial for a given value.			

(b) If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on.

Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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12.2: Use Multiplication Properties of Exponents (Part 1)

Learning Objectives

- Simplify expressions with exponents
- Simplify expressions using the Product Property of Exponents
- Simplify expressions using the Power Property of Exponents
- Simplify expressions using the Product to a Power Property
- Simplify expressions by applying several properties
- Multiply monomials

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{3}{4} \cdot \frac{3}{4}$. If you missed the problem, review Example 4.3.7.
2. Simplify: $(-2)(-2)(-2)$. If you missed the problem, review Example 3.7.6.

Simplify Expressions with Exponents

Remember that an exponent indicates repeated multiplication of the same quantity. For example, 2^4 means to multiply four factors of 2, so 2^4 means $2 \cdot 2 \cdot 2 \cdot 2$. This format is known as **exponential notation**.

Definition: Exponential Notation

a^m ← m ← exponent; a ← base

a^m means multiply m factors of a

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}}$$

This is read a to the mth power.

In the expression a^m, the exponent tells us how many times we use the base a as a factor.

$$7^3 \qquad (-8)^5$$

$$\underbrace{7 \cdot 7 \cdot 7}_{3 \text{ factors}} \qquad \underbrace{(-8)(-8)(-8)(-8)(-8)}_{5 \text{ factors}}$$

Before we begin working with variable expressions containing exponents, let's simplify a few expressions involving only numbers.

Example 12.2.1:

Simplify: (a) 5^3 (b) 9^1

Solution

(a) 5^3

Multiply 3 factors of 5.	$5 \cdot 5 \cdot 5$
Simplify.	125

(b) 9^1

Multiply 1 factor of 9.	9

Exercise 12.2.1:

Simplify: (a) 4^3 (b) 11^1

Answer a

64

Answer b

11

Exercise 12.2.2:

Simplify: (a) 3^4 (b) 21^1

Answer a

81

Answer b

21

Example 12.2.2:

Simplify: (a) $\left(\frac{7}{8}\right)^2$ (b) $(0.74)^2$

Solution

(a) $\left(\frac{7}{8}\right)^2$

Multiply two factors.	$\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)$	(12.2.1)
Simplify.	$\frac{49}{64}$	(12.2.2)

(b) $(0.74)^2$

Multiply two factors.	(0.74)(0.74)
Simplify.	0.5476

Exercise 12.2.3:

Simplify: (a) $\left(\frac{5}{8}\right)^2$ (b) $(0.67)^2$

Answer a

$\frac{25}{64}$

Answer b

0.4489

Exercise 12.2.4:

Simplify: (a) $\left(\frac{2}{5}\right)^3$ (b) $(0.127)^2$

Answer a

$\frac{8}{125}$

Answer b

0.016129

Example 12.2.3:

Simplify: (a) $(-3)^4$ (b) -3^4

Solution

(a) $(-3)^4$

Multiply four factors of −3.	$(-3)(-3)(-3)(-3)$
Simplify.	81

(b) -3^4

Multiply two factors.	$-(3 \cdot 3 \cdot 3 \cdot 3)$
Simplify.	−81

Notice the similarities and differences in parts (a) and (b). Why are the answers different? In part (a) the parentheses tell us to raise the (−3) to the 4th power. In part (b) we raise only the 3 to the 4th power and then find the opposite.

Exercise 12.2.5:

Simplify: (a) $(-2)^4$ (b) -2^4

Answer a

16

Answer b

-16

Exercise 12.2.6:

Simplify: (a) $(-8)^2$ (b) -8^2

Answer a

64

Answer b

-64

Simplify Expressions Using the Product Property of Exponents

You have seen that when you combine like terms by adding and subtracting, you need to have the same base with the same exponent. But when you multiply and divide, the exponents may be different, and sometimes the bases may be different, too. We'll derive the properties of exponents by looking for patterns in several examples. All the exponent properties hold true for any real numbers, but right now we will only use whole number exponents.

First, we will look at an example that leads to the Product Property.

	$x^2 \cdot x^2$	(12.2.3)
What does this mean? How many factors altogether?	$\underbrace{x \cdot x}_{2 \text{ factors}} \cdot \underbrace{x \cdot x \cdot x}_{3 \text{ factors}}$ 5 factors	
So, we have	x^5	(12.2.4)
Notice that 5 is the sum of the exponents, 2 and 3.	$x^2 \cdot x^3 \text{ is } x^{2+3}, \text{ or } x^5$	(12.2.5)
We write:	$x^2 \cdot x^3$ x^{2+3} x^5	

The base stayed the same and we added the exponents. This leads to the Product Property for Exponents.

Definition: Product Property of Exponents

If a is a real number and m, n are counting numbers, then

$$a^m \cdot a^n = a^{m+n} \tag{12.2.6}$$

To multiply with like bases, add the exponents.

An example with numbers helps to verify this property.

$$\begin{aligned} 2^2 \cdot 2^3 &\stackrel{?}{=} 2^{2+3} \\ 4 \cdot 8 &\stackrel{?}{=} 2^5 \\ 32 &= 32 \checkmark \end{aligned}$$

Example 12.2.4:

Simplify: $x^5 \cdot x^7$.

Solution

Use the product property, $a^m \cdot a^n = a^{m+n}$.	x^{5+7}	(12.2.7)
Simplify.	x^{12}	(12.2.8)

Exercise 12.2.7:

Simplify: $x^7 \cdot x^8$.

Answer

x^{15}

Exercise 12.2.8:

Simplify: $x^5 \cdot x^{11}$.

Answer

x^{16}

Example 12.2.5:

Simplify: $b^4 \cdot b$.

Solution

Rewrite, $b = b^1$.	$b^4 \cdot b^1$	(12.2.9)
Use the product property, $a^m \cdot a^n = a^{m+n}$.	b^{4+1}	(12.2.10)
Simplify.	b^5	(12.2.11)

Exercise 12.2.9:

Simplify: $p^9 \cdot p$.

Answer

p^{10}

Exercise 12.2.10:

Simplify: $m \cdot m^7$.

Answer

m^8

Example 12.2.6:

Simplify: $2^7 \cdot 2^9$.

Solution

Use the product property, $a^m \cdot a^n = a^{m+n}$.	2^{7+9}	(12.2.12)
Simplify.	2^{16}	(12.2.13)

Exercise 12.2.11:

Simplify: $6 \cdot 6^9$.

Answer

6^{10}

Exercise 12.2.12:

Simplify: $9^6 \cdot 9^9$.

Answer

9^{15}

Example 12.2.7:

Simplify: $y^{17} \cdot y^{23}$.

Solution

Notice, the bases are the same, so add the exponents.	y^{17+23}	(12.2.14)
Simplify.	y^{40}	(12.2.15)

Exercise 12.2.13:

Simplify: $y^{24} \cdot y^{19}$.

Answer

y^{43}

Exercise 12.2.14:

Simplify: $z^{15} \cdot z^{24}$.

Answer

z^{39}

We can extend the Product Property of Exponents to more than two factors.

Example 12.2.8:

Simplify: $x^3 \cdot x^4 \cdot x^2$.

Solution

Add the exponents, since the bases are the same.	x^{3+4+2}	(12.2.16)
Simplify.	x^9	(12.2.17)

Exercise 12.2.15:

Simplify: $x^7 \cdot x^5 \cdot x^9$.

Answer

x^{21}

Exercise 12.2.16:

Simplify: $y^3 \cdot y^8 \cdot y^4$.

Answer

y^{15}

Simplify Expressions Using the Power Property of Exponents

Now let's look at an exponential expression that contains a power raised to a power. See if you can discover a general property.

	$(x^2)^3$	(12.2.18)
What does this mean?	$x^2 \cdot x^2 \cdot x^2$	(12.2.19)
How many factors altogether?	$\underbrace{x \cdot x}_{2\ factors} \cdot \underbrace{x \cdot x}_{2\ factors} \cdot \underbrace{x \cdot x}_{2\ factors}$ $6\ factors$	
So, we have	x^6	(12.2.20)
Notice that 6 is the product of the exponents, 2 and 3.	$(x^2)^3\ is\ x^{2 \cdot 3}\ or\ x^6$	(12.2.21)
We write:	$(x^2)^3$ $x^{2 \cdot 3}$ x^6	

We multiplied the exponents. This leads to the Power Property for Exponents.

Definition: Power Property of Exponents

If a is a real number and m, n are whole numbers, then

$$(a^m)^n = a^{m \cdot n} \quad (12.2.22)$$

To raise a power to a power, multiply the exponents.

An example with numbers helps to verify this property.

$$(5^2)^3 \stackrel{?}{=} 5^{2 \cdot 3}$$
$$(25)^3 \stackrel{?}{=} 5^6$$
$$15,625 = 15,625\ \checkmark$$

Example 12.2.9:

Simplify: (a) $(x^5)^7$ (b) $(3^6)^8$

Solution

(a) $(x^5)^7$

Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$x^{5 \cdot 7}$	(12.2.23)
Simplify.	x^{35}	(12.2.24)

(b) $(3^6)^8$

Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$3^{6 \cdot 8}$	(12.2.25)
Simplify.	x^{48}	(12.2.26)

Exercise 12.2.17:

Simplify: (a) $(x^7)^4$ (b) $(7^4)^8$

Answer a

x^{28}

Answer b

7^{32}

Exercise 12.2.18:

Simplify: (a) $(x^6)^9$ (b) $(8^6)^7$

Answer a

y^{54}

Answer b

8^{42}

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12.3: Use Multiplication Properties of Exponents (Part 2)

Simplify Expressions Using the Product to a Power Property

We will now look at an expression containing a product that is raised to a power. Look for a pattern.

	$(2x)^3$
What does this mean?	2x • 2x • 2x
We group the like factors together.	2 • 2 • 2 • x • x • x
How many factors of 2 and of x?	$2^3 \cdot x^3$
Notice that each factor was raised to the power.	$(2x)^3$ is $2^3 \cdot x^3$
We write:	$(2x)^3$ $2^3 \cdot x^3$

The exponent applies to each of the factors. This leads to the Product to a Power Property for Exponents.

Definition: Product to a Power Property of Exponents

If a and b are real numbers and m is a whole number, then

$$(ab)^m = a^m b^m \tag{10.2.27}$$

To raise a product to a power, raise each factor to that power.

An example with numbers helps to verify this property:

$$(2 \cdot 3)^2 \stackrel{?}{=} 2^2 \cdot 3^2$$
$$6^2 \stackrel{?}{=} 4 \cdot 9$$
$$36 \stackrel{?}{=} 36 \checkmark$$

Example 12.3.10:

Simplify: $(-11x)^2$.

Solution

Use the Power of a Product Property, $(ab)^m = a^m b^m$.	$(-11)^2 x^2$	(10.2.28)
Simplify.	$121x^2$	(10.2.29)

Exercise 12.3.19:

Simplify: $(-14x)^2$.

Answer

$196x^2$

Exercise 12.3.20:

Simplify: $(-12a)^2$.

Answer

$144a^2$

Example 12.3.11:

Simplify: $(3xy)^3$.

Solution

Raise each factor to the third power.	$3^3x^3y^3$	(10.2.30)
Simplify.	$27x^3y^3$	(10.2.31)

Exercise 12.3.21:

Simplify: $(-4xy)^4$.

Answer

$256x^4y^4$

Exercise 12.3.22:

Simplify: $(6xy)^3$.

Answer

$216x^3y^3$

Simplify Expressions by Applying Several Properties

We now have three properties for multiplying expressions with exponents. Let's summarize them and then we'll do some examples that use more than one of the properties.

Definition: Properties of Exponents

If a, b are real numbers and m, n are whole numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power Property	$(ab)^m = a^m b^m$

Example 12.3.12:

Simplify: $(x^2)^6(x^5)^4$.

Solution

Use the Power Property.	$x^{12} \cdot x^{20}$

Add the exponents.	x^{32}

Exercise 12.3.23:

Simplify: $(x^4)^3(x^7)^4$.

Answer

x^{40}

Exercise 12.3.24:

Simplify: $(y^9)^2(y^8)^3$.

Answer

y^{42}

Example 12.3.13:

Simplify: $(-7x^3y^4)^2$.

Solution

Take each factor to the second power.	$(-7)^2(x^3)^2(y^4)^2$
Use the Power Property.	$49x^6y^8$

Exercise 12.3.25:

Simplify: $(-8x^4y^7)^3$.

Answer

$-512x^{12}y^{21}$

Exercise 12.3.26:

Simplify: $(-3a^5b^6)^4$.

Answer

$81a^{20}b^{24}$

Example 12.3.14:

Simplify: $(6n)^2(4n^3)$.

Solution

Raise 6n to the second power.	$6^2n^2 \cdot 4n^3$
Simplify.	$36n^2 \cdot 4n^3$
Use the Commutative Property.	$36 \cdot 4 \cdot n^2 \cdot n^3$
Multiply the constants and add the exponents.	$144n^5$

Notice that in the first monomial, the exponent was outside the parentheses and it applied to both factors inside. In the second monomial, the exponent was inside the parentheses and so it only applied to the n.

Exercise 12.3.27:

Simplify: $(7n)^2\ (2n^{12})$.

Answer

$98n^{14}$

Exercise 12.3.28:

Simplify: $(4m)^2(3m^3)$.

Answer

$48m^5$

Example 12.3.15:

Simplify: $(3p^2q)^4(2pq^2)^3$.

Solution

Use the Power of a Product Property.	$3^4(p^2)^4q^4 \cdot 2^3p^3(q^2)^3$
Use the Power Property.	$81p^8q^4 \cdot 8p^3q^6$
Use the Commutative Property.	$81 \cdot 8 \cdot p^8 \cdot p^3 \cdot q^4 \cdot q^6$
Multiply the constants and add the exponents for each variable.	$648p^{11}q^{10}$

Exercise 12.3.29:

Simplify: $(u^3v^2)^5(4uv^4)^3$.

Answer

$64u^{18}v^{22}$

Exercise 12.3.30:

Simplify: $(5x^2y^3)^2(3xy^4)^3$.

Answer

$675x^7y^{18}$

Multiply Monomials

Since a monomial is an algebraic expression, we can use the properties for simplifying expressions with exponents to multiply the monomials.

Example 12.3.16:

Multiply: $(4x^2)(-5x^3)$.

Solution

Use the Commutative Property to rearrange the factors.	$4 \cdot (-5) \cdot x^2 \cdot x^3$
Multiply.	$-20x^5$

Exercise 12.3.31:

Multiply: $(7x^7)(-8x^4)$.

Answer

$-56x^{11}$

Exercise 12.3.32:

Multiply: $(-9y^4)(-6y^5)$.

Answer

$54y^9$

Example 12.3.17:

Multiply: $\left(\frac{3}{4}c^3d\right)(12cd^2)$.

Solution

Use the Commutative Property to rearrange the factors.	$\frac{3}{4} \cdot 12 \cdot c^3 \cdot c \cdot d \cdot d^2$
Multiply.	$9c^4d^3$

Exercise 12.3.33:

Multiply: $\left(\frac{4}{5}m^4n^3d\right)(15mn^3)$.

Answer

$12m^5n^6$

Exercise 12.3.34:

Multiply: $\left(\frac{2}{3}p^5qd\right)(18p^6q^7)$.

Answer

$12p^{11}q^8$

ACCESS ADDITIONAL ONLINE RESOURCES

Exponent Properties

Exponent Properties 2

Practice Makes Perfect

Simplify Expressions with Exponents

In the following exercises, simplify each expression with exponents.

55. 4^5
56. 10^3
57. $\left(\frac{1}{2}\right)^2$
58. $\left(\frac{3}{5}\right)^2$
59. $(0.2)^3$
60. $(0.4)^3$
61. $(-5)^4$
62. $(-3)^5$
63. -5^4
64. -3^5
65. -10^4
66. -2^6
67. $-\left(\frac{2}{3}\right)^3$
68. $-\left(\frac{1}{4}\right)^4$
69. -0.5^2
70. -0.1^4

Simplify Expressions Using the Product Property of Exponents

In the following exercises, simplify each expression using the Product Property of Exponents.

71. $x^3 \cdot x^6$
72. $m^4 \cdot m^2$
73. $a \cdot a^4$
74. $y^{12} \cdot y$
75. $3^5 \cdot 3^9$
76. $5^{10} \cdot 5^6$
77. $z \cdot z^2 \cdot z^3$
78. $a \cdot a^3 \cdot a^5$
79. $x^a \cdot x^2$
80. $y^p \cdot y^3$
81. $y^a \cdot y^b$
82. $x^p \cdot x^q$

Simplify Expressions Using the Power Property of Exponents

In the following exercises, simplify each expression using the Power Property of Exponents.

83. $(u^4)^2$
84. $(x^2)^7$
85. $(y^5)^4$
86. $(a^3)^2$

87. $(10^2)^6$
88. $(2^8)^3$
89. $(x^{15})^6$
90. $(y^{12})^8$
91. $(x^2)^y$
92. $(y^3)^x$
93. $(5^x)^y$
94. $(7^a)^b$

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression using the Product to a Power Property.

95. $(5a)^2$
96. $(7x)^2$
97. $(-6m)^3$
98. $(-9n)^3$
99. $(4rs)^2$
100. $(5ab)^3$
101. $(4xyz)^4$
102. $(-5abc)^3$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.

103. $(x^2)^4 \cdot (x^3)^2$
104. $(y^4)^3 \cdot (y^5)^2$
105. $(a^2)^6 \cdot (a^3)^8$
106. $(b^7)^5 \cdot (b^2)^6$
107. $(3x)^2(5x)$
108. $(2y)^3\,(6y)$
109. $(5a)^2(2a)^3$
110. $(4b)^2(3b)^3$
111. $(2m^6)^3$
112. $(3y^2)^4$
113. $(10x^2y)^3$
114. $(2mn^4)^5$
115. $(-2a^3b^2)^4$
116. $(-10u^2v^4)^3$
117. $\left(\frac{2}{3}x^2y\right)^3$
118. $\left(\frac{7}{9}pq^4\right)^2$
119. $(8a^3)^2(2a)^4$
120. $(5r^2)^3(3r)^2$
121. $(10p^4)^3(5p^6)^2$
122. $(4x^3)^3(2x^5)^4$
123. $\left(\frac{1}{2}x^2y^3\right)^4\,(4x^5y^3)^2$
124. $\left(\frac{1}{3}m^3n^2\right)^4\,(9m^8n^3)^2$
125. $(3m^2n)^2(2mn^5)^4$
126. $(2pq^4)^3(5p^6q)^2$

Multiply Monomials

In the following exercises, multiply the following monomials.

127. $(12x^2)(-5x^4)$
128. $(-10y^3)(7y^2)$
129. $(-8u^6)(-9u)$
130. $(-6c^4)(-12c)$
131. $\left(\frac{1}{5}r^8\right)(20r^3)$
132. $\left(\frac{1}{4}a^5\right)(36a^2)$
133. $(4a^3b)(9a^2b^6)$
134. $(6m^4n^3)(7mn^5)$
135. $\left(\frac{4}{7}xy^2\right)(14xy^3)$
136. $\left(\frac{5}{8}u^3v\right)^3(24u^5v)$
137. $\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$
138. $\left(\frac{3}{5}m^3n^2\right)\left(\frac{5}{9}m^2n^3\right)$

Everyday Math

139. **Email** Janet emails a joke to six of her friends and tells them to forward it to six of their friends, who forward it to six of their friends, and so on. The number of people who receive the email on the second round is 6^2, on the third round is 6^3, as shown in the table. How many people will receive the email on the eighth round? Simplify the expression to show the number of people who receive the email.

Round	Number of people
1	6
2	6^2
3	6^3
…	…
8	?

140. **Salary** Raul's boss gives him a 5% raise every year on his birthday. This means that each year, Raul's salary is 1.05 times his last year's salary. If his original salary was \$40,000, his salary after 1 year was \$40,000(1.05), after 2 years was $\$40,000(1.05)^2$, after 3 years was $\$40,000(1.05)^3$, as shown in the table below. What will Raul's salary be after 10 years? Simplify the expression, to show Raul's salary in dollars.

Year	Salary
1	\$40,000(1.05)
2	$\$40,000(1.05)^2$
3	$\$40,000(1.05)^3$
…	…
10	?

Writing Exercises

141. Use the Product Property for Exponents to explain why $x \cdot x = x^2$.
142. Explain why $-5^3 = (-5)^3$ but $-5^4 \neq (-5)^4$.
143. Jorge thinks $\left(\frac{1}{2}\right)^2$ is 1. What is wrong with his reasoning?
144. Explain why $x^3 \cdot x^5$ is x^8, and not x^{15}.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with exponents.			
simplify expressions using the Product Property for Exponents.			
simplify expressions using the Power Property for Exponents.			
simplify expressions using the Product to a Power Property.			
simplify expressions by applying several properties.			
multiply monomials.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

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12.4: Multiply Polynomials (Part 1)

Learning Objectives

- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a trinomial by a binomial

be prepared!

Before you get started, take this readiness quiz.

1. Distribute: 2(x + 3). If you missed the problem, review Example 7.4.1.
2. Distribute: −11(4 − 3a). If you missed the problem, review Example 7.4.10.
3. Combine like terms: x^2 + 9x + 7x + 63. If you missed the problem, review Example 2.3.9.

Multiply a Polynomial by a Monomial

In Distributive Property you learned to use the Distributive Property to simplify expressions such as 2(x − 3). You multiplied both terms in the parentheses, x and 3, by 2, to get 2x − 6. With this chapter's new vocabulary, you can say you were multiplying a binomial, x − 3, by a monomial, 2. Multiplying a binomial by a monomial is nothing new for you!

Example 12.4.1:

Multiply: 3(x + 7).

Solution

Distribute.	$3(x+7)$
	3 • x + 3 • 7
Simplify.	3x + 21

Exercise 12.4.1:

Multiply: 6(x + 8).

Answer

6x + 48

Exercise 12.4.2:

Multiply: 2(y + 12).

Answer

2y + 24

Example 12.4.2:

Multiply: x(x − 8).

Solution

Distribute.	$x(x-8)$
	x^2 - 8x
Simplify.	x^2 - 8x

Exercise 12.4.3:

Multiply: y(y − 9).

Answer

$y^2 - 9y$

Exercise 12.4.4:

Multiply: p(p − 13).

Answer

$p^2 - 13p$

Example 12.4.3:

Multiply: 10x(4x + y).

Solution

Distribute.	$10x(4x+y)$
	10x • 4x + 10x • y
Simplify.	$40x^2$ + 10xy

Exercise 12.4.5:

Multiply: 8x(x + 3y).

Answer

$8x^2 + 24xy$

Exercise 12.4.6:

Multiply: 3r(6r + s).

Answer

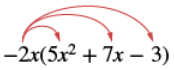
$18r^2 + 3rs$

Multiplying a monomial by a trinomial works in much the same way.

Example 12.4.4:

Multiply: $-2x(5x^2 + 7x - 3)$.

Solution

Distribute.	
	$-2x \cdot 5x^2 + (-2x) \cdot 7x - (-2x) \cdot 3$
Simplify.	$-10x^3 - 14x^2 + 6x$

Exercise 12.4.7:

Multiply: $-4y(8y^2 + 5y - 9)$.

Answer

$-32y^3 - 20y^2 + 36y$

Exercise 12.4.8:

Multiply: $-6x(9x^2 + x - 1)$.

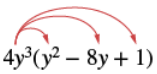
Answer

$-54x^3 - 6x^2 + 6x$

Example 12.4.5:

Multiply: $4y^3(y^2 - 8y + 1)$.

Solution

Distribute.	
	$4y^3 \cdot y^2 - 4y^3 \cdot 8y + 4y^3 \cdot 1$
Simplify.	$4y^5 - 32y^4 + 4y^3$

Exercise 12.4.9:

Multiply: $3x^2(4x^2 - 3x + 9)$.

Answer

$12x^4 - 9x^3 + 27x^2$

Exercise 12.4.10:

Multiply: $8y^2(3y^2 - 2y - 4)$.

Answer

$24y^4 - 16y^3 - 32y^2$

Now we will have the monomial as the second factor.

Example 12.4.6:

Multiply: (x + 3)p.

Solution

Distribute.	$(x+3)p$
	x • p + 3 • p
Simplify.	xp + 3p

Exercise 12.4.11:

Multiply: (x + 8)p.

Answer

$xp+8p$

Exercise 12.4.12:

Multiply: (a + 4)p.

Answer

$ap+4p$

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial.

Using the Distributive Property

We will start by using the Distributive Property. Look again at Example 12.4.6

	$(x+3)p$
We distributed the p to get	$xp+3\,p$
What if we have (x + 7) instead of p? Think of the (x + 7) as the p above.	$(x+3)(x+7)$
Distribute (x + 7).	$x(x+7)+3(x+7)$
Distribute again.	$x^2 + 7x + 3x + 21$
Combine like terms.	$x^2 + 10x + 21$

Notice that before combining like terms, we had four terms. We multiplied the two terms of the first binomial by the two terms of the second binomial—four multiplications.

Be careful to distinguish between a sum and a product.

Sum	Product
$x+x$	$x \cdot x$
$2x$	x^2
combine like terms	*add exponents of like bases*

Example 12.4.7:

Multiply: (x + 6)(x + 8).

Solution

	$(x+6)(x+8)$
Distribute (x + 8).	$x(x+8)+6(x+8)$
Distribute again.	$x^2 + 8x + 6x + 48$
Simplify.	$x^2 + 14x + 48$

Exercise 12.4.13:

Multiply: (x + 8)(x + 9).

Answer

$x^2+17x+72$

Exercise 12.4.14:

Multiply: (a + 4)(a + 5).

Answer

$a^2+9a+20$

Now we'll see how to multiply binomials where the variable has a coefficient.

Example 12.4.8:

Multiply: (2x + 9)(3x + 4).

Solution

Distribute (3x + 4).	$2x(3x+4)+9(3x+4)$
Distribute again.	$6x^2 + 8x + 27x + 36$
Simplify.	$6x^2 + 35x + 36$

Exercise 12.4.15:

Multiply: (5x + 9)(4x + 3).

Answer

$20x^2+51x+27$

Exercise 12.4.16:

Multiply: (10m + 9)(8m + 7).

Answer

$80m^2 + 142m + 63$

In the previous examples, the binomials were sums. When there are differences, we pay special attention to make sure the signs of the product are correct.

Example 12.4.9:

Multiply: (4y + 3)(6y − 5).

Solution

Distribute.	$4y(6y - 5) + 3(6y - 5)$
Distribute again.	$24y^2 - 20y + 18y - 15$
Simplify.	$24y^2 - 2y - 15$

Exercise 12.4.17:

Multiply: (7y + 1)(8y − 3).

Answer

$56y^2 - 13y - 3$

Exercise 12.4.18:

Multiply: (3x + 2)(5x − 8).

Answer

$15x^2 - 14x - 16$

Up to this point, the product of two binomials has been a trinomial. This is not always the case.

Example 12.4.10:

Multiply: (x + 2)(x − y).

Solution

Distribute.	$x(x - y) + 2(x - y)$
Distribute again.	x^2 - xy + 2x - 2y
Simplify.	There are no like terms to combine.

Exercise 12.4.19:

Multiply: (x + 5)(x − y).

Answer

$$x^2 - xy + 5x - 5y$$

Exercise 12.4.20:

Multiply: (x + 2y)(x − 1).

Answer

$$x^2 - x + 2xy - 2y$$

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12.5: Multiply Polynomials (Part 2)

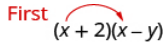
Using the FOIL Method

Remember that when you multiply a binomial by a binomial you get four terms. Sometimes you can combine like terms to get a trinomial, but sometimes there are no like terms to combine. Let's look at the last example again and pay particular attention to how we got the four terms.

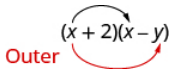
$$(x+2)(x-y)$$
$$x^2 - xy + 2x - 2y$$

Where did the first term, x^2, come from?

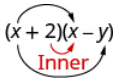
It is the product of x and x, the **first** terms in (x + 2) and (x − y).



The next term, −xy, is the product of x and − y, the two **outer** terms.

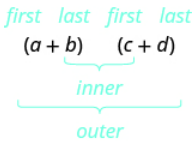


The third term, +2x, is the product of 2 and x, the two **inner** terms.



And the last term, −2y, came from multiplying the two **last** terms.

We abbreviate “First, Outer, Inner, Last” as FOIL. The letters stand for ‘First, Outer, Inner, Last’. The word FOIL is easy to remember and ensures we find all four products. We might say we use the FOIL method to multiply two binomials.



Let's look at (x + 3)(x + 7) again. Now we will work through an example where we use the FOIL pattern to multiply two binomials.

Distributive Property	FOIL
$(x + 3)(x + 7)$	$(x + 3)(x + 7)$
$x(x + 7) + 3(x + 7)$	
$x^2 + 7x + 3x + 21$	$x^2 + 7x + 3x + 21$
$F \quad O \quad I \quad L$	$F \quad O \quad I \quad L$
$x^2 + 10x + 21$	$x^2 + 10x + 21$

Example 12.5.11:

Multiply using the FOIL method: (x + 6)(x + 9).

Solution

Step 1: Multiply the **First** terms.	$(x+6)(x+9)$ $x^2+\_\_+\_\_+\_\_$ F O I L
Step 2: Multiply the **Outer** terms.	$(x+6)(x+9)$ $x^2+9x+\_\_+\_\_$ F O I L
Step 3: Multiply the **Inner** terms.	$(x+6)(x+9)$ $x^2+9x+6x+\_\_$ F O I L
Step 4: Multiply the **Last** terms.	$(x+6)(x+9)$ $x^2+9x+6x+54$ F O I L
Step 5: Combine like terms, when possible.	$x^2 + 15x + 54$

Exercise 12.5.21:

Multiply using the FOIL method: (x + 7)(x + 8).

Answer

$x^2+15x+56$

Exercise 12.5.22:

Multiply using the FOIL method: (y + 14)(y + 2).

Answer

$y^2+16y+28$

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

HOW TO: USE THE FOIL METHOD FOR MULTIPLYING TWO BINOMIALS

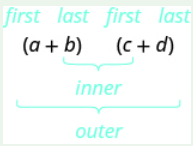
Step 1. Multiply the **First** terms.

Step 2. Multiply the **Outer** terms.

Step 3. Multiply the **Inner** terms.

Step 4. Multiply the **Last** terms.

Step 5. Combine like terms, when possible.



Example 12.5.12:

Multiply: (y − 8)(y + 6).

Solution

Step 1: Multiply the **First** terms.	$(y-8)(y+6)$ $y^2+\_\_+\_\_+\_\_$ F O I L

Step 2: Multiply the **Outer** terms.	$(y-8)(y+6)$ $y^2+6y+\_\_+\_\_$ F O I L
Step 3: Multiply the **Inner** terms.	$(y-8)(y+6)$ $y^2+6y-8y+\_\_$ F O I L
Step 4: Multiply the **Last** terms.	$(y-8)(y+6)$ $y^2+6y-8y-48$ F O I L
Step 5: Combine like terms, when possible.	$y^2 - 2y - 48$

? Exercise 12.5.23:

Multiply: (y − 3)(y + 8).

Answer

$y^2+5y-24$

? Exercise 12.5.24:

Multiply: (q − 4)(q + 5).

Answer

q^2+q-20

✓ Example 12.5.13:

Multiply: (2a + 3)(3a − 1).

Solution

	$(2a+3)(3a-1)$
Multiply the **First** terms.	$2a \cdot 3a$ $6a^2+\_\_+\_\_+\_\_$ F O I L
Multiply the **Outer** terms.	$2a \cdot (-1)$ $6a^2-2a+\_\_+\_\_$ F O I L
Multiply the **Inner** terms.	$3 \cdot 3a$ $6a^2-2a+9a+\_\_$ F O I L
Multiply the **Last** terms.	$3 \cdot (-1)$ $6a^2-2a+9a-3$ F O I L
Combine like terms.	$6a^2 + 7a - 3$

? Exercise 12.5.25:

Multiply: (4a + 9)(5a − 2).

Answer

$20a^2 + 37a - 18$

Exercise 12.5.26:

Multiply: (7x + 4)(7x − 8).

Answer

$49x^2 - 28x - 32$

Example 12.5.14:

Multiply: (5x − y)(2x − 7).

Solution

Multiply the **First** terms.	$10x^2 + \_\_ + \_\_ + \_\_$ F O I L
Multiply the **Outer** terms.	$10x^2 - 35x + \_\_ + \_\_$ F O I L
Multiply the **Inner** terms.	$10x^2 - 35x - 2xy + \_\_$ F O I L
Multiply the **Last** terms.	$10x^2 - 35x - 2xy + 7y$ F O I L
Combine like terms. There are none.	$10x^2$ - 35x - 2xy + 7y

Exercise 12.5.27:

Multiply: (12x − y)(x − 5).

Answer

$12x^2 - 60x - xy + 5y$

Exercise 12.5.28:

Multiply: (6a − b)(2a − 9).

Answer

$12a^2 - 54a - 2ab + 9b$

Using the Vertical Method

The FOIL method is usually the quickest method for multiplying two binomials, but it works only for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

23	
×46	
138	partial product
92	partial product
1058	product

You start by multiplying 23 by 6 to get 138. Then you multiply 23 by 4, lining up the partial product in the correct columns. Last, you add the partial products. Now we'll apply this same method to multiply two binomials.

Example 12.5.15:

Multiply using the vertical method: (5x − 1)(2x − 7).

Solution

It does not matter which binomial goes on the top. Line up the columns when you multiply as we did when we multiplied 23(46).

	$2x-7$ $\times\ 5x-1$
Multiply 2x − 7 by −1.	$-2x+7$ partial product
Multiply 2x − 7 by 5x.	$10x^2-35x$ partial product
Add like terms.	$10x^2-37x+7$ product

Notice the partial products are the same as the terms in the FOIL method.

$(5x-1)(2x-7)$	$2x-7$ $\times\ 5x-1$
$10x^2-35x-2x+7$	$-2x+7$
$10x^2-37x+7$	$10x^2-35x$ $10x^2-37x+7$

Exercise 12.5.29:

Multiply using the vertical method: (4m − 9)(3m − 7).

Answer

$12m^2-55m+63$

Exercise 12.5.30:

Multiply using the vertical method: (6n − 5)(7n − 2).

Answer

$42n^2-47n+10$

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The three methods are listed here to help you remember them.

Definition: Multiplying Two Binomials

To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Remember, FOIL only works when multiplying two binomials.

Multiply a Trinomial by a Binomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a trinomial by a binomial. Remember, the FOIL method will not work in this case, but we can use either the Distributive Property or the Vertical Method. We first look at an example using the Distributive Property.

Example 12.5.16:

Multiply using the Distributive Property: $(x + 3)(2x^2 - 5x + 8)$.

Solution

	$(x + 3)(2x^2 - 5x + 8)$
Distribute.	$x(2x^2 - 5x + 8) + 3(2x^2 - 5x + 8)$
Multiply.	$2x^3 - 5x^2 + 8x + 6x^2 - 15x + 24$
Combine like terms.	$2x^3 + x^2 - 7x + 24$

Exercise 12.5.31:

Multiply using the Distributive Property: $(y - 1)(y^2 - 7y + 2)$.

Answer

$y^3 - 8y^2 + 9y - 2$

Exercise 12.5.32:

Multiply using the Distributive Property: $(x + 2)(3x^2 - 4x + 5)$.

Answer

$3x^3 + 2x^2 - 3x + 10$

Now let's do this same multiplication using the Vertical Method.

Example 12.5.17:

Multiply using the Vertical Method: $(x + 3)(2x^2 - 5x + 8)$.

Solution

It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

	$2x^2 - 5x + 8$ $\times \quad x + 3$
Multiply $(2x^2 - 5x + 8)$ by 3.	$6x^2 - 15x + 24$
Multiply $(2x^2 - 5x + 8)$ by x.	$2x^3 - 5x^2 + 8x$
Add like terms.	$2x^3 + x^2 - 7x + 24$

Exercise 12.5.33:

Multiply using the Vertical Method: $(y - 1)(y^2 - 7y + 2)$.

Answer

$y^3 - 8y^2 + 9y - 2$

Exercise 12.5.34:

Multiply using the Vertical Method: $(x + 2)(3x^2 - 4x + 5)$.

Answer

$3x^3 + 2x^2 - 3x + 10$

ACCESS ADDITIONAL ONLINE RESOURCES

Multiply Monomials

Multiply Polynomials

Multiply Polynomials 2

Multiply Polynomials Review

Multiply Polynomials Using the Distributive Property

Multiply Binomials

Practice Makes Perfect

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

145. $4(x + 10)$
146. $6(y + 8)$
147. $15(r - 24)$
148. $12(v - 30)$
149. $-3(m + 11)$
150. $-4(p + 15)$
151. $-8(z - 5)$
152. $-3(x - 9)$
153. $u(u + 5)$
154. $q(q + 7)$
155. $n(n^2 - 3n)$
156. $s(s^2 - 6s)$
157. $12x(x - 10)$
158. $9m(m - 11)$
159. $-9a(3a + 5)$
160. $-4p(2p + 7)$
161. $6x(4x + y)$
162. $5a(9a + b)$
163. $5p(11p - 5q)$
164. $12u(3u - 4v)$
165. $3(v^2 + 10v + 25)$
166. $6(x^2 + 8x + 16)$
167. $2n(4n^2 - 4n + 1)$

168. $3r(2r^2 - 6r + 2)$
169. $-8y(y^2 + 2y - 15)$
170. $-5m(m^2 + 3m - 18)$
171. $5q^3(q^2 - 2q + 6)$
172. $9r^3(r^2 - 3r + 5)$
173. $-4z^2(3z^2 + 12z - 1)$
174. $-3x^2(7x^2 + 10x - 1)$
175. $(2y - 9)y$
176. $(8b - 1)b$
177. $(w - 6) \cdot 8$
178. $(k - 4) \cdot 5$

Multiply a Binomial by a Binomial

In the following exercises, multiply the following binomials using: (a) the Distributive Property (b) the FOIL method (c) the Vertical method

179. $(x + 4)(x + 6)$
180. $(u + 8)(u + 2)$
181. $(n + 12)(n - 3)$
182. $(y + 3)(y - 9)$

In the following exercises, multiply the following binomials. Use any method.

183. $(y + 8)(y + 3)$
184. $(x + 5)(x + 9)$
185. $(a + 6)(a + 16)$
186. $(q + 8)(q + 12)$
187. $(u - 5)(u - 9)$
188. $(r - 6)(r - 2)$
189. $(z - 10)(z - 22)$
190. $(b - 5)(b - 24)$
191. $(x - 4)(x + 7)$
192. $(s - 3)(s + 8)$
193. $(v + 12)(v - 5)$
194. $(d + 15)(d - 4)$
195. $(6n + 5)(n + 1)$
196. $(7y + 1)(y + 3)$
197. $(2m - 9)(10m + 1)$
198. $(5r - 4)(12r + 1)$
199. $(4c - 1)(4c + 1)$
200. $(8n - 1)(8n + 1)$
201. $(3u - 8)(5u - 14)$
202. $(2q - 5)(7q - 11)$
203. $(a + b)(2a + 3b)$
204. $(r + s)(3r + 2s)$
205. $(5x - y)(x - 4)$
206. $(4z - y)(z - 6)$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using (a) the Distributive Property and (b) the Vertical Method.

207. $(u + 4)(u^2 + 3u + 2)$
208. $(x + 5)(x^2 + 8x + 3)$
209. $(a + 10)(3a^2 + a - 5)$
210. $(n + 8)(4n^2 + n - 7)$

In the following exercises, multiply. Use either method.

211. $(y - 6)(y^2 - 10y + 9)$
212. $(k - 3)(k^2 - 8k + 7)$
213. $(2x + 1)(x^2 - 5x - 6)$
214. $(5v + 1)(v^2 - 6v - 10)$

Everyday Math

215. **Mental math** You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 13 times 15. Think of 13 as 10 + 3 and 15 as 10 + 5.
    a. Multiply (10 + 3)(10 + 5) by the FOIL method.
    b. Multiply 13 • 15 without using a calculator.
    c. Which way is easier for you? Why?
216. **Mental math** You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 18 times 17. Think of 18 as 20 − 2 and 17 as 20 − 3.
    a. Multiply (20 − 2)(20 − 3) by the FOIL method.
    b. Multiply 18 • 17 without using a calculator.
    c. Which way is easier for you? Why?

Writing Exercises

217. Which method do you prefer to use when multiplying two binomials—the Distributive Property, the FOIL method, or the Vertical Method? Why?
218. Which method do you prefer to use when multiplying a trinomial by a binomial—the Distributive Property or the Vertical Method? Why?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
multiply a polynomial by a monomial.			
multiply a binomial by a binomial.			
multiply a trinomial by a binomial.			

(b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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12.6: Divide Monomials (Part 1)

Learning Objectives

- Simplify expressions using the Quotient Property of Exponents
- Simplify expressions with zero exponents
- Simplify expressions using the Quotient to a Power Property
- Simplify expressions by applying several properties
- Divide monomials

be prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{8}{24}$. If you missed the problem, review Example 4.3.1.
2. Simplify:$(2m^3)^5$. If you missed the problem, review Example 10.3.13.
3. Simplify: $\frac{12x}{12y}$. If you missed the problem, review Example 4.3.5.

Simplify Expressions Using the Quotient Property of Exponents

Earlier in this chapter, we developed the properties of exponents for multiplication. We summarize these properties here.

Summary of Exponent Properties for Multiplication

If a, b are real numbers and m, n are whole numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$

Now we will look at the exponent properties for division. A quick memory refresher may help before we get started. In Fractions you learned that fractions may be simplified by dividing out common factors from the numerator and denominator using the Equivalent Fractions Property. This property will also help us work with algebraic fractions—which are also quotients.

Definition: Equivalent Fractions Property

If a, b, c are whole numbers where $b \neq 0$, $c \neq 0$, then

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \quad and \quad \frac{a \cdot c}{b \cdot c} = \frac{a}{b} \tag{12.6.1}$$

As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$ (12.6.2)	and	$\frac{x^2}{x^3}$ (12.6.3)
What do they mean?	$\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$ (12.6.4)		$\frac{x \cdot x}{x \cdot x \cdot x}$ (12.6.5)
Use the Equivalent Fractions Property	$\frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot 1}$ (12.6.6)		$\frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x}$ (12.6.7)

Simplify.	x^3 (12.6.8)		$\frac{1}{x}$ (12.6.9)

Notice that in each case the bases were the same and we subtracted the exponents.

- When the larger exponent was in the numerator, we were left with factors in the numerator and 1 in the denominator, which we simplified.
- When the larger exponent was in the denominator, we were left with factors in the denominator, and 1 in the numerator, which could not be simplified.

We write:

$$\begin{array}{ll} \frac{x^5}{x^2} & \frac{x^2}{x^3} \\ x^{5-2} & \frac{1}{x^{3-2}} \\ x^3 & \frac{1}{x} \end{array}$$

Definition: Quotient Property of Exponents

If a is a real number, a ≠ 0, and m, n are whole numbers, then

$$\frac{a^m}{a^n} = a^{m-n},\ m > n \quad and \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}},\ n > m \tag{12.6.10}$$

A couple of examples with numbers may help to verify this property.

$$\begin{array}{ll} \frac{3^4}{3^2} \stackrel{?}{=} 3^{4-2} & \frac{5^2}{5^3} \stackrel{?}{=} \frac{1}{5^{3-2}} \\ \frac{81}{9} \stackrel{?}{=} 3^2 & \frac{25}{125} \stackrel{?}{=} \frac{1}{5^1} \\ 9 = 9\ \checkmark & \frac{1}{5} = \frac{1}{5}\ \checkmark \end{array}$$

When we work with numbers and the exponent is less than or equal to 3, we will apply the exponent. When the exponent is greater than 3, we leave the answer in exponential form.

Example 12.6.1:

Simplify: (a) $\frac{x^{10}}{x^8}$ (b) $\frac{2^9}{2^2}$

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

(a)

Since 10 > 8, there are more factors of x in the numerator.	$\frac{x^{10}}{x^8}$	(12.6.11)
Use the quotient property with m > n, $\frac{a^m}{a^n} = a^{m-n}$.	x^{10-8}	(12.6.12)
Simplify.	x^2	(12.6.13)

(b)

Since 9 > 2, there are more factors of 2 in the numerator.	$\frac{2^9}{2^2}$	(12.6.14)
Use the quotient property with m > n, $\frac{a^m}{a^n} = a^{m-n}$.	2^{9-2}	(12.6.15)
Simplify.	2^7	(12.6.16)

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

? Exercise 12.6.1:

Simplify: (a) $\frac{x^{12}}{x^9}$ (b) $\frac{7^{14}}{7^5}$

Answer a

x^3

Answer b

7^9

? Exercise 12.6.2:

Simplify: (a) $\frac{y^{23}}{y^{17}}$ (b) $\frac{8^{15}}{8^7}$

Answer a

y^6

Answer b

8^8

✓ Example 12.6.2:

Simplify: (a) $\frac{b^{10}}{b^{15}}$ (b) $\frac{3^3}{3^5}$

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

(a)

Since 15 > 10, there are more factors of b in the denominator.	$\frac{b^{10}}{b^{15}}$	(12.6.17)
Use the quotient property with n > m, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{b^{15-10}}$	(12.6.18)
Simplify.	$\frac{1}{b^5}$	(12.6.19)

(b)

Since 5 > 3, there are more factors of 3 in the denominator.	$\frac{3^3}{3^5}$	(12.6.20)
Use the quotient property with n > m, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{3^{5-3}}$	(12.6.21)
Simplify.	$\frac{1}{3^2}$	(12.6.22)
Apply the exponent.	$\frac{1}{9}$	(12.6.23)

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator and 1 in the numerator.

? Exercise 12.6.3:

Simplify: (a) $\frac{x^8}{x^{15}}$ (b) $\frac{12^{11}}{12^{21}}$

Answer a

$\frac{1}{x^7}$

Answer b

$\frac{1}{12^10}$

? Exercise 12.6.4:

Simplify: (a) $\frac{m^{17}}{m^{26}}$ (b) $\frac{7^8}{7^{14}}$

Answer a

$\frac{1}{m^9}$

Answer b

$\frac{1}{7^6}$

✓ Example 12.6.3:

Simplify: (a) $\frac{a^5}{a^9}$ (b) $\frac{x^{11}}{x^7}$

Solution

(a)

Since 9 > 5, there are more a's in the denominator and so we will end up with factors in the denominator.	$\frac{a^5}{a^9}$	(12.6.24)

Use the quotient property with n > m, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{a^{9-5}}$	(12.6.25)
Simplify.	$\frac{1}{a^4}$	(12.6.26)

(b)

Notice there are more factors of x in the numerator, since 11 > 7. So we will end up with factors in the numerator.	$\frac{x^{11}}{x^{97}}$	(12.6.27)
Use the quotient property with m > n, $\frac{a^m}{a^n} = a^{m-n}$.	a^{11-7}	(12.6.28)
Simplify.	x^4	(12.6.29)

? Exercise 12.6.5:

Simplify: (a) $\frac{b^{19}}{b^{11}}$ (b) $\frac{z^5}{z^{11}}$

Answer a

b^8

Answer b

$\frac{1}{z^6}$

? Exercise 12.6.6:

Simplify: (a) $\frac{p^9}{p^{17}}$ (b) $\frac{w^{13}}{w^9}$

Answer a

$\frac{1}{p^8}$

Answer b

w^4

Simplify Expressions with Zero Exponents

A special case of the Quotient Property is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. From earlier work with fractions, we know that

$$\frac{2}{2} = 1 \qquad \frac{17}{17} = 1 \qquad \frac{-43}{-43} = 1 \tag{12.6.30}$$

In words, a number divided by itself is 1. So $\frac{x}{x}$ = 1, for any x (x ≠ 0), since any number divided by itself is 1.

The Quotient Property of Exponents shows us how to simplify $\frac{a^m}{a^n}$ when m > n and when n < m by subtracting exponents. What if m = n?

Now we will simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the **zero exponent**. Consider first $\frac{8}{8}$, which we know is 1.

	$\frac{8}{8}=1$	(12.6.31)
Write 8 as 2^3.	$\frac{2^3}{2^3}=1$	(12.6.32)
Subtract exponents.	$2^{3-3}=1$	(12.6.33)
Simplify.	$2^0=1$	(12.6.34)

In general, for $a \neq 0$:

$\frac{a^m}{a^m}$	$\frac{a^m}{a^m}$
a^{m-m}	$\frac{\overbrace{a \cdot a \cdot \ldots \cdot a}^{m \text{ factors}}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}}}$
a^0	1

We see $\frac{a^m}{a^n}$ simplifies to a^0 and to 1. So $a^0 = 1$.

Definition: Zero Exponent

If a is a non-zero number, then $a^0 = 1$. Any nonzero number raised to the zero power is 1.

In this text, we assume any variable that we raise to the zero power is not zero.

Example 12.6.4:

Simplify: (a) 12^0 (b) y^0

Solution

The definition says any non-zero number raised to the zero power is 1.

(a) 12^0

Use the definition of the zero exponent.	1

(b) y^0

Use the definition of the zero exponent.	1

Exercise 12.6.7:

Simplify: (a) 17^0 (b) m^0

Answer a

1

Answer b

1

Exercise 12.6.8:

Simplify: (a) k^0 (b) 29^0

Answer a

1

Answer b

1

Now that we have defined the zero exponent, we can expand all the Properties of Exponents to include whole number exponents.

What about raising an expression to the zero power? Let's look at $(2x)^0$. We can use the product to a power rule to rewrite this expression.

	$(2x)^0$
Use the Product to a Power Rule.	2^0x^0
Use the Zero Exponent Property.	$1 \cdot 1$
Simplify.	1

This tells us that any non-zero expression raised to the zero power is one.

Example 12.6.5:

Simplify: $(7z)^0$.

Solution

Use the definition of the zero exponent.	1

Exercise 12.6.9:

Simplify: $(-4y)^0$.

Answer

1

Exercise 12.6.10:

Simplify: $\left(\frac{2}{3}x\right)^0$.

Answer

1

Example 12.6.6:

Simplify: (a) $(-3x^2y)^0$ (b) $-3x^2y^0$

Solution

(a) $(-3x^2y)^0$

The product is raised to the zero power.	$(-3x^2y)^0$
Use the definition of the zero exponent.	1

(b) $-3x^2y^0$

Notice that only the variable y is being raised to the zero power.	$-3x^2y^0$
Use the definition of the zero exponent.	$-3x^2 \cdot 1$
Simplify.	$-3x^2$

Exercise 12.6.11:

Simplify: (a) $(7x^2y)^0$ (b) $7x^2y^0$

Answer a

1

Answer b

$7x^2$

Exercise 12.6.12:

Simplify: (a) $-23x^2y^0$ (b) $(-23x^2y)^0$

Answer a

$-23x^2$

Answer b

1

Simplify Expressions Using the Quotient to a Power Property

Now we will look at an example that will lead us to the Quotient to a Power Property.

	$\left(\frac{x}{y}\right)^3$	(12.6.35)
This means	$\frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y}$	(12.6.36)
Multiply the fractions.	$\frac{x \cdot x \cdot x}{y \cdot y \cdot y}$	(12.6.37)

Write with exponents.	$\frac{x^3}{y^3}$	(12.6.38)

Notice that the exponent applies to both the numerator and the denominator. We see that $\left(\frac{x}{y}\right)^3$ is $\frac{x^3}{y^3}$. We write:

$$\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3} \tag{12.6.39}$$

This leads to the Quotient to a Power Property for Exponents.

Definition: Quotient to a Power Property of Exponents

If a and b are real numbers, $b \neq 0$, and m is a counting number, then

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \tag{12.6.40}$$

To raise a fraction to a power, raise the numerator and denominator to that power.

An example with numbers may help you understand this property:

$$\left(\frac{2}{3}\right)^3 \stackrel{?}{=} \frac{2^3}{3^3}$$
$$\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} \stackrel{?}{=} \frac{8}{27}$$
$$\frac{8}{27} = \frac{8}{27} \checkmark$$

Example 12.6.7:

Simplify: (a) $\left(\frac{5}{8}\right)^2$ (b) $\left(\frac{x}{3}\right)^4$ (c) $\left(\frac{y}{m}\right)^3$

Solution

(a) $\left(\frac{5}{8}\right)^2$

Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{5^2}{8^2}$	(12.6.41)
Simplify.	$\frac{25}{64}$	(12.6.42)

(b) $\left(\frac{x}{3}\right)^4$

Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{x^4}{3^4}$	(12.6.43)
Simplify.	$\frac{x^4}{81}$	(12.6.44)

(c) $\left(\frac{y}{m}\right)^3$

Raise the numerator and denominator to the third power.	$\frac{y^3}{m^3}$	(12.6.45)

? Exercise 12.6.13:

Simplify: (a) $\left(\frac{7}{9}\right)^2$ (b) $\left(\frac{y}{8}\right)^3$ (c) $\left(\frac{p}{q}\right)^6$

Answer a

$\frac{49}{81}$

Answer b

$\frac{y^3}{512}$

Answer c

$\frac{p^6}{q^6}$

? Exercise 12.6.14:

Simplify: (a) $\left(\frac{1}{8}\right)^2$ (b) $\left(\frac{-5}{m}\right)^3$ (c) $\left(\frac{r}{s}\right)^4$

Answer a

$\frac{1}{64}$

Answer b

$-\frac{125}{m^3}$

Answer c

$\frac{r^4}{s^4}$

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12.7: Divide Monomials (Part 2)

Simplify Expressions by Applying Several Properties

We'll now summarize all the properties of exponents so they are all together to refer to as we simplify expressions using several properties. Notice that they are now defined for whole number exponents.

Summary of Exponent Properties

If a, b are real numbers and m, n are whole numbers, then

Product Property	$a^m \bullet a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \bullet n}$
Product to a Power Property	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0,\ m > n$
	$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad a \neq 0,\ n > m$
Zero Exponent Property	$a^0 = 1, \quad a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$

Example 12.7.8:

Simplify: $\frac{(x^2)^3}{x^5}$.

Solution

Multiply the exponents in the numerator, using the Power Property.	$\frac{x^6}{x^5}$
Subtract the exponents.	x

Exercise 12.7.15:

Simplify: $\frac{(a^4)^5}{a^9}$.

Answer

a^{11}

Exercise 12.7.16:

Simplify: $\frac{(b^5)^6}{b^{11}}$.

Answer

b^{19}

Example 12.7.9:

Simplify: $\frac{(m^8)}{(m^2)^4}$.

Solution

Multiply the exponents in the numerator, using the Power Property.	$\frac{m^8}{m^8}$
Subtract the exponents.	m^0
Zero power property	1

Exercise 12.7.17:

Simplify: $\frac{(k^{11}}{(k^3)^3}$.

Answer

k^2

Exercise 12.7.18:

Simplify: $\frac{(d^{23}}{(d^4)^6}$.

Answer

$\frac{1}{d}$

Example 12.7.10:

Simplify: $\left(\frac{x^7}{x^3}\right)^2$.

Solution

Remember parentheses come before exponents, and the bases are the same so we can simplify inside the parentheses. Subtract the exponents.	$(x^{7-3})^2$
Simplify.	$(x^4)^2$
Multiply the exponents.	x^8

Exercise 12.7.19:

Simplify: $\left(\frac{f^{14}}{f^8}\right)^2$.

Answer

f^{12}

Exercise 12.7.20:

Simplify: $\left(\frac{b^6}{b^{11}}\right)^2$.

Answer

$\frac{1}{b^{10}}$

Example 12.7.11:

Simplify: $\left(\frac{p^2}{q^5}\right)^3$.

Solution

Here we cannot simplify inside the parentheses first, since the bases are not the same.

Raise the numerator and denominator to the third power using the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\frac{(p^2)^3}{(q^5)^3}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$\frac{p^6}{q^{15}}$

Exercise 12.7.21:

Simplify: $\left(\frac{m^3}{n^8}\right)^5$.

Answer

$\frac{m^{15}}{n^{40}}$

Exercise 12.7.22:

Simplify: $\left(\frac{t^{10}}{u^7}\right)^2$.

Answer

$\frac{t^{20}}{u^{14}}$

Example 12.7.12:

Simplify: $\left(\frac{2x^3}{3y}\right)^4$.

Solution

Raise the numerator and denominator to the fourth power using the Quotient to a Power Property.	$\frac{(2x^3)^4}{(3y)^4}$
Raise each factor to the fourth power, using the Power to a Power Property.	$\frac{2^4(x^3)^4}{3^4y^4}$

Use the Power Property and simplify.	$\frac{16x^{12}}{81y^4}$

? Exercise 12.7.23:

Simplify: $\left(\frac{5b}{9c^3}\right)^2$.

Answer

$\frac{25b^2}{81c^6}$

? Exercise 12.7.24:

Simplify: $\left(\frac{4p^4}{7q^5}\right)^3$.

Answer

$\frac{64p^{12}}{343q^{15}}$

✓ Example 12.7.13:

Simplify: $\frac{(y^2)^3(y^2)^4}{(y^5)^4}$.

Solution

Use the Power Property.	$\frac{(y^6)(y^8)}{y^{20}}$
Add the exponents in the numerator, using the Product Property.	$\frac{y^{14}}{y^{20}}$
Use the Quotient Property.	$\frac{1}{y^6}$

? Exercise 12.7.25

Simplify: $\frac{(y^4)^4(y^3)^5}{(y^7)^6}$.

Answer

$\frac{1}{y^{11}}$

? Exercise 12.7.26

Simplify: $\frac{(3x^4)^2(x^3)^4}{(x^5)^3}$.

Answer

$9x^5$

Divide Monomials

We have now seen all the properties of exponents. We'll use them to divide monomials. Later, you'll use them to divide polynomials.

Example 12.7.14:

Find the quotient: $56x^5 \div 7x^2$.

Solution

Rewrite as a fraction.	$\frac{56x^5}{7x^2}$
Use fraction multiplication to separate the number part from the variable part.	$\frac{56}{7} \cdot \frac{x^5}{x^2}$
Use the Quotient Property.	$8x^3$

Exercise 12.7.27:

Find the quotient: $63x^8 \div 9x^4$.

Answer

$7x^4$

Exercise 12.7.28:

Find the quotient: $96y^{11} \div 6y^8$.

Answer

$16y^3$

When we divide monomials with more than one variable, we write one fraction for each variable.

Example 12.7.15:

Find the quotient: $\frac{42x^2y^3}{-7xy^5}$.

Solution

Use fraction multiplication.	$\frac{42}{-7} \cdot \frac{x^2}{x} \cdot \frac{y^3}{y^5}$
Simplify and use the Quotient Property.	$-6 \cdot x \cdot \frac{1}{y^2}$
Multiply.	$-\frac{6x}{y^2}$

Exercise 12.7.29:

Find the quotient: $\frac{-84x^8y^3}{7x^{10}y^2}$.

Answer

$-\frac{12y}{x^2}$

Exercise 12.7.30:

Find the quotient: $\frac{-72a^4b^5}{-8a^9b^5}$.

Answer

$\frac{9}{a^5}$

Example 12.7.16:

Find the quotient: $\frac{24a^5b^3}{48ab^4}$.

Solution

Use fraction multiplication.	$\frac{24}{48} \cdot \frac{a^5}{a} \cdot \frac{b^3}{b^4}$
Simplify and use the Quotient Property.	$\frac{1}{2} \cdot a^4 \cdot \frac{1}{b}$
Multiply.	$\frac{a^4}{2b}$

Exercise 12.7.31:

Find the quotient: $\frac{16a^7b^6}{24ab^8}$.

Answer

$\frac{2a^6}{3b^2}$

Exercise 12.7.32:

Find the quotient: $\frac{27p^4q^7}{-45p^{12}q}$.

Answer

$-\frac{3q^6}{5p^8}$

Once you become familiar with the process and have practiced it step by step several times, you may be able to simplify a fraction in one step.

Example 12.7.17:

Find the quotient: $\frac{14x^7y^{12}}{21x^{11}y^6}$.

Solution

Simplify and use the Quotient Property.	$\frac{2y^6}{3x^4}$

Be very careful to simplify $\frac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

Exercise 12.7.33:

Find the quotient: $\frac{28x^5y^{14}}{49x^9y^{12}}$.

Answer

$\frac{4y^2}{7x^4}$

Exercise 12.7.34:

Find the quotient: $\frac{30m^5n^{11}}{48m^{10}n^{14}}$.

Answer

$\frac{5}{8m^5n^3}$

In all examples so far, there was no work to do in the numerator or denominator before simplifying the fraction. In the next example, we'll first find the product of two monomials in the numerator before we simplify the fraction.

Example 12.7.18:

Find the quotient: $\frac{(3x^3y^2)(10x^2y^3)}{6x^4y^5}$.

Solution

Remember, the fraction bar is a grouping symbol. We will simplify the numerator first.

Simplify the numerator.	$\frac{30x^5y^5}{6x^4y^5}$
Simplify, using the Quotient Rule.	$5x$

Exercise 12.7.35:

Find the quotient: $\frac{(3x^4y^5)(8x^2y^5)}{12x^5y^8}$.

Answer

$2xy^2$

Exercise 12.7.36:

Find the quotient: $\frac{(-6a^6b^9)(-8a^5b^8)}{-12a^{10}b^{12}}$.

Answer

$-4ab^5$

ACCESS ADDITIONAL ONLINE RESOURCES

Simplify a Quotient

Zero Exponent

Quotient Rule

Polynomial Division

Polynomial Division 2

Practice Makes Perfect

Simplify Expressions Using the Quotient Property of Exponents

In the following exercises, simplify.

219. $\frac{4^8}{4^2}$
220. $\frac{3^{12}}{3^4}$
221. $\frac{x^{12}}{x^3}$
222. $\frac{u^9}{u^3}$
223. $\frac{r^5}{r}$
224. $\frac{y^4}{y}$
225. $\frac{y^4}{y^{20}}$
226. $\frac{x^{10}}{x^{30}}$
227. $\frac{10^3}{10^{15}}$
228. $\frac{r^2}{r^8}$
229. $\frac{a}{a^9}$
230. $\frac{2}{2^5}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.

231. 5^0
232. 10^0
233. a^0
234. x^0
235. -7^0
236. -4^0
237. (a) $(10p)^0$ (b) $10p^0$
238. (a) $(3a)^0$ (b) $3a^0$
239. (a) $(-27x^5y)^0$ (b) $-27x^5y^0$
240. (a) $(-92y^8z)^0$ (b) $-92y^8z^0$
241. (a) 15^0 (b) 15^1
242. (a) -6^0 (b) -6^1
243. $2 \cdot x^0 + 5 \cdot y^0$

244. $8 \cdot m^0 - 4 \cdot n^0$

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

245. $\left(\frac{3}{2}\right)^5$

246. $\left(\frac{4}{5}\right)^3$

247. $\left(\frac{m}{6}\right)^3$

248. $\left(\frac{p}{2}\right)^5$

249. $\left(\frac{x}{y}\right)^{10}$

250. $\left(\frac{a}{b}\right)^8$

251. $\left(\frac{a}{3b}\right)^2$

252. $\left(\frac{2x}{y}\right)^4$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

253. $\frac{(x^2)^4}{x^5}$

254. $\frac{(y^4)^3}{y^7}$

255. $\frac{(u^3)^4}{u^{10}}$

256. $\frac{(y^2)^5}{y^6}$

257. $\frac{y^8}{(y^5)^2}$

258. $\frac{p^{11}}{(p^5)^3}$

259. $\frac{r^5}{(r^4 \cdot r}$

260. $\frac{a^3 \cdot a^4}{(a^7}$

261. $\left(\frac{x^2}{x^8}\right)^3$

262. $\left(\frac{u}{u^{10}}\right)^2$

263. $\left(\frac{a^4 \cdot a^6}{a^3}\right)^2$

264. $\left(\frac{x^{3 \cdot x^8}}{x^4}\right)^3$

265. $\frac{(y^3)^5}{(y^4)^3}$

266. $\frac{(z^6)^2}{(z^2)^4}$

267. $\frac{(x^3)^6}{(x^4)^7}$
268. $\frac{(x^4)^8}{(x^5)^7}$
269. $\left(\frac{2r^3}{5s}\right)^4$
270. $\left(\frac{3m^2}{4n}\right)^3$
271. $\left(\frac{3y^2 \cdot y^5}{y^{15} \cdot y^8}\right)^0$
272. $\left(\frac{15z^4 \cdot z^9}{0.3z^2}\right)^0$
273. $\frac{(r^2)^5(r^4)^2}{(r^3)^7}$
274. $\frac{(p^4)^2(p^3)^5}{(p^2)^9}$
275. $\frac{(3x^4)^3(2x^3)^2}{(6x^5)^2}$
276. $\frac{(-2y^3)^4(3y^4)^2}{(-6y^3)^2}$

Divide Monomials

In the following exercises, divide the monomials.

277. $48b^8 \div 6b^2$
278. $42a^{14} \div 6a^2$
279. $36x^3 \div (-2x^9)$
280. $20u^8 \div (-4u^6)$
281. $\frac{18x^3}{9x^2}$
282. $\frac{36y^9}{4y^7}$
283. $\frac{-35x^7}{-42x^{13}}$
284. $\frac{18x^5}{-27x^9}$
285. $\frac{18r^5s}{3r^3s^9}$
286. $\frac{24p^7q}{6p^2q^5}$
287. $\frac{8mn^{10}}{64mn^4}$
288. $\frac{10a^4b}{50a^2b^6}$
289. $\frac{-12x^4y^9}{15x^6y^3}$
290. $\frac{48x^{11}y^9z^3}{36x^6y^8z^5}$
291. $\frac{64x^5y^9z^7}{48x^7y^{12}z^6}$
292. $\frac{(10u^2v)(4u^3v^6)}{5u^9v^2}$

293. $\frac{(6m^2n)(5m^4n^3)}{3m^{10}n^2}$
294. $\frac{(6a^4b^3)(4ab^5)}{(12a^8b)(a^3b)}$
295. $\frac{(4u^5v^4)(15u^8v)}{(12u^3v)(u^6v)}$

Mixed Practice

296. (a) $24a^5 + 2a^5$ (b) $24a^5 - 2a^5$ (c) $24a^5 \cdot 2a^5$ (d) $24a^5 \div 2a^5$
297. (a) $15n^{10} + 3n^{10}$ (b) $15n^{10} - 3n^{10}$ (c) $15n^{10} \cdot 3n^{10}$ (d) $15n^{10} \div 3n^{10}$
298. (a) $p^4 \cdot p^6$ (b) $(p^4)^6$
299. (a) $q^5 \cdot q^3$ (b) $(q^5)^3$
300. (a) $\frac{y^3}{y}$ (b) $\frac{y}{y^3}$
301. (a) $\frac{z^6}{z^5}$ (b) $\frac{z^5}{z^6}$
302. $(8x^5)(9x) \div 6x^3$
303. $(4y^5)(12y^7) \div 8y^2$
304. $\frac{27a^7}{3a^3} + \frac{54a^9}{9a^5}$
305. $\frac{32c^{11}}{4c^5} + \frac{42c^9}{6c^3}$
306. $\frac{32y^5}{8y^2} - \frac{60y^{10}}{5y^7}$
307. $\frac{48x^6}{6x^4} - \frac{35x^9}{7x^7}$
308. $\frac{63r^6s^3}{9r^4s^2} - \frac{72r^2s^2}{6s}$
309. $\frac{56y^4z^5}{7y^3z^3} - \frac{45y^2z^2}{5y}$

Everyday Math

310. **Memory** One megabyte is approximately 10^6 bytes. One gigabyte is approximately 10^9 bytes. How many megabytes are in one gigabyte?
311. **Memory** One megabyte is approximately 10^6 bytes. One terabyte is approximately 10^{12} bytes. How many megabytes are in one terabyte?

Writing Exercises

312. Vic thinks the quotient $\frac{x^{20}}{x^4}$ simplifies to x^5. What is wrong with his reasoning?
313. Mai simplifies the quotient $\frac{y^3}{y}$ by writing $\frac{y^3}{y} = 3$. What is wrong with her reasoning?
314. When Dimple simplified -3^0 and $(-3)^0$ she got the same answer. Explain how using the Order of Operations correctly gives different answers.
315. Roxie thinks n^0 simplifies to 0. What would you say to convince Roxie she is wrong?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions using the Quotient Property for Exponents.			
simplify expressions with zero exponents.			
simplify expressions using the Quotient to a Power Property.			
simplify expressions by applying several properties.			
divide monomials.			

(b) On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Contributors and Attributions

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12.8: Integer Exponents and Scientific Notation (Part 1)

Learning Objectives

- Use the definition of a negative exponent
- Simplify expressions with integer exponents
- Convert from decimal notation to scientific notation
- Convert scientific notation to decimal form
- Multiply and divide using scientific notation

be prepared!

Before you get started, take this readiness quiz.

1. What is the place value of the 6 in the number 64,891? If you missed this problem, review Example 1.1.3.
2. Name the decimal 0.0012. If you missed this problem, review Exercise 5.1.1.
3. Subtract: 5 − (−3). If you missed this problem, review Example 3.5.8.

Use the Definition of a Negative Exponent

The Quotient Property of Exponents, introduced in Divide Monomials, had two forms depending on whether the exponent in the numerator or denominator was larger.

Definition: Quotient Property of Exponents

If a is a real number, a ≠ 0, and m, n are whole numbers, then

$$\frac{a^m}{a^n} = a^{m-n},\ m > n \quad and \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}},\ n > m \tag{12.8.1}$$

What if we just subtract exponents, regardless of which is larger? Let's consider $\frac{x^2}{x^5}$. We subtract the exponent in the denominator from the exponent in the numerator.

$$\frac{x^2}{x^5}$$
$$x^{2-5}$$
$$x^{-3}$$

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors: $\frac{x^2}{x^5}$.

$$\frac{\cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}$$
$$\frac{1}{x^3}$$

This implies that $x^{-3} = \frac{1}{x^3}$ and it leads us to the definition of a **negative exponent**.

Definition: negative exponent

If n is a positive integer and a ≠ 0, then $a^{-n} = \frac{1}{a^n}$.

The negative exponent tells us to re-write the expression by taking the reciprocal of the base and then changing the sign of the exponent. Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write an expression with only positive exponents.

Example 12.8.1:

Simplify: (a) 4^{-2} (b) 10^{-3}

Solution

(a) 4^{-2}

Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{4^2}$	(12.8.2)
Simplify.	$\frac{1}{16}$	(12.8.3)

(b) 10^{-3}

Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{10^3}$	(12.8.4)
Simplify.	$\frac{1}{1000}$	(12.8.5)

Exercise 12.8.1:

Simplify: (a) 2^{-3} (b) 10^{-2}

Answer a

$\frac{1}{8}$

Answer b

$\frac{1}{100}$

Exercise 12.8.2:

Simplify: (a) 3^{-2} (b) 10^{-4}

Answer a

$\frac{1}{9}$

Answer b

$\frac{1}{10,000}$

When simplifying any expression with exponents, we must be careful to correctly identify the base that is raised to each exponent.

Example 12.8.2:

Simplify: (a) $(-3)^{-2}$ (b) -3^{-2}

Solution

The negative in the exponent does not affect the sign of the base.

a) $(-3)^{-2}$

The exponent applies to the base, −3.	$(-3)^{-2}$	(12.8.6)
Take the reciprocal of the base and change the sign of the exponent.	$\frac{1}{(-3)^2}$	(12.8.7)
Simplify.	$\frac{1}{9}$	(12.8.8)

(b) -3^{-2}

The expression -3^{-2} means "find the opposite of 3^{-2}. The exponent applies only to the base, 3.	-3^{-2}	(12.8.9)
Rewrite as a product with −1.	$-1 \cdot 3^{-2}$	(12.8.10)
Take the reciprocal of the base and change the sign of the exponent.	$-1 \cdot \frac{1}{3^2}$	(12.8.11)
Simplify.	$-\frac{1}{9}$	(12.8.12)

? Exercise 12.8.3:

Simplify: (a) $(-5)^{-2}$ (b) -5^{-2}

Answer a

$\frac{1}{25}$

Answer b

$-\frac{1}{25}$

? Exercise 12.8.4:

Simplify: (a) $(-2)^{-2}$ (b) -2^{-2}

Answer a

$\frac{1}{4}$

Answer b

$-\frac{1}{4}$

We must be careful to follow the order of operations. In the next example, parts (a) and (b) look similar, but we get different results.

✓ Example 12.8.3:

Simplify: (a) $4 \cdot 2^{-1}$ (b) $(4 \cdot 2)^{-1}$

Solution

Remember to always follow the order of operations.

(a) $4 \cdot 2^{-1}$

Do exponents before multiplication.	$4 \cdot 2^{-1}$	(12.8.13)
Use $a^{-n} = \frac{1}{a^n}$.	$4 \cdot \frac{1}{2^1}$	(12.8.14)
Simplify.	2	(12.8.15)

(b) $(4 \cdot 2)^{-1}$

Simplify inside the parentheses first.	$(8)^{-1}$	(12.8.16)
Use $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{8^1}$	(12.8.17)
Simplify.	$\frac{1}{8}$	(12.8.18)

? Exercise 12.8.5:

Simplify: (a) $6 \cdot 3^{-1}$ (b) $(6 \cdot 3)^{-1}$

Answer a

2

Answer b

$\frac{1}{18}$

? Exercise 12.8.6:

Simplify: (a) $8 \cdot 2^{-2}$ (b) $(8 \cdot 2)^{-2}$

Answer a

2

Answer b

$\frac{1}{256}$

When a variable is raised to a negative exponent, we apply the definition the same way we did with numbers.

✓ Example 12.8.4:

Simplify: x^{-6}.

Solution

Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{x^6}$	(12.8.19)

Exercise 12.8.7:

Simplify: y^{-7}.

Answer

$\frac{1}{y^7}$

Exercise 12.8.8:

Simplify: z^{-8}.

Answer

$\frac{1}{z^8}$

When there is a product and an exponent we have to be careful to apply the exponent to the correct quantity. According to the order of operations, expressions in parentheses are simplified before exponents are applied. We'll see how this works in the next example.

Example 12.8.5:

Simplify: (a) $5y^{-1}$ (b) $(5y)^{-1}$ (c) $(-5y)^{-1}$

Solution

(a) $5y^{-1}$

Notice the exponent applies to just the base y .	$5y^{-1}$	(12.8.20)
Take the reciprocal of y and change the sign of the exponent.	$5 \cdot \frac{1}{y^1}$	(12.8.21)
Simplify.	$\frac{5}{y}$	(12.8.22)

(b) $(5y)^{-1}$

Here the parentheses make the exponent apply to the base 5y.	$(5y)^{-1}$	(12.8.23)
Take the reciprocal of 5y and change the sign of the exponent.	$\frac{1}{(5y)^1}$	(12.8.24)
Simplify.	$\frac{1}{5y}$	(12.8.25)

(c) $(-5y)^{-1}$

The base is $-5y$. Take the reciprocal of $-5y$ and change the sign of the exponent.	$\frac{1}{(-5y)^1}$	(12.8.26)

Simplify.	$\frac{1}{-5y}$	(12.8.27)
Use $\frac{a}{-b}=-\frac{a}{b}$.	$-\frac{1}{5y}$	(12.8.28)

Exercise 12.8.9:

Simplify: (a) $8p^{-1}$ (b) $(8p)^{-1}$ (c) $(-8p)^{-1}$

Answer a

$\frac{8}{p}$

Answer b

$\frac{1}{8p}$

Answer c

$-\frac{1}{8p}$

Exercise 12.8.10:

Simplify: (a) $11q^{-1}$ (b) $(11q)^{-1}$ (c) $(-11q)^{-1}$

Answer a

$\frac{11}{q}$

Answer b

$\frac{1}{11q}$

Answer c

$-\frac{1}{11q}$

Now that we have defined negative exponents, the Quotient Property of Exponents needs only one form, $\frac{a^m}{a^n}=a^{m-n}$, where a ≠ 0 and m and n are integers.

When the exponent in the denominator is larger than the exponent in the numerator, the exponent of the quotient will be negative. If the result gives us a negative exponent, we will rewrite it by using the definition of negative exponents, $a^{-n}=\frac{1}{a^n}$.

Simplify Expressions with Integer Exponents

All the exponent properties we developed earlier in this chapter with whole number exponents apply to integer exponents, too. We restate them here for reference.

Summary of Exponent Properties

If a, b are real numbers and m, n are integers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$

Product to a Power Property	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}$, $a \neq 0$, $m > n$
	$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$, $a \neq 0$, $n > m$
Zero Exponent Property	$a^0 = 1$, $a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, $b \neq 0$
Definition of a Negative Exponent	$a^{-n} = \frac{1}{a^n}$

Example 12.8.6:

Simplify: (a) $x^{-4} \cdot x^6$ (b) $y^{-6} \cdot y^4$ (c) $z^{-5} \cdot z^{-3}$

Solution

(a) $x^{-4} \cdot x^6$

Use the Product Property, $a^m \cdot a^n = a^{m+n}$.	x^{-4+6}	(12.8.29)
Simplify.	x^2	(12.8.30)

(b) $y^{-6} \cdot y^4$

The bases are the same, so add the exponents.	y^{-6+4}	(12.8.31)
Simplify.	y^{-2}	(12.8.32)
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{y^2}$	(12.8.33)

(c) $z^{-5} \cdot z^{-3}$

The bases are the same, so add the exponents.	z^{-5-3}	(12.8.34)
Simplify.	z^{-8}	(12.8.35)
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{z^8}$	(12.8.36)

Exercise 12.8.11:

Simplify: (a) $x^{-3} \cdot x^7$ (b) $y^{-7} \cdot y^2$ (c) $z^{-4} \cdot z^{-5}$

Answer a

x^4

Answer b

$\frac{1}{y^5}$

Answer c

$\frac{1}{z^9}$

Exercise 12.8.12:

Simplify: (a) $a^{-1} \cdot a^6$ (b) $b^{-6} \cdot b^4$ (c) $c^{-8} \cdot c^{-7}$

Answer a

a^5

Answer b

$\frac{1}{b^4}$

Answer c

$\frac{1}{c^{15}}$

In the next two examples, we'll start by using the Commutative Property to group the same variables together. This makes it easier to identify the like bases before using the Product Property of Exponents.

Example 12.8.7:

Simplify: $(m^4n^{-3})(m^{-5}n^{-2})$.

Solution

Use the Commutative Property to get like bases together.	$m^4m^{-5} \cdot n^{-2}n^{-3}$	(12.8.37)
Add the exponents for each base.	$m^{-1} \cdot n^{-5}$	(12.8.38)
Take reciprocals and change the signs of the exponents.	$\frac{1}{m^1} \cdot \frac{1}{n^5}$	(12.8.39)
Simplify.	$\frac{1}{mn^5}$	(12.8.40)

Exercise 12.8.13:

Simplify: $(p^6q^{-2})(p^{-9}q^{-1})$.

Answer

$\frac{1}{p^3q^3}$

Exercise 12.8.14:

Simplify: $(r^5s^{-3})(r^{-7}s^{-5})$.

Answer

$\frac{1}{r^2 s^8}$

If the monomials have numerical coefficients, we multiply the coefficients, just as we did in Use Multiplication Properties of Exponents.

Example 12.8.8:

Simplify: $(2x^{-6}y^8)(-5x^5\,y^{-3})$.

Solution

Rewrite with the like bases together.	$2(-5)\cdot(x^{-6}x^5)\cdot(y^8y^{-3})$	(12.8.41)
Simplify.	$-10\cdot x^{-1}\cdot y^5$	(12.8.42)
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$-10\cdot\frac{1}{x^1}\cdot y^5$	(12.8.43)
Simplify.	$\frac{-10y^5}{x}$	(12.8.44)

Exercise 12.8.15:

Simplify: $(3u^{-5}v^7)(-4u^4v^{-2})$.

Answer

$-\frac{12v^5}{u}$

Exercise 12.8.16:

Simplify: $(-6c^{-6}d^4)(-5c^{-2}d^{-1})$.

Answer

$\frac{30d^3}{c^8}$

In the next two examples, we'll use the Power Property and the Product to a Power Property.

Example 12.8.9:

Simplify: $(k^3)^{-2}$.

Solution

Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$k^{3(-2)}$	(12.8.45)
Simplify.	k^{-6}	(12.8.46)
Rewrite with a positive exponent.	$\frac{1}{k^6}$	(12.8.47)

Exercise 12.8.17:

Simplify: $(x^4)^{-1}$.

Answer

$\frac{1}{x^4}$

Exercise 12.8.18:

Simplify: $(y^2)^{-2}$.

Answer

$\frac{1}{y^4}$

Example 12.8.10:

Simplify: $(5x^{-3})^2$.

Solution

Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$5^2(x^{-3})^2$	(12.8.48)
Simplify 5^2 and multiply the exponents of x using the Power Property, $(a^m)^n = a^{m \cdot n}$.	$25k^{-6}$	(12.8.49)
Rewrite x^{-6} by using the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$25 \cdot \frac{1}{x^6}$	(12.8.50)
Simplify.	$\frac{25}{x^6}$	(12.8.51)

Exercise 12.8.19:

Simplify: $(8a^{-4})^2$.

Answer

$\frac{64}{a^8}$

Exercise 12.8.20:

Simplify: $(2c^{-4})^3$.

Answer

$\frac{8}{c^12}$

To simplify a fraction, we use the Quotient Property.

Example 12.8.11:

Simplify: $\frac{r^5}{r^{-4}}$.

Solution

Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	$r^{5-(-4)}$	(12.8.52)
Be careful to subtract 5 - (\textcolor{red}{-4}).		
Simplify.	r^9	(12.8.53)

Exercise 12.8.21:

Simplify: $\frac{x^8}{x^{-3}}$.

Answer

x^{11}

Exercise 12.8.22:

Simplify: $\frac{y^7}{y^{-6}}$.

Answer

y^{13}

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12.9: Integer Exponents and Scientific Notation (Part 2)

Convert from Decimal Notation to Scientific Notation

Remember working with place value for whole numbers and decimals? Our number system is based on powers of 10. We use tens, hundreds, thousands, and so on. Our decimal numbers are also based on powers of tens—tenths, hundredths, thousandths, and so on.

Consider the numbers 4000 and 0.004. We know that 4000 means 4×1000 and 0.004 means $4 \times \frac{1}{1000}$. If we write the 1000 as a power of ten in exponential form, we can rewrite these numbers in this way:

4000	0.004
4×1000	$4 \times \frac{1}{1000}$
4×10^3	$4 \times \frac{1}{10^3}$
	4×10^{-3}

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than 10, and the second factor is a power of 10 written in exponential form, it is said to be in *scientific notation*.

Definition: Scientific Notation

A number is expressed in scientific notation when it is of the form $a \times 10^n$ where $a \geq 1$ and $a < 10$ and n is an integer.

It is customary in scientific notation to use × as the multiplication sign, even though we avoid using this sign elsewhere in algebra.

Scientific notation is a useful way of writing very large or very small numbers. It is used often in the sciences to make calculations easier.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.

$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
Moved the decimal point 3 places to the left.	Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor, 4, by itself.

- The power of 10 is positive when the number is larger than 1: $4000 = 4 \times 10^3$.
- The power of 10 is negative when the number is between 0 and 1: $0.004 = 4 \times 10^{-3}$.

Example 12.9.12:

Write 37,000 in scientific notation.

Solution

Step 1: Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.	37000.
Step 2: Count the number of decimal places, n, that the decimal point was moved.	3.70000 4 places
Step 3: Write the number as a product with a power of 10.	3.7×10^4

If the original number is: • greater than 1, the power of 10 will be 10^n • between 0 and 1, the power of 10 will be 10^{-n}	
Step 4: Check.10^4 is 10,000 and 10,000 times 3.7 will be 37,000.	$37{,}000 = 3.7\times10^4$

? Exercise 12.9.23:

Write in scientific notation: 96,000.

Answer

9.6×10^4

? Exercise 12.9.24:

Write in scientific notation: 48,300.

Answer

4.83×10^4

HOW TO: CONVERT FROM DECIMAL NOTATION TO SCIENTIFIC NOTATION

Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.

Step 2. Count the number of decimal places, n, that the decimal point was moved.

Step 3. Write the number as a product with a power of 10.

If the original number is:

- greater than 1, the power of 10 will be 10^n.
- between 0 and 1, the power of 10 will be 10^{-n}.

Step 4. Check.

✓ Example 12.9.13:

Write in scientific notation: 0.0052.

Solution

Move the decimal point to get 5.2, a number between 1 and 10.	0.0052
Count the number of decimal places the point was moved.	3 places
Write as a product with a power of 10.	5.2×10^{-3}
Check your answer:	5.2×10^{-3} $5.2 \times \frac{1}{10^3}$ $5.2 \times \frac{1}{1000}$ 5.2×0.001 0.0052
	$0.0052 = 5.2 \times 10^{-3}$

Exercise 12.9.25:

Write in scientific notation: 0.0078.

Answer

7.8×10^{-3}

Exercise 12.9.26:

Write in scientific notation: 0.0129.

Answer

1.29×10^{-2}

Convert Scientific Notation to Decimal Form

How can we convert from scientific notation to decimal form? Let's look at two numbers written in scientific notation and see.

9.12×10^{4}	9.12×10^{-4}
$9.12 \times 10,000$	9.12×0.0001
$91,200$	0.000912

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.

$9.12 \times 10^{4} = 91,200$	$9.12 \times 10^{-4} = 0.000912$
$9.12\_\_\_ \times 10^{4} = 91,200$	$\_\_\_9.12 \times 10^{-4} = 0.000912$

In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

Example 12.9.14:

Convert to decimal form: 6.2×10^{3}.

Solution

Step 1: Determine the exponent, n, on the factor 10.	6.2×10^{3}
Step 2: Move the decimal point n places, adding zeros if needed.	6,200.
• If the exponent is positive, move the decimal point n places to the right. • If the exponent is negative, move the decimal point |n| places to the left.	6,200
Step 3: Check to see if your answer makes sense.	
10^{3} is 1000 and 1000 times 6.2 will be 6,200.	$6.2 \times 10^{3} = 6,200$

Exercise 12.9.27:

Convert to decimal form: 1.3×10^{3}.

Answer

1,300

Exercise 12.9.28:

Convert to decimal form: 9.25×10^4.

Answer

92,500

HOW TO: CONVERT SCIENTIFIC NOTATION TO DECIMAL FORM

Step 1. Determine the exponent, n, on the factor 10.

Step 2. Move the decimal n places, adding zeros if needed.

- If the exponent is positive, move the decimal point n places to the right.
- If the exponent is negative, move the decimal point |n| places to the left.

Step 3. Check.

Example 12.9.15:

Convert to decimal form: 8.9×10^{-2}.

Solution

Determine the exponent n, on the factor 10.	The exponent is -2.
Move the decimal point 2 places to the left.	$\underbrace{.8.9}$
Add zeros as needed for placeholders.	0.089
	$8.9 \times 10^{-2} = 0.089$
The Check is left to you.	

Exercise 12.9.29:

Convert to decimal form: 1.2×10^{-4}.

Answer

0.00012

Exercise 12.9.30:

Convert to decimal form: 7.5×10^{-2}.

Answer

0.075

Multiply and Divide Using Scientific Notation

We use the Properties of Exponents to multiply and divide numbers in scientific notation.

Example 12.9.16:

Multiply. Write answers in decimal form: $(4 \times 10^5)(2 \times 10^{-7})$.

Solution

Use the Commutative Property to rearrange the factors.	$4 \cdot 2 \cdot 10^5 \cdot 10^{-7}$
Multiply 4 by 2 and use the Product Property to multiply 10^5 by 10^{-7}.	8×10^{-2}
Change to decimal form by moving the decimal two places left.	0.08

Exercise 12.9.31:

Multiply. Write answers in decimal form: $(3 \times 10^6)(2 \times 10^{-8})$.

Answer

0.06

Exercise 12.9.32:

Multiply. Write answers in decimal form: $(3 \times 10^{-2})(3 \times 10^{-1})$.

Answer

0.009

Example 12.9.17:

Divide. Write answers in decimal form: $\dfrac{9 \times 10^3}{3 \times 10^{-2}}$.

Solution

Separate the factors.	$\dfrac{9}{3} \times \dfrac{10^3}{10^{-2}}$	(12.9.1)
Divide 9 by 3 and use the Quotient Property to divide 10^3 by 10^{-2}.	3×10^5	
Change to decimal form by moving the decimal five places right.	300,000	

Exercise 12.9.33:

Divide. Write answers in decimal form:\dfrac{8 \times 10^{4}}{2 \times 10^{-1}}.

Answer

400,000

Exercise 12.9.34:

Divide. Write answers in decimal form:\dfrac{8 \times 10^{2}}{4 \times 10^{-2}}.

Answer

20,000

ACCESS ADDITIONAL ONLINE RESOURCES

Negative Exponents

Examples of Simplifying Expressions with Negative Exponents

Scientific Notation

Practice Makes Perfect

Use the Definition of a Negative Exponent

In the following exercises, simplify.

316. 5^{-3}
317. 8^{-2}
318. 3^{-4}
319. 2^{-5}
320. 7^{-1}
321. 10^{-1}
322. $2^{-3} + 2^{-2}$
323. $3^{-2} + 3^{-1}$
324. $3^{-1} + 4^{-1}$
325. $10^{-1} + 2^{-1}$
326. $10^{0} - 10^{-1} + 10^{-2}$
327. $2^{0} - 2^{-1} + 2^{-2}$
328. (a) $(-6)^{-2}$ (b) -6^{-2}
329. (a) $(-8)^{-2}$ (b) -8^{-2}
330. (a) $(-10)^{-4}$ (b) -10^{-4}
331. (a) $(-4)^{-6}$ (b) -4^{-6}
332. (a) $5 \cdot 2^{-1}$ (b) $(5 \cdot 2)^{-1}$
333. (a) $10 \cdot 3^{-1}$ (b) $(10 \cdot 3)^{-1}$
334. (a) $4 \cdot 10^{-3}$ (b) $(4 \cdot 10)^{-3}$
335. (a) $3 \cdot 5^{-2}$ (b) $(3 \cdot 5)^{-2}$
336. n^{-4}
337. p^{-3}
338. c^{-10}
339. m^{-5}
340. (a) $4x^{-1}$ (b) $(4x)^{-1}$ (c) $(-4x)^{-1}$
341. (a) $3q^{-1}$ (b) $(3q)^{-1}$ (c) $(-3q)^{-1}$
342. (a) $6m^{-1}$ (b) $(6m)^{-1}$ (c) $(-6m)^{-1}$
343. (a) $10k^{-1}$ (b) $(10k)^{-1}$ (c) $(-10k)^{-1}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

344. $p^{-4} \cdot p^{8}$
345. $r^{-2} \cdot r^{5}$
346. $n^{-10} \cdot n^{2}$

347. $q^{-8} \cdot q^{3}$
348. $k^{-3} \cdot k^{-2}$
349. $z^{-6} \cdot z^{-2}$
350. $a \cdot a^{-4}$
351. $m \cdot m^{-2}$
352. $p^{5} \cdot p^{-2} \cdot p^{-4}$
353. $x^{4} \cdot x^{-2} \cdot x^{-3}$
354. $a^{3}b^{-3}$
355. $u^{2}\, v^{-2}$
356. $(x^{5}y^{-1})(x^{-10}\, y^{-3})$
357. $(a^{3}b^{-3})(a^{-5}b^{-1})$
358. $(uv^{-2})(u^{-5}v^{-4})$
359. $(pq^{-4})(p^{-6}q^{-3})$
360. $(-2r^{-3}s^{9})(6r^{4}s^{-5})$
361. $(-3p^{-5}q^{8})(7p^{2}q^{-3})$
362. $(-6m^{-8}n^{-5})(-9m^{4}n^{2})$
363. $(-8a^{-5}b^{-4})(-4a^{2}b^{3})$
364. $(a^{3})^{-3}$
365. $(q^{10})^{-10}$
366. $(n^{2})^{-1}$
367. $(x^{4})^{-1}$
368. $(y^{-5})^{4}$
369. $(p^{-3})^{2}$
370. $(q^{-5})^{-2}$
371. $(m^{-2})^{-3}$
372. $(4y^{-3})^{2}$
373. $(3q^{-5})^{2}$
374. $(10p^{-2})^{-5}$
375. $(2n^{-3})^{-6}$
376. $u^{9}u^{-2}$
377. $b^{5}b^{-3}$
378. $x^{-6}x^{4}$
379. $m^{5}m^{-2}$
380. $q^{3}q^{12}$
381. $r^{6}r^{9}$
382. $n^{-4}n^{-10}$
383. $p^{-3}p^{-6}$

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

384. 45,000
385. 280,000
386. 8,750,000
387. 1,290,000
388. 0.036
389. 0.041
390. 0.00000924
391. 0.0000103
392. The population of the United States on July 4, 2010 was almost 310,000,000.
393. The population of the world on July 4, 2010 was more than 6,850,000,000.
394. The average width of a human hair is 0.0018 centimeters.
395. The probability of winning the 2010 Megamillions lottery is about 0.0000000057.

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

396. 4.1×10^{2}
397. 8.3×10^{2}
398. 5.5×10^{8}
399. 1.6×10^{10}
400. 3.5×10^{-2}
401. 2.8×10^{-2}
402. $1.93 \times 10{-}5$
403. $6.15 \times 10{-}8$
404. In 2010, the number of Facebook users each day who changed their status to 'engaged' was 2×10^{4}.
405. At the start of 2012, the US federal budget had a deficit of more than $\$1.5 \times 10^{13}$.
406. The concentration of carbon dioxide in the atmosphere is 3.9×10^{-4}.
407. The width of a proton is 1×10^{-5} of the width of an atom.

Multiply and Divide Using Scientific Notation

In the following exercises, multiply or divide and write your answer in decimal form.

408. $(2 \times 10^{5})(2 \times 10^{-9})$
409. $(3 \times 10^{2})(1 \times 10^{-5})$
410. $(1.6 \times 10^{-2})(5.2 \times 10^{-6})$
411. $(2.1 \times 10^{-4})(3.5 \times 10^{-2})$
412. $\dfrac{6 \times 10^{4}}{3 \times 10^{-2}}$
413. $\dfrac{8 \times 10^{6}}{4 \times 10^{-1}}$
414. $\dfrac{7 \times 10^{-2}}{1 \times 10^{-8}}$
415. $\dfrac{5 \times 10^{-3}}{1 \times 10^{-10}}$

Everyday Math

416. **Calories** In May 2010 the Food and Beverage Manufacturers pledged to reduce their products by 1.5 trillion calories by the end of 2015.
    1. Write 1.5 trillion in decimal notation.
    2. Write 1.5 trillion in scientific notation.
417. **Length of a year** The difference between the calendar year and the astronomical year is 0.000125 day.
    1. Write this number in scientific notation.
    2. How many years does it take for the difference to become 1 day?
418. **Calculator display** Many calculators automatically show answers in scientific notation if there are more digits than can fit in the calculator's display. To find the probability of getting a particular 5-card hand from a deck of cards, Mario divided 1 by 2,598,960 and saw the answer 3.848×10^{-7}. Write the number in decimal notation.
419. **Calculator display** Many calculators automatically show answers in scientific notation if there are more digits than can fit in the calculator's display. To find the number of ways Barbara could make a collage with 6 of her 50 favorite photographs, she multiplied $50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45$. Her calculator gave the answer 1.1441304×10^{10}. Write the number in decimal notation.

Writing Exercises

420. (a) Explain the meaning of the exponent in the expression 2^{3}. (b) Explain the meaning of the exponent in the expression 2^{-3}.
421. When you convert a number from decimal notation to scientific notation, how do you know if the exponent will be positive or negative?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the definition of a negative exponent.			
simplify expressions with integer exponents.			
convert from decimal notation to scientific notation.			
convert scientific notation to decimal form.			
multiply and divide using scientific notation.			

(b) After looking at the checklist, do you think you are well prepared for the next section? Why or why not?

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12.10: Introduction to Factoring Polynomials

Learning Objectives

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial

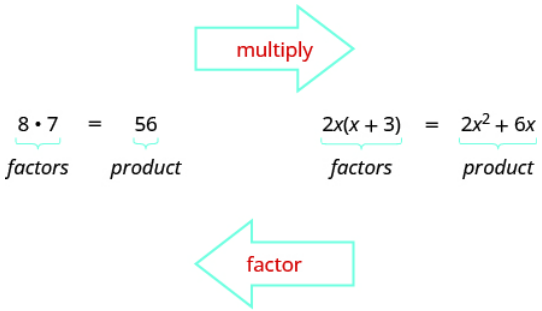
be prepared!

Before you get started, take this readiness quiz.

1. Factor 56 into primes. If you missed this problem, review Example 2.9.1.
2. Multiply: $-3(6a + 11)$. If you missed this problem, review Example 7.4.9.
3. Multiply: $4x^2(x^2 + 3x - 1)$. If you missed this problem, review Example 10.4.5.

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will be reversing this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called factoring.



In The Language of Algebra we factored numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the *greatest common factor* of two or more expressions. The method we use is similar to what we used to find the LCM.

Definition: Greatest Common Factor

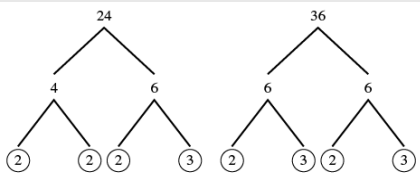
The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

First we will find the greatest common factor of two numbers

Example 12.10.1:

Find the greatest common factor of 24 and 36.

Solution

Step 1: Factor each coefficient into primes. Write all variables with exponents in expanded form.	Factor 24 and 36.	
Step 2: List all factors—matching common factors in a column.		$24 = 2 \cdot 2 \cdot 2 \cdot 3$ $36 = 2 \cdot 2 \cdot \quad 3 \cdot 3$

In each column, circle the common factors.	Circle the 2, 2, and 3 that are shared by both numbers.	$24 = 2 \cdot 2 \cdot 2 \cdot 3$ $36 = 2 \cdot 2 \cdot 3 \cdot 3$ $GCF = 2 \cdot 2 \cdot 3$ $GCF = 12$
Step 3: Bring down the common factors that all expressions share.	Bring down the 2, 2, 3 and then multiply.	
Step 4: Multiply the factors.		The GCF of 24 and 36 is 12.

Notice that since the GCF is a factor of both numbers, 24 and 36 can be written as multiples of 12.

$$24 = 12 \cdot 2$$
$$36 = 12 \cdot 3$$

Exercise 12.10.1:

Find the greatest common factor: 54, 36.

Answer

18

Exercise 12.10.2:

Find the greatest common factor: 48, 80.

Answer

16

In the previous example, we found the greatest common factor of constants. The greatest common factor of an algebraic expression can contain variables raised to powers along with coefficients. We summarize the steps we use to find the greatest common factor.

HOW TO: FIND THE GREATEST COMMON FACTOR

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.

Step 2. List all factors—matching common factors in a column. In each column, circle the common factors.

Step 3. Bring down the common factors that all expressions share.

Step 4. Multiply the factors.

Example 12.10.2:

Find the greatest common factor of 5x and 15.

Solution

Factor each number into primes. Circle the common factors in each column. Bring down the common factors.	$5x = 5 \cdot x$ $15 = 3 \cdot 5$ $GCF = 5$

The GCF of 5x and 15 is 5.

Exercise 12.10.3:

Find the greatest common factor: 7y, 14.

Answer

7

Exercise 12.10.4:

Find the greatest common factor: 22, 11m.

Answer

11

In the examples so far, the greatest common factor was a constant. In the next two examples we will get variables in the greatest common factor.

Example 12.10.3:

Find the greatest common factor of $12x^2$ and $18x^3$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors. Multiply the factors.	$12x^2 = 2 \cdot 2 \cdot 3 \cdot x \cdot x$ $18x^3 = 2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x$ $\text{GCF} = 2 \cdot 3 \cdot x \cdot x$ $\text{GCF} = 6x^2$

The GCF of $12x^2$ and $18x^3$ is $6x^2$.

Exercise 12.10.5:

Find the greatest common factor: $16x^2$, $24x^3$.

Answer

$8x^2$

Exercise 12.10.6:

Find the greatest common factor: $27y^3$, $18y^4$.

Answer

$9y^3$

Example 12.10.4:

Find the greatest common factor of $14x^3$, $8x^2$, $10x$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors. Multiply the factors.	$14x^3 = 2 \cdot 7 \cdot x \cdot x \cdot x$ $8x^2 = 2 \cdot 2 \cdot 2 \cdot x \cdot x$ $10x = 2 \cdot 5 \cdot x$ $GCF = 2 \cdot x$ $GCF = 2x$

The GCF of $14x^3$ and $8x^2$, and $10x$ is $2x$.

Exercise 12.10.7:

Find the greatest common factor: $21x^3$, $9x^2$, $15x$.

Answer

3x

Exercise 12.10.8:

Find the greatest common factor: $25m^4$, $35m^3$, $20m^2$.

Answer

$5m^2$

Factor the Greatest Common Factor from a Polynomial

Just like in arithmetic, where it is sometimes useful to represent a number in factored form (for example, 12 as 2 • 6 or 3 • 4), in algebra it can be useful to represent a polynomial in factored form. One way to do this is by finding the greatest common factor of all the terms. Remember that you can multiply a polynomial by a monomial as follows:

$$\begin{aligned} &2(x+7) && factors \\ &2 \cdot x + 2 \cdot 7 && \\ &2x + 14 && product \end{aligned}$$

Here, we will start with a product, like 2x + 14, and end with its factors, 2(x + 7). To do this we apply the Distributive Property "in reverse".

Definition: Distributive Property

If a, b, c are real numbers, then a(b + c) = ab + ac and ab + ac = a(b + c).

The form on the left is used to multiply. The form on the right is used to factor.

So how do we use the Distributive Property to factor a polynomial? We find the GCF of all the terms and write the polynomial as a product!

Example 12.10.5:

Factor: 2x + 14.

Solution

Step 1: Find the GCF of all the terms of the polynomial.	Find the GCF of 2x and 14.	$2x = 2 \cdot x$ $14 = 2 \cdot 7$ $GCF = 2$

Step 2: Rewrite each term as a product using the GCF.	Rewrite 2x and 14 as products of their GCF, 2. $2x = 2 \cdot x$ $14 = 2 \cdot 7$	$2x + 14$ $2 \cdot x + 2 \cdot 7$
Step 3: Use the Distributive Property 'in reverse' to factor the expression.		2(x + 7)
Step 4: Check by multiplying the factors.	Check.	$2(x + 7)$ $2 \cdot x + 2 \cdot 7$ $2x + 14$ ✓

? Exercise 12.10.9:

Factor: 4x + 12.

Answer

4(x + 3)

? Exercise 12.10.10:

Factor: 6a + 24.

Answer

6(a + 4)

Notice that in Example 10.84, we used the word *factor* as both a noun and a verb:

Noun	7 is a factor of 14
Verb	Factor 2 from 2x + 14

HOW TO: FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL

Step 1. Find the GCF of all the terms of the polynomial.

Step 2. Rewrite each term as a product using the GCF.

Step 3. Use the Distributive Property 'in reverse' to factor the expression.

Step 4. Check by multiplying the factors.

✓ Example 12.10.6:

Factor: 3a + 3.

Solution

Find the GCF of $3a$, 3.	$3a = 3 \cdot a$ $3 = 3$ GCF = 3	
Rewrite each term as a product using the GCF.	$3 \cdot a + 3 \cdot 1$	(12.10.1)

Use the Distributive Property 'in reverse' to factor the GCF.	$3(a+1)$	(12.10.2)
Check by multiplying the factors to get the original polynomial.	$3(a+1)$ $3\cdot a = 3\cdot 1$ $3a+3$ ✓	

Exercise 12.10.11:

Factor: 9a + 9.

Answer

9(a + 1)

Exercise 12.10.12:

Factor: 11x + 11.

Answer

11(x + 1)

The expressions in the next example have several factors in common. Remember to write the GCF as the product of all the common factors.

Example 12.10.7:

Factor: 12x − 60.

Find the GCF of $12x$ and 60.	$12x = 2\cdot 2\cdot 3\cdot x$ $60 = 2\cdot 2\cdot 3\cdot 5$ $\text{GCF} = 2\cdot 2\cdot 3$ $\text{GCF} = 12$	
Rewrite each term as a product using the GCF.	$12\cdot x - 12\cdot 5$	(12.10.3)
Factor the GCF.	$12(x-5)$	(12.10.4)
Check by multiplying the factors.	$12(x-5)$ $12\cdot x - 12\cdot 5$ $12x - 60$ ✓	

Exercise 12.10.13:

Factor: 11x − 44.

Answer

11(x - 4)

Exercise 12.10.14:

Factor: 13y − 52.

Answer

13(y - 4)

Now we'll factor the greatest common factor from a trinomial. We start by finding the GCF of all three terms.

Example 12.10.8:

Factor: $3y^2 + 6y + 9$.

Solution

Find the GCF of $3y^2$, $6y$, and 9.

$$\begin{aligned} 3y^2 &= 3 \cdot y \cdot y \\ 6y &= 2 \cdot 3 \cdot y \\ 9 &= 3 \cdot 3 \\ \text{GCF} &= 3 \end{aligned}$$

Rewrite each term as a product using the GCF.	$3 \cdot y^2 + 3 \cdot 2y + 3 \cdot 3$	(12.10.5)
Factor the GCF.	$3(y^2 + 2y + 3)$	(12.10.6)
Check by multiplying.	$3(y^2 + 2y + 3)$ $3 \cdot y^2 + 3 \cdot 2y + 3 \cdot 3$ $3y^2 + 6y + 9$ ✓	

Exercise 12.10.15:

Factor: $4y^2 + 8y + 12$.

Answer

$4\left(y^2 + 2y + 3\right)$

Exercise 12.10.16:

Factor: $6x^2 + 42x - 12$.

Answer

$6\left(x^2 + 7x - 2\right)$

In the next example, we factor a variable from a binomial.

Example 12.10.9:

Factor: $6x^2 + 5x$.

Solution

Find the GCF of $6x^2$ and $5x$ and the math that goes with it.	$6x^2 = 2 \cdot 3 \cdot x \cdot x$ $5x = 5 \cdot x$ $\text{GCF} = x$	
Rewrite each term as a product.	$x \cdot 6x + x \cdot 5$	(12.10.7)
Factor the GCF.	$x(6x + 5)$	(12.10.8)
Check by multiplying.	$x(6x + 5)$ $x \cdot 6x + x \cdot 5$ $6x^2 + 5x$ ✓	

? Exercise 12.10.17:

Factor: $9x^2 + 7x$.

Answer

$x(9x + 7)$

? Exercise 12.10.18:

Factor: $5a^2 - 12a$.

Answer

a(5a - 12)

When there are several common factors, as we'll see in the next two examples, good organization and neat work helps!

✓ Example 12.10.10:

Factor: $4x^3 - 20x^2$.

Solution

Find the GCF of $4x^3$ and $20x^2$.	$4x^3 = 2 \cdot 2 \cdot x \cdot x \cdot x$ $20x^2 = 2 \cdot 2 \cdot 5 \cdot x \cdot x$ $\text{GCF} = 2 \cdot 2 \cdot x \cdot x$ $\text{GCF} = 4x^2$	
Rewrite each term.	$4x^2 \cdot x - 4x^2 \cdot 5$	(12.10.9)
Factor the GCF.	$4x^2(x - 5)$	(12.10.10)
Check.	$4x^2(x - 5)$ $4x^2 \cdot x - 4x^2 \cdot 5$ $4x^3 - 20x^2$ ✓	

Exercise 12.10.19:

Factor: $2x^3 + 12x^2$.

Answer

$2x^2(x+6)$

Exercise 12.10.20:

Factor: $6y^3 - 15y^2$.

Answer

$3y^2(2y-5)$

Example 12.10.11:

Factor: $21y^2 + 35y$.

Solution

Find the GCF of $21y^2$ and $35y$.	$21y^2 = 3 \cdot 7 \cdot y \cdot y$ $35y = 5 \cdot 7 \cdot y$ $GCF = 7 \cdot y$ $GCF = 7y$	
Rewrite each term.	$7y \cdot 3y + 7y \cdot 5$	(12.10.11)
Factor the GCF.	$7y(3y+5)$	(12.10.12)

Exercise 12.10.21:

Factor: $18y^2 + 63y$.

Answer

$9y(2y + 7)$

Exercise 12.10.22:

Factor: $32k^2 + 56k$.

Answer

$8k(4k + 7)$

Example 12.10.12:

Factor: $14x^3 + 8x^2 - 10x$.

Solution

Previously, we found the GCF of $14x^3$, $8x^2$, and $10x$ to be $2x$.

Rewrite each term using the GCF, 2x.	$2x \cdot 7x^2 + 2x \cdot 4x - 2x \cdot 5$	(12.10.13)
Factor the GCF.	$2x(7x^2 + 4x - 5)$	(12.10.14)
Check.	$2x(7x^2 + 4x - 5)$ $2x \cdot 7x^2 + 2x \cdot 4x - 2x \cdot 5$ $14x^3 + 8x^2 - 10x$ ✓	

? Exercise 12.10.23:

Factor: $18y^3 - 6y^2 - 24y$.

Answer

$6y\left(3y^2 - y - 4\right)$

? Exercise 12.10.24:

Factor: $16x^3 + 8x^2 - 12x$.

Answer

$4x\left(4x^2 + 2x - 3\right)$

When the leading coefficient, the coefficient of the first term, is negative, we factor the negative out as part of the GCF.

✓ Example 12.10.13:

Factor: $-9y - 27$.

Solution

When the leading coefficient is negative, the GCF will be negative. Ignoring the signs of the terms, we first find the GCF of 9y and 27 is 9.	$9y = 3 \cdot 3 \cdot y$ $27 = 3 \cdot 3 \cdot 3$ $\text{GCF} = 3 \cdot 3$ $\text{GCF} = 9$	
Since the expression $-9y - 27$ has a negative leading coefficient, we use -9 as the GCF.		
Rewrite each term using the GCF.	$-9 \cdot y + (-9) \cdot 3$	(12.10.15)
Factor the GCF.	$-9(y + 3)$	(12.10.16)
Check.	$-9(y + 3)$ $-9 \cdot y + (-9) \cdot 3$ $-9y - 27$ ✓	

Exercise 12.10.25:

Factor: −5y − 35.

Answer

-5(y + 7)

Exercise 12.10.26:

Factor: −16z − 56.

Answer

-8(2z + 7)

Pay close attention to the signs of the terms in the next example.

Example 12.10.14:

Factor: $-4a^2 + 16a$.

Solution

The leading coefficient is negative, so the GCF will be negative.	$4a^2 = 2 \cdot 2 \cdot a \cdot a$ $16a = 2 \cdot 2 \cdot 2 \cdot 2 \cdot a$ $\text{GCF} = 2 \cdot 2 \cdot a$ $\text{GCF} = 4a$	
Since the leading coefficient is negative, the GCF is negative, −4a.		
Rewrite each term.	$-4a \cdot a - (-4a) \cdot 4$	(12.10.17)
Factor the GCF.	$-4a(a - 4)$	(12.10.18)
Check on your own by multiplying.		

Exercise 12.10.27:

Factor: $-7a^2 + 21a$.

Answer

-7a(a - 3)

Exercise 12.10.28:

Factor: $-6x^2 + x$.

Answer

-x(6x - 1)

ACCESS ADDITIONAL ONLINE RESOURCES

Factor GCF

Factor a Binomial

Identify GCF

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

422. 40, 56
423. 45, 75
424. 72, 162
425. 150, 275
426. 3x, 12
427. 4y, 28
428. 10a, 50
429. 5b, 30
430. 16y, $24y^2$
431. 9x, $15x^2$
432. $18m^3$, $36m^2$
433. $12p^4$, $48p^3$
434. 10x, $25x^2$, $15x^3$
435. 18a, $6a^2$, $22a^3$
436. 24u, $6u^2$, $30u^3$
437. 40y, $10y^2$, $90y^3$
438. $15a^4$, $9a^5$, $21a^6$
439. $35x^3$, $10x^4$, $5x^5$
440. $27y^2$, $45y^3$, $9y^4$
441. $14b^2$, $35b^3$, $63b^4$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

442. $2x + 8$
443. $5y + 15$
444. $3a - 24$
445. $4b - 20$
446. $9y - 9$
447. $7x - 7$
448. $5m^2 + 20m + 35$
449. $3n^2 + 21n + 12$
450. $8p^2 + 32p + 48$
451. $6q^2 + 30q + 42$
452. $8q^2 + 15q$
453. $9c^2 + 22c$
454. $13k^2 + 5k$
455. $17x^2 + 7x$
456. $5c^2 + 9c$
457. $4q^2 + 7q$
458. $5p^2 + 25p$
459. $3r^2 + 27r$

460. $24q^2 - 12q$
461. $30u^2 - 10u$
462. $yz + 4z$
463. $ab + 8b$
464. $60x - 6x^3$
465. $55y - 11y^4$
466. $48r^4 - 12r^3$
467. $45c^3 - 15c^2$
468. $4a^3 - 4ab^2$
469. $6c^3 - 6cd^2$
470. $30u^3 + 80u^2$
471. $48x^3 + 72x^2$
472. $120y^6 + 48y^4$
473. $144a^6 + 90a^3$
474. $4q^2 + 24q + 28$
475. $10y^2 + 50y + 40$
476. $15z^2 - 30z - 90$
477. $12u^2 - 36u - 108$
478. $3a^4 - 24a^3 + 18a^2$
479. $5p^4 - 20p^3 - 15p^2$
480. $11x^6 + 44x^5 - 121x^4$
481. $8c^5 + 40c^4 - 56c^3$
482. $-3n - 24$
483. $-7p - 84$
484. $-15a^2 - 40a$
485. $-18b^2 - 66b$
486. $-10y^3 + 60y^2$
487. $-8a^3 + 32a^2$
488. $-4u^5 + 56u^3$
489. $-9b^5 + 63b^3$

Everyday Math

490. **Revenue** A manufacturer of microwave ovens has found that the revenue received from selling microwaves a cost of p dollars each is given by the polynomial $-5p^2 + 150p$. Factor the greatest common factor from this polynomial.
491. **Height of a baseball** The height of a baseball hit with velocity 80 feet/second at 4 feet above ground level is $-16t^2 + 80t + 4$, with t = the number of seconds since it was hit. Factor the greatest common factor from this polynomial.

Writing Exercises

492. The greatest common factor of 36 and 60 is 12. Explain what this means.
493. What is the GCF of y^4, y^5, and y^{10}? Write a general rule that tells how to find the GCF of y^a, y^b, and y^c.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			

(b) Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

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12.E: Polynomials (Exercises)

10.1 - Add and Subtract Polynomials

Identify Polynomials, Monomials, Binomials and Trinomials

In the following exercises, determine if each of the following polynomials is a monomial, binomial, trinomial, or other polynomial.

494. $y^2 + 8y - 20$
495. $-6a^4$
496. $9x^3 - 1$
497. $n^3 - 3n^2 + 3n - 1$

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

498. $16x^2 - 40x - 25$
499. $5m + 9$
500. -15
501. $y^2 + 6y^3 + 9y^4$

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

502. $4p + 11p$
503. $-8y^3 - 5y^3$
504. Add $4n^5$, $-n^5$, $-6n^5$
505. Subtract $10x^2$ from $3x^2$

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

506. $(4a^2 + 9a - 11) + (6a^2 - 5a + 10)$
507. $(8m^2 + 12m - 5) - (2m^2 - 7m - 1)$
508. $(y^2 - 3y + 12) + (5y^2 - 9)$
509. $(5u^2 + 8u) - (4u - 7)$
510. Find the sum of $8q^3 - 27$ and $q^2 + 6q - 2$
511. Find the difference of $x^2 + 6x + 8$ and $x^2 - 8x + 15$

Evaluate a Polynomial for a Given Value of the Variable

In the following exercises, evaluate each polynomial for the given value.

512. $200x - \frac{1}{5}x^2$ when $x = 5$
513. $200x - \frac{1}{5}x^2$ when $x = 0$
514. $200x - \frac{1}{5}x^2$ when $x = 15$
515. $5 + 40x - \frac{1}{2}x^2$ when $x = 10$
516. $5 + 40x - \frac{1}{2}x^2$ when $x = -4$
517. $5 + 40x - \frac{1}{2}x^2$ when $x = 0$
518. A pair of glasses is dropped off a bridge 640 feet above a river. The polynomial $-16t^2 + 640$ gives the height of the glasses t seconds after they were dropped. Find the height of the glasses when $t = 6$.
519. The fuel efficiency (in miles per gallon) of a bus going at a speed of x miles per hour is given by the polynomial $-\frac{1}{160}x^2 + \frac{1}{2}x$. Find the fuel efficiency when $x = 20$ mph.

10.2 - Use Multiplication Properties of Exponents

Simplify Expressions with Exponents

In the following exercises, simplify.

520. 6^3

521. $\left(\frac{1}{2}\right)^4$

522. $(-0.5)^2$

523. -3^2

Simplify Expressions Using the Product Property of Exponents

In the following exercises, simplify each expression.

524. $p^3 \cdot p^{10}$

525. $2 \cdot 2^6$

526. $a \cdot a^2 \cdot a^3$

527. $x \cdot x^8$

Simplify Expressions Using the Power Property of Exponents

In the following exercises, simplify each expression.

528. $(y^4)^3$

529. $(r^3)^2$

530. $(3^2)^5$

531. $(a^{10})^y$

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression.

532. $(8n)^2$

533. $(-5x)^3$

534. $(2ab)^8$

535. $(-10mnp)^4$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.

536. $(3a^5)^3$

537. $(4y)^2(8y)$

538. $(x^3)^5(x^2)^3$

539. $(5st^2)^3(2s^3t^4)^2$

Multiply Monomials

In the following exercises, multiply the monomials.

540. $(-6p^4)(9p)$

541. $\left(\frac{1}{3}c^2\right)(30c^8)$

542. $(8x^2y^5)(7xy^6)$

543. $\left(\frac{2}{3}m^3n^6\right)\left(\frac{1}{6}m^4n^4\right)$

10.3 - Multiply Polynomials

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

544. $7(10 - x)$
545. $a^2(a^2 - 9a - 36)$
546. $-5y(125y^3 - 1)$
547. $(4n - 5)(2n^3)$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using various methods.

548. $(a + 5)(a + 2)$
549. $(y - 4)(y + 12)$
550. $(3x + 1)(2x - 7)$
551. $(6p - 11)(3p - 10)$
552. $(n + 8)(n + 1)$
553. $(k + 6)(k - 9)$
554. $(5u - 3)(u + 8)$
555. $(2y - 9)(5y - 7)$
556. $(p + 4)(p + 7)$
557. $(x - 8)(x + 9)$
558. $(3c + 1)(9c - 4)$
559. $(10a - 1)(3a - 3)$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using any method.

560. $(x + 1)(x^2 - 3x - 21)$
561. $(5b - 2)(3b^2 + b - 9)$
562. $(m + 6)(m^2 - 7m - 30)$
563. $(4y - 1)(6y^2 - 12y + 5)$

10.4 - Divide Monomials

Simplify Expressions Using the Quotient Property of Exponents

In the following exercises, simplify.

564. $\frac{2^8}{2^2}$
565. $\frac{a^6}{a}$
566. $\frac{n^3}{n^{12}}$
567. $\frac{x}{x^5}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.

568. 3^0
569. y^0
570. $(14t)^0$
571. $12a^0 - 15b^0$

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

572. $\left(\frac{3}{5}\right)^2$
573. $\left(\frac{x}{2}\right)^5$
574. $\left(\frac{5m}{n}\right)^3$
575. $\left(\frac{s}{10t}\right)^2$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

576. $\frac{(a^3)^2}{a^4}$
577. $\frac{u^3}{u^2 \cdot u^4}$
578. $\left(\frac{x}{x^9}\right)^5$
579. $\left(\frac{p^4 \cdot p^5}{p^3}\right)^2$
580. $\frac{(n^5)^3}{(n^2)^8}$
581. $\left(\frac{5s^2}{4t}\right)^3$

Divide Monomials

In the following exercises, divide the monomials.

582. $72p^{12} \div 8p^3$
583. $-26a^8 \div (2a^2)$
584. $\frac{45y^6}{-15y^{10}}$
585. $\frac{-30x^8}{-36x^9}$
586. $\frac{28a^9b}{7a^4b^3}$
587. $\frac{11u^6v^3}{55u^2v^8}$
588. $\frac{(5m^9n^3)(8m^3n^2)}{(10mn^4)(m^2n^5)}$
589. $\frac{42r^2s^4}{6rs^3} - \frac{54rs^2}{9s}$

10.5 - Integer Exponents and Scientific Notation

Use the Definition of a Negative Exponent

In the following exercises, simplify.

590. 6^{-2}
591. $(-10)^{-3}$
592. $5 \cdot 2^{-4}$
593. $(8n)^{-1}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

594. $x^{-3} \cdot x^{9}$
595. $r^{-5} \cdot r^{-4}$
596. $(uv^{-3})(u^{-4}v^{-2})$
597. $(m^{5})^{-1}$
598. $(k^{-2})^{-3}$
599. $\frac{q^{4}}{q^{20}}$
600. $\frac{b^{8}}{b^{-2}}$
601. $\frac{n^{-3}}{n^{-5}}$

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

602. 5,300,000
603. 0.00814
604. The thickness of a piece of paper is about 0.097 millimeter.
605. According to www.cleanair.com, U.S. businesses use about 21,000,000 tons of paper per year.

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

606. 2.9×10^{4}
607. 1.5×10^{8}
608. 3.75×10^{-1}
609. 9.413×10^{-5}

Multiply and Divide Using Scientific Notation

In the following exercises, multiply and write your answer in decimal form.

610. $(3 \times 10^{7})(2 \times 10^{-4})$
611. $(1.5 \times 10^{-3})(4.8 \times 10^{-1})$
612. $\frac{6 \times 10^{9}}{2 \times 10^{-1}}$
613. $\frac{9 \times 10^{-3}}{1 \times 10^{-6}}$

10.6 - Introduction to Factoring Polynomials

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

614. 5n, 45
615. 8a, 72
616. $12x^{2}$, $20x^{3}$, $36x^{4}$
617. $9y^{4}$, $21y^{5}$, $15y^{6}$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

618. $16u - 24$
619. $15r + 35$
620. $6p^{2} + 6p$
621. $10c^{2} - 10c$
622. $-9a^{5} - 9a^{3}$
623. $-7x^{8} - 28x^{3}$

624. $5y^2 - 55y + 45$
625. $2q^5 - 16q^3 + 30q^2$

PRACTICE TEST

626. For the polynomial $8y^4 - 3y^2 + 1$
    a. Is it a monomial, binomial, or trinomial?
    b. What is its degree?

In the following exercises, simplify each expression.

627. $(5a^2 + 2a - 12) + (9a^2 + 8a - 4)$
628. $(10x^2 - 3x + 5) - (4x^2 - 6)$
629. $\left(-\frac{3}{4}\right)^3$
630. $n \cdot n^4$
631. $(10p^3q^5)^2$
632. $(8xy^3)(-6x^4y^6)$
633. $4u(u^2 - 9u + 1)$
634. $(s + 8)(s + 9)$
635. $(m + 3)(7m - 2)$
636. $(11a - 6)(5a - 1)$
637. $(n - 8)(n\ 2 - 4n + 11)$
638. $(4a + 9b)(6a - 5b)$
639. $\frac{5^6}{5^8}$
640. $\left(\frac{x^3 \cdot x^9}{x^5}\right)^2$
641. $(47a^{18}b^{23}c^5)^0$
642. $\frac{24r^3s}{6r^2s^7}$
643. $\frac{8y^2 - 16y + 20}{4y}$
644. $(15xy^3 - 35x^2y) \div 5xy$
645. 4^{-1}
646. $(2y)^{-3}$
647. $p^{-3} \cdot p^{-8}$
648. $\frac{x^4}{x^{-5}}$
649. $(2.4 \times 10^8)(2 \times 10^{-5})$

In the following exercises, factor the greatest common factor from each polynomial.

650. $80a^3 + 120a^2 + 40a$
651. $-6x^2 - 30x$
652. Convert 5.25×10^{-4} to decimal form.

In the following exercises, simplify, and write your answer in decimal form.

653. $\frac{9 \times 10^4}{3 \times 10^{-1}}$
654. A hiker drops a pebble from a bridge 240 feet above a canyon. The polynomial $-16t^2 + 240$ gives the height of the pebble t seconds a after it was dropped. Find the height when t = 3.
655. According to www.cleanair.org, the amount of trash generated in the US in one year averages out to 112,000 pounds of trash per person. Write this number in scientific notation.

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12.S: Polynomials (Summary)

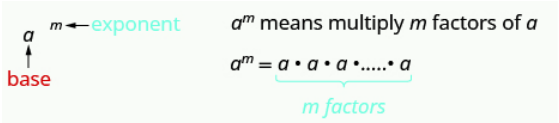
Key Terms

binomial	A polynomial with exactly two terms
degree of a constant	The degree of a constant is 0.
degree of a polynomial	The degree of a polynomial is the highest degree of all its terms.
degree of a term	The degree of a term of a polynomial is the exponent of its variable.
greatest common factor	The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.
monomial	A term of the form ax^m, where a is a constant and m is a whole number, is called a monomial.
negative exponent	If n is a positive integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.
polynomial	A polynomial is a monomial, or two or more monomials, combined by addition or subtraction.
scientific notation	A number expressed in scientific notation when it is of the form $a \times 10^n$, where $a \geq 1$ and $a < 10$, and n is an integer.
trinomial	A trinomial is a polynomial with exactly three terms.
zero exponent	If a is a non-zero number, then $a^0 = 1$. Any nonzero number raised to the zero power is 1.

Key Concepts

10.2 - Use Multiplication Properties of Exponents

- **Exponential Notation**

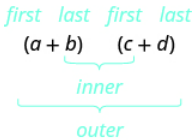


This is read a to the mth power.

- **Product Property of Exponents**
  - If a is a real number and m, n are counting numbers, then $a^m \cdot a^n = a^{m+n}$
  - To multiply with like bases, add the exponents.
- **Power Property for Exponents**
  - If a is a real number and m, n are counting numbers, then $(a^m)^n = a^{m \cdot n}$
- **Product to a Power Property for Exponents**
  - If a and b are real numbers and m is a whole number, then $(ab)^m = a^m b^m$

10.3 - Multiply Polynomials

• Use the FOIL method for multiplying two binomials.

Step 1. Multiply the **First** terms.	
Step 2. Multiply the **Outer** terms.	
Step 3. Multiply the **Inner** terms.	

Step 4. Multiply the **Last** terms.
Step 5. Combine like terms, when possible.

- **Multiplying Two Binomials**: To multiply binomials, use the:
  - Distributive Property
  - FOIL Method
  - Vertical Method
- **Multiplying a Trinomial by a Binomial**: To multiply a trinomial by a binomial, use the:
  - Distributive Property
  - Vertical Method

10.4 - Divide Monomials

- **Equivalent Fractions Property**
  - If a, b, c are whole numbers where b ≠ 0, c ≠ 0, then

$$\frac{a}{b}=\frac{a\cdot c}{b\cdot c} \quad and \quad \frac{a\cdot c}{b\cdot c}=\frac{a}{b} \tag{12.S.1}$$

- **Zero Exponent**
  - If a is a non-zero number, then $a^0 = 1$.
  - Any nonzero number raised to the zero power is 1.
- **Quotient Property for Exponents**
  - If a is a real number, a ≠ 0, and m, n are whole numbers, then

$$\frac{a^m}{a^n}=a^{m-n},\ m>n \quad and \quad \frac{a^m}{a^n}=\frac{1}{a^{n-m}},\ n>m \tag{12.S.2}$$

- **Quotient to a Power Property for Exponents**
  - If a and b are real numbers, b ≠ 0, and m is a counting number, then

$$\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m} \tag{12.S.3}$$

  - To raise a fraction to a power, raise the numerator and denominator to that power.

10.5 - Integer Exponents and Scientific Notation

- **Summary of Exponent Properties**
  - If a, b are real numbers and m, n are integers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m\cdot n}$
Product to a Power Property	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}$, a ≠ 0, m > n
	$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$, a ≠ 0, n > m
Zero Exponent Property	$a^0 = 1$, a ≠ 0
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, b ≠ 0
Definition of a Negative Exponent	$a^{-n} = \frac{1}{a^n}$

- **Convert from Decimal Notation to Scientific Notation**: To convert a decimal to scientific notation:

1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
2. Count the number of decimal places, n, that the decimal point was moved. Write the number as a product with a power of 10.
   - If the original number is greater than 1, the power of 10 will be 10^n.
   - If the original number is between 0 and 1, the power of 10 will be 10^n.
3. Check.

- **Convert Scientific Notation to Decimal Form**: To convert scientific notation to decimal form:
  1. Determine the exponent, n, on the factor 10.
  2. Move the decimal n places, adding zeros if needed.
     - If the exponent is positive, move the decimal point n places to the right.
     - If the exponent is negative, move the decimal point |n| places to the left.
  3. Check.

10.6 - Introduction to Factoring Polynomials

- **Find the greatest common factor.**
  1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
  2. List all factors—matching common factors in a column. In each column, circle the common factors.
  3. Bring down the common factors that all expressions share.
  4. Multiply the factors.
- **Distributive Property**
  - If a , b , c are real numbers, then a(b + c) = ab + ac and ab + ac = a(b + c).
- **Factor the greatest common factor from a polynomial.**
  1. Find the GCF of all the terms of the polynomial.
  2. Rewrite each term as a product using the GCF.
  3. Use the Distributive Property ‘in reverse’ to factor the expression.
  4. Check by multiplying the factors.

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13.1: Cumulative Review

> **Note**
>
> Answers to the Cumulative Review can be found in the Supplemental Resources. Please visit http://openstax college.org to view an updated list of the Learning Resources for this title and how to access them.

Chapter 1 - Whole Numbers

No exercises.

Chapter 2 - The Language of Algebra

Simplify:

1. $5(3 + 2 \cdot 6) - 8^2$

Solve:

2. $17 = y - 13$
3. $p + 14 = 23$

Translate into an algebraic expression.

4. 11 less than the product of 7 and x.

Translate into an algebraic equation and solve.

5. Twice the difference of y and 7 gives 84.
6. Find all the factors of 72.
7. Find the prime factorization of 132.
8. Find the least common multiple of 12 and 20.

Chapter 3 - Integers

Simplify:

9. $|8 - 9| - |3 - 8|$
10. $-2 + 4(-3 + 7)$
11. $27 - (-4 - 7)$
12. $28 \div (-4) - 7$

Translate into an algebraic expression or equation.

13. The sum of −5 and 13, increased by 11.
14. The product of −11 and 8.
15. The quotient of 7 and the sum of −4 and m.
16. The product of −3 and is −51.

Solve:

17. $-6r = 24$

Chapter 4 - Fractions

18. Locate the numbers on a number line. \(\dfrac{7}{8}, \dfrac{5}{3}, 3\dfrac{1}{4}, 5.

Simplify:

19. $\dfrac{21p}{57q}$
20. $\dfrac{3}{7} \cdot \left(-\dfrac{28}{45}\right)$

21. $-6\frac{3}{4} \div \frac{9}{2}$
22. $-3\frac{3}{5} \div 6$
23. $-4\frac{2}{3}\left(-\frac{6}{7}\right)$
24. $\dfrac{-2\frac{1}{4}}{-\frac{3}{8}}$
25. $\dfrac{7 \cdot 8 + 4(7-12)}{9 \cdot 6 - 2 \cdot 9}$
26. $-\frac{23}{36} + \frac{17}{20}$
27. $\dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{3}{4} - \frac{1}{3}}$
28. $3\frac{5}{8} - 2\frac{1}{2}$
29. $-\frac{2}{3}$r = 24

Chapter 5 - Decimals

Simplify:

30. 24.76 − 7.28
31. 12.9 + 15.633
32. (−5.6)(0.25)
33. \$6.29 ÷ 12
34. 3 4 (13.44 − 9.6)
35. $\sqrt{64} + \sqrt{225}$
36. $\sqrt{121x^2y^2}$
37. Write in order from smallest to largest: $\frac{5}{8}$, 0.75, $\frac{8}{15}$

Solve:

38. −8.6x = 34.4
39. Using 3.14 as the estimate for pi, approximate the (a) circumference and (b) area of a circle whose radius is 8 inches.
40. Find the mean of the numbers, 18, 16, 20, 12
41. Find the median of the numbers, 24, 29, 27, 28, 30
42. Identify the mode of the numbers, 6, 4, 4, 5, 6, 6, 4, 4, 4, 3, 5
43. Find the unit price of one t-shirt if they are sold at 3 for \$28.97.

Chapter 6 - Percents

44. Convert 14.7% to (a) a fraction and (b) a decimal.

Translate and solve.

45. 63 is 35% of what number?
46. The nutrition label on a package of granola bars says that each granola bar has 180 calories, and 81 calories are from fat. What percent of the total calories is from fat?
47. Elliot received \$510 commission when he sold a \$3,400 painting at the art gallery where he works. What was the rate of commission?
48. Nandita bought a set of towels on sale for \$67.50. The original price of the towels was \$90. What was the discount rate?
49. Alan invested \$23,000 in a friend's business. In 5 years the friend paid him the \$23,000 plus \$9,200 interest. What was the rate of interest?

Solve:

50. $\frac{9}{p} = \frac{-6}{14}$

Chapter 7 - The Properties of Real Numbers

51. List the (a) whole numbers, (b) integers, (c) r ational numbers, (d) irrational numbers, (e) real numbers -5, $-2\frac{1}{4}, -\sqrt{4}, 0.2\bar{5}, \frac{13}{5}, 4$

Simplify:

52. $\left(\frac{8}{15} + \frac{4}{7}\right) + \frac{3}{7}$
53. $3(y + 3) - 8(y - 4)$
54. $\frac{8}{17} \cdot 49 \cdot \frac{17}{8}$
55. A playground is 55 feet wide. Convert the width to yards.
56. Every day last week Amit recorded the number of minutes he spent reading. The recorded number of minutes he read each day was 48, 26, 81, 54, 43, 62, 106. How many hours did Amit spend reading last week?
57. June walked 2.8 kilometers. Convert this length to miles knowing 1 mile is 1.61 kilometer.

Chapter 8 - Solve Linear Equations

Solve:

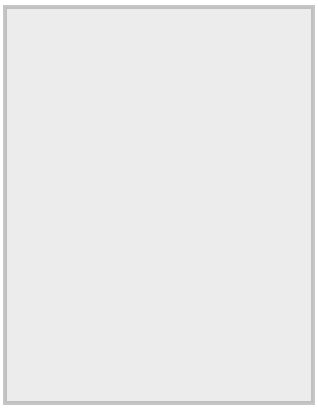
58. $y + 13 = -8$
59. $p + \frac{2}{5} = \frac{8}{5}$
60. $48 = \frac{2}{3}x$
61. $4(a - 3) - 6a = -18$
62. $7q + 14 = -35$
63. $4v - 27 = 7v$
64. $\frac{7}{8}y - 6 = \frac{3}{8}y - 8$
65. $26 - 4(z - 2) = 6$
66. $\frac{3}{4}x - \frac{2}{3} = \frac{1}{2}x - \frac{5}{6}$
67. $0.7y + 4.8 = 0.84y - 5.3$

Translate and solve.

68. Four less than n is 13.

Chapter 9 - Math Models and Geometry

69. One number is 8 less than another. Their sum is negative twenty-two. Find the numbers.
70. The sum of two consecutive integers is -95. Find the numbers.
71. Wilma has $3.65 in dimes and quarters. The number of dimes is 2 less than the number of quarters. How many of each coin does she have?
72. Two angles are supplementary. The larger angle is 24° more than the smaller angle. Find the measurements of both angles.
73. One angle of a triangle is 20° more than the smallest angle. The largest angle is the sum of the other angles. Find the measurements of all three angles.
74. Erik needs to attach a wire to hold the antenna to the roof of his house, as shown in the figure. The antenna is 12 feet tall and Erik has 15 feet of wire. How far from the base of the antenna can he attach the wire?



75. The width of a rectangle is 4 less than the length. The perimeter is 96 inches. Find the length and the width.
76. Find the (a) volume and (b) surface area of a rectangular carton with length 24 inches, width 18 inches, and height 6 inches.

Chapter 10 - Polynomials

Simplify:

77. $(8m^2 + 12m - 5) - (2m^2 - 7m - 1)$
78. $p^3 \cdot p^{10}$
79. $(y^4)^3$
80. $(3a^5)^3$
81. $(x^3)^5(x^2)3$
82. $\left(\frac{2}{3}m^3n^6\right)\left(\frac{1}{6}m^4n^4\right)$
83. $(y - 4)(y + 12)$
84. $(3c + 1)(9c - 4)$
85. $(x - 1)(x^2 - 3x - 2)$
86. $(8x)^0$
87. $\frac{(x^3)^5}{(x^2)^4}$
88. $\frac{32a^7b^2}{12a^3b^6}$
89. $(ab^{-3})(a^{-3}b^6)$
90. Write in scientific notation: (a) 4,800,000 (b) 0.00637

Factor the greatest common factor from the polynomial.

91. $3x^4 - 6x^3 - 18x^2$

Chapter 11 - Graphs

Graph:

92. $y = 4x - 3$
93. $y = -3x$
94. $y = \frac{1}{2}x + 3$
95. $x - y = 6$
96. $y = -2$
97. Find the intercepts. $2x + 3y = 12$

Graph using the intercepts.

98. $2x - 4y = 8$
99. Find the slope of the line shown.

100. Use the slope formula to find the slope of the line between the points (−5, −2), (3, 2).

101. Graph the line passing through the point (−3, 4) and with slope $m = -\frac{1}{3}$.

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13.2: Powers and Roots

Table B1

n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$
1	1	1	1	1
2	4	1.414214	8	1.259921
3	9	1.732051	27	1.442250
4	16	2	64	1.587401
5	25	2.236068	125	1.709976
6	36	2.449490	216	1.817121
7	49	2.645751	343	1.912931
8	64	2.828427	512	2
9	81	3	729	2.080084
10	100	3.162278	1,000	2.154435
11	121	3316625	1,331	1.223980
12	144	3.464102	1,728	2.289428
13	169	3.605551	2,197	2.351335
14	196	3.741657	2,744	2.410142
15	225	3.872983	3,375	2,466212
16	256	4	4,096	2.519842
17	289	4.123106	4,913	2.571282
18	324	4.242641	5,832	2.620741
19	361	4.358899	6,859	2.668402
20	400	4.472136	8,000	2.714418
21	441	4.582576	9,261	2.758924
22	484	4.690416	10,648	2.802039
23	529	4.795832	12,167	2.843867
24	576	4.898979	13,824	2.884499
25	625	5	15,625	2.924018
26	676	5.099020	17,576	2.962496
27	729	5.196152	19,683	3
28	784	5.291503	21,952	3.036589
29	841	5.385165	24,389	3.072317
30	900	5.477226	27,000	3.107233
31	961	5.567764	29,791	3.141381
32	1,024	5.656854	32,768	3.17482

n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$
33	1,089	5.744563	35,937	3.207534
34	1,156	5.830952	39,304	3.239612
35	1,225	5.916080	42,875	3.271066
36	1,296	6	46,656	3.301927
37	1,369	6.082763	50,653	3.332222
38	1,444	6164414	54,872	3.361975
39	1,521	6.244998	59,319	3.391211
40	1,600	6.324555	64,000	3.419952
41	1,681	6.403124	68,921	3.448217
42	1.764	6.480741	74,088	3.476027
43	1.849	6.557439	79,507	3.503398
44	1,936	6.633250	85,184	3.530348
45	2,025	6.708204	91,125	3.556893
46	2,116	6.782330	97,336	3.583048
47	2,209	6.855655	103,823	3.608826
48	2,304	6.928203	110,592	3.6324241
49	2,401	7	117,649	3.659306
50	2,500	7.071068	125,000	3.684031
51	2,601	7.141428	132,651	3.708430
52	2,704	7.211103	140,608	3.732511
53	2,809	7.280110	148,877	3.756286
54	2,916	7.348469	157,464	3.779763
55	3,025	7.416198	166,375	3.802952
56	3,136	7.483315	175,616	3.825862
57	3,249	7.549834	185,193	3.848501
58	3,364	7.615773	195,112	3.870877
59	3,481	7.681146	205,379	3.892996
60	3,600	7.745967	216,000	3.914868
61	3,721	7.810250	226,981	3.936497
62	3,844	7.874008	238,328	3.957892
63	3,969	7.937254	250,047	3.979057
64	4,096	8	262,144	4
65	4,225	8.062258	274,625	4.020726
66	6,356	8.124038	287,496	4.041240

n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$
67	4,489	8.185353	300,763	4.061548
68	4,624	8.246211	314,432	4.081655
69	4,761	8.306624	328,509	4.101566
70	4,900	8.366600	343,000	4.121285
71	5,041	8.426150	357,911	4.140818
72	5,184	8.485281	389,017	4.179339
73	5,329	8.544004	389,017	4.179339
74	5,476	8.602325	405,224	4.198336
75	5,625	8.660254	421,875	4.217163
76	5,776	8.17798	438,976	4.235824
77	5.929	8774964	456,533	4.254321
78	6,084	8.831761	474,552	4.272659
79	6,241	8.888194	493,039	4.290840
80	6,400	8.944272	512,000	4.308869
81	6,561	9	531,441	4.326749
82	6,724	9.055385	551,368	4.344481
83	6,889	9.110434	571,787	4.362071
84	7,056	9.165151	592,704	4.379519
85	7,225	9.219544	614,125	4.396830
86	7,396	9.273618	636,056	4.414005
87	7,569	9.327379	658,503	4.431048
88	7,744	9.380832	681,472	4.447960
89	7,821	8.433981	704,969	4.464745
90	8,100	9.486833	729,000	4.481405
91	8,281	9.539392	753,571	4.497941
92	8,464	9.591663	778,688	4.514357
93	8,649	9.643651	804,357	4.530655
94	8,836	9.695360	830,584	4.546836
95	9,025	9.746794	857,375	4.562903
96	9,216	9.797959	884,736	4.578857
97	9,409	9.848858	912,673	4.594701
98	9,604	9.899495	941,192	4.610436
99	9,801	9.949874	970,299	4.62065
100	10,000	10	1,000,000	4.641589

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13.3: Geometric Formulas

Table C1 - 2 Dimensions

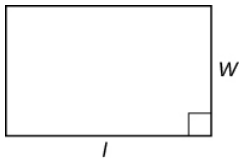
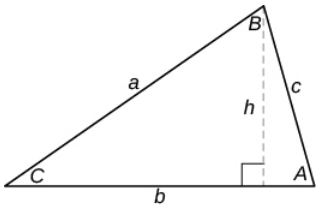
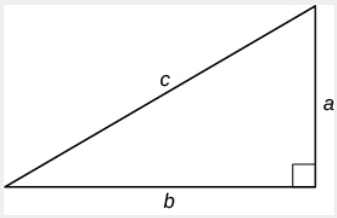
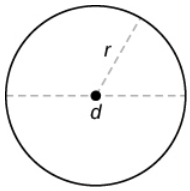
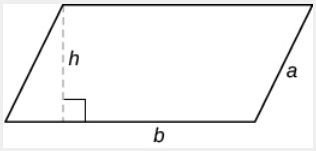
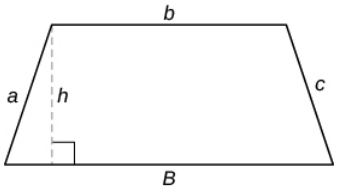
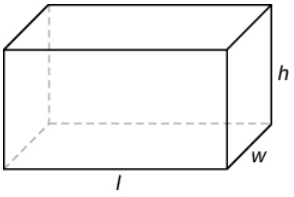
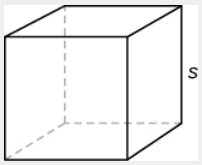
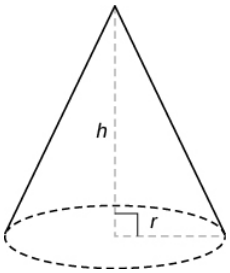
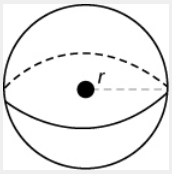
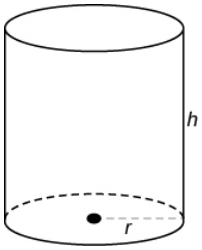
Name	Shape	Formulas
Rectangle		Perimeter: P = $2l + 2w$ Area: A = lw
Square		Perimeter: P = 4s Area: A = s^2
Triangle		Perimeter: P = a + b + c Area: A = $\frac{1}{2}$bh Sum of Angles: A + B + C = 180°
Right Triangle		Pythagorean Theorem: $a^2 + b^2 = c^2$ Area: A = $\frac{1}{2}$ab
Circle		Circumference: C = $2\pi r$ or C = πd Area: A = πr^2
Parallelogram		Perimeter: P = 2a + 2b Area: A = bh
Trapezoid		Perimeter: P = a + b + c + B Area: A = $\frac{1}{2}$(B + b)h

Table C2 - 3 Dimensions

Name	Shape	Formulas

Name	Shape	Formulas
Rectangular Solid		Volume: V = lwh Surface Area: SA = $2lw + 2wh + 2hl$
Cube		Volume: V = s^3 Surface Area: SA = $6s^2$
Cone		Volume: V = $\frac{1}{3}\pi r^2 h$ Surface Area: SA = $\pi r^2 + \pi r\sqrt{h^2 + r^2}$
Sphere		Volume: V = $\frac{4}{3}\pi r^3$ Surface Area: SA = $4\pi r^2$
Right Circular Cylinder		Volume: V = $\pi r^2 h$ Surface Area: SA = $2\pi r^2 + 2\pi rh$

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SECTION OVERVIEW

13.4: Math Models and Geometry

We are surrounded by all sorts of geometry. Architects use geometry to design buildings. Artists create vivid images out of colorful geometric shapes. Street signs, automobiles, and product packaging all take advantage of geometric properties. In this chapter, we will begin by considering a formal approach to solving problems and use it to solve a variety of common problems, including making decisions about money. Then we will explore geometry and relate it to everyday situations, using the problem-solving strategy we develop.

13.4.1: Solve Money Applications

13.4.2: Use Properties of Angles, Triangles, and the Pythagorean Theorem (Part 1)

13.4.3: Use Properties of Angles, Triangles, and the Pythagorean Theorem (Part 2)

13.4.4: Use Properties of Rectangles, Triangles, and Trapezoids (Part 1)

13.4.5: Use Properties of Rectangles, Triangles, and Trapezoids (Part 2)

Figure 9.1 - Note the many individual shapes in this building. (credit: Bert Kaufmann, Flickr)

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13.4.1: Solve Money Applications

Learning Objectives

- Solve coin word problems
- Solve ticket and stamp word problems

be prepared!

Before you get started, take this readiness quiz.

1. Multiply: 14(0.25). If you missed this problem, review Example 5.3.5.
2. Simplify: 100(0.2 + 0.05n). If you missed this problem, review Example 7.4.6.
3. Solve: 0.25x + 0.10(x + 4) = 2.5 If you missed this problem, review Example 8.6.8.

Solve Coin Word Problems

Imagine taking a handful of coins from your pocket or purse and placing them on your desk. How would you determine the value of that pile of coins?

If you can form a step-by-step plan for finding the total value of the coins, it will help you as you begin solving coin word problems.

One way to bring some order to the mess of coins would be to separate the coins into stacks according to their value. Quarters would go with quarters, dimes with dimes, nickels with nickels, and so on. To get the total value of all the coins, you would add the total value of each pile.

Figure 13.4.1.1- To determine the total value of a stack of nickels, multiply the number of nickels times the value of one nickel. (Credit: Darren Hester via ppdigital)

How would you determine the value of each pile? Think about the dime pile—how much is it worth? If you count the number of dimes, you'll know how many you have—the *number* of dimes.

But this does not tell you the *value* of all the dimes. Say you counted 17 dimes, how much are they worth? Each dime is worth \$0.10 —that is the *value* of one dime. To find the total value of the pile of 17 dimes, multiply 17 by \$0.10 to get \$1.70. This is the total value of all 17 dimes.

$$17 \cdot \$0.10 = \$1.70$$
$$number \cdot value = total\ value$$

Definition: Finding the Total Value for Coins of the Same Type

For coins of the same type, the total value can be found as follows:

$$number \cdot value = total\ value \tag{13.4.1.1}$$

where *number* is the number of coins, *value* is the value of each coin, and *total value* is the total value of all the coins.

You could continue this process for each type of coin, and then you would know the total value of each type of coin. To get the total value of *all* the coins, add the total value of each type of coin.

Let's look at a specific case. Suppose there are 14 quarters, 17 dimes, 21 nickels, and 39 pennies. We'll make a table to organize the information – the type of coin, the number of each, and the value.

Table 13.4.1.1

Type	Number	Value ($)	Total Value ($)
Quarters	14	0.25	3.50
Dimes	17	0.10	1.70
Nickels	21	0.05	1.05
Pennies	39	0.01	0.39
			6.64

The total value of all the coins is $6.64. Notice how Table 13.4.1.1 helped us organize all the information. Let's see how this method is used to solve a coin word problem.

Example 13.4.1.1:

Adalberto has $2.25 in dimes and nickels in his pocket. He has nine more nickels than dimes. How many of each type of coin does he have?

Solution

Step 1. **Read** the problem. Make sure you understand all the words and ideas.

- Determine the types of coins involved.

Think about the strategy we used to find the value of the handful of coins. The first thing you need is to notice what types of coins are involved. Adalberto has dimes and nickels.

- Create a table to organize the information.
  - Label the columns ‘type’, ‘number’, ‘value’, ‘total value’.
  - List the types of coins.
  - Write in the value of each type of coin.
  - Write in the total value of all the coins.

We can work this problem all in cents or in dollars. Here we will do it in dollars and put in the dollar sign ($) in the table as a reminder.

The value of a dime is $0.10 and the value of a nickel is $0.05. The total value of all the coins is $2.25.

Type	Number	Value ($)	Total Value ($)
Dimes		0.10	
Nickels		0.05	
			2.25

Step 2. **Identify** what you are looking for.

- We are asked to find the number of dimes and nickels Adalberto has.

Step 3. **Name** what you are looking for.

- Use variable expressions to represent the number of each type of coin.
- Multiply the number times the value to get the total value of each type of coin. In this problem you cannot count each type of coin—that is what you are looking for—but you have a clue. There are nine more nickels than dimes. The number of nickels is nine more than the number of dimes.
  - Let d = number of dimes.
  - d + 9 = number of nickels
- Fill in the "number" column to help get everything organized.

Type	Number	Value ($)	Total Value ($)
Dimes	d	0.10	
Nickels	d + 9	0.05	
			2.25

Now we have all the information we need from the problem!

You multiply the number times the value to get the total value of each type of coin. While you do not know the actual number, you do have an expression to represent it.

And so now multiply *number* • *value* and write the results in the Total Value column.

Type	Number	Value ($)	Total Value ($)
Dimes	d	0.10	0.10d
Nickels	d + 9	0.05	0.05(d + 9)
			2.25

Step 4. **Translate** into an equation. Restate the problem in one sentence. Then translate into an equation.

value of the dimes	+	value of the nickels	=	total value of the coins
$0.10d$	+	$0.05(d+9)$	=	2.25

Step 5. **Solve** the equation using good algebra techniques.

Write the equation.	$0.10d+0.05(d+9)=2.25$	$(13.4.1.2)$
Distribute.	$0.10d+0.05d+0.45=2.25$	$(13.4.1.3)$
Combine like terms.	$0.15d+0.45=2.25$	$(13.4.1.4)$
Subtract 0.45 from each side.	$0.15d=1.80$	$(13.4.1.5)$
Divide to find the number of dimes.	$d=12$	$(13.4.1.6)$

The number of nickels is d + 9.	$d+9$ $12+9$ 21

Step 6. **Check**.

$$\begin{aligned} 12\ dimes:\ 12(0.10) &= 1.20 \\ 21\ nickels:\ 21(0.05) &= 1.05 \\ \hline \$2.25\ \checkmark \end{aligned}$$

Step 7. Answer the question.

Adalberto has twelve dimes and twenty-one nickels.

If this were a homework exercise, our work might look like this:

Adalberto has $2.25 in dimes and nickels in his pocket. He has nine more nickels than dimes. How many of each type does he have?

Type	Number • Value($)		= Total Value ($)
Dimes	d	0.10	0.10d
Nickels	d + 9	0.05	0.05(d + 9)
			2.25

$0.10d + 0.05d + 0.45 = 2.25$
$0.15d + 0.45 = 2.25$
$0.15d = 1.80$
$d = 12$ dimes

$d + 9$
$12 + 9$
21 nickles

Check:

$$\begin{aligned} 12\ dimes \quad & 12(0.10) = 1.20 \\ 21\ nickels \quad & 21(0.05) = 1.05 \\ \hline & \$2.25 \end{aligned}$$

? Exercise 13.4.1.1:

Michaela has $2.05 in dimes and nickels in her change purse. She has seven more dimes than nickels. How many coins of each type does she have?

Answer

9 nickels, 16 dimes

? Exercise 13.4.1.2:

Liliana has $2.10 in nickels and quarters in her backpack. She has 12 more nickels than quarters. How many coins of each type does she have?

Answer

17 nickels, 5 quarters

HOW TO: SOLVE A COIN WORD PROBLEM

Step 1. **Read** the problem. Make sure you understand all the words and ideas, and create a table to organize the information.

Step 2. **Identify** what you are looking for.

Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.

- Use variable expressions to represent the number of each type of coin and write them in the table.
- Multiply the number times the value to get the total value of each type of coin.

Step 4. **Translate** into an equation. Write the equation by adding the total values of all the types of coins.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

You may find it helpful to put all the numbers into the table to make sure they check.

Table 13.4.1.2

	Type			Number			Value ($)			Total Value ($)	

Example 13.4.1.2:

Maria has $2.43 in quarters and pennies in her wallet. She has twice as many pennies as quarters. How many coins of each type does she have?

Solution

Step 1. Read the problem.

- Determine the types of coins involved. We know that Maria has quarters and pennies.
- Create a table to organize the information.
  - Label the columns type, number, value, total value.
  - List the types of coins.
  - Write in the value of each type of coin.
  - Write in the total value of all the coins.

Type	Number	Value ($)	Total Value ($)
Quarters		0.25	
Pennies		0.01	
			2.43

Step 2. Identify what you are looking for.

- We are looking for the number of quarters and pennies.

Step 3. Name: Represent the number of quarters and pennies using variables.

- We know Maria has twice as many pennies as quarters. The number of pennies is defined in terms of quarters.
  - Let q represent the number of quarters. Then the number of pennies is 2q.

Type	Number	Value ($)	Total Value ($)
Quarters	q	0.25	
Pennies	2q	0.01	
			2.43

Multiply the ‘number’ and the ‘value’ to get the ‘total value’ of each type of coin.

Type	Number	Value ($)	Total Value ($)
Quarters	q	0.25	0.25q
Pennies	2q	0.01	0.01(2q)
			2.43

Step 4. **Translate**. Write the equation by adding the 'total value’ of all the types of coins.

Step 5. **Solve** the equation.

Write the equation.	$0.25q+0.01(2q)=2.43$	$(13.4.1.7)$
Multiply.	$0.25q+0.02q=2.43$	$(13.4.1.8)$
Combine like terms.	$0.27q=2.43$	$(13.4.1.9)$
Divide by 0.27.	$q=9\ quarters$	$(13.4.1.10)$
The number of pennies is 2q.	$2q$ $2\cdot 9$ $18\ pennies$	

Step 6. **Check** the answer in the problem. Maria has 9 quarters and 18 pennies. Does this make $2.43?

$$\begin{array}{ll} 9\ quarters & 9(0.25)=2.25 \\ 18\ pennies & 18(0.01)=0.18 \\ \hline Total & \$2.43\ \checkmark \end{array}$$

Step 7. **Answer** the question. Maria has nine quarters and eighteen pennies.

? Exercise 13.4.1.3:

Sumanta has $4.20 in nickels and dimes in her desk drawer. She has twice as many nickels as dimes. How many coins of each type does she have?

Answer

42 nickels, 21 dimes

? Exercise 13.4.1.4:

Alison has three times as many dimes as quarters in her purse. She has $9.35 altogether. How many coins of each type does she have?

Answer

51 dimes, 17 quarters

In the next example, we'll show only the completed table—make sure you understand how to fill it in step by step.

Example 13.4.1.3:

Danny has $2.14 worth of pennies and nickels in his piggy bank. The number of nickels is two more than ten times the number of pennies. How many nickels and how many pennies does Danny have?

Solution

Step 1: **Read** the problem.	
Determine the types of coins involved. Create a table.	Pennies and nickels
Write in the value of each type of coin.	Pennies are worth $0.01. Nickels are worth $0.05.
Step 2: **Identify** what you are looking for.	the number of pennies and nickels
Step 3: **Name**. Represent the number of each type of coin using variables. The number of nickels is defined in terms of the number of pennies, so start with pennies.	Let p = number of pennies
The number of nickels is two more than then times the number of pennies.	10p + 2 = number of nickels

Multiply the number and the value to get the total value of each type of coin.

Type	Number	Value ($)	Total Value ($)
pennies	p	0.01	0.01p
nickels	10p + 2	0.05	0.05(10p + 2)
			$2.14

Step 4. **Translate**: Write the equation by adding the total value of all the types of coins.

Step 5. **Solve** the equation.

	$0.01p+0.50p+0.10=2.14$	(13.4.1.11)
	$0.51p+0.10=2.14$	(13.4.1.12)
	$0.51p=2.04$	(13.4.1.13)
	$p=4\ pennies$	(13.4.1.14)
How many nickels?	$10p+2$	(13.4.1.15)
	$10(4)+2$	(13.4.1.16)
	$42\ nickels$	(13.4.1.17)

Step 6. **Check**. Is the total value of 4 pennies and 42 nickels equal to $2.14?

$$4(0.01)+42(0.05) \stackrel{?}{=} 2.14$$
$$2.14 = 2.14 \checkmark$$

Step 7. **Answer** the question. Danny has 4 pennies and 42 nickels.

? Exercise 13.4.1.5:

Jesse has $6.55 worth of quarters and nickels in his pocket. The number of nickels is five more than two times the number of quarters. How many nickels and how many quarters does Jesse have?

Answer

41 nickels, 18 quarters

? Exercise 13.4.1.6:

Elaine has $7.00 in dimes and nickels in her coin jar. The number of dimes that Elaine has is seven less than three times the number of nickels. How many of each coin does Elaine have?

Answer

22 nickels, 59 dimes

Solve Ticket and Stamp Word Problems

The strategies we used for coin problems can be easily applied to some other kinds of problems too. Problems involving tickets or stamps are very similar to coin problems, for example. Like coins, tickets and stamps have different values; so we can organize the information in tables much like we did for coin problems.

✔ Example 13.4.1.4:

At a school concert, the total value of tickets sold was $1,506. Student tickets sold for $6 each and adult tickets sold for $9 each. The number of adult tickets sold was 5 less than three times the number of student tickets sold. How many student tickets and how many adult tickets were sold?

Solution

Step 1: **Read** the problem.

- Determine the types of tickets involved. There are student tickets and adult tickets.
- Create a table to organize the information.

Type	Number	Value ($)	Total Value ($)
Student		6	
Adult		9	
			1,506

Step 2. **Identify** what you are looking for. We are looking for the number of student and adult tickets.

Step 3. **Name**. Represent the number of each type of ticket using variables.

- We know the number of adult tickets sold was 5 less than three times the number of student tickets sold. Let s be the number of student tickets.
- Then 3s − 5 is the number of adult tickets.
- Multiply the number times the value to get the total value of each type of ticket.

Type	Number	Value ($)	Total Value ($)
Student	s	6	6s
Adult	3s - 5	9	9(9s - 5)
			1,506

Step 4. **Translate**: Write the equation by adding the total values of each type of ticket.

$$6s+9(3s-5)=1506 \qquad (13.4.1.18)$$

Step 5. **Solve** the equation.

$$\begin{aligned} 6s+27s-45 &= 1506 \\ 33s-45 &= 1506 \\ 33s &= 1551 \\ s &= 47\ students \end{aligned}$$

Substitute to find the number of adults.

$$\begin{aligned} 3s-5 &= number\ of\ adults \\ 3(47)-5 &= 136\ adults \end{aligned}$$

Step 6. **Check**. There were 47 student tickets at $6 each and 136 adult tickets at $9 each. Is the total value $1506? We find the total value of each type of ticket by multiplying the number of tickets times its value; we then add to get the total value of all the tickets sold.

$$\begin{aligned} 47\cdot 6 &= 282 \\ 136\cdot 9 &= 1224 \\ \hline & 1506 \end{aligned}$$

Step 7. **Answer** the question. They sold 47 student tickets and 136 adult tickets.

? Exercise 13.4.1.7:

The first day of a water polo tournament, the total value of tickets sold was $17,610. One-day passes sold for $20 and tournament passes sold for $30. The number of tournament passes sold was 37 more than the number of day passes sold. How many day passes and how many tournament passes were sold?

Answer

330 day passes, 367 tournament passes

? Exercise 13.4.1.8:

At the movie theater, the total value of tickets sold was $2,612.50. Adult tickets sold for $10 each and senior/ child tickets sold for $7.50 each. The number of senior/child tickets sold was 25 less than twice the number of adult tickets sold. How many senior/child tickets and how many adult tickets were sold?

Answer

112 adult tickets, 199 senior/child tickets

Now we'll do one where we fill in the table all at once.

Example 13.4.1.5:

Monica paid $10.44 for stamps she needed to mail the invitations to her sister's baby shower. The number of 49-cent stamps was four more than twice the number of 8-cent stamps. How many 49-cent stamps and how many 8-cent stamps did Monica buy?

Solution

The type of stamps are 49-cent stamps and 8-cent stamps. Their names also give the value. “The number of 49 cent stamps was four more than twice the number of 8 cent stamps.”

Let x = number of 8-cent stamps

2x + 4 = number of 49-cent stamps

Type	Number	Value ($)	Total Value ($)
49-cent stamps	2x + 4	0.49	0.49(2x + 4)
8-cent stamps	x	0.08	0.08x
			10.44

Write the equation from the total values.	$0.49(2x+4)+0.08x=10.44 \quad (13.4.1.19)$
Solve the equation.	$\begin{aligned} 0.98x+1.96+0.08x &= 10.44 \\ 1.06x+1.96 &= 10.44 \\ 1.06x &= 8.48 \\ x &= 8 \end{aligned}$
Monica bought 8 eight-cent stamps.	
Find the number of 49-cent stamps she bought by evaluating.	2x + 4 for x = 8. $\begin{aligned} &2x+4 \\ &2\cdot 8+4 \\ &16+4 \\ &20 \end{aligned}$
Check.	$\begin{aligned} 8(0.08)+20(0.49) &\stackrel{?}{=} 10.44 \\ 0.64+9.80 &\stackrel{?}{=} 10.44 \\ 10.44 &= 10.44 \checkmark \end{aligned}$

Monica bought eight 8-cent stamps and twenty 49-cent stamps.

Exercise 13.4.1.9:

Eric paid $16.64 for stamps so he could mail thank you notes for his wedding gifts. The number of 49-cent stamps was eight more than twice the number of 8-cent stamps. How many 49-cent stamps and how many 8-cent stamps did Eric buy?

Answer

32 at 49 cents, 12 at 8 cents

Exercise 13.4.1.10:

Kailee paid $14.84 for stamps. The number of 49-cent stamps was four less than three times the number of 21-cent stamps. How many 49-cent stamps and how many 21-cent stamps did Kailee buy?

Answer

26 at 49 cents, 10 at 21 cents

Practice Makes Perfect

Solve Coin Word Problems

In the following exercises, solve the coin word problems.

51. Jaime has $2.60 in dimes and nickels. The number of dimes is 14 more than the number of nickels. How many of each coin does he have?
52. Lee has $1.75 in dimes and nickels. The number of nickels is 11 more than the number of dimes. How many of each coin does he have?
53. Ngo has a collection of dimes and quarters with a total value of $3.50. The number of dimes is 7 more than the number of quarters. How many of each coin does he have?
54. Connor has a collection of dimes and quarters with a total value of $6.30. The number of dimes is 14 more than the number of quarters. How many of each coin does he have?
55. Carolyn has $2.55 in her purse in nickels and dimes. The number of nickels is 9 less than three times the number of dimes. Find the number of each type of coin.
56. Julio has $2.75 in his pocket in nickels and dimes. The number of dimes is 10 less than twice the number of nickels. Find the number of each type of coin.
57. Chi has $11.30 in dimes and quarters. The number of dimes is 3 more than three times the number of quarters. How many dimes and nickels does Chi have?
58. Tyler has $9.70 in dimes and quarters. The number of quarters is 8 more than four times the number of dimes. How many of each coin does he have?
59. A cash box of $1 and $5 bills is worth $45. The number of $1 bills is 3 more than the number of $5 bills. How many of each bill does it contain?
60. Joe's wallet contains $1 and $5 bills worth $47. The number of $1 bills is 5 more than the number of $5 bills. How many of each bill does he have?
61. In a cash drawer there is $125 in $5 and $10 bills. The number of $10 bills is twice the number of $5 bills. How many of each are in the drawer?
62. John has $175 in $5 and $10 bills in his drawer. The number of $5 bills is three times the number of $10 bills. How many of each are in the drawer?
63. Mukul has $3.75 in quarters, dimes and nickels in his pocket. He has five more dimes than quarters and nine more nickels than quarters. How many of each coin are in his pocket?
64. Vina has $4.70 in quarters, dimes and nickels in her purse. She has eight more dimes than quarters and six more nickels than quarters. How many of each coin are in her purse?

Solve Ticket and Stamp Word Problems

In the following exercises, solve the ticket and stamp word problems.

65. The play took in $550 one night. The number of $8 adult tickets was 10 less than twice the number of $5 child tickets. How many of each ticket were sold?
66. If the number of $8 child tickets is seventeen less than three times the number of $12 adult tickets and the theater took in $584, how many of each ticket were sold?
67. The movie theater took in $1,220 one Monday night. The number of $7 child tickets was ten more than twice the number of $9 adult tickets. How many of each were sold?
68. The ball game took in $1,340 one Saturday. The number of $12 adult tickets was 15 more than twice the number of $5 child tickets. How many of each were sold?

69. Julie went to the post office and bought both $0.49 stamps and $0.34 postcards for her office's bills She spent $62.60. The number of stamps was 20 more than twice the number of postcards. How many of each did she buy?
70. Before he left for college out of state, Jason went to the post office and bought both $0.49 stamps and $0.34 postcards and spent $12.52. The number of stamps was 4 more than twice the number of postcards. How many of each did he buy?
71. Maria spent $16.80 at the post office. She bought three times as many $0.49 stamps as $0.21 stamps. How many of each did she buy?
72. Hector spent $43.40 at the post office. He bought four times as many $0.49 stamps as $0.21 stamps. How many of each did he buy?
73. Hilda has $210 worth of $10 and $12 stock shares. The numbers of $10 shares is 5 more than twice the number of $12 shares. How many of each does she have?
74. Mario invested $475 in $45 and $25 stock shares. The number of $25 shares was 5 less than three times the number of $45 shares. How many of each type of share did he buy?

Everyday Math

75. **Parent Volunteer** As the treasurer of her daughter's Girl Scout troop, Laney collected money for some girls and adults to go to a 3-day camp. Each girl paid $75 and each adult paid $30. The total amount of money collected for camp was $765. If the number of girls is three times the number of adults, how many girls and how many adults paid for camp?
76. **Parent Volunteer** Laurie was completing the treasurer's report for her son's Boy Scout troop at the end of the school year. She didn't remember how many boys had paid the $24 full-year registration fee and how many had paid a $16 partial-year fee. She knew that the number of boys who paid for a full-year was ten more than the number who paid for a partial-year. If $400 was collected for all the registrations, how many boys had paid the full-year fee and how many had paid the partial-year fee?

Writing Exercises

77. Suppose you have 6 quarters, 9 dimes, and 4 pennies. Explain how you find the total value of all the coins.
78. Do you find it helpful to use a table when solving coin problems? Why or why not?
79. In the table used to solve coin problems, one column is labeled “number” and another column is labeled ‘“value.” What is the difference between the number and the value?
80. What similarities and differences did you see between solving the coin problems and the ticket and stamp problems?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve coin word problems.			
solve ticket and stamp word problems.			

(b) After reviewing this checklist, what will you do to become confident for all objectives?

Contributors and Attributions

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13.4.2: Use Properties of Angles, Triangles, and the Pythagorean Theorem (Part 1)

Learning Objectives

- Use the properties of angles
- Use the properties of triangles
- Use the Pythagorean Theorem

be prepared!

Before you get started, take this readiness quiz.

1. Solve: x + 3 + 6 = 11. If you missed this problem, review Example 8.1.6.
2. Solve: $\frac{a}{45} = \frac{4}{3}$. If you missed this problem, review Example 6.5.3.
3. Simplify: $\sqrt{36+64}$. If you missed this problem, review Example 5.12.4.

So far in this chapter, we have focused on solving word problems, which are similar to many real-world applications of algebra. In the next few sections, we will apply our problem-solving strategies to some common geometry problems.

Use the Properties of Angles

Are you familiar with the phrase 'do a 180'? It means to turn so that you face the opposite direction. It comes from the fact that the measure of an angle that makes a straight line is 180 degrees. See Figure 13.4.2.1.

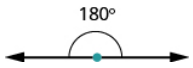


Figure 13.4.2.1

An **angle** is formed by two rays that share a common endpoint. Each ray is called a side of the angle and the common endpoint is called the **vertex**. An angle is named by its vertex. In Figure 13.4.2.2, ∠A is the angle with vertex at point A. The measure of ∠A is written m ∠ A.

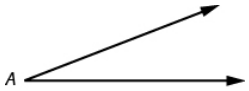


Figure 13.4.2.2- ∠ A is the angle with vertex at point A.

We measure angles in degrees, and use the symbol ° to represent degrees. We use the abbreviation m for the *measure* of an angle. So if ∠A is 27°, we would write m ∠ A = 27.

If the sum of the measures of two angles is 180°, then they are called **supplementary angles**. In Figure 13.4.2.3, each pair of angles is supplementary because their measures add to 180°. Each angle is the *supplement* of the other.

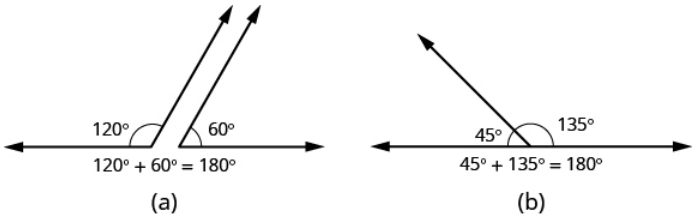


Figure 13.4.2.3- The sum of the measures of supplementary angles is 180°.

If the sum of the measures of two angles is 90°, then the angles are **complementary angles**. In Figure 13.4.2.4, each pair of angles is complementary, because their measures add to 90°. Each angle is the *complement* of the other.

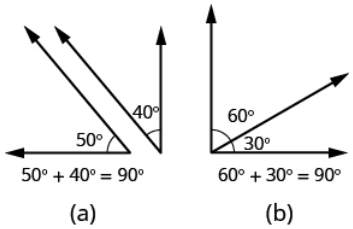


Figure 13.4.2.4- The sum of the measures of complementary angles is 90°.

Definition: Supplementary and Complementary Angles

If the sum of the measures of two angles is 180°, then the angles are supplementary.

If ∠A and ∠B are supplementary, then m∠A + m∠B = 180°.

If the sum of the measures of two angles is 90°, then the angles are complementary.

If ∠A and ∠B are complementary, then m∠A + m∠B = 90°.

In this section and the next, you will be introduced to some common geometry formulas. We will adapt our Problem Solving Strategy for Geometry Applications. The geometry formula will name the variables and give us the equation to solve.

In addition, since these applications will all involve geometric shapes, it will be helpful to draw a figure and then label it with the information from the problem. We will include this step in the Problem Solving Strategy for Geometry Applications.

HOW TO: USE A PROBLEM SOLVING STRATEGY FOR GEOMETRY APPLICATIONS

Step 1. **Read** the problem and make sure you understand all the words and ideas. Draw a figure and label it with the given information.

Step 2. **Identify** what you are looking for

Step 3. **Name** what you are looking for and choose a variable to represent it.

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

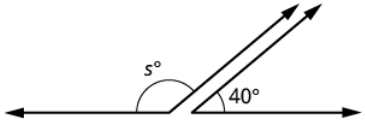
The next example will show how you can use the Problem Solving Strategy for Geometry Applications to answer questions about supplementary and complementary angles.

Example 13.4.2.1:

An angle measures 40°. Find (a) its supplement, and (b) its complement.

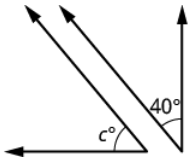
Solution

(a)

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the supplement of a 40°
Step 3. **Name**. Choose a variable to represent it.	let s = the measure of the supplement

Step 4. **Translate**. Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B = 180$ (13.4.2.1)
Step 5. **Solve** the equation.	$s + 40 = 180$ $s = 140$
Step 6. **Check.**	$140 + 40 \stackrel{?}{=} 180$ $180 = 180$ ✓
Step 7. **Answer** the question.	The supplement of the 40° angle is 140°.

(b)

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the complement of a 40°
Step 3. **Name**. Choose a variable to represent it.	let c = the measure of the complement
Step 4. **Translate**. Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B = 90$ (13.4.2.2)
Step 5. **Solve** the equation.	$c + 40 = 90$ $c = 50$
Step 6. **Check.**	$50 + 40 \stackrel{?}{=} 90$ $90 = 90$ ✓
Step 7. **Answer** the question.	The supplement of the 40° angle is 50°.

? Exercise 13.4.2.1:

An angle measures 25°. Find (a) its supplement, and (b) its complement.

Answer a

155°

Answer b

65°

? Exercise 13.4.2.2:

An angle measures 77°. Find (a) its supplement, and (b) its complement.

Answer a

103°

Answer b

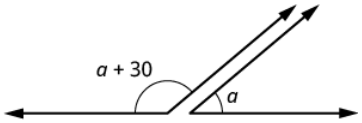
13°

Did you notice that the words complementary and supplementary are in alphabetical order just like 90 and 180 are in numerical order?

Example 13.4.2.2:

Two angles are supplementary. The larger angle is 30° more than the smaller angle. Find the measure of both angles.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the measures of both angles
Step 3. **Name**. Choose a variable to represent it.	let a = measure of the smaller angle a + 30 = measure of larger angle
Step 4. **Translate**. Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B = 180$ (13.4.2.3)
Step 5. **Solve** the equation.	$(a+30)+a = 180$ $2a+30 = 180$ $2a = 150$ $a = 75$ *measure of smaller angle* $a+30$ *measure of larger angle* $75+30$ 105
Step 6. **Check.**	$m\angle A + m\angle B = 180$ $75+105 \stackrel{?}{=} 180$ $180 = 180$ ✓
Step 7. **Answer** the question.	The measures of the angles are 75° and 105°.

Exercise 13.4.2.3:

Two angles are supplementary. The larger angle is 100° more than the smaller angle. Find the measures of both angles.

Answer

40°, 140°

Exercise 13.4.2.4:

Two angles are complementary. The larger angle is 40° more than the smaller angle. Find the measures of both angles.

Answer

25°, 65°

Use the Properties of Triangles

What do you already know about triangles? Triangle have three sides and three angles. Triangles are named by their vertices. The **triangle** in Figure 13.4.2.5is called ΔABC, read ‘triangle ABC’. We label each side with a lower case letter to match the upper

case letter of the opposite vertex.

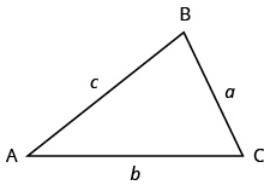


Figure 13.4.2.5- ΔABC has vertices A, B, and C and sides a, b, and c.

The three angles of a triangle are related in a special way. The sum of their measures is 180°.

$$m\angle A + m\angle B + m\angle C = 180^\circ \qquad (13.4.2.4)$$

Definition: Sum of the Measures of the Angles of a Triangle

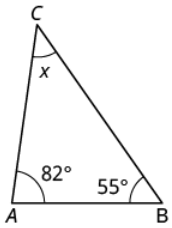
For any ΔABC, the sum of the measures of the angles is 180°.

$$m\angle A + m\angle B + m\angle C = 180^\circ \qquad (13.4.2.5)$$

Example 13.4.2.3:

The measures of two angles of a triangle are 55° and 82°. Find the measure of the third angle.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the measure of the third angle in a triangle
Step 3. **Name**. Choose a variable to represent it.	let x = the measure of the angle
Step 4. **Translate**. Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B + m\angle C = 180 \qquad (13.4.2.6)$
Step 5. **Solve** the equation.	$55 + 82 + x = 180$ $137 + x = 180$ $x = 43$
Step 6. **Check.**	$55 + 82 + 43 \stackrel{?}{=} 180$ $180 = 180$ ✓
Step 7. **Answer** the question.	The measure of the third angle is 43 degrees.

Exercise 13.4.2.5:

The measures of two angles of a triangle are 31° and 128°. Find the measure of the third angle.

Answer

21°

Exercise 13.4.2.6:

A triangle has angles of 49° and 75°. Find the measure of the third angle.

Answer

56°

Right Triangles

Some triangles have special names. We will look first at the **right triangle**. A right triangle has one 90° angle, which is often marked with the symbol shown in Figure 13.4.2.6

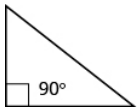


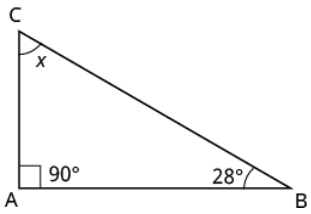
Figure 13.4.2.6

If we know that a triangle is a right triangle, we know that one angle measures 90° so we only need the measure of one of the other angles in order to determine the measure of the third angle.

Example 13.4.2.4:

One angle of a right triangle measures 28°. What is the measure of the third angle?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the measure of an angle
Step 3. **Name**. Choose a variable to represent it.	let x = the measure of the angle
Step 4. **Translate**. Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B + m\angle C = 180$ (13.4.2.7)
Step 5. **Solve** the equation.	$x + 90 + 28 = 180$ $x + 118 = 180$ $x = 62$
Step 6. **Check.**	$180 \stackrel{?}{=} 90 + 28 + 62$ $180 = 180$ ✓
Step 7. **Answer** the question.	The measure of the third angle is 62°.

Exercise 13.4.2.7:

One angle of a right triangle measures 56°. What is the measure of the other angle?

Answer

34°

Exercise 13.4.2.8:

One angle of a right triangle measures 45°. What is the measure of the other angle?

Answer

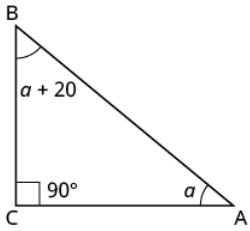
45°

In the examples so far, we could draw a figure and label it directly after reading the problem. In the next example, we will have to define one angle in terms of another. So we will wait to draw the figure until we write expressions for all the angles we are looking for.

Example 13.4.2.5:

The measure of one angle of a right triangle is 20° more than the measure of the smallest angle. Find the measures of all three angles.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the measures of all three angles
Step 3. **Name**. Choose a variable to represent it. Now draw the figure and label it with the given information.	Let a = 1st angle a + 20 = 2nd angle 90 = 3rd angle (the right angle) 
Step 4. **Translate**. Write the appropriate formula for the situation and substitute in the given information.	$m\angle A + m\angle B + m\angle C = 180$ $a + (a + 20) + 90 = 180$
Step 5. **Solve** the equation.	$2a + 110 = 180$ $2a = 70$ $a = 35$ *first angle* $a+20$ *second angle* $35+20$ 55 90 *third angle*
Step 6. **Check.**	$35 + 55 + 90 \stackrel{?}{=} 180$ $180 = 180$ ✓
Step 7. **Answer** the question.	The three angles measure 35°, 55°, and 90°.

Exercise 13.4.2.9:

The measure of one angle of a right triangle is 50° more than the measure of the smallest angle. Find the measures of all three angles.

Answer

20°, 70°, 90°

Exercise 13.4.2.10:

The measure of one angle of a right triangle is 30° more than the measure of the smallest angle. Find the measures of all three angles.

Answer

30°, 60°, 90°

Similar Triangles

When we use a map to plan a trip, a sketch to build a bookcase, or a pattern to sew a dress, we are working with similar figures. In geometry, if two figures have exactly the same shape but different sizes, we say they are **similar figures**. One is a scale model of the other. The corresponding sides of the two figures have the same ratio, and all their corresponding angles are have the same measures.

The two triangles in Figure 13.4.2.7 are similar. Each side of ΔABC is four times the length of the corresponding side of ΔXYZ and their corresponding angles have equal measures.

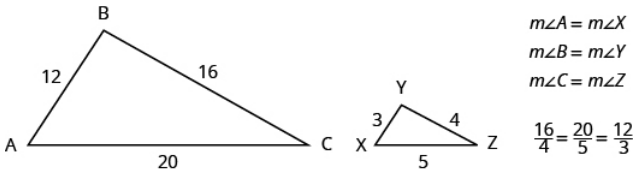
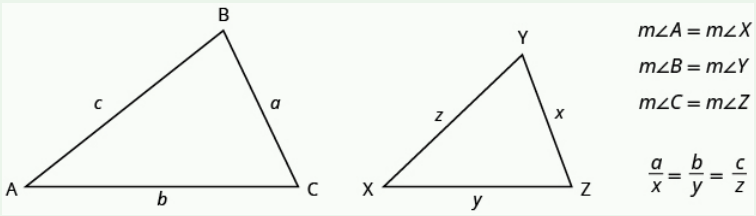


Figure 13.4.2.7- ΔABC and ΔXYZ are similar triangles. Their corresponding sides have the same ratio and the corresponding angles have the same measure.

Definition: Properties of Similar Triangles

If two triangles are similar, then their corresponding angle measures are equal and their corresponding side lengths are in the same ratio.



The length of a side of a triangle may be referred to by its endpoints, two vertices of the triangle. For example, in ΔABC:

the length a can also be written BC

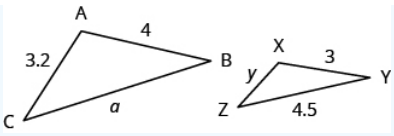
the length b can also be written AC

the length c can also be written AB

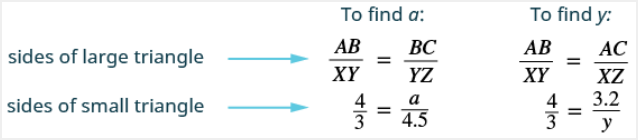
We will often use this notation when we solve similar triangles because it will help us match up the corresponding side lengths.

Example 13.4.2.6:

ΔABC and ΔXYZ are similar triangles. The lengths of two sides of each triangle are shown. Find the lengths of the third side of each triangle.

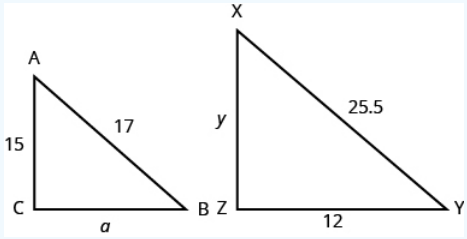


Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	The figure is provided.
Step 2. **Identify** what you are looking for.	The length of the sides of similar triangles
Step 3. **Name**. Choose a variable to represent it.	Let a = length of the third side of ΔABC, y = length of the third side ΔXYZ
Step 4. **Translate**.	The triangles are similar, so the corresponding sides are in the same ratio. So $$\frac{AB}{XY}=\frac{BC}{YZ}=\frac{AC}{XZ} \quad (13.4.2.8)$$ Since the side AB = 4 corresponds to the side XY = 3 , we will use the ratio $\frac{AB}{XY}=\frac{4}{3}$ to find the other sides. Be careful to match up corresponding sides correctly. 
Step 5. **Solve** the equation.	$3a=4(4.5)$ $3a=18$ $a=6$ $4y=3(3.2)$ $4y=9.6$ $y=2.4$
Step 6. **Check.**	$\frac{4}{3}\stackrel{?}{=}\frac{6}{4.5}$ $4(4.5)\stackrel{?}{=}6(3)$ $18=18$ ✓ $\frac{4}{3}\stackrel{?}{=}\frac{3.2}{2.4}$ $4(2.4)\stackrel{?}{=}3.2(3)$ $9.6=9.6$ ✓
Step 7. **Answer** the question.	The third side of ΔABC is 6 and the third side of ΔXYZ is 2.4.

? Exercise 13.4.2.11:

ΔABC is similar to ΔXYZ. Find a.

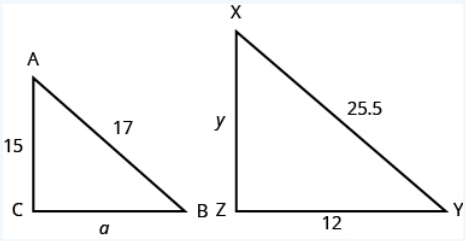


Answer

a = 8

Exercise 13.4.2.12:

ΔABC is similar to ΔXYZ. Find y.



Answer

y = 22.5

Contributors and Attributions

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13.4.3: Use Properties of Angles, Triangles, and the Pythagorean Theorem (Part 2)

Use the Pythagorean Theorem

The **Pythagorean Theorem** is a special property of right triangles that has been used since ancient times. It is named after the Greek philosopher and mathematician Pythagoras who lived around 500 BCE.

Remember that a right triangle has a 90° angle, which we usually mark with a small square in the corner. The side of the triangle opposite the 90° angle is called the **hypotenuse**, and the other two sides are called the **legs**. See Figure 13.4.3.8

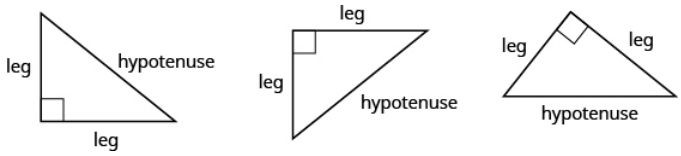


Figure 13.4.3.8- In a right triangle, the side opposite the 90° angle is called the hypotenuse and each of the other sides is called a leg.

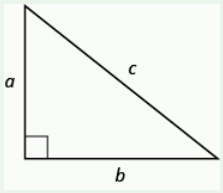
The Pythagorean Theorem tells how the lengths of the three sides of a right triangle relate to each other. It states that in any right triangle, the sum of the squares of the two legs equals the square of the hypotenuse.

Definition: The Pythagorean Theorem

In any right triangle ΔABC,

$$a^2 + b^2 = c^2 \tag{13.4.3.1}$$

where c is the length of the hypotenuse a and b are the lengths of the legs.



To solve problems that use the Pythagorean Theorem, we will need to find square roots. In Simplify and Use Square Roots we introduced the notation m and defined it in this way:

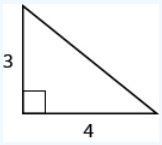
$$If\ m = n^2, then\ \sqrt{m} = n\ for\ n \geq 0 \tag{13.4.3.2}$$

For example, we found that $\sqrt{25}$ is 5 because 5^2 = 25.

We will use this definition of square roots to solve for the length of a side in a right triangle.

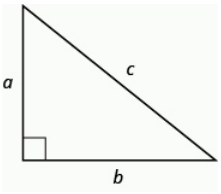
Example 13.4.3.7:

Use the Pythagorean Theorem to find the length of the hypotenuse.



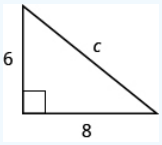
Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the length of the hypotenuse of the triangle

Step 3. **Name**. Choose a variable to represent it.	Let c = the length of the hypotenuse 
Step 4. **Translate**. Write the appropriate formula. Substitute.	$a^2+b^2=c^2$ $3^2+4^2=c^2$
Step 5. **Solve** the equation.	$9+16=c^2$ $25=c^2$ $\sqrt{25}=c^2$ $5=c$
Step 6. **Check**.	$3^2+4^2=5^2$ $9+16\stackrel{?}{=}25$ $25=25$ ✓
Step 7. **Answer** the question.	The length of the hypotenuse is 5.

? Exercise 13.4.3.13:

Use the Pythagorean Theorem to find the length of the hypotenuse.

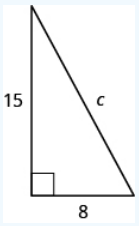


Answer

c = 10

? Exercise 13.4.3.14:

Use the Pythagorean Theorem to find the length of the hypotenuse.

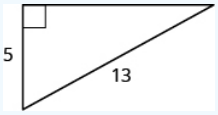


Answer

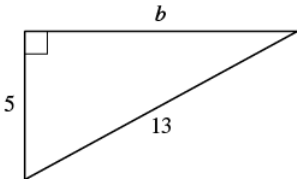
c = 17

Example 13.4.3.8:

Use the Pythagorean Theorem to find the length of the longer leg.

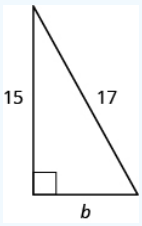


Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	The length of the leg of the triangle
Step 3. **Name**. Choose a variable to represent it.	Let b = the leg of the triangle Label side b 
Step 4. **Translate**. Write the appropriate formula. Substitute.	$a^2+b^2=c^2$ $5^2+b^2=13^2$
Step 5. **Solve** the equation.	$25+b^2=169$ $b^2=144$ $b^2=\sqrt{144}$ $b=12$
Step 6. **Check**.	$5^2+12^2 \stackrel{?}{=} 13^2$ $25+144 \stackrel{?}{=} 169$ $169=169$ ✓
Step 7. **Answer** the question.	The length of the leg is 12.

Exercise 13.4.3.15:

Use the Pythagorean Theorem to find the length of the leg.

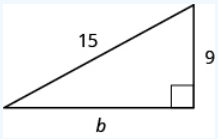


Answer

b = 8

Exercise 13.4.3.16:

Use the Pythagorean Theorem to find the length of the leg.

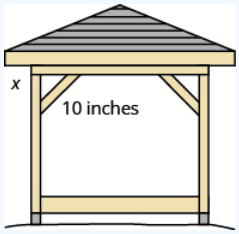


Answer

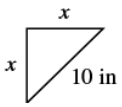
b = 12

Example 13.4.3.9:

Kelvin is building a gazebo and wants to brace each corner by placing a 10-inch wooden bracket diagonally as shown. How far below the corner should he fasten the bracket if he wants the distances from the corner to each end of the bracket to be equal? Approximate to the nearest tenth of an inch.



Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the distance from the corner that the bracket should be attached
Step 3. **Name**. Choose a variable to represent it.	Let x = the distance from the corner 
Step 4. **Translate**. Write the appropriate formula. Substitute.	$a^2+b^2=c^2$ $x^2+x^2=10^2$
Step 5. **Solve** the equation. Isolate the variable. Use the definition of the square root. Simplify. Approximate to the nearest tenth.	$2x^2=100$ $x^2=50$ $x=\sqrt{50}$ $b\approx 7.1$
Step 6. **Check**.	$a^2+b^2=c^2$ $(7.1)^2+(7.1)^2 \stackrel{?}{\approx} 10^2$ $Yes.$
Step 7. **Answer** the question.	Kelvin should fasten each piece of wood approximately 7.1" from the corner.

Exercise 13.4.3.17:

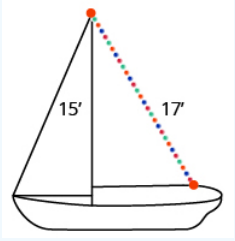
John puts the base of a 13-ft ladder 5 feet from the wall of his house. How far up the wall does the ladder reach?

Answer

12 feet

Exercise 13.4.3.18:

Randy wants to attach a 17-ft string of lights to the top of the 15-ft mast of his sailboat. How far from the base of the mast should he attach the end of the light string?



Answer

8 feet

ACCESS ADDITIONAL ONLINE RESOURCES

Animation: The Sum of the Interior Angles of a Triangle

Similar Polygons

Example: Determine the Length of the Hypotenuse of a Right Triangle

Practice Makes Perfect

Use the Properties of Angles

In the following exercises, find (a) the supplement and (b) the complement of the given angle.

81. 53°
82. 16°
83. 29°
84. 72°

In the following exercises, use the properties of angles to solve.

85. Find the supplement of a 135° angle.
86. Find the complement of a 38° angle.
87. Find the complement of a 27.5° angle.
88. Find the supplement of a 109.5° angle.
89. Two angles are supplementary. The larger angle is 56° more than the smaller angle. Find the measures of both angles.

90. Two angles are supplementary. The smaller angle is 36° less than the larger angle. Find the measures of both angles.
91. Two angles are complementary. The smaller angle is 34° less than the larger angle. Find the measures of both angles.
92. Two angles are complementary. The larger angle is 52° more than the smaller angle. Find the measures of both angles.

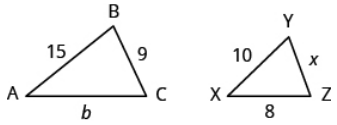
Use the Properties of Triangles

In the following exercises, solve using properties of triangles.

93. The measures of two angles of a triangle are 26° and 98°. Find the measure of the third angle.
94. The measures of two angles of a triangle are 61° and 84°. Find the measure of the third angle.
95. The measures of two angles of a triangle are 105° and 31°. Find the measure of the third angle.
96. The measures of two angles of a triangle are 47° and 72°. Find the measure of the third angle.
97. One angle of a right triangle measures 33°. What is the measure of the other angle?
98. One angle of a right triangle measures 51°. What is the measure of the other angle?
99. One angle of a right triangle measures 22.5°. What is the measure of the other angle?
100. One angle of a right triangle measures 36.5°. What is the measure of the other angle?
101. The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.
102. The measure of the smallest angle of a right triangle is 20° less than the measure of the other small angle. Find the measures of all three angles.
103. The angles in a triangle are such that the measure of one angle is twice the measure of the smallest angle, while the measure of the third angle is three times the measure of the smallest angle. Find the measures of all three angles.
104. The angles in a triangle are such that the measure of one angle is 20° more than the measure of the smallest angle, while the measure of the third angle is three times the measure of the smallest angle. Find the measures of all three angles.

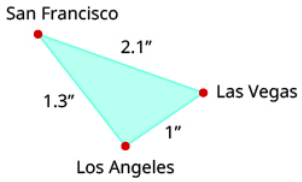
Find the Length of the Missing Side

In the following exercises, Δ ABC is similar to Δ XYZ. Find the length of the indicated side.



105. side b
106. side x

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle whose sides are shown in the figure below. The actual distance from Los Angeles to Las Vegas is 270 miles.

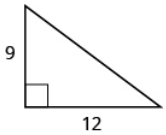


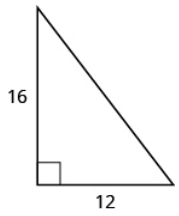
107. Find the distance from Los Angeles to San Francisco.
108. Find the distance from San Francisco to Las Vegas.

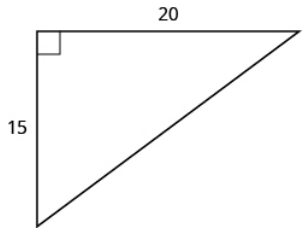
Use the Pythagorean Theorem

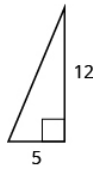
In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.

109.



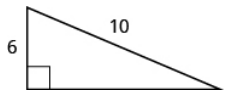
110. 

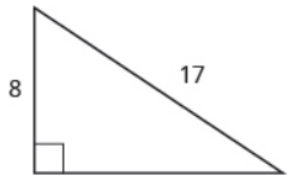
111. 

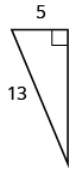
112. 

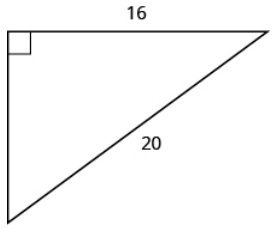
Find the Length of the Missing Side

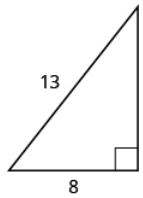
In the following exercises, use the Pythagorean Theorem to find the length of the missing side. Round to the nearest tenth, if necessary.

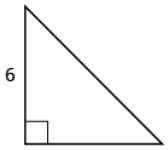
113. 

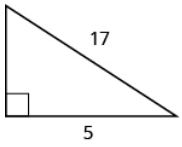
114. 

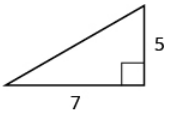
115. 

116. 

117. 

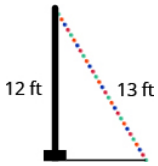
118. 

119. 

120. 

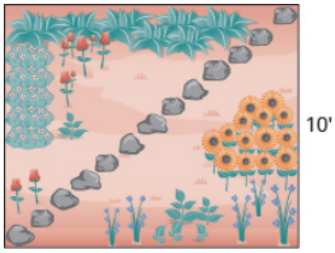
In the following exercises, solve. Approximate to the nearest tenth, if necessary.

121. A 13-foot string of lights will be attached to the top of a 12-foot pole for a holiday display. How far from the base of the pole should the end of the string of lights be anchored?



122. Pam wants to put a banner across her garage door to congratulate her son on his college graduation. The garage door is 12 feet high and 16 feet wide. How long should the banner be to fit the garage door?

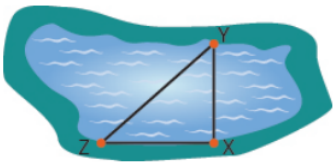
123. Chi is planning to put a path of paving stones through her flower garden. The flower garden is a square with sides of 10 feet. What will the length of the path be?



124. Brian borrowed a 20-foot extension ladder to paint his house. If he sets the base of the ladder 6 feet from the house, how far up will the top of the ladder reach?

Everyday Math

125. **Building a scale model** Joe wants to build a doll house for his daughter. He wants the doll house to look just like his house. His house is 30 feet wide and 35 feet tall at the highest point of the roof. If the dollhouse will be 2.5 feet wide, how tall will its highest point be?
126. **Measurement** A city engineer plans to build a footbridge across a lake from point X to point Y, as shown in the picture below. To find the length of the footbridge, she draws a right triangle XYZ, with right angle at X. She measures the distance from X to Z, 800 feet, and from Y to Z, 1,000 feet. How long will the bridge be?



Writing Exercises

127. Write three of the properties of triangles from this section and then explain each in your own words.
128. Explain how the figure below illustrates the Pythagorean Theorem for a triangle with legs of length 3 and 4.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the properties of angles.			
use the properties of triangles.			
use the Pythagorean Theorem.			

(b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Contributors and Attributions

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13.4.4: Use Properties of Rectangles, Triangles, and Trapezoids (Part 1)

Learning Objectives

- Understand linear, square, and cubic measure
- Use properties of rectangles
- Use properties of triangles
- Use properties of trapezoids

be prepared!

Before you get started, take this readiness quiz.

1. The length of a rectangle is 3 less than the width. Let w represent the width. Write an expression for the length of the rectangle. If you missed this problem, review Example 2.4.14.
2. Simplify: $\frac{1}{2}(6h)$. If you missed this problem, review Example 7.2.3.
3. Simplify: $\frac{5}{2}(10.3 - 7.9)$. If you missed this problem, review Example 5.5.9.

In this section, we'll continue working with geometry applications. We will add some more properties of triangles, and we'll learn about the properties of rectangles and trapezoids.

Understand Linear, Square, and Cubic Measure

When you measure your height or the length of a garden hose, you use a ruler or tape measure (Figure 13.4.4.1). A tape measure might remind you of a line—you use it for **linear measure**, which measures length. Inch, foot, yard, mile, centimeter and meter are units of linear measure.

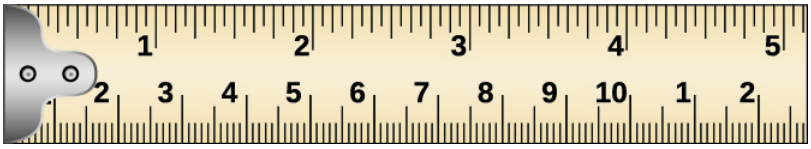


Figure 13.4.4.1- This tape measure measures inches along the top and centimeters along the bottom.

When you want to know how much tile is needed to cover a floor, or the size of a wall to be painted, you need to know the **area**, a measure of the region needed to cover a surface. Area is measured is **square units**. We often use square inches, square feet, square centimeters, or square miles to measure area. A square centimeter is a square that is one centimeter (cm) on each side. A square inch is a square that is one inch on each side (Figure 13.4.4.2).

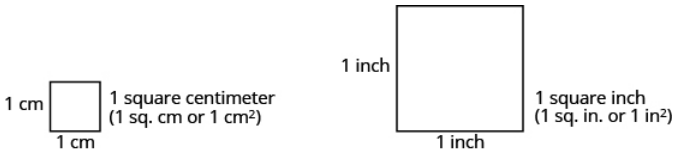


Figure 13.4.4.2- Square measures have sides that are each 1 unit in length.

Figure 13.4.4.3 shows a rectangular rug that is 2 feet long by 3 feet wide. Each square is 1 foot wide by 1 foot long, or 1 square foot. The rug is made of 6 squares. The area of the rug is 6 square feet.

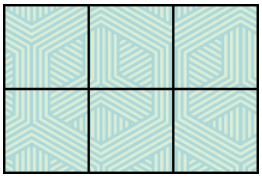


Figure 13.4.4.3- The rug contains six squares of 1 square foot each, so the total area of the rug is 6 square feet.

When you measure how much it takes to fill a container, such as the amount of gasoline that can fit in a tank, or the amount of medicine in a syringe, you are measuring **volume**. Volume is measured in **cubic units** such as cubic inches or cubic centimeters. When measuring the volume of a rectangular solid, you measure how many cubes fill the container. We often use cubic centimeters, cubic inches, and cubic feet. A cubic centimeter is a cube that measures one centimeter on each side, while a cubic inch is a cube that measures one inch on each side (Figure 13.4.4.4).

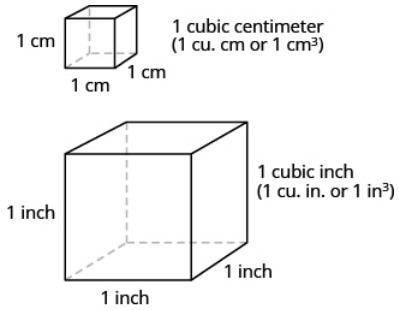


Figure 13.4.4.4- Cubic measures have sides that are 1 unit in length.

Suppose the cube in Figure 13.4.4.5 measures 3 inches on each side and is cut on the lines shown. How many little cubes does it contain? If we were to take the big cube apart, we would find 27 little cubes, with each one measuring one inch on all sides. So each little cube has a volume of 1 cubic inch, and the volume of the big cube is 27 cubic inches.

Figure 13.4.4.5- A cube that measures 3 inches on each side is made up of 27 one-inch cubes, or 27 cubic inches.

Example 13.4.4.1:

For each item, state whether you would use linear, square, or cubic measure: (a) amount of carpeting needed in a room (b) extension cord length (c) amount of sand in a sandbox (d) length of a curtain rod (e) amount of flour in a canister (f) size of the roof of a doghouse.

Solution

(a) You are measuring how much surface the carpet covers, which is the area.	square measure
(b) You are measuring how long the extension cord is, which is the length.	linear measure
(c) You are measuring the volume of the sand.	cubic measure
(d) You are measuring the length of the curtain rod.	linear measure
(e) You are measuring the volume of the flour.	cubic measure
(f) You are measuring the area of the roof.	square measure

Exercise 13.4.4.1:

Determine whether you would use linear, square, or cubic measure for each item. (a) amount of paint in a can (b) height of a tree (c) floor of your bedroom (d) diameter of bike wheel (e) size of a piece of sod (f) amount of water in a swimming pool

Answer a

cubic

Answer b

linear

Answer c

square

Answer d

linear

Answer e

square

Answer f

cubic

Exercise 13.4.4.2:

Determine whether you would use linear, square, or cubic measure for each item. (a) volume of a packing box (b) size of patio (c) amount of medicine in a syringe (d) length of a piece of yarn (e) size of housing lot (f) height of a flagpole

Answer a

cubic

Answer b

square

Answer c

cubic

Answer d

linear

Answer e

square

Answer f

linear

Many geometry applications will involve finding the perimeter or the area of a figure. There are also many applications of perimeter and area in everyday life, so it is important to make sure you understand what they each mean.

Picture a room that needs new floor tiles. The tiles come in squares that are a foot on each side—one square foot. How many of those squares are needed to cover the floor? This is the area of the floor.

Next, think about putting new baseboard around the room, once the tiles have been laid. To figure out how many strips are needed, you must know the distance around the room. You would use a tape measure to measure the number of feet around the room. This distance is the perimeter.

Definition: Perimeter and Area

The perimeter is a measure of the distance around a figure.

The area is a measure of the surface covered by a figure.

Figure 13.4.4.6shows a square tile that is 1 inch on each side. If an ant walked around the edge of the tile, it would walk 4 inches. This distance is the perimeter of the tile.

Since the tile is a square that is 1 inch on each side, its area is one square inch. The area of a shape is measured by determining how many square units cover the shape.

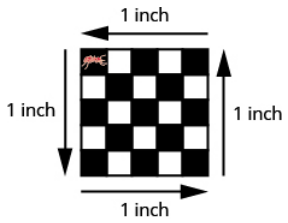
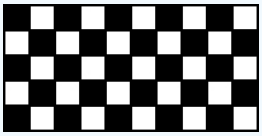


Figure 13.4.4.6- Perimeter = 4 inches, Area = 1 square inch. When the ant walks completely around the tile on its edge, it is tracing the perimeter of the tile. The area of the tile is 1 square inch.

Example 13.4.4.2:

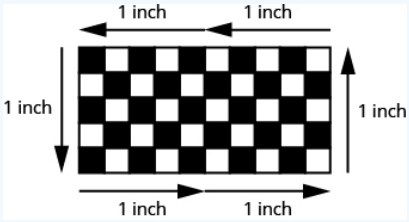
Each of two square tiles is 1 square inch. Two tiles are shown together. (a) What is the perimeter of the figure? (b) What is the area?



Solution

(a) The perimeter is the distance around the figure. The perimeter is 6 inches.

(b) The area is the surface covered by the figure. There are 2 square inch tiles so the area is 2 square inches.



Exercise 13.4.4.3:

Find the (a) perimeter and (b) area of the figure:

Answer a

8 inches

Answer b

3 sq. inches

Exercise 13.4.4.4:

Find the (a) perimeter and (b) area of the figure:

Answer a

8 centimeters

Answer b

4 sq. centimeters

Use the Properties of Rectangles

A **rectangle** has four sides and four right angles. The opposite sides of a rectangle are the same length. We refer to one side of the rectangle as the length, L, and the adjacent side as the width, W. See Figure 13.4.4.7.

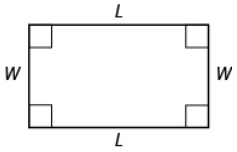


Figure 13.4.4.7- A rectangle has four sides, and four right angles. The sides are labeled L for length and W for width.

The perimeter, P, of the rectangle is the distance around the rectangle. If you started at one corner and walked around the rectangle, you would walk L + W + L + W units, or two lengths and two widths. The perimeter then is

$$P = L + W + L + W$$
$$or$$
$$P = 2L + 2W$$

What about the area of a rectangle? Remember the rectangular rug from the beginning of this section. It was 2 feet long by 3 feet wide, and its area was 6 square feet. See Figure 13.4.4.8 Since A = 2 • 3, we see that the area, A, is the length, L, times the width, W, so the area of a rectangle is A = L • W.

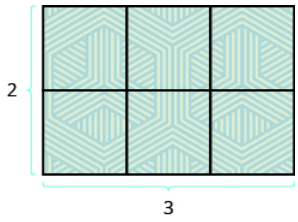


Figure 13.4.4.8- The area of this rectangular rug is 6 square feet, its length times its width.

Definition: Properties of Rectangles

- Rectangles have four sides and four right (90°) angles.
- The lengths of opposite sides are equal.
- The perimeter, P, of a rectangle is the sum of twice the length and twice the width. See Figure 9.19.

$$P = 2L + 2W \tag{13.4.4.1}$$

- The area, A, of a rectangle is the length times the width.

$$A = L \cdot W \tag{13.4.4.2}$$

For easy reference as we work the examples in this section, we will restate the Problem Solving Strategy for Geometry Applications here.

HOW TO: USE A PROBLEM SOLVING STRATEGY FOR GEOMETRY APPLICATIONS

Step 1. **Read** the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.

Step 2. **Identify** what you are looking for.

Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

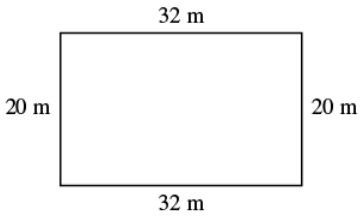
Step 7. **Answer** the question with a complete sentence.

Example 13.4.4.3:

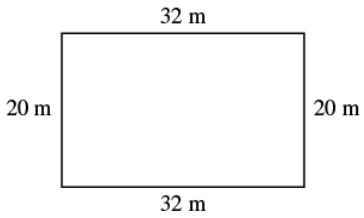
The length of a rectangle is 32 meters and the width is 20 meters. Find (a) the perimeter, and (b) the area.

Solution

(a)

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the perimeter of a rectangle
Step 3. **Name**. Choose a variable to represent it.	Let P = the perimeter
Step 4. **Translate**. Write the appropriate formula. Substitute.	$\underbrace{P}_{P} \quad \underbrace{=}_{=} \quad \underbrace{2\,L}_{2(32)} \quad \underbrace{+}_{+} \quad \underbrace{2\,W}_{2(20)}$
Step 5. **Solve** the equation.	$P = 64 + 40$ $P = 104$
Step 6. **Check**.	$P \stackrel{?}{=} 104$ $20 + 32 + 20 + 32 \stackrel{?}{=} 104$ $104 = 104$ ✓
Step 7. **Answer** the question.	The perimeter of the rectangle is 104 meters.

(b)

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of a rectangle
Step 3. **Name**. Choose a variable to represent it.	Let A = the area
Step 4. **Translate**. Write the appropriate formula. Substitute.	$\underbrace{A}_{A} \quad \underbrace{=}_{=} \quad \underbrace{L}_{32\text{ m}} \quad \underbrace{\cdot}_{\cdot} \quad \underbrace{W}_{20\text{ m}}$
Step 5. **Solve** the equation.	$A = 640 \qquad (13.4.4.3)$
Step 6. **Check**.	$A \stackrel{?}{=} 640$ $32 \cdot 20 \stackrel{?}{=} 640$ $640 = 640$ ✓
Step 7. **Answer** the question.	The area of the rectangle is 60 square meters.

? Exercise 13.4.4.5:

The length of a rectangle is 120 yards and the width is 50 yards. Find (a) the perimeter and (b) the area.

Answer a

340 yd

Answer b

6000 sq. yd

? Exercise 13.4.4.6:

The length of a rectangle is 62 feet and the width is 48 feet. Find (a) the perimeter and (b) the area.

Answer a

220 ft

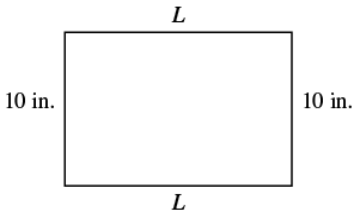
Answer b

2976 sq. ft

✓ Example 13.4.4.4:

Find the length of a rectangle with perimeter 50 inches and width 10 inches.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the length of the rectangle
Step 3. **Name**. Choose a variable to represent it.	Let L = the length
Step 4. **Translate**. Write the appropriate formula. Substitute.	$\underbrace{P}_{50} \underbrace{=}_{=} \underbrace{2\,L}_{2L} \underbrace{+}_{+} \underbrace{2\,W}_{2(10)}$
Step 5. **Solve** the equation.	$\begin{aligned} 50-20 &= 2L+20-20 \\ 30 &= 2L \\ \frac{30}{2} &= \frac{2L}{2} \\ 15 &= L \end{aligned}$
Step 6. **Check**.	$\begin{aligned} P &\stackrel{?}{=} 50 \\ 15+10+15+10 &\stackrel{?}{=} 50 \\ 50 &= 50 \checkmark \end{aligned}$
Step 7. **Answer** the question.	The length is 15 inches.

? Exercise 13.4.4.7:

Find the length of a rectangle with a perimeter of 80 inches and width of 25 inches.

Answer

15 inches

? Exercise 13.4.4.8:

Find the length of a rectangle with a perimeter of 30 yards and width of 6 yards.

Answer

9 yd

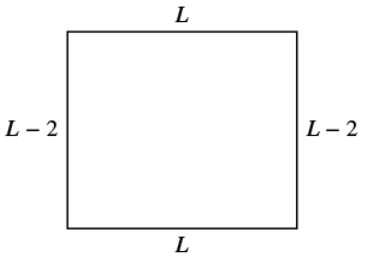
In the next example, the width is defined in terms of the length. We'll wait to draw the figure until we write an expression for the width so that we can label one side with that expression.

✓ Example 13.4.4.5:

The width of a rectangle is two inches less than the length. The perimeter is 52 inches. Find the length and width.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the length and width of the rectangle

Step 3. **Name**. Choose a variable to represent it. Now we can draw a figure using these expressions for the length and width.	Since the width is defined in terms of the length, we let L = length. The width is two feet less that the length, so we let L − 2 = width. 
Step 4. **Translate**. Write the appropriate formula. The formula for the perimeter of a rectangle relates all the information. Substitute in the given information.	$\underbrace{P}_{52} \underbrace{=}_{=} \underbrace{2L}_{2L} \underbrace{+}_{+} \underbrace{2W}_{2(L-2)}$
Step 5. **Solve** the equation.	$52 = 2L + 2L - 4$ (13.4.4.4)
Combine like terms.	$52 = 4L - 4$ (13.4.4.5)
Add 4 to each side.	$56 = 4L$ (13.4.4.6)
Divide by 4.	$\frac{56}{4} = \frac{4L}{4}$ $14 = L$ $14 = L$ The length is 14 inches.
Now we need to find the width. The width is L − 2.	$L - 2$ $14 - 2$ 12 The width is 12 inches.
Step 6. **Check**.	Since 14 + 12 + 14 + 12 = 52 , this works!
Step 7. **Answer** the question.	The length is 14 feet and the width is 12 feet.

? Exercise 13.4.4.9:

The width of a rectangle is seven meters less than the length. The perimeter is 58 meters. Find the length and width.

Answer

18 m, 11 m

? Exercise 13.4.4.10:

The length of a rectangle is eight feet more than the width. The perimeter is 60 feet. Find the length and width.

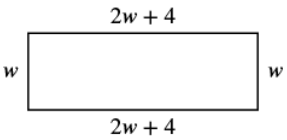
Answer

11 ft, 19 ft

Example 13.4.4.6:

The length of a rectangle is four centimeters more than twice the width. The perimeter is 32 centimeters. Find the length and width.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the length and width
Step 3. **Name**. Choose a variable to represent it.	let W = width The length is four more than twice the width. 2w + 4 = length 
Step 4. **Translate**. Write the appropriate formula and substitute in the given information.	$\underbrace{P}_{32} \quad \underbrace{=}_{=} \quad \underbrace{2L}_{2(2w+4)} \quad \underbrace{+}_{+} \quad \underbrace{2W}_{2w}$
Step 5. **Solve** the equation.	$32 = 4w + 8 + 2w$ $32 = 6w + 8$ $24 = 6w$ $4 = w \quad width$ $2w + 4 \quad length$ $2(4) + 4$ $12 \quad The\ length\ is\ 12\ cm.$
Step 6. **Check**.	$p = 2L + 2W$ $32 \stackrel{?}{=} 2 \cdot 12 + 2 \cdot 4$ $32 = 32\ \checkmark$
Step 7. **Answer** the question.	The length is 12 cm and the width is 4 cm.

Exercise 13.4.4.11:

The length of a rectangle is eight more than twice the width. The perimeter is 64 feet. Find the length and width.

Answer

8 ft, 24 ft

Exercise 13.4.4.12:

The width of a rectangle is six less than twice the length. The perimeter is 18 centimeters. Find the length and width.

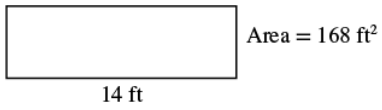
Answer

5 cm, 4 cm

Example 13.4.4.7:

The area of a rectangular room is 168 square feet. The length is 14 feet. What is the width?

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what you are looking for.	the width of a rectangular room
Step 3. **Name**. Choose a variable to represent it.	Let W = width
Step 4. **Translate**. Write the appropriate formula and substitute in the given information.	$A = LW$ $168 = 14W$
Step 5. **Solve** the equation.	$\frac{168}{14} = \frac{14W}{14}$ $12 = W$
Step 6. **Check**.	$A = LW$ $168 \stackrel{?}{=} 14 \cdot 12$ $168 = 168$ ✓
Step 7. **Answer** the question.	The width of the room is 12 feet.

Exercise 13.4.4.13:

The area of a rectangle is 598 square feet. The length is 23 feet. What is the width?

Answer

26 ft

Exercise 13.4.4.14:

The width of a rectangle is 21 meters. The area is 609 square meters. What is the length?

Answer

29 m

Example 13.4.4.8:

The perimeter of a rectangular swimming pool is 150 feet. The length is 15 feet more than the width. Find the length and width.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the length and width of the pool
Step 3. **Name**. Choose a variable to represent it. The length is 15 feet more than the width.	Let W = width W + 15 = length
Step 4. **Translate**. Write the appropriate formula and substitute.	$\underbrace{150}_{P} \quad \underbrace{=} \quad \underbrace{2(w+15)}_{2L} \quad \underbrace{+} \quad \underbrace{2w}_{2W}$
Step 5. **Solve** the equation.	$150 = 2w + 30 + 2w$ $150 = 4w + 30$ $120 = 4w$ $30 = w \quad the\ width\ of\ the\ pool$ $w + 15 \quad the\ length\ of\ the\ pool$ $30 + 15$ 45
Step 6. **Check**.	$p = 2L + 2W$ $150 \stackrel{?}{=} 2(45) + 2(30)$ $150 = 150$ ✓
Step 7. **Answer** the question.	The length of the pool is 45 feet and the width is 30 feet.

? Exercise 13.4.4.15:

The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.

Answer

30 ft, 70 ft

? Exercise 13.4.4.16:

The length of a rectangular garden is 30 yards more than the width. The perimeter is 300 yards. Find the length and width.

Answer

60 yd, 90 yd

Contributors and Attributions

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13.4.5: Use Properties of Rectangles, Triangles, and Trapezoids (Part 2)

Use the Properties of Triangles

We now know how to find the area of a rectangle. We can use this fact to help us visualize the formula for the area of a triangle. In the rectangle in Figure 13.4.5.9 we've labeled the length b and the width h, so it's area is bh.

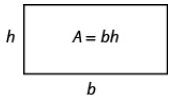


Figure 13.4.5.9- The area of a rectangle is the base, b, times the height, h.

We can divide this rectangle into two **congruent** triangles (Figure 13.4.5.10). Triangles that are congruent have identical side lengths and angles, and so their areas are equal. The area of each triangle is one-half the area of the rectangle, or $\frac{1}{2}$bh. This example helps us see why the formula for the area of a triangle is A = $\frac{1}{2}$bh.

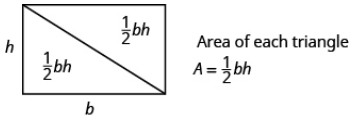


Figure 13.4.5.10- A rectangle can be divided into two triangles of equal area. The area of each triangle is one-half the area of the rectangle.

The formula for the area of a triangle is A = $\frac{1}{2}$bh, where b is the base and h is the height. To find the area of the triangle, you need to know its base and height. The base is the length of one side of the triangle, usually the side at the bottom. The height is the length of the line that connects the base to the opposite vertex, and makes a 90° angle with the base. Figure 13.4.5.11shows three triangles with the base and height of each marked.

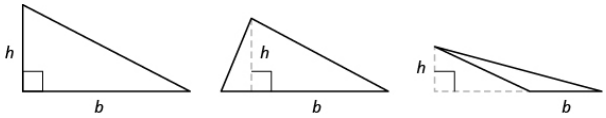


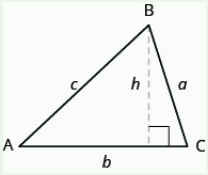
Figure 13.4.5.11- The height h of a triangle is the length of a line segment that connects the the base to the opposite vertex and makes a 90° angle with the base.

Definition: Triangle Properties

For any triangle ΔABC, the sum of the measures of the angles is 180°.

$$m\angle A + m\angle B + m\angle C = 180^\circ \$\$The perimeter of a triangle is the sum of the lengths of the sides. \$\$P = a + b + c\$ \quad (13.4.5.1)$$

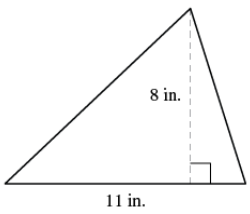
$$\$The area of a triangle is one-half the base, b, times the height, h. \$\$A = \frac{1}{2}bh$$



Example 13.4.5.9:

Find the area of a triangle whose base is 11 inches and whose height is 8 inches.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	

Step 2. **Identify** what you are looking for.	the area of the triangle
Step 3. **Name**. Choose a variable to represent it.	let A = area of the triangle
Step 4.**Translate**. Write the appropriate formula. Substitute.	$\underbrace{A}_{} = \underbrace{\frac{1}{2}}_{} \cdot \underbrace{b}_{} \cdot \underbrace{h}_{}$ $A = \frac{1}{2} \cdot 11 \cdot 8$
Step 5. **Solve** the equation.	A = 44 square inches
Step 6. **Check**.	$A = \frac{1}{2}bh$ $44 \stackrel{?}{=} \frac{1}{2}(11)8$ $44 = 44$ ✓
Step 7. **Answer** the question.	The area is 44 square inches.

Exercise 13.4.5.17:

Find the area of a triangle with base 13 inches and height 2 inches.

Answer

13 sq. in.

Exercise 13.4.5.18:

Find the area of a triangle with base 14 inches and height 7 inches.

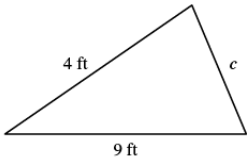
Answer

49 sq. in.

Example 13.4.5.10:

The perimeter of a triangular garden is 24 feet. The lengths of two sides are 4 feet and 9 feet. How long is the third side?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	 $P = 24$ ft
Step 2. **Identify** what you are looking for.	length of the third side of a triangle
Step 3. **Name**. Choose a variable to represent it.	Let c = the third side
Step 4.**Translate**. Write the appropriate formula. Substitute in the given information.	$\underbrace{P}_{} = \underbrace{a}_{} + \underbrace{b}_{} + \underbrace{c}_{}$ $24 = 4 + 9 + c$
Step 5. **Solve** the equation.	$24 = 13 + c$ $11 = c$
Step 6. **Check**.	$P = a + b + c$ $24 \stackrel{?}{=} 4 + 9 + 11$ $24 = 24$ ✓
Step 7. **Answer** the question.	The third side is 11 feet long.

Exercise 13.4.5.19:

The perimeter of a triangular garden is 48 feet. The lengths of two sides are 18 feet and 22 feet. How long is the third side?

Answer

8 ft

Exercise 13.4.5.20:

The lengths of two sides of a triangular window are 7 feet and 5 feet. The perimeter is 18 feet. How long is the third side?

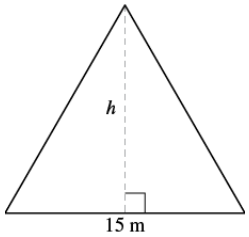
Answer

6 ft

Example 13.4.5.11:

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window's height?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	height of a triangle
Step 3. **Name**. Choose a variable to represent it.	Let h = the height
Step 4.**Translate**. Write the appropriate formula. Substitute in the given information.	$\underbrace{A}\ \underbrace{=}\ \underbrace{\frac{1}{2}}\ \underbrace{\cdot}\ \underbrace{b}\ \underbrace{\cdot}\ \underbrace{h}$ $90 = \frac{1}{2} \cdot 15 \cdot h$
Step 5. **Solve** the equation.	$90 = \frac{15}{2}h$ $12 = h$
Step 6. **Check**.	$A = \frac{1}{2}bh$ $90 \stackrel{?}{=} \frac{1}{2} \cdot 15 \cdot 12$ $90 = 90$ ✓
Step 7. **Answer** the question.	The height of the triangle is 12 meters.

Exercise 13.4.5.21:

The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Answer

14 in.

Exercise 13.4.5.22:

A triangular tent door has an area of 15 square feet. The height is 5 feet. What is the base?

Answer

6 ft

Isosceles and Equilateral Triangles

Besides the right triangle, some other triangles have special names. A triangle with two sides of equal length is called an **isosceles triangle**. A triangle that has three sides of equal length is called an **equilateral triangle**. Figure 13.4.5.12shows both types of triangles.

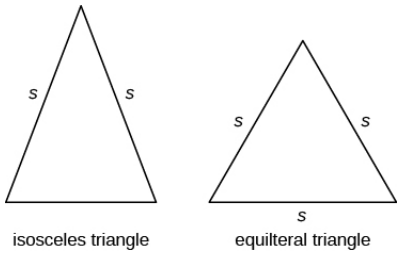


Figure 13.4.5.12- In an isosceles triangle, two sides have the same length, and the third side is the base. In an equilateral triangle, all three sides have the same length.

Definition: Isosceles and Equilateral Triangles

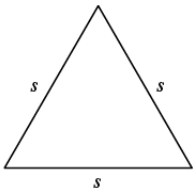
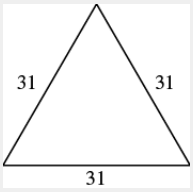
An isosceles triangle has two sides the same length.

An equilateral triangle has three sides of equal length.

Example 13.4.5.12:

The perimeter of an equilateral triangle is 93 inches. Find the length of each side.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	 Perimeter = 93 in.
Step 2. **Identify** what you are looking for.	length of the sides of an equilateral triangle
Step 3. **Name**. Choose a variable to represent it.	Let s = length of each side
Step 4.**Translate**. Write the appropriate formula. Substitute.	$\underbrace{P}_{93} \underbrace{=}_{=} \underbrace{a}_{s} \underbrace{+}_{+} \underbrace{b}_{s} \underbrace{+}_{+} \underbrace{c}_{s}$
Step 5. **Solve** the equation.	$93 = 3s$ $31 = s$
Step 6. **Check**.	 $93 = 31 + 31 + 31$ $93 = 93$ ✓

Step 7. **Answer** the question.	Each side is 31 inches.

Exercise 13.4.5.23:

Find the length of each side of an equilateral triangle with perimeter 39 inches.

Answer

13 in.

Exercise 13.4.5.24:

Find the length of each side of an equilateral triangle with perimeter 51 centimeters.

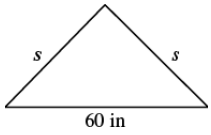
Answer

17 cm

Example 13.4.5.13:

Arianna has 156 inches of beading to use as trim around a scarf. The scarf will be an isosceles triangle with a base of 60 inches. How long can she make the two equal sides?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	 P = 156 in.
Step 2. **Identify** what you are looking for.	the lengths of the two equal sides
Step 3. **Name**. Choose a variable to represent it.	Let s = the length of each side
Step 4.**Translate**. Write the appropriate formula. Substitute in the given information.	$\underbrace{P}_{156} \underbrace{=}_{=} \underbrace{a}_{s} \underbrace{+}_{+} \underbrace{b}_{60} \underbrace{+}_{+} \underbrace{c}_{s}$
Step 5. **Solve** the equation.	$156 = 2s + 60$ $96 = 2s$ $48 = s$
Step 6. **Check**.	$p = a + b + c$ $156 \stackrel{?}{=} 48 + 60 + 48$ $156 = 156$ ✓
Step 7. **Answer** the question.	Arianna can make each of the two equal sides 48 inches long.

Exercise 13.4.5.25:

A backyard deck is in the shape of an isosceles triangle with a base of 20 feet. The perimeter of the deck is 48 feet. How long is each of the equal sides of the deck?

Answer

14 ft

Exercise 13.4.5.26:

A boat's sail is an isosceles triangle with base of 8 meters. The perimeter is 22 meters. How long is each of the equal sides of the sail?

Answer

7 m

Use the Properties of Trapezoids

A **trapezoid** is four-sided figure, a *quadrilateral*, with two sides that are parallel and two sides that are not. The parallel sides are called the bases. We call the length of the smaller base b, and the length of the bigger base B. The height, h, of a trapezoid is the distance between the two bases as shown in Figure 13.4.5.13

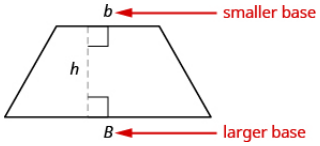


Figure 13.4.5.13- A trapezoid has a larger base, B, and a smaller base, b. The height h is the distance between the bases.

The formula for the area of a trapezoid is:

$$Area_{trapezoid} = \frac{1}{2}h(b+B) \tag{13.4.5.2}$$

Splitting the trapezoid into two triangles may help us understand the formula. The area of the trapezoid is the sum of the areas of the two triangles. See Figure 13.4.5.14

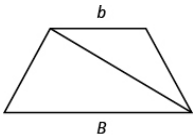


Figure 13.4.5.14- Splitting a trapezoid into two triangles may help you understand the formula for its area.

The height of the trapezoid is also the height of each of the two triangles. See Figure 13.4.5.15

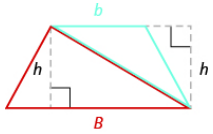


Figure 13.4.5.15

The formula for the area of a trapezoid is

$$Area_{trapezoid} = \frac{1}{2}h(b+B) \tag{13.4.5.3}$$

If we distribute, we get,

$$\text{Area}_{trapezoid} = \frac{1}{2}bh + \frac{1}{2}Bh$$

$$\text{Area}_{trapezoid} = A_{blue\triangle} + A_{red\triangle}$$

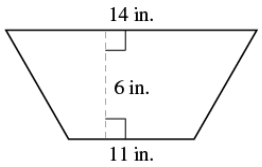
Definition: Properties of Trapezoids

- A trapezoid has four sides. See Figure 9.25.
- Two of its sides are parallel and two sides are not.
- The area, A, of a trapezoid is A = $\frac{1}{2}$h(b + B).

Example 13.4.5.14:

Find the area of a trapezoid whose height is 6 inches and whose bases are 14 and 11 inches.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of the trapezoid
Step 3. **Name**. Choose a variable to represent it.	Let A = the area
Step 4.**Translate**. Write the appropriate formula. Substitute.	$A = \frac{1}{2} \cdot h \cdot (b+B)$ $A = \frac{1}{2} \cdot 6 \cdot (11+14)$
Step 5. **Solve** the equation.	$A = \frac{1}{2} \cdot 6(25)$ $A = 3(25)$ $A = 75\ square\ inches$
Step 6. **Check**: Is this answer reasonable?	

If we draw a rectangle around the trapezoid that has the same big base B and a height h, its area should be greater than that of the trapezoid.

If we draw a rectangle inside the trapezoid that has the same little base b and a height h, its area should be smaller than that of the trapezoid.

$A_{rectangle} = bh$	$A_{trapezoid} = \frac{1}{2}h(b+B)$	$A_{rectangle} = bh$
$A_{rectangle} = 14 \cdot 6$	$A_{trapezoid} = \frac{1}{2} \cdot 6(11+14)$	$A_{rectangle} = 11 \cdot 6$
$A_{rectangle} = 84$ sq. in.	$A_{trapezoid} = 75$ sq. in.	$A_{rectangle} = 66$ sq. in.

The area of the larger rectangle is 84 square inches and the area of the smaller rectangle is 66 square inches. So it makes sense that the area of the trapezoid is between 84 and 66 square inches

Step 7. **Answer** the question.	The area of the trapezoid is 75 square inches.

? Exercise 13.4.5.27:

The height of a trapezoid is 14 yards and the bases are 7 and 16 yards. What is the area?

Answer

161 sq. yd

? Exercise 13.4.5.28:

The height of a trapezoid is 18 centimeters and the bases are 17 and 8 centimeters. What is the area?

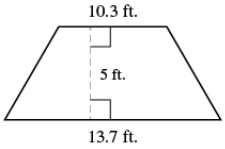
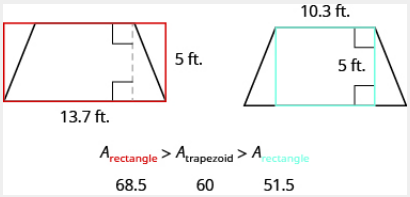
Answer

255 sq. cm

✓ Example 13.4.5.15:

Find the area of a trapezoid whose height is 5 feet and whose bases are 10.3 and 13.7 feet.

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of the trapezoid
Step 3. **Name**. Choose a variable to represent it.	Let A = the area
Step 4.**Translate**. Write the appropriate formula. Substitute.	$\underbrace{A}_{A} \quad \underbrace{=}_{=} \quad \underbrace{\frac{1}{2}}_{\frac{1}{2}} \quad \underbrace{\cdot}_{\cdot} \quad \underbrace{h}_{5} \quad \underbrace{\cdot}_{\cdot} \quad \underbrace{(b+B)}_{(10.3+13.7)}$
Step 5. **Solve** the equation.	$A = \frac{1}{2} \cdot 5(24)$ $A = 12 \cdot 5$ $A = 60 \; square \; feet$
Step 6. **Check**: Is this answer reasonable? The area of the trapezoid should be less than the area of a rectangle with base 13.7 and height 5, but more than the area of a rectangle with base 10.3 and height 5.	
Step 7. **Answer** the question.	The area of the trapezoid is 60 square feet.

? Exercise 13.4.5.29:

The height of a trapezoid is 7 centimeters and the bases are 4.6 and 7.4 centimeters. What is the area?

Answer

42 sq. cm

? Exercise 13.4.5.30:

The height of a trapezoid is 9 meters and the bases are 6.2 and 7.8 meters. What is the area?

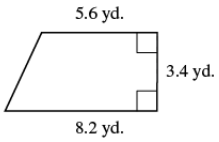
Answer

63 sq. m

✓ Example 13.4.5.16:

Vinny has a garden that is shaped like a trapezoid. The trapezoid has a height of 3.4 yards and the bases are 8.2 and 5.6 yards. How many square yards will be available to plant?

Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	
Step 2. **Identify** what you are looking for.	the area of the trapezoid
Step 3. **Name**. Choose a variable to represent it.	Let A = the area

Step 4.**Translate**. Write the appropriate formula. Substitute.	$\underbrace{A}_{A} \underbrace{=}_{=} \underbrace{\frac{1}{2}}_{\frac{1}{2}} \underbrace{\cdot}_{\cdot} \underbrace{h}_{3.4} \underbrace{\cdot}_{\cdot} \underbrace{(b+B)}_{(5.6+8.2)}$ $A = \frac{1}{2} \cdot 3.4 \cdot (5.6+8.2)$
Step 5. **Solve** the equation.	$A = \frac{1}{2} \cdot (3.4)(13.8)$ $A = 23.46 \; square \; yards$

Step 6. **Check**: Is this answer reasonable? Yes. The area of the trapezoid is less than the area of a rectangle with a base of 8.2 yd and height 3.4 yd, but more than the area of a rectangle with base 5.6 yd and height 3.4 yd.

$A_{rectangle} = Bh$ $= (8.2)(3.4)$ $= 27.88 \text{ yd}^2$	$A_{trapezoid} = \frac{1}{2}(3.4 \text{ yd})(5.6 = 8.2)$ $= 23.46 \text{ yd}^2$	$A_{rectangle} = bh$ $= (5.6)(3.4)$ $= 19.04 \text{ yd}^2$
$A_{rectangle} > A_{trapezoid} > A_{rectangle}$ 27.88 23.46 19.04		

Step 7. **Answer** the question.	Vinny has 23.46 square yards in which he can plant.

? Exercise 13.4.5.31:

Lin wants to sod his lawn, which is shaped like a trapezoid. The bases are 10.8 yards and 6.7 yards, and the height is 4.6 yards. How many square yards of sod does he need?

Answer

40.25 sq. yd

? Exercise 13.4.5.32:

Kira wants cover his patio with concrete pavers. If the patio is shaped like a trapezoid whose bases are 18 feet and 14 feet and whose height is 15 feet, how many square feet of pavers will he need?

Answer

240 sq. ft

ACCESS ADDITIONAL ONLINE RESOURCES

Perimeter of a Rectangle

Area of a Rectangle

Perimeter and Area Formulas

Area of a Triangle

Area of a Triangle with Fractions

Area of a Trapezoid

Practice Makes Perfect

Understand Linear, Square, and Cubic Measure

In the following exercises, determine whether you would measure each item using linear, square, or cubic units.

129. amount of water in a fish tank
130. length of dental floss
131. living area of an apartment
132. floor space of a bathroom tile
133. height of a doorway
134. capacity of a truck trailer

In the following exercises, find the (a) perimeter and (b) area of each figure. Assume each side of the square is 1 cm.

135.

136.

137.

138.

139.

140.

Use the Properties of Rectangles

In the following exercises, find the (a) perimeter and (b) area of each rectangle.

141. The length of a rectangle is 85 feet and the width is 45 feet.
142. The length of a rectangle is 26 inches and the width is 58 inches.
143. A rectangular room is 15 feet wide by 14 feet long.
144. A driveway is in the shape of a rectangle 20 feet wide by 35 feet long.

In the following exercises, solve.

145. Find the length of a rectangle with perimeter 124 inches and width 38 inches.
146. Find the length of a rectangle with perimeter 20.2 yards and width of 7.8 yards.
147. Find the width of a rectangle with perimeter 92 meters and length 19 meters.
148. Find the width of a rectangle with perimeter 16.2 meters and length 3.2 meters.
149. The area of a rectangle is 414 square meters. The length is 18 meters. What is the width?
150. The area of a rectangle is 782 square centimeters. The width is 17 centimeters. What is the length?
151. The length of a rectangle is 9 inches more than the width. The perimeter is 46 inches. Find the length and the width.
152. The width of a rectangle is 8 inches more than the length. The perimeter is 52 inches. Find the length and the width.
153. The perimeter of a rectangle is 58 meters. The width of the rectangle is 5 meters less than the length. Find the length and the width of the rectangle.
154. The perimeter of a rectangle is 62 feet. The width is 7 feet less than the length. Find the length and the width.
155. The width of the rectangle is 0.7 meters less than the length. The perimeter of a rectangle is 52.6 meters. Find the dimensions of the rectangle.
156. The length of the rectangle is 1.1 meters less than the width. The perimeter of a rectangle is 49.4 meters. Find the dimensions of the rectangle.
157. The perimeter of a rectangle of 150 feet. The length of the rectangle is twice the width. Find the length and width of the rectangle.
158. The length of a rectangle is three times the width. The perimeter is 72 feet. Find the length and width of the rectangle.
159. The length of a rectangle is 3 meters less than twice the width. The perimeter is 36 meters. Find the length and width.
160. The length of a rectangle is 5 inches more than twice the width. The perimeter is 34 inches. Find the length and width.
161. The width of a rectangular window is 24 inches. The area is 624 square inches. What is the length?
162. The length of a rectangular poster is 28 inches. The area is 1316 square inches. What is the width?
163. The area of a rectangular roof is 2310 square meters. The length is 42 meters. What is the width?
164. The area of a rectangular tarp is 132 square feet. The width is 12 feet. What is the length?
165. The perimeter of a rectangular courtyard is 160 feet. The length is 10 feet more than the width. Find the length and the width.

166. The perimeter of a rectangular painting is 306 centimeters. The length is 17 centimeters more than the width. Find the length and the width.
167. The width of a rectangular window is 40 inches less than the height. The perimeter of the doorway is 224 inches. Find the length and the width.
168. The width of a rectangular playground is 7 meters less than the length. The perimeter of the playground is 46 meters. Find the length and the width.

Use the Properties of Triangles

In the following exercises, solve using the properties of triangles.

169. Find the area of a triangle with base 12 inches and height 5 inches.
170. Find the area of a triangle with base 45 centimeters and height 30 centimeters.
171. Find the area of a triangle with base 8.3 meters and height 6.1 meters.
172. Find the area of a triangle with base 24.2 feet and height 20.5 feet.
173. A triangular flag has base of 1 foot and height of 1.5 feet. What is its area?
174. A triangular window has base of 8 feet and height of 6 feet. What is its area?
175. If a triangle has sides of 6 feet and 9 feet and the perimeter is 23 feet, how long is the third side?
176. If a triangle has sides of 14 centimeters and 18 centimeters and the perimeter is 49 centimeters, how long is the third side?
177. What is the base of a triangle with an area of 207 square inches and height of 18 inches?
178. What is the height of a triangle with an area of 893 square inches and base of 38 inches?
179. The perimeter of a triangular reflecting pool is 36 yards. The lengths of two sides are 10 yards and 15 yards. How long is the third side?
180. A triangular courtyard has perimeter of 120 meters. The lengths of two sides are 30 meters and 50 meters. How long is the third side?
181. An isosceles triangle has a base of 20 centimeters. If the perimeter is 76 centimeters, find the length of each of the other sides.
182. An isosceles triangle has a base of 25 inches. If the perimeter is 95 inches, find the length of each of the other sides.
183. Find the length of each side of an equilateral triangle with a perimeter of 51 yards.
184. Find the length of each side of an equilateral triangle with a perimeter of 54 meters.
185. The perimeter of an equilateral triangle is 18 meters. Find the length of each side.
186. The perimeter of an equilateral triangle is 42 miles. Find the length of each side.
187. The perimeter of an isosceles triangle is 42 feet. The length of the shortest side is 12 feet. Find the length of the other two sides.
188. The perimeter of an isosceles triangle is 83 inches. The length of the shortest side is 24 inches. Find the length of the other two sides.
189. A dish is in the shape of an equilateral triangle. Each side is 8 inches long. Find the perimeter.
190. A floor tile is in the shape of an equilateral triangle. Each side is 1.5 feet long. Find the perimeter.
191. A road sign in the shape of an isosceles triangle has a base of 36 inches. If the perimeter is 91 inches, find the length of each of the other sides.
192. A scarf in the shape of an isosceles triangle has a base of 0.75 meters. If the perimeter is 2 meters, find the length of each of the other sides.
193. The perimeter of a triangle is 39 feet. One side of the triangle is 1 foot longer than the second side. The third side is 2 feet longer than the second side. Find the length of each side.
194. The perimeter of a triangle is 35 feet. One side of the triangle is 5 feet longer than the second side. The third side is 3 feet longer than the second side. Find the length of each side.
195. One side of a triangle is twice the smallest side. The third side is 5 feet more than the shortest side. The perimeter is 17 feet. Find the lengths of all three sides.
196. One side of a triangle is three times the smallest side. The third side is 3 feet more than the shortest side. The perimeter is 13 feet. Find the lengths of all three sides.

Use the Properties of Trapezoids

In the following exercises, solve using the properties of trapezoids.

197. The height of a trapezoid is 12 feet and the bases are 9 and 15 feet. What is the area?
198. The height of a trapezoid is 24 yards and the bases are 18 and 30 yards. What is the area?
199. Find the area of a trapezoid with a height of 51 meters and bases of 43 and 67 meters.
200. Find the area of a trapezoid with a height of 62 inches and bases of 58 and 75 inches.
201. The height of a trapezoid is 15 centimeters and the bases are 12.5 and 18.3 centimeters. What is the area?
202. The height of a trapezoid is 48 feet and the bases are 38.6 and 60.2 feet. What is the area?
203. Find the area of a trapezoid with a height of 4.2 meters and bases of 8.1 and 5.5 meters.
204. Find the area of a trapezoid with a height of 32.5 centimeters and bases of 54.6 and 41.4 centimeters.
205. Laurel is making a banner shaped like a trapezoid. The height of the banner is 3 feet and the bases are 4 and 5 feet. What is the area of the banner?
206. Niko wants to tile the floor of his bathroom. The floor is shaped like a trapezoid with width 5 feet and lengths 5 feet and 8 feet. What is the area of the floor?
207. Theresa needs a new top for her kitchen counter. The counter is shaped like a trapezoid with width 18.5 inches and lengths 62 and 50 inches. What is the area of the counter?
208. Elena is knitting a scarf. The scarf will be shaped like a trapezoid with width 8 inches and lengths 48.2 inches and 56.2 inches. What is the area of the scarf?

Everyday Math

209. **Fence** Jose just removed the children's playset from his back yard to make room for a rectangular garden. He wants to put a fence around the garden to keep out the dog. He has a 50 foot roll of fence in his garage that he plans to use. To fit in the backyard, the width of the garden must be 10 feet. How long can he make the other side if he wants to use the entire roll of fence?
210. **Gardening** Lupita wants to fence in her tomato garden. The garden is rectangular and the length is twice the width. It will take 48 feet of fencing to enclose the garden. Find the length and width of her garden.
211. **Fence** Christa wants to put a fence around her triangular flowerbed. The sides of the flowerbed are 6 feet, 8 feet, and 10 feet. The fence costs $10 per foot. How much will it cost for Christa to fence in her flowerbed?
212. **Painting** Caleb wants to paint one wall of his attic. The wall is shaped like a trapezoid with height 8 feet and bases 20 feet and 12 feet. The cost of the painting one square foot of wall is about $0.05. About how much will it cost for Caleb to paint the attic wall?

Writing Exercises

213. If you need to put tile on your kitchen floor, do you need to know the perimeter or the area of the kitchen? Explain your reasoning.
214. If you need to put a fence around your backyard, do you need to know the perimeter or the area of the backyard? Explain your reasoning.
215. Look at the two figures. (a) Which figure looks like it has the larger area? Which looks like it has the larger perimeter? (b) Now calculate the area and perimeter of each figure. Which has the larger area? Which has the larger perimeter?

216. The length of a rectangle is 5 feet more than the width. The area is 50 square feet. Find the length and the width. (a) Write the equation you would use to solve the problem. (b) Why can't you solve this equation with the methods you learned in the previous chapter?

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
understand linear, square, and cubic measure.			
use the properties of rectangles.			
use the properties of triangles.			
use the properties of trapezoids.			

(b) On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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13.5.1: Prelude to Foundations of Algebra

Just like a building needs a firm foundation to support it, your study of algebra needs to have a firm foundation. To ensure this, we begin this book with a review of arithmetic operations with whole numbers, integers, fractions, and decimals, so that you have a solid base that will support your study of algebra.

Figure 13.5.1.1: In order to be structurally sound, the foundation of a building must be carefully constructed.

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13.5.2: Introduction to Whole Numbers

Learning Objectives

- By the end of this section, you will be able to:
- Use place value with whole numbers
- Identify multiples and and apply divisibility tests
- Find prime factorizations and least common multiples

As we begin our study of elementary algebra, we need to refresh some of our skills and vocabulary. This chapter will focus on whole numbers, integers, fractions, decimals, and real numbers. We will also begin our use of algebraic notation and vocabulary.

Use Place Value with Whole Numbers

The most basic numbers used in algebra are the numbers we use to count objects in our world: $1, 2, 3, 4$, and so on. These are called the **counting numbers**. Counting numbers are also called *natural numbers*. If we add zero to the counting numbers, we get the set of **whole numbers**.

- Counting Numbers: $1, 2, 3, \ldots$
- Whole Numbers: $0, 1, 2, 3, \ldots$

The notation "$\ldots$" is called ellipsis and means "and so on," or that the pattern continues endlessly.

We can visualize counting numbers and whole numbers on a **number line** (see Figure $13.5.2.1$).

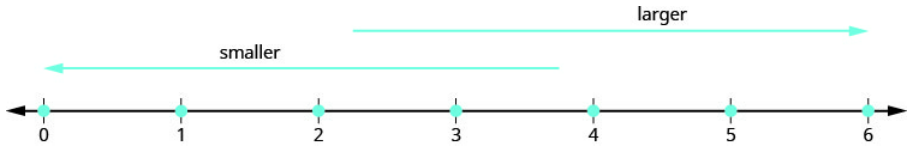


Figure 13.5.2.1: The numbers on the number line get larger as they go from left to right, and smaller as they go from right to left. While this number line shows only the whole numbers 0 through 6, the numbers keep going without end.

Doing the Manipulative Mathematics activity "Number Line-Part 1" will help you develop a better understanding of the counting numbers and the whole numbers.

Our **number system** is called a place value system, because the value of a digit depends on its position in a number. Figure $13.5.2.2$ shows the **place values**. The place values are separated into groups of three, which are called periods. The periods are *ones*, *thousands*, *millions*, *billions*, *trillions*, and so on. In a written number, commas separate the periods.

Place Value														
Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
								5	2	7	8	1	9	4

Figure 13.5.2.2: The number 5278194 is shown in the chart. The digit 5 is in the millions place. The digit 2 is in the hundred-thousands place. The digit 7 is in the ten-thousands place. The digit 8 is in the thousands place. The digit 1 is in the hundreds place. The digit 9 is in the tens place. The digit 4 is in the ones place.

Exercise 13.5.2.1

In the number 63407218, find the place value of each digit:

1. 7

2. 0
3. 1
4. 6
5. 3

Answer

Place the number in the place value chart:

Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
							6	3	4	0	7	2	1	8

1. The 7 is in the thousands place.
2. The 0 is in the ten thousands place.
3. The 1 is in the tens place.
4. The 6 is in the ten-millions place.
5. The 3 is in the millions place.

Exercise 13.5.2.2

For the number 27493615 find the place value of each digit:

1. 2
2. 1
3. 4
4. 7
5. 5

Answer

1. ten millions
2. tens
3. hundred thousands
4. millions
5. ones

Exercise 13.5.2.3

For the number 519711641328 find the place value of each digit:

1. 9
2. 4
3. 2
4. 6
5. 7

Answer

1. billions
2. ten thousands

3. tens
4. hundred thousands
5. hundred millions

When you write a check, you write out the number in words as well as in digits. To write a number in words, write the number in each period, followed by the name of the period, without the *s* at the end. Start at the left, where the periods have the largest value. The ones period is not named. The commas separate the periods, so wherever there is a comma in the number, put a comma between the words (see Figure 13.5.2.3). The number 74218369is written as seventy-four million, two hundred eighteen thousand, three hundred sixty-nine.

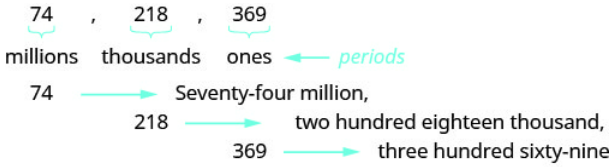


Figure 13.5.2.3

NAME A WHOLE NUMBER IN WORDS.

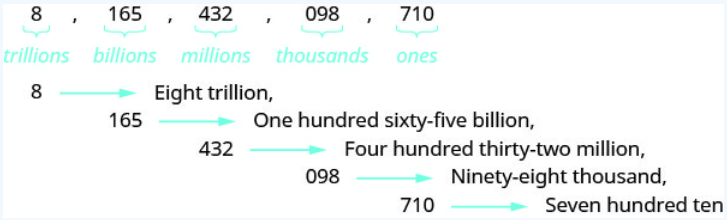
1. Start at the left and name the number in each period, followed by the period name.
2. Put commas in the number to separate the periods.
3. Do not name the ones period.

Exercise 13.5.2.4

Name the number 8165432098710using words.

Answer

Name the number in each period, followed by the period name.



Put the commas in to separate the periods.

So, 8165432098710is named as eight trillion, one hundred sixty-five billion, four hundred thirty-two million, ninety-eight thousand, seven hundred ten.

Exercise 13.5.2.5

Name the number 9,258,137,904,0619,258,137,904,061 using words.

Answer

nine trillion, two hundred fifty-eight billion, one hundred thirty-seven million, nine hundred four thousand, sixty-one

Exercise 13.5.2.6

Name the number 17,864,325,619,00417,864,325,619,004 using words.

Answer

seventeen trillion, eight hundred sixty-four billion, three hundred twenty-five million, six hundred nineteen thousand four

We are now going to reverse the process by writing the digits from the name of the number. To write the number in digits, we first look for the clue words that indicate the periods. It is helpful to draw three blanks for the needed periods and then fill in the blanks with the numbers, separating the periods with commas.

WRITE A WHOLE NUMBER USING DIGITS.

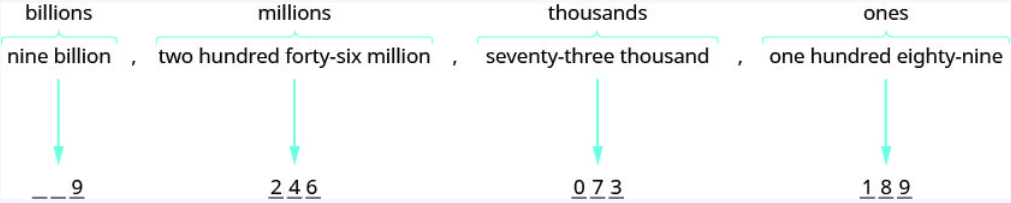
1. Identify the words that indicate periods. (Remember, the ones period is never named.)
2. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.
3. Name the number in each period and place the digits in the correct place value position.

Exercise 13.5.2.7

Write *nine billion, two hundred forty-six million, seventy-three thousand, one hundred eighty-nine* as a whole number using digits.

Answer

Identify the words that indicate periods.
Except for the first period, all other periods must have three places. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.
Then write the digits in each period.



The number is 9,246,073,189.

Exercise 13.5.2.8

Write the number two billion, four hundred sixty-six million, seven hundred fourteen thousand, fifty-one as a whole number using digits.

Answer

2,466,714,051

Exercise 13.5.2.9

Write the number eleven billion, nine hundred twenty-one million, eight hundred thirty thousand, one hundred six as a whole number using digits.

Answer

11,921,830,106

In 2013, the U.S. Census Bureau estimated the population of the state of New York as 19,651,127. We could say the population of New York was approximately 20 million. In many cases, you don't need the exact value; an approximate number is good enough.

The process of approximating a number is called **rounding**. Numbers are rounded to a specific place value, depending on how much accuracy is needed. Saying that the population of New York is approximately 20 million means that we rounded to the

millions place.

Exercise 13.5.2.10 How to Round Whole Numbers

Round 23,658 to the nearest hundred.

Answer

Step 1. Locate the given place value with an arrow. All digits to the left do not change.	Locate the hundreds place in 23,658.	hundredths place ↓ 23,658
Step 2. Underline the digit to the right of the given place value.	Underline the 5, which is to the right of the hundreds place.	hundredths place ↓ 23,658
Step 3. Is this digit greater than or equal to 5? Yes–add 1 to the digit in the given place value. No–do not change the digit in the given place value.	Add 1 to the 6 in the hundreds place, since 5 is greater than or equal to 5.	23,658 add 1 ↗
Step 4. Replace all digits to the right of the given place value with zeros.	Replace all digits to the right of the hundreds place with zeros.	23,700 add 1 ↗ replace with 0s ↖ So, 23,700 is rounded to the nearest hundred.

Exercise 13.5.2.11

Round to the nearest hundred: 17,852.

Answer

17,900

Exercise 13.5.2.12

Round to the nearest hundred: 468,751.

Answer

468,800

ROUND WHOLE NUMBERS.

1. Locate the given place value and mark it with an arrow. All digits to the left of the arrow do not change.
2. Underline the digit to the right of the given place value.
3. Is this digit greater than or equal to 5?
   - Yes–add 11 to the digit in the given place value.
   - No–do not change the digit in the given place value.
4. Replace all digits to the right of the given place value with zeros.

Exercise 13.5.2.13

Round 103,978103,978 to the nearest:

1. hundred
2. thousand
3. ten thousand

Answer

1.

Locate the hundreds place in 103,978.	hundreds place ↓ 103,978
Underline the digit to the right of the hundreds place.	hundreds place ↓ 103,978
Since 7 is greater than or equal to 5, add 1 to the 9. Replace all digits to the right of the hundreds place with zeros.	hundreds place ↓ 103,978 add 1 9 + 1 = 10 replace 9 with 0 and carry the 1 replace with 0s 104,000
	So, 104,000 is 103,978 rounded to the nearest hundred.

2.

Locate the thousands place and underline the digit to the right of the thousands place.	thousands place ↓ 103,978
Since 9 is greater than or equal to 5, add 1 to the 3. Replace all digits to the right of the hundreds place with zeros.	thousands place ↓ 103,978 add 1 3 + 1 = 4 replace 3 with 4 replace with 0s 104,000
	So, 104,000 is 103,978 rounded to the nearest thousand.

3.

Locate the ten thousands place and underline the digit to the right of the ten thousands place.	ten thousands place ↓ 103,978
Since 3 is less than 5, we leave the 0 as is, and then replace the digits to the right with zeros.	100,000
	So, 100,000 is 103,978 rounded to the nearest ten thousand.

Exercise 13.5.2.14

Round 206,981 to the nearest: 1. hundred 2. thousand 3. ten thousand.

Answer

1. 207,000
2. 207,000

3. 210,000

Exercise 13.5.2.15

Round 784,951 to the nearest: 1. hundred 2. thousand 3. ten thousand.

Answer

1. 785,000
2. 785,000
3. 780,000

Identify Multiples and Apply Divisibility Tests

The numbers 2, 4, 6, 8, 10, and 12 are called **multiples** of 2. A multiple of 2 can be written as the **product** of a counting number and 2.

Figure 13.5.2.4

Similarly, a multiple of 3 would be the product of a counting number and 3.

Figure 13.5.2.5

We could find the multiples of any number by continuing this process.

Note

Doing the Manipulative Mathematics activity “Multiples” will help you develop a better understanding of multiples.

Table 13.5.2.1shows the multiples of 2 through 9 for the first 12 counting numbers.

Table 13.5.2.1

Counting Number	1	2	3	4	5	6	7	8	9	10	11	12
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96

Counting Number	1	2	3	4	5	6	7	8	9	10	11	12
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108
Multiples of 10	10	20	30	40	50	60	70	80	90	100	110	120

MULTIPLE OF A NUMBER

A number is a **multiple** of n if it is the product of a counting number and n.

Another way to say that 15 is a multiple of 3 is to say that 15 is **divisible** by 3. That means that when we divide 3 into 15, we get a counting number. In fact, $15 \div 3$ is 5, so 15 is $5 \cdot 3$.

DIVISIBLE BY A NUMBER

If a number m is a multiple of n, then m is **divisible** by n

Look at the multiples of 5 in Table 13.5.2.1. They all end in 5 or 0. Numbers with last digit of 5 or 0 are divisible by 5. Looking for other patterns in Table 13.5.2.1that shows multiples of the numbers 2 through 9, we can discover the following divisibility tests:

DIVISIBILITY TESTS

A number is divisible by:

- 2 if the last digit is 0, 2, 4, 6, or 8.
- 3 if the sum of the digits is divisible by 3.
- 5 if the last digit is 5 or 0.
- 6 if it is divisible by both 2 and 3.
- 10 if it ends with 0.

Exercise 13.5.2.16

Is 5625 divisible by 2? By 3? By 5? By 6? By 10?

Answer

Is 5625 divisible by 2?
Does it end in 0, 2, 4, 6, or 8? No. 5625 is not divisible by 2. (13.5.2.1)

Is 5625 divisible by 3?
What is the sum of the digits? $5+6+2+5=18$
Is the sum divisible by 3? Yes, 5625 is divisible by 3. (13.5.2.2)

Is 5625 divisible by 5 or 10?
What is the last digit? It is 5. 5625 is divisible by 5 but not by 10. (13.5.2.3)

Is 5625 divisible by 6?
Is it divisible by both 2 and 3? No, 5625 is not divisible by 2, so 5625 is not divisible by 6. (13.5.2.4)

Exercise 13.5.2.17

Determine whether 4,962 is divisible by 2, by 3, by 5, by 6, and by 10.

Answer

by 2, 3, and 6

Exercise 13.5.2.18

Determine whether 3,765 is divisible by 2, by 3, by 5, by 6, and by 10.

Answer

by 3 and 5

Find Prime Factorizations and Least Common Multiples

In mathematics, there are often several ways to talk about the same ideas. So far, we've seen that if m is a multiple of n, we can say that m is divisible by n. For example, since 72 is a multiple of 8, we say 72 is divisible by 8. Since 72 is a multiple of 9, we say 72 is divisible by 9. We can express this still another way.

Since $8 \cdot 9 = 72$, we say that 8 and 9 are **factors** of 72. When we write $72 = 8 \cdot 9$, we say we have factored 72.

$$\underbrace{8 \cdot 9}_{\text{factors}} = \underbrace{72}_{\text{product}}$$

Figure 13.5.2.6

Other ways to factor 72 are $1 \cdot 72$, $2 \cdot 36$, $3 \cdot 24$, $4 \cdot 18$ and $6 \cdot 12$. Seventy-two has many factors: 1, 2, 3, 4, 6, 8, 9, 12, 18, 36, and 72.

FACTORS

If $a \cdot b = m$, then a and b are **factors** of m.

Some numbers, like 72, have many factors. Other numbers have only two factors.

Note

Doing the Manipulative Mathematics activity "Model Multiplication and Factoring" will help you develop a better understanding of multiplication and factoring.

PRIME NUMBER AND COMPOSITE NUMBER

A **prime number** is a counting number greater than 1, whose only factors are 1 and itself.

A **composite number** is a counting number that is not prime. A composite number has factors other than 1 and itself.

Note

Doing the Manipulative Mathematics activity "Prime Numbers" will help you develop a better understanding of prime numbers.

The counting numbers from 2 to 19 are listed in Figure 13.5.2.7, with their factors. Make sure to agree with the "prime" or "composite" label for each!

Number	Factors	Prime or Composite?
2	1,2	Prime
3	1,3	Prime
4	1,2,4	Composite
5	1,5	Prime
6	1,2,3,6	Composite
7	1,7	Prime
8	1,2,4,8	Composite
9	1,3,9	Composite
10	1,2,5,10	Composite

Number	Factors	Prime or Composite?
11	1,11	Prime
12	1,2,3,4,6,12	Composite
13	1,13	Prime
14	1,2,7,14	Composite
15	1,3,5,15	Composite
16	1,2,4,8,16	Composite
17	1,17	Prime
18	1,2,3,6,9,18	Composite
19	1,19	Prime

Figure 13.5.2.7

The **prime numbers** less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. Notice that the only even prime number is 2.

A composite number can be written as a unique **product** of primes. This is called the **prime factorization** of the number. Finding the prime factorization of a composite number will be useful later in this course.

PRIME FACTORIZATION

The **prime factorization** of a number is the product of prime numbers that equals the number.

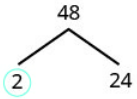
To find the prime factorization of a composite number, find any two factors of the number and use them to create two branches. If a factor is prime, that branch is complete. Circle that prime!

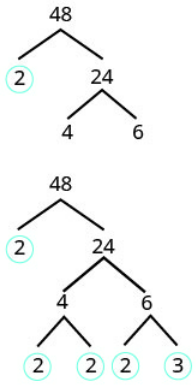
If the factor is not prime, find two factors of the number and continue the process. Once all the branches have circled primes at the end, the factorization is complete. The composite number can now be written as a product of prime numbers.

Exercise 13.5.2.19

Factor 48.

Answer

Step 1. Find two factors whose product is the given number. Use these numbers to create two branches.	$48 = 2 \cdot 24$	
Step 2. If a factor is prime, that branch is complete. Circle the prime.	2 is prime. Circle the prime.	

Step 3. If a factor is not prime, write it as the product of two factors and continue the process.	24 is not prime. Break it into 2 more factors. 4 and 6 are not prime. Break them each into two factors. 2 and 3 are prime, so circle them.	
Step 4. Write the composite number as the product of all the circled primes.		$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

We say $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ is the prime factorization of 48. We generally write the primes in ascending order. Be sure to multiply the factors to verify your answer!

If we first factored 48 in a different way, for example as $6 \cdot 8$, the result would still be the same. Finish the prime factorization and verify this for yourself.

? Exercise 13.5.2.20

Find the prime factorization of 80.

Answer

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

? Exercise 13.5.2.21

Find the prime factorization of 60.

Answer

$2 \cdot 2 \cdot 3 \cdot 5$

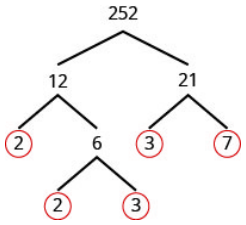
FIND THE PRIME FACTORIZATION OF A COMPOSITE NUMBER.

1. Find two factors whose product is the given number, and use these numbers to create two branches.
2. If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
3. If a factor is not prime, write it as the product of two factors and continue the process.
4. Write the composite number as the product of all the circled primes.

? Exercise 13.5.2.22

Find the prime factorization of 252.

Answer

Step 1. Find two factors whose product is 252. 12 and 21 are not prime. Break 12 and 21 into two more factors. Continue until all primes are factored.	
Step 2. Write 252 as the product of all the circled primes.	$252 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$

? Exercise 13.5.2.23

Find the prime factorization of 126.

Answer

$2 \cdot 3 \cdot 3 \cdot 7$

? Exercise 13.5.2.24

Find the prime factorization of 294.

Answer

$2 \cdot 3 \cdot 7 \cdot 7$

One of the reasons we look at multiples and primes is to use these techniques to find the **least common multiple** of two numbers. This will be useful when we add and subtract fractions with different **denominator**s. Two methods are used most often to find the least common multiple and we will look at both of them.

The first method is the Listing Multiples Method. To find the least common multiple of 12 and 18, we list the first few multiples of 12 and 18:

12: 12, 24, **36**, 48, 60, **72**, 84, 96, **108**...

18: 18, **36**, 54, **72**, 90, **108**...

Common Multiples: 36, 72, 108...

Least Common Multiple: 36

Figure 13.5.2.8

Notice that some numbers appear in both lists. They are the **common multiples** of 12 and 18.

We see that the first few common multiples of 12 and 18 are 36, 72, and 108. Since 36 is the smallest of the common multiples, we call it the *least common multiple*. We often use the abbreviation LCM.

LEAST COMMON MULTIPLE

The **least common multiple** (LCM) of two numbers is the smallest number that is a multiple of both numbers.

The procedure box lists the steps to take to find the LCM using the prime factors method we used above for 12 and 18.

FIND THE LEAST COMMON MULTIPLE BY LISTING MULTIPLES.

1. List several multiples of each number.
2. Look for the smallest number that appears on both lists.
3. This number is the LCM.

Exercise 13.5.2.25

Find the least common multiple of 15 and 20 by listing multiples.

Answer

Make lists of the first few multiples of 15 and of 20, and use them to find the least common multiple.	**15:** 15, 30, 45, **60**, 75, 90, 105, 120 **20:** 20, 40, **60**, 80, 100, 120, 140, 160
Look for the smallest number that appears in both lists.	The first number to appear on both lists is 60, so 60 is the least common multiple of 15 and 20.

Notice that 120 is in both lists, too. It is a common multiple, but it is not the *least* common multiple.

Exercise 13.5.2.26

Find the least common multiple by listing multiples: 9 and 12.

Answer

36

Exercise 13.5.2.27

Find the least common multiple by listing multiples: 18 and 24.

Answer

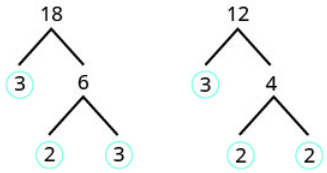
72

Our second method to find the least common multiple of two numbers is to use The Prime Factors Method. Let's find the LCM of 12 and 18 again, this time using their prime factors.

Exercise 13.5.2.28

Find the Least Common Multiple (LCM) of 12 and 18 using the prime factors method.

Answer

Step 1. Write each number as a product of primes.		
Step 2. List the primes of each number. Match primes vertically when possible.	List the primes of 12. List the primes of 18. Line up with the primes of 12 when possible. If not create a new column.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$
Step 3. Bring down the number from each column.		$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 3 \cdot 3$
Step 4. Multiply the factors.		$\text{LCM} = 36$

Notice that the prime factors of $12(2 \cdot 2 \cdot 3)$ and the prime factors of $18(2 \cdot 3 \cdot 3)$ are included in the LCM $(2 \cdot 2 \cdot 3 \cdot 3)$. So 36 is the least common multiple of 12 and 18.

By matching up the common primes, each common prime factor is used only once. This way you are sure that 36 is the *least* common multiple.

Exercise 13.5.2.29

Find the LCM using the prime factors method: 9 and 12.

Answer

36

Exercise 13.5.2.30

Find the LCM using the prime factors method: 18 and 24.

Answer

72

FIND THE LEAST COMMON MULTIPLE USING THE PRIME FACTORS METHOD.

1. Write each number as a product of primes.
2. List the primes of each number. Match primes vertically when possible.
3. Bring down the columns.
4. Multiply the factors.

Exercise 13.5.2.31

Find the Least Common Multiple (LCM) of 24 and 36 using the prime factors method.

Answer

Find the primes of 24 and 36. Match primes vertically when possible. Bring down all columns.	$24 = 2 \cdot 2 \cdot 2 \cdot 3$ $36 = 2 \cdot 2 \cdot \quad 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$
Multiply the factors.	$\text{LCM} = 72$
	The LCM of 24 and 36 is 72.

Exercise 13.5.2.32

Find the LCM using the prime factors method: 21 and 28.

Answer

84

Exercise 13.5.2.33

Find the LCM using the prime factors method: 24 and 32.

Answer

96

Note

Access this online resource for additional instruction and practice with using whole numbers. You will need to enable Java in your web browser to use the application.

- Sieve of Eratosthenes

Key Concepts

- **Place Value** as in Figure.
- **Name a Whole Number in Words**
  1. Start at the left and name the number in each period, followed by the period name.
  2. Put commas in the number to separate the periods.
  3. Do not name the ones period.
- **Write a Whole Number Using Digits**
  1. Identify the words that indicate periods. (Remember the ones period is never named.)
  2. Draw 3 blanks to indicate the number of places needed in each period. Separate the periods by commas.
  3. Name the number in each period and place the digits in the correct place value position.
- **Round Whole Numbers**
  1. Locate the given place value and mark it with an arrow. All digits to the left of the arrow do not change.
  2. Underline the digit to the right of the given place value.
  3. Is this digit greater than or equal to 5?
     - Yes—add 1 to the digit in the given place value.
     - No—do not change the digit in the given place value.
  4. Replace all digits to the right of the given place value with zeros.
- **Divisibility Tests:** A number is divisible by:
  - 2 if the last digit is 0, 2, 4, 6, or 8.
  - 3 if the sum of the digits is divisible by 3.
  - 5 if the last digit is 5 or 0.
  - 6 if it is divisible by both 2 and 3.
  - 10 if it ends with 0.
- **Find the Prime Factorization of a Composite Number**
  1. Find two factors whose product is the given number, and use these numbers to create two branches.
  2. If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
  3. If a factor is not prime, write it as the product of two factors and continue the process.
  4. Write the composite number as the product of all the circled primes.
- **Find the Least Common Multiple by Listing Multiples**
  1. List several multiples of each number.
  2. Look for the smallest number that appears on both lists.
  3. This number is the LCM.
- **Find the Least Common Multiple Using the Prime Factors Method**
  1. Write each number as a product of primes.
  2. List the primes of each number. Match primes vertically when possible.
  3. Bring down the columns.

4. Multiply the factors.

Glossary

composite number

A composite number is a counting number that is not prime. A composite number has factors other than 1 and itself.

counting numbers

The counting numbers are the numbers 1, 2, 3, …

divisible by a number

If a number m is a multiple of n, then m is divisible by n. (If 6 is a multiple of 3, then 6 is divisible by 3.)

factors

If $a \cdot b = m$, then a and b are factors of m. Since $3 \cdot 4 = 12$, then 3 and 4 are factors of 12.

least common multiple

The least common multiple of two numbers is the smallest number that is a multiple of both numbers.

multiple of a number

A number is a multiple of n if it is the product of a counting number and n.

number line

A number line is used to visualize numbers. The numbers on the number line get larger as they go from left to right, and smaller as they go from right to left.

origin

The origin is the point labeled 0 on a number line.

prime factorization

The prime factorization of a number is the product of prime numbers that equals the number.

prime number

A prime number is a counting number greater than 1, whose only factors are 1 and itself.

whole numbers

The whole numbers are the numbers 0, 1, 2, 3, ...

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13.5.2E: Exercises

Practice Makes Perfect

Use Place Value with Whole Numbers

In the following exercises, find the place value of each digit in the given numbers.

Exercise 13.5.2E.34

51,493

1. 1
2. 4
3. 9
4. 5
5. 3

Answer

1. thousands
2. hundreds
3. tens
4. ten thousands
5. ones

Exercise 13.5.2E.35

87,210

1. 2
2. 8
3. 0
4. 7
5. 1

Exercise 13.5.2E.36

164,285

1. 5
2. 6
3. 1
4. 8
5. 2

Answer

1. ones
2. ten thousands
3. hundred thousands
4. tens
5. hundreds

Exercise 13.5.2E. 37

395,076

1. 5
2. 3
3. 7
4. 0
5. 9

Exercise 13.5.2E. 38

93,285,170

1. 9
2. 8
3. 7
4. 5
5. 3

Answer

1. ten millions
2. ten thousands
3. tens
4. thousands
5. millions

Exercise 13.5.2E. 39

36,084,215

1. 8
2. 6
3. 5
4. 4
5. 3

Exercise 13.5.2E. 40

7,284,915,860,132

1. 7
2. 4
3. 5
4. 3
5. 0

Answer

1. trillions
2. billions
3. millions
4. tens
5. thousands

Exercise 13.5.2E.41

2,850,361,159,433

1. 9
2. 8
3. 6
4. 4
5. 2

In the following exercises, name each number using words.

Exercise 13.5.2E.42

1,078

Answer

one thousand, seventy-eight

Exercise 13.5.2E.43

5,902

Exercise 13.5.2E.44

364,510

Answer

three hundred sixty-four thousand, five hundred ten

Exercise 13.5.2E.45

146,023

Exercise 13.5.2E.46

5,846,103

Answer

five million, eight hundred forty-six thousand, one hundred three

Exercise 13.5.2E.47

1,458,398

Exercise 13.5.2E.48

37,889,005

Answer

thirty-seven million, eight hundred eighty-nine thousand, five

Exercise 13.5.2E.49

62,008,465

In the following exercises, write each number as a whole number using digits.

Exercise 13.5.2E.50

four hundred twelve

Answer

412

Exercise 13.5.2E.51

two hundred fifty-three

Exercise 13.5.2E.52

thirty-five thousand, nine hundred seventy-five

Answer

35,975

Exercise 13.5.2E.53

sixty-one thousand, four hundred fifteen

Exercise 13.5.2E.54

eleven million, forty-four thousand, one hundred sixty-seven

Answer

11,044,167

Exercise 13.5.2E.55

eighteen million, one hundred two thousand, seven hundred eighty-three

Exercise 13.5.2E.56

three billion, two hundred twenty-six million, five hundred twelve thousand, seventeen

Answer

3,226,512,017

Exercise 13.5.2E.57

eleven billion, four hundred seventy-one million, thirty-six thousand, one hundred six

In the following, round to the indicated place value.

Exercise 13.5.2E.58

Round to the nearest ten.

1. 386
2. 2,931

Answer

1. 390
2. 2,930

Exercise 13.5.2E.59

Round to the nearest ten.

1. 792
2. 5,647

Exercise 13.5.2E.60

Round to the nearest hundred.

1. 13,748
2. 391,794

Answer

1. 13,700
2. 391,800

Exercise 13.5.2E.61

Round to the nearest hundred.

1. 28,166
2. 481,628

Exercise 13.5.2E.62

Round to the nearest ten.

1. 1,492
2. 1,497

Answer

1. 1,490
2. 1,500

Exercise 13.5.2E.63

Round to the nearest ten.

1. 2,791
2. 2,795

Exercise 13.5.2E.64

Round to the nearest hundred.

1. 63,994
2. 63,940

Answer

1. 64,000
2. 63,900

Exercise 13.5.2E.65

Round to the nearest hundred.

1. 49,584
2. 49,548

In the following exercises, round each number to the nearest ⓐ hundred, ⓑ thousand, ⓒ ten thousand.

Exercise 13.5.2E.66

392,546

Answer

1. 392,500
2. 393,000
3. 390,000

Exercise 13.5.2E.67

619,348

Exercise 13.5.2E.68

2,586,991

Answer

1. 2,587,000
2. 2,587,000
3. 2,590,000

Exercise 13.5.2E.69

4,287,965

Identify Multiples and Factors

In the following exercises, use the divisibility tests to determine whether each number is divisible by 2, 3, 5, 6, and 10.

Exercise 13.5.2E.70

84

Answer

divisible by 2, 3, and 6

Exercise 13.5.2E.71

9,696

Exercise 13.5.2E.72

75

Answer

divisible by 3 and 5

Exercise 13.5.2E.73

78

Exercise 13.5.2E.74

900

Answer

divisible by 2, 3, 5, 6, and 10

Exercise 13.5.2E.75

800

Exercise 13.5.2E.76

986

Answer

divisible by 2

Exercise 13.5.2E.77

942

Exercise 13.5.2E.78

350

Answer

divisible by 2, 5, and 10

Exercise 13.5.2E.79

550

Exercise 13.5.2E.80

22,335

Answer

divisible by 3 and 5

Exercise 13.5.2E.81

39,075

Find Prime Factorizations and Least Common Multiples

In the following exercises, find the prime factorization.

Exercise 13.5.2E.82

86

Answer

$2 \cdot 43$

Exercise 13.5.2E.83

78

Exercise 13.5.2E.84

132

Answer

$2 \cdot 2 \cdot 3 \cdot 11$

Exercise 13.5.2E.85

455

Exercise 13.5.2E.86

693

Answer

$3 \cdot 3 \cdot 7 \cdot 11$

Exercise 13.5.2E.87

400

Exercise 13.5.2E.88

432

Answer

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

Exercise 13.5.2E.89

627

Exercise 13.5.2E.90

2,160

Answer

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5$

Exercise 13.5.2E.91

2,520

In the following exercises, find the least common multiple of the each pair of numbers using the multiples method.

Exercise 13.5.2E.92

8, 12

Answer

24

Exercise 13.5.2E.93

4, 3

Exercise 13.5.2E.94

12, 16

Answer

48

Exercise 13.5.2E.95

30, 40

Exercise 13.5.2E.96

20, 30

Answer

60

Exercise 13.5.2E.97

44, 55

In the following exercises, find the least common multiple of each pair of numbers using the prime factors method.

Exercise 13.5.2E.98

8, 12

Answer

24

Exercise 13.5.2E.99

12, 16

Exercise 13.5.2E.100

28, 40

Answer

280

Exercise 13.5.2E.101

84, 90

Exercise 13.5.2E.102

55, 88

Answer

440

Exercise 13.5.2E.103

60, 72

Everyday Math

Exercise 13.5.2E.104

Writing a Check Jorge bought a car for $24,493. He paid for the car with a check. Write the purchase price in words.

Answer

twenty-four thousand, four hundred ninety-three dollars

Exercise 13.5.2E.105

Writing a Check Marissa's kitchen remodeling cost $18,549. She wrote a check to the contractor. Write the amount paid in words.

Exercise 13.5.2E.106

Buying a Car Jorge bought a car for $24,493. Round the price to the nearest

1. ten
2. hundred
3. thousand; and
4. ten-thousand.

Answer

1. $24,490
2. $24,500
3. $24,000
4. $20,000

Exercise 13.5.2E.107

Remodeling a Kitchen Marissa's kitchen remodeling cost $18,549, Round the cost to the nearest

1. ten
2. hundred
3. thousand and
4. ten-thousand.

Exercise 13.5.2E.108

Population The population of China was 1,339,724,852 on November 1, 2010. Round the population to the nearest

1. billion
2. hundred-million; and
3. million.

Answer

1. 1,000,000,000
2. 1,300,000,000
3. 1,340,000,000

Exercise 13.5.2E.109

Astronomy The average distance between Earth and the sun is 149,597,888 kilometers. Round the distance to the nearest

1. hundred-million
2. ten-million; and
3. million.

Exercise 13.5.2E.110

Grocery Shopping Hot dogs are sold in packages of 10, but hot dog buns come in packs of eight. What is the smallest number that makes the hot dogs and buns come out even?

Answer

40

Exercise 13.5.2E.111

Grocery Shopping Paper plates are sold in packages of 12 and party cups come in packs of eight. What is the smallest number that makes the plates and cups come out even?

Writing Exercises

Exercise 13.5.2E.112

Give an everyday example where it helps to round numbers.

Exercise 13.5.2E.113

If a number is divisible by 2 and by 3 why is it also divisible by 6?

Exercise 13.5.2E.114

What is the difference between prime numbers and composite numbers?

Answer

Answers may vary.

Exercise 13.5.2E.115

Explain in your own words how to find the prime factorization of a composite number, using any method you prefer.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use place value with whole numbers.			
identify multiples and apply divisibility tests.			
find prime factorizations and least common multiples.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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13.5.3: Use the Language of Algebra

Learning Objectives

By the end of this section, you will be able to:

- Use variables and algebraic symbols
- Simplify expressions using the order of operations
- Evaluate an expression
- Identify and combine like terms
- Translate an English phrase to an algebraic expression

Use Variables and Algebraic Symbols

Suppose this year Greg is 20 years old and Alex is 23. You know that Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more, right? In the language of algebra, we say that Greg's age and Alex's age are variables and the 3 is a constant. The ages change ("vary") but the 3 years between them always stays the same ("constant"). Since Greg's age and Alex's age will always differ by 3 years, 3 is the constant. In algebra, we use letters of the alphabet to represent variables. So if we call Greg's age g, then we could use $g+3g+3$ to represent Alex's age. See Table 13.5.3.1.

Table 13.5.3.1

Greg's age	Alex's age
12	15
20	23
35	38
g	$g+3$

The letters used to represent these changing ages are called variables. The letters most commonly used for variables are $x, y, a, b,$ and c.

Definition: VARIABLE

A **variable** is a letter that represents a number whose value may change.

Definition: CONSTANT

A **constant** is a number whose value always stays the same.

To write algebraically, we need some operation symbols as well as numbers and variables. There are several types of symbols we will be using.

There are four basic arithmetic **operations: addition, subtraction, multiplication,** and **division**. We'll list the symbols used to indicate these operations below (Table 13.5.3.2). You'll probably recognize some of them.

Table 13.5.3.2

Operation	Notation	Say:	The result is...
Addition	$a+b$	a plus b	the sum of a and b
Subtraction	$a-b$	a minus b	the difference of a and b
Multiplication	$a \cdot b, ab, (a)(b), (a)b, a(b)$	a times b	the product of a and b
Division	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$	a divided by b	the quotient of a and b, a is called the dividend, and b is called the divisor

We perform these operations on two numbers. When translating from symbolic form to English, or from English to symbolic form, pay attention to the words “of” and “and.”

- The *difference* of 9 and 2 means subtract 9 and 2, in other words, 9 minus 2, which we write symbolically as $9-2$.
- The *product* of 4 and 8 means multiply 4 and 8, in other words 4 times 8, which we write symbolically as $4 \cdot 8$.

In algebra, the cross symbol, $\times$, is not used to show multiplication because that symbol may cause confusion. Does $3xy$ mean $3 \times y$ (‘three times y’) or $3 \cdot x \cdot y$ (three times x times y)? To make it clear, use $\cdot$ or parentheses for multiplication.

When two quantities have the same value, we say they are equal and connect them with an **equal sign**.

EQUALITY SYMBOL

$a = b$ is read “a is equal to b”

The symbol “$=$” is called the **equal sign**.

On the number line, the numbers get larger as they go from left to right. The number line can be used to explain the symbols “$<$” and “$>$”.

INEQUALITY

$a < b$ is read “a is less than b”

a is to the left of b on the number line

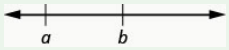

Figure 13.5.3.1

$a > b$ is read "a is greater than b”

a is to the right of b on the number line

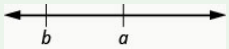

Figure 13.5.3.2

The expressions $a < b$ or $a > b$ can be read from left to right or right to left, though in English we usually read from left to right Table 13.5.3.3 In general, $a < b$ is equivalent to $b > a$. For example $7 < 11$ is equivalent to $11 > 7$. And $a > b$ is equivalent to $b < a$. For example $17 > 4$ is equivalent to $4 < 17$.

Table 13.5.3.3

Inequality Symbols	Words
$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a \leq b$	a is less than or equal to b
$a > b$	a is greater than b
$a \geq b$	a is greater than or not equal to b

Example 13.5.3.1

Translate from algebra into English:

1. $17 \leq 26$
2. $8 \neq 17 - 3$
3. $12 > 27 \div 3$
4. $y + 7 < 19$

Solution

1. $17 \leq 26$, 17 is less than or equal to 26
2. $8 \neq 17 - 3$, 8 is not equal to 17 minus 3

3. $12 > 27 \div 3$, 12 is greater than 27 divided by 3
4. $y + 7 < 19$, y plus 7 is less than 19

Try It 13.5.3.2

Translate from algebra into English:

1. $14 \leq 27$
2. $19 - 2 \neq 8$
3. $12 > 4 \div 2$
4. $x - 7 < 1$

Answer

1. 14 is less than or equal to 27
2. 19 minus 2 is not equal to 8
3. 12 is greater than 4 divided by 2
4. x minus 7 is less than 1

Try It 13.5.3.3

Translate from algebra into English:

1. $19 \leq 15$
2. $7 = 12 - 5$
3. $15 \div 3 < 8$
4. $y + 3 < 6$

Answer

1. 19 is greater then or equal to 15
2. 7 is equal to 12 minus 5
3. 15 divided by 3 is less than 8
4. y plus 3 is greater than 6

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in English. They help to make clear which expressions are to be kept together and separate from other expressions. We will introduce three types now.

GROUPING SYMBOLS

Parentheses	$()$
Brackets	$[]$
Braces	$\{\}$

Here are some examples of expressions that include **grouping symbols**. We will simplify expressions like these later in this section.

$$8(14-8) \qquad 21-3[2+4(9-8)] \qquad 24 \div \{13-2[1(6-5)+4]\}$$

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. “Running very fast” is a phrase, but “The football player was running very fast” is a sentence. A sentence has a subject and a verb. In algebra, we have expressions and *equations*.

EXPRESSION

An **expression** is a number, a variable, or a combination of numbers and variables using operation symbols.

An **expression** is like an English phrase. Here are some examples of expressions:

Table 13.5.3.4

Expression	Words	English Phrase
$3+5$	3 plus 5	the sum of three and five
$n-1$	n minus one	the difference of n and one
$6 \cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Notice that the English phrases do not form a complete sentence because the phrase does not have a verb. An **equation** is two expressions linked with an **equal sign**. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb.

Definition: EQUATION

An **equation** is two expressions connected by an equal sign.

Here are some examples of equations.

Table 13.5.3.5

Equation	English Sentence
$3+5=8$	the sum of three and five is equal to eight
$n-1=14$	n minus one equals fourteen
$6 \cdot 7=42$	The product of six ans seven is equal to forty-two
$x=53$	x is equal to fifty-three
$y+9=2y-3$	y plus nine is equal to two y minus three

Example 13.5.3.4

Determine if each is an expression or an equation:

1. $2(x+3)=10$
2. $4(y-1)+1$
3. $x \div 25$
4. $y+8=40$

Solution

1. $2(x+3)=10$. This is an equation — two expressions are connected with an equal sign.
2. $4(y-1)+1$. This is an expression — no equal sign.
3. $x \div 25$. This is an expression — no equal sign.
4. $y+8=40$. This is an equation — two expressions are connected with an equal sign.

Try It 13.5.3.5

Determine if each is an expression or an equation:

1. $3(x-7)=27$
2. $5(4y-2)-7$

Answer

1. equation
2. expression

Try It 13.5.3.6

Determine if each is an expression or an equation:

1. $y^3 \div 14$
2. $4x - 6 = 22$

Answer

1. expression
2. equation

Suppose we need to multiply nine factors of 2. We could write this as $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. This is tedious and it can be hard to keep track of all those 2s, so we use exponents. We write $2 \cdot 2 \cdot 2$ as 2^3 and $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ as 2^9. In expressions such as 2^3, the 2 is called the base and the 3 is called the *exponent*. The exponent tells us how many times we need to multiply the base.

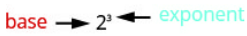

Figure 13.5.3.3

We read 2^3 as "two to the third power" or "two cubed."

We say 2^3 is in exponential notation and $2 \cdot 2 \cdot 2$ is in expanded notation.

EXPONENTIAL NOTATION

a^n means the product of n factors of a.

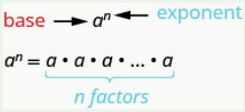

Figure 13.5.3.4

The expression a^n is read a to the nth power.

While we read a^n as "a to the nth power," we usually read:

- a^2 "a squared"
- a^3 "a cubed"

We'll see later why a^2 and a^3 have special names.

Table 13.5.3.6shows how we read some expressions with exponents.

Table 13.5.3.6

Expression	In Words
7^2	7 to the second power or 7 squared
5^3	5 to the third power or 5 cubed
9^4	9 to the fourth power
12^5	12 to the fifth power

Example 13.5.3.7

Simplify: 3^4

Solution

3^4

Expand the expression	$3 \cdot 3 \cdot 3 \cdot 3$
Multiply left to right	$9 \cdot 3 \cdot 3$
Multiply	$27 \cdot 3$
Multiply	81

? Try It 13.5.3.8

Simplify:

1. 5^3
2. 1^7

Answer

1. 125
2. 1

? Try It 13.5.3.9

1. 7^2
2. 0^5

Answer

1. 49
2. 0

Simplify Expressions Using the Order of Operations

To **simplify an expression** means to do all the math possible. For example, to simplify $4 \cdot 2 + 1$ we'd first multiply $4 \cdot 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:

$$4 \cdot 2 + 1$$

$$8 + 1$$

$$9$$

By not using an **equal sign** when you simplify an expression, you may avoid confusing expressions with equations.

SIMPLIFY AN EXPRESSION

To **simplify an expression**, do all operations in the expression.

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values. For example, consider the expression:

$$4 + 3 \cdot 7$$

If you simplify this expression, what do you get?

Some students say 49,

$$4 + 3 \cdot 7$$

Since $4 + 3$ gives 7.

$$7 \cdot 7$$

And $7 \cdot 7$ is 49

$$49$$

Others say 25,

$$4+3\cdot 7$$

Since $3\cdot 7$ is 21.

$$4+21$$

And $21+4$ makes 25.

$$25$$

Imagine the confusion in our banking system if every problem had several different correct answers!

The same expression should give the same result. So mathematicians early on established some guidelines that are called the **Order of Operations**.

PERFORM THE ORDER OF OPERATIONS.

1. Parentheses and Other Grouping Symbols
   - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
2. Exponents
   - Simplify all expressions with exponents.
3. Multiplication and Division
   - Perform all multiplication and division in order from left to right. These operations have equal priority.
4. Addition and Subtraction
   - Perform all addition and subtraction in order from left to right. These operations have equal priority.

Note

Doing the Manipulative Mathematics activity "Game of 24" will give you practice using the order of operations.

Students often ask, "How will I remember the order?" Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase: "Please Excuse My Dear Aunt Sally."

Parentheses	**P**lease
Exponents	**E**xcuse
Multiplication **D**ivision	**M**y **D**ear
Addition **S**ubtraction	**A**unt **S**ally

It's good that "**M**y **D**ear" goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, "**A**unt **S**ally" goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

Let's try an example.

Example 13.5.3.10

Simplify:

1. $4+3\cdot 7$
2. $(4+3)\cdot 7$

Solution

1.

	$4+3\cdot 7$
Are there any **p**arentheses? No.	

Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply first.	$4+3\cdot 7$
Add.	$4+21$
	25

2.

	$(4+3)\cdot 7$
Are there any **p**arentheses? Yes.	$(4+3)\cdot 7$
Simplify inside the parentheses.	$(7)7$
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply.	49

Try It 13.5.3.11

Simplify:

1. $12-5\cdot 2$
2. $(12-5)\cdot 2$

Answer

1. 2
2. 14

Try It 13.5.3.12

Simplify:

1. $8+3\cdot 9$
2. $(8+3)\cdot 9$

Answer

1. 35
2. 99

Example 13.5.3.13

Simplify: $18\div 6+4(5-2)$

Solution

Parentheses? Yes, subtract first.	$18\div 6+4(5-2)$ $18\div 6+4(3)$
Exponents? No.	
Multiplication or division? Yes.	$18\div 6+4(3)$
Divide first because we multiply and divide left to right.	$3+4(3)$
Any other multiplication or division? Yes.	

Multiply.	$3+12$
Any other multiplication or division? No.	
Any addition or subtraction? Yes.	15

Try It 13.5.3.14

Simplify: $30 \div 5+10(3-2)$

Answer

16

Try It 13.5.3.15

Simplify: $70 \div 10+4(6-2)$

Answer

23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

Example 13.5.3.16

Simplify: $5+2^3+3[6-3(4-2)]$.

Solution

	$5+2^3+3[6-3(4-2)]$
Are there any parentheses (or other grouping symbol)? Yes.	
Focus on the parentheses that are inside the brackets.	$5+2^3+3[6-3(4-2)]$
Subtract.	$5+2^3+3[6-3(2)]$
Continue inside the brackets and multiply.	$5+2^3+3[6-6]$
Continue inside the brackets and subtract.	$5+2^3+3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes.	$5+2^3+3[0]$
Simplify exponents.	$5+8+3[0]$
Is there any multiplication or division? Yes.	
Multiply.	$5+8+0$
Is there any addition or subtraction? Yes.	
Add.	$13+0$
Add.	13

Try It 13.5.3.17

Simplify: $9+5^3-[4(9+3)]$.

Answer

86

Try It 13.5.3.18

Simplify: $7^2 - 2[4(5+1)]$.

Answer

1

Evaluate an Expression

In the last few examples, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

EVALUATE AN EXPRESSION

To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression, substitute that number for the variable in the expression and then simplify the expression.

Example 13.5.3.19

Evaluate $7x - 4$, when

1. $x = 5$
2. $x = 1$

Solution

1.

when $x = 5$	$7x - 4$
	$7(5) - 4$
Multiply.	$35 - 4$
Subtract.	31

2.

when $x = 1$	$7x - 4$
	$7(1) - 4$
Multiply.	$7 - 4$
Subtract.	3

Try It 13.5.3.20

Evaluate $8x - 3$, when

1. $x = 2$
2. $x = 1$

Answer

1. 13
2. 5

Try It 13.5.3.21

Evaluate $4y - 4$, when

1. $y = 3$
2. $y = 5$

Answer

1. 8
2. 16

Example 13.5.3.22

Evaluate $x = 4$, when

1. x^2
2. 3^x

Solution

1.

	x^2
Replace x with 4.	$(4)^2$
Use definition of exponent.	$4 \cdot 4$
Simplify.	16

2.

	3^x
Replace x with 4.	3^4
Use definition of exponent.	$3 \cdot 3 \cdot 3 \cdot 3$
Simplify.	81

Try It 13.5.3.23

Evaluate $x = 3$, when

1. x^2
2. 4^x

Answer

1. 9
2. 64

Try It 13.5.3.24

Evaluate $x = 6$, when

1. x^3
2. 2^x

Answer

1. 216
2. 64

Example 13.5.3.25

Evaluate $2x^2+3x+8$ when $x=4$.

Solution

	$2x^2+3x+8$
Substitute $x=4$.	$2x^2+3x+8$ $2(4)^2+3(4)+8$
Follow the order of operations.	$2(16)+3(4)+8$
	$32+12+8$
	52

Try It 13.5.3.26

Evaluate $3x^2+4x+1$ when $x=3$.

Answer

40

Try It 13.5.3.27

Evaluate $6x^2-4x-7$ when $x=2$.

Answer

9

Identify and Combine Like Terms

Algebraic expressions are made up of terms. A **term** is a constant, or the product of a constant and one or more variables.

TERM

A **term** is a constant, or the product of a constant and one or more variables.

Examples of terms are $7, y, 5x^2, 9a$, and b^5.

The constant that multiplies the variable is called the **coefficient**.

COEFFICIENT

The **coefficient** of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x=1\cdot x$.

Example 13.5.3.28

Identify the coefficient of each term:

1. $14y$
2. $15x^2$
3. a

Solution

1. The coefficient of $14y$ is 14
2. The coefficient of $15x^2$ is 15
3. The coefficient of a is 1 since $a = 1a$.

Try It 13.5.3.29

Identify the coefficient of each term:

1. $17x$
2. $41b^2$
3. z

Answer

1. 14
2. 41
3. 1

Try It 13.5.3.30

Identify the coefficient of each term:

1. $9p$
2. $13a^2$
3. y^3

Answer

1. 9
2. 13
3. 1

Some terms share common traits. Look at the following 6 terms. Which ones seem to have traits in common?

$$5x \quad 7 \quad n^2 \quad 4 \quad 3x \quad 9n^2$$

The 7 and the 4 are both constant terms.

The $5x$ and the $3x$ are both terms with x.

The n^2 and the $9n^2$ are both terms with n^2.

When two terms are constants or have the same variable and exponent, we say they are **like terms**.

- 7 and 4 are like terms.
- $5x$ and $3x$ are like terms.
- x^2 and $9x^2$ are like terms.

LIKE TERMS

Terms that are either constants or have the same variables raised to the same powers are called **like terms**.

Example 13.5.3.31

Identify the like terms: $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$.

Solution

y^3 and $4y^3$ are like terms because both have y^3; the variable and the exponent match.

$7x^2$ and $5x^2$ are like terms because both have x^2; the variable and the exponent match.

14 and 23 are like terms because both are constants.

There is no other term like $9x$.

Try It 13.5.3.32

Identify the like terms: $9, 2x^3, y^2, 8x^3, 15, 9y, 11y^2$.

Answer

9 and 15, y^2 and $11y^2$, $2x^3$ and $8x^3$

Try It 13.5.3.33

Identify the like terms: $4x^3, 8x^2, 19, 3x^3, 24, 6x^3$.

Answer

19 and 24, $8x^2$ and $3x^2$, $4x^3$ and $6x^3$

Adding or subtracting terms forms an expression. In the expression $2x^2 + 3x + 8$, from Example, the three terms are $2x^2$, $3x$, and 8.

Example 13.5.3.34

Identify the terms in each expression.

1. $9x^2 + 7x + 12$
2. $8x + 3y$

Solution

1. The terms of $9x^2 + 7x + 12$ are $9x^2$, $7x$, and 12.
2. The terms of $8x + 3y$ are $8x$ and $3y$.

Try It 13.5.3.35

Identify the terms in the expression $4x^2 + 5x + 17$.

Answer

$4x^2, 5x, 17$

Try It 13.5.3.36

Identify the terms in the expression $5x + 2y$.

Answer

$5x, 2y$

If there are like terms in an expression, you can simplify the expression by combining the like terms. What do you think $4x + 7x + x$ would simplify to? If you thought $12x$, you would be right!

$$\begin{array}{c} 4x + 7x + x \\ x + x + x + x \quad + x + x + x + x + x + x + x \quad + x \\ 12x \end{array} \tag{13.5.3.1}$$

Add the coefficients and keep the same variable. It doesn't matter what x is—if you have 4 of something and add 7 more of the same thing and then add 1 more, the result is 12 of them. For example, 4 oranges plus 7 oranges plus 1 orange is 12 oranges. We will discuss the mathematical properties behind this later.

Simplify: $4x + 7x + x$

Add the coefficients. $12x$

Example 13.5.3.37: How To Combine Like Terms

Simplify: $2x^2+3x+7+x^2+4x+5$

Solution

Step 1. Identify the like terms.	$2x^2+3x+7+x^2+4x+5$ $2x^2+3x+7+x^2+4x+5$
Step 2. Rearrange the expression so the like terms are together.	$2x^2+x^2+3x+4x+7+5$
Step 3. Combine like terms.	$3x^2+7x+12$

Try It 13.5.3.38

Simplify: $3x^2+7x+9+7x^2+9x+8$.

Answer

$10x^2+16x+17$

Try It 13.5.3.39

Simplify: $4y^2+5y+2+8y^2+4y+5$.

Answer

$12y^2+9y+7$

COMBINE LIKE TERMS.

1. Identify like terms.
2. Rearrange the expression so like terms are together.
3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Translate an English Phrase to an Algebraic Expression

In the last section, we listed many operation symbols that are used in algebra, then we translated expressions and equations into English phrases and sentences. Now we'll reverse the process. We'll translate English phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. Table 13.5.3.7summarizes them.

Table 13.5.3.7

Operation	Phrase	Expression	
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a+b$	(13.5.3.2)
Subtraction	a minus b the difference of a and b a decreased by b b less than a b subtracted from a	$a-b$	(13.5.3.3)

Operation	Phrase	Expression
Multiplication	a times b the product of a and b twice a	$a \cdot b, ab, a(b), (a)(b)$ (13.5.3.4) $2a$ (13.5.3.5)
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$ (13.5.3.6)

Look closely at these phrases using the four operations:

the **sum** *of a and b*

the **difference** *of a and b*

the **product** *of a and b*

the **quotient** *of a and b*

Figure 13.5.3.5

Each phrase tells us to operate on two numbers. Look for the words *of* and *and* to find the numbers.

Example 13.5.3.40

Translate each English phrase into an algebraic expression:

1. the difference of $17x$ and 5
2. the quotient of $10x^2$ and 7.

Solution

1. The key word is *difference*, which tells us the operation is subtraction. Look for the words *of* and *and* to find the numbers to subtract.

   the *difference of* 17*x and* 5

   17*x* minus 5

   17*x* – 5

2. The key word is "quotient," which tells us the operation is division.

   the *quotient of* 10*x*² *and* 7

   divide 10*x*² by 7

   10*x*² ÷ 7

This can also be written $10x^2/7$ or $\frac{10x^2}{7}$.

Try It 13.5.3.41

Translate each English phrase into an algebraic expression:

1. the difference of $14x^2$ and 13
2. the quotient of $12x$ and 2.

Answer

1. $14x^2 - 13$
2. $12x \div 2$

Try It 13.5.3.42

Translate each English phrase into an algebraic expression:

1. the sum of $17y^2$ and 19
2. the product of 7 and y.

Answer

1. $17y^2+19$
2. $7y$

How old will you be in eight years? What age is eight more years than your age now? Did you add 8 to your present age? Eight "more than" means 8 added to your present age. How old were you seven years ago? This is 7 years less than your age now. You subtract 7 from your present age. Seven "less than" means 7 subtracted from your present age.

Example 13.5.3.43

Translate the English phrase into an algebraic expression:

1. Seventeen more than y
2. Nine less than $9x^2$.

Solution

1. The key words are *more than*. They tell us the operation is addition. *More than* means "added to."

$$\begin{aligned}&\text{Seventeen more than } y\\&\text{Seventeen added to } y\\&y+17\end{aligned}$$

2. The key words are *less than*. They tell us to subtract. *Less than* means "subtracted from."

$$\begin{aligned}&\text{Nine less than } 9x^2\\&\text{Nine subtracted from } 9x^2\\&9x^2-9\end{aligned}$$

Try It 13.5.3.44

Translate the English phrase into an algebraic expression:

1. Eleven more than x
2. Fourteen less than $11a$.

Answer

1. $x+11$
2. $11a-14$

Try It 13.5.3.45

Translate the English phrase into an algebraic expression:

1. 13 more than z
2. 18 less than $8x$.

Answer

1. $z+13$
2. $8x-18$

Example 13.5.3.46

Translate the English phrase into an algebraic expression:

1. five times the sum of m and n
2. the sum of five times m and n.

Solution

There are two operation words—*times* tells us to multiply and *sum* tells us to add.
1. Because we are multiplying 5 times the sum we need parentheses around the sum of m and n, $(m+n)$. This forces us to determine the sum first. (Remember the order of operations.)

$$\begin{array}{c}\text{five times the sum of } m \text{ and } n \\ 5(m+n)\end{array} \tag{13.5.3.7}$$

2. To take a sum, we look for the words "of" and "and" to see what is being added. Here we are taking the sum *of* five times m and \ (n\.)

$$\begin{array}{c}\text{the sum of five times } m \text{ and } n \\ 5m+n\end{array} \tag{13.5.3.8}$$

Try It 13.5.3.47

Translate the English phrase into an algebraic expression:

1. four times the sum of p and q
2. the sum of four times p and q.

Answer

1. $4(p+q)$
2. $4p+q$

Try It 13.5.3.48

Translate the English phrase into an algebraic expression:

1. the difference of two times x and 8,
2. two times the difference of x and 8.

Answer

1. $2x-8$
2. $2(x-8)$

Later in this course, we'll apply our skills in algebra to solving applications. The first step will be to translate an English phrase to an algebraic expression. We'll see how to do this in the next two examples.

Example 13.5.3.49

The length of a rectangle is 6 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Solution

$$\begin{array}{ll} \text{Write a phrase about the length of the rectangle.} & \text{6 less than the width} \\ \text{Substitute } w \text{ for "the width."} & \text{6 less then w} \\ \text{Rewrite "less than" as "subtracted from."} & \text{6 subtracted from w} \\ \text{Translate the phrase into algebra.} & w-6 \end{array} \tag{13.5.3.9}$$

Try It 13.5.3.50

The length of a rectangle is 7 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Answer

$w-7$

Try It 13.5.3.51

The width of a rectangle is 6 less than the length. Let l represent the length of the rectangle. Write an expression for the width of the rectangle.

Answer

$l-6$

Example 13.5.3.52

June has dimes and quarters in her purse. The number of dimes is three less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution

Write the phrase about the number of dimes.	three less than four times the number of quarters
Substitute q for the number of quarters.	3 less than 4 times q
Translate "4 times q ."	3 less than 4q
Translate the phrase into algebra.	4q - 3

(13.5.3.10)

Try It 13.5.3.53

Geoffrey has dimes and quarters in his pocket. The number of dimes is eight less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Answer

$4q-8$

Try It 13.5.3.54

Lauren has dimes and nickels in her purse. The number of dimes is three more than seven times the number of nickels. Let n represent the number of nickels. Write an expression for the number of dimes.

Answer

$7n+3$

Key Concepts

- **Notation** — **The result is…**
  - $a+b$ — the sum of a and b
  - $a-b$ — the difference of a and b
  - $a\cdot b, ab, (a)(b)(a)b, a(b)$ — the product of a and b
  - $a\div b, a/b, \frac{a}{b}, b\overline{)a}$ — the quotient of a and b
- **Inequality**
  - $a<b$ is read "a is less than b" — a is to the left of b on the number line
  - $a>b$ is read "a is greater than b" — a is to the right of b on the number line

- **Inequality Symbols** **Words**
  - $a \neq b$ a is not equal to b
  - $a < b$ a is less than b
  - $a \leq b$ a is less than or equal to b
  - $a > b$ a is greater than b
  - $a \geq b$ a is greater than or equal to b
- **Grouping Symbols**
  - Parentheses ()
  - Brackets []
  - Braces {}
- **Exponential Notation**
  - a^n means the product of n factors of a. The expression a^n is read a to the nth power.
- **Order of Operations:** When simplifying mathematical expressions perform the operations in the following order:
  1. Parentheses and other Grouping Symbols: Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
  2. Exponents: Simplify all expressions with exponents.
  3. Multiplication and Division: Perform all multiplication and division in order from left to right. These operations have equal priority.
  4. Addition and Subtraction: Perform all addition and subtraction in order from left to right. These operations have equal priority.
- **Combine Like Terms**
  1. Identify like terms.
  2. Rearrange the expression so like terms are together.
  3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Glossary

coefficient

The coefficient of a term is the constant that multiplies the variable in a term.

constant

A constant is a number whose value always stays the same.

equality symbol

The symbol "$=$" is called the equal sign. We read $a = b$ as "a is equal to b."

equation

An equation is two expressions connected by an equal sign.

evaluate an expression

To evaluate an expression means to find the value of the expression when the variable is replaced by a given number.

expression

An expression is a number, a variable, or a combination of numbers and variables using operation symbols.

like terms

Terms that are either constants or have the same variables raised to the same powers are called like terms.

simplify an expression

To simplify an expression, do all operations in the expression.

term

A term is a constant or the product of a constant and one or more variables.

variable

A variable is a letter that represents a number whose value may change.

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13.5.3E: Exercises

Practice Makes Perfect

Use Variables and Algebraic Symbols

In the following exercises, translate from algebra to English.

Exercise 13.5.3E.55

$16 - 9$

Answer

16 minus 9, the difference of sixteen and nine

Exercise 13.5.3E.56

$3 \cdot 9$

Exercise 13.5.3E.57

$28 \div 4$

Answer

28 divided by 4, the quotient of twenty-eight and four

Exercise 13.5.3E.58

$x + 11$

Exercise 13.5.3E.59

$(2)(7)$

Answer

2 times 7, the product of two and seven

Exercise 13.5.3E.60

$(4)(8)$

Exercise 13.5.3E.61

$14 < 21$

Answer

fourteen is less than twenty-one

Exercise 13.5.3E.62

$17 < 35$

Exercise 13.5.3E.63

$36 \geq 19$

Answer

thirty-six is greater than or equal to nineteen

Exercise 13.5.3E.64

$6n = 36$

Exercise 13.5.3E.65

$y - 1 > 6$

Answer

y minus 1 is greater than 6, the difference of y and one is greater than six

Exercise 13.5.3E.66

$y - 4 > 8$

Exercise 13.5.3E.67

$2 \leq 18 \div 6$

Answer

2 is less than or equal to 18 divided by 6; 2 is less than or equal to the quotient of eighteen and six

Exercise 13.5.3E.68

$a \neq 1 \cdot 12$

In the following exercises, determine if each is an expression or an equation.

Exercise 13.5.3E.69

$9 \cdot 6 = 54$

Answer

equation

Exercise 13.5.3E.70

$7 \cdot 9 = 63$

Exercise 13.5.3E.71

$5 \cdot 4 + 3$

Answer

expression

Exercise 13.5.3E.72

$x+7$

Exercise 13.5.3E.73

$x+9$

Answer

expression

Exercise 13.5.3E.74

$y-5=25$

Simplify Expressions Using the Order of Operations

In the following exercises, simplify each expression.

Exercise 13.5.3E.75

5^3

Answer

125

Exercise 13.5.3E.76

8^3

Exercise 13.5.3E.77

2^8

Answer

256

Exercise 13.5.3E.78

10^5

In the following exercises, simplify using the order of operations.

Exercise 13.5.3E.79

1. $3+8\cdot 5$
2. $(3+8)\cdot 5$

Answer

1. 43
2. 55

Exercise 13.5.3E.80

1. $2+6\cdot 3$
2. $(2+6)\cdot 3$

Exercise 13.5.3E.81

$2^3-12\div(9-5)$

Answer

5

Exercise 13.5.3E.82

$3^2-18\div(11-5)$

Exercise 13.5.3E.83

$3\cdot 8+5\cdot 2$

Answer

34

Exercise 13.5.3E.84

$4\cdot 7+3\cdot 5$

Exercise 13.5.3E.85

$2+8(6+1)$

Answer

58

Exercise 13.5.3E.86

$4+6(3+6)$

Exercise 13.5.3E.87

$4\cdot 12/8$

Answer

6

Exercise 13.5.3E.88

$2\cdot 36/6$

Exercise 13.5.3E.89

$(6+10)\div(2+2)$

Answer

4

Exercise 13.5.3E.90

$(9+12)\div(3+4)$

Exercise 13.5.3E.91

$20\div4+6\cdot5$

Answer

35

Exercise 13.5.3E.92

$33\div3+8\cdot2$

Exercise 13.5.3E.93

3^2+7^2

Answer

58

Exercise 13.5.3E.94

$(3+7)^2$

Exercise 13.5.3E.95

$3(1+9\cdot6)-4^2$

Answer

149

Exercise 13.5.3E.96

$5(2+8\cdot4)-7^2$

Exercise 13.5.3E.97

$2[1+3(10-2)]$

Answer

50

Exercise 13.5.3E.98

$5[2+4(3-2)]$

Evaluate an Expression

In the following exercises, evaluate the following expressions.

Exercise 13.5.3E.99

$7x+8$ when $x=2$

Answer

22

Exercise 13.5.3E.100

$8x-6$ when $x=7$

Exercise 13.5.3E.101

x^2 when $x=12$

Answer

144

Exercise 13.5.3E.102

x^3 when $x=5$

Exercise 13.5.3E.103

x^5 when $x=2$

Answer

32

Exercise 13.5.3E.104

4^x when $x=2$

Exercise 13.5.3E.105

x^2+3x-7 when $x=4$

Answer

21

Exercise 13.5.3E.106

$6x+3y-9$ when $x=10, y=7$

Answer

9

Exercise 13.5.3E.107

$(x+y)^2$ when $x=6, y=9$

Exercise 13.5.3E.108

a^2+b^2 when $a=3, b=8$

Answer

73

Exercise 13.5.3E.109

r^2-s^2 when $r=12, s=5$

Exercise 13.5.3E.110

$2l+2w$ when $l=15, w=12$

Answer

54

Exercise 13.5.3E.111

$2l+2w$ when $l=18, w=14$

Simplify Expressions by Combining Like Terms

In the following exercises, identify the coefficient of each term.

Exercise 13.5.3E.112

$8a$

Answer

8

Exercise 13.5.3E.113

$13m$

Exercise 13.5.3E.114

$5r^2$

Answer

5

Exercise 13.5.3E.115

$6x^3$

In the following exercises, identify the like terms.

Exercise 13.5.3E.116

$x^3, 8x, 14, 8y, 5, 8x^3$

Answer

x^3 and $8x^3$, 14 and 5

Exercise 13.5.3E.117

$6z, 3w^2, 1, 6z^2, 4z, w^2$

Exercise 13.5.3E.118

$9a, a^2, 16, 16b^2, 4, 9b^2$

Answer

16 and 4, $16b^2$ and $9b^2$

Exercise 13.5.3E.119

$3, 25r^2, 10s, 10r, 4r^2, 3s$

In the following exercises, identify the terms in each expression.

Exercise 13.5.3E.120

$15x^2 + 6x + 2$

Answer

$15x^2, 6x, 2$

Exercise 13.5.3E.121

$11x^2 + 8x + 5$

Exercise 13.5.3E.122

$10y^3 + y + 2$

Answer

$10y^3, y, 2$

Exercise 13.5.3E.123

$9y^3 + y + 5$

In the following exercises, simplify the following expressions by combining like terms.

Exercise 13.5.3E.124

$10x + 3x$

Answer

$13x$

Exercise 13.5.3E.125

$15x + 4x$

Exercise 13.5.3E.126

$4c + 2c + c$

Answer

$7c$

Exercise 13.5.3E.127

$6y + 4y + y$

Exercise 13.5.3E.128

$7u + 2 + 3u + 1$

Answer

$10u + 3$

Exercise 13.5.3E.129

$8d + 6 + 2d + 5$

Exercise 13.5.3E.130

$10a + 7 + 5a - 2 + 7a - 4$

Answer

$22a + 1$

Exercise 13.5.3E.131

$7c + 4 + 6c - 3 + 9c - 1$

Exercise 13.5.3E.132

$3x^2 + 12x + 11 + 14x^2 + 8x + 5$

Answer

$17x^2 + 20x + 16$

Exercise 13.5.3E.133

$5b^2+9b+10+2b^2+3b-4$

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the phrases into algebraic expressions.

Exercise 13.5.3E.134

the difference of 14 and 9

Answer

$14-9$

Exercise 13.5.3E.135

the difference of 19 and 8

Exercise 13.5.3E.136

the product of 9 and 7

Answer

$9 \cdot 7$

Exercise 13.5.3E.137

the product of 8 and 7

Exercise 13.5.3E.138

the quotient of 36 and 9

Answer

$36 \div 9$

Exercise 13.5.3E.139

the quotient of 42 and 7

Exercise 13.5.3E.140

the sum of $8x$ and $3x$

Answer

$8x+3x$

Exercise 13.5.3E.141

the sum of $13x$ and $3x$

Exercise 13.5.3E.142

the quotient of y and 3

Answer

$\frac{y}{3}$

Exercise 13.5.3E.143

the quotient of y and 8

Exercise 13.5.3E.144

eight times the difference of y and nine

Answer

$8(y-9)$

Exercise 13.5.3E.145

seven times the difference of y and one

Exercise 13.5.3E.146

Eric has rock and classical CDs in his car. The number of rock CDs is 3 more than the number of classical CDs. Let c represent the number of classical CDs. Write an expression for the number of rock CDs.

Answer

$c+3$

Exercise 13.5.3E.147

The number of girls in a second-grade class is 4 less than the number of boys. Let b represent the number of boys. Write an expression for the number of girls.

Exercise 13.5.3E.148

Greg has nickels and pennies in his pocket. The number of pennies is seven less than twice the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

Answer

$2n-7$

Exercise 13.5.3E.149

Jeannette has $5 and $10 bills in her wallet. The number of fives is three more than six times the number of tens. Let t represent the number of tens. Write an expression for the number of fives.

Everyday Math

Exercise 13.5.3E.150

Car insurance Justin's car insurance has a $750 deductible per incident. This means that he pays $750 and his insurance company will pay all costs beyond $750. If Justin files a claim for $2,100.

1. how much will he pay?
2. how much will his insurance company pay?

Answer

1. $750
2. $1,350

Exercise 13.5.3E.151

Home insurance Armando's home insurance has a $2,500 deductible per incident. This means that he pays $2,500 and the insurance company will pay all costs beyond $2,500. If Armando files a claim for $19,400.

1. how much will he pay?
2. how much will the insurance company pay?

Writing Exercises

Exercise 13.5.3E.152

Explain the difference between an expression and an equation.

Answer

Answers may vary

Exercise 13.5.3E.153

Why is it important to use the order of operations to simplify an expression?

Exercise 13.5.3E.154

Explain how you identify the like terms in the expression $8a^2+4a+9-a^2-1$

Answer

Answers may vary

Exercise 13.5.3E.155

Explain the difference between the phrases "4 times the sum of x and y" and "the sum of 4 times x and y."

Self Check

ⓐ Use this checklist to evaluate your mastery of the objectives of this section.

the *quotient of* $10x^2$ *and* 7
divide $10x^2$ by 7
$10x^2 \div 7$

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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13.5.4: Add and Subtract Integers

Learning Objectives

- Use negatives and opposites
- Simplify: expressions with absolute value
- Add integers
- Subtract integers

A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **Integers**.

Use Negatives and Opposites

Our work so far has only included the counting numbers and the whole numbers. But if you have ever experienced a temperature below zero or accidentally overdrawn your checking account, you are already familiar with negative numbers. **Negative numbers** are numbers less than 0. The **negative numbers** are to the left of zero on the number line. See Figure 13.5.4.1.

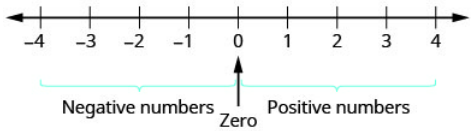


Figure 13.5.4.1. The number line shows the location of positive and negative numbers.

The arrows on the ends of the number line indicate that the numbers keep going forever. There is no biggest positive number, and there is no smallest negative number.

Is zero a positive or a negative number? Numbers larger than zero are positive, and numbers smaller than zero are negative. Zero is neither positive nor negative.

Consider how numbers are ordered on the number line. Going from left to right, the numbers increase in value. Going from right to left, the numbers decrease in value. See Figure 13.5.4.2

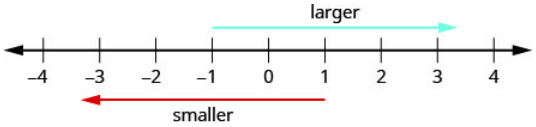


Figure 13.5.4.2. The numbers on a number line increase in value going from left to right and decrease in value going from right to left.

Doing the Manipulative Mathematics activity "Number Line-part 2" will help you develop a better understanding of integers.

Remember that we use the notation:

$a < b$ (read "a is less than b") when a is to the left of b on the number line.

$a > b$ (read "a is greater than b") when a is to the right of b on the number line.

Now we need to extend the number line which showed the whole numbers to include negative numbers, too. The numbers marked by points in Figure 13.5.4.3 are called the integers. The integers are the numbers $\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$

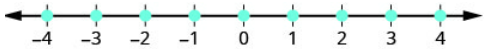


Figure 13.5.4.3. All the marked numbers are called *integers*.

Example 13.5.4.1

Order each of the following pairs of numbers, using $<$ or $>$:

1. 14 ___ 6
2. -1 ___ 9
3. -1 ___ -4
4. 2 ___ -20

Solution:

It may be helpful to refer to the number line shown.

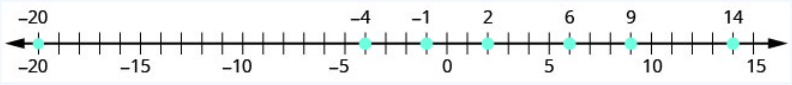


Figure 13.5.4.4

1. $14 \_\_\_ 6$

   $14 > 6$ 14 is to the right of 6 on the number line.

2. $-1 \_\_\_ 9$

   $-1 < 9$ -1 is to the left of 9 on the number line.

3. $-1 \_\_\_ -4$

   $-1 > -4$ -1 is to the right of -4 on the number line.

4. $2 \_\_\_ -20$

   $2 > -20$ 2 is to the right of -20 on the number line.

Try It 13.5.4.1

Order each of the following pairs of numbers, using $<$ or $>$:

1. $15 \_\_\_ 7$
2. $-2 \_\_\_ 5$
3. $-3 \_\_\_ -7$
4. $5 \_\_\_ -17$

Answer

1. $<$
2. $>$
3. $<$
4. $>$

Try It 13.5.4.2

Order each of the following pairs of numbers, using $<$ or $>$:

1. $8 \_\_\_ 13$
2. $3 \_\_\_ -4$
3. $-5 \_\_\_ -2$
4. $9 \_\_\_ -21$

Answer

1. $<$
2. $>$
3. $<$
4. $>$

You may have noticed that, on the **number line**, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, they are called **opposites**. The opposite of 2 is -2, and

the opposite of -2 is 2.

OPPOSITE

The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

Figure 13.5.4.5illustrates the definition.

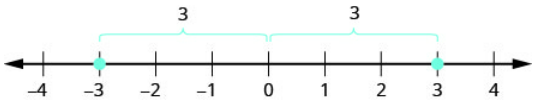


Figure 13.5.4.5: The opposite of 3 is -3.

Sometimes in algebra the same symbol has different meanings. Just like some words in English, the specific meaning becomes clear by looking at how it is used. You have seen the symbol "$-$" used in three different ways.

$10-4$ Between two numbers, it indicates the operation of *subtraction*.
We read $10-4$ as "10 minus 4."

-8 In front of a number, it indicates a *negative* number.
We read -8 as "negative eight."

$-x$ In front of a variable, it indicates the *opposite*.
We read $-x$ as "the opposite of x"

$-(-2)$ Here there are two "$-$" signs.
The one in the parentheses tells us the number is negative 2.
The one outside the parentheses tells us to take the *opposite* of -2.
We read $-(-2)$ as "the opposite of negative two."

OPPOSITE NOTATION

$-a$ means the opposite of the number a.

The notation $-a$ is read as "the opposite of a."

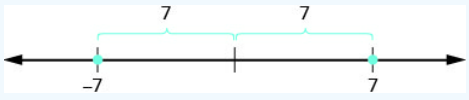
Example 13.5.4.2

Find:

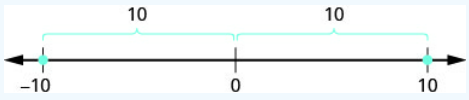
1. the opposite of 7
2. the opposite of -10
3. the opposite of $-(-6)$

Solution:

1. -7 is the same distance from 0 as 7, but on the opposite side of 0. The opposite of 7 is -7.



2. 10 is the same distance from 0 as -10, but on the opposite side of 0. The opposite of -10 is 10.



3. The opposite of $-(-6)$ is -6.

Try It 13.5.4.3

Find:

1. the opposite of 4
2. the opposite of -3
3. $-(-1)$

Answer

1. -4
2. 3
3. 1

Try It 13.5.4.4

Find:

1. the opposite of 8
2. the opposite of -5
3. $-(-5)$

Answer

1. -8
2. 5
3. 5

Our work with opposites gives us a way to define the integers. The whole numbers and their opposites are called the **integers**. The integers are the numbers $\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$

INTEGERS

The whole numbers and their opposites are called the **integers**.

The integers are the numbers

$$\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$$

When evaluating the opposite of a **variable**, we must be very careful. Without knowing whether the variable represents a positive or negative number, we don't know whether −x−x is positive or negative. We can see this in Example $13.5.4.1$.

Example 13.5.4.3:

Evaluate

1. $-x$, when $x = 8$
2. $-x$, when $x = -8$

Solution:

1.

To evaluate when $x = 8$ means to substitute 8 for x.	
	-x

Substitute 8 for x.	$-(8)$
Write the opposite of 8.	-8

2.

To evaluate when $x = -8$ means to substitute -8 for $-x$.	
	-x
Substitute -8 for x.	$-(-8)$
Write the opposite of -8.	8

Try It 13.5.4.5

Evaluate $-n$, when

1. $n = 4$
2. $n = -4$

Answer

1. -4
2. 4

Try It 13.5.4.6

Evaluate $-m$, when

1. $m = 11$
2. $m = -11$

Answer

1. -11
2. 11

Simplify: Expressions with Absolute Value

We saw that numbers such as 2 and -2 are opposites because they are the same distance from 0 on the number line. They are both two units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

ABSOLUTE VALUE

The **absolute value** of a number is its distance from 0 on the number line.

The absolute value of a number \(n\) is written as $|n|$.

For example,

- -5 is 5 units away from 0, so $|-5| = 5$.
- 5 is 5 units away from 0, so $|5| = 5$.

Figure $13.5.4.6$illustrates this idea.

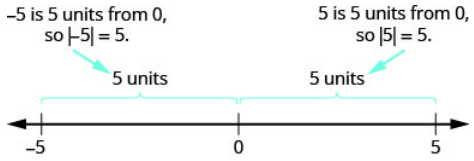


Figure 13.5.4.6: The integers 5 and are 5 units away from 0.

The absolute value of a number is never negative (because distance cannot be negative). The only number with absolute value equal to zero is the number zero itself, because the distance from 0 to 0 on the number line is zero units.

PROPERTY OF ABSOLUTE VALUE

$|n| \geq 0$ for all numbers

Absolute values are always greater than or equal to zero!

Mathematicians say it more precisely, "absolute values are always non-negative." Non-negative means greater than or equal to zero.

Example 13.5.4.4:

Simplify:

1. $|3|$
2. $|-44|$
3. $|0|$

Solution:

The absolute value of a number is the distance between the number and zero. Distance is never negative, so the absolute value is never negative.

1. $|3| = 3$
2. $|-44| = 44$
3. $|0| = 0$

Try It 13.5.4.7

Simplify:

1. $|4|$
2. $|-28|$
3. $|0|$

Answer

1. 4
2. 28
3. 0

Try It 13.5.4.8

Simplify:

1. $|-13|$
2. $|47|$

Answer

1. 13

2. 47

In the next example, we'll order expressions with absolute values. Remember, positive numbers are always greater than negative numbers!

Example 13.5.4.5:

Fill in $<, >,$ or $=$ for each of the following pairs of numbers:

1. $|-5|$ ____ $-|-5|$
2. 8 ____ $-|-8|$
3. -9 ____ $-|-9|$
4. $-(-16)$ ____ $-|-16|$

Solution:

1. Simplification $|-5| \quad \_\_\_\_ \quad -|-5|$
   Order. $5 \quad \_\_\_\_ \quad -5$
   $5 \quad > \quad -5$
   $|-5| \quad > \quad -|-5|$
2. Simplification $8 \quad \_\_\_\_ \quad -|-8|$
   Order. $8 \quad \_\_\_\_ \quad -8$
   $8 \quad > \quad -8$
   so
   $|8| \quad > \quad -|-8|$
3. Simplification $-9 \quad \_\_\_\_ \quad -|-9|$
   Order. $-9 \quad \_\_\_\_ \quad -9$
   $-9 \quad = \quad -9$
   so
   $-9 \quad = \quad -|-9|$
4. Simplification $-(-16) \quad \_\_\_\_ \quad -|16|$
   Order. $16 \quad \_\_\_\_ \quad -16$
   $16 \quad > \quad -16$
   so
   $-(-16) \quad > \quad -|-16|$

Try It 13.5.4.9

Fill in $<, >,$ or $=$ for each of the following pairs of numbers:

1. $|-9|$ ____ $-|-9|$
2. 2 ____ $-|-2|$
3. -8 ____ $-|-8|$
4. $-(-9)$ ____ $-|-9|$

Answer

1. $>$
2. $>$
3. $<$
4. $>$

Try It 13.5.4.10

Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

1. 7 ____ $-|-7|$
2. $-(-10)$ ____ $-|-10|$
3. $|-4|$ ____ $-|-4|$
4. -1 ____ $|-1|$

Answer

1. $>$
2. $>$
3. $>$
4. $<$

We now add absolute value bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the **absolute value** of the resulting number.

GROUPING SYMBOLS

Parentheses	$()$	Braces	$\{\}$
Brackets	$[\,]$	Absolute	$\|\,\|$

In the next example, we simplify the expressions inside absolute value bars first, just like we do with parentheses.

Example 13.5.4.6:

Simplify: $24-|19-3(6-2)|$

Solution:

	$24-\|19-3(6-2)\|$
Work inside parentheses first: subtract 2 from 6	$24-\|19-3(4)\|$
Multiply $3(4)$	$24-\|19-12\|$
Subtract inside the absolute value bars.	$24-\|7\|$
Take the absolute value.	$24-7$
Subtract.	17

Try It 13.5.4.11

Simplify: $19-|11-4(3-1)|$

Answer

16

Try It 13.5.4.12

Simplify: $9-|8-4(7-5)|$

Answer

9

Example 13.5.4.7

Evaluate:

1. $|x|$ when $x = -35$
2. $|y|$ when $y = -20$
3. $-|u|$ when $u = 12$
4. $-|p|$ when $p = -14$

Solution:

1. $|x|$ when $x = -35$

	$\|x\|$
Substitute -35 for x	$\|-35\|$
Take the absolute value.	35

2. $|y|$ when $y = -20$

	$\|-y\|$
Substitute -20 for y	$\|-(-20)\|$
Simplify	$\|20\|$
Take the absolute value.	20

3. $-|u|$ when $u = 12$

	$-\|u\|$
Substitute 12 for u	$\|-12\|$
Take the absolute value.	-12

4. $-|p|$ when $p = -14$

	$-\|p\|$
Substitute -14 for p	$-\|-14\|$
Take the absolute value.	-14

Try It 13.5.4.13

Evaluate:

1. $|x|$ when $x = -17$
2. $|y|$ when $y = -39$
3. $-|m|$ when $m = 22$
4. $-|p|$ when $p = -11$

Answer

1. 17
2. 39
3. -22
4. -11

Try It 13.5.4.14

Evaluate:

1. $|y|$ when $y = -23$
2. $|-y|$ when $y = -21$
3. $-|n|$ when $n = 37$
4. $-|q|$ when $q = -49$

Answer

1. 23
2. 21
3. -37
4. -49

Add Integers

Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more challenging.

Doing the Manipulative Mathematics activity "Addition of Signed Numbers" will help you develop a better understanding of adding integers."

We will use two color counters to model addition and subtraction of negatives so that you can visualize the procedures instead of memorizing the rules.

We let one color (blue) represent positive. The other color (red) will represent the negatives. If we have one positive counter and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero.

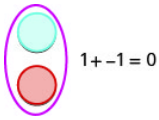
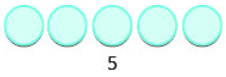


Figure 13.5.4.7

We will use the counters to show how to add the four addition facts using the numbers $5, -5$ and $3, -3$.

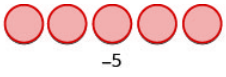
$$5+3 \quad -5+(-3) \quad -5+3 \quad 5+(-3)$$

To add $5+3$, we realize that $5+3$ means the sum of 5 and 3.

We start with 5 positives.	
And then we add 3 positives.	
We now have 8 positives. The sum of 5 and 3 is 8.	

Now we will add $-5+(-3)$. Watch for similarities to the last example $5+3=8$.

To add $-5+(-3)$, we realize this means the sum of -5 and -3.

We start with 5 negatives.	
And then we add 3 negatives.	
We now have 8 negative. The sum of -5 and -3 is -8.	

In what ways were these first two examples similar?

- The first example adds 5 positives and 3 positives—both positives.
- The second example adds 5 negatives and 3 negatives—both negatives.

In each case we got 8 — either 8 positives or 8 negatives.

When the signs were the same, the counters were all the same color, and so we added them.

Figure 13.5.4.7

? Example 13.5.4.8

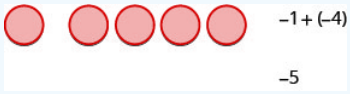
Add:

1. $1+4$
2. $-1+(-4)$

Solution:

1.

1 positive plus 4 positives is 5 positives.

2. 

1 negative plus 4 negatives is 5 negatives.

? Try It 13.5.4.15

Add:

1. $2+4$
2. $-2+(-4)$

Answer

1. 6
2. -6

? Try It 13.5.4.16

Add:

1. $2+5$
2. $-2+(-5)$

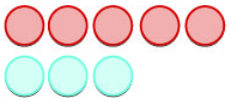
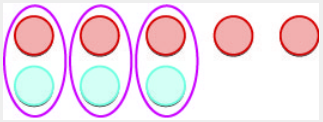
Answer

1. 7
2. -7

So what happens when the signs are different? Let's add $-5+3$. We realize this means the sum of -5 and 3. When the counters were the same color, we put them in a row. When the counters are a different color, we line them up under each other.

Table 13.5.4.1

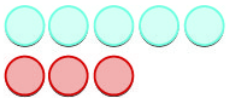
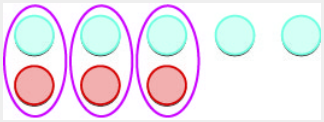
	$-5+3$ means the sum of -5 and 3.
We start with 5 negatives.	

And then we add 3 positives.	
We remove any neutral pairs.	
We have 2 negatives left.	
The sum of -5 and 3 is -2.	$-5+3=2$

Notice that there were more negatives than positives, so the result was negative.

Let's now add the last combination, $5+(-3)$.

Table 13.5.4.2

	$5+(-3)$ means the sum of -5 and -3.
We start with 5 positives.	
And then we add 3 negatives.	
We remove any neutral pairs.	
We have 2 positives left.	
The sum of 5 and -3 is 2.	$5+(-3)=2$

When we use counters to model addition of positive and negative integers, it is easy to see whether there are more positive or more negative counters. So we know whether the sum will be positive or negative.

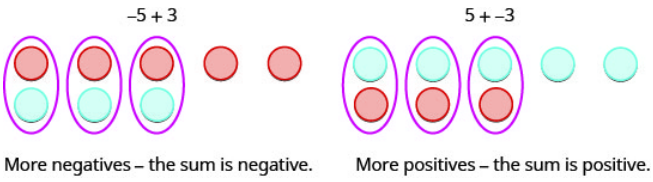


Figure 13.5.4.8

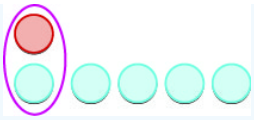
? Example 13.5.4.9

Add:

1. $-1+5$
2. $1+(-5)$

Solution:

1. $-1+5$



There are more positives, so the sum is positive.

So, $-1+5=4$.

2. $1+(-5)$

There are more negatives, so the sum is negative.

So, $1+(-5)=-4$

Try It 13.5.4.17

Add:

1. $-2+4$
2. $2+(-4)$

Answer

1. 2
2. -2

Try It 13.5.4.18

Add:

1. $-2+5$
2. $2+(-5)$

Answer

1. 3
2. -3

Now that we have added small positive and negative integers with a model, we can visualize the model in our minds to simplify problems with any numbers.

When you need to add numbers such as $37+(-53)$, you really don't want to have to count out 37 blue counters and 53 red counters. With the model in your mind, can you visualize what you would do to solve the problem?

Picture 37 blue counters with 53 red counters lined up underneath. Since there would be more red (negative) counters than blue (positive) counters, the sum would be *negative*. How many more red counters would there be? Because $53-37=16$, there are 16 more red counters.

Therefore, the sum of $37+(-53)$ is -16.

$$37+(-53)=-16$$

Let's try another one. We'll add $-74+(-27)$. Again, imagine 74 red counters and 27 more red counters, so we'd have 101 red counters. This means the sum is -101.

$$-74+(-27)=-101$$

Let's look again at the results of adding the different combinations of $5, -5$ and $3, -3$.

ADDITION OF POSITIVE AND NEGATIVE INTEGERS

$5+3=8$ — both positive, sum positive

$-5+(-3)=-8$ — both negative, sum negative

When the signs are the same, the counters would be all the same color, so add them.

$-5+3=-2$ — different signs, more negatives, sum negative

$5+(-3)=2$ — different signs, more positives, sum positive

When the signs are different, some of the counters would make neutral pairs, so subtract to see how many are left.

Visualize the model as you simplify the expressions in the following examples.

Example 13.5.4.10

Simplify:

1. $19+(-47)$
2. $-14+(-36)$

Solution:

1. Since the signs are different, we subtract 19 from 47. The answer will be negative because there are more negatives than positives.

Add. $19+(-47)=-28$

2. Since the signs are same, we add. The answer will be negative because there are more negatives than positives.

Add. $-14+(-36)=-50$

Try It 13.5.4.19

Simplify:

1. $-31+(-19)$
2. $15+(-32)$

Answer

1. -50
2. -17

Try It 13.5.4.20

Simplify:

1. $-42+(-28)$
2. $25+(-61)$

Answer

1. -70
2. -36

The techniques used up to now extend to more complicated problems, like the ones we've seen before. Remember to follow the order of operations!

Example 13.5.4.11

Simplify:

$-5+3(-2+7)$

Solution:

	$-5+3(-2+7)$
Simplify inside the parenthesis	$-5+3(5)$
Multiply	$-5+15$
add left to right	10

Try It 13.5.4.21

Simplify:

$-2+5(-4+7)$

Answer

13

Try It 13.5.4.22

Simplify:

$-4+2(-3+5)$

Answer

0

Subtract Integers

Note

Doing the Manipulative Mathematics activity "Subtraction of Signed Numbers" will help you develop a better understanding of subtracting integers.

We will continue to use counters to model the subtraction. Remember, the blue counters represent positive numbers and the red counters represent negative numbers.

Perhaps when you were younger, you read "$5-3$" as "5 take away 3." When you use counters, you can think of subtraction the same way!

We will model the four subtraction facts using the numbers 5 and 3.

$$5-3 \quad -5-(-3)) \quad -5-3 \quad 5-(-3)$$

To subtract $5-3$, we restate the problem as "5 take away 3."

Table 13.5.4.3

We start with 5 positives.	
We 'take away' 3 positives.	

We have 2 positives left.	
The difference of 5 and 3 is 2.	2

Now we will subtract $-5-(-3)$. Watch for similarities to the last example $5-3=2$.

To subtract $-5-(-3)$, we restate this as "-5 take away -3"

Table 13.5.4.4

We start with 5 negatives.	
We 'take away' 3 negatives.	
We have 2 negatives left.	
The difference of -5 and -3 is -2.	-2

Notice that these two examples are much alike: The first example, we subtract 3 positives from 5 positives and end up with 2 positives.

In the second example, we subtract 3 negatives from 5 negatives and end up with 2 negatives.

Each example used counters of only one color, and the "take away" model of subtraction was easy to apply.

Figure 13.5.4.9

Example 13.5.4.12

Subtract:

1. $7-5$
2. $-7-(-5)$

Solution:

1. $7-5$
   2 Take 5 positives from 7 positives and get 2 positives
2. $-7-(-5)$
   -2 Take 5 negatives from 7 negatives and get 2 negatives

Try It 13.5.4.23

Subtract:

1. $6-4$
2. $-6-(-4)$

Answer

1. 2
2. -2

Try It 13.5.4.24

Subtract:

1. $7 - 4$
2. $-7 - (-4)$

Answer

1. 3
2. -3

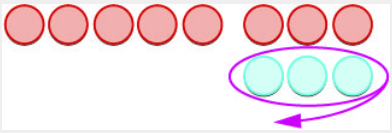
What happens when we have to subtract one positive and one negative number? We'll need to use both white and red counters as well as some **neutral pairs**. Adding a neutral pair does not change the value. It is like changing quarters to nickels—the value is the same, but it looks different.

- To subtract $-5 - 3$, we restate it as -5 take away 3.

We start with 5 negatives. We need to take away 3 positives, but we do not have any positives to take away.

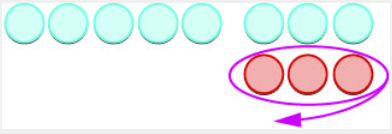
Remember, a neutral pair has value zero. If we add 0 to 5 its value is still 5. We add neutral pairs to the 5 negatives until we get 3 positives to take away.

Table 13.5.4.5

	$-5 - 3$ means -5 take away 3
We start with 5 negatives.	
We now add the neutrals needed to get 3 positives.	
We remove the 3 positives.	
We are left with 8 negatives.	
The difference of -5 and 3 is -8.	$-5 - 3 = -8$

And now, the fourth case, $5 - (-3)$. We start with 5 positives. We need to take away 3 negatives, but there are no negatives to take away. So we add neutral pairs until we have 3 negatives to take away.

Table 13.5.4.6

	$5 - (-3)$ means 5 take away -3
We start with 5 positives.	
We now add the needed neutrals pairs.	
We remove the 3 negatives.	

We are left with 8 positives.	8 positives
The difference of 5 and -3 is 8.	$5-(-3)=8$

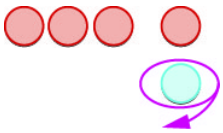
Example 13.5.4.13

Subtract:

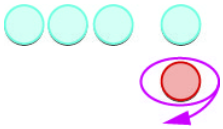
1. $-3-1$
2. $3-(-1)$

Solution:

1.

Take 1 positive from the one added neutral pair.		$-3-1$ -4

2.

Take 1 negative from the one added neutral pair.		$3-(-1)$ 4

Try It 13.5.4.25

Subtract:

1. $-6-4$
2. $6-(-4)$

Answer

1. -10
2. 10

Try It 13.5.4.26

Subtract:

1. $-7-4$
2. $7-(-4)$

Answer

1. -11
2. 11

Have you noticed that *subtraction of signed numbers can be done by adding the opposite*? In Exercise 13.5.4.33 $-3-1$ is the same as $-3+(-1)$ and $3-(-1)$ is the same as $3+1$. You will often see this idea, the **subtraction property**, written as follows:

SUBTRACTION PROPERTY

$$a-b=a+(-b)$$

Subtracting a number is the same as adding its opposite.

Look at these two examples.

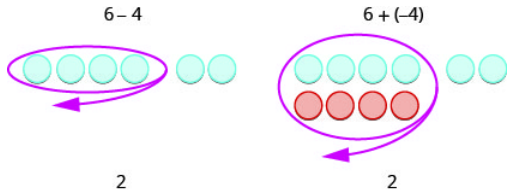


Figure 13.5.4.10: $6-4$ gives the same answer as $6+(-4)$.

Of course, when you have a subtraction problem that has only positive numbers, like $6-4$, you just do the subtraction. You already knew how to subtract $6-4$ long ago. But *knowing* that $6-4$ gives the same answer as $6+(-4)$ helps when you are subtracting negative numbers. Make sure that you understand how $6-4$ and $6+(-4)$ give the same results!

Example 13.5.4.14

Simplify:

1. $13-8$ and $13+(-8)$
2. $-17-9$ and $-17+(-9)$

Solution:

1. Subtract. $13-8$ and $13+(-8)$
   5 5
2. Subtract. $-17-9$ and $-17+(-9)$
   -26 -26

Try It 13.5.4.27

Simplify:

1. $21-13$ and $21+(-13)$
2. $-11-7$ and $-11+(-7)$

Answer

1. 8
2. -18

Try It 13.5.4.28

Simplify:

1. $15-7$ and $15+(-7)$
2. $-14-8$ and $-14+(-8)$

Answer

1. 8
2. -22

Look at what happens when we subtract a negative.

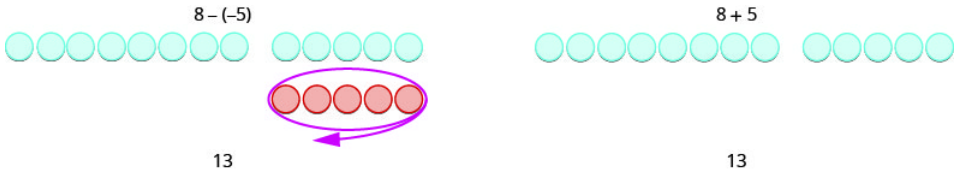


Figure 13.5.4.11: $8-(-5)$ gives the same answer as $8+5$

Subtracting a negative number is like adding a positive!

You will often see this written as $a-(-b)=a+b$.

Does that work for other numbers, too? Let's do the following example and see.

Example 13.5.4.15

Simplify:

1. $9-(-15)$ and $9+15$
2. $-7-(-4)$ and $-7+4$

Solution:

1.		$9-(-15)$	$9+15$
	Subtract	24	24
2.		$-7-(-4)$	$-7+4$
	Subtract	-3	-3

Try It 13.5.4.29

Simplify:

1. $6-(-13)$ and $6+13$
2. $-5-(-1)$ and $-5+1$

Answer

1. 19
2. -4

Try It 13.5.4.30

Simplify:

1. $4-(-19)$ and $4+19$
2. $-4-(-7)$ and $-4+7$

Answer

1. 23
2. 3

Let's look again at the results of subtracting the different combinations of $5, -5$ and $3, -3$.

SUBTRACTION OF INTEGERS

$5-3$	$-5-(-3)$
2	-2
5 positives take away 3 positives	5 negatives take away 3 negatives
2 positives	2 negatives

When there would be enough counters of the color to take away, subtract.

$-5-3$	$5-(-3)$
-8	8
5 negatives, want to take away 3 positives need neutral pairs	5 positives, want to take away 3 negatives need neutral pairs

When there would be not enough counters of the color to take away, add.

What happens when there are more than three integers? We just use the order of operations as usual.

? Example 13.5.4.16

Simplify:

$7-(-4-3)-9$

Solution:

	$7-(-4-3)-9$
Simplify inside the parenthesis first.	$7-(-7)-9$
Subtract left to right.	$14-9$
Subtract	5

? Try It 13.5.4.31

Simplify:

$8-(-3-1)-9$

Answer

3

? Try It 13.5.4.32

Simplify:

$12-(-9-6)-14$

Answer

12

Access these online resources for additional instruction and practice with adding and subtracting integers. You will need to enable Java in your web browser to use the applications.

- Add Colored Chip
- Subtract Colored Chip

Key Concepts

- **Addition of Positive and Negative Integers**

$5+3$	$-5+(-3)$
8	-8
both positive, sum positive	both negative, sum negative

$-5+3$	$5+(-3)$
-2	2
different signs, more negatives sum negative	different signs, more positives sum positive

- **Property of Absolute Value**:

$|n| \geq 0$ for all numbers. Absolute values are always greater than or equal to zero!

- **Subtraction of Integers**

$5-3$	$-5-(-3)$
2	-2
5 positives take away 3 positives 2 positives	5 negatives take away 3 negatives 2 negatives
$-5-3$	$5-(-3)$
-8	8
5 negatives, want to subtract 3 positives need neutral pairs	5 positives, want to subtract 3 negatives need neutral pairs

- **Subtraction Property:** Subtracting a number is the same as adding its opposite.

Glossary

absolute value

The absolute value of a number is its distance from 0 on the number line. The absolute value of a number nn is written as |n|.

integers

The whole numbers and their opposites are called the integers: $\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$

opposite

The opposite of a number is the number that is the same distance from zero on the number line but on the opposite side of zero: $-a$ means the opposite of the number. The notation $-a$ is read “the opposite of a.”

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13.5.4E: Exercises

Practice Makes Perfect

Use Negatives and Opposites of Integers

In the following exercises, order each of the following pairs of numbers, using < or >.

Exercise 13.5.4E.1

1. 9___4
2. −3___6
3. −8___−2
4. 1___−10

Answer

1. >
2. <
3. <
4. >

Exercise 13.5.4E.2

1. −7___3
2. −10___−5
3. 2___−6
4. 8___9

In the following exercises, find the opposite of each number.

Exercise 13.5.4E.3

1. 2
2. −6

Answer

1. −2
2. 6

Exercise 13.5.4E.4

1. 9
2. −4

In the following exercises, simplify.

Exercise 13.5.4E.5

−(−4)

Answer

4

Exercise 13.5.4E.6

$-(-8)$

Exercise 13.5.4E.7

$-(-15)$

Answer

15

Exercise 13.5.4E.8

$-(-11)$

In the following exercises, evaluate.

Exercise 13.5.4E.9

$-c$ when

1. $c=12$
2. $c=-12$

Answer

1. -12
2. 12

1. Exercise 13.5.4E.10

$-d$ when

1. $d=21$
2. $d=-21$

Simplify Expressions with Absolute Value

In the following exercises, simplify.

Exercise 13.5.4E.11

1. $|-32|$
2. $|0|$
3. $|16|$

Answer

1. 32
2. 0
3. 16

Exercise 13.5.4E.12

1. $|0|$
2. $|-40|$
3. $|22|$

In the following exercises, fill in <, >, or = for each of the following pairs of numbers.

Exercise 13.5.4E.13

1. $-6 \_\_\_ |-6|$
2. $-|-3| \_\_\_ -3$

Answer

1. <
2. =

Exercise 13.5.4E.14

1. $|-5| \_\_\_ -|-5|$
2. $9 \_\_\_ -|-9|$

In the following exercises, simplify.

Exercise 13.5.4E.15

$-(-5)$ and $-|-5|$

Answer

5,−5

Exercise 13.5.4E.16

$-|-9|$ and $-(-9)$

Exercise 13.5.4E.17

$8|-7|$

Answer

56

Exercise 13.5.4E.18

$5|-5|$

Exercise 13.5.4E.19

$|15-7|-|14-6|$

Answer

0

Exercise 13.5.4E.20

$|17-8|-|13-4|$

Exercise 13.5.4E.21

$18-|2(8-3)|$

Answer

8

Exercise 13.5.4E.22

$18-|3(8-5)|$

In the following exercises, evaluate.

Exercise 13.5.4E.23

1. $-|p|$ when $p=19$
2. $-|q|$ when $q=-33$

Answer

1. -19
2. -33

Exercise 13.5.4E.24

1. $-|a|$ when $a=60$
2. $-|b|$ when $b=-12$

Add Integers

In the following exercises, simplify each expression.

Exercise 13.5.4E.25

$-21+(-59)$

Answer

-80

Exercise 13.5.4E.26

$-35+(-47)$

Exercise 13.5.4E.27

$48+(-16)$

Answer

32

Exercise 13.5.4E.28

$34+(-19)$

Exercise 13.5.4E.29

$-14+(-12)+4$

Answer

-22

Exercise 13.5.4E.30

$-17+(-18)+6$

Exercise 13.5.4E.31

$135+(-110)+83$

Answer

108

Exercise 13.5.4E.32

$-38+27+(-8)+12$

Exercise 13.5.4E.33

$19+2(-3+8)$

Answer

29

Exercise 13.5.4E.34

$24+3(-5+9)$

Subtract Integers

In the following exercises, simplify.

Exercise 13.5.4E.35

$8-2$

Answer

6

Exercise 13.5.4E.36

$-6-(-4)$

Exercise 13.5.4E.37

$-5-4$

Answer

-9

Exercise 13.5.4E.38

$-7-2$

Exercise 13.5.4E.39

$8-(-4)$

Answer

12

Exercise 13.5.4E.40

$7-(-3)$

Exercise 13.5.4E.41

1. $44-28$
2. $44+(-28)$

Answer

1. 16
2. 16

Exercise 13.5.4E.42

1. $35-16$
2. $35+(-16)$

Exercise 13.5.4E.43

1. $27-(-18)$
2. $27+18$

Answer

1. 45
2. 45

Exercise 13.5.4E.44

1. $46-(-37)$
2. $46+37$

In the following exercises, simplify each expression.

Exercise 13.5.4E.45

$15-(-12)$

Answer

27

Exercise 13.5.4E.46

14−(−11)

Exercise 13.5.4E.47

48−87

Answer

-39

Exercise 13.5.4E.48

45−69

Exercise 13.5.4E.49

−17−42

Answer

-59

Exercise 13.5.4E.50

−19−46

Exercise 13.5.4E.51

−103−(−52)

Answer

-51

Exercise 13.5.4E.52

−105−(−68)

Exercise 13.5.4E.53

−45−(−54)

Answer

9

Exercise 13.5.4E.54

−58−(−67)

Exercise 13.5.4E.55

8−3−7

Answer

-2

Exercise 13.5.4E.56

9−6−5

Exercise 13.5.4E.57

−5−4+7

Answer

-2

Exercise 13.5.4E.58

−3−8+4

Exercise 13.5.4E.59

−14−(−27)+9

Answer

22

Exercise 13.5.4E.60

64+(−17)−9

Exercise 13.5.4E.61

(2−7)−(3−8)

Answer

0

Exercise 13.5.4E.62

(1−8)−(2−9)

Exercise 13.5.4E.63

−(6−8)−(2−4)

Answer

4

Exercise 13.5.4E.64

−(4−5)−(7−8)

Exercise 13.5.4E.65

25−[10−(3−12)]

Answer

6

Exercise 13.5.4E.66

32−[5−(15−20)]

Exercise 13.5.4E.67

6.3−4.3−7.2

Answer

-5.2

Exercise 13.5.4E.68

5.7−8.2−4.9

Exercise 13.5.4E.69

$5^2 - 6^2$

Answer

-11

Exercise 13.5.4E.70

$6^2 - 7^2$

Everyday Math

Exercise 13.5.4E.71

Elevation The highest elevation in the United States is Mount McKinley, Alaska, at 20,320 feet above sea level. The lowest elevation is Death Valley, California, at 282 feet below sea level.

Use integers to write the elevation of:

1. Mount McKinley.
2. Death Valley.

Answer

1. 20,329
2. −282

Exercise 13.5.4E.72

Extreme temperatures The highest recorded temperature on Earth was 58° Celsius, recorded in the Sahara Desert in 1922. The lowest recorded temperature was 90° below 0° Celsius, recorded in Antarctica in 1983.

Use integers to write the:

1. highest recorded temperature.
2. lowest recorded temperature.

Exercise 13.5.4E.73

State budgets In June, 2011, the state of Pennsylvania estimated it would have a budget surplus of $540 million. That same month, Texas estimated it would have a budget deficit of $27 billion.

Use integers to write the budget of:

1. Pennsylvania.
2. Texas.

Answer

1. $540 million
2. −$27 billion

Exercise 13.5.4E.74

College enrollments Across the United States, community college enrollment grew by 1,400,000 students from Fall 2007 to Fall 2010. In California, community college enrollment declined by 110,171 students from Fall 2009 to Fall 2010.

Use integers to write the change in enrollment:

1. in the U.S. from Fall 2007 to Fall 2010.
2. in California from Fall 2009 to Fall 2010.

Exercise 13.5.4E.75

Stock Market The week of September 15, 2008 was one of the most volatile weeks ever for the US stock market. The closing numbers of the Dow Jones Industrial Average each day were:

Monday	−504
Tuesday	+142
Wednesday	−449
Thursday	+410
Friday	+369

What was the overall change for the week? Was it positive or negative?

Answer

-32

Exercise 13.5.4E.76

Stock Market During the week of June 22, 2009, the closing numbers of the Dow Jones Industrial Average each day were:

Monday	−201
Tuesday	−16
Wednesday	−23
Thursday	+172
Friday	−34

What was the overall change for the week? Was it positive or negative?

Writing Exercises

Exercise 13.5.4E.77

Give an example of a negative number from your life experience.

Answer

Answers may vary

Exercise 13.5.4E.78

What are the three uses of the "−" sign in algebra? Explain how they differ.

Exercise 13.5.4E.79

Explain why the sum of −8 and 2 is negative, but the sum of 8 and −2 is positive.

Answer

Answers may vary

Exercise 13.5.4E.80

Give an example from your life experience of adding two negative numbers.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use negatives and opposites of integers.			
simplify expressions with absolute value.			
add integers.			
subtract integers.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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13.5.5: Multiply and Divide Integers

Learning Objectives

By the end of this section, you will be able to:

- Multiply integers
- Divide integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate English phrases to algebraic expressions
- Use integers in applications

A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **Integers**.

Multiply Integers

Since multiplication is mathematical shorthand for repeated addition, our model can easily be applied to show **multiplication** of **integers**. Let's look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction. Here, we will use the model just to help us discover the pattern.

We remember that $a \cdot b$ means add a, b times. Here, we are using the model just to help us discover the pattern.

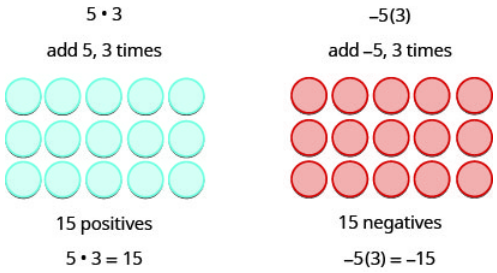

Figure 13.5.5.1

The next two examples are more interesting.

What does it mean to multiply 5 by -3? It means subtract $5, 3$ times. Looking at subtraction as "taking away," it means to take away $5, 3$ times. But there is nothing to take away, so we start by adding neutral pairs on the workspace. Then we take away 5 three times.

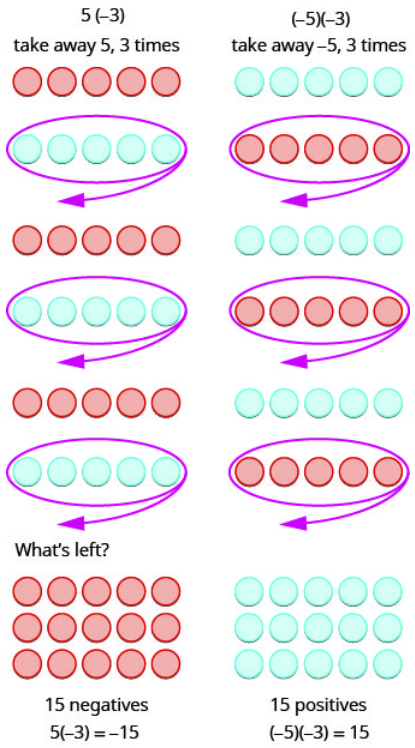

Figure 13.5.5.2

In summary:

$$5 \cdot 3 = 15 \qquad -5(3) = -15$$
$$5(-3) = -15 \qquad (-5)(-3) = 15$$

Notice that for multiplication of two signed numbers, when the:

- signs are the *same*, the product is *positive*.
- signs are *different*, the product is *negative*.

We'll put this all together in the chart below.

MULTIPLICATION OF SIGNED NUMBERS

For multiplication of two signed numbers:

Table 13.5.5.1

Same signs	Product	Example
Two positives	Positive	$7 \cdot 4 = 28$
Two negatives	Positive	$-8(-6) = 48$

Table 13.5.5.2

Different signs	Product	Example
Positive · negative	Negative	$7(-9) = -63$
Negative · positive	Negative	$-5 \cdot 10 = -50$

Example 13.5.5.1

Multiply:

1. $-9 \cdot 3$
2. $-2(-5)$
3. $4(-8)$
4. $7 \cdot 6$

Solution

1. $-9 \cdot 3 = -27$ Multiply, noting that the signs are different, so the product is negative.
2. $-2(-5) = 10$ Multiply, noting that the signs are same, so the product is positive.
3. $4(-8) = -32$ Multiply, with different signs.
4. $7 \cdot 6 = 42$ Multiply, with the same signs.

Try It 13.5.5.2

Multiply:

1. $-6 \cdot 8$
2. $-4(-7)$
3. $9(-7)$
4. $5 \cdot 12$

Answer

1. -48
2. 28
3. -63
4. 60

Try It 13.5.5.3

Multiply:

1. $-8 \cdot 7$
2. $-6(-9)$
3. $7(-4)$
4. $3 \cdot 13$

Answer

1. -56
2. 54
3. -28
4. 39

When we multiply a number by 1, the result is the same number. What happens when we multiply a number by -1? Let's multiply a positive number and then a negative number by -1 to see what we get.

Multiply.	Multiply.
$-1 \cdot 4 = -4$	$-1(-3) = 3$
Note that -4 is the opposite of 4.	Note that 3 is the opposite of -3

Each time we multiply a number by -1, we get its opposite!

MULTIPLICATION BY −1

$$-1a = -a$$

Multiplying a number by -1 gives its opposite.

Example 13.5.5.4

Multiply:

1. $-1 \cdot 7$
2. $-1(-11)$

Solution

1.		$-1 \cdot 7$
	Multiply, noting that the signs are different so the product is negative.	-7
		-7 is the opposite of 7.
2.		$-1(-11)$
	Multiply, noting that the signs are the same so the product is positive.	11
		11 is the opposite of -11.

Try It 13.5.5.5

Multiply:

1. $-1 \cdot 9$
2. $-1 \cdot (-17)$

Answer

1. -9
2. 17

Try It 13.5.5.6

Multiply:

1. $-1 \cdot 8$
2. $-1 \cdot (-16)$

Answer

1. -8
2. 16

Divide Integers

What about **division**? Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $5 \cdot 3 = 15$. In words, this expression says that 15 can be divided into three groups of five each because adding five three times gives 15. Look at some examples of multiplying integers, to figure out the rules for dividing integers.

$$\begin{aligned} 5 \cdot 3 &= 15 \quad \text{so} \quad 15 \div 3 = 5 \\ -5 \cdot 3 &= -15 \quad \text{so} \quad -15 \div 3 = -5 \\ (5)(-3) &= -15 \quad \text{so} \quad -15 \div (-3) = 5 \\ (-5)(-3) &= 15 \quad \text{so} \quad 15 \div (-3) = -5 \end{aligned}$$

Division follows the same rules as multiplication!

For division of two signed numbers, when the:

- signs are the *same*, the quotient is *positive*.
- signs are *different*, the quotient is *negative*.

And remember that we can always check the answer of a division problem by multiplying.

MULTIPLICATION AND DIVISION OF SIGNED NUMBERS

For multiplication and division of two signed numbers:

- If the signs are the same, the result is positive.
- If the signs are different, the result is negative.

Table 13.5.5.3

Same signs	Result
Two positives	Positive
Two negatives	Positive
If the signs are the same, the result is positive.	

Table 13.5.5.4

Different signs	Result
Positive and negative	Negative
Negative and positive	Negative
If the signs are different, the result is negative.	

Example 13.5.5.7

1. $-27 \div 3$
2. $-100 \div (-4)$

Solution

1. $-27 \div 3$
   Divide, with different signs, the quotient is negative. -9
2. $-100 \div (-4)$
   Divide, with signs that are the same the quotient is negative. 25

Try It 13.5.5.8

Divide:

1. $-42 \div 6$
2. $-117 \div (-3)$

Answer

1. -7
2. 39

Try It 13.5.5.9

Divide:

1. $-63 \div 7$
2. $-115 \div (-5)$

Answer

1. -9
2. 23

Simplify Expressions with Integers

What happens when there are more than two numbers in an expression? The order of operations still applies when negatives are included. Remember My Dear Aunt Sally?

Let's try some examples. We'll simplify expressions that use all four operations with integers—addition, subtraction, multiplication, and division. Remember to follow the order of operations.

Example 13.5.5.10

Simplify:

$7(-2)+4(-7)-6$

Solution

	$7(-2)+4(-7)-6$
Multiply first.	$-14+(-28)-6$
Add.	$-42-6$
Subtract	-48

Try It 13.5.5.11

Simplify:

$8(-3)+5(-7)-4$

Answer

-63

Try It 13.5.5.12

Simplify:

$9(-3)+7(-8)-1$

Answer

-84

Example 13.5.5.13

Simplify:

1. $(-2)^4$
2. -2^4

Solution

1. $(-2)^4$

Write in expanded form.	$(-2)(-2)(-2)(-2)$
Multiply	$4(-2)(-2)$
Multiply	$-8(-2)$
Multiply	16

2. -2^4

Write in expanded form. We are asked to find the opposite of 2^4.	$-(2\cdot 2\cdot 2\cdot 2)$
Multiply	$-(4\cdot 2\cdot 2)$
Multiply	$-(8\cdot 2)$
Multiply	-16

Notice the difference in parts (1) and (2). In part (1), the exponent means to raise what is in the parentheses, the (-2) to the 4th power. In part (2), the exponent means to raise just the 2 to the 4th power and then take the opposite.

? Try It 13.5.5.14

Simplify:

1. $(-3)^4$
2. -3^4

Answer

1. 81
2. -81

? Try It 13.5.5.15

Simplify:

1. $(-7)^2$
2. -7^2

Answer

1. 49
2. -49

The next example reminds us to simplify inside parentheses first.

✓ Example 13.5.5.16

Simplify:

$12-3(9-12)$

Solution

$12-3(9-12)$	
$=12-3(-3)$	Subtract in parentheses first.
$=12-(-9)$	Multiply.
$=12+9$	Add the opposite.
$=21$	

Try It 13.5.5.17

Simplify:

$17-4(8-11)$

Answer

29

Try It 13.5.5.18

Simplify:

$16-6(7-13)$

Answer

52

Example 13.5.5.19

Simplify:

$8(-9)\div(-2)^3$

Solution

	$8(-9)\div(-2)^3$
Exponents first	$8(-9)\div(-8)$
Multiply.	$-72\div(-8)$
Divide	9

Try It 13.5.5.20

Simplify:

$12(-9)\div(-3)^3$

Answer

4

Try It 13.5.5.21

Simplify:

$18(-4)\div(-2)^3$

Answer

9

Example 13.5.5.22

Simplify:

$-30\div 2+(-3)(-7)$

Solution

	$-30 \div 2 + (-3)(-7)$
Multiply and divide left to right, so divide first.	$-15 + (-3)(-7)$
Multiply.	$-15 + 21$
Add	6

Try It 13.5.5.23

Simplify:

$-27 \div 3 + (-5)(-6)$

Answer

21

Try It 13.5.5.24

Simplify:

$-32 \div 4 + (-2)(-7)$

Answer

6

Evaluate Variable Expressions with Integers

Remember that to evaluate an expression means to substitute a number for the variable in the expression. Now we can use negative numbers as well as positive numbers.

Example 13.5.5.25

When $n = -5$, evaluate:

1. $n + 1$
2. $-n + 1$.

Solution

1.

	$n + 1$
Substitute -5 for n	$-5 + 1$
Simplify.	-4

2.

	$n + 1$
Substitute -5 for n	$-(-5) + 1$
Simplify.	$5 + 1$
Add.	6

Try It 13.5.5.26

When $n = -8$, evaluate:

1. $n + 2$
2. $-n + 2$.

Answer

1. -6
2. 10

Try It 13.5.5.27

When $y = -9$, evaluate:

1. $y + 8$
2. $-y + 8$.

Answer

1. -1
2. 17

Example 13.5.5.28

Evaluate $(x + y)^2$ when $x = -18$ and $y = 24$.

Solution

	$(x + y)^2$
Substitute -18 for x and 24 for y	$(-18 + 24)^2$
Add inside parentheses	$(6)^2$
Simplify.	36

Try It 13.5.5.29

Evaluate $(x + y)^2$ when $x = -15$ and $y = 29$.

Answer

196

Try It 13.5.5.30

Evaluate $(x + y)^3$ when $x = -8$ and $y = 10$.

Answer

8

Example 13.5.5.31

Evaluate $20 - z$ when

1. $z = 12$
2. $z = -12$

Solution

1.

	$20 - z$
Substitute 12 for z.	$20 - 12$
Subtract	8

2.

	$20 - z$
Substitute -12 for z.	$20 - (-12)$
Subtract	32

Try It 13.5.5.32

Evaluate $17 - k$ when

1. $k = 19$
2. $k = -19$

Answer

1. -2
2. 36

Try It 13.5.5.33

Evaluate $-5 - b$ when

1. $b = 14$
2. $b = -14$

Answer

1. -19
2. 9

Example 13.5.5.34

Evaluate:

$2x^2 + 3x + 8$ when $x = 4$.

Solution

Substitute 4 for x. Use parentheses to show multiplication.

	$2x^2 + 3x + 8$
Substitute	$2(4)^2 + 3(4) + 8$
Evaluate exponents.	$2(16) + 3(4) + 8$
Multiply.	$32 + 12 + 8$
Add.	52

Try It 13.5.5.35

Evaluate:

$3x^2 - 2x + 6$ when $x = -3$.

Answer

39

Try It 13.5.5.36

Evaluate:

$4x^2 - x - 5$ when $x = -2$.

Answer

13

Translate Phrases to Expressions with Integers

Our earlier work translating English to algebra also applies to phrases that include both positive and negative numbers.

Example 13.5.5.37

Translate and simplify: the sum of 8 and -12, increased by 3.

Solution

	the **sum** of 8 and -12, increased by 3
Translate.	$[8+(-12)]+3$
Simplify. Be careful not to confuse the brackets with an absolute value sign.	$(-4)+3$
Add.	-1

Try It 13.5.5.38

Translate and simplify: the sum of 9 and -16, increased by 4.

Answer

$(9+(-16))+4-3$

Try It 13.5.5.39

Translate and simplify: the sum of -8 and -12, increased by 7.

Answer

$(-8+(-12))+7-13$

When we first introduced the operation symbols, we saw that the expression may be read in several ways. They are listed in the chart below.

Table 13.5.5.5

$a-b$
a minus b
the difference of a and b
b subtracted from a
b less than a

Be careful to get a and b in the right order!

Example 13.5.5.40

Translate and then simplify

1. the difference of 13 and -21
2. subtract 24 from -19.

Solution

1.

	the **difference** of 13 and -21
Translate.	$13-(-21)$
Simplify.	34

2.

	subtract 24 **from** -19
Translate.	$-19-24$
Remember, subtract b from a means $a-b$	
Simplify.	-43

Try It 13.5.5.41

Translate and simplify

1. the difference of 14 and -23
2. subtract 21 from -17.

Answer

1. $14-(-23); 37$
2. $-17-21; -38$

Try It 13.5.5.42

Translate and simplify

1. the difference of 11 and -19
2. subtract 18 from -11.

Answer

1. $11-(-19); 30$
2. $-11-18; -29$

Once again, our prior work translating English to algebra transfers to phrases that include both multiplying and dividing integers. Remember that the key word for **multiplication** is "**product**" and for **division** is "**quotient**."

Example 13.5.5.43

Translate to an algebraic expression and simplify if possible: the product of -2 and 14.

Solution

	the product of -2 and 14
Translate.	$(-2)(14)$
Simplify.	-28

Try It 13.5.5.44

Translate to an algebraic expression and simplify if possible: the product of -5 and 12.

Answer

$-5(12); -60$

Try It 13.5.5.45

Translate to an algebraic expression and simplify if possible: the product of 8 and -13.

Answer

$-8(13); -104$

Example 13.5.5.46

Translate to an algebraic expression and simplify if possible: the quotient of -56 and -7.

Solution

$$\begin{array}{ll} & \text{the quotient of } -56 \text{ and } -7 \\ \text{Translate.} & -56 \div (-7) \\ \text{Simplify.} & 8 \end{array} \qquad (13.5.5.1)$$

Try It 13.5.5.47

Translate to an algebraic expression and simplify if possible: the quotient of -63 and -9.

Answer

$-63 \div (-9); 7$

Try It 13.5.5.48

Translate to an algebraic expression and simplify if possible: the quotient of -72 and -9.

Answer

$-72 \div (-9); 8$

Use Integers in Applications

We'll outline a plan to solve applications. It's hard to find something if we don't know what we're looking for or what to call it! So when we solve an application, we first need to determine what the problem is asking us to find. Then we'll write a phrase that gives the information to find it. We'll translate the phrase into an expression and then simplify the expression to get the answer. Finally, we summarize the answer in a sentence to make sure it makes sense.

How to Apply a Strategy to Solve Applications with Integers

Example 13.5.5.49

The temperature in Urbana, Illinois one morning was 11 degrees. By mid-afternoon, the temperature had dropped to -9 degrees. What was the difference of the morning and afternoon temperatures?

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.	
Step 2. Identify what we are asked to find.	the difference of the morning and afternoon temperatures
Step 3. Write a phrase that gives the information to find it.	the difference of 11 and -9
Step 4. Translate the phrase to an expression.	$11 - (-9)$
Step 5. Simplify the expression.	20
Step 6. Write a complete sentence that answers the question.	The difference in temperatures was 20 degrees.

Try It 13.5.5.50

The temperature in Anchorage, Alaska one morning was 15 degrees. By mid-afternoon the temperature had dropped to 30 degrees below zero. What was the difference in the morning and afternoon temperatures?

Answer

The difference in temperatures was 45 degrees.

Try It 13.5.5.51

The temperature in Denver was -6 degrees at lunchtime. By sunset the temperature had dropped to -15 degrees. What was the difference in the lunchtime and sunset temperatures?

Answer

The difference in temperatures was 9 degrees.

APPLY A STRATEGY TO SOLVE APPLICATIONS WITH INTEGERS.

1. Read the problem. Make sure all the words and ideas are understood
2. Identify what we are asked to find.
3. Write a phrase that gives the information to find it.
4. Translate the phrase to an expression.
5. Simplify the expression.
6. Answer the question with a complete sentence.

Example 13.5.5.52

The Mustangs football team received three penalties in the third quarter. Each penalty gave them a loss of fifteen yards. What is the number of yards lost?

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.	
Step 2. Identify what we are asked to find.	the number of yards lost
Step 3. Write a phrase that gives the information to find it.	three times a 15-yard penalty
Step 4. Translate the phrase to an expression.	$3(-15)$
Step 5. Simplify the expression.	-45
Step 6. Write a complete sentence that answers the question.	The team lost 45 yards.

Try It 13.5.5.53

The Bears played poorly and had seven penalties in the game. Each penalty resulted in a loss of 15 yards. What is the number of yards lost due to penalties?

Answer

The Bears lost 105 yards.

? Try It 13.5.5.54

Bill uses the ATM on campus because it is convenient. However, each time he uses it he is charged a $2 fee. Last month he used the ATM eight times. How much was his total fee for using the ATM?

Answer

A $16 fee was deducted from his checking account.

Key Concepts

- **Multiplication and Division of Two Signed Numbers**
  - Same signs—Product is positive
  - Different signs—Product is negative
- **Strategy for Applications**
  1. Identify what you are asked to find.
  2. Write a phrase that gives the information to find it.
  3. Translate the phrase to an expression.
  4. Simplify the expression.
  5. Answer the question with a complete sentence.

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13.5.5E: Exercises

Practice Makes Perfect

Multiply Integers

In the following exercises, multiply.

Exercise 13.5.5E.55

$-4 \cdot 8$

Answer

-32

Exercise 13.5.5E.56

$-3 \cdot 9$

Exercise 13.5.5E.57

$9(-7)$

Answer

-63

Exercise 13.5.5E.58

$13(-5)$

Exercise 13.5.5E.59

$-1 \cdot 6$

Answer

-6

Exercise 13.5.5E.60

$-1 \cdot 3$

Exercise 13.5.5E.61

$-1(-14)$

Answer

14

Exercise 13.5.5E.62

$-1(-19)$

Divide Integers

In the following exercises, divide.

Exercise 13.5.5E.63

$-24 \div 6$

Answer

-4

Exercise 13.5.5E.64

$35 \div (-7)$

Exercise 13.5.5E.65

$-52 \div (-4)$

Answer

13

Exercise 13.5.5E.66

$-84 \div (-6)$

Exercise 13.5.5E.67

$-180 \div 15$

Answer

-12

Exercise 13.5.5E.68

$-192 \div 12$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

Exercise 13.5.5E.69

$5(-6)+7(-2)-3$

Answer

-47

Exercise 13.5.5E.70

$8(-4)+5(-4)-6$

Exercise 13.5.5E.71

$(-2)^6$

Answer

64

Exercise 13.5.5E.72

$(-3)^5$

Exercise 13.5.5E.73

$(-4)^2$

Answer

-16

Exercise 13.5.5E.74

$(-6)^2$

Exercise 13.5.5E.75

$-3(-5)(6)$

Answer

90

Exercise 13.5.5E.76

$-4(-6)(3)$

Exercise 13.5.5E.77

$(8-11)(9-12)$

Answer

9

Exercise 13.5.5E.78

$(6-11)(8-13)$

Exercise 13.5.5E.79

$26-3(2-7)$

Answer

41

Exercise 13.5.5E.80

23−2(4−6)

Exercise 13.5.5E.81

$65 \div (-5) + (-28) \div (-7)$

Answer

-9

Exercise 13.5.5E.82

$52 \div (-4) + (-32) \div (-8)$

Exercise 13.5.5E.83

9−2[3−8(−2)]

Answer

-29

Exercise 13.5.5E.84

11−3[7−4(−20)]

Exercise 13.5.5E.85

$(-3)^2 - 24 \div (8 - 2)$

Answer

5

Exercise 13.5.5E.86

$(-4)^2 - 32 \div (12 - 4)$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

Exercise 13.5.5E.87

y+(−14) when

1. y=−33
2. y=30

Answer

1. −47
2. 16

Exercise 13.5.5E.88

x+(−21) when

1. x=−27
2. x=44

Exercise 13.5.5E.89

1. a+3 when a=−7
2. −a+3 when a=−7

Answer

1. −4
2. 10

Exercise 13.5.5E.90

1. d+(−9) when d=−8
2. −d+(−9) when d=−8

Exercise 13.5.5E.91

m+n when
m=−15,n=7

Answer

-8

Exercise 13.5.5E.92

p+q when
p=−9,q=17

Exercise 13.5.5E.93

r+s when r=−9,s=−7

Answer

-16

Exercise 13.5.5E.94

t+u when t=−6,u=−5

Exercise 13.5.5E.95

$(x+y)^2$ when
x=−3,y=14

Answer

121

Exercise 13.5.5E.96

$(y+z)^2$ when
y=−3, z=15

Exercise 13.5.5E.97

−2x+17 when

1. x=8
2. x=−8

Answer

1. 1
2. 33

Exercise 13.5.5E.98

−5y+14 when

1. y=9
2. y=−9

Exercise 13.5.5E.99

10−3m when

1. m=5
2. m=−5

Answer

1. −5
2. 25

Exercise 13.5.5E.100

18−4n when

1. n=3
2. n=−3

Exercise 13.5.5E.101

$2w^2-3w+7$ when
w=−2

Answer

21

Exercise 13.5.5E.102

$3u^2-4u+5$

Exercise 13.5.5E.103

9a−2b−8 when
a=−6 and b=−3

Answer

-56

Exercise 13.5.5E.104

7m−4n−2 when
m=−4 and n=−9

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

Exercise 13.5.5E.105

the sum of 3 and −15, increased by 7

Answer

(3+(−15))+7;−5

Exercise 13.5.5E.106

the sum of −8 and −9, increased by 23

Exercise 13.5.5E.107

the difference of 10 and −18

Answer

10−(−18);28

Exercise 13.5.5E.108

subtract 11 from −25

Exercise 13.5.5E.109

the difference of −5 and −30

Answer

−5−(−30);25

Exercise 13.5.5E.110

subtract −6 from −13

Exercise 13.5.5E.111

the product of −3 and 15

Answer

$-3 \cdot 15; -45$

Exercise 13.5.5E.112

the product of −4 and 16

Exercise 13.5.5E.113

the quotient of −60 and −20

Answer

$-60 \div (-20); 3$

Exercise 13.5.5E.114

the quotient of −40 and −20

Exercise 13.5.5E.115

the quotient of −6 and the sum of *a* and *b*

Answer

$\frac{-6}{a+b}$

Exercise 13.5.5E.116

the quotient of −6 and the sum of *a* and *b*

Exercise 13.5.5E.117

the product of −10 and the difference of p and q

Answer

−10(p−q)

Exercise 13.5.5E.118

the product of −13 and the difference of c and d

Use Integers in Applications

In the following exercises, solve.

Exercise 13.5.5E.119

Temperature On January 15, the high temperature in Anaheim, California, was 84°. That same day, the high temperature in Embarrass, Minnesota was −12°. What was the difference between the temperature in Anaheim and the temperature in Embarrass?

Answer

96°

Exercise 13.5.5E.120

Temperature On January 21, the high temperature in Palm Springs, California, was 89°, and the high temperature in Whitefield, New Hampshire was −31°. What was the difference between the temperature in Palm Springs and the temperature in Whitefield?

Exercise 13.5.5E.121

Football At the first down, the Chargers had the ball on their 25 yard line. On the next three downs, they lost 6 yards, gained 10 yards, and lost 8 yards. What was the yard line at the end of the fourth down?

Answer

21

Exercise 13.5.5E.122

Football At the first down, the Steelers had the ball on their 30 yard line. On the next three downs, they gained 9 yards, lost 14 yards, and lost 2 yards. What was the yard line at the end of the fourth down?

Exercise 13.5.5E.123

Checking Account Mayra has $124 in her checking account. She writes a check for $152. What is the new balance in her checking account?

Answer

−$28

Exercise 13.5.5E.124

Checking Account Selina has $165 in her checking account. She writes a check for $207. What is the new balance in her checking account?

Exercise 13.5.5E.125

Checking Account Diontre has a balance of −$38 in his checking account. He deposits $225 to the account. What is the new balance?

Answer

$187

Exercise 13.5.5E.126

Checking Account Reymonte has a balance of −$49 in his checking account. He deposits $281 to the account. What is the new balance?

Everyday Math

Exercise 13.5.5E.127

Stock market Javier owns 300 shares of stock in one company. On Tuesday, the stock price dropped $12 per share. What was the total effect on Javier's portfolio?

Answer

Weight loss In the first week of a diet program, eight women lost an average of 3 pounds each. What was the total weight change for the eight women?

Exercise 13.5.5E.128

Weight loss In the first week of a diet program, eight women lost an average of 3 pounds each. What was the total weight change for the eight women?

Writing Exercises

Exercise 13.5.5E.129

In your own words, state the rules for multiplying integers.

Answer

Answers may vary

Exercise 13.5.5E.130

In your own words, state the rules for dividing integers.

Exercise 13.5.5E.131

Why is $-2^4 \neq (-2)^4$?

Answer

Answers may vary

Exercise 13.5.5E.132

Why is $-4^3 \neq (-4)^3$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
multiply integers.			
divide integers.			
simplify expressions with integers.			
evaluate variable expressions with integers.			
translate English phrases to algebraic expressions.			
use integers in applications.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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13.5.6: Visualize Fractions

Learning Objectives

By the end of this section, you will be able to:

- Find equivalent fractions
- Simplify fractions
- Multiply fractions
- Divide fractions
- Simplify expressions written with a fraction bar
- Translate phrases to expressions with fractions

Note

A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **Fractions**.

Find Equivalent Fractions

Fractions are a way to represent parts of a whole. The fraction $\frac{1}{3}$ means that one whole has been divided into 3 equal parts and each part is one of the three equal parts. See Figure 13.5.6.1. The fraction $\frac{2}{3}$ represents two of three equal parts. In the fraction $\frac{2}{3}$, the 2 is called the **numerator** and the 3 is called the **denominator**.

Figure 13.5.6.1: The circle on the left has been divided into 3 equal parts. Each part is $\frac{1}{3}$ of the 3 equal parts. In the circle on the right, $\frac{2}{3}$ of the circle is shaded (2 of the 3 equal parts).

Doing the Manipulative Mathematics activity "Model Fractions" will help you develop a better understanding of fractions, their numerators and denominators.

FRACTION

A **fraction** is written $\frac{a}{b}$, where $b \neq 0$ and

- a is the **numerator** and b is the **denominator**.

A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

If a whole pie has been cut into 6 pieces and we eat all 6 pieces, we ate $\frac{6}{6}$ pieces, or, in other words, one whole pie.

Figure 13.5.6.2

So $\frac{6}{6} = 1$. This leads us to the property of one that tells us that any number, except zero, divided by itself is 1.

PROPERTY OF ONE

$$\frac{a}{a} = 1 \quad (a \neq 0) \tag{13.5.6.1}$$

Any number, except zero, divided by itself is one.

Note

Doing the Manipulative Mathematics activity “Fractions Equivalent to One” will help you develop a better understanding of fractions that are equivalent to one.

If a pie was cut in 6 pieces and we ate all 6, we ate $\frac{6}{6}$ pieces, or, in other words, one whole pie. If the pie was cut into 8 pieces and we ate all 8, we ate $\frac{8}{8}$ pieces, or one whole pie. We ate the same amount—one whole pie.

The fractions $\frac{6}{6}$ and $\frac{8}{8}$ have the same value, 1, and so they are called equivalent fractions. **Equivalent fractions** are fractions that have the same value.

Let’s think of pizzas this time. Figure $13.5.6.3$shows two images: a single pizza on the left, cut into two equal pieces, and a second pizza of the same size, cut into eight pieces on the right. This is a way to show that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$. In other words, they are **equivalent fractions**.

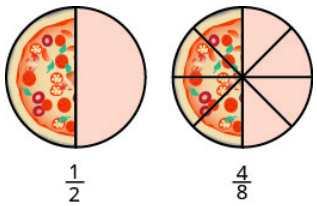

Figure $13.5.6.3$: Since the same amount is of each pizza is shaded, we see that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$. They are equivalent fractions.

EQUIVALENT FRACTIONS

Equivalent fractions are fractions that have the same value.

How can we use mathematics to change $\frac{1}{2}$ into $\frac{4}{8}$? How could we take a pizza that is cut into 2 pieces and cut it into 8 pieces? We could cut each of the 2 larger pieces into 4 smaller pieces! The whole pizza would then be cut into 88 pieces instead of just 2. Mathematically, what we’ve described could be written like this as $\frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$. See Figure $13.5.6.4$

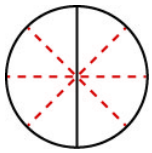

Figure $13.5.6.4$: Cutting each half of the pizza into 4 pieces, gives us pizza cut into 8 pieces: $\frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$

This model leads to the following property:

EQUIVALENT FRACTIONS PROPERTY

If a, b, c are numbers where $b \neq 0, c \neq 0$, then

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \tag{13.5.6.2}$$

If we had cut the pizza differently, we could get

$$\frac{1 \cdot 2}{2 \cdot 2} = \frac{2}{4} \text{ so } \frac{1}{2} = \frac{2}{4}$$

$$\frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6} \text{ so } \frac{1}{2} = \frac{3}{6}$$

$$\frac{1 \cdot 10}{2 \cdot 10} = \frac{10}{20} \text{ so } \frac{1}{2} = \frac{10}{20}$$

Figure 13.5.6.5

So, we say $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{10}{20}$ are equivalent fractions.

Note

Doing the Manipulative Mathematics activity "Equivalent Fractions" will help you develop a better understanding of what it means when two fractions are equivalent.

Example 13.5.6.1

Find three fractions equivalent to $\frac{2}{5}$.

Solution

To find a fraction equivalent to $\frac{2}{5}$, we multiply the numerator and denominator by the same number. We can choose any number, except for zero. Let's multiply them by 2, 3, and then 5.

$$\frac{2 \cdot 2}{5 \cdot 2} = \frac{4}{10} \quad \frac{2 \cdot 3}{5 \cdot 3} = \frac{6}{15} \quad \frac{2 \cdot 5}{5 \cdot 5} = \frac{10}{25}$$

So, $\frac{4}{10}$, $\frac{6}{15}$, and $\frac{10}{25}$ are equivalent to $\frac{2}{5}$.

Try It 13.5.6.2

Find three fractions equivalent to $\frac{3}{5}$.

Answer

$\frac{6}{10}$, $\frac{9}{15}$, $\frac{12}{20}$; answers may vary

Try It 13.5.6.3

Find three fractions equivalent to $\frac{4}{5}$.

Answer

$\frac{8}{10}$, $\frac{12}{15}$, $\frac{16}{20}$; answers may vary

Simplify Fractions

A fraction is considered **simplified** if there are no common factors, other than 1, in its numerator and denominator.

For example,

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

SIMPLIFIED FRACTION

A fraction is considered **simplified** if there are no common factors in its numerator and denominator.

The phrase *reduce a fraction* means to simplify the fraction. We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. If an expression has fractions, it is not completely simplified until the fractions are simplified.

In Exercise 13.5.6.4 we used the equivalent fractions property to find equivalent fractions. Now we'll use the equivalent fractions property in reverse to simplify fractions. We can rewrite the property to show both forms together.

EQUIVALENT FRACTIONS PROPERTY

If a, b, c are numbers where $b \neq 0, c \neq 0$,

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b} \tag{13.5.6.3}$$

Example 13.5.6.4

Simplify: $-\frac{32}{56}$

Solution

	$-\frac{32}{56}$
Rewrite the numerator and denominator showing the common factors.	$-\frac{4 \cdot 8}{7 \cdot 8}$
Simplify using the equivalent fractions property.	$-\frac{4}{7}$

Notice that the fraction $-\frac{4}{7}$ is simplified because there are no more common factors.

p

Try It 13.5.6.5

Simplify: $-\frac{42}{54}$

Answer

$-\frac{7}{9}$

Try It 13.5.6.6

Simplify: $-\frac{30}{54}$

Answer

$-\frac{5}{9}$

Sometimes it may not be easy to find common factors of the numerator and denominator. When this happens, a good idea is to factor the numerator and the denominator into **prime numbers**. Then divide out the common factors using the equivalent fractions property.

Example 13.5.6.7

Simplify: $-\frac{210}{385}$

Solution

Step 1. Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.	Rewrite 210 and 385 as the product of the primes.	$-\frac{210}{385}$ $-\frac{2\cdot 3\cdot 5\cdot 7}{5\cdot 7\cdot 11}$
Step 2. Simplify using the equivalent fractions property by dividing out common factors.	Mark the common factors 5 and 7. Divide out the common factors.	$-\frac{2\cdot 3\cdot \cancel{5}\cdot \cancel{7}}{\cancel{5}\cdot \cancel{7}\cdot 11}$ $-\frac{2\cdot 3}{11}$
Step 3. Multiplify the remaining factors, if necessary.		$-\frac{6}{11}$

Try It 13.5.6.8

Simplify: $-\frac{69}{120}$

Answer

$-\frac{23}{40}$

Try It 13.5.6.9

Simplify: $-\frac{120}{192}$

Answer

$-\frac{5}{8}$

We now summarize the steps you should follow to simplify fractions.

SIMPLIFY A FRACTION.

1. Rewrite the numerator and denominator to show the common factors.
   If needed, factor the numerator and denominator into prime numbers first.
2. Simplify using the equivalent fractions property by dividing out common factors.
3. Multiply any remaining factors, if needed.

Example 13.5.6.10

Simplify: $\frac{5x}{5y}$

Solution

$\frac{5x}{5y}$

Rewrite showing the common factors, then divide out the common factors.	Rewrite showing the common factors, then divide out the common factors. $\frac{5x}{5y}$ $\frac{5 \cdot x}{5 \cdot y}$ Simplify. $\frac{x}{y}$
Simplify.	$\frac{x}{y}$

Try It 13.5.6.11

Simplify: $\frac{7x}{7y}$

Answer

$\frac{x}{y}$

Try It 13.5.6.12

Simplify: $\frac{3a}{3b}$

Answer

$\frac{a}{b}$

Multiply Fractions

Many people find multiplying and dividing fractions easier than adding and subtracting fractions. So we will start with fraction multiplication.

Doing the Manipulative Mathematics activity "Model Fraction Multiplication" will help you develop a better understanding of multiplying fractions.

We'll use a model to show you how to multiply two fractions and to help you remember the procedure. Let's start with $\frac{3}{4}$.

Figure 13.5.6.6

Now we'll take $\frac{1}{2}$ of $\frac{3}{4}$.

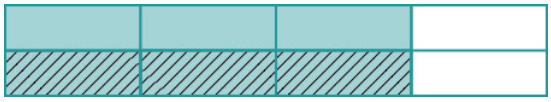

Figure 13.5.6.6

Notice that now, the whole is divided into 8 equal parts. So $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$.

To multiply fractions, we multiply the numerators and multiply the denominators.

FRACTION MULTIPLICATION

If a, b, c and d are numbers where $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \tag{13.5.6.4}$$

To multiply fractions, multiply the numerators and multiply the denominators.

When **multiplying fractions**, the properties of positive and negative numbers still apply, of course. It is a good idea to determine the sign of the **product** as the first step. In Exercise 13.5.6.13 we will multiply negative and a positive, so the product will be negative.

Example 13.5.6.13

Multiply: $-\frac{11}{12} \cdot \frac{5}{7}$

Solution

The first step is to find the sign of the product. Since the signs are the different, the product is negative.

	$-\frac{11}{12} \cdot \frac{5}{7}$
Determine the sign of the product; multiply.	$-\frac{11 \cdot 5}{12 \cdot 7}$
Are there any common factors in the numerator and the denominator? No	$-\frac{55}{84}$

(13.5.6.5)

Try It 13.5.6.14

Multiply: $-\frac{10}{28} \cdot \frac{8}{15}$

Answer

$-\frac{4}{21}$

Try It 13.5.6.15

Multiply: $-\frac{9}{20} \cdot \frac{5}{12}$

Answer

$-\frac{3}{16}$

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $\frac{a}{1}$. So, for example, $3 = \frac{3}{1}$.

Example 13.5.6.16

Multiply: $-\frac{12}{5}(-20x)$

Solution

Determine the sign of the product. The signs are the same, so the product is positive.

	$-\frac{12}{5}(-20x)$
Write $20x$ as a fraction.	$\frac{12}{5}(\frac{20x}{1})$
Multiply.	
Rewrite 20 to show the common factor 5 and divide it out.	$\frac{12 \cdot 4 \cdot \cancel{5}x}{\cancel{5} \cdot 1}$
Simplify.	$48x$

Try It 13.5.6.17

Multiply: $\frac{11}{3}(-9a)$

Answer

$-33a$

Try It 13.5.6.18

Multiply: $\frac{13}{7}(-14b)$

Answer

$-26b$

Divide Fractions

Now that we know how to multiply fractions, we are almost ready to divide. Before we can do that, that we need some vocabulary.

The **reciprocal** of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

Notice that $\frac{2}{3} \cdot \frac{3}{2} = 1$. A number and its reciprocal multiply to 1.

To get a **product** of positive 1 when multiplying two numbers, the numbers must have the same sign. So reciprocals must have the same sign.

The reciprocal of $-\frac{10}{7}$ is $-\frac{7}{10}$, since $-\frac{10}{7}(-\frac{7}{10}) = 1$.

RECIPROCAL

The **reciprocal** of $\frac{a}{b}$ is $\frac{b}{a}$.

A number and its reciprocal multiply to one $\frac{a}{b} \cdot \frac{b}{a} = 1$

Note

Doing the Manipulative Mathematics activity “Model Fraction Division” will help you develop a better understanding of dividing fractions.

To divide fractions, we multiply the first fraction by the reciprocal of the second.

FRACTION DIVISION

If a, b, c and d are numbers where $b \neq 0, c \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \tag{13.5.6.6}$$

To divide fractions, we multiply the first fraction by the reciprocal of the second.

We need to say $b \neq 0, c \neq 0$ and $d \neq 0$ to be sure we don't divide by zero!

Example 13.5.6.19

Divide: $-\frac{2}{3} \div \frac{n}{5}$

Solution

	$-\frac{2}{3} \div \frac{n}{5}$
To divide, multiply the first fraction by the reciprocal of the second.	$-\frac{2}{3} \cdot \frac{5}{n}$
Multiply.	$-\frac{10}{3n}$

(13.5.6.7)

Try It 13.5.6.20

Divide: $-\frac{3}{5} \div \frac{p}{7}$.

Answer

$-\frac{21}{5p}$

Try It 13.5.6.21

Divide: $-\frac{5}{8} \div \frac{q}{3}$.

Answer

$-\frac{15}{8q}$

Example 13.5.6.22

Find the quotient:

$-\frac{7}{18} \div (-\frac{14}{27})$

Solution

	$-\frac{7}{18} \div (-\frac{14}{27})$

To divide, multiply the first fraction by the reciprocal of the second.	$-\frac{7}{18} \cdot -\frac{27}{14}$
Determine the sign of the product, and then multiply..	$\frac{7 \cdot 27}{18 \cdot 14}$
Rewrite showing common factors.	$\frac{7 \cdot 9 \cdot 3}{9 \cdot 2 \cdot 7 \cdot 2}$
Remove common factors.	$\frac{3}{2 \cdot 2}$
Simplify.	$\frac{3}{4}$

Try It 13.5.6.23

Find the quotient:

$-\frac{7}{8} \div (-\frac{14}{27})$

Answer

$\frac{4}{15}$

Try It 13.5.6.24

Find the quotient:

$-\frac{7}{8} \div (-\frac{14}{27})$

Answer

$\frac{2}{3}$

There are several ways to remember which steps to take to multiply or divide fractions. One way is to repeat the call outs to yourself. If you do this each time you do an exercise, you will have the steps memorized.

- “To multiply fractions, multiply the numerators and multiply the denominators.”
- “To divide fractions, multiply the first fraction by the reciprocal of the second.”

Another way is to keep two examples in mind:

One fourth of two pizzas is one half of a pizza.

$$2 \cdot \frac{1}{4}$$

$$\frac{2}{1} \cdot \frac{1}{4}$$

$$\frac{2}{4}$$

$$\frac{1}{2}$$

There are eight quarters in $2.00.

$$2 \div \frac{1}{4}$$

$$\frac{2}{1} \div \frac{1}{4}$$

$$\frac{2}{1} \cdot \frac{4}{1}$$

$$8$$

Figure 13.5.6.7

The numerators or denominators of some fractions contain fractions themselves. A fraction in which the numerator or the denominator is a fraction is called a **complex fraction**.

COMPLEX FRACTION

A **complex fraction** is a fraction in which the numerator or the denominator contains a fraction.

Some examples of complex fractions are:

$$\frac{\frac{6}{7}}{3} \quad \frac{\frac{3}{4}}{\frac{5}{8}} \quad \frac{\frac{x}{2}}{\frac{5}{6}} \tag{13.5.6.8}$$

To simplify a complex fraction, we remember that the fraction bar means **division**. For example, the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ means $\frac{3}{4} \div \frac{5}{8}$.

Example 13.5.6.25

Simplify: $\frac{\frac{3}{4}}{\frac{5}{8}}$

Solution

	$\frac{\frac{3}{4}}{\frac{5}{8}}$
Rewrite as division.	$\frac{3}{4} \div \frac{5}{8}$
Multiply the first fraction by the reciprocal of the second.	$\frac{3}{4} \cdot \frac{8}{5}$
Multiply.	$\frac{3 \cdot 8}{4 \cdot 5}$
Look for common factors.	$\frac{3 \cdot 4 \cdot 2}{4 \cdot 5}$
Divide out common factors and simplify.	$\frac{6}{5}$

Try It 13.5.6.26

Simplify: $\frac{\frac{2}{3}}{\frac{5}{6}}$

Answer

$\frac{4}{5}$

Try It 13.5.6.27

Simplify: $\frac{\frac{3}{7}}{\frac{6}{11}}$

Answer

$\frac{11}{14}$

Example 13.5.6.28

Simplify: $\frac{\frac{x}{2}}{\frac{xy}{6}}$

Solution

	$\frac{\frac{x}{2}}{\frac{xy}{6}}$
Rewrite as division.	$\frac{x}{2} \div \frac{xy}{6}$
Multiply the first fraction by the reciprocal of the second.	$\frac{x}{2} \cdot \frac{6}{xy}$
Multiply.	$\frac{x \cdot 6}{2 \cdot xy}$
Look for common factors.	$\frac{x \cdot 3 \cdot 2}{2 \cdot x \cdot y}$
Divide out common factors and simplify.	$\frac{3}{y}$

Try It 13.5.6.29

Simplify: $\frac{\frac{a}{8}}{\frac{ab}{6}}$

Answer

$\frac{3}{4b}$

Try It 13.5.6.30

Simplify: $\dfrac{\frac{p}{2}}{\frac{pq}{8}}$

Answer

$\dfrac{4}{q}$

Simplify Expressions with a Fraction Bar

The line that separates the numerator from the denominator in a fraction is called a fraction bar. A **fraction bar** acts as grouping symbol. The order of operations then tells us to simplify the numerator and then the denominator. Then we divide.

To simplify the expression $\dfrac{5-3}{7+1}$, we first simplify the numerator and the denominator separately. Then we divide.

$$\begin{array}{l} \dfrac{5-3}{7+1} \\ \dfrac{2}{8} \\ \dfrac{1}{4} \end{array} \tag{13.5.6.9}$$

SIMPLIFY AN EXPRESSION WITH A FRACTION BAR.

1. Simplify the expression in the numerator. Simplify the expression in the denominator.
2. Simplify the fraction.

Example 13.5.6.31

Simplify: $\dfrac{4-2(3)}{2^2+2}$

Solution

	$\dfrac{4-2(3)}{2^2+2}$
Use the order of operations to simplify the numerator and the denominator.	$\dfrac{4-6}{4+2}$
Simplify the numerator and the denominator	$\dfrac{-2}{6}$
Simplify. A negative divided by a positive is negative.	$-\dfrac{1}{3}$

(13.5.6.10)

Try It 13.5.6.32

Simplify: $\dfrac{6-3(5)}{3^2+3}$

Answer

$-\dfrac{3}{4}$

Try It 13.5.6.33

Simplify: $\dfrac{4-4(6)}{3^2+3}$

Answer

$-\dfrac{5}{3}$

Where does the negative sign go in a fraction? Usually the negative sign is in front of the fraction, but you will sometimes see a fraction with a negative numerator, or sometimes with a negative denominator. Remember that fractions represent division. When the numerator and denominator have different signs, the quotient is negative.

$$\begin{aligned} \tfrac{-1}{3} &= -\tfrac{1}{3} \quad \tfrac{\text{negative}}{\text{positive}} = \text{negative} \\ \tfrac{1}{-3} &= -\tfrac{1}{3} \quad \tfrac{\text{positive}}{\text{negative}} = \text{negative} \end{aligned} \tag{13.5.6.11}$$

PLACEMENT OF NEGATIVE SIGN IN A FRACTION

For any positive numbers a and b,

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b} \tag{13.5.6.12}$$

Example 13.5.6.34

Simplify: $\frac{4(-3)+6(-2)}{-3(2)-2}$

Solution

The fraction bar acts like a grouping symbol. So completely simplify the numerator and the denominator separately.

$$\begin{aligned} & \tfrac{4(-3)+6(-2)}{-3(2)-2} \\ \text{Multiply.} \quad & \tfrac{-12+(-12)}{-6-2} \\ \text{Simplify.} \quad & \tfrac{-24}{-8} \\ \text{Divide.} \quad & 3 \end{aligned} \tag{13.5.6.13}$$

Try It 13.5.6.35

Simplify: $\frac{8(-2)+4(-3)}{-5(2)+3}$

Answer

4

Try It 13.5.6.36

Simplify: $\frac{7(-1)+9(-3)}{-5(3)-2}$

Answer

2

Translate Phrases to Expressions with Fractions

Now that we have done some work with fractions, we are ready to translate phrases that would result in expressions with fractions.

The English words **quotient** and **ratio** are often used to describe fractions. Remember that "quotient" means **division**. The quotient of aa and bb is the result we get from dividing a by b, or $\frac{a}{b}$.

Example 13.5.6.37

Translate the English phrase into an algebraic expression: the quotient of the difference of m and n, and p.

Solution

We are looking for the *quotient of* the difference of m and n, and p. This means we want to divide the difference of m and n, and p.

$$\frac{m-n}{p} \tag{13.5.6.14}$$

Try It 13.5.6.38

Translate the English phrase into an algebraic expression: the quotient of the difference of a and b, and cd.

Answer

$\frac{a-b}{cd}$

Try It 13.5.6.39

Translate the English phrase into an algebraic expression: the quotient of the sum of p and q, and r.

Answer

$\frac{p+q}{r}$

Key Concepts

- **Equivalent Fractions Property:** If a, b, c are numbers where $b \neq 0, c \neq 0$, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$
- **Fraction Division:** If a, b, c and d are numbers where $b \neq 0, c \neq 0$ and $d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. To divide fractions, multiply the first fraction by the reciprocal of the second.
- **Fraction Multiplication:** If a, b, c and d are numbers where $b \neq 0, d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. To multiply fractions, multiply the numerators and multiply the denominators.
- **Placement of Negative Sign in a Fraction:** For any positive numbers a and b, $\frac{-a}{a} = \frac{a}{-a} = -\frac{a}{b}$
- **Property of One:** $\frac{a}{a} = 1$; Any number, except zero, divided by itself is one.
- **Simplify a Fraction**
  1. Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.
  2. Simplify using the equivalent fractions property by dividing out common factors.
  3. Multiply any remaining factors.
- **Simplify an Expression with a Fraction Bar**
  1. Simplify the expression in the numerator. Simplify the expression in the denominator.
  2. Simplify the fraction.

Glossary

complex fraction

A complex fraction is a fraction in which the numerator or the denominator contains a fraction.

denominator

The denominator is the value on the bottom part of the fraction that indicates the number of equal parts into which the whole has been divided.

equivalent fractions

Equivalent fractions are fractions that have the same value.

fraction

A fraction is written $\frac{a}{b}$, where $b \neq 0$, a is the numerator and b is the denominator. A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator aa indicates how many parts are included.

numerator

The numerator is the value on the top part of the fraction that indicates how many parts of the whole are included.

reciprocal

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. A number and its reciprocal multiply to one: $\frac{a}{b} \cdot \frac{b}{a} = 1$.

simplified fraction

A fraction is considered simplified if there are no common factors in its numerator and denominator.

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13.5.6E: Exercises

Practice Makes Perfect

Find Equivalent Fractions

In the following exercises, find three fractions equivalent to the given fraction. Show your work, using figures or algebra.

Exercise 13.5.6E.1

$\frac{3}{8}$

Answer

$\frac{6}{16}, \frac{9}{24}, \frac{12}{32}$, answers may vary

Exercise 13.5.6E.2

$\frac{5}{8}$

Exercise 13.5.6E.3

$\frac{5}{9}$

Answer

$\frac{10}{18}, \frac{15}{27}, \frac{20}{36}$, answers may vary

Exercise 13.5.6E.4

$\frac{1}{8}$

Simplify Fractions

In the following exercises, simplify.

Exercise 13.5.6E.5

$-\frac{40}{88}$

Answer

$-\frac{5}{11}$

Exercise 13.5.6E.6

$-\frac{63}{99}$

Exercise 13.5.6E.7

$$-\frac{108}{63}$$

Answer

$$-\frac{12}{7}$$

Exercise 13.5.6E.8

$$-\frac{104}{48}$$

Exercise 13.5.6E.9

$$\frac{120}{252}$$

Answer

$$\frac{10}{21}$$

Exercise 13.5.6E.10

$$\frac{182}{294}$$

Exercise 13.5.6E.11

$$-\frac{3x}{12y}$$

Answer

$$-\frac{x}{4y}$$

Exercise 13.5.6E.12

$$-\frac{4x}{32y}$$

Exercise 13.5.6E.13

$$\frac{14x^2}{21y}$$

Answer

$$\frac{2x^2}{3y}$$

Exercise 13.5.6E.14

$$\frac{24a}{32b^2}$$

Multiply Fractions

In the following exercises, multiply.

Exercise 13.5.6E.15

$$\frac{3}{4} \cdot \frac{9}{10}$$

Answer

$$\frac{27}{40}$$

Exercise 13.5.6E.16

$$\frac{4}{5} \cdot \frac{2}{7}$$

Exercise 13.5.6E.17

$$-\frac{2}{3} \cdot -\frac{3}{8}$$

Answer

$$\frac{1}{4}$$

Exercise 13.5.6E.18

$$-\frac{3}{4}\left(-\frac{4}{9}\right)$$

Exercise 13.5.6E.19

$$-\frac{5}{9} \cdot \frac{3}{10}$$

Answer

$$-\frac{1}{6}$$

Exercise 13.5.6E.20

$$-\frac{3}{8} \cdot \frac{4}{15}$$

Exercise 13.5.6E.21

$$\left(-\frac{14}{15}\right)\left(\frac{9}{20}\right)$$

Answer

$$-\frac{21}{50}$$

Exercise 13.5.6E.22

$$\left(-\frac{9}{10}\right)\left(\frac{25}{33}\right)$$

Exercise 13.5.6E.23

$$\left(-\frac{63}{84}\right)\left(-\frac{44}{90}\right)$$

Answer

$$\frac{11}{30}$$

Exercise 13.5.6E.24

$$\left(-\frac{63}{60}\right)\left(-\frac{40}{88}\right)$$

Exercise 13.5.6E.25

$$4 \cdot \frac{5}{11}$$

Answer

$$\frac{20}{11}$$

Exercise 13.5.6E.26

$$5 \cdot \frac{8}{3}$$

Exercise 13.5.6E.27

$$\frac{3}{7} \cdot 21n$$

Answer

9n

Exercise 13.5.6E.28

$$\frac{5}{6} \cdot 30m$$

Exercise 13.5.6E.29

$-8 \cdot \frac{17}{4}$

Answer

-34

Exercise 13.5.6E.30

$(-1)\left(-\frac{6}{7}\right)$

Divide Fractions

In the following exercises, divide.

Exercise 13.5.6E.31

$\frac{3}{4} \div \frac{2}{3}$

Answer

$\frac{9}{8}$

Exercise 13.5.6E.32

$\frac{4}{5} \div \frac{3}{4}$

Exercise 13.5.6E.33

$-\frac{7}{9} \div \left(-\frac{7}{4}\right)$

Answer

$\frac{4}{9}$

Exercise 13.5.6E.34

$-\frac{5}{6} \div \left(-\frac{5}{6}\right)$

Exercise 13.5.6E.35

$\frac{3}{4} \div \frac{x}{11}$

Answer

$\frac{33}{4x}$

Exercise 13.5.6E.36

$\frac{2}{5} \div \frac{y}{9}$

Exercise 13.5.6E.37

$\frac{5}{18} \div -\frac{15}{24}$

Answer

$-\frac{4}{9}$

Exercise 13.5.6E.38

$\frac{7}{18} \div \left(-\frac{14}{27}\right)$

Exercise 13.5.6E.39

$\frac{8u}{15} \div \frac{12v}{25}$

Answer

$\frac{10u}{9v}$

Exercise 13.5.6E.40

$\frac{12r}{25} \div \frac{18s}{35}$

Exercise 13.5.6E.41

$-5 \div \frac{1}{2}$

Answer

-10

Exercise 13.5.6E.42

$-3 \div \frac{1}{4}$

Exercise 13.5.6E.43

$\frac{3}{4} \div (-12)$

Answer

$\frac{1}{16}$

Exercise 13.5.6E.44

$-15 \div -\frac{5}{3}$

In the following exercises, simplify.

Exercise 13.5.6E.45

$$\frac{-\frac{8}{21}}{\frac{12}{35}}$$

Answer

$-\frac{10}{9}$

Exercise 13.5.6E.46

$$\frac{-\frac{9}{16}}{\frac{33}{40}}$$

Exercise 13.5.6E.47

$$\frac{-\frac{4}{5}}{2}$$

Answer

$-\frac{2}{5}$

Exercise 13.5.6E.48

$$\frac{5}{\frac{3}{10}}$$

Exercise 13.5.6E.49

$$\frac{\frac{m}{3}}{\frac{n}{2}}$$

Answer

$\frac{2m}{3n}$

Exercise 13.5.6E.50

$$\frac{-\frac{3}{8}}{-\frac{y}{12}}$$

Simplify Expressions Written with a Fraction Bar

In the following exercises, simplify.

Exercise 13.5.6E.51

$$\frac{22+3}{10}$$

Answer

$$\frac{5}{2}$$

Exercise 13.5.6E.52

$$\frac{19-4}{6}$$

Exercise 13.5.6E.53

$$\frac{48}{24-15}$$

Answer

$$\frac{16}{3}$$

Exercise 13.5.6E.54

$$\frac{46}{4+4}$$

Exercise 13.5.6E.55

$$\frac{-6+6}{8+4}$$

Answer

$$0$$

Exercise 13.5.6E.56

$$\frac{-6+3}{17-8}$$

Exercise 13.5.6E.57

$$\frac{4 \cdot 3}{6 \cdot 6}$$

Answer

$$\frac{1}{3}$$

Exercise 13.5.6E.58

$$\frac{6 \cdot 6}{9 \cdot 2}$$

Exercise 13.5.6E.59

$$\frac{4^2 - 1}{25}$$

Answer

$$\frac{3}{5}$$

Exercise 13.5.6E.60

$$\frac{7^2 + 1}{60}$$

Exercise 13.5.6E.61

$$\frac{8 \cdot 3 + 2 \cdot 9}{14 + 3}$$

Answer

$$\frac{42}{17}$$

Exercise 13.5.6E.62

$$\frac{9 \cdot 6 - 4 \cdot 7}{22 + 3}$$

Exercise 13.5.6E.63

$$\frac{5 \cdot 6 - 3 \cdot 4}{4 \cdot 5 - 2 \cdot 3}$$

Answer

$$\frac{9}{7}$$

Exercise 13.5.6E.64

$$\frac{8 \cdot 9 - 7 \cdot 6}{5 \cdot 6 - 9 \cdot 2}$$

Exercise 13.5.6E.65

$$\frac{5^2 - 3^2}{3 - 5}$$

Answer

-8

Exercise 13.5.6E.66

$$\frac{6^2 - 4^2}{4 - 6}$$

Exercise 13.5.6E.67

$$\frac{7 \cdot 4 - 2(8 - 5)}{9 \cdot 3 - 3 \cdot 5}$$

Answer

$\frac{11}{6}$

Exercise 13.5.6E.68

$$\frac{9 \cdot 7 - 3(12 - 8)}{8 \cdot 7 - 6 \cdot 6}$$

Exercise 13.5.6E.69

$$\frac{9(8 - 2) - 3(15 - 7)}{6(7 - 1) - 3(17 - 9)}$$

Answer

$\frac{5}{2}$

Exercise 13.5.6E.70

$$\frac{8(9 - 2) - 4(14 - 9)}{7(8 - 3) - 3(16 - 9)}$$

Translate Phrases to Expressions with Fractions

In the following exercises, translate each English phrase into an algebraic expression.

Exercise 13.5.6E.71

the quotient of r and the sum of s and 10

Answer

$$\frac{r}{s+10}$$

Exercise 13.5.6E. 72

the quotient of A and the difference of 3 and B

Exercise 13.5.6E. 73

the quotient of the difference of x and y, and -3

Answer

$$\frac{x-y}{-3}$$

Exercise 13.5.6E. 74

the quotient of the sum of m and n, and $4q$

Everyday Math

Exercise 13.5.6E. 75

Baking. A recipe for chocolate chip cookies calls for $\frac{3}{4}$ cup brown sugar. Imelda wants to double the recipe.

1. How much brown sugar will Imelda need? Show your calculation.
2. Measuring cups usually come in sets of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, and 1 cup. Draw a diagram to show two different ways that Imelda could measure the brown sugar needed to double the cookie recipe.

Answer

1. $1\frac{1}{2}$ cups
2. answers will vary

Exercise 13.5.6E. 76

Baking. Nina is making 4 pans of fudge to serve after a music recital. For each pan, she needs $\frac{2}{3}$ cup of condensed milk.

1. How much condensed milk will Nina need? Show your calculation.
2. Measuring cups usually come in sets of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, and 1 cup. Draw a diagram to show two different ways that Nina could measure the condensed milk needed for 4 pans of fudge.

Exercise 13.5.6E. 77

Portions Don purchased a bulk package of candy that weighs 5 pounds. He wants to sell the candy in little bags that hold $\frac{1}{4}$ pound. How many little bags of candy can he fill from the bulk package?

Answer

20 bags

Exercise 13.5.6E.78

Portions Kristen has $\frac{3}{4}$ yards of ribbon that she wants to cut into 6 equal parts to make hair ribbons for her daughter's 6 dolls. How long will each doll's hair ribbon be?

Writing Exercises

Exercise 13.5.6E.79

Rafael wanted to order half a medium pizza at a restaurant. The waiter told him that a medium pizza could be cut into 6 or 8 slices. Would he prefer 3 out of 6 slices or 4 out of 8 slices? Rafael replied that since he wasn't very hungry, he would prefer 3 out of 6 slices. Explain what is wrong with Rafael's reasoning.

Answer

Answers may vary

Exercise 13.5.6E.80

Give an example from everyday life that demonstrates how $\frac{1}{2} \cdot \frac{2}{3}$ is $\frac{1}{3}$.

Exercise 13.5.6E.81

Explain how you find the reciprocal of a fraction.

Answer

Answers may vary

Exercise 13.5.6E.82

Explain how you find the reciprocal of a negative number.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find equivalent fractions.			
simplify fractions.			
multiply fractions.			
divide fractions.			
simplify expressions written with a fraction bar.			
translate phrases to expressions with fractions.			

ⓑ After looking at the checklist, do you think you are well prepared for the next section? Why or why not?

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13.5.7: Add and Subtract Fractions

Learning Objectives

By the end of this section, you will be able to:

- Add or subtract fractions with a common denominator
- Add or subtract fractions with different denominators
- Use the order of operations to simplify complex fractions
- Evaluate variable expressions with fractions

Note

A more thorough introduction to the topics covered in this section can be found in the *Prealgebra*chapter, **Fractions**.

Add or Subtract Fractions with a Common Denominator

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators right straight across. To add or subtract fractions, they must have a common denominator.

FRACTION ADDITION AND SUBTRACTION

If a, b, and c are numbers where $c \neq 0$, then

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \quad \text{and} \quad \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c} \tag{13.5.7.1}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

Manipulative Mathematics

Doing the Manipulative Mathematics activities "Model Fraction Addition" and "Model Fraction Subtraction" will help you develop a better understanding of adding and subtracting fractions.

Example 13.5.7.1

Find the sum: $\frac{x}{3}+\frac{2}{3}$.

Solution

$$\begin{array}{ll} & \frac{x}{3}+\frac{2}{3} \\ \text{Add the numerators and place the sum over the common denominator} & \frac{x+2}{3} \end{array} \tag{13.5.7.2}$$

Try It 13.5.7.2

Find the sum: $\frac{x}{4}+\frac{3}{4}$.

Answer

$\frac{x+3}{4}$

Try It 13.5.7.3

Find the sum: $\frac{y}{8}+\frac{5}{8}$.

Answer

$\frac{y+5}{8}$

Example 13.5.7.4

Find the difference: $-\frac{23}{24}-\frac{13}{24}$

Solution

	$-\frac{23}{24}-\frac{13}{24}$
Subtract the numerators and place the difference over the common denominator	$\frac{-23-13}{24}$ (13.5.7.3)
Simplify.	$\frac{-36}{24}$
Simplify. Remember, $-\frac{a}{b}=\frac{-a}{b}$	$-\frac{3}{2}$

Try It 13.5.7.5

Find the difference: $-\frac{19}{28}-\frac{7}{28}$

Answer

$-\frac{26}{28}$

Try It 13.5.7.6

Find the difference: $-\frac{27}{32}-\frac{1}{32}$

Answer

$-\frac{7}{8}$

Example 13.5.7.7

Find the difference: $-\frac{10}{x}-\frac{4}{x}$

Solution

	$-\frac{10}{x}-\frac{4}{x}$	
Subtract the numerators and place the difference over the common denominator	$\frac{-14}{x}$	(13.5.7.4)
Rewrite with the sign in front of the fraction.	$-\frac{14}{x}$	

Try It 13.5.7.8

Find the difference: $-\frac{9}{x}-\frac{7}{x}$

Answer

$-\frac{16}{x}$

Try It 13.5.7.9

Find the difference: $-\frac{17}{a}-\frac{5}{a}$

Answer

$-\frac{22}{a}$

Now we will do an example that has both addition and subtraction.

Example 13.5.7.10

Simplify: $\frac{3}{8}+(-\frac{5}{8})-\frac{1}{8}$

Solution

Add and Subtract fractions — do they have a common denominator? Yes.	$\frac{3}{8}+(-\frac{5}{8})-\frac{1}{8}$	
Add and subtract the numerators and place the result over the common denominator.	$\frac{3+(-5)-1}{8}$	(13.5.7.5)
Simplify left to right.	$\frac{-2-1}{8}$	
Simplify.	$-\frac{3}{8}$	

Try It 13.5.7.11

Simplify: $\frac{2}{9}+(-\frac{4}{9})-\frac{7}{9}$

Answer

-1

Try It 13.5.7.12

Simplify: $\frac{2}{5}+(-\frac{4}{9})-\frac{7}{9}$

Answer

$-\frac{2}{3}$

Add or Subtract Fractions with Different Denominators

As we have seen, to add or subtract fractions, their denominators must be the same. The **least common denominator** (LCD) of two fractions is the smallest number that can be used as a common **denominator** of the fractions. The LCD of the two fractions is the least common multiple (LCM) of their denominators.

LEAST COMMON DENOMINATOR

The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

Note

Doing the Manipulative Mathematics activity "Finding the Least Common Denominator" will help you develop a better understanding of the LCD.

After we find the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD. Putting these steps together allows us to add and subtract fractions because their denominators will be the same!

Example 13.5.7.13

Add: $\frac{7}{12}+\frac{5}{18}$

Solution

Step 1. Do they have a common denominator? No—rewrite each fraction with the LCD (least common denominator).	No. Find the LCD of 12, 18. Change into equivalent fractions with the LCD, 36. Do not simplify the equivalent fractions! If you do, you'll get back to the original fractions and lose the common denominator!	$12=2\cdot2\cdot3$ $18=2\cdot\quad3\cdot3$ $\text{LCD}=2\cdot2\cdot3\cdot3$ $\text{LCD}=36$ $\frac{7}{12}+\frac{5}{18}$ $\frac{7\cdot3}{12\cdot3}+\frac{5\cdot2}{18\cdot2}$ $\frac{21}{36}+\frac{10}{36}$
Step 2. Add or subtract the fractions.	Add.	$\frac{31}{36}$
Step 3. Simplify, if possible.	Because 31 is a prime number, it has no factors in common with 36. The answer is simplified.	

Try It 13.5.7.14

Add: $\frac{7}{12}+\frac{11}{15}$

Answer

$\frac{79}{60}$

Try It 13.5.7.15

Add: $\frac{13}{15}+\frac{17}{20}$

Answer

$\frac{103}{60}$

ADD OR SUBTRACT FRACTIONS.

1. Do they have a common denominator?
   - Yes—go to step 2.
   - No—rewrite each fraction with the LCD (least common denominator). Find the LCD. Change each fraction into an equivalent fraction with the LCD as its denominator.
2. Add or subtract the fractions.
3. Simplify, if possible.

When finding the equivalent fractions needed to create the common denominators, there is a quick way to find the number we need to multiply both the numerator and denominator. This method works if we found the LCD by factoring into primes.

Look at the factors of the LCD and then at each column above those factors. The "missing" factors of each denominator are the numbers we need.

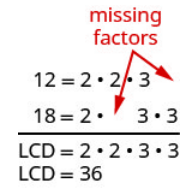

Figure: 13.5.7.1

In Exercise 13.5.7.13, the LCD, 36, has two factors of 2 and two factors of 3.

The numerator 12 has two factors of 2 but only one of 3—so it is "missing" one 3—we multiply the numerator and denominator by 3.

The numerator 18 is missing one factor of 2—so we multiply the numerator and denominator by 2.

We will apply this method as we subtract the fractions in Exercise 13.5.7.16

Example 13.5.7.16

Subtract: $\frac{7}{15}-\frac{19}{24}$

Solution

Do the fractions have a common denominator? No, so we need to find the LCD.

Find the LCD. $\frac{7}{15}-\frac{19}{24}$ $15=3\cdot 5$ $24=2\cdot 2\cdot 2\cdot 3$ $LCD=2\cdot 2\cdot 2\cdot 3\cdot 5$ $LCD=120$	
Notice, 15 is “missing” three factors of 2 and 24 is “missing” the 5 from the factors of the LCD. So we multiply 8 in the first fraction and 5 in the second fraction to get the LCD.	
Rewrite as equivalent fractions with the LCD.	$\frac{7\cdot 8}{15\cdot 8}-\frac{19\cdot 5}{24\cdot 5}$
Simplify.	$\frac{56}{120}-\frac{95}{120}$
Subtract.	$-\frac{39}{120}$
Check to see if the answer can be simplified.	$-\frac{13\cdot 3}{40\cdot 3}$
Both 39 and 120 have a factor of 3.	
Simplify.	$-\frac{13}{40}$

Do not simplify the equivalent fractions! If you do, you’ll get back to the original fractions and lose the common denominator!

? Try It 13.5.7.17

Subtract: $\frac{13}{24}-\frac{17}{32}$

Answer

$\frac{1}{96}$

? Try It 13.5.7.18

Subtract: $\frac{7}{15}-\frac{19}{24}$

Answer

$\frac{75}{224}$

In the next example, one of the fractions has a variable in its numerator. Notice that we do the same steps as when both numerators are numbers.

✓ Example 13.5.7.19

Add: $\frac{3}{5}+\frac{x}{8}$

Solution

The fractions have different denominators.

	$\frac{3}{5}+\frac{x}{8}$

Find the LCD. $\begin{array}{l} 5= \quad 5 \\ 8=2\cdot 2\cdot 2 \\ \text{LCD}=2\cdot 2\cdot 2\cdot 5 \\ \text{LCD}=40 \end{array}$	
Rewrite as equivalent fractions with the LCD.	$\frac{3\cdot 8}{5\cdot 8}+\frac{x\cdot 5}{8\cdot 5}$
Simplify.	$\frac{24}{40}+\frac{5x}{40}$
Add.	$\frac{24+5x}{40}$

Remember, we can only add like terms: 24 and $5x$ are not like terms.

Try It 13.5.7.20

Add: $\frac{y}{6}+\frac{7}{9}$

Answer

$\frac{3y+14}{18}$

Try It 13.5.7.21

Add: $\frac{x}{6}+\frac{7}{15}$

Answer

$\frac{15x+42}{153}$

We now have all four operations for fractions. Table $13.5.7.1$summarizes **fraction operations**.

Table 13.5.7.1

Fraction Multiplication	**Fraction Division**
$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$ Multiply the numerators and multiply the denominators	$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$ Multiply the first fraction by the reciprocal of the second.
Fraction Addition	**Fraction Subtraction**
$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ Add the numerators and place the sum over the common denominator.	$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$ Subtract the numerators and place the difference over the common denominator.
To multiply or divide fractions, and LCD is NOT needed. To add or subtract fractions, an LCD is needed.	

Example 13.5.7.22

Simplify:

1. $\frac{5x}{6}-\frac{3}{10}$
2. $\frac{5x}{6}\cdot\frac{3}{10}$.

Solution

First ask, “What is the operation?” Once we identify the operation that will determine whether we need a common denominator. Remember, we need a common denominator to add or subtract, but not to multiply or divide.

1. What is the operation? The operation is subtraction.

Do the fractions have a common denominator? No.	$\frac{5x}{6}-\frac{3}{10}$	(13.5.7.6)
Rewrite each fractions as an equivalent fraction with the LCD.	$\frac{5x\cdot 5}{6\cdot 5}-\frac{3\cdot 3}{10\cdot 3}$	
	$\frac{25x}{30}-\frac{9}{30}$	
Subtract the numerators and place the difference over the common denominators.	$\frac{25x-9}{30}$	
Simplify, if possible. There are no common factors.		
The fraction is simplified.		

2. What is the operation? Multiplication.

	$\frac{5x}{6}\cdot\frac{3}{10}$	(13.5.7.7)
To multiply fractions, multiply the numerators and multiply the denominators	$\frac{5x\cdot 3}{6\cdot 10}$	
Rewrite, showing common factors.	$\frac{5x\cdot 3}{2\cdot 3\cdot 2\cdot 5}$	
common denominators.		
Simplify.	$\frac{x}{4}$	

? Try It 13.5.7.23

Simplify:

1. $\frac{3a}{4}-\frac{8}{9}$
2. $\frac{3a}{4}\cdot\frac{8}{9}$

Answer

1. $\frac{27a-32}{36}$
2. $\frac{2a}{3}$

? Try It 13.5.7.24

Simplify:

1. $\frac{4k}{5}-\frac{1}{6}$
2. $\frac{4k}{5}\cdot\frac{1}{6}$

Answer

1. $\frac{24k-5}{30}$
2. $\frac{2k}{15}$

Use the Order of Operations to Simplify Complex Fractions

We have seen that a complex fraction is a fraction in which the numerat or or denominator contains a fraction. The fraction bar indicates **division**. We simplified the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ by dividing $\frac{3}{4}$ by $\frac{5}{8}$.

Now we'll look at complex fractions where the numerator or denominator contains an expression that can be simplified. So we first must completely simplify the numerator and denominator separately using the order of operations. Then we divide the numerator by the denominator.

Example 13.5.7.25: How to simplify complex fractions

Simplify: $\frac{(\frac{1}{2})^2}{4+3^2}$

Solution

Step 1. Simplify the numerator. * Remember, $\left(\frac{1}{2}\right)^2$ means $\frac{1}{2}\cdot\frac{1}{2}$.	$\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$ $\frac{\frac{1}{4}}{4+3^2}$
Step 2. Simplify the denominator.	$\frac{\frac{1}{4}}{4+9}$ $\frac{\frac{1}{4}}{13}$
Step 3. Divide the numerator by the denominator. Simplify if possible. * Remember, $13=\frac{13}{1}$	$\frac{1}{4}\div 13$ $\frac{1}{4}\cdot\frac{1}{13}$ $\frac{1}{52}$

Try It 13.5.7.26

Simplify: $\frac{(\frac{1}{3})^2}{2^3+2}$

Answer

$\frac{1}{90}$

Try It 13.5.7.27

Simplify: $\frac{1+4^2}{(\frac{1}{4})^2}$

Answer

272

SIMPLIFY COMPLEX FRACTIONS.

1. Simplify the numerator.
2. Simplify the denominator.
3. Divide the numerator by the denominator. Simplify if possible.

Example 13.5.7.28

Simplify: $\dfrac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$

Solution

	$\frac{(\frac{1}{2}+\frac{2}{3})}{(\frac{3}{4}-\frac{1}{6})}$	
Simplify the numerator (LCD $= 6$) and simplify the denominator (LCD $= 12$).	$\frac{(\frac{3}{6}+\frac{4}{6})}{(\frac{9}{12}-\frac{2}{12})}$	
Simplify.	$\frac{(\frac{7}{6})}{(\frac{7}{12})}$	(13.5.7.8)
Divide the numerator by the denominator.	$\frac{7}{6} \div \frac{7}{12}$	
Simplify.	$\frac{7}{6} \cdot \frac{12}{7}$	
Divide out common factors.	$\frac{7 \cdot 6 \cdot 2}{6 \cdot 7}$	
Simplify.	2	

Try It 13.5.7.29

Simplify: $\dfrac{\frac{1}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$

Answer

2

Try It 13.5.7.30

Simplify: $\dfrac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{4}+\frac{1}{3}}$

Answer

$\frac{2}{7}$

Evaluate Variable Expressions with Fractions

We have evaluated expressions before, but now we can evaluate expressions with fractions. Remember, to evaluate an expression, we substitute the value of the variable into the expression and then simplify.

Example 13.5.7.31

Evaluate $x+\dfrac{1}{3}$ when

1. $x=-\dfrac{1}{3}$

2. $x = -\frac{3}{4}$

Solution

1. To evaluate $x + \frac{1}{3}$ when $x = -\frac{1}{3}$, substitute $-\frac{1}{3}$ for x in the expression.

	$x+\frac{1}{3}$
Substitute $-\frac{1}{3}$ for x.	$-\frac{1}{3}+\frac{1}{3}$
Simplify.	0

2. To evaluate $x + \frac{1}{3}$ when $x = -\frac{3}{4}$, substitute $-\frac{3}{4}$ for x in the expression.

	$x+\frac{1}{3}$
Substitute $-\frac{3}{4}$ for x.	$-\frac{3}{4}+\frac{1}{3}$
Rewrite as equivalent fractions with the LCD, 12.	$-\frac{3\cdot 3}{4\cdot 3}+\frac{1\cdot 4}{3\cdot 4}$
Simplify.	$-\frac{9}{12}+\frac{4}{12}$
Add.	$-\frac{5}{12}$

? Try It 13.5.7.32

Evaluate $x + \frac{3}{4}$ when

1. $x = -\frac{7}{4}$
2. $x = -\frac{5}{4}$

Answer

1. -1
2. $-\frac{1}{2}$

? Try It 13.5.7.33

Evaluate $y + \frac{1}{2}$ when

1. $y = \frac{2}{3}$
2. $y = -\frac{3}{4}$

Answer

1. $\frac{7}{6}$
2. $-\frac{1}{12}$

Example 13.5.7.34

Evaluate $-\frac{5}{6}-y$ when $y=-\frac{2}{3}$

Solution

	$-\frac{5}{6}-y$
Substitute $-\frac{2}{3}$ for y.	$-\frac{5}{6}-\left(-\frac{2}{3}\right)$
Rewrite as equivalent fractions with the LCD, 6.	$-\frac{5}{6}-\left(-\frac{4}{6}\right)$
Subtract.	$\frac{-5-(-4)}{6}$
Simplify.	$-\frac{1}{6}$

Try It 13.5.7.35

Evaluate $y+\frac{1}{2}$ when $y=\frac{2}{3}$

Answer

$-\frac{1}{4}$

Try It 13.5.7.36

Evaluate $y+\frac{1}{2}$ when $y=\frac{2}{3}$

Answer

$-\frac{17}{8}$

Example 13.5.7.37

Evaluate $2x^2y$ when $x=\frac{1}{4}$ and $y=-\frac{2}{3}$.

Solution

Substitute the values into the expression.

	$2x^2y$
Substitute $\frac{1}{4}$ for x and $-\frac{2}{3}$ for y.	$2\left(\frac{1}{4}\right)^2\left(-\frac{2}{3}\right)$
Simplify exponents first.	$2\left(\frac{1}{16}\right)\left(-\frac{2}{3}\right)$
Multiply. Divide out the common factors. Notice we write 16 as $2\cdot 2\cdot 4$ to make it easy to remove	$-\frac{\cancel{2}\cdot 1\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot 4\cdot 3}$
Simplify.	$-\frac{1}{12}$

Try It 13.5.7.38

Evaluate $3ab^2$ when $a=-\frac{2}{3}$ and $b=-\frac{1}{2}$.

Answer

$-\frac{1}{2}$

Try It 13.5.7.39

Evaluate $4c^3d$ when $c=-\frac{1}{2}$ and $d=-\frac{4}{3}$.

Answer

$\frac{2}{3}$

The next example will have only variables, no constants.

Example 13.5.7.40

Evaluate $\frac{p+q}{r}$ when $p=-4, q=-2$, and $r=8$.

Solution

To evaluate $\frac{p+q}{r}$ when $p=-4, q=-2$, and $r=8$, we substitute the values into the expression.

	$\frac{p+q}{r}$
Substitute –4 for p, –2 for q and 8 for r.	$\frac{-4+(-2)}{8}$
Add in the numerator first.	$\frac{-6}{8}$
Simplify.	$-\frac{3}{4}$

Try It 13.5.7.41

Evaluate $\frac{a+b}{c}$ when $a=-8, b=-7$, and $c=6$.

Answer

$-\frac{5}{2}$

Try It 13.5.7.42

Evaluate $\frac{x+y}{z}$ when $x=9, y=-18$, and $z=-6$.

Answer

$\frac{3}{2}$

Key Concepts

- **Fraction Addition and Subtraction:** If a, b, and c are numbers where $c \neq 0$, then
  $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ and $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
  To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.
- **Strategy for Adding or Subtracting Fractions**
  1. Do they have a common denominator?
     Yes—go to step 2.
     No—Rewrite each fraction with the LCD (Least Common Denominator). Find the LCD. Change each fraction into an equivalent fraction with the LCD as its denominator.
  2. Add or subtract the fractions.
  3. Simplify, if possible. To multiply or divide fractions, an LCD IS NOT needed. To add or subtract fractions, an LCD IS needed.
- **Simplify Complex Fractions**
  1. Simplify the numerator.
  2. Simplify the denominator.
  3. Divide the numerator by the denominator. Simplify if possible.

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13.5.7E: Exercises

Practice Makes Perfect

Add and Subtract Fractions with a Common Denominator

In the following exercises, add.

Exercise 13.5.7E.1

$$\frac{6}{13}+\frac{5}{13}$$

Answer

$\frac{11}{13}$

Exercise 13.5.7E.2

$$\frac{4}{15}+\frac{7}{15}$$

Exercise 13.5.7E.3

$$\frac{x}{4}+\frac{3}{4}$$

Answer

$\frac{x+3}{4}$

Exercise 13.5.7E.4

$$\frac{8}{q}+\frac{6}{q}$$

Exercise 13.5.7E.5

$$-\frac{3}{16}+\left(-\frac{7}{16}\right)$$

Answer

$-\frac{5}{8}$

Exercise 13.5.7E.6

$$-\frac{5}{16}+\left(-\frac{9}{16}\right)$$

Exercise 13.5.7E.7

$$-\frac{8}{17}+\frac{15}{17}$$

Answer

$\frac{7}{17}$

Exercise 13.5.7E.8

$$-\frac{9}{19}+\frac{17}{19}$$

Exercise 13.5.7E.9

$$\frac{6}{13}+\left(-\frac{10}{13}\right)+\left(-\frac{12}{13}\right)$$

Answer

$-\frac{16}{13}$

Exercise 13.5.7E.10

$$\frac{5}{12}+\left(-\frac{7}{12}\right)+\left(-\frac{11}{12}\right)$$

In the following exercises, subtract.

Exercise 13.5.7E.11

$$\frac{11}{15}-\frac{7}{15}$$

Answer

$\frac{4}{15}$

Exercise 13.5.7E.12

$$\frac{9}{13}-\frac{4}{13}$$

Exercise 13.5.7E.13

$$\frac{11}{12}-\frac{5}{12}$$

Answer

$\frac{1}{2}$

Exercise 13.5.7E.14

$$\frac{7}{12}-\frac{5}{12}$$

Exercise 13.5.7E.15

$$\frac{19}{21}-\frac{4}{21}$$

Answer

$\frac{5}{7}$

Exercise 13.5.7E.16

$$\frac{17}{21} - \frac{8}{21}$$

Exercise 13.5.7E.17

$$\frac{5y}{8} - \frac{7}{8}$$

Answer

$\frac{5y-7}{8}$

Exercise 13.5.7E.18

$$\frac{11z}{13} - \frac{8}{13}$$

Exercise 13.5.7E.19

$$-\frac{23}{u} - \frac{15}{u}$$

Answer

$-\frac{38}{u}$

Exercise 13.5.7E.20

$$-\frac{29}{v} - \frac{26}{v}$$

Exercise 13.5.7E.21

$$-\frac{3}{5} - \left(-\frac{4}{5}\right)$$

Answer

$\frac{1}{5}$

Exercise 13.5.7E.22

$$-\frac{3}{7} - \left(-\frac{5}{7}\right)$$

Exercise 13.5.7E.23

$$-\frac{7}{9} - \left(-\frac{5}{9}\right)$$

Answer

$-\frac{2}{9}$

Exercise 13.5.7E.24

$$-\frac{8}{11}-\left(-\frac{5}{11}\right)$$

Mixed Practice

In the following exercises, simplify.

Exercise 13.5.7E.25

$$-\frac{5}{18}\cdot\frac{9}{10}$$

Answer

$-\frac{1}{4}$

Exercise 13.5.7E.26

$$-\frac{3}{14}\cdot\frac{7}{12}$$

Exercise 13.5.7E.27

$$\frac{n}{5}-\frac{4}{5}$$

Answer

$\frac{n-4}{5}$

Exercise 13.5.7E.28

$$\frac{6}{11}-\frac{s}{11}$$

Exercise 13.5.7E.29

$$-\frac{7}{24}+\frac{2}{24}$$

Answer

$-frac524$

Exercise 13.5.7E.30

$$-\frac{5}{18}+\frac{1}{18}$$

Exercise 13.5.7E.31

$$\frac{8}{15}\div\frac{12}{5}$$

Answer

$\frac{2}{9}$

Exercise 13.5.7E.32

$\frac{7}{12} \div \frac{9}{28}$

Add or Subtract Fractions with Different Denominators

In the following exercises, add or subtract.

Exercise 13.5.7E.33

$\frac{1}{2} + \frac{1}{7}$

Answer

$\frac{9}{14}$

Exercise 13.5.7E.34

$\frac{1}{3} + \frac{1}{8}$

Exercise 13.5.7E.35

$\frac{1}{3} - \left(-\frac{1}{9}\right)$

Answer

$\frac{4}{9}$

Exercise 13.5.7E.36

$\frac{1}{4} - \left(-\frac{1}{8}\right)$

Exercise 13.5.7E.37

$\frac{7}{12} + \frac{5}{12}$

Answer

$\frac{29}{24}$

Exercise 13.5.7E.38

$\frac{5}{12} + \frac{3}{8}$

Exercise 13.5.7E.39

$\frac{7}{12} - \frac{9}{16}$

Answer

$\frac{1}{48}$

Exercise 13.5.7E.40

$\frac{7}{16} - \frac{5}{12}$

Exercise 13.5.7E.41

$\frac{2}{3} - \frac{3}{8}$

Answer

$\frac{7}{24}$

Exercise 13.5.7E.42

$\frac{5}{6} - \frac{3}{4}$

Exercise 13.5.7E.43

$-\frac{11}{30} + \frac{27}{40}$

Answer

$\frac{37}{120}$

Exercise 13.5.7E.44

$-\frac{9}{20} + \frac{17}{30}$

Exercise 13.5.7E.45

$-\frac{13}{30} + \frac{25}{42}$

Answer

$\frac{17}{105}$

Exercise 13.5.7E.46

$-\frac{23}{30} + \frac{5}{48}$

Exercise 13.5.7E.47

$-\frac{39}{56} - \frac{22}{35}$

Answer

$-\frac{53}{40}$

Exercise 13.5.7E.48

$-\frac{33}{49} - \frac{18}{35}$

Exercise 13.5.7E.49

$-\frac{2}{3}-\left(-\frac{3}{4}\right)$

Answer

$\frac{1}{12}$

Exercise 13.5.7E.50

$-\frac{3}{4}-\left(-\frac{4}{5}\right)$

Exercise 13.5.7E.51

$1+\frac{7}{8}$

Answer

$\frac{15}{8}$

Exercise 13.5.7E.52

$1-\frac{3}{10}$

Exercise 13.5.7E.53

$\frac{x}{3}+\frac{1}{4}$

Answer

$\frac{4x+3}{12}$

Exercise 13.5.7E.54

$\frac{y}{2}+\frac{2}{3}$

Exercise 13.5.7E.55

$\frac{y}{4}-\frac{3}{5}$

Answer

$\frac{5y-12}{20}$

Exercise 13.5.7E.56

$\frac{x}{5}-\frac{1}{4}$

Mixed Practice

In the following exercises, simplify.

Exercise 13.5.7E.57

1. $\frac{2}{3}+\frac{1}{6}$
2. $\frac{2}{3} \div \frac{1}{6}$

Answer

1. $\frac{5}{6}$
2. 4

Exercise 13.5.7E.58

1. $-\frac{2}{5}-\frac{1}{8}$
2. $-\frac{2}{5} \cdot \frac{1}{8}$

Exercise 13.5.7E.59

1. $\frac{5n}{6} \div \frac{8}{15}$
2. $\frac{5n}{6}-\frac{8}{15}$

Answer

1. $\frac{25n}{16}$
2. $\frac{25n-16}{30}$

Exercise 13.5.7E.60

1. $\frac{3a}{8} \div \frac{7}{12}$
2. $\frac{3a}{8}-\frac{7}{12}$

Exercise 13.5.7E.61

$-\frac{3}{8} \div \left(-\frac{3}{10}\right)$

Answer

$\frac{5}{4}$

Exercise 13.5.7E.62

$-\frac{5}{12} \div \left(-\frac{5}{9}\right)$

Exercise 13.5.7E.63

$-\frac{3}{8}+\frac{5}{12}$

Answer

$\frac{1}{24}$

Exercise 13.5.7E.64

$-\frac{1}{8}+\frac{7}{12}$

Exercise 13.5.7E.65

$\frac{5}{6} - \frac{1}{9}$

Answer

$\frac{13}{18}$

Exercise 13.5.7E.66

$\frac{5}{9} - \frac{1}{6}$

Exercise 13.5.7E.67

$-\frac{7}{15} - \frac{y}{4}$

Answer

$\frac{-28-15y}{60}$

Exercise 13.5.7E.68

$-\frac{3}{8} - \frac{x}{11}$

Exercise 13.5.7E.69

$\frac{11}{12a} \cdot \frac{9a}{16}$

Answer

$\frac{33}{64}$

Exercise 13.5.7E.70

$\frac{10y}{13} \cdot \frac{8}{15y}$

Use the Order of Operations to Simplify Complex Fractions

In the following exercises, simplify.

Exercise 13.5.7E.71

$\frac{2^3+4^2}{\left(\frac{2}{3}\right)^2}$

Answer

54

Exercise 13.5.7E.72

$\frac{3^3-3^2}{\left(\frac{3}{4}\right)^2}$

Exercise 13.5.7E.73

$$\frac{\left(\frac{3}{5}\right)^2}{\left(\frac{3}{7}\right)^2}$$

Answer

$\frac{49}{25}$

Exercise 13.5.7E.74

$$\frac{\left(\frac{3}{4}\right)^2}{\left(\frac{5}{8}\right)^2}$$

Exercise 13.5.7E.75

$$\frac{2}{\frac{1}{3}+\frac{1}{5}}$$

Answer

$\frac{15}{4}$

Exercise 13.5.7E.76

$$\frac{5}{\frac{1}{4}+\frac{1}{3}}$$

Exercise 13.5.7E.77

$$\frac{\frac{7}{8}-\frac{2}{3}}{\frac{1}{2}+\frac{3}{8}}$$

Answer

$\frac{5}{21}$

Exercise 13.5.7E.78

$$\frac{\frac{3}{4}-\frac{3}{5}}{\frac{1}{4}+\frac{2}{5}}$$

Exercise 13.5.7E.79

$\frac{1}{2}+\frac{2}{3}\cdot\frac{5}{12}$

Answer

$\frac{7}{9}$

Exercise 13.5.7E.80

$\frac{1}{3}+\frac{2}{5}\cdot\frac{3}{4}$

Exercise 13.5.7E.81

$1-\frac{3}{5} \div \frac{1}{10}$

Answer

-5

Exercise 13.5.7E.82

$1-\frac{5}{6} \div \frac{1}{12}$

Exercise 13.5.7E.83

$\frac{2}{3}+\frac{1}{6}+\frac{3}{4}$

Answer

$\frac{19}{12}$

Exercise 13.5.7E.84

$\frac{2}{3}+\frac{1}{4}+\frac{3}{5}$

Exercise 13.5.7E.85

$\frac{3}{8}-\frac{1}{6}+\frac{3}{4}$

Answer

$\frac{23}{24}$

Exercise 13.5.7E.86

$\frac{2}{5}+\frac{5}{8}-\frac{3}{4}$

Exercise 13.5.7E.87

$12\left(\frac{9}{20}-\frac{4}{15}\right)$

Answer

$\frac{11}{5}$

Exercise 13.5.7E.88

$8\left(\frac{15}{16}-\frac{5}{6}\right)$

Exercise 13.5.7E.89

$\frac{\frac{5}{8}+\frac{1}{6}}{\frac{19}{24}}$

Answer

1

Exercise 13.5.7E.90

$\frac{\frac{1}{6}+\frac{3}{10}}{\frac{14}{30}}$

Exercise 13.5.7E.91

$\left(\frac{5}{9}+\frac{1}{6}\right) \div \left(\frac{2}{3}-\frac{1}{2}\right)$

Answer

$\frac{13}{3}$

Exercise 13.5.7E.92

$\left(\frac{3}{4}+\frac{1}{6}\right) \div \left(\frac{5}{8}-\frac{1}{3}\right)$

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

Exercise 13.5.7E.93

$x+\left(-\frac{5}{6}\right)$ when

1. $x=\frac{1}{3}$
2. $x=-\frac{1}{6}$

Answer

1. $-\frac{1}{2}$
2. -1

Exercise 13.5.7E.94

$x+\left(-\frac{11}{12}\right)$ when

1. $x=\frac{11}{12}$
2. $x=-\frac{3}{4}$

Exercise 13.5.7E.95

$x-\frac{2}{5}$ when

1. $x=\frac{3}{5}$
2. $x=-\frac{3}{5}$

Answer

1. $\frac{1}{5}$
2. -1

Exercise 13.5.7E.96

$x - \frac{1}{3}$ when

1. $x = \frac{2}{3}$
2. $x = -\frac{2}{3}$

Exercise 13.5.7E.97

$\frac{7}{10} - w$ when

1. $w = \frac{1}{2}$
2. $w = -\frac{1}{2}$

Answer

1. $\frac{1}{5}$
2. $\frac{6}{5}$

Exercise 13.5.7E.98

$\frac{5}{12} - w$ when

1. $w = \frac{1}{4}$
2. $w = -\frac{1}{4}$

Exercise 13.5.7E.99

$2x^2y^3$ when $x = -\frac{2}{3}$ and $y = -\frac{1}{2}$

Answer

$-\frac{1}{9}$

Exercise 13.5.7E.100

$8u^2v^3$ when $u = -\frac{3}{4}$ and $v = -\frac{1}{2}$

Exercise 13.5.7E.101

$\frac{a+b}{a-b}$ when $a = -3, b = 8$

Answer

$-\frac{5}{11}$

Exercise 13.5.7E.102

$\frac{r-s}{r+s}$ when $r = 10, s = -5$

Everyday Math

Exercise 13.5.7E.103

Decorating Laronda is making covers for the throw pillows on her sofa. For each pillow cover, she needs $\frac{1}{2}$ yard of print fabric and \frac{3}{8}\) yard of solid fabric. What is the total amount of fabric Laronda needs for each pillow cover?

Answer

$\frac{7}{8}$ yard

Exercise 13.5.7E.104

Baking Vanessa is baking chocolate chip cookies and oatmeal cookies. She needs $\frac{1}{2}$ cup of sugar for the chocolate chip cookies and $\frac{1}{4}$ of sugar for the oatmeal cookies. How much sugar does she need altogether?

Writing Exercises

Exercise 13.5.7E.105

Why do you need a common denominator to add or subtract fractions? Explain.

Answer

Answers may vary

Exercise 13.5.7E.106

How do you find the LCD of 2 fractions?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract fractions with different denominators.			
identify and use fraction operations.			
use the order of operations to simplify complex fractions.			
evaluate variable expressions with fractions.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next chapter? Why or why not?

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13.5.8: Decimals

Learning Objectives

By the end of this section, you will be able to:

- Name and write decimals
- Round decimals
- Add and subtract decimals
- Multiply and divide decimals
- Convert decimals, fractions, and percents

Note

A more thorough introduction to the topics covered in this section can be found in the *Prealgebra*chapter, **Decimals**.

Name and Write Decimals

Decimals are another way of writing **fraction**s whose denominators are powers of 10.

$$\begin{aligned} 0.1 &= \tfrac{1}{10} && 0.1 \text{ is "one tenth"} \\ 0.01 &= \tfrac{1}{100} && 0.01 \text{ is "one hundredth} \\ 0.001 &= \tfrac{1}{1{,}000} && 0.001 \text{ is "one thousandth} \\ 0.0001 &= \tfrac{1}{10{,}000} && 0.0001 \text{ is "one ten-thousandth"} \end{aligned} \tag{13.5.8.1}$$

Notice that "ten thousand" is a number larger than one, but "one ten-thousand**th**" is a number smaller than one. The "th" at the end of the name tells you that the number is smaller than one.

When we name a whole number, the name corresponds to the place value based on the powers of ten. We read 10,000 as "ten thousand" and 10,000,000 as "ten million." Likewise, the names of the **decimal** places correspond to their fraction values. Figure 13.5.8.1shows the names of the **place values** to the left and right of the decimal point.

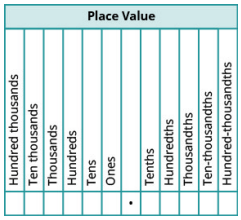

Figure 13.5.8.1: Place value of decimal numbers are shown to the left and right of the decimal point.

Exercise 13.5.8.1

Name the decimal 4.3.

Answer

Step 1. Name the number to the left of the decimal point.	4 is to the left of the decimal point.	4.3 four _____	**Step 2.** Write 'and' for the decimal point.		four and _____
Step 3. Name the 'number' part to the right of the decimal point as if it were a whole number.	3 is to the right of the decimal point.	four and three _____	**Step 4.** Name the decimal place.		four and three tenths

Exercise 13.5.8.2

Name the decimal 6.7.

Answer

six and seven tenths

Exercise 13.5.8.3

Name the decimal 5.8.

Answer

five and eight tenths

We summarize the steps needed to name a **decimal** below.

NAME A DECIMAL.

1. Name the number to the left of the decimal point.
2. Write "and" for the decimal point.
3. Name the "number" part to the right of the decimal point as if it were a whole number.
4. Name the decimal place of the last digit.

Exercise 13.5.8.4

Name the decimal: -15.571.

Answer

	-15.571
Name the number to the left of the decimal point.	negative fifteen _____
Write “and” for the decimal point.	negative fifteen and _____
Name the number to the right of the decimal point.	negative fifteen and five hundred seventy-one _____
The 1 is in the thousandths place.	negative fifteen and five hundred seventy-one thousandths

Exercise 13.5.8.5

Name the decimal: -13.461.

Answer

negative thirteen and four hundred sixty-one thousandths

Exercise 13.5.8.6

Name the decimal: -2.053.

Answer

negative two and fifty-three thousandths

When we write a check we write both the numerals and the name of the number. Let’s see how to write the decimal from the name.

Exercise 13.5.8.7: How to Write decimals

Write “fourteen and twenty-four thousandths” as a decimal.

Answer

Step 1. Look for the word ‘and’; it locates the decimal point. Place a decimal point under the word ‘and’. Translate the words before ‘and’ into the whole number and place to the left of the decimal point.		fourteen and twenty-four thousandths fourteen and twenty-four thousandths ____ . _____ 14 . _____
Step 2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.	The last word is ‘thousandths’.	14 . ____ ____ ____ tenths hundredths thousandths
Step 3. Translate the words after ‘and’ into the number to the right of the decimal point. Write the number in the spaces – putting the final digit in the last place.		14 . ____ 2 4
Step 4. Fill in zeros for empty place holders as needed.	Zeros are needed in the tenths place.	14 . 0 2 4 Fourteen and twenty-four thousandths is written 14.024.

Exercise 13.5.8.8

Write as a decimal: thirteen and sixty-eight thousandths.

Answer

13.068

Exercise 13.5.8.9

Write as a decimal: five and ninety-four thousandths.

Answer

5.094

We summarize the steps to writing a **decimal**.

WRITE A DECIMAL.

1. Look for the word “and”—it locates the decimal point.
   - Place a decimal point under the word “and.” Translate the words before “and” into the whole number and place it to the left of the decimal point.
   - If there is no “and,” write a “0” with a decimal point to its right.
2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
3. Translate the words after “and” into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.
4. Fill in zeros for place holders as needed.

Round Decimals

Rounding decimals is very much like rounding whole numbers. We will round decimals with a method based on the one we used to round whole numbers.

Exercise 13.5.8.10

Round 18.379 to the nearest hundredth.

Answer

Step 1. Locate the given place value and mark it with an arrow.		hundredths place ↓ 18.379	**Step 2.** Underline the digit to the right of the given place value.		hundredths place ↓ 18.379
Step 3. Is this digit greater than or equal to 5? **Yes:** Add 1 to the digit in the given place value. **No:** Do not change the digit in the given place value.	Because 9 is greater than or equal to 5, add 1 to the 7.	18.37 9 add 1 ↗ delete	**Step 4.** Rewrite the number, removing all digits to the right of the rounding digit.		18.38 18.38 is 18.379 rounded to the nearest hundredth.

Exercise 13.5.8.11

Round to the nearest hundredth: 1.047.

Answer

1.05

Exercise 13.5.8.12

Round to the nearest hundredth: 9.173.

Answer

9.17

We summarize the steps for rounding a **decimal** here.

ROUND DECIMALS.

1. Locate the given place value and mark it with an arrow.
2. Underline the digit to the right of the place value.
3. Is this digit greater than or equal to 5?
   - Yes—add 1 to the digit in the given place value.
   - No—do not change the digit in the given place value.
4. Rewrite the number, deleting all digits to the right of the rounding digit.

Exercise 13.5.8.13

Round 18.379 to the nearest

1. tenth
2. whole number.

Answer

Round 18.379

1. to the nearest tenth

Locate the tenths place with an arrow.	tenths place ↓ 18.379
Underline the digit to the right of the given place value.	tenths place ↓ 18.379
Because 7 is greater than or equal to 5, add 1 to the 3.	18.379 add 1 ↗ delete
Rewrite the number, deleting all digits to the right of the rounding digit.	18.4
Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest tenth is 18.4.

2. to the nearest whole number

Locate the ones place with an arrow.	ones place ↓ 18.379
Underline the digit to the right of the given place value.	ones place ↓ 18.379

Since 3 is not greater than or equal to 5, do not add 1 to the 8.	18.379 (do not add 1, delete)
Rewrite the number, deleting all digits to the right of the rounding digit.	18
	So, 18.379 rounded to the nearest whole number is 18.

Exercise 13.5.8.14

Round 6.582 to the nearest

1. hundredth
2. tenth
3. whole number.

Answer

1. 6.58
2. 6.6
3. 7

Exercise 13.5.8.15

Round 15.2175 to the nearest

1. thousandth
2. hundredth
3. tenth.

Answer

1. 15.218
2. 15.22
3. 15.2

Add and Subtract Decimals

To add or subtract decimals, we line up the decimal points. By lining up the decimal points this way, we can add or subtract the corresponding **place values**. We then add or subtract the numbers as if they were whole numbers and then place the decimal point in the sum.

ADD OR SUBTRACT DECIMALS.

1. Write the numbers so the decimal points line up vertically.
2. Use zeros as place holders, as needed.
3. Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.

Exercise 13.5.8.16

Add: $23.5+41.38$.

Answer

Write the numbers so that the decimal points line up vertically.

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline \end{array} \tag{13.5.8.2}$$

Put 0 as a placeholder after the 5 in 23.5. Remember, $\frac{5}{10}=\frac{50}{100}$, so $0.5=0.50$

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline \end{array} \tag{13.5.8.3}$$

Add the numbers as if they were whole numbers . Then place the decimal point in the sum.

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline 64.88 \end{array} \tag{13.5.8.4}$$

Exercise 13.5.8.17

Add: $4.8+11.69$.

Answer

16.49

Exercise 13.5.8.18

Add: $5.123+18.47$.

Answer

23.593

Exercise 13.5.8.19

Subtract: $20 - 14.65$.

Answer

Write the numbers so that the decimal points line up vertically.

$$\begin{array}{r} 20 - 14.65 \\ 20. \\ -14.65 \end{array}$$

Remember, 20 is a whole number, so place the decimal point after the 0.

Put zeros to the right as placeholders.

$$\begin{array}{r} 20.00 \\ -14.65 \end{array} \qquad \begin{array}{r} (13.5.8.8) \\ (13.5.8.9) \end{array}$$

Write the numbers so that the decimal points line up vertically.

$$\begin{array}{r} {}^{9}\ {}^{9} \\ 1\ \cancel{1}0\ \cancel{1}0\ \cancel{1}0 \\ \cancel{2}\ \cancel{0}.\ \cancel{0}\ \cancel{0} \\ -14.65 \\ \hline 5.35 \end{array}$$

Exercise 13.5.8.20

Subtract: $10 - 9.58$.

Answer

0.42

Exercise 13.5.8.21

Subtract: $50 - 37.42$.

Answer

12.58

Multiply and Divide Decimals

Multiplying decimals is very much like multiplying whole numbers—we just have to determine where to place the decimal point. The procedure for multiplying decimals will make sense if we first convert them to fractions and then multiply.

So let's see what we would get as the **product** of decimals by converting them to fractions first. We will do two examples side-by-side. Look for a pattern!

Table 13.5.8.1

	$\underbrace{(0.3)}_{1\text{ place}}\ \underbrace{(0.7)}_{1\text{ place}}$	$\underbrace{(0.2)}_{1\text{ place}}\ \underbrace{(0.46)}_{2\text{ places}}$
Convert to fractions.	$\frac{3}{10} \cdot \frac{7}{10}$	$\frac{2}{10} \cdot \frac{46}{100}$
Multiply.	$\frac{21}{100}$	$\frac{92}{1000}$
Convert to decimals.	$\underbrace{0.21}_{2\text{ places}}$	$\underbrace{0.092}_{3\text{ places}}$

Notice, in the first example, we multiplied two numbers that each had one digit after the decimal point and the product had two decimal places. In the second example, we multiplied a number with one decimal place by a number with two decimal places and the product had three decimal places.

We multiply the numbers just as we do whole numbers, temporarily ignoring the decimal point. We then count the number of decimal points in the factors and that sum tells us the number of decimal places in the product.

The rules for multiplying positive and negative numbers apply to decimals, too, of course!

When *multiplying* two numbers,

- if their signs are the *same* the product is *positive*.
- if their signs are *different* the product is *negative*.

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. Finally, we write the product with the appropriate sign.

MULTIPLY DECIMALS.

1. Determine the sign of the product.
2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
3. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
4. Write the product with the appropriate sign.

Exercise 13.5.8.22

Multiply: $(-3.9)(4.075)$.

Answer

	$(-3.9)(4.075)$
The signs are different. The product will be negative.	
Write in vertical format, lining up the numbers on the right.	4.075 × 3.9
Multiply.	4.075 × 3.9 36675 12225 158925
Add the number of decimal places in the factors $(1+3)$. (–3.9) (4.075) 1 place 3 places Place the decimal point 4 places from the right.	4.075 × 3.9 36675 12225 15.8925 4 places
The signs are different, so the product is negative.	$(-3.9)(4.075) = -15.8925$

Exercise 13.5.8.23

Multiply: $-4.5(6.107)$

Answer

-27.4815

Exercise 13.5.8.24

Multiply: $-10.79(8.12)$.

Answer

-87.6148

In many of your other classes, especially in the sciences, you will multiply decimals by powers of 10 (10, 100, 1000, etc.). If you multiply a few products on paper, you may notice a pattern relating the number of zeros in the power of 10 to number of decimal places we move the decimal point to the right to get the product.

MULTIPLY A DECIMAL BY A POWER OF TEN.

1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
2. Add zeros at the end of the number as needed.

Exercise 13.5.8.25

Multiply 5.63

1. by 10
2. by 100
3. by 1,000.

Answer

By looking at the number of zeros in the multiple of ten, we see the number of places we need to move the decimal to the right.

ⓐ

	$5.63(10)$
There is 1 zero in 10, so move the decimal point 1 place to the right.	5.63 56.3

ⓑ

	$5.63(100)$
There are 2 zeros in 100, so move the decimal point 2 places to the right.	There are 2 zeros in 100, so move the decimal point 2 places to the right. 5.63 (100) 5.63 563

ⓒ

There are 3 zeros in 1,000, so move the decimal point 3 places to the right.	5.63
A zero must be added at the end.	5,630

Exercise 13.5.8.26

Multiply 2.58

1. by 10
2. by 100
3. by 1,000.

Answer

1. 25.8
2. 258
3. 2,580

Exercise 13.5.8.27

Multiply 14.2

1. by 10
2. by 100
3. by 1,000.

Answer

1. 142
2. 1,420
3. 14,200

Just as with multiplication, **division** of decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed.

To divide decimals, determine what power of 10 to multiply the denominator by to make it a whole number. Then multiply the numerator by that same power of 10. Because of the equivalent fractions property, we haven't changed the value of the fraction! The effect is to move the decimal points in the numerator and denominator the same number of places to the right. For example:

$$\begin{array}{c} \frac{0.8}{0.4} \\ \frac{0.8(10)}{0.4(10)} \\ \frac{8}{4} \end{array} \tag{13.5.8.16}$$

We use the rules for dividing positive and negative numbers with decimals, too. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

We review the notation and vocabulary for division:

$$\underset{\text{dividend}}{a} \div \underset{\text{divisor}}{b} = \underset{\text{quotient}}{c} \qquad \begin{array}{r} \overset{\text{quotient}}{c} \\ \underset{\text{divisor}}{b}\,\overline{)\,\underset{\text{dividend}}{a}} \end{array} \tag{13.5.8.17}$$

We'll write the steps to take when dividing decimals, for easy reference.

DIVIDE DECIMALS.

1. Determine the sign of the quotient.
2. Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places—adding zeros as needed.
3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
4. Write the quotient with the appropriate sign.

Exercise 13.5.8.28

Divide: $-25.65 \div (-0.06)$.

Answer

Remember, you can "move" the decimals in the divisor and dividend because of the Equivalent Fractions Property.

	$-25.65 \div (-0.06)$
The signs are the same.	The quotient is positive.
Make the divisor a whole number by "moving" the decimal point all the way to the right.	
"Move" the decimal point in the dividend the same number of places.	$0.06\overline{)25.65}$
Divide. Place the decimal point in the quotient above the decimal point in the dividend.	$\begin{array}{r} 427.5 \\ 006\overline{)2565.0} \\ \underline{-24} \\ 16 \\ \underline{-12} \\ 45 \\ \underline{-42} \\ 30 \\ \underline{30} \end{array}$
Write the quotient with the appropriate sign.	$-25.65 \div (-0.06) = 427.5$

Exercise 13.5.8.29

Divide: $-23.492 \div (-0.04)$.

Answer

687.3

Exercise 13.5.8.30

Divide: $-4.11 \div (-0.12)$.

Answer

34.25

A common application of dividing whole numbers into decimals is when we want to find the price of one item that is sold as part of a multi-pack. For example, suppose a case of 24 water bottles costs $3.99. To find the price of one water bottle, we would divide $3.99 by 24. We show this division in Exercise 13.5.8.31 In calculations with money, we will round the answer to the nearest cent (hundredth).

Exercise 13.5.8.31

Divide: $3.99 \div 24$.

Answer

.99 divided by 24 is given. A long division problem is set up with 24 dividing 3.99. A table is given with directions on the left and the mathematical steps on the right. The first step reads “Place the decimal point in the quotient above the decimal point in the dividend. Divide as usual. When do we stop? Since this division involves money, we round it to the nearest cent (hundredth). To do this, we must carry the division to the thousandths place.” To the right of this, we have a long division problem set up with 24 dividing 3.990. The quotient is given as 0.166. To show the work, below 3.990 it reads 24, solid horizontal line, 159, 144, solid horizontal line, 150, 144, solid horizontal line, and finally 6. The fifth step reads “Round to the nearest cent.” To the right of this, we have $0.166 is approximately equal to $0.17 and hence >.99 divided by 24 is $0.17.">

	$3.99 \div 24$
Place the decimal point in the quotient above the decimal point in the dividend.	
Divide as usual. When do we stop? Since this division involves money, we round it to the nearest cent (hundredth.) To do this, we must carry the division to the thousandths place.	$$\begin{array}{r} 0.166 \\ 24\overline{)3.990} \\ \underline{24} \\ 159 \\ \underline{144} \\ 150 \\ \underline{144} \\ 6 \end{array}$$
Round to the nearest cent.	$0.166 \approx $0.17 $3.99 \div 2 \approx $0.17

Exercise 13.5.8.32

Divide: $6.99 \div 36$.

Answer

$0.19

Exercise 13.5.8.33

Divide: $4.99 \div 12$.

Answer

$0.42

Convert Decimals, Fractions, and Percents

We convert decimals into fractions by identifying the place value of the last (farthest right) digit. In the decimal 0.03 the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03.

$$00.03 = \frac{3}{100} \tag{13.5.8.18}$$

Notice, when the number to the left of the decimal is zero, we get a fraction whose numerator is less than its denominator. Fractions like this are called **proper fractions**.

The steps to take to convert a decimal to a fraction are summarized in the procedure box.

CONVERT A DECIMAL TO A PROPER FRACTION.

1. Determine the place value of the final digit.
2. Write the fraction.
   - numerator—the “numbers” to the right of the decimal point
   - denominator—the place value corresponding to the final digit

Exercise 13.5.8.34

Write 0.374 as a fraction.

Answer

	0.374
Determine the place value of the final digit.	$$\begin{array}{lll} 0.3 & 7 & 4 \\ \text{tenths} & \text{hundredths} & \text{thousandths} \end{array}$$
Write the fraction for 0.374: • The numerator is 374. • The denominator is 1,000.	$\frac{374}{1000}$
Simplify the fraction.	$\frac{2 \cdot 187}{2 \cdot 500}$
Divide out the common factors.	$\frac{187}{500}$ so, $0.374 = \frac{187}{500}$

Did you notice that the number of zeros in the denominator of $\frac{374}{1000}$ is the same as the number of decimal places in 0.374?

Exercise 13.5.8.35

Write 0.234 as a fraction.

Answer

$\frac{117}{500}$

Exercise 13.5.8.36

Write 0.024 as a fraction.

Answer

$\frac{3}{125}$

We've learned to convert decimals to fractions. Now we will do the reverse—convert fractions to decimals. Remember that the fraction bar means division. So $\frac{4}{5}$ can be written $4 \div 5$ or $5\overline{)4}$. This leads to the following method for converting a fraction to a decimal.

CONVERT A FRACTION TO A DECIMAL.

To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

Exercise 13.5.8.37

Write $-\frac{5}{8}$ as a decimal.

Answer

Since a fraction bar means division, we begin by writing $\frac{5}{8}$ as $8\overline{)5}$. Now divide.

$$\begin{array}{r} 0.625 \\ 8\overline{)5.000} \\ \underline{48} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ \end{array}$$

so, $-\frac{5}{8} = -0.625$

Exercise 13.5.8.38

Write $-\frac{7}{8}$ as a decimal.

Answer

−0.875

Exercise 13.5.8.39

Write $-\frac{3}{8}$ as a decimal.

Answer

−0.375

When we divide, we will not always get a zero remainder. Sometimes the quotient ends up with a decimal that repeats. A **repeating decimal** is a decimal in which the last digit or group of digits repeats endlessly. A bar is placed over the repeating block of digits to indicate it repeats.

REPEATING DECIMAL

A **repeating decimal** is a decimal in which the last digit or group of digits repeats endlessly.

A bar is placed over the repeating block of digits to indicate it repeats.

Exercise 13.5.8.40

Write $\frac{43}{22}$ as a decimal.

Answer

Divide 43 by 22.

$$\begin{array}{r} 1.95454 \\ 22\overline{)43.00000} \\ 22 \\ \hline 210 \\ 198 \\ \hline 120 \\ 110 \\ \hline 100 \\ 88 \\ \hline 120 \\ 110 \\ \hline 100 \\ 88 \\ \hline \cdots \end{array}$$

120 repeats

The pattern repeats, so the numbers in the quotient will repeat as well.

100 repeats

so, $\frac{43}{22} = 1.95\overline{4}$

Exercise 13.5.8.41

Write $\frac{27}{11}$ as a decimal.

Answer

$2.\overline{45}$

Exercise 13.5.8.42

Write $\frac{51}{22}$ as a decimal.

Answer

$2.3\overline{18}$

Sometimes we may have to simplify expressions with fractions and decimals together.

Exercise 13.5.8.43

Simplify: $\frac{7}{8} + 6.4$.

Answer

First we must change one number so both numbers are in the same form. We can change the fraction to a decimal, or change the decimal to a fraction. Usually it is easier to change the fraction to a decimal.

		$\frac{7}{8} + 6.4$
Change $\frac{7}{8}$ to a decimal.	$\begin{array}{r} 0.875 \\ 8\overline{)7.000} \\ 64 \\ \hline 60 \\ 56 \\ \hline 40 \\ 40 \\ \hline 0 \end{array}$	
Add.		$0.875 + 6.4$
		7.275
		So, $\frac{7}{8} + 6.4 = 7.275$

Exercise 13.5.8.44

Simplify: $\frac{3}{8} + 4.9$.

Answer

5.275

Exercise 13.5.8.45

Simplify: $5.7 + \frac{13}{20}$.

Answer

6.35

A **percent** is a ratio whose denominator is 100. Percent means per hundred. We use the percent symbol, %, to show percent.

PERCENT

A **percent** is a ratio whose denominator is 100.

Since a percent is a ratio, it can easily be expressed as a fraction. Percent means per 100, so the denominator of the fraction is 100. We then change the fraction to a decimal by dividing the numerator by the denominator.

	6%	78%	135%
Write as a ratio with denominator 100.	$\frac{6}{100}$	$\frac{78}{100}$	$\frac{135}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.06	0.78	1.35

(13.5.8.19)

Do you see the pattern? *To convert a percent number to a decimal number, we move the decimal point two places to the left.*

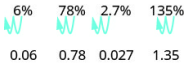

Figure 13.5.8.2

Exercise 13.5.8.46

Convert each percent to a decimal:

1. 62%
2. 135%
3. 35.7%.

Answer

1.	
	62%
Move the decimal point two places to the left.	0.62
2.	
	135%
Move the decimal point two places to the left.	1.35
3.	
	5.7%
Move the decimal point two places to the left.	0.057

Exercise 13.5.8.47

Convert each percent to a decimal:

1. 9%
2. 87%
3. 3.9%.

Answer

1. 0.09
2. 0.87
3. 0.039

Exercise 13.5.8.48

Convert each percent to a decimal:

1. 3%
2. 91%
3. 8.3%.

Answer

1. 0.03
2. 0.91
3. 0.083

Converting a decimal to a percent makes sense if we remember the definition of percent and keep place value in mind.

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent.

	0.83	1.05	0.075
Write as a fraction	$\frac{83}{100}$	$1\frac{5}{100}$	$\frac{75}{1000}$
The denominator is 100.		$\frac{105}{100}$	$\frac{7.5}{100}$
Write the ratio as a percent.	83%	105%	7.5%

(13.5.8.20)

Recognize the pattern? *To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.*

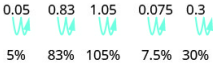

Figure 13.5.8.3

Exercise 13.5.8.49

Convert each decimal to a percent:

1. 0.51
2. 1.25
3. 0.093.

Answer

1.	
	0.51
Move the decimal point two places to the right.	51

2.	
	1.25
Move the decimal point two places to the right.	125

3.	
	0.093
Move the decimal point two places to the right.	9.3

Exercise 13.5.8.50

Convert each decimal to a percent:

1. 0.17
2. 1.75
3. 0.0825

Answer

1. 17%
2. 175%
3. 8.25%

Exercise 13.5.8.51

Convert each decimal to a percent:

1. 0.41
2. 2.25
3. 0.0925.

Answer

1. 41%
2. 225%
3. 9.25%

Key Concepts

- **Name a Decimal**
  1. Name the number to the left of the decimal point.
  2. Write ”and” for the decimal point.
  3. Name the “number” part to the right of the decimal point as if it were a whole number.
  4. Name the decimal place of the last digit.
- **Write a Decimal**
  1. Look for the word ‘and’—it locates the decimal point. Place a decimal point under the word ‘and.’ Translate the words before ‘and’ into the whole number and place it to the left of the decimal point. If there is no “and,” write a “0” with a decimal point to its right.
  2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
  3. Translate the words after ‘and’ into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.
  4. Fill in zeros for place holders as needed.
- **Round a Decimal**
  1. Locate the given place value and mark it with an arrow.
  2. Underline the digit to the right of the place value.
  3. Is this digit greater than or equal to 5? Yes—add 1 to the digit in the given place value. No—do not change the digit in the given place value.
  4. Rewrite the number, deleting all digits to the right of the rounding digit.
- **Add or Subtract Decimals**
  1. Write the numbers so the decimal points line up vertically.
  2. Use zeros as place holders, as needed.
  3. Add or subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.
- **Multiply Decimals**
  1. Determine the sign of the product.

2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
3. Place the decimal point. The number of decimal places in the product is the sum of the decimal places in the factors.
4. Write the product with the appropriate sign.

- **Multiply a Decimal by a Power of Ten**
  1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
  2. Add zeros at the end of the number as needed.
- **Divide Decimals**
  1. Determine the sign of the quotient.
  2. Make the divisor a whole number by “moving” the decimal point all the way to the right. “Move” the decimal point in the dividend the same number of places - adding zeros as needed.
  3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
  4. Write the quotient with the appropriate sign.
- **Convert a Decimal to a Proper Fraction**
  1. Determine the place value of the final digit.
  2. Write the fraction: numerator—the ‘numbers’ to the right of the decimal point; denominator—the place value corresponding to the final digit.
- **Convert a Fraction to a Decimal** Divide the numerator of the fraction by the denominator.

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13.5.8E: Exercises

Practice Makes Perfect

Needs to be integrated from https://cnx.org/contents/CImQfPDv@8.46:NciJ1qa0@19/1-7-Decimals

Name and Write Decimals

In the following exercises, write as a decimal.

531.

Twenty-nine and eighty-one hundredths

532.

Sixty-one and seventy-four hundredths

533.

Seven tenths

534.

Six tenths

535.

Twenty-nine thousandth

536.

Thirty-five thousandths

537.

Negative eleven and nine ten-thousandths

538.

Negative fifty-nine and two ten-thousandths

In the following exercises, name each decimal.

539.

5.5

540.

14.02

541.

8.71

542.

2.64

543.

0.002

544.

0.479

545.

−17.9−17.9

546.

−31.4−31.4

Round Decimals

In the following exercises, round each number to the nearest tenth.

547.

0.67

548.

0.49

549.

2.84

550.

4.63

In the following exercises, round each number to the nearest hundredth.

551.

0.845

552.

0.761

553.

0.299

554.

0.697

555.

4.098

556.

7.096

In the following exercises, round each number to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

557.

5.781

558.

1.6381

559.

63.479

560.

84.28184.281

Add and Subtract Decimals

In the following exercises, add or subtract.

561.

16.92+7.5616.92+7.56

562.

248.25−91.29248.25−91.29

563.

21.76−30.9921.76−30.99

564.

38.6+13.6738.6+13.67

565.

−16.53−24.38−16.53−24.38

566.

−19.47−32.58−19.47−32.58

567.

−38.69+31.47−38.69+31.47

568.

29.83+19.7629.83+19.76

569.

72.5−10072.5−100

570.

86.2−10086.2−100

571.

15+0.7315+0.73

572.

27+0.8727+0.87

573.

91.95−(−10.462)91.95−(−10.462)

574.

94.69−(−12.678)94.69−(−12.678)

575.

55.01−3.755.01−3.7

576.

59.08−4.659.08−4.6

577.

2.51−7.42.51−7.4

578.

3.84−6.13.84−6.1

Multiply and Divide Decimals

In the following exercises, multiply.

579.

(0.24)(0.6)(0.24)(0.6)

580.

(0.81)(0.3)(0.81)(0.3)

581.

(5.9)(7.12)(5.9)(7.12)

582.

(2.3)(9.41)(2.3)(9.41)

583.

(−4.3)(2.71)(−4.3)(2.71)

584.

(−8.5)(1.69)(−8.5)(1.69)

585.

(−5.18)(−65.23)(−5.18)(−65.23)

586.

(−9.16)(−68.34)(−9.16)(−68.34)

587.

(0.06)(21.75)(0.06)(21.75)

588.

(0.08)(52.45)(0.08)(52.45)

589.

(9.24)(10)(9.24)(10)

590.

(6.531)(10)(6.531)(10)

591.

(55.2)(1000)(55.2)(1000)

592.

(99.4)(1000)(99.4)(1000)

In the following exercises, divide.

593.

4.75÷254.75÷25

594.

12.04÷4312.04÷43

595.

$117.25÷48$117.25÷48

596.

$109.24÷36$109.24÷36

597.

0.6÷0.20.6÷0.2

598.

0.8÷0.40.8÷0.4

599.

1.44÷(−0.3)1.44÷(−0.3)

600.

1.25÷(−0.5)1.25÷(−0.5)

601.

−1.75÷(−0.05)−1.75÷(−0.05)

602.

−1.15÷(−0.05)−1.15÷(−0.05)

603.

5.2÷2.55.2÷2.5

604.

6.5÷3.256.5÷3.25

605.

11÷0.5511÷0.55

606.

14÷0.3514÷0.35

Convert Decimals, Fractions and Percents

In the following exercises, write each decimal as a fraction.

607.

0.04

608.

0.19

609.

0.52

610.

0.78

611.

1.25

612.

1.35

613.

0.375

614.

0.464

615.

0.095

616.

0.085

In the following exercises, convert each fraction to a decimal.

617.

17201720

618.

13201320

619.

114114

620.

174174

621.

−31025−31025

622.

−28425−28425

623.

15111511

624.

18111811

625.

1511115111

626.

2511125111

627.

2.4+582.4+58

628.

3.9+9203.9+920

In the following exercises, convert each percent to a decimal.

629.

1%

630.

2%

631.

63%

632.

71%

633.

150%

634.

250%

635.

21.4%

636.

39.3%

637.

7.8%

638.

6.4%

In the following exercises, convert each decimal to a percent.

639.

0.01

640.

0.03

641.

1.35

642.

1.56

643.

3

644.

4

645.

0.0875

646.

0.0625

647.

2.254

648.

2.317

Everyday Math

649.

Salary Increase Danny got a raise and now makes $58,965.95 a year. Round this number to the nearest
ⓐ dollar
ⓑ thousand dollars
ⓒ ten thousand dollars.

650.

New Car Purchase Selena's new car cost $23,795.95. Round this number to the nearest
ⓐ dollar
ⓑ thousand dollars
ⓒ ten thousand dollars.

651.

Sales Tax Hyo Jin lives in San Diego. She bought a refrigerator for $1,624.99 and when the clerk calculated the sales tax it came out to exactly $142.186625. Round the sales tax to the nearest
ⓐ penny and
ⓑ dollar.

652.

Sales Tax Jennifer bought a $1,038.99 dining room set for her home in Cincinnati. She calculated the sales tax to be exactly $67.53435. Round the sales tax to the nearest
ⓐ penny and
ⓑ dollar.

653.

Paycheck Annie has two jobs. She gets paid $14.04 per hour for tutoring at City College and $8.75 per hour at a coffee shop. Last week she tutored for 8 hours and worked at the coffee shop for 15 hours.
ⓐ How much did she earn?
ⓑ If she had worked all 23 hours as a tutor instead of working both jobs, how much more would she have earned?

654.

Paycheck Jake has two jobs. He gets paid $7.95 per hour at the college cafeteria and $20.25 at the art gallery. Last week he worked 12 hours at the cafeteria and 5 hours at the art gallery.

ⓐ How much did he earn?

ⓑ If he had worked all 17 hours at the art gallery instead of working both jobs, how much more would he have earned?

Writing Exercises

655.

How does knowing about US money help you learn about decimals?

656.

Explain how you write “three and nine hundredths” as a decimal.

657.

Without solving the problem “44 is 80% of what number” think about what the solution might be. Should it be a number that is greater than 44 or less than 44? Explain your reasoning.

658.

When the Szetos sold their home, the selling price was 500% of what they had paid for the house 30 years ago. Explain what 500% means in this context.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
name and write decimals.			
round decimals.			
add and subtract decimals.			
multiply and divide decimals.			
convert decimals, fractions, and percents.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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13.5.9: The Real Numbers

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with square roots
- Identify integers, rational numbers, irrational numbers, and real numbers
- Locate fractions on the number line
- Locate decimals on the number line

Note

A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapters, **Decimals** and **Properties of Real Numbers**.

Simplify Expressions with Square Roots

Remember that when a number n is multiplied by itself, we write n^2 and read it "n squared." The result is called the **square** of n. For example,

$$\begin{array}{ll} 8^2 & \text{read '8 squared'} \\ 64 & 64 \text{ is called the square of } 8\text{ .} \end{array}$$

Similarly, 121 is the square of 11, because 11^2 is 121.

SQUARE OF A NUMBER

If $n^2 = m$, then m is the **square** of n.

Note

Doing the Manipulative Mathematics activity "Square Numbers" will help you develop a better understanding of perfect square numbers.

Complete the following table to show the squares of the counting numbers 1 through 15.

Number	n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Square	n^2								64			121				

Figure 13.5.9.1

The numbers in the second row are called perfect square numbers. It will be helpful to learn to recognize the perfect square numbers.

The squares of the counting numbers are positive numbers. What about the squares of negative numbers? We know that when the signs of two numbers are the same, their product is positive. So the square of any negative number is also positive.

$$(-3)^2 = 9 \quad (-8)^2 = 64 \quad (-11)^2 = 121 \quad (-15)^2 = 225$$

Did you notice that these squares are the same as the squares of the positive numbers?

Sometimes we will need to look at the relationship between numbers and their squares in reverse. Because $10^2 = 100$, we say 100 is the square of 10. We also say that 10 is a *square root* of 100. A number whose square is mm is called a **square root** of m.

SQUARE ROOT OF A NUMBER

If $n^2 = m$, then n is a **square root** of m.

Notice $(-10)^2 = 100$ also, so -10 is also a square root of 100. Therefore, both 10 and -10 are square roots of 100.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? The **radical sign**, $\sqrt{m}$, denotes the positive square root. The positive square root is called the **principal square root**. When we use the radical sign that always means we want the principal square root.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

SQUARE ROOT NOTATION

$\sqrt{m}$ is read "the square root of m"

radical sign $\rightarrow \sqrt{m} \leftarrow$ radicand

If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

The square root of m, $\sqrt{m}$, is the positive number whose square is m.

Since 10 is the principal square root of 100, we write $\sqrt{100} = 10$. You may want to complete the following table to help you recognize square roots.

$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$
									10					

Figure 13.5.9.2

Exercise 13.5.9.1

Simplify:

1. $\sqrt{25}$
2. $\sqrt{121}$

Answer

1. $\sqrt{25} = 5$, since $5^2 = 25$.
2. $\sqrt{121} = 11$, since $11^2 = 121$.

Exercise 13.5.9.2

Simplify:

1. $\sqrt{36}$
2. $\sqrt{169}$

Answer

1. $\sqrt{36} = 6$
2. $\sqrt{169} = 13$

Exercise 13.5.9.3

Simplify:

1. $\sqrt{16}$
2. $\sqrt{196}$

Answer

1. $\sqrt{16} = 4$
2. $\sqrt{196} = 14$

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{100}=10$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{100}=-10$. We read $-\sqrt{100}$ as "the opposite of the square root of 10."

Exercise 13.5.9.4

Simplify:

1. $-\sqrt{9}$
2. $-\sqrt{144}$

Answer

1. $-\sqrt{9}=-3$, since the negative is in front of the radical sign.
2. $-\sqrt{144}=-12$, since the negative is in front of the radical sign.

Exercise 13.5.9.5

Simplify:

1. $-\sqrt{4}$
2. $-\sqrt{225}$

Answer

1. $-\sqrt{4}=-2$
2. $-\sqrt{225}=-15$

Exercise 13.5.9.6

Simplify:

1. $-\sqrt{81}$
2. $-\sqrt{100}$

Answer

1. $-\sqrt{81}=-9$
2. $-\sqrt{100}=-10$

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

We have already described numbers as ***counting numbers***, ***whole numbers***, and ***integers***. What is the difference between these types of numbers?

Counting numbers	$1, 2, 3, 4, \ldots$
Whole numbers	$0, 1, 2, 3, 4, \ldots$
Integers	$\cdots -3, -2, -1, 0, 1, 2, 3, \ldots$

What type of numbers would we get if we started with all the integers and then included all the fractions? The numbers we would have form the set of rational numbers. A **rational number** is a number that can be written as a ratio of two integers.

RATIONAL NUMBER

A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$

A rational number can be written as the ratio of two integers.

All signed fractions, such as $\frac{4}{5}, -\frac{7}{8}, \frac{13}{4}, -\frac{20}{3}$ are rational numbers. Each numerator and each denominator is an integer.

Are integers rational numbers? To decide if an integer is a rational number, we try to write it as a ratio of two integers. Each integer can be written as a ratio of integers in many ways. For example, 3 is equivalent to $\frac{3}{1}, -\frac{6}{2}, \frac{9}{3}, \frac{12}{4}, -\frac{15}{5} \ldots$

An easy way to write an integer as a ratio of integers is to write it as a fraction with denominator one.

$$3 = \frac{3}{1} \quad -8 = -\frac{8}{1} \quad 0 = \frac{0}{1}$$

Since any integer can be written as the ratio of two integers, *all integers are rational numbers*! Remember that the counting numbers and the whole numbers are also integers, and so they, too, are rational.

What about decimals? Are they rational? Let's look at a few to see if we can write each of them as the ratio of two integers.

We've already seen that integers are rational numbers. The integer -8 could be written as the decimal -8.0. So, clearly, some decimals are rational.

Think about the decimal 7.3. Can we write it as a ratio of two integers? Because 7.3 means $7\frac{3}{10}$, we can write it as an improper fraction, $\frac{73}{10}$. So 7.3 is the ratio of the integers 73 and 10. It is a rational number.

In general, any decimal that ends after a number of digits (such as 7.3 or -1.2684) is a rational number. We can use the place value of the last digit as the denominator when writing the decimal as a fraction.

? Exercise 13.5.9.7

Write as the ratio of two integers:

1. -27
2. 7.31

Answer

1. Writing -27 as a fraction with denominator 1, we get $\frac{-27}{1}$.
2. For 7.31, we have two ways to think of it. We could see that it is equal to 731 hundredths or $\frac{731}{100}$,

   or we could start by rewriting 7.31 as a mixed number. That is, $7.31 = 7\frac{31}{100}$. Then we can convert this mixed number to an improper fraction to get $\frac{731}{100}$.

Since both numbers can be written as the ratio of two integers, we see that -27 and 7.31 are both rational numbers.

? Exercise 13.5.9.8

Write as the ratio of two integers:

1. -24
2. 3.57

Answer

1. $-24 = \frac{-24}{1}$
2. $3.57 = \frac{357}{100}$

Exercise 13.5.9.9

Write as the ratio of two integers:

1. -19
2. 8.41

Answer

1. $-19 = \dfrac{-19}{1}$
2. $8.41 = \dfrac{841}{100}$

Let's look at the decimal form of the numbers we know are rational.

We have seen that *every integer is a rational number*, since $a = \dfrac{a}{1}$ for any integer, a. We can also change any integer to a decimal by adding a decimal point and a zero.

Integer	-2	-1	0	1	2	3
Decimal form	-2.0	-1.0	0.0	1.0	2.0	3.0

These decimal numbers stop.

We have also seen that *every fraction is a rational number*. Look at the decimal form of the fractions we considered above.

Ratio of integers	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$
The decimal form	0.8	-0.875	3.25	$-6.666\ldots$
			$-6.\overline{6}$	

These decimal numbers either stop or repeat.

What do these examples tell us?

Every rational number can be written both as a ratio of integers, ($\dfrac{p}{q}$, where p and q are integers and $q \neq 0$), and as a decimal that either stops or repeats.

Here are the numbers we looked at above expressed as a ratio of integers and as a decimal:

Table 13.5.9.1

Fractions					Integers					
Number	$\frac{4}{5}$	$\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$	-2	-1	0	1	2	3
Ratio of Integers	$\frac{4}{5}$	$\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$	$-\frac{2}{1}$	$-\frac{1}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$
Decimal Form	0.8	-0.875	3.25	$-6.\overline{6}$	-2.0	-1.0	0.0	1.0	2.0	3.0

RATIONAL NUMBER

A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$

Its decimal form stops or repeats.

Are there any decimals that do not stop or repeat? Yes!

The number π (the Greek letter *pi*, pronounced "pie"), which is very important in describing circles, has a decimal form that does not stop or repeat.

$$\pi = 3.141592654\ldots \tag{13.5.9.1}$$

We can even create a decimal pattern that does not stop or repeat, such as

$$2.01001000100001\ldots \tag{13.5.9.2}$$

Numbers whose decimal form does not stop or repeat cannot be written as a fraction of integers. We call these numbers irrational.

IRRATIONAL NUMBER

An **irrational number** is a number that cannot be written as the ratio of two integers.

Its decimal form does not stop and does not repeat.

Let's summarize a method we can use to determine whether a number is rational or irrational.

RATIONAL OR IRRATIONAL?

If the decimal form of a number

- *repeats or stops*, the number is **rational**.
- *does not repeat and does not stop*, the number is **irrational**.

Exercise 13.5.9.10

Given the numbers $0.58\overline{3}, 0.47, 3.605551275\ldots$ list the

1. rational numbers
2. irrational numbers.

Answer

1.	Look for decimals that repeat or stop	The 3 repeats in $0.58\overline{3}$. The decimal 0.47 stops after the 7. So $0.58\overline{3}$ and $0.47$are rational
2.	Look for decimals that repeat or stop	$3.605551275\ldots$ has no repeating block of digits and it does not stop. So $3.605551275\ldots$ is irrational.

Exercise 13.5.9.11

For the given numbers list the

1. rational numbers
2. irrational numbers: $0.29, 0.81\overline{6}, 2.515115111\ldots.$

Answer

1. $0.29, 0.81\overline{6}$
2. $2.515115111\ldots.$

Exercise 13.5.9.12

For the given numbers list the

1. rational numbers
2. irrational numbers: $2.6\overline{3}, 0.125, 0.418302\ldots$

Answer

1. $2.6\overline{3}, 0.125$

2. $0.418302\ldots$

Exercise 13.5.9.13

For each number given, identify whether it is rational or irrational:

1. $\sqrt{36}$
2. $\sqrt{44}$

Answer

1. Recognize that 36 is a perfect square, since $6^2 = 36$. So $\sqrt{36} = 6$, therefore $\sqrt{36}$ is rational.
2. Remember that $6^2 = 36$ and $7^2 = 49$, so 44 is not a perfect square. Therefore, the decimal form of $\sqrt{44}$ will never repeat and never stop, so $\sqrt{44}$ is irrational.

Exercise 13.5.9.14

For each number given, identify whether it is rational or irrational:

1. $\sqrt{81}$
2. $\sqrt{17}$

Answer

1. rational
2. irrational

Exercise 13.5.9.15

For each number given, identify whether it is rational or irrational:

1. $\sqrt{116}$
2. $\sqrt{121}$

Answer

1. irrational
2. rational

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. The irrational numbers are numbers whose decimal form does not stop and does not repeat. When we put together the rational numbers and the irrational numbers, we get the set of **real numbers**.

REAL NUMBER

A **real number** is a number that is either rational or irrational.

All the numbers we use in elementary algebra are real numbers. Figure 13.5.9.3illustrates how the number sets we've discussed in this section fit together.

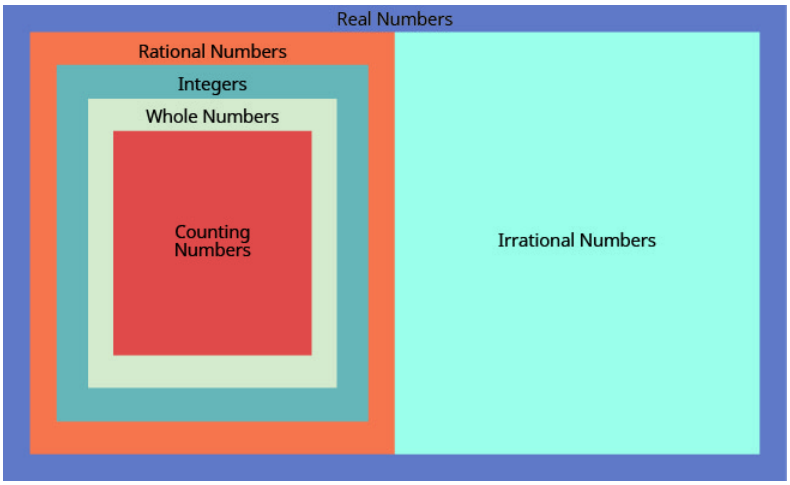


Figure 13.5.9.3: This chart shows the number sets that make up the set of real numbers. Does the term "real numbers" seem strange to you? Are there any numbers that are not "real," and, if so, what could they be?

Can we simplify $\sqrt{-25}$? Is there a number whose square is -25?

$$(\quad)^2 = -25?$$

None of the numbers that we have dealt with so far has a square that is -25. Why? Any positive number squared is positive. Any negative number squared is positive. So we say there is no real number equal to $\sqrt{-25}$.

The square root of a negative number is not a real number.

? Exercise 13.5.9.16

For each number given, identify whether it is a real number or not a real number:

1. $\sqrt{-169}$
2. $-\sqrt{64}$

Answer

1. There is no real number whose square is -169. Therefore, $\sqrt{-169}$ is not a real number.
2. Since the negative is in front of the radical, $-\sqrt{64}$ is -8, Since -8 is a real number, $-\sqrt{64}$ is a real number.

? Exercise 13.5.9.17

For each number given, identify whether it is a real number or not a real number:

1. $\sqrt{-196}$
2. $-\sqrt{81}$

Answer

1. not a real number
2. real number

? Exercise 13.5.9.18

For each number given, identify whether it is a real number or not a real number:

1. $-\sqrt{49}$
2. $\sqrt{-121}$

Answer

1. real number

2. not a real number

Exercise 13.5.9.19

Given the numbers $-7, \frac{14}{5}, 8, \sqrt{5}, 5.9, \sqrt{64}$ list the

1. whole numbers
2. integers
3. rational numbers
4. irrational numbers
5. real numbers

Answer

1. Remember, the whole numbers are 0, 1, 2, 3, … and 8 is the only whole number given.
2. The integers are the whole numbers, their opposites, and 0. So the whole number 8 is an integer, and -7 is the opposite of a whole number so it is an integer, too. Also, notice that 64 is the square of 8 so $-\sqrt{64} = -8$. So the integers are $-7, 8, \sqrt{64}$.
3. Since all integers are rational, then $-7, 8, -\sqrt{64}$ are rational. Rational numbers also include fractions and decimals that repeat or stop, so $\frac{14}{5}$ and 5.9 are rational. So the list of rational numbers is $-7, \frac{14}{5}, 8, 5.9, \sqrt{64}$
4. Remember that 5 is not a perfect square, so $\sqrt{5}$ is irrational.
5. All the numbers listed are real numbers.

Exercise 13.5.9.20

For the given numbers, list the

1. whole numbers
2. integers
3. rational numbers
4. irrational numbers
5. real numbers: $-3, -\sqrt{2}, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$

Answer

1. $4, \sqrt{49}$.
2. $-3, 4, \sqrt{49}$
3. $-3, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$
4. $-\sqrt{2}$
5. $-3, \sqrt{2}, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$

Exercise 13.5.9.21

For the given numbers, list the

1. whole numbers
2. integers
3. rational numbers
4. irrational numbers
5. real numbers: $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975\ldots$

Answer

1. $6, \sqrt{121}$.
2. $-\sqrt{25}, -1, 6, \sqrt{121}$
3. $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}$

4. $2.041975\ldots$
5. $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975\ldots$

Locate Fractions on the Number Line

The last time we looked at the **number line**, it only had positive and negative integers on it. We now want to include **fraction**s and decimals on it.

Note

Doing the Manipulative Mathematics activity "Number Line Part 3" will help you develop a better understanding of the location of fractions on the number line.

Let's start with fractions and locate $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5$ and $\frac{8}{3}$ on the number line.

We'll start with the whole numbers 3 and −5. because they are the easiest to plot. See Figure 13.5.9.4

The **proper fractions** listed are $\frac{1}{5}$ and $-\frac{4}{5}$. We know the proper fraction $\frac{1}{5}$ has value less than one and so would be located between 0 and 1. The denominator is 5, so we divide the unit from 0 to 1 into 5 equal parts $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. We plot $\frac{1}{5}$. See Figure 13.5.9.4

Similarly, $-\frac{4}{5}$ is between 0 and −1. After dividing the unit into 5 equal parts we plot $-\frac{4}{5}$. See Figure 13.5.9.4

Finally, look at the **improper fractions** $\frac{7}{4}, -\frac{9}{2}, \frac{8}{3}$. These are fractions in which the numerator is greater than the denominator. Locating these points may be easier if you change each of them to a mixed number. See Figure 13.5.9.4

$$\frac{7}{4} = 1\frac{3}{4} \quad -\frac{9}{2} = -4\frac{1}{2} \quad \frac{8}{3} = 2\frac{2}{3} \tag{13.5.9.3}$$

Figure 13.5.9.4shows the number line with all the points plotted.

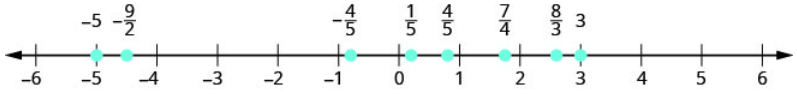


Figure 13.5.9.4

Exercise 13.5.9.22

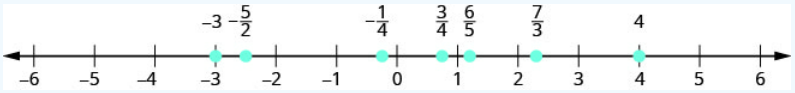
Locate and label the following on a number line: $4, \frac{3}{4}, -\frac{1}{4}, -3, \frac{6}{5}, -\frac{5}{2}$ and $\frac{7}{3}$.

Answer

Locate and plot the integers, 4,−3.

Locate the proper fraction $\frac{3}{4}$ first. The fraction $\frac{3}{4}$ is between 0 and 1. Divide the distance between 0 and 1 into four equal parts then, we plot $\frac{3}{4}$. Similarly plot $-\frac{1}{4}$.

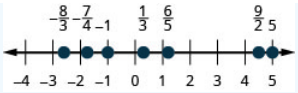
Now locate the improper fractions $\frac{6}{5}, -\frac{5}{2}, \frac{7}{3}$. It is easier to plot them if we convert them to mixed numbers and then plot them as described above: $\frac{6}{5} = 1\frac{1}{5}, -\frac{5}{2} = -2\frac{1}{2}, \frac{7}{3} = 2\frac{1}{3}$.



Exercise 13.5.9.23

Locate and label the following on a number line: $-1, \frac{1}{3}, \frac{6}{5}, -\frac{7}{4}, \frac{9}{2}, 5$ and $-\frac{8}{3}$.

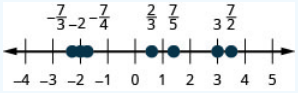
Answer



Exercise 13.5.9.24

Locate and label the following on a number line: $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5$ and $\frac{8}{3}$.

Answer



In Exercise 13.5.9.25 we'll use the inequality symbols to order fractions. In previous chapters we used the number line to order numbers.

- $a < b$ "a is less than b" when a is to the left of b on the number line.
- $a < b$ " a is greater than b" when a is to the right of b on the number line.

As we move from left to right on a number line, the values increase.

Exercise 13.5.9.25

Order each of the following pairs of numbers, using $<$ or $>$. It may be helpful to refer Figure 13.5.9.5

1. $-\frac{2}{3}\_\_\_ -1$
2. $-3\frac{1}{2}\_\_\_ -3$
3. $-\frac{3}{4}\_\_\_ -\frac{1}{4}$
4. $-2\_\_\_ -\frac{8}{3}$

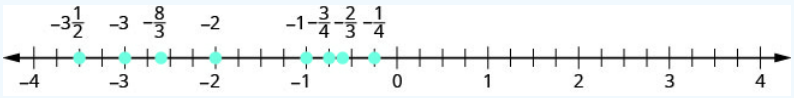


Figure 13.5.9.5

Answer

Be careful when ordering negative numbers.

1. Since $-\frac{2}{3}$ is to the right of -1 on the number line, we know that $-\frac{2}{3} > -1$.
2. Since $-3\frac{1}{2}$ is to the left of -3 on the number line, we know that $-3\frac{1}{2} < -3$.
3. Since $-\frac{3}{4}$ is to the left of $-\frac{1}{4}$ on the number line, we know that $-\frac{3}{4} < -\frac{1}{4}$.
4. Since -2 is to the right of $-\frac{8}{3}$ on the number line, we know that $-2 > -\frac{8}{3}$.

Exercise 13.5.9.26

Order each of the following pairs of numbers, using $<$ or $>$.

1. $-\frac{1}{3}\_\_\_ -1$
2. $-1\frac{1}{2}\_\_\_ -2$
3. $-\frac{2}{3}\_\_\_ -\frac{1}{3}$
4. $-3\_\_\_ -\frac{7}{3}$

Answer

1. $-\frac{1}{3} > -1$
2. $-1\frac{1}{2} > -2$

3. $-\frac{2}{3} < -\frac{1}{3}$
4. $-3 < -\frac{7}{3}$

Exercise 13.5.9.27

Order each of the following pairs of numbers, using $<$ or $>$.

1. $-1 \_\_\_ -\frac{2}{3}$
2. $-2\frac{1}{4} \_\_\_ -2$
3. $-\frac{3}{5} \_\_\_ -\frac{4}{5}$
4. $-4 \_\_\_ -\frac{10}{3}$

Answer

1. $-1 < -\frac{2}{3}$
2. $-2\frac{1}{4} < -2$
3. $-\frac{3}{5} > -\frac{4}{5}$
4. $-4 < -\frac{10}{3}$

Locate Decimals on the Number Line

Since decimals are forms of fractions, locating decimals on the number line is similar to locating fractions on the number line.

Exercise 13.5.9.28

Locate 0.4 on the number line.

Answer

A proper fraction has value less than one. The decimal number 0.4 is equivalent to $\frac{4}{10}$, a proper fraction, so 0.4 is located between 0 and 1. On a number line, divide the interval between 0 and 1 into 10 equal parts. Now label the parts $0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$. We write 0 as 0.0 and 1 and 1.0, so that the numbers are consistently in tenths. Finally, mark 0.4 on the number line. See Figure 13.5.9.6

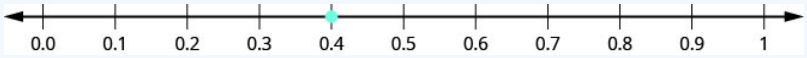
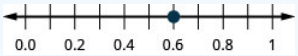


Figure 13.5.9.6

Exercise 13.5.9.29

Locate on the number line: 0.6.

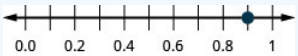
Answer



Exercise 13.5.9.30

Locate on the number line: 0.9.

Answer



Exercise 13.5.9.31

Locate -0.74 on the number line.

Answer

The decimal (−0.74\) is equivalent to $-\frac{74}{100}$, so it is located between 0 and −1. On a number line, mark off and label the hundredths in the interval between 0 and −1. See Figure 13.5.9.7

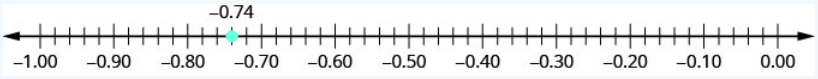
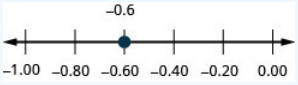


Figure 13.5.9.7

Exercise 13.5.9.32

Locate on the number line: −0.6.

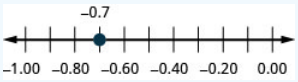
Answer



Exercise 13.5.9.33

Locate on the number line: −0.7.

Answer



Which is larger, 0.04 or 0.40? If you think of this as money, you know that \$0.40 (forty cents) is greater than \$0.04 (four cents). So, $0.40 > 0.04$

Again, we can use the number line to order numbers.

- $a < b$ "a is less than b" when a is to the left of b on the number line.
- $a < b$ " a is greater than b" when a is to the right of b on the number line.

Where are 0.04 and 0.40 located on the number line? See Figure 13.5.9.8

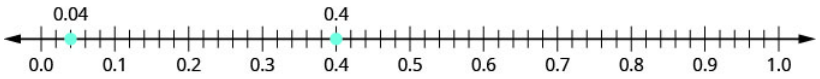


Figure 13.5.9.8

We see that 0.40 is to the right of 0.04 on the number line. This is another way to demonstrate that $0.40 > 0.04$.

How does 0.31 compare to 0.308? This doesn't translate into money to make it easy to compare. But if we convert 0.31 and 0.308 into fractions, we can tell which is larger.

Table 13.5.9.2

	0.31	0.308
Convert to fractions.	$\frac{31}{100}$	$\frac{308}{1000}$
We need a common denominator to compare them.	$\frac{31 \cdot 10}{100 \cdot 10}$	$\frac{308}{1000}$
	$\frac{310}{1000}$	$\frac{308}{1000}$

Because $310 > 308$, we know that $\frac{310}{1000} > \frac{308}{1000}$. Therefore, $0.31 > 0.308$.

Notice what we did in converting 0.31 to a fraction—we started with the fraction $\frac{31}{100}$ and ended with the equivalent fraction $\frac{310}{1000}$.Converting $\frac{310}{1000}$ back to a decimal gives 0.310. So 0.31 is equivalent to 0.310. Writing zeros at the end of a decimal does not change its value!

$$\frac{31}{100} = \frac{310}{1000} \quad \text{and} \quad 0.31 = 0.310$$

We say 0.31 and 0.310 are **equivalent decimals**.

EQUIVALENT DECIMALS

Two decimals are equivalent if they convert to equivalent fractions.

We use equivalent decimals when we order decimals.

The steps we take to order decimals are summarized here.

ORDER DECIMALS.

1. Write the numbers one under the other, lining up the decimal points.
2. Check to see if both numbers have the same number of digits. If not, write zeros at the end of the one with fewer digits to make them match.
3. Compare the numbers as if they were whole numbers.
4. Order the numbers using the appropriate inequality sign.

Exercise 13.5.9.34

Order 0.64 ___ 0.6using $<$ or $>$.

Answer

Write the numbers one under the other, lining up the decimal points.	0.64 0.6
Add a zero to 0.6 to make it a decimal with 2 decimal places. Now they are both hundredths.	0.64 0.60
64 is greater than 60.	$64 > 60$
64 hundredths is greater than 60 hundredths.	$0.64 > 0.60$
	$0.64 > 0.6$

Exercise 13.5.9.35

Order each of the following pairs of numbers, using $<$ or $>$: 0.42 ___ 0.4

Answer

$0.42 > 0.4$

Exercise 13.5.9.36

Order each of the following pairs of numbers, using $<$ or $>$: 0.18 ___ 0.1

Answer

$0.18 > 0.1$

Exercise 13.5.9.37

Order 0.83 ___ 0.803using $<$ or $>$.

Answer

0.83 ___ 0.803

Write the numbers one under the other, lining up the decimal points.	0.83 0.803
They do not have the same number of digits. Write one zero at the end of 0.83.	0.830 0.803
Since $830 > 803$, 830 hundredths is greater than 803 thousandths.	$0.830 > 0.803$
	$0.83 > 0.803$

Exercise 13.5.9.38

Order each of the following pairs of numbers, using $<$ or $>$: 0.76 ___ 0.706

Answer

$0.76 > 0.706$

Exercise 13.5.9.39

Order each of the following pairs of numbers, using $<$ or $>$: 0.305 ___ 0.35

Answer

$0.305 < 0.35$

When we order negative decimals, it is important to remember how to order negative integers. Recall that larger numbers are to the right on the number line. For example, because -2 lies to the right of -3 on the number line, we know that $-2 > -3$. Similarly, smaller numbers lie to the left on the number line. For example, because -9 lies to the left of -6 on the number line, we know that $-9 < -6$. See Figure 13.5.9.9

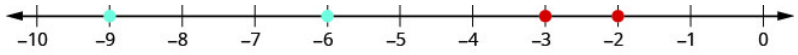


Figure 13.5.9.9

If we zoomed in on the interval between 0 and -1, as shown in Exercise 13.5.9.40, we would see in the same way that $-0.2 > -0.3$ and $-0.9 < -0.6$.

Exercise 13.5.9.40

Use $<$ or $>$ to order -0.1 ___ -0.8.

Answer

	-0.1 ___ -0.8
Write the numbers one under the other, lining up the decimal points. They have the same number of digits.	-0.1 -0.8
Since $-1 > -8$, we know that -1 tenth is greater than -8 tenths. So, we know that	$-0.1 > -0.8$

Exercise 13.5.9.41

Order the following pair of numbers, using $<$ or $>$: -0.3 ___ -0.5.

Answer

$-0.3 > -0.5$

Exercise 13.5.9.42

Order the following pair of numbers, using $<$ or $>$: -0.6 ___ -0.7.

Answer

$-0.6 > -0.7$

Key Concepts

- **Square Root Notation**
  $\sqrt{m}$ is read 'the square root of m.' If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.
- **Order Decimals**
  1. Write the numbers one under the other, lining up the decimal points.
  2. Check to see if both numbers have the same number of digits. If not, write zeros at the end of the one with fewer digits to make them match.
  3. Compare the numbers as if they were whole numbers.
  4. Order the numbers using the appropriate inequality sign.

Practice Makes Perfect

Simplify Expressions with Square Roots

In the following exercises, simplify.

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13.5.9E: Exercises

Practice Makes Perfect

Simplify Expressions with Square Roots

In the following exercises, simplify.

1. $\sqrt{36}$

Answer

$\sqrt{36} = 6$

2. $\sqrt{4}$

3. $\sqrt{64}$

Answer

$\sqrt{64} = 8$

4. $\sqrt{169}$

5. $\sqrt{9}$

Answer

$\sqrt{9} = 3$

6. $\sqrt{16}$

7. $\sqrt{100}$

Answer

$\sqrt{100} = 10$

8. $\sqrt{144}$

9. $\sqrt{-4}$

Answer

$\sqrt{-4}$ is not a real number.

10. $\sqrt{-100}$

11. $\sqrt{-1}$

Answer

$\sqrt{-1}$ is not a real number.

12. $\sqrt{-121}$

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercises, write each rational number as the ratio of two integers.

13. ⓐ 5 ⓑ 3.19

Answer

a. $5 = \frac{5}{1}$ b. $3.19 = \frac{319}{100}$

14. ⓐ 8 ⓑ 1.61

15. ⓐ -12 ⓑ 9.279

Answer

a. $-12 = \frac{-12}{1}$ b. $9.279 = \frac{9279}{1000}$

16. ⓐ -16 ⓑ 4.399

In the following exercises, list the ⓐ rational numbers, ⓑ irrational numbers

17. $\{0.75, 0.22\overline{3}, 1.39174\}$

Answer

All of these numbers are rational.

18. $\{0.36, 0.94729\ldots, 2.52\overline{8}\}$

19. $\{0.4\overline{5}, 1.919293\ldots, 3.59\}$

Answer

a. $0.4\overline{5}$and 3.59 are rational numbers. b. 1.919293… is irrational, since it does not terminate or repeat.

20. $\{0.1\overline{3}, 0.42982\ldots, 1.875\}$

In the following exercises, identify whether each number is rational or irrational.

21. ⓐ $\sqrt{25}$ ⓑ $\sqrt{30}$

Answer

a. $\sqrt{25} = 5$ is rational. b. $\sqrt{30}$ is irrational.

22. ⓐ $\sqrt{44}$ ⓑ $\sqrt{49}$

23. ⓐ $\sqrt{164}$ ⓑ $\sqrt{169}$

Answer

a. $\sqrt{164}$ is irrational. b. $\sqrt{169} = 13$ is rational.

24. ⓐ $\sqrt{225}$ ⓑ $\sqrt{216}$

In the following exercises, identify whether each number is a real number or not a real number.

25. ⓐ $-\sqrt{81}$ ⓑ $\sqrt{-121}$

Answer

a. $-\sqrt{81} = -9$ is a real number. b. $\sqrt{-121}$ is not a real number.

26. ⓐ $-\sqrt{64}$ ⓑ $\sqrt{-9}$

27. ⓐ $\sqrt{-36}$ ⓑ $-\sqrt{144}$

Answer

a. $\sqrt{-36}$ is not a real number. b. $-\sqrt{144} = -12$ is a real number.

28. ⓐ $\sqrt{-49}$ ⓑ $-\sqrt{144}$

In the following exercises, for each set of numbers, list the ⓐ whole numbers, ⓑ integers, ⓒ rational numbers, ⓓ irrational numbers, ⓔ real numbers.

29. $\{-8, 0, 1.95286\ldots, \frac{12}{5}, \sqrt{36}, 9\}$

Answer

a. whole numbers: 0, $\sqrt{36}=6$, 9
b. integers: -8, 0, $\sqrt{36}=6$, 9
c. rational numbers: -8, 0, $\sqrt{36}=6$, 9, $\frac{12}{5}$
d. irrational numbers: 1.95286…
e. real numbers: All of these numbers are real numbers.

30. $\{-9, -3\frac{4}{9}, -\sqrt{9}, 0.40\overline{9}, 116, 7\}$

31. $\{-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}\}$

Answer

a. whole numbers: None of these numbers are whole numbers.
b. integers: $-\sqrt{100}=-10$, −7, -1
c. rational numbers: All of these numbers are rational.
d. irrational numbers: None of these numbers is irrational.
e. real numbers: All of these numbers are real numbers.

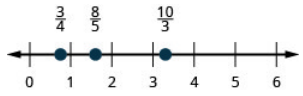
32. $\{-6, -\frac{5}{2}, 0, 0.\overline{714285}, 2\frac{1}{5}, \sqrt{14}\}$

Locate Fractions on the Number Line

In the following exercises, locate the numbers on a number line.

33. $\frac{3}{4}, \frac{8}{5}, \frac{10}{3}$

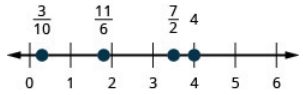
Answer



34. $\frac{1}{4}, \frac{9}{5}, \frac{11}{3}$

35. $\frac{3}{10}, \frac{7}{2}, \frac{11}{6}, 4$

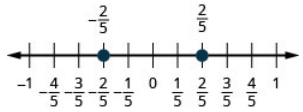
Answer



36. $\frac{7}{10}, \frac{5}{2}, \frac{13}{8}, 3$

37. $\frac{2}{5}, -\frac{2}{5}$

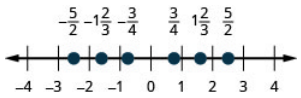
Answer



38. $\frac{3}{4}, -\frac{3}{4}$

39. $\frac{3}{4}, -\frac{3}{4}, 1\frac{2}{3}, -1\frac{2}{3}, \frac{5}{2}, -\frac{5}{2}$

Answer



40. $\frac{1}{5}, -\frac{2}{5}, 1\frac{3}{4}, -1\frac{3}{4}, \frac{8}{3}, -\frac{8}{3}$

In the following exercises, order each of the pairs of numbers, using < or >.

41. -1 ___ $-\frac{1}{4}$

Answer

$-1 < -\frac{1}{4}$

42. -1 ___ $-\frac{1}{3}$

43. $-2\frac{1}{2}$ ___ -3

Answer

$-2\frac{1}{2} > -3$

44. $-1\frac{3}{4}$ ___ -2

45. $-\frac{5}{12}$ ___ $-\frac{7}{12}$

Answer

$-\frac{5}{12} > -\frac{7}{12}$

46. $-\frac{9}{10}$ ___ $-\frac{3}{10}$

47. -3 ___ $-\frac{13}{5}$

Answer

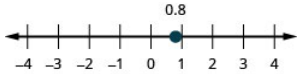
$-3 < -\frac{13}{5}$

48. -4 ___ $-\frac{23}{6}$

Locating Decimals on the Number Line In the following exercises, locate the number on the number line.

49. 0.8

Answer



50. −0.9

51. −1.6

Answer

52. 3.1

In the following exercises, order each pair of numbers, using < or >.

53. 0.37 ___ 0.63

Answer

$0.37 < 0.63$

54. 0.86 ___ 0.69

55. 0.91 ___ 0.901

Answer

$0.91 > 0.901$

56. 0.415 ___ 0.41

57. -0.5 ___ -0.3

Answer

$-0.5 < -0.3$

58. -0.1 ___ -0.4

59. -0.62 ___ -0.619

Answer

$-0.62 < -0.619$

60. -7.31 ___ -7.3

Everyday Math

61. **Field trip** All the 5th graders at Lincoln Elementary School will go on a field trip to the science museum. Counting all the children, teachers, and chaperones, there will be 147 people. Each bus holds 44 people.

ⓐ How many busses will be needed?
ⓑ Why must the answer be a whole number?
ⓒ Why shouldn't you round the answer the usual way, by choosing the whole number closest to the exact answer?

62. **Child care** Serena wants to open a licensed child care center. Her state requires there be no more than 12 children for each teacher. She would like her child care center to serve 40 children.

ⓐ How many teachers will be needed?
ⓑ Why must the answer be a whole number?
ⓒ Why shouldn't you round the answer the usual way, by choosing the whole number closest to the exact answer?

Writing Exercises

63. In your own words, explain the difference between a rational number and an irrational number.

64. Explain how the sets of numbers (counting, whole, integer, rational, irrationals, reals) are related to each other.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objective of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with square roots.			
identify integers, rational numbers, irrational numbers, and real numbers.			
locate fractions on the number line.			
locate decimals on the number line.			

ⓑ On a scale of 1−10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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13.5.10: Properties of Real Numbers

Learning Objectives

By the end of this section, you will be able to:

- Use the commutative and associative properties
- Use the identity and inverse properties of addition and multiplication
- Use the properties of zero
- Simplify expressions using the distributive property

Note

A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **The Properties of Real Numbers**.

Use the Commutative and Associative Properties

Think about adding two numbers, say 5 and 3. The order we add them doesn't affect the result, does it?

$$\begin{array}{ll} 5+3 & 3+5 \\ 8 & 8 \\ 5+3= & 3+5 \end{array}$$

The results are the same.

As we can see, the order in which we add does not matter!

What about multiplying 5 and 3?

$$\begin{array}{ll} 5\cdot 3 & 3\cdot 5 \\ 15 & 15 \\ 5\cdot 3= & 3\cdot 5 \end{array}$$

Again, the results are the same!

The order in which we multiply does not matter!

These examples illustrate the **commutative property**. When adding or multiplying, changing the *order* gives the same result.

COMMUTATIVE PROPERTY

of Addition	If a, b are real numbers, then	$a+b=b+a$
of Multiplication	If a, b are real numbers, then	$a\cdot b=b\cdot a$

When adding or multiplying, changing the *order* gives the same result.

The commutative property has to do with order. If you change the order of the numbers when adding or multiplying, the result is the same.

What about subtraction? Does order matter when we subtract numbers? Does $7-3$ give the same result as $3-7$?

$$\begin{array}{ll} 7-3 & 3-7 \\ 4 & -4 \end{array}$$

$$4 \neq -4$$

$$7-3 \neq 3-7$$

The results are not the same.

Since changing the order of the subtraction did not give the same result, we know that *subtraction is not commutative*.

Let's see what happens when we divide two numbers. Is **division** commutative?

$$\begin{array}{ll} 12 \div 4 & 4 \div 12 \\ \frac{12}{4} & \frac{4}{12} \\ 3 & \frac{1}{3} \end{array}$$

$$3 \neq \frac{1}{3}$$
$$12 \div 4 \neq 4 \div 12$$

The results are not the same.

Since changing the order of the division did not give the same result, *division is not commutative*. The commutative properties only apply to addition and multiplication!

- Addition and multiplication *are* commutative.
- Subtraction and Division *are not* commutative.

If you were asked to simplify this expression, how would you do it and what would your answer be?

$$7+8+2$$

Some people would think $7+8$ is 15 and then $15+2$ is 17. Others might start with $8+2$ makes 10 and then $7+10$ makes 17.

Either way gives the same result. Remember, we use parentheses as grouping symbols to indicate which operation should be done first.

Add $7+8$.	$(7+8)+2$
Add.	$15+2$
Add.	17

	$7+(8+2)$
Add $8+2$.	$7+10$
Add.	77

$$(7+8)+2=7+(8+2)$$

When adding three numbers, changing the grouping of the numbers gives the same result.

This is true for multiplication, too.

	$(5 \cdot \frac{1}{3}) \cdot 3$
Multiply. $5 \cdot \frac{1}{3}$	$\frac{5}{3} \cdot 3$
Multiply.	5

	$5 \cdot (\frac{1}{3} \cdot 3)$
Multiply. $\frac{1}{3} \cdot 3$	$5 \cdot 1$
Multiply.	5

$$(5 \cdot \tfrac{1}{3}) \cdot 3 = 5 \cdot (\tfrac{1}{3} \cdot 3)$$

When multiplying three numbers, changing the grouping of the numbers gives the same result.

You probably know this, but the terminology may be new to you. These examples illustrate the **associative property**.

ASSOCIATIVE PROPERTY

of Addition	If a, b, c are real numbers, then $(a+b)+c=a+(b+c)$
of Multiplication	If a, b, c are real numbers, then $(a \cdot b) \cdot c=a \cdot(b \cdot c)$

When adding or multiplying, changing the *grouping* gives the same result.

Let's think again about multiplying $5 \cdot \frac{1}{3} \cdot 3$. We got the same result both ways, but which way was easier? Multiplying $\frac{1}{3}$ and 3 first, as shown above on the right side, eliminates the fraction in the first step. Using the associative property can make the math easier!

The **associative property** has to do with grouping. If we change how the numbers are grouped, the result will be the same. Notice it is the same three numbers in the same order—the only difference is the grouping.

We saw that subtraction and division were not commutative. They are not associative either.

When simplifying an expression, it is always a good idea to plan what the steps will be. In order to combine like terms in the next example, we will use the commutative property of addition to write the like terms together.

Example 13.5.10.1

Simplify: $18p+6q+15p+5q$.

Solution

	$18p+6q+15p+5q$
Use the commutative property of addition to re-order so that like terms are together.	$18p+15p+6q+5q$
Add like terms.	$33p+11q$

Try It 13.5.10.2

Simplify: $23r+14s+9r+15s$.

Answer

$32r+29s$

Try It 13.5.10.3

Simplify: $37m+21n+4m-15n$.

Answer

$41m+6n$

When we have to simplify algebraic **expressions**, we can often make the work easier by applying the commutative or associative property first, instead of automatically following the order of operations. When adding or subtracting fractions, combine those with a common denominator first.

Example 13.5.10.4

Simplify: $\left(\frac{5}{13}+\frac{3}{4}\right)+\frac{1}{4}$

Solution

	$(\frac{5}{13}+\frac{3}{4})+\frac{1}{4}$
Notice that the last 2 terms have a common denominator, so change the grouping.	$\frac{5}{13}+(\frac{3}{4}+\frac{1}{4})$
Add in parentheses first.	$\frac{5}{13}+(\frac{4}{4})$
Simplify the fraction.	$\frac{5}{13}+1$
Add.	$1\frac{5}{13}$
Convert to an improper fraction.	$\frac{18}{13}$

Try It 13.5.10.5

Simplify: $(\frac{7}{15}+\frac{5}{8})+\frac{3}{8}$

Answer

$1\frac{7}{15}$

Try It 13.5.10.6

Simplify: $(\frac{2}{9}+\frac{7}{12})+\frac{5}{12}$

Answer

$1\frac{2}{9}$

Example 13.5.10.7

Use the associative property to simplify $6(3x)$.

Solution

Use the associative property of multiplication, $(a\cdot b)\cdot c=a\cdot(b\cdot c)$, to change the grouping.

	$6(3x)$
Change the grouping.	$(6\cdot 3)x$
Multiply in the parentheses.	18

Notice that we can multiply $6\cdot 3$ but we could not multiply $3x$ without having a value for x.

Try It 13.5.10.8

Use the associative property to simplify $8(4x)$.

Answer

$32x$

Try It 13.5.10.9

Use the associative property to simplify $-9(7y)$.

Answer

$-63y$

Use the Identity and Inverse Properties of Addition and Multiplication

What happens when we add 0 to any number? Adding 0 doesn't change the value. For this reason, we call 0 the **additive identity**.

For example,

$$\begin{array}{ccc} 13+0 & -14+0 & 0+(-8) \\ 13 & -14 & -8 \end{array}$$

These examples illustrate the **Identity Property of Addition** that states that for any real number a, $a+0=a$ and $0+a=a$.

What happens when we multiply any number by one? Multiplying by 1 doesn't change the value. So we call 1 the **multiplicative identity**.

For example,

$$\begin{array}{ccc} 43\cdot 1 & -27\cdot 1 & 1\cdot\frac{3}{5} \\ 43 & -27 & \frac{3}{5} \end{array}$$

These examples illustrate the **Identity Property of Multiplication** that states that for any real number a, $a\cdot 1=a$ and $1\cdot a=a$.

We summarize the Identity Properties below.

IDENTITY PROPERTY

of addition For any real number a: $a+0=a \quad 0+a=a$
0 is the **additive identity**
of multiplication For any real number a: $a\cdot 1=a \quad 1\cdot a=a$
1 is the **multiplicative identity**

What number added to 5 gives the additive identity, 0?
$5+\_\_\_\_\_=0$ We know $5+(-5)=0$
What number added to −6 gives the additive identity, 0?
$-6+\_\_\_\_\_=0$ We know $-6+6=0$

Figure 13.5.10.1

Notice that in each case, the missing number was the opposite of the number!

We call $-a$. the **additive inverse** of a. *The opposite of a number is its additive inverse.* A number and its opposite add to zero, which is the additive identity. This leads to the **Inverse Property of Addition** that states for any real number $a, a+(-a)=0$. Remember, a number and its opposite add to zero.

What number multiplied by $\frac{2}{3}$ gives the multiplicative identity, 1? In other words, $\frac{2}{3}$ times what results in 1?

$\frac{2}{3}\cdot\_\_\_=1$ We know $\frac{2}{3}\cdot\frac{3}{2}=1$

Figure 13.5.10.2

What number multiplied by 2 gives the multiplicative identity, 1? In other words 2 times what results in 1?

$2\cdot\_\_\_=1$ We know $2\cdot\frac{1}{2}=1$

Figure 13.5.10.3

Notice that in each case, the missing number was the **reciprocal** of the number!

We call $\frac{1}{a}$ the **multiplicative inverse** of a. *The reciprocal of aa number is its multiplicative inverse.* A number and its reciprocal multiply to one, which is the multiplicative identity. This leads to the **Inverse Property of Multiplication** that states that for any real number $a, a \neq 0, a \cdot \frac{1}{a} = 1$.

We'll formally state the inverse properties here:

INVERSE PROPERTY

of addition	For any real number a, $-a$. is the **additive inverse** of a A number and its opposite add to zero.	$a + (-a) = 0$
of multiplication	For any real number $a, a \neq 0$ $\frac{1}{a}$. is the **multiplicative inverse** of a A number and its reciprocal multiply to one.	$a \cdot \frac{1}{a} = 1$

Example 13.5.10.10

Find the additive inverse of

1. $\frac{5}{8}$
2. 0.6
3. -8
4. $-\frac{4}{3}$

Solution

To find the additive inverse, we find the opposite.

1. The additive inverse of $\frac{5}{8}$ is the opposite of $\frac{5}{8}$. The additive inverse of $\frac{5}{8}$ is $-\frac{5}{8}$
2. The additive inverse of 0.6 is the opposite of 0.6. The additive inverse of 0.6 is -0.6.
3. The additive inverse of -8 is the opposite of -8. We write the opposite of -8 as $-(-8)$, and then simplify it to 8. Therefore, the additive inverse of -8 is 8.
4. The additive inverse of $-\frac{4}{3}$ is the opposite of $-\frac{4}{3}$. We write this as $-(-\frac{4}{3})$, and then simplify to $\frac{4}{3}$. Thus, the additive inverse of $-\frac{4}{3}$ is $\frac{4}{3}$.

Try It 13.5.10.11

Find the additive inverse of

1. $\frac{7}{9}$
2. 1.2
3. -14
4. $-\frac{9}{4}$

Answer

1. $-\frac{7}{9}$
2. -1.2
3. 14
4. $\frac{9}{4}$

Try It 13.5.10.12

Find the additive inverse of

1. $\frac{7}{13}$
2. 8.4
3. -46
4. $-\frac{5}{2}$

Answer

1. $-\frac{7}{13}$
2. -8.4
3. 46
4. $\frac{5}{2}$

Example 13.5.10.13

Find the multiplicative inverse of

1. 9
2. $-\frac{1}{9}$
3. 0.9

Solution

To find the multiplicative inverse, we find the reciprocal.

1. The multiplicative inverse of 9 is the reciprocal of 9, which is $\frac{1}{9}$. Therefore, the multiplicative inverse of 9 is $\frac{1}{9}$.
2. The multiplicative inverse of $-\frac{1}{9}$ is the reciprocal of $-\frac{1}{9}$, which is -9. Thus, the multiplicative inverse of $-\frac{1}{9}$ is -9.
3. To find the multiplicative inverse of 0.9, we first convert 0.9 to a fraction, $\frac{9}{10}$. Then we find the reciprocal of the fraction. The reciprocal of $\frac{9}{10}$ is $\frac{10}{9}$. So the multiplicative inverse of 0.9 is $\frac{10}{9}$.

Try It 13.5.10.14

Find the multiplicative inverse of

1. 4
2. $-\frac{1}{7}$
3. 0.3

Answer

1. $\frac{1}{4}$
2. -7
3. $\frac{10}{3}$

Try It 13.5.10.15

Find the multiplicative inverse of

1. 18
2. $-\frac{4}{5}$
3. 0.6

Answer

1. $\frac{1}{18}$

2. $-\frac{5}{4}$
3. $\frac{5}{3}$

Use the Properties of Zero

The identity property of addition says that when we add 0 to any number, the result is that same number. What happens when we multiply a number by 0? Multiplying by 0 makes the **product** equal zero.

MULTIPLICATION BY ZERO

For any real number a.

$$a \cdot 0 = 0 \quad 0 \cdot a = 0$$

The product of any real number and 0 is 0.

What about **division** involving zero? What is $0 \div 3$? Think about a real example: If there are no cookies in the cookie jar and 3 people are to share them, how many cookies does each person get? There are no cookies to share, so each person gets 0 cookies. So,

$$0 \div 3 = 0$$

We can check division with the related multiplication fact.

$$12 \div 6 = 2 \text{ because } 2 \cdot 6 = 12$$

So we know $0 \div 3 = 0$ because $0 \cdot 3 = 0$.

DIVISION OF ZERO

For any real number a, except 0, $\frac{0}{a} = 0$ and $0 \div a = 0$.

Zero divided by any real number except zero is zero.

Now think about dividing *by* zero. What is the result of dividing 4 by 0? Think about the related multiplication fact: $4 \div 0 = ?$ means $? \cdot 0 = 4$. Is there a number that multiplied by 0 gives 4? Since any real number multiplied by 0 gives 0, there is no real number that can be multiplied by 0 to obtain 4.

We conclude that there is no answer to $4 \div 0$ and so we say that division by 0 is undefined.

DIVISION BY ZERO

For any real number a, $\frac{a}{0}$ and $a \div 0$ are undefined.

Division by zero is undefined.

We summarize the properties of zero below.

PROPERTIES OF ZERO

Multiplication by Zero: For any real number a,

$$a \cdot 0 = 0 \quad 0 \cdot a = 0 \quad \text{The product of any number and 0 is 0}$$

Division of Zero, Division by Zero: For any real number $a, a \neq 0$

$\frac{0}{a} = 0$	Zero divided by any real number, except itself is zero.
$\frac{a}{0}$ is undefined	Division by zero is undefined.

Example 13.5.10.16

Simplify:

1. $-8 \cdot 0$
2. $\frac{0}{-2}$
3. $\frac{-32}{0}$

Solution

1. $-8 \cdot 0$

   The product of any real number and 0 is 0 — 0

2. $\frac{0}{-2}$

   Zero divided by any real number, except itself, is 0 — 0

3. $\frac{-32}{0}$

   Division by 0 is undefined. — undefined

Try It 13.5.10.17

Simplify:

1. $-14 \cdot 0$
2. $\frac{0}{-6}$
3. $\frac{-2}{0}$

Answer

1. 0
2. 0
3. undefined

Try It 13.5.10.18

Simplify:

1. $0(-17)$
2. $\frac{0}{-10}$
3. $\frac{-5}{0}$

Answer

1. 0
2. 0
3. undefined

We will now practice using the properties of identities, inverses, and zero to simplify expressions.

Example 13.5.10.19

Simplify:

1. $\frac{0}{n+5}$, where $n \neq -5$
2. $\frac{10-3p}{0}$ where $10-3p \neq 0$

Solution

1. $\frac{0}{n+5}$

   Zero divided by any real number except itself is 0. 0

2. $\frac{10-3p}{0}$

   Division by 0 is undefined undefined

Try It 13.5.10.26

Simplify:

1. $\frac{0}{m+7}$, where $m \neq -7$
2. $\frac{18-6c}{0}$, where $18-6c \neq 0$

Answer

1. 0
2. undefined

Try It 13.5.10.27

Simplify:

1. $\frac{0}{d-4}$, where $d \neq 4$
2. $\frac{15-4q}{0}$, where $15-4q \neq 0$

Answer

1. 0
2. undefined

Example 13.5.10.20

Simplify: $-84n+(-73n)+84n$.

Solution

	$-84n+(-73n)+84n$
Notice that the first and third terms are opposites; use the commutative property of addition to re-order the terms.	$-84n+84n+(-73n)$
Add left to right.	$0+(-73)$
Add.	$-73n$

Try It 13.5.10.21

Simplify: $-27a+(-48a)+27a$.

Answer

$-48a$

? Try It 13.5.10.22

Simplify: $39x+(-92x)+(-39x)$.

Answer

$-92x$

Now we will see how recognizing reciprocals is helpful. Before multiplying left to right, look for reciprocals—their product is 1.

✓ Example 13.5.10.23

Simplify: $\frac{7}{15}\cdot\frac{8}{23}\cdot\frac{15}{7}$

Solution

	$\frac{7}{15}\cdot\frac{8}{23}\cdot\frac{15}{7}$
Notice that the first and third terms are reciprocals, so use the commutative property of multiplication to re-order the factors.	$\frac{7}{15}\cdot\frac{15}{7}\cdot\frac{8}{23}$
Multiply left to right.	$1\cdot\frac{8}{23}$
Multiply.	$\frac{8}{23}$

? Exercise 13.5.10.24

Simplify: $\frac{9}{16}\cdot\frac{5}{49}\cdot\frac{16}{9}$

Answer

$\frac{5}{49}$

? Try It 13.5.10.25

Simplify: $\frac{6}{17}\cdot\frac{11}{25}\cdot\frac{17}{6}$

Answer

$\frac{11}{25}$

✓ Example 13.5.10.28

Simplify: $\frac{3}{4}\cdot\frac{4}{3}(6x+12)$

Solution

	$\frac{3}{4} \cdot \frac{4}{3}(6x+12)$
There is nothing to do in the parentheses, so multiply the two fractions first—notice, they are reciprocals.	$1(6x+12)$
Simplify by recognizing the multiplicative identity.	$6x+12$

? Try It 13.5.10.29

Simplify: $\frac{2}{5} \cdot \frac{5}{2}(20y+50)$

Answer

$20y+50$

? Try It 13.5.10.30

Simplify: $\frac{3}{8} \cdot \frac{8}{3}(12z+16)$

Answer

$12z+16$

Simplify Expressions Using the Distributive Property

Suppose that three friends are going to the movies. They each need $9.25—that's 9 dollars and 1 quarter—to pay for their tickets. How much money do they need all together?

You can think about the dollars separately from the quarters. They need 3 times $9 so $27, and 3 times 1 quarter, so 75 cents. In total, they need $27.75. If you think about doing the math in this way, you are using the **distributive property**.

DISTRIBUTIVE PROPERTY

If a, b, c are real numbers, then	$a(b+c) = ab+ac$
Also,	$(b+c)a = ba+ca$
	$a(b-c) = ab-ac$
	$(b-c)a = ba-ca$

Back to our friends at the movies, we could find the total amount of money they need like this:

$$3(9.25)$$
$$3(9+0.25)$$
$$3(9)+3(0.25)$$
$$27+0.75$$
$$27.75$$

In algebra, we use the **distributive property** to remove parentheses as we simplify expressions.

For example, if we are asked to simplify the expression $3(x+4)$, the order of operations says to work in the parentheses first. But we cannot add *x* and 4, since they are not like terms. So we use the distributive property, as shown in Exercise 13.5.10.31

Example 13.5.10.31

Simplify: $3(x+4)$.

Solution

	$3(x+4)$
Distribute.	$3\cdot x+3\cdot 4$
Multiply.	$3x+12$

Try It 13.5.10.32

Simplify: $4(x+2)$.

Answer

$4x+8$

Try It 13.5.10.33

Simplify: $6(x+7)$.

Answer

$6x+42$

Some students find it helpful to draw in arrows to remind them how to use the distributive property. Then the first step in Exercise 13.5.10.31would look like this:

Example 13.5.10.34

Simplify: $8(\frac{3}{8}x+\frac{1}{4})$.

Solution

	$8\left(\frac{3}{8}x+\frac{1}{4}\right)$
Distribute.	$8\cdot\frac{3}{8}x+8\cdot\frac{1}{4}$
Multiply.	$3x+2$

Try It 13.5.10.35

Simplify: $6(\frac{5}{6}y+\frac{1}{2})$.

Answer

$5y+3$

Try It 13.5.10.36

Simplify: $12(\frac{1}{3}n+\frac{3}{4})$.

Answer

$4n+9$

Using the distributive property as shown in Exercise $13.5.10.37$will be very useful when we solve money applications in later chapters.

Example 13.5.10.37

Simplify: $100(0.3+0.25q)$.

Solution

	$100(0.3+0.25q)$
Distribute.	$100(0.3)+100(0.25q)$
Multiply.	$30+25q$

Try It 13.5.10.38

Simplify: $100(0.7+0.15p)$.

Answer

$70+15p$

Try It 13.5.10.39

Simplify: $100(0.04+0.35d)$.

Answer

$4+35d$

When we distribute a negative number, we need to be extra careful to get the signs correct!

Example 13.5.10.40

Simplify: $-2(4y+1)$.

Solution

	$-2(4y+1)$
Distribute.	$-2 \cdot 4y+(-2) \cdot 1$
Multiply.	$-8y-2$

Try It 13.5.10.41

Simplify: $-3(6m+5)$.

Answer

$-18m-15)$

Try It 13.5.10.42

Simplify: $-6(8n+11)$.

Answer

$-48n-66)$

Example 13.5.10.43

Simplify: $-11(4-3a)$.

Solution

Distribute.	$-11(4-3a)$
Multiply.	$-11 \cdot 4-(-11) \cdot 3a$ $-44-(-33a)$
Simplify.	$-44+33a$

Notice that you could also write the result as $33a-44$. Do you know why?

Try It 13.5.10.44

Simplify: $-5(2-3a)$.

Answer

$10+15a$

Try It 13.5.10.45

Simplify: $-7(8-15y)$.

Answer

$-56+105y$

Exercise 13.5.10.46will show how to use the distributive property to find the opposite of an expression.

Example 13.5.10.46

Simplify: $-(y+5)$.

Solution

	$-(y+5)$
Multiplying by -1 results in the opposite.	$-1(y+5)$
Distribute.	$-1 \cdot y+(-1) \cdot 5$
Simplify.	$-y+(-5)$
	$-y-5$

Try It 13.5.10.47

Simplify: $-(z-11)$.

Answer

$-z+11$

Try It 13.5.10.48

Simplify: $-(x-4)$.

Answer

$-x+4$

There will be times when we'll need to use the **distributive property** as part of the order of operations. Start by looking at the parentheses. If the expression inside the parentheses cannot be simplified, the next step would be multiply using the distributive property, which removes the parentheses. The next two examples will illustrate this.

Example 13.5.10.49

Simplify: $8-2(x+3)$.

Be sure to follow the order of operations. Multiplication comes before subtraction, so we will distribute the 2 first and then subtract.

Solution

	$8-2(x+3)$
Distribute.	$8-2\cdot x-2\cdot 3$
Multiply.	$8-2x-6$
Combine like terms.	$-2x+2$

Try It 13.5.10.50

Simplify: $9-3(x+2)$.

Answer

$3-3x$

Try It 13.5.10.51

Simplify: $7x-5(x+4)$.

Answer

$2x-20$

Example 13.5.10.52

Simplify: $4(x-8)-(x+3)$.

Solution

	$4(x-8)-(x+3)$
Distribute.	$4x-32-x-3$
Combine like terms.	$3x-35$

Try It 13.5.10.53

Simplify: $6(x-9)-(x+12)$.

Answer

$5x-66$

Try It 13.5.10.54

Simplify: $8(x-1)-(x+5)$.

Answer

$7x-13$

All the properties of real numbers we have used in this chapter are summarized in Table 13.5.10.1

Table 13.5.10.1

Commutative Property	
of addition If a,b are real numbers, then	$a+b=b+a$
of multiplication If a,b are real numbers, then	$a\cdot b=b\cdot a$
Associative Property	
of addition If a,b,c are real numbers, then	$(a+b)+c=a+(b+c)$
of multiplication If a,b,c are real numbers, then	$(a\cdot b)\cdot c=a\cdot(b\cdot c)$
Distributive Property	
If a,b,c are real numbers, then	$a(b+c)=ab+ac$
Identity Property	
of addition For any real number a: 0 is the **additive identity** **of multiplication** For any real number a: **1** is the **multiplicative identity**	$a+0=a$ $0+a=a$ $a\cdot 1=a$ $1\cdot a=a$
Inverse Property	

of addition For any real number a, $-a$ is the **additive inverse** of a	$a+(-a)=0$
of multiplication For any real number $a, a \neq 0$ $\frac{1}{a}$ is the **multiplicative inverse** of a	$a \cdot \frac{1}{a}=1$
Properties of Zero	
For any real number a,	$a \cdot 0=0$
For any real number $a, a \neq 0$	$0 \cdot a=0$
	$\frac{0}{a}=0$
For any real number $a, a \neq 0$	$\frac{a}{0}$ is undefined

Key Concepts

- **Commutative Property of**
  - **Addition:** If a,b are real numbers, then $a+b=b+a$.
  - **Multiplication:** If a,b are real numbers, then $a \cdot b=b \cdot a$. When adding or multiplying, changing the *order* gives the same result.
- **Associative Property of**
  - **Addition:** If a,b,c are real numbers, then $(a+b)+c=a+(b+c)$.
  - **Multiplication:** If a,b,c are real numbers, then $(a \cdot b) \cdot c=a \cdot(b \cdot c)$. When adding or multiplying, changing the *grouping* gives the same result.
- **Distributive Property:** If a,b,c are real numbers, then
  - $a(b+c)=ab+ac$
  - $(b+c)a=ba+ca$
  - $a(b-c)=ab-ac$
  - $(b+c)a=ba+ca$
- **Identity Property**
  - **of Addition:** For any real number a: $a+0=a$ **0** is the **additive identity**
  - **of Multiplication:** For any real number a: $a \cdot 1=a \quad 1 \cdot a=a$ **1**1 is the **multiplicative identity**
- **Inverse Property**
  - **of Addition:** For any real number $a, a+(-a)=0$. A number and its *opposite* add to zero. $-a$ is the **additive inverse** of a.
  - **of Multiplication:** For any real number $a,(a \neq 0) a \cdot \frac{1}{a}=1$. A number and its *reciprocal* multiply to one. $\frac{1}{a}$ is the **multiplicative inverse** of a.
- **Properties of Zero**
  - For any real number a, $a \cdot 0=0 \quad 0 \cdot a=0$ – The product of any real number and 0 is 0.
  - $\frac{0}{a}=0$ for $a \neq 0$ – Zero divided by any real number except zero is zero.
  - $\frac{a}{0}$ is undefined – Division by zero is undefined.

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13.5.10E: Exercises

Practice Makes Perfect

Use the Commutative and Associative Properties

In the following exercises, use the associative property to simplify.

Exercise 1

$3(4x)$

Answer

$12x$

Exercise 2

$4(7m)$

Exercise 3

$(y+12)+28$

Answer

$y+40$

Exercise 4

$(n+17)+33$

In the following exercises, simplify.

Exercise 5

$\frac{1}{2}+\frac{7}{8}+\left(-\frac{1}{2}\right)$

Answer

$\frac{7}{8}$

Exercise 6

$\frac{2}{5}+\frac{5}{12}+\left(-\frac{2}{5}\right)$

Exercise 7

$\frac{3}{20}\cdot\frac{49}{11}\cdot\frac{20}{3}$

Answer

$\frac{49}{11}$

Exercise 8

$\frac{13}{18} \cdot \frac{25}{7} \cdot \frac{18}{13}$

Exercise 9

$-24 \cdot 7 \cdot \frac{3}{8}$

Answer

-63

Exercise 10

$-36 \cdot 11 \cdot \frac{4}{9}$

Exercise 11

$\left(\frac{5}{6} + \frac{8}{15}\right) + \frac{7}{15}$

Answer

$1\frac{5}{6}$

Exercise 12

$\left(\frac{11}{12} + \frac{4}{9}\right) + \frac{5}{9}$

Exercise 13

$17(0.25)(4)$

Answer

17

Exercise 14

$36(0.2)(5)$

Exercise 15

$[2.48(12)](0.5)$

Answer

14.88

Exercise 16

$[9.731(4)](0.75)$

Exercise 17

$7(4a)$

Answer

$28a$

Exercise 18

$9(8w)$

Exercise 19

$-15(5m)$

Answer

$-75m$

Exercise 20

$-23(2n)$

Exercise 21

$12(\frac{5}{6}p)$

Answer

$10p$

Exercise 22

$20(\frac{3}{5}q)$

Exercise 23

$43m+(-12n)+(-16m)+(-9n)$

Answer

$27m+(-21n)$

Exercise 24

$-22p+17q+(-35p)+(-27q)$

Exercise 25

$\frac{3}{8}g+\frac{1}{12}h+\frac{7}{8}g+\frac{5}{12}h$

Answer

$\frac{5}{4}g+\frac{1}{2}h$

Exercise 26

$\frac{5}{6}a+\frac{3}{10}b+\frac{1}{6}a+\frac{9}{10}b$

Exercise 27

$6.8p+9.14q+(-4.37p)+(-0.88q)$

Answer

$2.43p+8.26q$

Exercise 28

$9.6m+7.22n+(-2.19m)+(-0.65n)$

Use the Identity and Inverse Properties of Addition and Multiplication

In the following exercises, find the additive inverse of each number

Exercise 29

1. $\frac{2}{5}$
2. 4.3
3. -8
4. $-\frac{10}{3}$

Answer

1. $-\frac{2}{5}$
2. -4.3
3. 8
4. $\frac{10}{3}$

Exercise 30

1. $\frac{5}{9}$
2. 2.1
3. -3
4. $-\frac{9}{5}$

Exercise 31

1. $-\frac{7}{6}$
2. -0.075
3. 23
4. $\frac{1}{4}$

Answer

1. $\frac{7}{6}$
2. 0.075
3. -23
4. $-\frac{1}{4}$

Exercise 32

1. $-\frac{8}{3}$
2. -0.019
3. 52
4. $\frac{5}{6}$

In the following exercises, find the multiplicative inverse of each number.

Exercise 33

1. 6
2. $-\frac{3}{4}$
3. 0.7

Answer

1. $\frac{1}{6}$
2. $-\frac{4}{3}$
3. $\frac{10}{7}$

Exercise 34

1. 12
2. $-\frac{9}{2}$
3. 0.13

Exercise 35

1. $\frac{11}{12}$
2. -1.1
3. -4

Answer

1. $\frac{12}{11}$
2. $-\frac{10}{11}$
3. $-\frac{1}{4}$

Exercise 36

1. $\frac{17}{20}$
2. -1.5
3. -3

Use the Properties of Zero

In the following exercises, simplify.

Exercise 37

$\frac{0}{6}$

Answer

0

Exercise 38

$\frac{3}{0}$

Exercise 39

$0 \div \frac{11}{12}$

Answer

0

Exercise 40

$\frac{0}{3}$

Exercise 41

$\frac{6}{0}$

Answer

undefined

Exercise 42

$0 \cdot \frac{8}{15}$

Exercise 43

$(-3.14)(0)$

Answer

0

Exercise 44

$\frac{\frac{1}{10}}{0}$

Mixed Practice

In the following exercises, simplify.

Exercise 45

$19a + 44 - 19a$

Answer

44

Exercise 46

$27c + 16 - 27c$

Exercise 47

$10(0.1d)$

Answer

$1d$

Exercise 48

$100(0.01p)$

Exercise 49

$\frac{0}{u-4.99}$, where $u \neq 4.99$

Answer

0

Exercise 50

$\frac{0}{v-65.1}$, where $v \neq 65.1$

Exercise 51

$0 \div \left(x - \frac{1}{2}\right)$, where $x \neq \frac{1}{2}$

Answer

0

Exercise 52

$0 \div \left(y - \frac{1}{6}\right)$, where $y \neq \frac{1}{6}$

Exercise 53

$\frac{32-5a}{0}$, where $32 - 5a \neq 0$

Answer

undefined

Exercise 54

$\frac{28-9b}{0}$, where $28 - 9b \neq 0$

Exercise 55

$\left(\frac{3}{4}+\frac{9}{10}m\right)\div 0$ where $\frac{3}{4}+\frac{9}{10}m\neq 0$

Answer

undefined

Exercise 56

$\left(\frac{5}{16}n-\frac{3}{7}\right)\div 0$ where $\frac{5}{16}n-\frac{3}{7}\neq 0$

Exercise 57

$15\cdot\frac{3}{5}(4d+10)$

Answer

$36d+90$

Exercise 58

$18\cdot\frac{5}{6}(15h+24)$

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the distributive property.

Exercise 59

$8(4y+9)$

Answer

$32y+72$

Exercise 60

$9(3w+7)$

Exercise 61

$6(c-13)$

Answer

$6c-78$

Exercise 62

$7(y-13)$

Exercise 63

$\frac{1}{4}(3q+12)$

Answer

$\frac{3}{4}q+3$

Exercise 64

$\frac{1}{5}(4m+20)$

Exercise 65

$9\left(\frac{5}{9}y-\frac{1}{3}\right)$

Answer

$5y-3$

Exercise 66

$10\left(\frac{3}{10}x-\frac{2}{5}\right)$

Exercise 67

$12\left(\frac{1}{4}+\frac{2}{3}r\right)$

Answer

$3+8r$

Exercise 68

$12\left(\frac{1}{6}+\frac{3}{4}s\right)$

Exercise 69

$r(s-18)$

Answer

$rs-18r$

Exercise 70

$u(v-10)$

Exercise 71

$(y+4)p$

Answer

$yp+4p$

Exercise 72

$(a+7)x$

Exercise 73

$-7(4p+1)$

Answer

$-28p-7$

Exercise 74

$-9(9a+4)$

Exercise 75

$-3(x-6)$

Answer

$-3x+18$

Exercise 76

$-4(q-7)$

Exercise 77

$-(3x-7)$

Answer

$-3x+7$

Exercise 78

$-(5p-4)$

Exercise 79

$16-3(y+8)$

Answer

$-3y-8$

Exercise 80

$18-4(x+2)$

Exercise 81

$4-11(3c-2)$

Answer

$-33c+26$

Exercise 82

$9-6(7n-5)$

Exercise 83

$22-(a+3)$

Answer

$-a+19$

Exercise 84

$8-(r-7)$

Exercise 85

$(5m-3)-(m+7)$

Answer

$4m-10$

Exercise 86

$(4y-1)-(y-2)$

Exercise 87

$5(2n+9)+12(n-3)$

Answer

$22n+9$

Exercise 88

$9(5u+8)+2(u-6)$

Exercise 89

$9(8x-3)-(-2)$

Answer

$72x-25$

Exercise 90

$4(6x-1)-(-8)$

Exercise 91

$14(c-1)-8(c-6)$

Answer

$6c+34$

Exercise 92

$11(n-7)-5(n-1)$

Exercise 93

$6(7y+8)-(30y-15)$

Answer

$12y+63$

Exercise 94

$7(3n+9)-(4n-13)$

Everyday Math

Exercise 95

Insurance copayment Carrie had to have 5 fillings done. Each filling cost $80. Her dental insurance required her to pay 20% of the cost as a copay. Calculate Carrie's copay:

1. First, by multiplying 0.20 by 80 to find her copay for each filling and then multiplying your answer by 5 to find her total copay for 5 fillings.
2. Next, by multiplying [5(0.20)](80)
3. Which of the properties of real numbers says that your answers to parts (a), where you multiplied 5[(0.20)(80)] and (b), where you multiplied [5(0.20)](80), should be equal?

Answer

1. $80
2. $80
3. answers will vary

Exercise 96

Cooking time Helen bought a 24-pound turkey for her family's Thanksgiving dinner and wants to know what time to put the turkey in to the oven. She wants to allow 20 minutes per pound cooking time. Calculate the length of time needed to roast the turkey:

1. First, by multiplying 24·20 to find the total number of minutes and then multiplying the answer by $\frac{1}{60}$ to convert minutes into hours.
2. Next, by multiplying $24(20\cdot\frac{1}{60})$.
3. Which of the properties of real numbers says that your answers to parts (a), where you multiplied $(24\cdot 20)\frac{1}{60}$, and (b), where you multiplied $24(20\cdot\frac{1}{60})$, should be equal?

Exercise 97

Buying by the case Trader Joe's grocery stores sold a bottle of wine they called "Two Buck Chuck" for $1.99. They sold a case of 12 bottles for $23.88. To find the cost of 12 bottles at $1.99, notice that 1.99 is 2−0.01.

1. Multiply 12(1.99) by using the distributive property to multiply 12(2−0.01).
2. Was it a bargain to buy "Two Buck Chuck" by the case?

Answer

1. $23.88
2. no, the price is the same

Exercise 98

Multi-pack purchase Adele's shampoo sells for $3.99 per bottle at the grocery store. At the warehouse store, the same shampoo is sold as a 3 pack for $10.49. To find the cost of 3 bottles at $3.99, notice that 3.99 is 4−0.01.

1. Multiply 3(3.99) by using the distributive property to multiply 3(4−0.01).
2. How much would Adele save by buying 3 bottles at the warehouse store instead of at the grocery store?

Writing Exercises

Exercise 99

In your own words, state the commutative property of addition.

Answer

$Answers may vary$

Exercise 100

What is the difference between the additive inverse and the multiplicative inverse of a number?

Exercise 101

Simplify $8\left(x-\frac{1}{4}\right)$ using the distributive property and explain each step.

Answer

$Answers may vary$

Exercise 102

Explain how you can multiply 4($5.97) without paper or calculator by thinking of $5.97 as 6−0.03 and then using the distributive property.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the commutative and associative properties.			
use the identity and inverse properties of addition and multiplication.			
use the properties of zero.			
simplify expressions using the distributive property.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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13.5.11: Systems of Measurement

Learning Objectives

By the end of this section, you will be able to:

- Make unit conversions in the US system
- Use mixed units of measurement in the US system
- Make unit conversions in the metric system
- Use mixed units of measurement in the metric system
- Convert between the US and the metric systems of measurement
- Convert between Fahrenheit and Celsius temperatures

Note

A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **The Properties of Real Numbers**.

Make Unit Conversions in the U.S. System

There are two systems of measurement commonly used around the world. Most countries use the metric system. The U.S. uses a different system of measurement, usually called the **U.S. system**. We will look at the U.S. system first.

The U.S. system of measurement uses units of inch, foot, yard, and mile to measure length and pound and ton to measure weight. For capacity, the units used are cup, pint, quart, and gallons. Both the U.S. system and the metric system measure time in seconds, minutes, and hours.

The equivalencies of measurements are shown in Table 13.5.11.1 The table also shows, in parentheses, the common abbreviations for each measurement.

Table 13.5.11.1

U.S. System of Measurement			
Length	1 foot (ft.)	=	12 inches (in.)
	1 yard (yd.)	=	3 feet (ft.)
	1 mile (mi.)	=	5280 feet (ft.)
Volume	3 teaspoons (t)	=	1 tablespoon (T)
	16 tablespoons (T)	=	1 cup (C)
	1 cup (C)	=	8 fluid ounces (fl.oz.)
	1 pint (pt.)	=	2 cups (C)
	1 quart (qt.)	=	2 pints (ft.)
	1 gallon (gal)	=	4 quarts (qt.)
Weight	1 pound (lb.)	=	16 ounces (oz.)
	1 ton	=	2000 pounds (lb.)
Time	1 minute (min)	=	60 seconds (sec)
	1 hour (hr)	=	60 minutes (min)
	1 day	=	24 hours (hr)
	1 week (wk)	=	7 days
	1 year (yr)	=	365 days

In many real-life applications, we need to convert between units of measurement, such as feet and yards, minutes and seconds, quarts and gallons, etc. We will use the identity property of multiplication to do these conversions. We'll restate the identity property of multiplication here for easy reference.

IDENTITY PROPERTY OF MULTIPLICATION

For any real number a : $a \cdot 1 = a \quad 1 \cdot a = a$

1 is the **multiplicative identity**

To use the identity property of multiplication, we write 1 in a form that will help us convert the units. For example, suppose we want to change inches to feet. We know that 1 foot is equal to 12 inches, so we will write 1 as the fraction $\frac{1 \text{ foot}}{12 \text{ inches}}$. When we multiply by this fraction we do not change the value, but just change the units.

But $\frac{12 \text{ inches}}{1 \text{ foot}}$ also equals 1. How do we decide whether to multiply by $\frac{1 \text{ foot}}{12 \text{ inches}}$ or $\frac{12 \text{ inches}}{1 \text{ foot}}$? We choose the fraction that will make the units we want to convert *from* divide out. Treat the unit words like **factors** and "divide out" common units like we do common factors. If we want to convert 6666 inches to feet, which multiplication will eliminate the inches?

$$66 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \quad \text{or} \quad 66 \text{ inches} \cdot \frac{12 \text{ inches}}{1 \text{ foot}}$$

The first form works since $66 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}}$.

Figure 13.5.11.1

The inches divide out and leave only feet. The second form does not have any units that will divide out and so will not help us.

Example 13.5.11.1

Mary Anne is 66 inches tall. Convert her height into feet.

Solution

Step 1. Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.	Multiply 66 inches by 1, writing 1 as a fraction relating inches and feet. We need inches in the denominator so that the inches will divide out!	$66 \text{ inches} \cdot 1$ $66 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}}$
Step 2. Multiply.	Think of 66 inches as $\frac{66 \text{ inches}}{1}$.	$\frac{66 \text{ inches} \cdot 1 \text{ foot}}{12 \text{ inches}}$
Step 3. Simplify the fraction.	Notice: inches divide out.	$\frac{66 \text{ inches} \cdot 1 \text{ foot}}{12 \text{ inches}}$ $\frac{66 \text{ feet}}{12}$
Step 4. Simplify.	Divide 66 by 12.	5.5 feet

Try It 13.5.11.2

Lexie is 30 inches tall. Convert her height to feet.

Answer

2.5 feet

Try It 13.5.11.3

Rene bought a hose that is 18 yards long. Convert the length to feet.

Answer

54 feet

MAKE UNIT CONVERSIONS.

1. Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.
2. Multiply.
3. Simplify the fraction.
4. Simplify.

When we use the identity property of multiplication to convert units, we need to make sure the units we want to change from will divide out. Usually this means we want the conversion fraction to have those units in the denominator.

Example 13.5.11.4

Ndula, an elephant at the San Diego Safari Park, weighs almost 3.2 tons. Convert her weight to pounds.

Solution

We will convert 3.2 tons into pounds. We will use the identity property of multiplication, writing 1 as the fraction $\frac{2000 \text{ pounds}}{1 \text{ ton}}$.

	3.2 tons
Multiply the measurement to be converted, by 1.	$3.2 \text{ tons} \cdot 1$
Write 1 as a fraction relating tons and pounds.	$3.2 \text{ tons} \cdot \frac{2000 \text{ pounds}}{1 \text{ ton}}$
Simplify.	$\frac{3.2 \cancel{\text{tons}} \cdot 2{,}000 \text{ pounds}}{1 \cancel{\text{ton}}}$
Multiply.	6400 pounds
	Ndula weighs almost 6400 pounds.

Try It 13.5.11.5

Arnold's SUV weighs about 4.3 tons. Convert the weight to pounds.

Answer

8600 pounds

Try It 13.5.11.6

The Carnival *Destiny* cruise ship weighs 51000 tons. Convert the weight to pounds.

Answer

102000000 pounds

Sometimes, to convert from one unit to another, we may need to use several other units in between, so we will need to multiply several fractions.

Example 13.5.11.7

Juliet is going with her family to their summer home. She will be away from her boyfriend for 9 weeks. Convert the time to minutes.

Solution

To convert weeks into minutes we will convert weeks into days, days into hours, and then hours into minutes. To do this we will multiply by conversion factors of 1.

	9 weeks
Write 1 as $\frac{7\text{ days}}{1\text{ week}}$, and $\frac{60\text{ minutes}}{1\text{ hour}}$.	$\frac{9\text{ wk}}{1}\cdot\frac{7\text{ days}}{1\text{ wk}}\cdot\frac{24\text{ hr}}{1\text{ day}}\cdot\frac{60\text{ min}}{1\text{ hr}}$
Divide out the common units.	9 weeks Write 1 as $\frac{7\text{ days}}{1\text{ week}}$, $\frac{24\text{ hours}}{1\text{ day}}$, and $\frac{60\text{ minutes}}{1\text{ hour}}$. $\frac{9\text{ wk}}{1}\cdot\frac{7\text{ days}}{1\text{ wk}}\cdot\frac{24\text{ hr}}{1\text{ day}}\cdot\frac{60\text{ min}}{1\text{ hr}}$ Divide out the common units. $\frac{9\,\cancel{\text{wk}}}{1}\cdot\frac{7\,\cancel{\text{days}}}{1\,\cancel{\text{wk}}}\cdot\frac{24\,\cancel{\text{hr}}}{1\,\cancel{\text{day}}}\cdot\frac{60\text{ min}}{1\,\cancel{\text{hr}}}$ Multiply. $\frac{9\cdot7\cdot24\cdot60\text{ min}}{1\cdot1\cdot1\cdot1}$ Multiply. 90,720 min
Multiply.	$\frac{9\cdot7\cdot24\cdot60\text{ min}}{1\cdot1\cdot1\cdot1}$
Multiply.	90,720 min

Juliet and her boyfriend will be apart for 90,720 minutes (although it may seem like an eternity!).

Try It 13.5.11.8

The distance between the earth and the moon is about 250,000 miles. Convert this length to yards.

Answer

440,000,000 yards

Try It 13.5.11.9

The astronauts of Expedition 28 on the International Space Station spend 15 weeks in space. Convert the time to minutes.

Answer

151,200 minutes

Example 13.5.11.10

How many ounces are in 1 gallon?

Solution

We will convert gallons to ounces by multiplying by several conversion factors. Refer to Table 13.5.11.1

	1 gallon
Multiply the measurement to be converted by 1.	$\frac{1\text{ gallon}}{1}\cdot\frac{4\text{ quarts}}{1\text{ gallon}}\cdot\frac{2\text{ pints}}{1\text{ quart}}\cdot\frac{2\text{ cups}}{1\text{ pint}}\cdot\frac{8\text{ ounces}}{1\text{ cup}}$
Use conversion factors to get to the right unit. Simplify.	1 gallon Multiply the measurement to be converted by 1. $\frac{1\text{ gallon}}{1}\cdot\frac{4\text{ quarts}}{1\text{ gallon}}\cdot\frac{2\text{ pints}}{1\text{ quart}}\cdot\frac{2\text{ cups}}{1\text{ pint}}\cdot\frac{8\text{ ounces}}{1\text{ cup}}$ Use conversion factors to get to the right unit. Simplify. $\frac{1\,\cancel{\text{gallon}}}{1}\cdot\frac{4\text{ quarts}}{1\,\cancel{\text{gallon}}}\cdot\frac{2\,\cancel{\text{pints}}}{1\text{ quart}}\cdot\frac{2\,\cancel{\text{cups}}}{1\,\cancel{\text{pint}}}\cdot\frac{8\text{ ounces}}{1\,\cancel{\text{cup}}}$ Multiply. $\frac{1\cdot4\cdot2\cdot2\cdot8\text{ ounces}}{1\cdot1\cdot1\cdot1\cdot1}$ Simplify. 128 ounces There are 128 ounces in a gallon.
Multiply.	$\frac{1\cdot4\cdot2\cdot2\cdot8\text{ ounces}}{1\cdot1\cdot1\cdot1\cdot1}$
Simplify.	128 ounces

Try It 13.5.11.11

How many cups are in 1 gallon?

Answer

16 cups

Try It 13.5.11.12

How many teaspoons are in 1 cup?

Answer

48 teaspoons

Use Mixed Units of Measurement in the U.S. System

We often use mixed units of measurement in everyday situations. Suppose Joe is 5 feet 10 inches tall, stays at work for 7 hours and 45 minutes, and then eats a 1 pound 2 ounce steak for dinner—all these measurements have mixed units.

Performing arithmetic operations on measurements with mixed units of measures requires care. Be sure to add or subtract like units!

Example 13.5.11.13

Seymour bought three steaks for a barbecue. Their weights were 14 ounces, 1 pound 2 ounces and 1 pound 6 ounces. How many total pounds of steak did he buy?

Solution

We will add the weights of the steaks to find the total weight of the steaks.

Add the ounces. Then add the pounds.	14 ounces 1 pound 2 ounces + 1 pound 6 ounces 2 pounds 22 ounces
Convert 22 ounces to pounds and ounces.	2 pounds + 1 pound, 6 ounces 3 pounds, 6 ounces
Add the pounds.	Seymour bought 3 pounds 6 ounces of steak.

Add the ounces. Then add the pounds.	
Convert 22 ounces to pounds and ounces.	2 pounds + 1 pound, 6 ounces
Add the pounds.	3 pounds, 6 ounces
	Seymour bought 3 pounds 6 ounces of steak.

Try It 13.5.11.14

Laura gave birth to triplets weighing 3 pounds 3 ounces, 3 pounds 3 ounces, and 2 pounds 9 ounces. What was the total birth weight of the three babies?

Answer

8 lbs. 15 oz

Try It 13.5.11.15

Stan cut two pieces of crown molding for his family room that were 8 feet 7 inches and 12 feet 11 inches. What was the total length of the molding?

Answer

21 ft. 6 in.

Example 13.5.11.16

Anthony bought four planks of wood that were each 6 feet 4 inches long. What is the total length of the wood he purchased?

Solution

We will multiply the length of one plank to find the total length.

Multiply the inches and then the feet.	6 feet 4 inches × 4 24 feet 16 inches
Convert the 16 inches to feet. Add the feet.	24 feet + 1 foot 4 inches 25 feet 4 inches
	Anthony bought 25 feet and 4 inches of wood.

Try It 13.5.11.17

Henri wants to triple his spaghetti sauce recipe that uses 1 pound 8 ounces of ground turkey. How many pounds of ground turkey will he need?

Answer

4 lbs. 8 oz.

Try It 13.5.11.18

Joellen wants to double a solution of 5 gallons 3 quarts. How many gallons of solution will she have in all?

Answer

11 gallons 2 qt.

Make Unit Conversions in the Metric System

In the **metric system**, units are related by powers of 10. The roots words of their names reflect this relation. For example, the basic unit for measuring length is a meter. One kilometer is 1,000 meters; the prefix *kilo* means *thousand*. One centimeter is $\frac{1}{100}$ of a meter, just like one cent is $\frac{1}{100}$ of one dollar.

The equivalencies of measurements in the metric system are shown in Table 13.5.11.2 The common abbreviations for each measurement are given in parentheses.

Table 13.5.11.2

Metric System of Measurement		
Length	Mass	Capacity

Metric System of Measurement		
Length	Mass	Capacity
1 kilometer (km) = 1,000 m	1 kilogram (kg) = 1,000 g	1 kiloliter (kL) = 1,000 L
1 hectometer (hm) = 100 m	1 hectogram (hg) = 100 g	1 hectoliter (hL) = 100 L
1 dekameter (dam) = 10 m	1 dekagram (dag) = 10 g	1 dekaliter (daL) = 10 L
1 meter (m) = 1 m	1 gram (g) = 1 g	1 liter (L) = 1 L
1 decimeter (dm) = 0.1 m	1 decigram (dg) = 0.1 g	1 deciliter (dL) = 0.1 L
1 centimeter (cm) = 0.01 m	1 centigram (cg) = 0.01 g	1 centiliter (cL) = 0.01 L
1 millimeter (mm) = 0.001 m	1 milligram (mg) = 0.001 g	1 milliliter (mL) = 0.001 L
1 meter = 100 centimeters	1 gram = 100 centigrams	1 liter = 100 centiliters
1 meter = 1,000 millimeters	1 gram = 1,000 milligrams	1 liter = 1,000 milliliters

To make conversions in the metric system, we will use the same technique we did in the US system. Using the identity property of multiplication, we will multiply by a conversion factor of one to get to the correct units.

Have you ever run a 5K or 10K race? The length of those races are measured in kilometers. The metric system is commonly used in the United States when talking about the length of a race.

Example 13.5.11.19

Nick ran a 10K race. How many meters did he run?

Solution

We will convert kilometers to meters using the identity property of multiplication.

	10 kilometers
Multiply the measurement to be converted by 1.	$10 \text{ kilometers} \cdot 1$
Write 1 as a fraction relating kilometers and meters.	$10 \text{ kilometers} \cdot \frac{1{,}000 \text{ meters}}{1 \text{ kilometers}}$
Simplify.	$\frac{10 \text{ kilometers} \cdot 1{,}000 \text{ m}}{1 \text{ kilometers}}$
Multiply.	10,000 meters
	Nick ran 10,000 meters.

Try It 13.5.11.20

Sandy completed her first 5K race! How many meters did she run?

Answer

5,000 meters

Try It 13.5.11.21

Herman bought a rug 2.5 meters in length. How many centimeters is the length?

Answer

250 centimeters

Example 13.5.11.22

Eleanor's newborn baby weighed 3,200 grams. How many kilograms did the baby weigh?

Solution

We will convert grams into kilograms.

	$3{,}200\ \text{grams}$
Multiply the measurement to be converted by 1.	$3{,}200\ \text{grams} \cdot 1$
Write 1 as a function relating kilograms and grams.	$3{,}200\ \text{grams} \cdot \frac{1\ \text{kg}}{1{,}000\ \text{grams}}$
Simplify.	$3{,}200\ \cancel{\text{grams}} \cdot \frac{1\ \text{kg}}{1{,}000\ \cancel{\text{grams}}}$
Multiply.	$\frac{3{,}200\ \text{kilograms}}{1{,}000}$
Divide.	3.2 kilograms The baby weighed 3.2 kilograms.

Try It 13.5.11.23

Kari's newborn baby weighed 2,800 grams. How many kilograms did the baby weigh?

Answer

2.8 kilograms

Try It 13.5.11.24

Anderson received a package that was marked 4,500 grams. How many kilograms did this package weigh?

Answer

4.5 kilograms

As you become familiar with the metric system you may see a pattern. Since the system is based on multiples of ten, the calculations involve multiplying by multiples of ten. We have learned how to simplify these calculations by just moving the decimal.

To multiply by 10, 100, or 1,000, we move the decimal to the right one, two, or three places, respectively. To multiply by 0.1, 0.01, or 0.001, we move the decimal to the left one, two, or three places, respectively.

We can apply this pattern when we make measurement conversions in the metric system. In Exercise 13.5.11.25 we changed 3,200 grams to kilograms by multiplying by $\frac{1}{1000}$ (or 0.001). This is the same as moving the decimal three places to the left.

$$3{,}200 \cdot \frac{1}{1{,}000} \qquad 3{,}200.$$
$$3.2 \qquad\qquad 3.2$$

Figure 13.5.11.2

Example 13.5.11.25

Convert

1. 350 L to kiloliters
2. 4.1 L to milliliters.

Solution

1. We will convert liters to kiloliters. In Table 13.5.11.2 we see that 1 kiloliter=1,000 liters.1kiloliter=1,000 liters.

	350 L
Multiply by 1, writing 1 as a fraction relating liters to kiloliters.	$350\ \text{L}\ \frac{1\ \text{kL}}{1000\text{L}}$
Simplify.	$350\ \cancel{\text{L}}\ \frac{1\ \text{kL}}{1000\ \cancel{\text{L}}}$
Multiply by 1, writing 1 as a fraction relating liters to kiloliters. Simplify. Move the decimal 3 units to the left. (350.)	350 L $350\ \cancel{\text{L}} \cdot \frac{1\ \text{kL}}{1{,}000\ \cancel{\text{L}}}$ 0.35 kL
	0.35 kL

2. We will convert liters to milliliters. From Table 13.5.11.2we see that 1 liter=1,000 milliliters.1 liter=1,000 milliliters.

	4.1 L
Multiply by 1, writing 1 as a fraction relating liters to milliliters.	$4.1\ \text{L} \cdot \frac{1{,}000\ \text{mL}}{1\ \text{L}}$
Simplify.	$4.1\ \cancel{\text{L}} \cdot \frac{1{,}000\ \text{mL}}{1\ \cancel{\text{L}}}$
Move the decimal 3 units to the right.	4.100 mL
	4,100 mL

Try It 13.5.11.26

Convert:

1. 725 L to kiloliters
2. 6.3 L to milliliters

Answer

1. 7,250 kiloliters
2. 6,300 milliliters

Try It 13.5.11.27

Convert:

1. 350 hL to liters
2. 4.1 L to centiliters

Answer

1. 35,000 liters
2. 410 centiliters

Use Mixed Units of Measurement in the Metric System

Performing arithmetic operations on measurements with mixed units of measures in the metric system requires the same care we used in the US system. But it may be easier because of the relation of the units to the powers of 10. Make sure to add or subtract like units.

Example 13.5.11.28

Ryland is 1.6 meters tall. His younger brother is 85 centimeters tall. How much taller is Ryland than his younger brother?

Solution

We can convert both measurements to either centimeters or meters. Since meters is the larger unit, we will subtract the lengths in meters. We convert 85 centimeters to meters by moving the decimal 2 places to the left.

$$\begin{array}{ll} \text{Write the 85 centimeters as meters.} & 1.60\text{m} \\ & \underline{-0.85\text{m}} \\ & 0.75\text{m} \end{array} \qquad (13.5.11.1)$$

Ryland is 0.75 m0.75 m taller than his brother.

Try It 13.5.11.29

Mariella is 1.58 meters tall. Her daughter is 75 centimeters tall. How much taller is Mariella than her daughter? Write the answer in centimeters.

Answer

83 centimeters

Try It 13.5.11.30

The fence around Hank's yard is 2 meters high. Hank is 96 centimeters tall. How much shorter than the fence is Hank? Write the answer in meters.

Answer

1.04 meters

Example 13.5.11.31

Dena's recipe for lentil soup calls for 150 milliliters of olive oil. Dena wants to triple the recipe. How many liters of olive oil will she need?

Solution

We will find the amount of olive oil in milliliters then convert to liters.

	Triple 150 mL
Translate to algebra.	$3 \cdot 150$ mL
Multiply.	450 mL
Convert to liters.	$450 \cdot \frac{0.001\text{ L}}{1\text{ ml}}$
Simplify.	0.45 L
	Dena needs 0.45 liters of olive oil.

Try It 13.5.11.32

A recipe for Alfredo sauce calls for 250 milliliters of milk. Renata is making pasta with Alfredo sauce for a big party and needs to multiply the recipe amounts by 8. How many liters of milk will she need?

Answer

2 liters

Try It 13.5.11.33

To make one pan of baklava, Dorothea needs 400 grams of filo pastry. If Dorothea plans to make 6 pans of baklava, how many kilograms of filo pastry will she need?

Answer

2.4 kilograms

Convert Between the U.S. and the Metric Systems of Measurement

Many measurements in the United States are made in metric units. Our soda may come in 2-liter bottles, our calcium may come in 500-mg capsules, and we may run a 5K race. To work easily in both systems, we need to be able to convert between the two systems.

Table 13.5.11.3shows some of the most common conversions.

Table 13.5.11.3

Conversion Factors Between U.S. and Metric Systems		
Length	Mass	Capacity
1 in. = 2.54 cm		
1 ft. = 0.305 m	1 lb. = 0.45 kg	1 qt. = 0.95 L
1 yd. = 0.914 m	1 oz. = 28 g	1 fl. oz. = 30 ml
1 mi. = 1.61 km	1 kg = 2.2 lb	1 L = 1.06 lb
1 m = 3.28 ft		

Figure 13.5.11.3shows how inches and centimeters are related on a ruler.

Figure 13.5.11.3: This ruler shows inches and centimeters.

Figure 13.5.11.4shows the ounce and milliliter markings on a measuring cup.

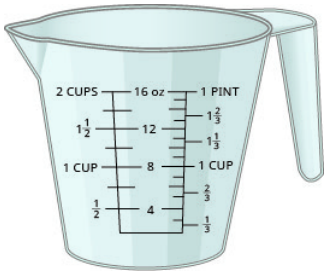

Figure 13.5.11.4: This measuring cup shows ounces and cups.

Figure 13.5.11.5shows how pounds and kilograms marked on a bathroom scale.

Figure 13.5.11.5: This scale shows pounds and kilograms.

We make conversions between the systems just as we do within the systems—by multiplying by unit conversion factors.

Example 13.5.11.34

Lee's water bottle holds 500 mL of water. How many ounces are in the bottle? Round to the nearest tenth of an ounce.

Solution

	500 mL
Multiplying by a unit conversion factor relating mL and ounces	$500 \text{ milliliters} \cdot \frac{1 \text{ ounce}}{30 \text{ milliliters}}$
Simplify.	$\frac{50 \text{ ounce}}{30}$
Divide.	16.7 ounces
	The water bottle has 16.7 ounces

Try It 13.5.11.35

How many quarts of soda are in a 2-L bottle?

Answer

2.12 quarts

Try It 13.5.11.36

How many liters are in 4 quarts of milk?

Answer

3.8 liters

Example 13.5.11.37

Soleil was on a road trip and saw a sign that said the next rest stop was in 100 kilometers. How many miles until the next rest stop?

Solution

	100 kilometers
Multiplying by a unit conversion factor relating km and mi.	$100 \text{ kilometers} \cdot \frac{1 \text{ mile}}{1.61 \text{ kilometers}}$
Simplify.	$\frac{100 \text{ miles}}{1.61}$
Divide.	62 miles
	Soleil will travel 62 miles.

Try It 13.5.11.38

The height of Mount Kilimanjaro is 5,895 meters. Convert the height to feet.

Answer

19,335.6 feet

Try It 13.5.11.39

The flight distance from New York City to London is 5,586 kilometers. Convert the distance to miles.

Answer

3,469.57 miles

Convert between Fahrenheit and Celsius Temperatures

Have you ever been in a foreign country and heard the weather forecast? If the forecast is for 22°C, what does that mean?

The U.S. and metric systems use different scales to measure temperature. The U.S. system uses degrees Fahrenheit, written °F. The metric system uses degrees Celsius, written °C. Figure 13.5.11.6shows the relationship between the two systems.

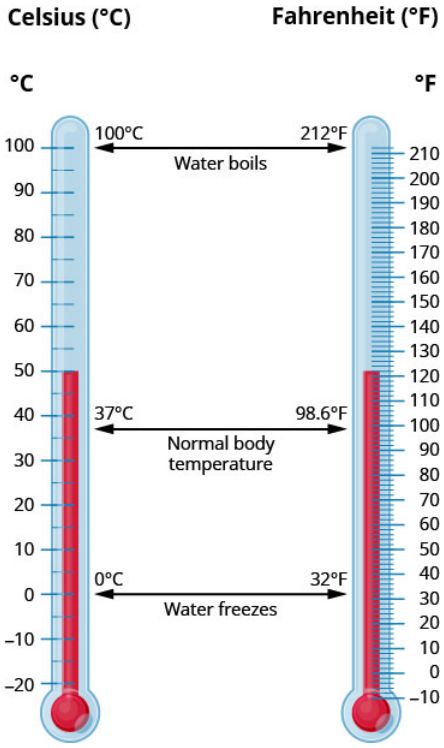

Figure 13.5.11.6: The diagram shows normal body temperature, along with the freezing and boiling temperatures of water in degrees Fahrenheit and degrees Celsius.

TEMPERATURE CONVERSION

To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula

$$C = \frac{5}{9}(F - 32) \tag{13.5.11.2}$$

To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula

$$F = \frac{9}{5}C + 32 \tag{13.5.11.3}$$

Example 13.5.11.40

Convert 50° Fahrenheit into degrees Celsius.

Solution

We will substitute 50°F into the formula to find C.

	$C = \frac{5}{9}(F - 32)$
Substitute 50 for F.	$C = \frac{5}{9}(50 - 32)$
Simplify in parentheses.	$C = \frac{5}{9}(18)$
Multiply.	$C = 10$
	So we found that 50°F is equivalent to 10°C.

Try It 13.5.11.41

Convert the Fahrenheit temperature to degrees Celsius: 59° Fahrenheit.

Answer

15°C

Try It 13.5.11.42

Convert the Fahrenheit temperature to degrees Celsius: 41° Fahrenheit.

Answer

5°C

Example 13.5.11.43

While visiting Paris, Woody saw the temperature was 20° Celsius. Convert the temperature into degrees Fahrenheit.

Solution

We will substitute 20°C into the formula to find F.

	$F = \frac{9}{5}C + 32$
Substitute 20 for C.	$F = \frac{9}{5}(20) + 32$
Multiply.	$F = 36 + 32$
Add.	$F = 68$
	So we found that 20°C is equivalent to 68°F.

Try It 13.5.11.44

Convert the Celsius temperature to degrees Fahrenheit: the temperature in Helsinki, Finland, was 15° Celsius.

Answer

59°F

Try It 13.5.11.45

Convert the Celsius temperature to degrees Fahrenheit: the temperature in Sydney, Australia, was 10° Celsius.

Answer

50°F

Key Concepts

- **Metric System of Measurement**
  - **Length**

    1 kilometer (km) = 1,000 m

    1 hectometer (hm) = 100 m

    1 dekameter (dam) = 10 m

    1 meter (m) = 1 m

    1 decimeter (dm) = 0.1 m

    1 centimeter (cm) = 0.01 m

    1 millimeter (mm) = 0.001 m

    1 meter = 100 centimeters

    1 meter = 1,000 millimeters
- **Mass**

  1 kilogram (kg) = 1,000 g

  1 hectogram (hg) = 100 g

  1 dekagram (dag) = 10 g

  1 gram (g) = 1 g

  1 decigram (dg) = 0.1 g

  1 centigram (cg) = 0.01 g

  1 milligram (mg) = 0.001 g

  1 gram = 100 centigrams

  1 gram = 1,000 milligrams
- **Capacity**

  1 kiloliter (kL) = 1,000 L

1 hectoliter (hL) = 100 L

1 dekaliter (daL) = 10 L

1 liter (L) = 1 L

1 deciliter (dL) = 0.1 L

1 centiliter (cL) = 0.01 L

1 milliliter (mL) = 0.001 L

1 liter = 100 centiliters

1 liter = 1,000 milliliters

- **Temperature Conversion**
  - To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula $C = \frac{5}{9}(F - 32)$
  - To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula $F = \frac{9}{5}C + 32$

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13.5.11E: Exercises

Practice Makes Perfect

Make Unit Conversions in the U.S. System

In the following exercises, convert the units.

Exercise 13.5.11E.1

A park bench is 6 feet long. Convert the length to inches.

Answer

72 inches

Exercise 13.5.11E.2

A floor tile is 2 feet wide. Convert the width to inches.

Exercise 13.5.11E.3

A ribbon is 18 inches long. Convert the length to feet.

Answer

1.5 feet

Exercise 13.5.11E.4

Carson is 45 inches tall. Convert his height to feet.

Exercise 13.5.11E.5

A football field is 160 feet wide. Convert the width to yards.

Answer

$53\frac{1}{3}$ yards

Exercise 13.5.11E.6

On a baseball diamond, the distance from home plate to first base is 30 yards. Convert the distance to feet.

Exercise 13.5.11E.7

On a baseball diamond, the distance from home plate to first base is 30 yards. Convert the distance to feet.

Answer

7,920 feet

Exercise 13.5.11E.8

Denver, Colorado, is 5,183 feet above sea level. Convert the height to miles.

Exercise 13.5.11E.9

A killer whale weighs 4.6 tons. Convert the weight to pounds.

Answer

9,200 pounds

Exercise 13.5.11E.10

Blue whales can weigh as much as 150 tons. Convert the weight to pounds.

Exercise 13.5.11E.11

An empty bus weighs 35,000 pounds. Convert the weight to tons.

Answer

$17\frac{1}{2}$ tons

Exercise 13.5.11E.12

At take-off, an airplane weighs 220,000 pounds. Convert the weight to tons.

Exercise 13.5.11E.13

Rocco waited $1\frac{1}{2}$ hours for his appointment. Convert the time to seconds.

Answer

5,400 s

Exercise 13.5.11E.14

Misty's surgery lasted $2\frac{1}{4}$ hours. Convert the time to seconds.

Exercise 13.5.11E.15

How many teaspoons are in a pint?

Answer

96 teaspoons

Exercise 13.5.11E.16

How many tablespoons are in a gallon?

Exercise 13.5.11E.17

JJ's cat, Posy, weighs 14 pounds. Convert her weight to ounces.

Answer

224 ounces

Exercise 13.5.11E.18

April's dog, Beans, weighs 8 pounds. Convert his weight to ounces.

Exercise 13.5.11E.19

Crista will serve 20 cups of juice at her son's party. Convert the volume to gallons.

Answer

$1\frac{1}{4}$ gallons

Exercise 13.5.11E.20

Lance needs 50 cups of water for the runners in a race. Convert the volume to gallons.

Exercise 13.5.11E.21

Jon is 6 feet 4 inches tall. Convert his height to inches.

Answer

26 in.

Exercise 13.5.11E.22

Faye is 4 feet 10 inches tall. Convert her height to inches.

Exercise 13.5.11E.23

The voyage of the *Mayflower* took 2 months and 5 days. Convert the time to days.

Answer

65 days

Exercise 13.5.11E.24

Lynn's cruise lasted 6 days and 18 hours. Convert the time to hours.

Exercise 13.5.11E.25

Baby Preston weighed 7 pounds 3 ounces at birth. Convert his weight to ounces.

Answer

115 ounces

Exercise 13.5.11E.26

Baby Audrey weighted 6 pounds 15 ounces at birth. Convert her weight to ounces.

Use Mixed Units of Measurement in the U.S. System

In the following exercises, solve.

Exercise 13.5.11E.27

Eli caught three fish. The weights of the fish were 2 pounds 4 ounces, 1 pound 11 ounces, and 4 pounds 14 ounces. What was the total weight of the three fish?

Answer

8 lbs. 13 oz.

Exercise 13.5.11E.28

Judy bought 1 pound 6 ounces of almonds, 2 pounds 3 ounces of walnuts, and 8 ounces of cashews. How many pounds of nuts did Judy buy?

Exercise 13.5.11E.29

One day Anya kept track of the number of minutes she spent driving. She recorded 45, 10, 8, 65, 20, and 35. How many hours did Anya spend driving?

Answer

3.05 hours

Exercise 13.5.11E.30

Last year Eric went on 6 business trips. The number of days of each was 5, 2, 8, 12, 6, and 3. How many weeks did Eric spend on business trips last year?

Exercise 13.5.11E.31

Renee attached a 6 feet 6 inch extension cord to her computer's 3 feet 8 inch power cord. What was the total length of the cords?

Answer

10 ft. 2 in.

Exercise 13.5.11E.32

Fawzi's SUV is 6 feet 4 inches tall. If he puts a 2 feet 10 inch box on top of his SUV, what is the total height of the SUV and the box?

Exercise 13.5.11E.33

Leilani wants to make 8 placemats. For each placemat she needs 18 inches of fabric. How many yards of fabric will she need for the 8 placemats?

Answer

4 yards

Exercise 13.5.11E.34

Mireille needs to cut 24 inches of ribbon for each of the 12 girls in her dance class. How many yards of ribbon will she need altogether?

Make Unit Conversions in the Metric System

In the following exercises, convert the units.

Exercise 13.5.11E.35

Ghalib ran 5 kilometers. Convert the length to meters.

Answer

5,000 meters

Exercise 13.5.11E.36

Kitaka hiked 8 kilometers. Convert the length to meters.

Exercise 13.5.11E.37

Estrella is 1.55 meters tall. Convert her height to centimeters.

Answer

155 centimeters

Exercise 13.5.11E.38

The width of the wading pool is 2.45 meters. Convert the width to centimeters.

Exercise 13.5.11E.39

Mount Whitney is 3,072 meters tall. Convert the height to kilometers.

Answer

3.072 kilometers

Exercise 13.5.11E.40

The depth of the Mariana Trench is 10,911 meters. Convert the depth to kilometers.

Exercise 13.5.11E.41

June’s multivitamin contains 1,500 milligrams of calcium. Convert this to grams.

Answer

1.5 grams

Exercise 13.5.11E.42

A typical ruby-throated hummingbird weights 3 grams. Convert this to milligrams.

Exercise 13.5.11E.43

One stick of butter contains 91.6 grams of fat. Convert this to milligrams.

Answer

91,600 milligrams

Exercise 13.5.11E.44

One serving of gourmet ice cream has 25 grams of fat. Convert this to milligrams.

Exercise 13.5.11E.45

The maximum mass of an airmail letter is 2 kilograms. Convert this to grams.

Answer

2,000 grams

Exercise 13.5.11E.46

Dimitri's daughter weighed 3.8 kilograms at birth. Convert this to grams.

Exercise 13.5.11E.47

A bottle of wine contained 750 milliliters. Convert this to liters.

Answer

0.75 liters

Exercise 13.5.11E.48

A bottle of medicine contained 300 milliliters. Convert this to liters.

Use Mixed Units of Measurement in the Metric System

In the following exercises, solve.

Exercise 13.5.11E.49

Matthias is 1.8 meters tall. His son is 89 centimeters tall. How much taller is Matthias than his son?

Answer

91 centimeters

Exercise 13.5.11E.50

Stavros is 1.6 meters tall. His sister is 95 centimeters tall. How much taller is Stavros than his sister?

Exercise 13.5.11E.51

A typical dove weighs 345 grams. A typical duck weighs 1.2 kilograms. What is the difference, in grams, of the weights of a duck and a dove?

Answer

855 grams

Exercise 13.5.11E.52

Concetta had a 2-kilogram bag of flour. She used 180 grams of flour to make biscotti. How many kilograms of flour are left in the bag?

Exercise 13.5.11E.53

Harry mailed 5 packages that weighed 420 grams each. What was the total weight of the packages in kilograms?

Answer

2.1 kilograms

Exercise 13.5.11E.54

One glass of orange juice provides 560 milligrams of potassium. Linda drinks one glass of orange juice every morning. How many grams of potassium does Linda get from her orange juice in 30 days?

Exercise 13.5.11E.55

Jonas drinks 200 milliliters of water 8 times a day. How many liters of water does Jonas drink in a day?

Answer

1.6 liters

Exercise 13.5.11E.56

One serving of whole grain sandwich bread provides 6 grams of protein. How many milligrams of protein are provided by 7 servings of whole grain sandwich bread?

Convert Between the U.S. and the Metric Systems of Measurement

In the following exercises, make the unit conversions. Round to the nearest tenth.

Exercise 13.5.11E.57

Bill is 75 inches tall. Convert his height to centimeters.

Answer

190.5 centimeters

Exercise 13.5.11E.58

Frankie is 42 inches tall. Convert his height to centimeters.

Exercise 13.5.11E.59

Marcus passed a football 24 yards. Convert the pass length to meters

Answer

21.9 meters

Exercise 13.5.11E.60

Connie bought 9 yards of fabric to make drapes. Convert the fabric length to meters.

Exercise 13.5.11E.61

Each American throws out an average of 1,650 pounds of garbage per year. Convert this weight to kilograms.

Answer

742.5 kilograms

Exercise 13.5.11E.62

An average American will throw away 90,000 pounds of trash over his or her lifetime. Convert this weight to kilograms.

Exercise 13.5.11E.63

A 5K run is 5 kilometers long. Convert this length to miles.

Answer

3.1 miles

Exercise 13.5.11E.64

Kathryn is 1.6 meters tall. Convert her height to feet.

Exercise 13.5.11E.65

Dawn's suitcase weighed 20 kilograms. Convert the weight to pounds.

Answer

44 pounds

Exercise 13.5.11E.66

Jackson's backpack weighed 15 kilograms. Convert the weight to pounds.

Exercise 13.5.11E.67

Ozzie put 14 gallons of gas in his truck. Convert the volume to liters.

Answer

53.2 liters

Exercise 13.5.11E.68

Bernard bought 8 gallons of paint. Convert the volume to liters.

Convert between Fahrenheit and Celsius Temperatures

In the following exercises, convert the Fahrenheit temperatures to degrees Celsius. Round to the nearest tenth.

Exercise 13.5.11E.69

86° Fahrenheit

Answer

30°C

Exercise 13.5.11E.70

77° Fahrenheit

Exercise 13.5.11E.71

104° Fahrenheit

Answer

40°C

Exercise 13.5.11E.72

14° Fahrenheit

Exercise 13.5.11E.73

72° Fahrenheit

Answer

22.2°C

Exercise 13.5.11E.74

4° Fahrenheit

Exercise 13.5.11E.75

0° Fahrenheit

Answer

−17.8°C

Exercise 13.5.11E.76

120° Fahrenheit

In the following exercises, convert the Celsius temperatures to degrees Fahrenheit. Round to the nearest tenth.

Exercise 13.5.11E.77

5° Celsius

Answer

41°F

Exercise 13.5.11E.78

25° Celsius

Exercise 13.5.11E.79

−10° Celsius

Answer

14°F

Exercise 13.5.11E.80

−15° Celsius

Exercise 13.5.11E.81

22° Celsius

Answer

71.6°F

Exercise 13.5.11E.82

8° Celsius

Exercise 13.5.11E.83

43° Celsius

Answer

109.4°F

Exercise 13.5.11E.84

16° Celsius

Everyday Math

Exercise 13.5.11E.85

Nutrition Julian drinks one can of soda every day. Each can of soda contains 40 grams of sugar. How many kilograms of sugar does Julian get from soda in 1 year?

Answer

14.6 kilograms

Exercise 13.5.11E.86

Reflectors The reflectors in each lane-marking stripe on a highway are spaced 16 yards apart. How many reflectors are needed for a one mile long lane-marking stripe?

Writing Exercises

Exercise 13.5.11E. 87

Some people think that 65° to 75° Fahrenheit is the ideal temperature range.

1. What is your ideal temperature range? Why do you think so?
2. Convert your ideal temperatures from Fahrenheit to Celsius.

Answer

Answers may vary.

Exercise 13.5.11E. 88

1. Did you grow up using the U.S. or the metric system of measurement?
2. Describe two examples in your life when you had to convert between the two systems of measurement.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
define U.S. units of measurement and convert from one unit to another.			
use U.S units of measurement.			
define metric units of measurement and convert from one unit to another.			
use metric units of measurement.			
convert between the U.S. and the metric system of measurement.			
convert between Fahrenheit and Celsius temperatures.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

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13.5.12: Chapter 1 Review Exercises

Chapter Review Exercises

Introduction to Whole Numbers

Use Place Value with Whole Number

In the following exercises find the place value of each digit.

Exercise 13.5.12.1

26,915

1. 1
2. 2
3. 9
4. 5
5. 6

Answer

1. tens
2. ten thousands
3. hundreds
4. ones
5. thousands

Exercise 13.5.12.2

359,417

1. 9
2. 3
3. 4
4. 7
5. 1

Exercise 13.5.12.3

58,129,304

1. 5
2. 0
3. 1
4. 8
5. 2

Answer

1. ten millions
2. tens
3. hundred thousands
4. millions
5. ten thousands

Exercise 13.5.12.4

9,430,286,157

1. 6
2. 4
3. 9
4. 0
5. 5

In the following exercises, name each number.

Exercise 13.5.12.5

6,104

Answer

six thousand, one hundred four

Exercise 13.5.12.6

493,068

Exercise 13.5.12.7

3,975,284

Answer

three million, nine hundred seventy-five thousand, two hundred eighty-four

Exercise 13.5.12.8

85,620,435

In the following exercises, write each number as a whole number using digits.

Exercise 13.5.12.9

three hundred fifteen

Answer

315

Exercise 13.5.12.10

sixty-five thousand, nine hundred twelve

Exercise 13.5.12.11

ninety million, four hundred twenty-five thousand, sixteen

Answer

90,425,016

Exercise 13.5.12.12

one billion, forty-three million, nine hundred twenty-two thousand, three hundred eleven

In the following exercises, round to the indicated place value.

Exercise 13.5.12.13

Round to the nearest ten.

1. 407
2. 8,564

Answer

1. 410
2. 8,560

Exercise 13.5.12.14

Round to the nearest hundred.

1. 25,846
2. 25,864

In the following exercises, round each number to the nearest 1. hundred 2. thousand 3. ten thousand.

Exercise 13.5.12.15

864,951

Answer

1. 865,000865,000
2. 865,000865,000
3. 860,000

Exercise 13.5.12.16

3,972,849

Identify Multiples and Factors

In the following exercises, use the divisibility tests to determine whether each number is divisible by 2, by 3, by 5, by 6, and by 10.

Exercise 13.5.12.17

168

Answer

by 2,3,6

Exercise 13.5.12.18

264

Exercise 13.5.12.19

375

Answer

by 3,5

Exercise 13.5.12.20

750

Exercise 13.5.12.21

1430

Answer

by 2,5,10

Exercise 13.5.12.22

1080

Find Prime Factorizations and Least Common Multiples

In the following exercises, find the prime factorization.

Exercise 13.5.12.23

420

Answer

$2 \cdot 2 \cdot 3 \cdot 5 \cdot 7$

Exercise 13.5.12.24

115

Exercise 13.5.12.25

225

Answer

$3 \cdot 3 \cdot 5 \cdot 5$

Exercise 13.5.12.26

2475

Exercise 13.5.12.27

1560

Answer

$2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 13$

Exercise 13.5.12.28

56

Exercise 13.5.12.29

72

Answer

$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

Exercise 13.5.12.30

168

Exercise 13.5.12.31

252

Answer

$2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$

Exercise 13.5.12.32

391

In the following exercises, find the least common multiple of the following numbers using the multiples method.

Exercise 13.5.12.33

6,15

Answer

30

Exercise 13.5.12.34

60, 75

In the following exercises, find the least common multiple of the following numbers using the prime factors method.

Exercise 13.5.12.35

24, 30

Answer

120

Exercise 13.5.12.36

70, 84

Use the Language of Algebra

Use Variables and Algebraic Symbols

In the following exercises, translate the following from algebra to English.

Exercise 13.5.12.37

$25-7$

Answer

25 minus 7, the difference of twenty-five and seven

Exercise 13.5.12.38

$5 \cdot 6$

Exercise 13.5.12.39

$45 \div 5$

Answer

45 divided by 5, the quotient of forty-five and five

Exercise 13.5.12.40

x+8

Exercise 13.5.12.41

$42 \geq 27$

Answer

forty-two is greater than or equal to twenty-seven

Exercise 13.5.12.42

3n=24

Exercise 13.5.12.43

$3 \leq 20 \div 4$

Answer

3 is less than or equal to 20 divided by 4, three is less than or equal to the quotient of twenty and four

Exercise 13.5.12.44

$a \neq 7 \cdot 4$

In the following exercises, determine if each is an expression or an equation.

Exercise 13.5.12.45

$6 \cdot 3 + 5$

Answer

expression

Exercise 13.5.12.46

y−8=32

Simplify Expressions Using the Order of Operations

In the following exercises, simplify each expression.

Exercise 13.5.12.47

3^5

Answer

243

Exercise 13.5.12.48

10^8

In the following exercises, simplify

Exercise 13.5.12.49

6+10/2+2

Answer

13

Exercise 13.5.12.50

9+12/3+4

Exercise 13.5.12.51

$20 \div (4 + 6) \cdot 5$

Answer

10

Exercise 13.5.12.52

$33 \div (3+8) \cdot 2$

Exercise 13.5.12.53

$4^2 + 5^2$

Answer

41

Exercise 13.5.12.54

$(4+5)^2$

Evaluate an Expression

In the following exercises, evaluate the following expressions.

Exercise 13.5.12.55

9x+7 when x=3

Answer

34

Exercise 13.5.12.56

5x−4 when x=6

Exercise 13.5.12.57

x^4 when $x = 3$

Answer

81

Exercise 13.5.12.58

3^x when $x = 3$

Exercise 13.5.12.59

$x^2 + 5x - 8$ when $x = 6$

Answer

58

Exercise 13.5.12.60

$2x + 4y - 5$ when
$x = 7, y = 8$

Simplify Expressions by Combining Like Terms

In the following exercises, identify the coefficient of each term.

Exercise 13.5.12.61

12n

Answer

12

Exercise 13.5.12.62

$9x^2$

In the following exercises, identify the like terms.

Exercise 13.5.12.63

$3n, n^2, 12, 12p^2, 3, 3n^2$

Answer

12 and 3, n^2 and $3n^2$

Exercise 13.5.12.64

$5, 18r^2, 9s, 9r, 5r^2, 5s$

In the following exercises, identify the terms in each expression.

Exercise 13.5.12.65

$11x^2 + 3x + 6$

Answer

$11x^2, 3x, 6$

Exercise 13.5.12.66

$22y^3 + y + 15$

In the following exercises, simplify the following expressions by combining like terms.

Exercise 13.5.12.67

17a+9a

Answer

26a

Exercise 13.5.12.68

18z+9z

Exercise 13.5.12.69

9x+3x+8

Answer

12x+8

Exercise 13.5.12.70

8a+5a+9

Exercise 13.5.12.71

7p+6+5p−4

Answer

12p+2

Exercise 13.5.12.72

8x+7+4x−5

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the following phrases into algebraic expressions.

Exercise 13.5.12.73

the sum of 8 and 12

Answer

8+12

Exercise 13.5.12.74

the sum of 9 and 1

Exercise 13.5.12.75

the difference of x and 4

Answer

x−4

Exercise 13.5.12.76

the difference of x and 3

Exercise 13.5.12.77

the product of 6 and y

Answer

6y

Exercise 13.5.12.78

the product of 9 and y

Exercise 13.5.12.79

Adele bought a skirt and a blouse. The skirt cost $15 more than the blouse. Let bb represent the cost of the blouse. Write an expression for the cost of the skirt.

Answer

b+15

Exercise 13.5.12.80

Marcella has 6 fewer boy cousins than girl cousins. Let g represent the number of girl cousins. Write an expression for the number of boy cousins.

Add and Subtract Integers

Use Negatives and Opposites of Integers

In the following exercises, order each of the following pairs of numbers, using < or >.

Exercise 13.5.12.81

1. 6___2
2. −7___4
3. −9___−1
4. 9___−3

Answer

1. >
2. <
3. <
4. >

Exercise 13.5.12.82

1. −5___1
2. −4___−9
3. 6___10
4. 3___−8

In the following exercises,, find the opposite of each number.

Exercise 13.5.12.83

1. −8
2. 1

Answer

1. 8
2. −1

Exercise 13.5.12.84

1. −2
2. 6

In the following exercises, simplify.

Exercise 13.5.12.85

−(−19)

Answer

19

Exercise 13.5.12.86

−(−53)

In the following exercises, simplify.

Exercise 13.5.12.87

−m when

1. m=3
2. m=−3

Answer

1. −3
2. 3

Exercise 13.5.12.88

−p when

1. p=6
2. p=−6

Simplify Expressions with Absolute Value

In the following exercises,, simplify.

Exercise 13.5.12.89

1. |7|
2. |−25|
3. |0|

Answer

1. 7
2. 25
3. 0

Exercise 13.5.12.90

1. |5|
2. |0|
3. |−19|

In the following exercises, fill in <, >, or = for each of the following pairs of numbers.

Exercise 13.5.12.91

1. −8___|−8|
2. −|−2|___−2

Answer

1. <
2. =

Exercise 13.5.12.92

1. |−3|___−|−3|
2. 4___−|−4|

In the following exercises, simplify.

Exercise 13.5.12.93

|8−4|

Answer

4

Exercise 13.5.12.94

|9−6|

Exercise 13.5.12.95

8(14−2|−2|)

Answer

80

Exercise 13.5.12.96

6(13−4|−2|)

In the following exercises, evaluate.

Exercise 13.5.12.97

1. $|x|$ when $x=-28$

Answer

1. 28
2. 15

Exercise 13.5.12.98

1. $|y|$ when $y=-37$
2. $|-z|$ when $z=-24$

Add Integers

In the following exercises, simplify each expression.

Exercise 13.5.12.99

$-200+65$

Answer

-135

Exercise 13.5.12.100

$-150+45$

Exercise 13.5.12.101

$2+(-8)+6$

Answer

0

Exercise 13.5.12.102

$4+(-9)+7$

Exercise 13.5.12.103

$140+(-75)+67$

Answer

132

Exercise 13.5.12.104

$-32+24+(-6)+10$

Subtract Integers

In the following exercises, simplify.

Exercise 13.5.12.105

$9-3$

Answer

6

Exercise 13.5.12.106

$-5-(-1)$

Exercise 13.5.12.107

1. $15-6$
2. $15+(-6)$

Answer

1. 9
2. 9

Exercise 13.5.12.108

1. $12-9$
2. $12+(-9)$

Exercise 13.5.12.109

1. $8-(-9)$
2. $8+9$

Answer

1. 17
2. 17

Exercise 13.5.12.110

1. $4-(-4)$
2. $4+4$

In the following exercises, simplify each expression.

Exercise 13.5.12.111

$10-(-19)$

Answer

29

Exercise 13.5.12.112

$11-(-18)$

Exercise 13.5.12.113

$31-79$

Answer

-48

Exercise 13.5.12.114

$39-81$

Exercise 13.5.12.115

$-31-11$

Answer

-42

Exercise 13.5.12.116

$-32-18$

Exercise 13.5.12.117

$-15-(-28)+5$

Answer

18

Exercise 13.5.12.118

$71+(-10)-8$

Exercise 13.5.12.119

$-16-(-4+1)-7$

Answer

-20

Exercise 13.5.12.120

$-15-(-6+4)-3$

Multiply Integers

In the following exercises, multiply.

Exercise 13.5.12.121

$-5(7)$

Answer

−35

Exercise 13.5.12.122

−8(6)

Exercise 13.5.12.123

−18(−2)

Answer

36

Exercise 13.5.12.124

−10(−6)

Divide Integers

In the following exercises, divide.

Exercise 13.5.12.125

$-28 \div 7$

Answer

-4

Exercise 13.5.12.126

$56 \div (-7)$

Exercise 13.5.12.127

$-120 \div (-20)$

Answer

6

Exercise 13.5.12.128

$-200 \div 25$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

Exercise 13.5.12.129

−8(−2)−3(−9)

Answer

43

Exercise 13.5.12.130

$-7(-4)-5(-3)$

Exercise 13.5.12.131

$(-5)^3$

Answer

-125

Exercise 13.5.12.132

$(-4)^3$

Exercise 13.5.12.133

$-4 \cdot 2 \cdot 11$

Answer

-88

Exercise 13.5.12.134

$-5 \cdot 3 \cdot 10$

Exercise 13.5.12.135

$-10(-4) \div (-8)$

Answer

-5

Exercise 13.5.12.136

$-8(-6) \div (-4)$

Exercise 13.5.12.137

$31-4(3-9)$

Answer

55

Exercise 13.5.12.138

$24-3(2-10)$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

Exercise 13.5.12.139

x+8 when

1. x=−26
2. x=−95

Answer

1. −18
2. −87

Exercise 13.5.12.140

y+9 when

1. y=−29
2. y=−84

Exercise 13.5.12.141

When b=−11, evaluate:

1. b+6
2. −b+6

Answer

1. −5
2. 17

Exercise 13.5.12.142

When c=−9, evaluate:

1. c+(−4)c+(−4)
2. −c+(−4)

Exercise 13.5.12.143

$p^2 - 5p + 2$ when
$p = -1$

Answer

8

Exercise 13.5.12.144

$q^2 - 2q + 9$ when $q = -2$

Exercise 13.5.12.145

$6x - 5y + 15$ when $x = 3$ and $y = -1$

Answer

38

Exercise 13.5.12.146

$3p-2q+9$ when $p=8$ and $q=-2$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

Exercise 13.5.12.147

the sum of −4 and −17, increased by 32

Answer

(−4+(−17))+32;11

Exercise 13.5.12.148

1. the difference of 15 and −7
2. subtract 15 from −7

Exercise 13.5.12.149

the quotient of −45 and −9

Answer

$\frac{-45}{-9}; 5$

Exercise 13.5.12.150

the product of −12 and the difference of c and d

Use Integers in Applications

In the following exercises, solve.

Exercise 13.5.12.151

Temperature The high temperature one day in Miami Beach, Florida, was 76°. That same day, the high temperature in Buffalo, New York was −8°. What was the difference between the temperature in Miami Beach and the temperature in Buffalo?

Answer

84 degrees

Exercise 13.5.12.152

Checking Account Adrianne has a balance of −$22 in her checking account. She deposits $301 to the account. What is the new balance?

Visualize Fractions

Find Equivalent Fractions

In the following exercises, find three fractions equivalent to the given fraction. Show your work, using figures or algebra.

Exercise 13.5.12.153

$\frac{1}{4}$

Answer

$\frac{2}{8}, \frac{3}{12}, \frac{4}{16}$ answers may vary

Exercise 13.5.12.154

$\frac{1}{3}$

Exercise 13.5.12.155

$\frac{5}{6}$

Answer

$\frac{10}{12}, \frac{15}{18}, \frac{20}{24}$ answers may vary

Exercise 13.5.12.156

$\frac{2}{7}$

Simplify Fractions

In the following exercises, simplify.

Exercise 13.5.12.157

$\frac{7}{21}$

Answer

$\frac{1}{3}$

Exercise 13.5.12.158

$\frac{8}{24}$

Exercise 13.5.12.159

$\frac{15}{20}$

Answer

$\frac{3}{4}$

Exercise 13.5.12.160

$\frac{12}{18}$

Exercise 13.5.12.161

$-\frac{168}{192}$

Answer

$-\frac{7}{8}$

Exercise 13.5.12.162

$-\frac{140}{224}$

Exercise 13.5.12.163

$\frac{11x}{11y}$

Answer

$\frac{x}{y}$

Exercise 13.5.12.164

$\frac{15a}{15b}$

Multiply Fractions

In the following exercises, multiply.

Exercise 13.5.12.165

$\frac{2}{5} \cdot \frac{1}{3}$

Answer

$\frac{2}{15}$

Exercise 13.5.12.166

$\frac{1}{2} \cdot \frac{3}{8}$

Exercise 13.5.12.167

$\frac{7}{12}\left(-\frac{8}{21}\right)$

Answer

$-\frac{2}{9}$

Exercise 13.5.12.168

$\frac{5}{12}\left(-\frac{8}{15}\right)$

Exercise 13.5.12.169

$-28p\left(-\frac{1}{4}\right)$

Answer

7p

Exercise 13.5.12.170

$-51q\left(-\frac{1}{3}\right)$

Exercise 13.5.12.172

$\frac{14}{5}(-15)$

Answer

-42

Exercise 13.5.12.173

$-1\left(-\frac{3}{8}\right)$

Divide Fractions

In the following exercises, divide

Exercise 13.5.12.174

$\frac{1}{2} \div \frac{1}{4}$

Answer

2

Exercise 13.5.12.175

$\frac{1}{2} \div \frac{1}{8}$

Exercise 13.5.12.176

$-\frac{4}{5} \div \frac{4}{7}$

Answer

$-\frac{7}{5}$

Exercise 13.5.12.177

$-\frac{3}{4} \div \frac{3}{5}$

Exercise 13.5.12.178

$\frac{5}{8} \div \frac{a}{10}$

Answer

$\frac{25}{4a}$

Exercise 13.5.12.179

$\frac{5}{6} \div \frac{c}{15}$

Exercise 13.5.12.180

$\frac{7p}{12} \div \frac{21p}{8}$

Answer

$\frac{2}{9}$

Exercise 13.5.12.181

$\frac{5q}{12} \div \frac{15q}{8}$

Exercise 13.5.12.182

$\frac{2}{5} \div (-10)$

Answer

$-\frac{1}{25}$

Exercise 13.5.12.183

$-18 \div -\left(\frac{9}{2}\right)$

In the following exercises, simplify.

Exercise 13.5.12.184

$\frac{\frac{2}{3}}{\frac{8}{9}}$

Answer

$\frac{3}{4}$

Exercise 13.5.12.185

$\frac{\frac{4}{5}}{\frac{8}{15}}$

Exercise 13.5.12.186

$\frac{-\frac{9}{10}}{3}$

Answer

$-\frac{3}{10}$

Exercise 13.5.12.187

$\frac{2}{\frac{5}{8}}$

Exercise 13.5.12.188

$\frac{\frac{r}{5}}{\frac{s}{3}}$

Answer

$\frac{3r}{5s}$

Exercise 13.5.12.189

$\frac{-\frac{x}{6}}{-\frac{8}{9}}$

Simplify Expressions Written with a Fraction Bar

In the following exercises, simplify.

Exercise 13.5.12.190

$\frac{4+11}{8}$

Answer

$\frac{15}{8}$

Exercise 13.5.12.191

$\frac{9+3}{7}$

Exercise 13.5.12.192

$\frac{30}{7-12}$

Answer

-6

Exercise 13.5.12.193

$\frac{15}{4-9}$

Exercise 13.5.12.194

$\frac{22-14}{19-13}$

Answer

$\frac{4}{3}$

Exercise 13.5.12.195

$\frac{15+9}{18+12}$

Exercise 13.5.12.196

$\frac{5 \cdot 8}{-10}$

Answer

-4

Exercise 13.5.12.197

$\frac{3 \cdot 4}{-24}$

Exercise 13.5.12.198

$\frac{15 \cdot 5-5^2}{2 \cdot 10}$

Answer

$\frac{5}{2}$

Exercise 13.5.12.199

$\frac{12 \cdot 9-3^2}{3 \cdot 18}$

Exercise 13.5.12.200

$\frac{2+4(3)}{-3-2^2}$

Answer

-2

Exercise 13.5.12.201

$\frac{7+3(5)}{-2-3^2}$

Translate Phrases to Expressions with Fractions

In the following exercises, translate each English phrase into an algebraic expression.

Exercise 13.5.12.202

the quotient of c and the sum of d and 9.

Answer

$\frac{c}{d+9}$

Exercise 13.5.12.203

the quotient of the difference of h and k, and -5.

Add and Subtract Fractions

Add and Subtract Fractions with a Common Denominator

In the following exercises, add.

Exercise 13.5.12.204

$\frac{4}{9}+\frac{1}{9}$

Answer

$\frac{5}{9}$

Exercise 13.5.12.205

$\frac{2}{9}+\frac{5}{9}$

Exercise 13.5.12.206

$\frac{y}{3}+\frac{2}{3}$

Answer

$\frac{y+2}{3}$

Exercise 13.5.12.207

$\frac{7}{p}+\frac{9}{p}$

Exercise 13.5.12.208

$-\frac{1}{8}+\left(-\frac{3}{8}\right)$

Answer

$-\frac{1}{2}$

Exercise 13.5.12.209

$-\frac{1}{8}+\left(-\frac{5}{8}\right)$

In the following exercises, subtract.

Exercise 13.5.12.210

$\frac{4}{5} - \frac{1}{5}$

Answer

$\frac{3}{5}$

Exercise 13.5.12.211

$\frac{4}{5} - \frac{3}{5}$

Exercise 13.5.12.212

$\frac{y}{17} - \frac{9}{17}$

Answer

$\frac{y-9}{17}$

Exercise 13.5.12.213

$\frac{x}{19} - \frac{8}{19}$

Exercise 13.5.12.214

$-\frac{8}{d} - \frac{3}{d}$

Answer

$-\frac{11}{d}$

Exercise 13.5.12.215

$-\frac{7}{c} - \frac{7}{c}$

Add or Subtract Fractions with Different Denominators

In the following exercises, add or subtract.

Exercise 13.5.12.216

$\frac{1}{3} + \frac{1}{5}$

Answer

$\frac{8}{15}$

Exercise 13.5.12.217

$\frac{1}{4} + \frac{1}{5}$

Exercise 13.5.12.218

$\frac{1}{5}-\left(-\frac{1}{10}\right)$

Answer

$\frac{3}{10}$

Exercise 13.5.12.219

$\frac{1}{2}-\left(-\frac{1}{6}\right)$

Exercise 13.5.12.220

$\frac{2}{3}+\frac{3}{4}$

Answer

$\frac{17}{12}$

Exercise 13.5.12.221

$\frac{3}{4}+\frac{2}{5}$

Exercise 13.5.12.222

$\frac{11}{12}-\frac{3}{8}$

Answer

$\frac{13}{24}$

Exercise 13.5.12.223

$\frac{5}{8}-\frac{7}{12}$

Exercise 13.5.12.224

$-\frac{9}{16}-\left(-\frac{4}{5}\right)$

Answer

$\frac{19}{80}$

Exercise 13.5.12.225

$-\frac{7}{20}-\left(-\frac{5}{8}\right)$

Exercise 13.5.12.226

$1+\frac{5}{6}$

Answer

$\frac{11}{6}$

Exercise 13.5.12.227

$1-\frac{5}{9}$

Use the Order of Operations to Simplify Complex Fractions

In the following exercises, simplify.

Exercise 13.5.12.228

$\frac{\left(\frac{1}{5}\right)^2}{2+3^2}$

Answer

$\frac{1}{275}$

Exercise 13.5.12.229

$\frac{\left(\frac{1}{3}\right)^2}{5+2^2}$

Exercise 13.5.12.230

$\frac{\frac{2}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{2}{3}}$

Answer

14

Exercise 13.5.12.231

$\frac{\frac{3}{4}+\frac{1}{2}}{\frac{5}{6}-\frac{2}{3}}$

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

Exercise 13.5.12.232

$x+\frac{1}{2}$ when

1. $x=-\frac{1}{8}$
2. $x=-\frac{1}{2}$

Answer

1. $\frac{3}{8}$
2. 0

Exercise 13.5.12.233

$x+\frac{2}{3}$ when

1. $x=-\frac{1}{6}$

2. $x = -\frac{5}{3}$

Exercise 13.5.12.234

$4p^2q$ when $p = -\frac{1}{2}$ and $q = \frac{5}{9}$

Answer

$\frac{5}{9}$

Exercise 13.5.12.235

$5m^2n$ when $m = -\frac{2}{5}$ and $n = \frac{1}{3}$

Exercise 13.5.12.236

$\frac{u+v}{w}$ when
$u = -4, v = -8, w = 2$

Answer

-6

Exercise 13.5.12.237

$\frac{m+n}{p}$ when
$m = -6, n = -2, p = 4$

Decimals

Name and Write Decimals

In the following exercises, write as a decimal.

Exercise 13.5.12.238

Eight and three hundredths

Answer

8.03

Exercise 13.5.12.239

Nine and seven hundredths

Exercise 13.5.12.240

One thousandth

Answer

0.001

Exercise 13.5.12.241

Nine thousandths

In the following exercises, name each decimal.

Exercise 13.5.12.242

7.8

Answer

seven and eight tenths

Exercise 13.5.12.243

5.01

Exercise 13.5.12.244

0.005

Answer

five thousandths

Exercise 13.5.12.245

0.381

Round Decimals

In the following exercises, round each number to the nearest

1. hundredth
2. tenth
3. whole number.

Exercise 13.5.12.246

5.7932

Answer

1. 5.79
2. 5.8
3. 6

Exercise 13.5.12.247

3.6284

Exercise 13.5.12.248

12.4768

Answer

1. 12.48
2. 12.5
3. 12

Exercise 13.5.12.249

25.8449

Add and Subtract Decimals

In the following exercises, add or subtract.

Exercise 13.5.12.250

18.37+9.36

Answer

27.73

Exercise 13.5.12.251

256.37−85.49

Exercise 13.5.12.252

15.35−20.88

Answer

−5.53

Exercise 13.5.12.253

37.5+12.23

Exercise 13.5.12.254

−4.2+(−9.3)

Answer

−13.5

Exercise 13.5.12.255

−8.6+(−8.6)

Exercise 13.5.12.256

100−64.2

Answer

35.8

Exercise 13.5.12.257

100−65.83

Exercise 13.5.12.258

2.51+40

Answer

42.51

Exercise 13.5.12.259

9.38+60

Multiply and Divide Decimals

In the following exercises, multiply.

Exercise 13.5.12.260

(0.3)(0.4)

Answer

0.12

Exercise 13.5.12.261

(0.6)(0.7)

Exercise 13.5.12.262

(8.52)(3.14)

Answer

26.7528

Exercise 13.5.12.263

(5.32)(4.86)

Exercise 13.5.12.264

(0.09)(24.78)

Answer

2.2302

Exercise 13.5.12.265

(0.04)(36.89)

In the following exercises, divide.

Exercise 13.5.12.266

$0.15 \div 5$

Answer

0.03

Exercise 13.5.12.267

$0.27 \div 3$

Exercise 13.5.12.268

$\$8.49 \div 12$

Answer

$0.71

Exercise 13.5.12.269

$\$16.99 \div 9$

Exercise 13.5.12.270

$12 \div 0.08$

Answer

150

Exercise 13.5.12.271

$5 \div 0.04$

Convert Decimals, Fractions, and Percents

In the following exercises, write each decimal as a fraction.

Exercise 13.5.12.272

0.08

Answer

$\frac{2}{25}$

Exercise 13.5.12.273

0.17

Exercise 13.5.12.274

0.425

Answer

$\frac{17}{40}$

Exercise 13.5.12.275

0.184

Exercise 13.5.12.276

1.75

Answer

$\frac{7}{4}$

Exercise 13.5.12.277

0.035

In the following exercises, convert each fraction to a decimal.

Exercise 13.5.12.278

$\frac{2}{5}$

Answer

0.4

Exercise 13.5.12.279

$\frac{4}{5}$

Exercise 13.5.12.280

$-\frac{3}{8}$

Answer

−0.375

Exercise 13.5.12.281

$-\frac{5}{8}$

Exercise 13.5.12.282

$\frac{5}{9}$

Answer

$0.\overline{5}$

Exercise 13.5.12.283

$\frac{2}{9}$

Exercise 13.5.12.284

$\frac{1}{2}+6.5$

Answer

7

Exercise 13.5.12.285

$\frac{1}{4}+10.75$

In the following exercises, convert each percent to a decimal.

Exercise 13.5.12.286

5%

Answer

0.05

Exercise 13.5.12.287

9%

Exercise 13.5.12.288

40%

Answer

0.4

Exercise 13.5.12.289

50%

Exercise 13.5.12.290

115%

Answer

1.15

Exercise 13.5.12.291

125%

In the following exercises, convert each decimal to a percent.

Exercise 13.5.12.292

0.18

Answer

18%

Exercise 13.5.12.293

0.15

Exercise 13.5.12.294

0.009

Answer

0.9%

Exercise 13.5.12.295

0.008

Exercise 13.5.12.296

1.5

Answer

150%

Exercise 13.5.12.297

2.2

The Real Numbers

Simplify Expressions with Square Roots

In the following exercises, simplify.

Exercise 13.5.12.298

$\sqrt{64}$

Answer

8

Exercise 13.5.12.299

$\sqrt{144}$

Exercise 13.5.12.300

$-\sqrt{25}$

Answer

-5

Exercise 13.5.12.301

$-\sqrt{81}$

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercises, write as the ratio of two integers.

Exercise 13.5.12.302

1. 9
2. 8.47

Answer

1. $\frac{9}{1}$
2. $\frac{847}{100}$

Exercise 13.5.12.303

1. -15
2. 3.591

In the following exercises, list the

1. rational numbers,
2. irrational numbers.

Exercise 13.5.12.304

$0.84, 0.79132\ldots, 1.\overline{3}$

Answer

1. $0.84, 1.3$
2. $0.79132\ldots$

Exercise 13.5.12.305

$2.3\overline{8}, 0.572, 4.93814\ldots$

In the following exercises, identify whether each number is rational or irrational.

Exercise 13.5.12.306

1. $\sqrt{121}$
2. $\sqrt{48}$

Answer

1. rational

2. irrational

Exercise 13.5.12.307

1. $\sqrt{56}$
2. $\sqrt{16}$

In the following exercises, identify whether each number is a real number or not a real number.

Exercise 13.5.12.308

1. $\sqrt{-9}$
2. $-\sqrt{169}$

Answer

1. not a real number
2. real number

Exercise 13.5.12.309

1. $\sqrt{-64}$
2. $-\sqrt{81}$

In the following exercises, list the

1. whole numbers,
2. integers,
3. rational numbers,
4. irrational numbers,
5. real numbers for each set of numbers.

Exercise 13.5.12.310

$-4, 0, \frac{5}{6}, \sqrt{16}, \sqrt{18}, 5.2537\ldots$

Answer

1. $0, \sqrt{16}$
2. $-4, 0, \sqrt{16}$
3. $-4, 0, \frac{5}{6}, \sqrt{16}$
4. $\sqrt{18}, 5.2537\ldots$
5. $-4, 0, \frac{5}{6}, \sqrt{16}, \sqrt{18}, 5.2537\ldots$

Exercise 13.5.12.311

$-\sqrt{4}, 0.\overline{36}, \frac{13}{3}, 6.9152\ldots, \sqrt{48}, 10\frac{1}{2}$

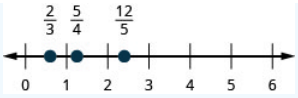
Locate Fractions on the Number Line

In the following exercises, locate the numbers on a number line.

Exercise 13.5.12.312

$\frac{2}{3}, \frac{5}{4}, \frac{12}{5}$

Answer



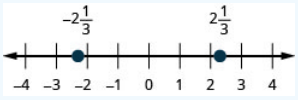
Exercise 13.5.12.313

$\frac{1}{3}, \frac{7}{4}, \frac{13}{5}$

Exercise 13.5.12.314

$2\frac{1}{3}, -2\frac{1}{3}$

Answer



Exercise 13.5.12.315

$1\frac{3}{5}, -1\frac{3}{5}$

In the following exercises, order each of the following pairs of numbers, using < or >.

Exercise 13.5.12.316

$-1 \_\_\_ -\frac{1}{8}$

Answer

$<$

Exercise 13.5.12.317

$-3\frac{1}{4} \_\_\_ -4$

Exercise 13.5.12.318

$-\frac{7}{9} \_\_\_ \frac{4}{9}$

Answer

$>$

Exercise 13.5.12.319

$-2 \_\_\_ \frac{19}{8}$

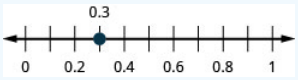
Locate Decimals on the Number Line

In the following exercises, locate on the number line.

Exercise 13.5.12.320

0.3

Answer



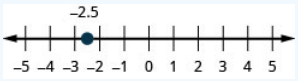
Exercise 13.5.12.321

−0.2

Exercise 13.5.12.322

−2.5

Answer



Exercise 13.5.12.323

2.7

In the following exercises, order each of the following pairs of numbers, using < or >.

Exercise 13.5.12.324

0.9___0.6

Answer

>

Exercise 13.5.12.325

0.7___0.8

Exercise 13.5.12.326

−0.6___−0.59

Answer

>

Exercise 13.5.12.327

−0.27___−0.3

Properties of Real Numbers

Use the Commutative and Associative Properties

In the following exercises, use the Associative Property to simplify.

Exercise 13.5.12.328

$-12(4m)$

Answer

$-48m$

Exercise 13.5.12.329

$30\left(\frac{5}{6}q\right)$

Exercise 13.5.12.330

(a+16)+31

Answer

a+47

Exercise 13.5.12.331

(c+0.2)+0.7

In the following exercises, simplify.

Exercise 13.5.12.332

6y+37+(−6y)

Answer

37

Exercise 13.5.12.333

$\frac{1}{4}+\frac{11}{15}+\left(-\frac{1}{4}\right)$

Exercise 13.5.12.334

$\frac{14}{11}\cdot\frac{35}{9}\cdot\frac{14}{11}$

Answer

$\frac{35}{9}$

Exercise 13.5.12.335

$-18\cdot 15\cdot\frac{2}{9}$

Exercise 13.5.12.336

$\left(\frac{7}{12}+\frac{4}{5}\right)+\frac{1}{5}$

Answer

$1\frac{7}{12}$

Exercise 13.5.12.337

(3.98d+0.75d)+1.25d

Exercise 13.5.12.338

11x+8y+16x+15y

Answer

27x+23y

Exercise 13.5.12.339

52m+(−20n)+(−18m)+(−5n)

Use the Identity and Inverse Properties of Addition and Multiplication

In the following exercises, find the additive inverse of each number.

Exercise 13.5.12.340

1. $\frac{1}{3}$
2. 5.1
3. -14
4. $-\frac{8}{5}$

Answer

1. $-\frac{1}{3}$
2. -5.1
3. -14
4. $-\frac{8}{5}$

Exercise 13.5.12.341

1. $-\frac{7}{8}$
2. -0.03
3. 17
4. $\frac{12}{5}$

In the following exercises, find the multiplicative inverse of each number.

Exercise 13.5.12.342

1. 10
2. $-\frac{4}{9}$
3. 0.6

Answer

1. $\frac{1}{10}$
2. $-\frac{9}{4}$
3. $\frac{5}{3}$

Exercise 13.5.12.343

1. $-\frac{9}{2}$
2. -7
3. 2.1

Use the Properties of Zero

In the following exercises, simplify.

Exercise 13.5.12.344

$83 \cdot 0$

Answer

0

Exercise 13.5.12.345

$\frac{0}{9}$

Exercise 13.5.12.346

$\frac{5}{0}$

Answer

undefined

Exercise 13.5.12.347

$0 \div \frac{2}{3}$

In the following exercises, simplify.

Exercise 13.5.12.348

$43+39+(-43)$

Answer

39

Exercise 13.5.12.349

$(n+6.75)+0.25$

Exercise 13.5.12.350

$\frac{5}{13} \cdot 57 \cdot \frac{13}{5}$

Answer

57

Exercise 13.5.12.351

$\frac{1}{6} \cdot 17 \cdot 12$

Exercise 13.5.12.352

$\frac{2}{3} \cdot 28 \cdot \frac{3}{7}$

Answer

8

Exercise 13.5.12.353

$9(6x - 11) + 15$

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the Distributive Property.

Exercise 13.5.12.354

7(x+9)

Answer

7x+63

Exercise 13.5.12.355

9(u−4)

Exercise 13.5.12.356

−3(6m−1)

Answer

−18m+3

Exercise 13.5.12.357

−8(−7a−12)

Exercise 13.5.12.358

$\frac{1}{3}(15n - 6)$

Answer

5n−2

Exercise 13.5.12.359

$(y + 10) \cdot p$

Exercise 13.5.12.360

$(a-4)-(6a+9)$

Answer

$-5a-13$

Exercise 13.5.12.361

$4(x+3)-8(x-7)$

Systems of Measurement

1.1 Define U.S. Units of Measurement and Convert from One Unit to Another

In the following exercises, convert the units. Round to the nearest tenth.

Exercise 13.5.12.362

A floral arbor is 7 feet tall. Convert the height to inches.

Answer

84 inches

Exercise 13.5.12.363

A picture frame is 42 inches wide. Convert the width to feet.

Exercise 13.5.12.364

Kelly is 5 feet 4 inches tall. Convert her height to inches.

Answer

64 inches

Exercise 13.5.12.365

A playground is 45 feet wide. Convert the width to yards.

Exercise 13.5.12.366

The height of Mount Shasta is 14,179 feet. Convert the height to miles.

Answer

2.7 miles

Exercise 13.5.12.367

Shamu weights 4.5 tons. Convert the weight to pounds.

Exercise 13.5.12.368

The play lasted $1\frac{3}{4}$ hours. Convert the time to minutes.

Answer

105 minutes

Exercise 13.5.12.369

How many tablespoons are in a quart?

Exercise 13.5.12.370

Naomi's baby weighed 5 pounds 14 ounces at birth. Convert the weight to ounces.

Answer

94 ounces

Exercise 13.5.12.371

Trinh needs 30 cups of paint for her class art project. Convert the volume to gallons.

Use Mixed Units of Measurement in the U.S. System.

In the following exercises, solve.

Exercise 13.5.12.372

John caught 4 lobsters. The weights of the lobsters were 1 pound 9 ounces, 1 pound 12 ounces, 4 pounds 2 ounces, and 2 pounds 15 ounces. What was the total weight of the lobsters?

Answer

10 lbs. 6 oz.

Exercise 13.5.12.373

Every day last week Pedro recorded the number of minutes he spent reading. The number of minutes were 50, 25, 83, 45, 32, 60, 135. How many hours did Pedro spend reading?

Exercise 13.5.12.374

Fouad is 6 feet 2 inches tall. If he stands on a rung of a ladder 8 feet 10 inches high, how high off the ground is the top of Fouad's head?

Answer

15 feet

Exercise 13.5.12.375

Dalila wants to make throw pillow covers. Each cover takes 30 inches of fabric. How many yards of fabric does she need for 4 covers?

Make Unit Conversions in the Metric System

In the following exercises, convert the units.

Exercise 13.5.12.376

Donna is 1.7 meters tall. Convert her height to centimeters.

Answer

170 centimeters

Exercise 13.5.12.377

Mount Everest is 8,850 meters tall. Convert the height to kilometers.

Exercise 13.5.12.378

One cup of yogurt contains 488 milligrams of calcium. Convert this to grams.

Answer

0.488 grams

Exercise 13.5.12.379

One cup of yogurt contains 13 grams of protein. Convert this to milligrams.

Exercise 13.5.12.380

Sergio weighed 2.9 kilograms at birth. Convert this to grams.

Answer

2,900 grams

Exercise 13.5.12.381

A bottle of water contained 650 milliliters. Convert this to liters.

Use Mixed Units of Measurement in the Metric System

In the following exerices, solve.

Exercise 13.5.12.382

Minh is 2 meters tall. His daughter is 88 centimeters tall. How much taller is Minh than his daughter?

Answer

1.12 meter

Exercise 13.5.12.383

Selma had a 1 liter bottle of water. If she drank 145 milliliters, how much water was left in the bottle?

Exercise 13.5.12.384

One serving of cranberry juice contains 30 grams of sugar. How many kilograms of sugar are in 30 servings of cranberry juice?

Answer

0.9 kilograms

Exercise 13.5.12.385

One ounce of tofu provided 2 grams of protein. How many milligrams of protein are provided by 5 ounces of tofu?

Convert between the U.S. and the Metric Systems of Measurement

In the following exercises, make the unit conversions. Round to the nearest tenth.

Exercise 13.5.12.386

Majid is 69 inches tall. Convert his height to centimeters.

Answer

175.3 centimeters

Exercise 13.5.12.387

A college basketball court is 84 feet long. Convert this length to meters.

Exercise 13.5.12.388

Caroline walked 2.5 kilometers. Convert this length to miles.

Answer

1.6 miles

Exercise 13.5.12.389

Lucas weighs 78 kilograms. Convert his weight to pounds.

Exercise 13.5.12.390

Steve’s car holds 55 liters of gas. Convert this to gallons.

Answer

14.6 gallons

Exercise 13.5.12.391

A box of books weighs 25 pounds. Convert the weight to kilograms.

Convert between Fahrenheit and Celsius Temperatures

In the following exercises, convert the Fahrenheit temperatures to degrees Celsius. Round to the nearest tenth.

Exercise 13.5.12.392

95° Fahrenheit

Answer

35° C

Exercise 13.5.12.393

23° Fahrenheit

Exercise 13.5.12.394

20° Fahrenheit

Answer

–6.7° C

Exercise 13.5.12.395

64° Fahrenheit

In the following exercises, convert the Celsius temperatures to degrees Fahrenheit. Round to the nearest tenth.

Exercise 13.5.12.396

30° Celsius

Answer

86° F

Exercise 13.5.12.397

–5° Celsius

Exercise 13.5.12.398

–12° Celsius

Answer

10.4° F

Exercise 13.5.12.399

24° Celsius

Chapter Practice Test

Exercise 13.5.12.1

Write as a whole number using digits: two hundred five thousand, six hundred seventeen.

Answer

205,617

Exercise 13.5.12.2

Find the prime factorization of 504.

Exercise 13.5.12.3

Find the Least Common Multiple of 18 and 24.

Answer

72

Exercise 13.5.12.4

Combine like terms: 5n+8+2n−1.

In the following exercises, evaluate.

Exercise 13.5.12.5

$-|x|$ when $x = -2$

Answer

−2

Exercise 13.5.12.6

11−a when a=−3

Exercise 13.5.12.7

Translate to an algebraic expression and simplify: twenty less than negative 7.

Answer

−7−20;−27

Exercise 13.5.12.8

Monique has a balance of −$18 in her checking account. She deposits $152 to the account. What is the new balance?

Exercise 13.5.12.9

Round 677.1348 to the nearest hundredth.

Answer

677.13

Exercise 13.5.12.10

Convert $\frac{4}{5}$ to a decimal.

Exercise 13.5.12.11

Convert 1.85 to a percent.

Answer

185%

Exercise 13.5.12.12

Locate $\frac{2}{3}, -1.5,$ and $\frac{9}{4}$ on a number line.

In the following exercises, simplify each expression.

Exercise 13.5.12.13

$4+10(3+9)-5^2$

Answer

99

Exercise 13.5.12.14

$-85+42$

Exercise 13.5.12.15

$-19-25$

Answer

-44

Exercise 13.5.12.16

$(-2)^4$

Exercise 13.5.12.17

$-5(-9)\div 15$

Answer

3

Exercise 13.5.12.18

$\frac{3}{8}\cdot\frac{11}{12}$

Exercise 13.5.12.19

$\frac{4}{5} \div \frac{9}{20}$

Answer

$\frac{16}{9}$

Exercise 13.5.12.20

$\frac{12+3 \cdot 5}{15-6}$

Exercise 13.5.12.21

$\frac{m}{7} + \frac{10}{7}$

Answer

$\frac{m+10}{7}$

Exercise 13.5.12.22

$\frac{7}{12} - \frac{3}{8}$

Exercise 13.5.12.23

$-5.8 + (-4.7)$

Answer

−10.5

Exercise 13.5.12.24

100−64.25

Exercise 13.5.12.25

(0.07)(31.95)

Answer

2.2365

Exercise 13.5.12.26

$9 \div 0.05$

Exercise 13.5.12.27

$-14\left(\frac{5}{7}p\right)$

Answer

−10p

Exercise 13.5.12.28

$(u+8)-9$

Exercise 13.5.12.29

$6x+(-4y)+9x+8y$

Answer

$15x+4y$

Exercise 13.5.12.30

$\frac{0}{23}$

Exercise 13.5.12.31

$\frac{75}{0}$

Answer

undefined

Exercise 13.5.12.32

$-2(13q-5)$

Exercise 13.5.12.33

A movie lasted $1\frac{2}{3}$ hours. How many minutes did it last? (1 hour $=$ 60 minutes)

Answer

100 minutes

Exercise 13.5.12.34

Mike's SUV is 5 feet 11 inches tall. He wants to put a rooftop cargo bag on the the SUV. The cargo bag is 1 foot 6 inches tall. What will the total height be of the SUV with the cargo bag on the roof? *(1 foot = 12 inches)*

Exercise 13.5.12.35

Jennifer ran 2.8 miles. Convert this length to kilometers. *(1 mile = 1.61 kilometers)*

Answer

4.508 km

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Thumbnail: https://www.wikihow.com/Make-a-Mathematical-Model

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13.6.1: Use a Problem-Solving Strategy

Learning Objectives

By the end of this section, you will be able to:

- Approach word problems with a positive attitude
- Use a problem-solving strategy for word problems
- Solve number problems

Note

Before you get started, take this readiness quiz.

1. Translate "6 less than twice x" into an algebraic expression.
   If you missed this problem, review Exercise 1.3.43.
2. Solve: $\frac{2}{3}x = 24$.
   If you missed this problem, review Exercise 2.2.10.
3. Solve: $3x + 8 = 14$.
   If you missed this problem, review Exercise 2.3.1.

Approach Word Problems with a Positive Attitude

"If you think you can… or think you can't… you're right."—Henry Ford

The world is full of word problems! Will my income qualify me to rent that apartment? How much punch do I need to make for the party? What size diamond can I afford to buy my girlfriend? Should I fly or drive to my family reunion? How much money do I need to fill the car with gas? How much tip should I leave at a restaurant? How many socks should I pack for vacation? What size turkey do I need to buy for Thanksgiving dinner, and then what time do I need to put it in the oven? If my sister and I buy our mother a present, how much does each of us pay?

Now that we can solve equations, we are ready to apply our new skills to word problems. Do you know anyone who has had negative experiences in the past with word problems? Have you ever had thoughts like the student below (Figure 13.6.1.1)?

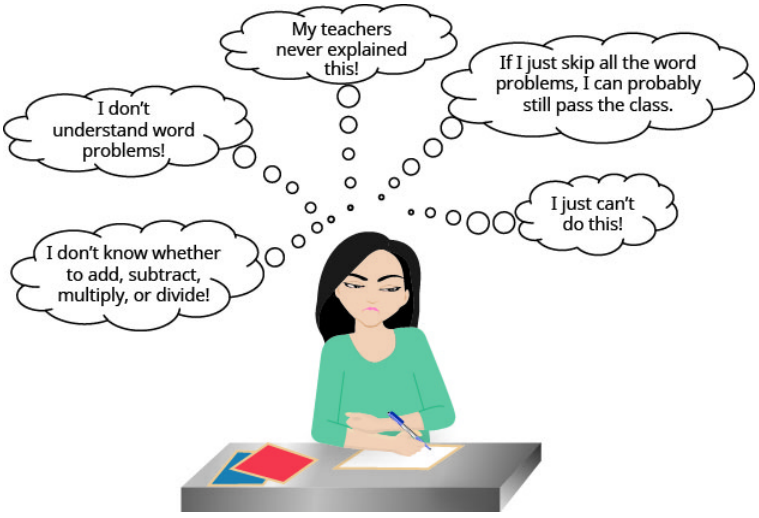

Figure 13.6.1.1: Negative thoughts can be barriers to success.

When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. We need to calm our fears and change our negative feelings.

Start with a fresh slate and begin to think positive thoughts. If we take control and believe we can be successful, we will be able to master word problems! Read the positive thoughts in Figure 13.6.1.2and say them out loud.

Figure 13.6.1.2: Thinking positive thoughts is a first step towards success.

Think of something, outside of school, that you can do now but couldn't do 3 years ago. Is it driving a car? Snowboarding? Cooking a gourmet meal? Speaking a new language? Your past experiences with word problems happened when you were younger—now you're older and ready to succeed!

Use a Problem-Solving Strategy for Word Problems

We have reviewed translating English phrases into algebraic expressions, using some basic mathematical vocabulary and symbols. We have also translated English sentences into algebraic equations and solved some word problems. The word problems applied math to everyday situations. We restated the situation in one sentence, assigned a variable, and then wrote an equation to solve the problem. This method works as long as the situation is familiar and the math is not too complicated.

Now, we'll expand our strategy so we can use it to successfully solve any word problem. We'll list the strategy here, and then we'll use it to solve some problems. We summarize below an effective strategy for problem solving.

USE A PROBLEM-SOLVING STRATEGY TO SOLVE WORD PROBLEMS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Example 13.6.1.1

Pilar bought a purse on sale for $18, which is one-half of the original price. What was the original price of the purse?

Solution

Step 1. Read the problem. Read the problem two or more times if necessary. Look up any unfamiliar words in a dictionary or on the internet.

In this problem, is it clear what is being discussed? Is every word familiar?

Let p = the original price of the purse.

Step 2. Identify what you are looking for. Did you ever go into your bedroom to get something and then forget what you were looking for? It's hard to find something if you are not sure what it is! Read the problem again and look for words that tell you what you are looking for!

In this problem, the words "what was the original price of the purse" tell us what we need to find.

Step 3. Name what we are looking for. Choose a variable to represent that quantity. We can use any letter for the variable, but choose one that makes it easy to remember what it represents.

Step 4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Translate the English sentence into an algebraic equation.

Reread the problem carefully to see how the given information is related. Often, there is one sentence that gives this information, or it may help to write one sentence with all the important information. Look for clue words to help translate the sentence into algebra. Translate the sentence into an equation.

Restate the problem in one sentence with all the important information.	**18** **is** **one-half the original price.**
Translate into an equation.	$18 = \frac{1}{2} \cdot p$

Step 5. Solve the equation using good algebraic techniques. Even if you know the solution right away, using good algebraic techniques here will better prepare you to solve problems that do not have obvious answers.

Solve the equation.	$18 = \frac{1}{2}p$
Multiply both sides by 2.	$2 \cdot 18 = 2 \cdot \frac{1}{2}p$
Simplify.	$36 = p$

Step 6. Check the answer in the problem to make sure it makes sense. We solved the equation and found that $p = 36$,which means "the original price" was $\$36$.

Does $36 make sense in the problem? Yes, because 18 is one-half of 36, and the purse was on sale at half the original price.

If this were a homework exercise, our work might look like this:

Pilar bought a purse on sale for $\$18$, which is one-half the original price. What was the original price of the purse?

Step 7. Answer the question with a complete sentence. The problem asked "What was the original price of the purse?"

The answer to the question is: "The original price of the purse was $36."

	Let $p =$ the original price.
	18 is one-half the original price.
	$18 = \frac{1}{2}p$
Multiply both sides by 2.	$2 \cdot 18 = 2 \cdot \frac{1}{2}p$
Simplify.	$36 = p$
Check. Is $36 a reasonable price for a purse?	
Yes.	
Is 18 one half of 36?	
$18 \stackrel{?}{=} \frac{1}{2} \cdot 36$	
$18 = 18$✓	
	The original price of the purse was $36.

Try It 13.6.1.2

Joaquin bought a bookcase on sale for $120, which was two-thirds of the original price. What was the original price of the bookcase?

Answer

$180

Try It 13.6.1.3

Two-fifths of the songs in Mariel's playlist are country. If there are 16 country songs, what is the total number of songs in the playlist?

Answer

40

Let's try this approach with another example.

Example 13.6.1.4

Ginny and her classmates formed a study group. The number of girls in the study group was three more than twice the number of boys. There were 11 girls in the study group. How many boys were in the study group?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	How many boys were in the study group?
Step 3. Name. Choose a variable to represent the number of boys.	Let $n =$ the number of boys.
Step 4. Translate. Restate the problem in one sentence with all the important information.	**The number of girls** (11) **was** **three more than twice the number of boys**
Translate into an equation.	$11 = 2b+3$
Step 5. Solve the equation.	$11 = 2b+3$
Subtract 3 from each side.	$11-3 = 2b+3-3$
Simplify.	$8 = 2b$
Divide each side by 2.	$\frac{8}{2} = \frac{2b}{2}$
Simplify.	$4 = b$
Step 6. Check. First, is our answer reasonable? Yes, having 4 boys in a study group seems OK. The problem says the number of girls was 3 more than twice the number of boys. If there are four boys, does that make eleven girls? Twice 4 boys is 8. Three more than 8 is 11.	
Step 7. Answer the question.	There were 4 boys in the study group.

Try It 13.6.1.5

Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was 3 more than twice the number of notebooks. He bought 7 textbooks. How many notebooks did he buy?

Answer

2

Try It 13.6.1.6

Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. How many crossword puzzles did he do?

Answer

7

Solve Number Problems

Now that we have a problem solving strategy, we will use it on several different types of word problems. The first type we will work on is “number problems.” Number problems give some clues about one or more numbers. We use these clues to write an equation. Number problems don’t usually arise on an everyday basis, but they provide a good introduction to practicing the problem solving strategy outlined above.

Example 13.6.1.7

The difference of a number and six is 13. Find the number.

Solution

Step 1. Read the problem. Are all the words familiar?	
Step 2. Identify what we are looking for.	the number
Step 3. Name. Choose a variable to represent the number.	Let $n =$ the number.
Step 4. Translate. Remember to look for clue words like "difference... of... and..."	
Restate the problem as one sentence.	$\underbrace{\textbf{The difference of the number and 6}}$ $\underbrace{\textbf{is}}$ $\underbrace{\textbf{13}}$
Translate into an equation.	$n - 6 \quad = \quad 13$
Step 5. Solve the equation.	$n - 6 = 13$
Simplify.	$n = 19$
Step 6. Check.	
The difference of 19 and 6 is 13. It checks!	
Step 7. Answer the question.	The number is 19.

Try It 13.6.1.8

The difference of a number and eight is 17. Find the number.

Answer

25

Try It 13.6.1.9

The difference of a number and eleven is -7. Find the number.

Answer

4

Example 13.6.1.10

The sum of twice a number and seven is 15. Find the number.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the number
Step 3. Name. Choose a variable to represent the number.	Let $n =$ the number.
Step 4. Translate.	
Restate the problem as one sentence.	The sum of twice a number and 7 is 15
Translate into an equation.	$2n + 7 = 15$
Step 5. Solve the equation.	$2n + 7 = 15$
Subtract 7 from each side and simplify.	$2n = 8$
Divide each side by 2 and simplify.	$n = 4$
Step 6. Check.	
Is the sum of twice 4 and 7 equal to 15?	
$2 \cdot 4 + 7 \stackrel{?}{=} 15$ $15 = 15\checkmark$	
Step 7. Answer the question.	The number is 4.

Did you notice that we left out some of the steps as we solved this equation? If you're not yet ready to leave out these steps, write down as many as you need.

Try It 13.6.1.11

The sum of four times a number and two is 14. Find the number.

Answer

3

Try It 13.6.1.12

The sum of three times a number and seven is 25. Find the number.

Answer

6

Some number word problems ask us to find two or more numbers. It may be tempting to name them all with different variables, but so far we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

Example 13.6.1.13

One number is five more than another. The sum of the numbers is 21. Find the numbers.

Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		We are looking for two numbers.
Step 3. Name. We have two numbers to name and need a name for each.		
Choose a variable to represent the first number.		Let $n = 1^{st}$ number.
What do we know about the second number?		One number is five more than another.
		$n + 5 = 2^{nd}$ number
Step 4. Translate. Restate the problem as one sentence with all the important information.		The sum of the 1st number and the 2nd number is 21.
Translate into an equation.		1st number + 2nd number = 21
Substitute the variable expressions.		$n + n + 5 = 21$
Step 5. Solve the equation.		$n + n + 5 = 21$
Combine like terms.		$2n + 5 = 21$
Subtract 5 from both sides and simplify.		$2n = 16$
Divide by 2 and simplify.		$n = 8$ 1st number
Find the second number, too.		$n + 5$ 2nd number
		$8 + 5$
		13
Step 6. Check.		
Do these numbers check in the problem?		
Is one number 5 more than the other?	$13 \stackrel{?}{=} 8 + 5$	
Is thirteen 5 more than 8? Yes.	$13 = 13$✓	
Is the sum of the two numbers 21?	$8 + 13 \stackrel{?}{=} 21$	
	$21 = 21$✓	
Step 7. Answer the question.		The numbers are 8 and 13.

Try It 13.6.1.14

One number is six more than another. The sum of the numbers is twenty-four. Find the numbers.

Answer

9, 15

Try It 13.6.1.15

The sum of two numbers is fifty-eight. One number is four more than the other. Find the numbers.

Answer

27, 31

Example 13.6.1.16

The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		We are looking for two numbers.
Step 3. Name.		
Choose a variable.		Let $n = 1^{st}$ number.
One number is 4 less than the other.		$n - 4 = 2^{nd}$ number
Step 4. Translate.		
Write as one sentence.		The sum of the 2 numbers is negative 14.
Translate into an equation.		1st number + 2nd number is negative fourteen
Step 5. Solve the equation.		$n + n - 4 = -14$
Combine like terms.		$n + n - 4 = -14$
Add 4 to each side and simplify.		$2n - 4 = -14$
Simplify.		$2n = -10$
		$n = -5$ 1st number
		$n - 4$ 2nd number
		$-5 - 4$
		-9
Step 6. Check.		
Is −9 four less than −5?	$-5 - 4 \stackrel{?}{=} -9$	
	$-9 = -9$✓	
Is their sum −14?	$-5 + (-9) \stackrel{?}{=} -14$	
	$-14 = -14$✓	
Step 7. Answer the question.		The numbers are −5 and −9.

Try It 13.6.1.17

The sum of two numbers is negative twenty-three. One number is seven less than the other. Find the numbers.

Answer

-15, -8

Try It 13.6.1.18

The sum of two numbers is -18. One number is 40 more than the other. Find the numbers.

Answer

-29, 11

Example 13.6.1.19

One number is ten more than twice another. Their sum is one. Find the numbers.

Solution

Step 1. Read the problem.		
Step 2. Identify what you are looking for.		We are looking for two numbers.
Step 3. Name.		
Choose a variable.		Let $x = 1^{st}$ number.
One number is 10 more than twice another.		$2x + 10 = 2^{nd}$ number
Step 4. Translate.		
Restate as one sentence.		Their sum is one.
		The sum of the two numbers is 1.
Translate into an equation.		$x + 2x + 10 = 1$
Step 5. Solve the equation.		
Combine like terms.		$x + 2x + 10 = 1$
Subtract 10 from each side.		$3x + 10 = 1$
Divide each side by 3.		$3x = -9$
		$x = -3$ 1st number
		$2x + 10$ 2nd number
		$2(-3) + 10$
		4
Step 6. Check.		
Is ten more than twice −3 equal to 4?	$2(-3) + 10 \stackrel{?}{=} 4$	
	$-6 + 10 \stackrel{?}{=} 4$	
	$4 = 4$✓	

Is their sum 1?	$-3+4\stackrel{?}{=}1$	
	$1=1\checkmark$	
Step 7. Answer the question.		The numbers are -3 and 4.

Try It 13.6.1.20

One number is eight more than twice another. Their sum is negative four. Find the numbers.

Answer

$-4,\ 0$

Try It 13.6.1.21

One number is three more than three times another. Their sum is -5. Find the numbers.

Answer

$-3,\ -2$

Some number problems involve consecutive integers. *Consecutive integers* are integers that immediately follow each other. Examples of consecutive integers are:

$$\begin{aligned} &1, 2, 3, 4 \\ &-10, -9, -8, -7 \\ &150, 151, 152, 153 \end{aligned} \tag{13.6.1.1}$$

Notice that each number is one more than the number preceding it. So if we define the first integer as n, the next consecutive integer is $n+1$. The one after that is one more than $n+1$, so it is $n+1+1$, which is $n+2$.

$$\begin{aligned} &n && 1^{\text{st}} \text{ integer} \\ &n+1 && 2^{\text{nd}} \text{ consecutive integer} \\ &n+2 && 3^{\text{rd}} \text{ consecutive integer } \ldots \text{ etc.} \end{aligned} \tag{13.6.1.2}$$

Example 13.6.1.22

The sum of two consecutive integers is 47. Find the numbers.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	two consecutive integers
Step 3. Name each number.	Let $n = 1^{st}$ integer.
	$n+1 =$ next consecutive integer
Step 4. Translate.	
Restate as one sentence.	The sum of the integers is 47.
Translate into an equation.	$n + n + 1 = 47$
Step 5. Solve the equation.	$n + n + 1 = 47$
Combine like terms.	$2n + 1 = 47$
Subtract 1 from each side.	$2n = 46$

Divide each side by 2.	$n = 23$ 1st integer
	$n + 1$ next consecutive integer
	$23 + 1$
	24
Step 6. Check.	
$23 + 24 \stackrel{?}{=} 47$ $47 = 47$✓	
Step 7. Answer the question.	The two consecutive integers are 23 and 24.

Try It 13.6.1.23

The sum of two consecutive integers is 95. Find the numbers.

Answer

47, 48

Try It 13.6.1.24

The sum of two consecutive integers is −31. Find the numbers.

Answer

-16, -15

Example 13.6.1.25

Find three consecutive integers whose sum is −42.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	three consecutive integers
Step 3. Name each of the three numbers.	Let $n = 1^{st}$ integer.
	$n + 1 = 2^{nd}$ consecutive integer
	$n + 2 = 3^{rd}$ consecutive integer
Step 4. Translate.	
Restate as one sentence.	The sum of the three integers is -42.
Translate into an equation.	$n + n + 1 + n + 2 = -42$
Step 5. Solve the equation.	$n + n + 1 + n + 2 = -42$
Combine like terms.	$3n + 3 = -42$
Subtract 3 from each side.	$3n = -45$
Divide each side by 3.	$n = -15$ 1st integer
	$n + 1$ 2nd integer

	$-15+1$
	-14
	$n+2$ 3rd integer
	$-15+2$
	-13
Step 6. Check.	
$-13+(-14)+(-15) \stackrel{?}{=} -42$ $-42 = -42\checkmark$	
Step 7. Answer the question.	The three consecutive integers are −13, −14, and −15.

Try It 13.6.1.26

Find three consecutive integers whose sum is −96.

Answer

-33, -32, -31

Try It 13.6.1.27

Find three consecutive integers whose sum is −36.

Answer

-13, -12, -11

Now that we have worked with consecutive integers, we will expand our work to include consecutive even integers and consecutive odd integers. *Consecutive even integers* are even integers that immediately follow one another. Examples of consecutive even integers are:

$$\begin{aligned} &18, 20, 22 \\ &64, 66, 68 \\ &-12, -10, -8 \end{aligned} \tag{13.6.1.3}$$

Notice each integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$.

$$\begin{aligned} n &\quad 1^{\text{st}} \text{ even integer} \\ n+2 &\quad 2^{\text{nd}} \text{ consecutive even integer} \\ n+4 &\quad 3^{\text{rd}} \text{ consecutive even integer } \ldots \text{ etc.} \end{aligned} \tag{13.6.1.4}$$

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 77, 79, and 81.

$$\begin{aligned} &77, 79, 81 \\ &n, n+2, n+4 \end{aligned} \tag{13.6.1.5}$$

$$\begin{aligned} n &\quad 1^{\text{st}} \text{ odd integer} \\ n+2 &\quad 2^{\text{nd}} \text{ consecutive odd integer} \\ n+4 &\quad 3^{\text{rd}} \text{ consecutive odd integer } \ldots \text{ etc.} \end{aligned} \tag{13.6.1.6}$$

Does it seem strange to add 2 (an even number) to get from one odd integer to the next? Do you get an odd number or an even number when we add 2 to 3? to 11? to 47?

Whether the problem asks for consecutive even numbers or odd numbers, you don't have to do anything different. The pattern is still the same—to get from one odd or one even integer to the next, add 2.

Example 13.6.1.28

Find three consecutive even integers whose sum is 84.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	three consecutive even integers
Step 3. Name the integers.	Let $n = 1^{st}$ even integers.
	$n+2 = 2^{nd}$ consecutive even integer
	$n+4 = 3^{rd}$ consecutive even integer
Step 4. Translate.	
Restate as one sentence.	The sum of the three even integers is 84.
Translate into an equation.	$n+n+2+n+4=84$
Step 5. Solve the equation.	
Combine like terms.	$n+n+2+n+4=84$
Subtract 6 from each side.	$3n+6=84$
Divide each side by 3.	$3n=78$
	$n=26$ 1st integer
	$n+2$ 2nd integer
	$26+2$
	28
	$n+4$ 3rd integer
	$26+4$
	30
Step 6. Check.	
	$26+28+30 \stackrel{?}{=} 84$
	$84=84$✓
Step 7. Answer the question.	The three consecutive integers are 26, 28, and 30.

(13.6.1.7)

Try It 13.6.1.29

Find three consecutive even integers whose sum is 102.

Answer

32, 34, 36

Try It 13.6.1.30

Find three consecutive even integers whose sum is −24.

Answer

$-10, -8, -6$

Example 13.6.1.31

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	How much does the husband earn?
Step 3. Name.	
Choose a variable to represent the amount the husband earns.	Let $h =$ the amount the husband earns.
The wife earns $\$16,000$ less than twice that.	$2h - 16,000$ the amount the wife earns.
Step 4. Translate.	Together the husband and wife earn $\$110,000$.
Restate the problem in one sentence with all the important information.	The amount the husband earns plus the amount the wife earns is $110,000
Translate into an equation.	$h + 2h - 16,000 = 110,000$
Step 5. Solve the equation.	$h + 2h - 16,000 = 110,000$
Combine like terms.	$3h - 16,000 = 110,000$
Add $16,000$ to both sides and simplify.	$3h = 126,000$
Divide each side by 3.	$h = 42,000$
	$\$42,000$ amount husband earns
	$2h - 16,000$ amount wife earns
	$2(42,000) - 16,000$
	$84,000 - 16,000$
	$68,000$
Step 6. Check.	
If the wife earns $\$68,000$ and the husband earns $\$42,000$ is the total $\$110,000$? Yes!	
Step 7. Answer the question.	The husband earns $\$42,000$ a year.

Try It 13.6.1.32

According to the National Automobile Dealers Association, the average cost of a car in 2014 was $28,500. This was $1,500 less than 6 times the cost in 1975. What was the average cost of a car in 1975?

Answer

$5000

? Try It 13.6.1.33

U.S. Census data shows that the median price of new home in the United States in November 2014 was $280,900. This was $10,700 more than 14 times the price in November 1964. What was the median price of a new home in November 1964?

Answer

$19300

Key Concepts

- **Problem-Solving Strategy**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- **Consecutive Integers**
  Consecutive integers are integers that immediately follow each other.

$$\begin{array}{ll} n & 1^{\text{st}} \text{ integer} \\ n+1 & 2^{\text{nd}} \text{ consecutive integer} \\ n+2 & 3^{\text{rd}} \text{ consecutive integer } \ldots \text{ etc.} \end{array} \tag{13.6.1.8}$$

  Consecutive even integers are even integers that immediately follow one another.

$$\begin{array}{ll} n & 1^{\text{st}} \text{ integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive even integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive even integer } \ldots \text{ etc.} \end{array} \tag{13.6.1.9}$$

  Consecutive odd integers are odd integers that immediately follow one another.

$$\begin{array}{ll} n & 1^{\text{st}} \text{ integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive odd integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive odd integer } \ldots \text{ etc.} \end{array} \tag{13.6.1.10}$$

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13.6.1E: Exercises

Practice Makes Perfect

Use the Approach Word Problems with a Positive Attitude

In the following exercises, prepare the lists described.

Exercise 13.6.1E.1

List five positive thoughts you can say to yourself that will help you approach word problems with a positive attitude. You may want to copy them on a sheet of paper and put it in the front of your notebook, where you can read them often.

Answer

Answers will vary

Exercise 13.6.1E.2

List five negative thoughts that you have said to yourself in the past that will hinder your progress on word problems. You may want to write each one on a small piece of paper and rip it up to symbolically destroy the negative thoughts.

Use a Problem-Solving Strategy for Word Problems

In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question.

Exercise 13.6.1E.3

Two-thirds of the children in the fourth-grade class are girls. If there are 20 girls, what is the total number of children in the class?

Answer

30

Exercise 13.6.1E.4

Three-fifths of the members of the school choir are women. If there are 24 women, what is the total number of choir members?

Exercise 13.6.1E.5

Zachary has 25 country music CDs, which is one-fifth of his CD collection. How many CDs does Zachary have?

Answer

125

Exercise 13.6.1E.6

One-fourth of the candies in a bag of M&M's are red. If there are 23 red candies, how many candies are in the bag?

Exercise 13.6.1E.7

There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.

Answer

6

Exercise 13.6.1E.8

There are 18 Cub Scouts in Pack 645. The number of scouts is three more than five times the number of adult leaders. Find the number of adult leaders.

Exercise 13.6.1E.9

Huong is organizing paperback and hardback books for her club's used book sale. The number of paperbacks is 12 less than three times the number of hardbacks. Huong had 162 paperbacks. How many hardback books were there?

Answer

58

Exercise 13.6.1E.10

Jeff is lining up children's and adult bicycles at the bike shop where he works. The number of children's bicycles is nine less than three times the number of adult bicycles. There are 42 adult bicycles. How many children's bicycles are there?

Exercise 13.6.1E.11

Philip pays $1,620 in rent every month. This amount is $120 more than twice what his brother Paul pays for rent. How much does Paul pay for rent?

Answer

$750

Exercise 13.6.1E.12

Marc just bought an SUV for $54,000. This is $7,400 less than twice what his wife paid for her car last year. How much did his wife pay for her car?

Exercise 13.6.1E.13

Laurie has $46,000 invested in stocks and bonds. The amount invested in stocks is $8,000 less than four times the amount invested in bonds. How much does Laurie have invested in bonds?

Answer

$13,500

Exercise 13.6.1E.14

Erica earned a total of $50,450 last year from her two jobs. The amount she earned from her job at the store was $1,250 more than three times the amount she earned from her job at the college. How much did she earn from her job at the college?

Solve Number Problems

In the following exercises, solve each number word problem.

Exercise 13.6.1E.15

The sum of a number and eight is 12. Find the number.

Answer

4

Exercise 13.6.1E.16

The sum of a number and nine is 17. Find the number.

Exercise 13.6.1E.17

The difference of a number and 12 is three. Find the number.

Answer

15

Exercise 13.6.1E.18

The difference of a number and eight is four. Find the number.

Exercise 13.6.1E.19

The sum of three times a number and eight is 23. Find the number.

Answer

5

Exercise 13.6.1E.20

The sum of twice a number and six is 14. Find the number.

Exercise 13.6.1E.21

The difference of twice a number and seven is 17. Find the number.

Answer

12

Exercise 13.6.1E.22

The difference of four times a number and seven is 21. Find the number.

Exercise 13.6.1E.23

Three times the sum of a number and nine is 12. Find the number.

Answer

-5

Exercise 13.6.1E. 24

Six times the sum of a number and eight is 30. Find the number.

Exercise 13.6.1E. 25

One number is six more than the other. Their sum is 42. Find the numbers.

Answer

18, 24

Exercise 13.6.1E. 26

One number is five more than the other. Their sum is 33. Find the numbers.

Exercise 13.6.1E. 27

The sum of two numbers is 20. One number is four less than the other. Find the numbers.

Answer

8, 12

Exercise 13.6.1E. 28

The sum of two numbers is 27. One number is seven less than the other. Find the numbers.

Exercise 13.6.1E. 29

The sum of two numbers is −45. One number is nine more than the other. Find the numbers.

Answer

−18,−27

Exercise 13.6.1E. 30

The sum of two numbers is −61. One number is 35 more than the other. Find the numbers.

Exercise 13.6.1E. 31

The sum of two numbers is −316. One number is 94 less than the other. Find the numbers.

Answer

−111,−205

Exercise 13.6.1E. 32

The sum of two numbers is −284. One number is 62 less than the other. Find the numbers.

Exercise 13.6.1E.33

One number is 14 less than another. If their sum is increased by seven, the result is 85. Find the numbers.

Answer

32, 46

Exercise 13.6.1E.34

One number is 11 less than another. If their sum is increased by eight, the result is 71. Find the numbers.

Exercise 13.6.1E.35

One number is five more than another. If their sum is increased by nine, the result is 60. Find the numbers.

Answer

23, 28

Exercise 13.6.1E.36

One number is eight more than another. If their sum is increased by 17, the result is 95. Find the numbers.

Exercise 13.6.1E.37

One number is one more than twice another. Their sum is −5. Find the numbers.

Answer

−2,−3

Exercise 13.6.1E.38

One number is six more than five times another. Their sum is six. Find the numbers.

Exercise 13.6.1E.39

The sum of two numbers is 14. One number is two less than three times the other. Find the numbers.

Answer

4, 10

Exercise 13.6.1E.40

The sum of two numbers is zero. One number is nine less than twice the other. Find the numbers.

Exercise 13.6.1E.41

The sum of two consecutive integers is 77. Find the integers.

Answer

38, 39

Exercise 13.6.1E.42

The sum of two consecutive integers is 89. Find the integers.

Exercise 13.6.1E.43

The sum of two consecutive integers is −23. Find the integers.

Answer

−11,−12

Exercise 13.6.1E.44

The sum of two consecutive integers is −37. Find the integers.

Exercise 13.6.1E.45

The sum of three consecutive integers is 78. Find the integers.

Answer

25, 26, 27

Exercise 13.6.1E.46

The sum of three consecutive integers is 60. Find the integers.

Exercise 13.6.1E.47

Find three consecutive integers whose sum is −36.

Answer

−11,−12,−13

Exercise 13.6.1E.48

Find three consecutive integers whose sum is −3.

Exercise 13.6.1E.49

Find three consecutive even integers whose sum is 258.

Answer

84, 86, 88

Exercise 13.6.1E.50

Find three consecutive even integers whose sum is 222.

Exercise 13.6.1E.51

Find three consecutive odd integers whose sum is 171.

Answer

55, 57, 59

Exercise 13.6.1E.52

Find three consecutive odd integers whose sum is 291.

Exercise 13.6.1E.53

Find three consecutive even integers whose sum is −36.

Answer

−10,−12,−14

Exercise 13.6.1E.54

Find three consecutive even integers whose sum is −84.

Exercise 13.6.1E.55

Find three consecutive odd integers whose sum is −213.

Answer

−69,−71,−73

Exercise 13.6.1E.56

Find three consecutive odd integers whose sum is −267.

Everyday Math

Exercise 13.6.1E.57

Sale Price Patty paid $35 for a purse on sale for $10 off the original price. What was the original price of the purse?

Answer

$45

Exercise 13.6.1E.58

Sale Price Travis bought a pair of boots on sale for $25 off the original price. He paid $60 for the boots. What was the original price of the boots?

Exercise 13.6.1E.59

Buying in Bulk Minh spent $6.25 on five sticker books to give his nephews. Find the cost of each sticker book.

Answer

$1.25

Exercise 13.6.1E.60

Buying in Bulk Alicia bought a package of eight peaches for $3.20. Find the cost of each peach.

Exercise 13.6.1E.61

Price before Sales Tax Tom paid $1,166.40 for a new refrigerator, including $86.40 tax. What was the price of the refrigerator?

Answer

$1080

Exercise 13.6.1E.62

Price before Sales Tax Kenji paid $2,279 for a new living room set, including $129 tax. What was the price of the living room set?

Writing Exercises

Exercise 13.6.1E.63

What has been your past experience solving word problems?

Answer

answers will vary

Exercise 13.6.1E.64

When you start to solve a word problem, how do you decide what to let the variable represent?

Exercise 13.6.1E.65

What are consecutive odd integers? Name three consecutive odd integers between 50 and 60.

Answer

Consecutive odd integers are odd numbers that immediately follow each other. An example of three consecutive odd integers between 50 and 60 would be 51, 53, and 55.

Exercise 13.6.1E.66

What are consecutive even integers? Name three consecutive even integers between −50 and −40.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
approach word problems with a positive attitude.			
use a problem solving strategy for word problems.			
solve number problems.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential—every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

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13.6.2: Solve Percent Applications

Learning Objectives

By the end of this section, you will be able to:

- Translate and solve basic percent equations
- Solve percent applications
- Find percent increase and percent decrease
- Solve simple interest applications
- Solve applications with discount or mark-up

Be prepared

Before you get started, take this readiness quiz.

1. Convert 4.5% to a decimal.
   If you missed this problem, review [link].
2. Convert 0.6 to a percent.
   If you missed this problem, review [link].
3. Round 0.875 to the nearest hundredth.
   If you missed this problem, review [link].
4. Multiply (4.5)(2.38).
   If you missed this problem, review [link].
5. Solve 3.5=0.7n.
   If you missed this problem, review [link].
6. Subtract 50−37.45.
   If you missed this problem, review [link].

Translate and Solve Basic Percent Equations

We will solve percent equations using the methods we used to solve equations with fractions or decimals. Without the tools of algebra, the best method available to solve percent problems was by setting them up as proportions. Now as an algebra student, you can just translate English sentences into algebraic equations and then solve the equations.

We can use any letter you like as a variable, but it is a good idea to choose a letter that will remind us of what you are looking for. We must be sure to change the given percent to a decimal when we put it in the equation.

Example 13.6.2.1

Translate and solve: What number is 35% of 90?

Solution

	What number is 35% of 90?
Translate into algebra. Let n= the number.	$n = 0.35 \cdot 90$
Remember "of" means multiply, "is" means equals.	
Multiply.	$n = 31.5$
	31.5 is 35% of 90

Try It 13.6.2.2

Translate and solve:

What number is 45% of 80?

Answer

36

Try It 13.6.2.3

Translate and solve:

What number is 55% of 60?

Answer

33

We must be very careful when we translate the words in the next example. The unknown quantity will not be isolated at first, like it was in Example. We will again use direct translation to write the equation.

Example 13.6.2.4

Translate and solve: 6.5% of what number is $1.17?

Solution

	6.5% of what number is $1.17?
Translate. Let n= the number.	$0.065 \cdot n = 1.17$
Multiply.	$0.065n = 1.17$
Divide both sides by 0.065 and simplify.	$n = 18$
	6.5% of $18 is $1.17

Try It 13.6.2.5

Translate and solve:

7.5% of what number is $1.95?

Answer

$26

Try It 13.6.2.6

Translate and solve:

8.5% of what number is $3.06?

Answer

$36

In the next example, we are looking for the percent.

Example 13.6.2.7

Translate and solve: 144 is what percent of 96?

Solution

	144 is what percent of 96?
Translate into algebra. Let p= the percent.	$144 = p \cdot 96$
Multiply.	$144 = 96p$
Divide by 96 and simplify.	$1.5 = p$
Convert to percent.	$150\% = p$
	144 is 150% of 96

Note that we are asked to find percent, so we must have our final result in percent form.

Try It 13.6.2.8

Translate and solve:

110 is what percent of 88?

Answer

125%

Try It 13.6.2.9

Translate and solve:

126 is what percent of 72?

Answer

175%

Solve Applications of Percent

Many applications of **percent**—such as tips, sales tax, discounts, and interest—occur in our daily lives. To solve these applications we'll translate to a basic percent equation, just like those we solved in previous examples. Once we translate the sentence into a percent equation, we know how to solve it.

We will restate the problem solving strategy we used earlier for easy reference.

USE A PROBLEM-SOLVING STRATEGY TO SOLVE AN APPLICATION.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Now that we have the strategy to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications will involve everyday situations, you can rely on your own experience.

Example 13.6.2.10

Dezohn and his girlfriend enjoyed a nice dinner at a restaurant and his bill was $68.50. He wants to leave an 18% tip. If the tip will be 18% of the total bill, how much tip should he leave?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the amount of tip should Dezohn leave
Step 3. Name what we are looking for.	
Choose a variable to represent it.	Let t = amount of tip.
Step 4. Translate into an equation.	The tip is 18% of the total bill.
Write a sentence that gives the information to find it.	$\underbrace{\text{The tip}}\ \underbrace{\text{is}}\ \underbrace{18\%}\ \underbrace{\text{of}}\ \underbrace{\$68.50}$
Translate the sentence into an equation.	$t = 0.18 \cdot 68.50$
Step 5. Solve the equation. Multiply.	$t = 12.33$
Step 6. Check. Does this make sense?	
Yes, 20% is close to 18%, and 20% of $70 is $14.	
Step 7. Answer the question with a complete sentence.	Dezohn should leave a tip of $12.33.

Notice that we used t to represent the unknown tip.

Try It 13.6.2.11

Cierra and her sister enjoyed a dinner in a restaurant and the bill was $81.50. If she wants to leave 18% of the total bill as her tip, how much should she leave?

Answer

$14.67

Try It 13.6.2.12

Kimngoc had lunch at her favorite restaurant. She wants to leave 15% of the total bill as her tip. If her bill was $14.40, how much will she leave for the tip?

Answer

$2.16

Example 13.6.2.13

The label on Masao's breakfast cereal said that one serving of cereal provides 85 milligrams (mg) of potassium, which is 2% of the recommended daily amount. What is the total recommended daily amount of potassium?

Solution

Step 1. Read the problem.	

Step 2. Identify what we are looking for.	the total amount of potassium that is recommended
Step 3. Name what we are looking for.	
Choose a variable to represent it.	Let a= total amount of potassium.
Step 4. Translate. Write a sentence that gives the information to find it.	85 mg is 2% of the total amount
Translate into an equation.	$85 = 0.02 \cdot a$
Step 5. Solve the equation.	$4{,}250 = a$
Step 6. Check. Does this make sense?	
Yes, 2% is a small percent and 85 is a small part of 4,250.	
Step 7. Answer the question with a complete sentence.	The amount of potassium that is recommended is 4,250 mg.

Try It 13.6.2.14

One serving of wheat square cereal has seven grams of fiber, which is 28% of the recommended daily amount. What is the total recommended daily amount of fiber?

Answer

25 grams

Try It 13.6.2.15

One serving of rice cereal has 190 mg of sodium, which is 8% of the recommended daily amount. What is the total recommended daily amount of sodium?

Answer

2,375 mg

Example 13.6.2.16

Mitzi received some gourmet brownies as a gift. The wrapper said each brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the percent of the total calories from fat
Step 3. Name what we are looking for.	
Choose a variable to represent it.	Let p= percent of fat.
Step 4. Translate. Write a sentence that gives the information to find it.	What percent of 480 is 240?
Translate into an equation.	$p \cdot 480 = 240$
Step 5. Solve the equation.	$480\,p = 240$
Divide by 480.	$p = 0.5$
Put in a percent form.	$p = 50\%$

Step 6. Check. Does this make sense?	
Yes, 240 is half of 480, so 50% makes sense.	
Step 7. Answer the question with a complete sentence.	Of the total calories in each brownie, 50% is fat.

Try It 13.6.2.17

Solve. Round to the nearest whole percent.

Veronica is planning to make muffins from a mix. The package says each muffin will be 230 calories and 60 calories will be from fat. What percent of the total calories is from fat?

Answer

26%

Try It 13.6.2.18

Solve. Round to the nearest whole percent.

The mix Ricardo plans to use to make brownies says that each brownie will be 190 calories, and 76 calories are from fat. What percent of the total calories are from fat?

Answer

40%

Find Percent Increase and Percent Decrease

People in the media often talk about how much an amount has increased or decreased over a certain period of time. They usually express this increase or decrease as a percent.

To find the **percent increase**, first we find the amount of increase, the difference of the new amount and the original amount. Then we find what percent the amount of increase is of the original amount.

FIND THE PERCENT INCREASE.

1. Find the amount of increase.

   $$\text{new amount} - \text{original amount} = \text{increase}$$
2. Find the percent increase.

   The increase is what percent of the original amount?

Example 13.6.2.19

In 2011, the California governor proposed raising community college fees from $26 a unit to $36 a unit. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the percent increase
Step 3. Name what we are looking for.	
Choose a variable to represent it.	Let p = the percent.

Step 4. Translate. Write a sentence that gives the information to find it.	
First find the amount of increase.	new amount − original amount = increase
	$36-26=10$
Find the percent.	Increase is what percent of the original amount?
	10 is what percent of 26?
Translate into an equation.	$10 = p \cdot 26$
Step 5. Solve the equation.	$10 = 26p$
Divide by 26.	$0.3846\ldots = p$
Change to percent form; round to the nearest tenth.	$38.5\% = p$
Step 6. Check. Does this make sense?	
Yes, 38.5% is close to $\frac{1}{3}$, and 10 is close to $\frac{1}{3}$ of 26.	
Step 7. Answer the question with a complete sentence.	The new fees represent a 38.5% increase over the old fees.

Try It 13.6.2.20

Find the percent increase. (Round to the nearest tenth of a percent.)

In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents.

Answer

8.8%

Try It 13.6.2.21

Find the percent increase.

In 1995, the standard bus fare in Chicago was $1.50. In 2008, the standard bus fare was $2.25.

Answer

50%

Finding the **percent decrease** is very similar to finding the percent increase, but now the amount of decrease is the difference of the original amount and the new amount. Then we find what percent the amount of decrease is of the original amount.

FIND THE PERCENT DECREASE.

1. Find the amount of decrease.

   $$\text{original amount} - \text{new amount} = \text{decrease}$$
2. Find the percent decrease.

   The decrease is what percent of the original amount?

Example 13.6.2.22

The average price of a gallon of gas in one city in June 2014 was $3.71. The average price in that city in July was $3.64. Find the percent decrease.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the percent decrease
Step 3. Name what we are looking for.	
Choose a variable to represent that quantity.	Let p= the percent decrease.
Step 4. Translate. Write a sentence that gives the information to find it.	
First find the amount of decrease.	3.71−3.64=0.07
Find the percent.	Decrease is what percent of the original amount?
	0.07 is what percent of 3.71?
Translate into an equation.	$0.07 = p \cdot 3.71$
Step 5. Solve the equation.	$0.07 = 3.71\,p$
Divide by 3.71.	$0.0188\ldots = p$
Change to percent form; round to the nearest tenth.	$1.9\% = p$
Step 6. Check. Does this make sense?	
Yes, if the original price was $4, a 2% decrease would be 8 cents.	
Step 7. Answer the question with a complete sentence.	The price of gas decreased 1.9%.

Try It 13.6.2.23

Find the percent decrease. (Round to the nearest tenth of a percent.)

The population of North Dakota was about 672,000 in 2010. The population is projected to be about 630,000 in 2020.

Answer

6.3%

Try It 13.6.2.24

Find the percent decrease.

Last year, Sheila's salary was $42,000. Because of furlough days, this year, her salary was $37,800.

Answer

10%

Solve Simple Interest Applications

Do you know that banks pay you to keep your money? The money a customer puts in the bank is called the **principal**, P, and the money the bank pays the customer is called the **interest**. The interest is computed as a certain percent of the principal; called the

rate of interest, *r*. We usually express rate of interest as a percent per year, and we calculate it by using the decimal equivalent of the percent. The variable *t*, (for *time*) represents the number of years the money is in the account.

To find the interest we use the simple interest formula, I=Prt.

SIMPLE INTEREST

If an amount of money, *P*, called the principal, is invested for a period of *t* years at an annual interest rate *r*, the amount of interest, *I*, earned is

$$I = Prt \quad \text{where} \quad \begin{aligned} I &= \text{interest} \\ P &= \text{principle} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned} \tag{13.6.2.1}$$

Interest earned according to this formula is called **simple interest**.

Interest may also be calculated another way, called compound interest. This type of interest will be covered in later math classes.

The formula we use to calculate simple interest is I=Prt. To use the formula, we substitute in the values the problem gives us for the variables, and then solve for the unknown variable. It may be helpful to organize the information in a chart.

Example 13.6.2.25

Nathaly deposited $12,500 in her bank account where it will earn 4% interest. How much interest will Nathaly earn in 5 years?

$$\begin{aligned} I &= ? \\ P &= \$12,500 \\ r &= 4\% \\ t &= 5 \text{ years} \end{aligned}$$

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for. — the amount of interest earned

Step 3. Name what we are looking for. Choose a variable to represent that quantity — Let I = the amount of interest.

Step 4. Translate into an equation.

Write the formula. — $I = Prt$

Substitute in the given information. — $I = (12500)(.04)(5)$

Step 5. Solve the equation. — $I = 2500$

Step 6. Check: Does this make sense?

Is $2,500 is a reasonable interest on $12,500? Yes.

Step 7. Answer the question with a complete sentence. — The interest is $2500

Try It 13.6.2.26

Areli invested a principal of $950 in her bank account with interest rate 3%. How much interest did she earn in 5 years?

Answer

$142.50

Try It 13.6.2.27

Susana invested a principal of $36,000 in her bank account with interest rate 6.5%. How much interest did she earn in 3 years?

Answer

$7020

There may be times when we know the amount of interest earned on a given principal over a certain length of time, but we don't know the rate. To find the rate, we use the simple interest formula, substitute in the given values for the principal and time, and then solve for the rate.

Example 13.6.2.28

Loren loaned his brother $3,000 to help him buy a car. In 4 years his brother paid him back the $3,000 plus $660 in interest. What was the rate of interest?

$$\begin{aligned} I &= \$660 \\ P &= \$3000 \\ r &= ? \\ t &= 4 \text{ years} \end{aligned} \tag{13.6.2.2}$$

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for.	the rate of interest
Step 3. Name what we are looking for. Choose a variable to represent that quantity	Let r = the rate of interest.
Step 4. Translate into an equation.	
Write the formula.	$I = Prt$
Substitute in the given information.	$660 = (3000)r(4)$
Step 5. Solve the equation.	
	$660 = (12000)r$
Divide.	$0.055 = r$
Change to percent form.	$5.5\% = r$

Step 6. Check: Does this make sense?

$I = Prt$

$660 \stackrel{?}{=} (3000)(0.055)(4)$

$660 = 660$✓

Step 7. Answer the question with a complete sentence.	The rate of interest was 5.5%

Notice that in this example, Loren's brother paid Loren interest, just like a bank would have paid interest if Loren invested his money there.

? Try It 13.6.2.29

Jim loaned his sister $5,000 to help her buy a house. In 3 years, she paid him the $5,000, plus $900 interest. What was the rate of interest?

Answer

6%

? Try It 13.6.2.30

Hang borrowed $7,500 from her parents to pay her tuition. In 5 years, she paid them $1,500 interest in addition to the $7,500 she borrowed. What was the rate of interest?

Answer

4%

✓ Example 13.6.2.31

Eduardo noticed that his new car loan papers stated that with a 7.5% interest rate, he would pay $6,596.25 in interest over 5 years. How much did he borrow to pay for his car?

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for. Choose a variable to represent that quantity.	the amount borrowed (the principal)
Step 3. Name what we are looking for. Choose a variable to represent that quantity	Let P = principal borrowed.
Step 4. Translate into an equation.	
Write the formula.	$I = Prt$
Substitute in the given information.	$6596.25 = P(0.075)(5)$
Step 5. Solve the equation.	
	$6596.25 = 0.375P$
Divide.	$17590 = P$

Step 6. Check: Does this make sense?

$I = Prt$

$6596.25 \stackrel{?}{=} (17590)(0.075)(5)$

$6596.25 = 6596.25$✓

Step 7. Answer the question with a complete sentence. The principal is $17590

? Try It 13.6.2.32

Sean's new car loan statement said he would pay $4,866.25 in interest from an interest rate of 8.5% over 5 years. How much did he borrow to buy his new car?

Answer

$11,450

? Try It 13.6.2.33

In 5 years, Gloria's bank account earned $2,400 interest at 5%. How much had she deposited in the account?

Answer

$9,600

Solve Applications with Discount or Mark-up

Applications of discount are very common in retail settings. When you buy an item on sale, the original price has been discounted by some dollar amount. The **discount rate**, usually given as a percent, is used to determine the amount of the discount. To determine the **amount of discount**, we multiply the discount rate by the original price.

We summarize the discount model in the box below.

DISCOUNT

$$\begin{aligned} \text{amount of discount} &= \text{discount rate} \times \text{original price} \\ \text{sale price} &= \text{original price - amount of discount} \end{aligned} \tag{13.6.2.3}$$

Keep in mind that the sale price should always be less than the original price.

Example 13.6.2.34

Elise bought a dress that was discounted 35% off of the original price of $140. What was (a) the amount of discount and (b) the sale price of the dress?

Solution

(a)
$$\begin{aligned} \text{Original price} &= \$140 \\ \text{Discount rate} &= 35\% \\ \text{Discount} &= ? \end{aligned}$$

Step 1. Read the problem.

Step 2. Identify what we are looking for. — the amount of discount

Step 3. Name what we are looking for. Choose a variable to represent that quantity. — Let d = the amount of discount.

Step 4. Translate into an equation. Write a sentence that gives the information to find it. Translate into an equation — $d = 0.35(140)$

Step 5. Solve the equation. — $d = 49$

Step 6. Check: Does this make sense?

Is a $49 discount reasonable for a $140 dress? Yes.

Step 7. Write a complete sentence to answer the question. — The amount of discount was $49

(b) Read the problem again.

Step 1. Identify what we are looking for.	the sale price of the dress
Step 2. Name what we are looking for.	
Choose a variable to represent that quantity.	Let s= the sale price.
Step 3. Translate into an equation.	
Write a sentence that gives the information to find it.	The sale price is the $140 minus the $49 discount
Translate into an equation.	$s = 140 - 49$
Step 4. Solve the equation.	$s = 91$
Step 5. Check. Does this make sense?	
Is the sale price less than the original price?	

Yes, $91 is less than $140.	
Step 6. Answer the question with a complete sentence.	The sale price of the dress was $91.

Try It 13.6.2.35

Find (a) the amount of discount and (b) the sale price:

Sergio bought a belt that was discounted 40% from an original price of $29.

Answer

(a) $11.60, (b) $17.40

Try It 13.6.2.36

Find (a) the amount of discount and (b) the sale price:

Oscar bought a barbecue that was discounted 65% from an original price of $395.

Answer

(a) $256.75, (b) $138.25

There may be times when we know the original price and the sale price, and we want to know the **discount rate**. To find the discount rate, first we will find the amount of discount and then use it to compute the rate as a percent of the original price. Example 13.6.2.37will show this case.

Example 13.6.2.37

Jeannette bought a swimsuit at a sale price of $13.95. The original price of the swimsuit was $31. Find the (a) amount of discount and (b) discount rate.

Solution

(a) Original price = $31
Discount rate = ?
Discount = $13.95

Step 1. Read the problem.

Step 2. Identify what we are looking for. — the amount of discount

Step 3. Name what we are looking for.
Choose a variable to represent that quantity. — Let d = the amount of discount.

Step 4. Translate into an equation. Write a sentence that gives the information to find it. — The discount is the difference between the original price and the sale price.

Translate into an equation — $d = 31 - 13.95$

Step 5. Solve the equation. — $d = 17.05$

Step 6. Check: Does this make sense?

Is 17.05 less than 31? Yes.

Step 7. Answer the question with a complete sentence. — The amount of discount was $17.05

(b) Read the problem again.

Step 1. Identify what we are looking for.	the discount rate
Step 2. Name what we are looking for.	
Choose a variable to represent it.	Let r = the discount rate.
Step 3. Translate into an equation.	
Write a sentence that gives the information to find it.	The discount of $17.05 is what percent of $31 ?
Translate into an equation.	$17.05 = r \cdot 31$
Step 4. Solve the equation.	$17.05 = 31r$
Divide both sides by 31.	$0.55 = r$
Change to percent form.	$r = 55\%$
Step 5. Check. Does this make sense?	
Is $17.05 equal to 55% of $31?	
$17.05 \stackrel{?}{=} 0.55(31)$	
$17.05 = 17.05$✓	
Step 6. Answer the question with a complete sentence.	The rate of discount was 55%.

Try It 13.6.2.38

Find

a. the amount of discount and
b. the discount rate.

Lena bought a kitchen table at the sale price of $375.20. The original price of the table was $560.

Answer

a. $184.80
b. 33%

Try It 13.6.2.39

Find

a. the amount of discount and
b. the discount rate.

Nick bought a multi-room air conditioner at a sale price of $340. The original price of the air conditioner was $400.

Answer

a. $60
b. 15%

Applications of mark-up are very common in retail settings. The price a retailer pays for an item is called the **original cost**. The retailer then adds a **mark-up** to the original cost to get the **list price**, the price he sells the item for. The mark-up is usually calculated as a percent of the original cost. To determine the amount of mark-up, multiply the mark-up rate by the original cost.

We summarize the mark-up model in the box below.

MARK-UP

$$\begin{aligned} \text{amount of mark-up} &= \text{mark-up rate} \times \text{original cost} \\ \text{list price} &= \text{original cost} + \text{amount of mark up} \end{aligned} \tag{13.6.2.4}$$

Keep in mind that the list price should always be more than the original cost.

Example 13.6.2.40

Adam's art gallery bought a photograph at original cost $250. Adam marked the price up 40%. Find the

a. amount of mark-up and
b. the list price of the photograph.

Solution

a.

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the amount of mark-up
Step 3. Name what we are looking for.	
Choose a variable to represent it.	Let m = the amount of markup.
Step 4. Translate into an equation.	
Write a sentence that gives the information to find it.	The mark-up is 40% of the $250 original cost

Translate into an equation.	$m = 0.40 \cdot 250$
Step 5. Solve the equation.	$m = 100$
Step 6. Check. Does this make sense?	
Yes, 40% is less than one-half and 100 is less than half of 250.	
Step 7. Answer the question with a complete sentence.	The mark-up on the photograph was $100.

b.

Step 1. Read the problem again.	
Step 2. Identify what we are looking for.	the list price
Step 3. Name what we are looking for.	
Choose a variable to represent it.	Let p= the list price.
Step 4. Translate into an equation.	
Write a sentence that gives the information to find it.	The list price is original cost plus the mark-up
Translate into an equation.	$p = 250 + 100$
Step 5. Solve the equation.	$p = 350$
Step 6. Check. Does this make sense?	
Is the list price more than the net price? Is $350 more than $250? Yes.	
Step 7. Answer the question with a complete sentence.	The list price of the photograph was $350.

Try It 13.6.2.41

Find

a. the amount of mark-up and
b. the list price.

Jim's music store bought a guitar at original cost $1,200. Jim marked the price up 50%.

Answer

a. $600
b. $1,800

Try It 13.6.2.42

Find

a. the amount of mark-up and
b. the list price.

The Auto Resale Store bought Pablo's Toyota for $8,500. They marked the price up 35%.

Answer

a. $2,975
b. $11,475

Key Concepts

- **Percent Increase** To find the percent increase:
  1. Find the amount of increase. increase=new amount−originalamount
  2. Find the percent increase. Increase is what percent of the original amount?
- **Percent Decrease** To find the percent decrease:
  1. Find the amount of decrease. decrease=original amount−newamount
  2. Find the percent decrease. Decrease is what percent of the original amount?
- **Simple Interest** If an amount of money, *P*, called the principal, is invested for a period of *t* years at an annual interest rate *r*, the amount of interest, *I*, earned is

$$I = Prt \quad \text{where} \quad \begin{aligned} I &= \text{interest} \\ P &= \text{principle} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned} \tag{13.6.2.5}$$

- **Discount**
  - amount of discount is discount rate × original price
  - sale price is original price – discount
- **Mark-up**
  - amount of mark-up is mark-up rate × original cost
  - list price is original cost + mark up

Glossary

amount of discount

The amount of discount is the amount resulting when a discount rate is multiplied by the original price of an item.

discount rate

The discount rate is the percent used to determine the amount of a discount, common in retail settings.

interest

Interest is the money that a bank pays its customers for keeping their money in the bank.

list price

The list price is the price a retailer sells an item for.

mark-up

A mark-up is a percentage of the original cost used to increase the price of an item.

original cost

The original cost in a retail setting, is the price that a retailer pays for an item.

principal

The principal is the original amount of money invested or borrowed for a period of time at a specific interest rate.

rate of interest

The rate of interest is a percent of the principal, usually expressed as a percent per year.

simple interest

Simple interest is the interest earned according to the formula I=Prt.

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13.6.2E: Exercises

Practice Makes Perfect

Translate and Solve Basic Percent Equations

In the following exercises, translate and solve.

Exercise 13.6.2E.1

What number is 45% of 120?

Answer

54

Exercise 13.6.2E.2

What number is 65% of 100?

Exercise 13.6.2E.3

What number is 24% of 112?

Answer

26.88

Exercise 13.6.2E.4

What number is 36% of 124?

Exercise 13.6.2E.5

250% of 65 is what number?

Answer

162.5

Exercise 13.6.2E.6

150% of 90 is what number?

Exercise 13.6.2E.7

800% of 2250 is what number?

Answer

18,000

Exercise 13.6.2E.8

600% of 1740 is what number?

Exercise 13.6.2E.9

28 is 25% of what number?

Answer

112

Exercise 13.6.2E.10

36 is 25% of what number?

Exercise 13.6.2E.11

81 is 75% of what number?

Answer

108

Exercise 13.6.2E.12

93 is 75% of what number?

Exercise 13.6.2E.13

8.2% of what number is $2.87?

Answer

$35

Exercise 13.6.2E.14

6.4% of what number is $2.88?

Exercise 13.6.2E.15

11.5% of what number is $108.10?

Answer

$940

Exercise 13.6.2E.16

12.3% of what number is $92.25?

Exercise 13.6.2E.17

What percent of 260 is 78?

Answer

30%

Exercise 13.6.2E.18

What percent of 215 is 86?

Exercise 13.6.2E.19

What percent of 1500 is 540?

Answer

36%

Exercise 13.6.2E.20

What percent of 1800 is 846?

Exercise 13.6.2E.21

30 is what percent of 20?

Answer

150%

Exercise 13.6.2E.22

50 is what percent of 40?

Exercise 13.6.2E.23

840 is what percent of 480?

Answer

175%

Exercise 13.6.2E.24

790 is what percent of 395?

Solve Percent Applications

In the following exercises, solve.

Exercise 13.6.2E.25

Geneva treated her parents to dinner at their favorite restaurant. The bill was $74.25. Geneva wants to leave 16% of the total bill as a tip. How much should the tip be?

Answer

$11.88

Exercise 13.6.2E.26

When Hiro and his co-workers had lunch at a restaurant near their work, the bill was $90.50. They want to leave 18% of the total bill as a tip. How much should the tip be?

Exercise 13.6.2E.27

Trong has 12% of each paycheck automatically deposited to his savings account. His last paycheck was $2165. How much money was deposited to Trong's savings account?

Answer

$259.80

Exercise 13.6.2E.28

Cherise deposits 8% of each paycheck into her retirement account. Her last paycheck was $1,485. How much did Cherise deposit into her retirement account?

Exercise 13.6.2E.29

One serving of oatmeal has eight grams of fiber, which is 33% of the recommended daily amount. What is the total recommended daily amount of fiber?

Answer

24.2 g

Exercise 13.6.2E.30

One serving of trail mix has 67 grams of carbohydrates, which is 22% of the recommended daily amount. What is the total recommended daily amount of carbohydrates?

Exercise 13.6.2E.31

A bacon cheeseburger at a popular fast food restaurant contains 2070 milligrams (mg) of sodium, which is 86% of the recommended daily amount. What is the total recommended daily amount of sodium?

Answer

2407 mg

Exercise 13.6.2E.32

A grilled chicken salad at a popular fast food restaurant contains 650 milligrams (mg) of sodium, which is 27% of the recommended daily amount. What is the total recommended daily amount of sodium?

Exercise 13.6.2E.33

After 3 months on a diet, Lisa had lost 12% of her original weight. She lost 21 pounds. What was Lisa's original weight?

Answer

175 lb.

Exercise 13.6.2E.34

Tricia got a 6% raise on her weekly salary. The raise was $30 per week. What was her original salary?

Exercise 13.6.2E.35

Yuki bought a dress on sale for $72. The sale price was 60% of the original price. What was the original price of the dress?

Answer

$120

Exercise 13.6.2E.36

Kim bought a pair of shoes on sale for $40.50. The sale price was 45% of the original price. What was the original price of the shoes?

Exercise 13.6.2E.37

Tim left a $9 tip for a $50 restaurant bill. What percent tip did he leave?

Answer

18%

Exercise 13.6.2E.38

Rashid left a $15 tip for a $75 restaurant bill. What percent tip did he leave?

Exercise 13.6.2E.39

The nutrition fact sheet at a fast food restaurant says the fish sandwich has 380 calories, and 171 calories are from fat. What percent of the total calories is from fat?

Answer

45%

Exercise 13.6.2E.40

The nutrition fact sheet at a fast food restaurant says a small portion of chicken nuggets has 190 calories, and 114 calories are from fat. What percent of the total calories is from fat?

Exercise 13.6.2E.41

Emma gets paid $3,000 per month. She pays $750 a month for rent. What percent of her monthly pay goes to rent?

Answer

25%

Exercise 13.6.2E.42

Dimple gets paid $3,200 per month. She pays $960 a month for rent. What percent of her monthly pay goes to rent?

Find Percent Increase and Percent Decrease

In the following exercises, solve.

Exercise 13.6.2E.43

Tamanika got a raise in her hourly pay, from $15.50 to $17.36. Find the percent increase.

Answer

12%

Exercise 13.6.2E.44

Ayodele got a raise in her hourly pay, from $24.50 to $25.48. Find the percent increase.

Exercise 13.6.2E.45

Annual student fees at the University of California rose from about $4,000 in 2000 to about $12,000 in 2010. Find the percent increase.

Answer

200%

Exercise 13.6.2E.46

The price of a share of one stock rose from $12.50 to $50. Find the percent increase.

Exercise 13.6.2E.47

According to *Time* magazine annual global seafood consumption rose from 22 pounds per person in the 1960s to 38 pounds per person in 2011. Find the percent increase. (Round to the nearest tenth of a percent.)

Answer

72.7%

Exercise 13.6.2E.48

In one month, the median home price in the Northeast rose from $225,400 to $241,500. Find the percent increase. (Round to the nearest tenth of a percent.)

Exercise 13.6.2E.49

A grocery store reduced the price of a loaf of bread from $2.80 to $2.73. Find the percent decrease.

Answer

2.5%

Exercise 13.6.2E.50

The price of a share of one stock fell from $8.75 to $8.54. Find the percent decrease.

Exercise 13.6.2E.51

Hernando's salary was $49,500 last year. This year his salary was cut to $44,055. Find the percent decrease.

Answer

11%

Exercise 13.6.2E.52

In 10 years, the population of Detroit fell from 950,000 to about 712,500. Find the percent decrease.

Exercise 13.6.2E.53

In 1 month, the median home price in the West fell from $203,400 to $192,300. Find the percent decrease. (Round to the nearest tenth of a percent.)

Answer

5.5%

Exercise 13.6.2E.54

Sales of video games and consoles fell from $1,150 million to $1,030 million in 1 year. Find the percent decrease. (Round to the nearest tenth of a percent.)

Solve Simple Interest Applications

In the following exercises, solve.

Exercise 13.6.2E.55

Casey deposited $1,450 in a bank account with interest rate 4%. How much interest was earned in two years?

Answer

$116

Exercise 13.6.2E.56

Terrence deposited $5,720 in a bank account with interest rate 6%. How much interest was earned in 4 years?

Exercise 13.6.2E.57

Robin deposited $31,000 in a bank account with interest rate 5.2%. How much interest was earned in 3 years?

Answer

$4,836

Exercise 13.6.2E.58

Carleen deposited $16,400 in a bank account with interest rate 3.9%. How much interest was earned in 8 years?

Exercise 13.6.2E.59

Hilaria borrowed $8,000 from her grandfather to pay for college. Five years later, she paid him back the $8,000, plus $1,200 interest. What was the rate of interest?

Answer

3%

Exercise 13.6.2E.60

Kenneth loaned his niece $1,200 to buy a computer. Two years later, she paid him back the $1,200, plus $96 interest. What was the rate of interest?

Exercise 13.6.2E.61

Lebron loaned his daughter $20,000 to help her buy a condominium. When she sold the condominium four years later, she paid him the $20,000, plus $3,000 interest. What was the rate of interest?

Answer

3.75%

Exercise 13.6.2E.62

Pablo borrowed $50,000 to start a business. Three years later, he repaid the $50,000, plus $9,375 interest. What was the rate of interest?

Exercise 13.6.2E.63

In 10 years, a bank account that paid 5.25% earned $18,375 interest. What was the principal of the account?

Answer

$35,000

Exercise 13.6.2E.64

In 25 years, a bond that paid 4.75% earned $2,375 interest. What was the principal of the bond?

Exercise 13.6.2E.65

Joshua's computer loan statement said he would pay $1,244.34 in interest for a 3-year loan at 12.4%. How much did Joshua borrow to buy the computer?

Answer

$3,345

Exercise 13.6.2E.66

Margaret's car loan statement said she would pay $7,683.20 in interest for a 5-year loan at 9.8%. How much did Margaret borrow to buy the car?

Solve Applications with Discount or Mark-up

In the following exercises, find the sale price.

Exercise 13.6.2E.67

Perla bought a cell phone that was on sale for $50 off. The original price of the cell phone was $189.

Answer

$139

Exercise 13.6.2E.68

Sophie saw a dress she liked on sale for $15 off. The original price of the dress was $96.

Exercise 13.6.2E.69

Rick wants to buy a tool set with original price $165. Next week the tool set will be on sale for $40 off.

Answer

$125

Exercise 13.6.2E.70

Angelo's store is having a sale on televisions. One television, with original price $859, is selling for $125 off.

In the following exercises, find

1. the amount of discount and
2. the sale price.

Exercise 13.6.2E.71

Janelle bought a beach chair on sale at 60% off. The original price was $44.95.

Answer

1. $26.97
2. $17.98

Exercise 13.6.2E.72

Errol bought a skateboard helmet on sale at 40% off. The original price was $49.95.

Exercise 13.6.2E.73

Kathy wants to buy a camera that lists for $389. The camera is on sale with a 33% discount.

Answer

1. $128.37
2. $260.63

Exercise 13.6.2E.74

Colleen bought a suit that was discounted 25% from an original price of $245.

Exercise 13.6.2E.75

Erys bought a treadmill on sale at 35% off. The original price was $949.95 (round to the nearest cent.)

Answer

1. $332.48
2. $617.47

Exercise 13.6.2E.76

Jay bought a guitar on sale at 45% off. The original price was $514.75 (round to the nearest cent.)

In the following exercises, find

1. the amount of discount and
2. the discount rate. (Round to the nearest tenth of a percent if needed.)

Exercise 13.6.2E.77

Larry and Donna bought a sofa at the sale price of $1,344. The original price of the sofa was $1,920.

Answer

1. $576
2. 30%

Exercise 13.6.2E.78

Hiroshi bought a lawnmower at the sale price of $240. The original price of the lawnmower is $300.

Exercise 13.6.2E.79

Patty bought a baby stroller on sale for $301.75. The original price of the stroller was $355.

Answer

1. $53.25
2. 15%

Exercise 13.6.2E.80

Bill found a book he wanted on sale for $20.80. The original price of the book was $32.

Exercise 13.6.2E.81

Nikki bought a patio set on sale for $480. The original price was $850. To the nearest tenth of a percent, what was the rate of discount?

Answer

1. $370
2. 43.5%

Exercise 13.6.2E.82

Stella bought a dinette set on sale for $725. The original price was $1,299. To the nearest tenth of a percent, what was the rate of discount?

In the following exercises, find ⓐ the amount of the mark-up and ⓑ the list price.

Exercise 13.6.2E.83

Daria bought a bracelet at original cost $16 to sell in her handicraft store. She marked the price up 45%.

Answer

1. $7.20
2. $23.20

Exercise 13.6.2E.84

Regina bought a handmade quilt at original cost $120 to sell in her quilt store. She marked the price up 55%.

Exercise 13.6.2E.85

Tom paid $0.60 a pound for tomatoes to sell at his produce store. He added a 33% mark-up.

Answer

1. $0.20
2. $0.80

Exercise 13.6.2E.86

Flora paid her supplier $0.74 a stem for roses to sell at her flower shop. She added an 85% mark-up.

Exercise 13.6.2E.87

Alan bought a used bicycle for $115. After re-conditioning it, he added 225% mark-up and then advertised it for sale.

Answer

1. $258.75
2. $373.75

Exercise 13.6.2E.88

Michael bought a classic car for $8,500. He restored it, then added 150% mark-up before advertising it for sale.

Everyday Math

Exercise 13.6.2E.89

Leaving a Tip At the campus coffee cart, a medium coffee costs $1.65. MaryAnne brings $2.00 with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

Answer

21.2%

Exercise 13.6.2E.90

Splitting a Bill Four friends went out to lunch and the bill came to $53.75. They decided to add enough tip to make a total of $64, so that they could easily split the bill evenly among themselves. What percent tip did they leave?

Writing Exercises

Exercise 13.6.2E.91

Without solving the problem “44 is 80% of what number” think about what the solution might be. Should it be a number that is greater than 44 or less than 44? Explain your reasoning.

Answer

The number should be greater than 44. Since 80% equals 0.8 in decimal form, 0.8 is less than one, and we must multiply the number by 0.8 to get 44, the number must be greater than 44.

Exercise 13.6.2E.92

Without solving the problem “What is 20% of 300?” think about what the solution might be. Should it be a number that is greater than 300 or less than 300? Explain your reasoning.

Exercise 13.6.2E.93

After returning from vacation, Alex said he should have packed 50% fewer shorts and 200% more shirts. Explain what Alex meant.

Answer

He meant that he should have packed half the shorts and twice the shirts.

Exercise 13.6.2E.94

Because of road construction in one city, commuters were advised to plan that their Monday morning commute would take 150% of their usual commuting time. Explain what this means.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
translate and solve basic percent equations.			
solve percent applications.			
find percent increase and percent decrease.			
solve simple interest applications.			
solve applications with discount or mark-up.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

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13.6.3: Solve Mixture Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve coin word problems
- Solve ticket and stamp word problems
- Solve mixture word problems
- Use the mixture model to solve investment problems using simple interest

Note

Before you get started, take this readiness quiz.

1. Multiply: $14(0.25)$.
   If you missed this problem, review Exercise 1.8.19.
2. Solve: $0.25x + 0.10(x + 4) = 2.5$.
   If you missed this problem, review Exercise 2.4.22.
3. The number of dimes is three more than the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.
   If you missed this problem, review Exercise 1.3.43.

Solve Coin Word Problems

In **mixture problems**, we will have two or more items with different values to combine together. The mixture model is used by grocers and bartenders to make sure they set fair prices for the products they sell. Many other professionals, like chemists, investment bankers, and landscapers also use the mixture model.

Note

Doing the Manipulative Mathematics activity *Coin Lab* will help you develop a better understanding of mixture word problems.

We will start by looking at an application everyone is familiar with—money!

Imagine that we take a handful of coins from a pocket or purse and place them on a desk. How would we determine the value of that pile of coins? If we can form a step-by-step plan for finding the total value of the coins, it will help us as we begin solving coin word problems.

So what would we do? To get some order to the mess of coins, we could separate the coins into piles according to their value. Quarters would go with quarters, dimes with dimes, nickels with nickels, and so on. To get the total value of all the coins, we would add the total value of each pile.

Figure 13.6.3.1

How would we determine the value of each pile? Think about the dime pile—how much is it worth? If we count the number of dimes, we'll know how many we have—the *number* of dimes.

But this does not tell us the *value* of all the dimes. Say we counted 17 dimes, how much are they worth? Each dime is worth \$0.10—that is the *value* of one dime. To find the total value of the pile of 17 dimes, multiply 17 by \$0.10 to get \$1.70. This is the total

value of all 17 dimes. This method leads to the following model.

TOTAL VALUE OF COINS

For the same type of coin, the total value of a number of coins is found by using the model

$$number \cdot value = total\ value \tag{13.6.3.1}$$

where

number is the number of coins

value is the value of each coin

total value is the total value of all the coins

The number of dimes times the value of each dime equals the total value of the dimes.

$$\begin{aligned} \text{number.} \cdot \text{ value } &= \text{ total value} \\ 17 \cdot \$0.10 &= \$1.70 \end{aligned}$$

We could continue this process for each type of coin, and then we would know the total value of each type of coin. To get the total value of *all* the coins, add the total value of each type of coin.

Let's look at a specific case. Suppose there are 14 quarters, 17 dimes, 21 nickels, and 39 pennies.

Type	Number	• Value($)	= Total Value($)
Quarters	14	0.25	3.50
Dimes	17	0.10	1.70
Nickels	21	0.05	1.05
Pennies	39	0.01	0.39
			6.64

Table 13.6.3.1

The total value of all the coins is $6.64.

Notice how the chart helps organize all the information! Let's see how we use this method to solve a coin word problem.

Example 13.6.3.1

Adalberto has $2.25 in dimes and nickels in his pocket. He has nine more nickels than dimes. How many of each type of coin does he have?

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.

Determine the types of coins involved.
Think about the strategy we used to find the value of the handful of coins. The first thing we need is to notice what types of coins are involved. Adalberto has dimes and nickels. **Create a table** to organize the information. See chart below.

- Label the columns "type," "number," "value," "total value."
- List the types of coins.
- Write in the value of each type of coin.
- Write in the total value of all the coins.

We can work this problem all in cents or in dollars. Here we will do it in dollars and put in the dollar sign ($) in the table as a reminder.
The value of a dime is $0.10 and the value of a nickel is $0.05. The total value of all the coins is $2.25. The table below shows

this information.

Type	Number •	Value($) =	Total Value($)
Dimes		0.10	
Nickels		0.05	
			2.25

Step 2. Identify what we are looking for.

We are asked to find the number of dimes and nickels Adalberto has.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

Use variable expressions to represent the number of each type of coin and write them in the table.

Multiply the number times the value to get the total value of each type of coin.

Next we counted the number of each type of coin. In this problem we cannot count each type of coin—that is what you are looking for—but we have a clue. There are nine more nickels than dimes. The number of nickels is nine more than the number of dimes.

$$\begin{aligned} \text{Let } d &= \text{ number of dimes.} \\ d+9 &= \text{ number of nickels} \end{aligned}$$

Fill in the "number" column in the table to help get everything organized.

Type	Number •	Value($) =	Total Value($)
Dimes	d	0.10	
Nickels	$d+9$	0.05	
			2.25

Now we have all the information we need from the problem!

We multiply the number times the value to get the total value of each type of coin. While we do not know the actual number, we do have an expression to represent it.

And so now multiply $\text{number} \cdot \text{value} = \text{total value}$. See how this is done in the table below.

Type	Number •	Value($) =	Total Value($)
Dimes	d	0.10	$0.10d$
Nickels	$d+9$	0.05	$0.05(d+9)$
			2.25

Notice that we made the heading of the table show the model.

Step 4. Translate into an equation. It may be helpful to restate the problem in one sentence. Translate the English sentence into an algebraic equation.

Write the equation by adding the total values of all the types of coins.

	Value of dimes	+	value of nickels	=	total value of coins
Translate to an equation.	$0.10d$	+	$0.05(d+9)$	=	2.25

Step 5. Solve the equation using good algebra techniques.

Now solve this equation.	$0.10d + 0.05(d + 9) = 2.25$
Distribute.	$0.10d + 0.05d + 0.45 = 2.25$
Combine like terms.	$0.15d + 0.45 = 2.25$
Subtract 0.45 from each side.	$0.15d = 1.80$
Divide.	$d = 12$

So there are 12 dimes.	
The number of nickels is d+9d+9.	$d + 9$
	$12 + 9$
	21

Step 6. Check the answer in the problem and make sure it makes sense.

Does this check?

$$\begin{array}{lrcl} 12 \text{ dimes} & 12(0.10) & = & 1.20 \\ 21 \text{ nickels} & 21(0.05) & = & \underline{1.05} \\ & & & \$2.25 \checkmark \end{array} \qquad (13.6.3.2)$$

Step 7. Answer the question with a complete sentence.

Adalberto has twelve dimes and twenty-one nickels.

If this were a homework exercise, our work might look like the following.

Adalberto has $2.25 in dimes and nickels in his pocket. He has nine more nickels than dimes.

How many of each type does he have?

Type	Number • Value($)		= Total Value ($)
Dimes	d	0.10	0.10d
Nickels	d + 9	0.05	0.05(d + 9)
			2.25 ✓

12 dimes 12(0.10) = 1.20

21 nickels 21(0.05) = 1.05

$2.25 ✓

Adalberto has twelve dimes and twenty-one nickels.

$0.10d + 0.05d + 0.45 = 2.25$

$0.15d + 0.45 = 2.25$

$0.15d = 1.80$

$d = 12$ dimes

$d + 9$

$12 + 9$

21 nickels

? Try It 13.6.3.1

Michaela has $2.05 in dimes and nickels in her change purse. She has seven more dimes than nickels. How many coins of each type does she have?

Answer

9 nickels, 16 dimes

? Try It 13.6.3.2

Liliana has $2.10 in nickels and quarters in her backpack. She has 12 more nickels than quarters. How many coins of each type does she have?

Answer

17 nickels, 5 quarters

SOLVE COIN WORD PROBLEMS.

1. **Read** the problem. Make sure all the words and ideas are understood.
   - Determine the types of coins involved.
   - Create a table to organize the information.
   - Label the columns “type,” “number,” “value,” “total value.”
   - List the types of coins.
   - Write in the value of each type of coin.
   - Write in the total value of all the coins.

Type	Number •	Value($) =	Total Value($)

2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
   - Use variable expressions to represent the number of each type of coin and write them in the table.
   - Multiply the number times the value to get the total value of each type of coin.
4. **Translate** into an equation.
   It may be helpful to restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.
   Write the equation by adding the total values of all the types of coins.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Example 13.6.3.2

Maria has $2.43 in quarters and pennies in her wallet. She has twice as many pennies as quarters. How many coins of each type does she have?

Solution

Step 1. Read the problem.

Determine the types of coins involved.

We know that Maria has quarters and pennies.

Create a table to organize the information.

- **Step 2. Identify** what you are looking for.
  - We are looking for the number of quarters and pennies.

  Step 3. Name. Represent the number of quarters and pennies using variables.
  - Multiply the ‘number’ and the ‘value’ to get the ‘total value’ of each type of coin.

Type	Number •	Value($) =	Total Value($)
Quarters	q	0.25	$0.25q$
Pennies	$2q$	0.01	$0.01(2q)$
			2.43

  Step 4. Translate. Write the equation by adding the ‘total value’ of all the types of coins.

Step5. Solve the equation.	$0.25q+0.01(2q)=2.43$
Multiply.	$0.25q+0.02q=2.43$
Combine like terms.	$0.27q=2.43$
Divide by 0.27	$q=9$ quarters
The number of pennies is 2q.	$2q$
	$2\cdot 9$
	18 pennies

Step 6. Check the answer in the problem.

Maria has 9 quarters and 18 pennies. Dies this make $2.43?

9 quarters	$9(0.25)$	$=$	2.25
18 pennies	$18(0.01)$	$=$	0.18
Total			$2.43✓

Step 7. Answer the question. Maria has nine quarters and eighteen pennies.

Try It 13.6.3.3

Sumanta has $4.20 in nickels and dimes in her piggy bank. She has twice as many nickels as dimes. How many coins of each type does she have?

Answer

42 nickels, 21 dimes

Try It 13.6.3.4

Alison has three times as many dimes as quarters in her purse. She has $9.35 altogether. How many coins of each type does she have?

Answer

51 dimes, 17 quarters

In the next example, we'll show only the completed table—remember the steps we take to fill in the table.

Example 13.6.3.3

Danny has $2.14 worth of pennies and nickels in his piggy bank. The number of nickels is two more than ten times the number of pennies. How many nickels and how many pennies does Danny have?

Solution

Step 1. Read the problem.	
Determine the types of coins involved.	pennies and nickels
Create a table.	
Write in the value of each type of coin.	Pennies are worth $0.01. Nickels are worth $0.05.
Step 2. Identify what we are looking for.	the number of pennies and nickels

Step 3. Name. Represent the number of each type of coin using variables.	
The number of nickels is defined in terms of the number of pennies, so start with pennies.	Let $p =$ number of pennies.
The number of nickels is two more than ten times the number of pennies.	And let $10p + 2 =$ number of nickels.
Multiply the number and the value to get the total value of each type of coin.	

Type	Number • Value ($) = Total Value ($)		
pennies	p	0.01	$0.01p$
nickels	$10p + 2$	0.05	$0.05(10p + 2)$
			$2.14

Step 4. Translate. Write the equation by adding the total value of all the types of coins.	$0.01p + 0.05(10p + 2) = 2.14$
Step 5. Solve the equation.	$0.01p + 0.50p + 0.10 = 2.14$ $0.51p + 0.10 = 2.14$ $0.51p = 2.04$ $p = 4$ pennies
How many nickels?	$10p + 2$ $10(4) + 2$ 42 nickels
Step 6. Check the answer in the problem and make sure it makes sense Danny has four pennies and 42 nickels. Is the total value $2.14? $4(0.01) + 42(0.05) \stackrel{?}{=} 2.14$ $2.14 = 2.14$✓	
Step 7. Answer the question.	Danny has four pennies and 42 nickels.

Try It 13.6.3.5

Jesse has $6.55 worth of quarters and nickels in his pocket. The number of nickels is five more than two times the number of quarters. How many nickels and how many quarters does Jesse have?

Answer

41 nickels, 18 quarters

Try It 13.6.3.6

Elane has $7.00 total in dimes and nickels in her coin jar. The number of dimes that Elane has is seven less than three times the number of nickels. How many of each coin does Elane have?

Answer

22 nickels, 59 dimes

Solve Ticket and Stamp Word Problems

Problems involving tickets or stamps are very much like coin problems. Each type of ticket and stamp has a value, just like each type of coin does. So to solve these problems, we will follow the same steps we used to solve coin problems.

Example 13.6.3.4

At a school concert, the total value of tickets sold was $1,506. Student tickets sold for $6 each and adult tickets sold for $9 each. The number of adult tickets sold was five less than three times the number of student tickets sold. How many student tickets and how many adult tickets were sold?

Solution

Step 1. Read the problem.

- Determine the types of tickets involved. There are student tickets and adult tickets.
- Create a table to organize the information.

Type	Number •	Value($) =	Total Value($)
Student		6	
Adult		9	
			1506

Step 2. Identify what we are looking for.

- We are looking for the number of student and adult tickets.

Step 3. Name. Represent the number of each type of ticket using variables.

We know the number of adult tickets sold was five less than three times the number of student tickets sold.

- Multiply the number times the value to get the total value of each type of ticket.

Type	Number •	Value($) =	Total Value($)
Student	s	6	$6s$
Adult	$3s-5$	9	$9(3s-5)$
			1506

Step 4. Translate. Write the equation by adding the total values of each type of ticket.

$$6s+9(3s-5)=1506$$

Step 5. Solve the equation.

$$\begin{aligned} 6s+27s-45 &= 1506 \\ 33s-45 &= 1506 \\ 33s &= 1551 \\ s &= 47 \text{ student tickets} \\ \text{Number of adult tickets} &= 3s-5 \\ &= 3(47)-5 \\ \text{So there were} \quad 136 &\quad \text{adult tickets} \end{aligned}$$

Step 6. Check the answer.

There were 47 student tickets at $6 each and 136 adult tickets at $9 each. Is the total value $1,506? We find the total value of each type of ticket by multiplying the number of tickets times its value then add to get the total value of all the tickets sold.

$$\begin{aligned} 47\cdot 6 &= 282 \\ 136\cdot 9 &= \underline{1224} \\ & \quad 1506\checkmark \end{aligned}$$

Step 7. Answer the question. They sold 47 student tickets and 136 adult tickets.

Try It 13.6.3.7

The first day of a water polo tournament the total value of tickets sold was $17,610. One-day passes sold for $20 and tournament passes sold for $30. The number of tournament passes sold was 37 more than the number of day passes sold. How many day passes and how many tournament passes were sold?

Answer

330 day passes, 367 tournament passes

Try It 13.6.3.8

At the movie theater, the total value of tickets sold was $2,612.50. Adult tickets sold for $10 each and senior/child tickets sold for $7.50 each. The number of senior/child tickets sold was 25 less than twice the number of adult tickets sold. How many senior/child tickets and how many adult tickets were sold?

Answer

112 adult tickets, 199 senior/child tickets

We have learned how to find the total number of tickets when the number of one type of ticket is based on the number of the other type. Next, we'll look at an example where we know the total number of tickets and have to figure out how the two types of tickets relate.

Suppose Bianca sold a total of 100 tickets. Each ticket was either an adult ticket or a child ticket. If she sold 20 child tickets, how many adult tickets did she sell?

- *Did you say '80'? How did you figure that out? Did you subtract 20 from 100?*

If she sold 45 child tickets, how many adult tickets did she sell?

- *Did you say '55'? How did you find it? By subtracting 45 from 100?*

What if she sold 75 child tickets? How many adult tickets did she sell?

- *The number of adult tickets must be 100−75. She sold 25 adult tickets.*

Now, suppose Bianca sold x child tickets. Then how many adult tickets did she sell? To find out, we would follow the same logic we used above. In each case, we subtracted the number of child tickets from 100 to get the number of adult tickets. We now do the same with x.

We have summarized this below.

Child tickets	Adult tickets
20	80
45	55
75	25
x	$100-x$

Table 13.6.3.2

We can apply these techniques to other examples

Example 13.6.3.5

Galen sold 810 tickets for his church's carnival for a total of $2,820. Children's tickets cost $3 each and adult tickets cost $5 each. How many children's tickets and how many adult tickets did he sell?

Solution

Step 1. Read the problem.

- Determine the types of tickets involved. There are children tickets and adult tickets.
- Create a table to organize the information.

Type	Number	• Value($)	= Total Value($)
Children		3	
Adult		5	
			2820

Step 2. Identify what we are looking for.

- We are looking for the number of children and adult tickets.

Step 3. Name. Represent the number of each type of ticket using variables.

- We know the total number of tickets sold was 810.
- This means the number of children's tickets plus the number of adult tickets must add up to 810.
- Let c be the number of children tickets.

Type	Number	• Value($)	= Total Value($)
Children	c	3	$3c$
Adult	$810-c$	5	$5(810-c)$
			2820

- Then $810-c$ is the number of adult tickets.
- Multiply the number times the value to get the total value of each type of ticket.

Step 4. Translate.

Write the equation by adding the total values of each type of ticket.

Step 5. Solve the equation.

$$\begin{aligned} 3c+5(810-c) &= 2,820 \\ 3c+4,050-5c &= 2,820 \\ -2c &= -1,230 \\ c &= 615 \text{ children tickets} \end{aligned}$$

How many adults?

$$\begin{aligned} &810-c \\ &810-615 \\ &195 \text{ adult tickets} \end{aligned}$$

Step 6. Check the answer. There were 615 children's tickets at $3 each and 195 adult tickets at $5 each. Is the total value $2,820?

$$\begin{aligned} 615\cdot 3 &= 1845 \\ 195\cdot 5 &= \underline{975} \\ & 2,820\checkmark \end{aligned}$$

Step 7. Answer the question. Galen sold 615 children's tickets and 195 adult tickets.

Try It 13.6.3.9

During her shift at the museum ticket booth, Leah sold 115 tickets for a total of $1,163. Adult tickets cost $12 and student tickets cost $5. How many adult tickets and how many student tickets did Leah sell?

Answer

84 adult tickets, 31 student tickets

Try It 13.6.3.10

A whale-watching ship had 40 paying passengers on board. The total collected from tickets was $1,196. Full-fare passengers paid $32 each and reduced-fare passengers paid $26 each. How many full-fare passengers and how many reduced-fare passengers were on the ship?

Answer

26 full-fare, 14 reduced fare

Now, we'll do one where we fill in the table all at once.

Example 13.6.3.6

Monica paid $8.36 for stamps. The number of 41-cent stamps was four more than twice the number of two-cent stamps. How many 41-cent stamps and how many two-cent stamps did Monica buy?

Solution

The types of stamps are 41-cent stamps and two-cent stamps. Their names also give the value!

"The number of 41-cent stamps was four more than twice the number of two-cent stamps."

$$\text{Let } x = \text{ number of 2 -cent stamps.}$$
$$2x+4 = \text{ number of 41 - cent stamps}$$

Type	Number •	Value($) =	Total Value($)
41 cent stamps	$2x+4$	0.41	$0.41(2x+4)$
2 cent stamps	x	0.02	$0.02x$
			8.36

Write the equation from the total values.

$$0.41(2x+4)+0.02x = 8.36$$
$$0.82x+1.64+0.02x = 8.36$$
$$0.84x+1.64 = 8.36$$

Solve the equation.

$$0.84x = 6.72$$
$$x = 8$$

Monica bought eight two-cent stamps.

Find the number of 41-cent stamps she bought by evaluating $2x+4$ for $x=8$

$$2x+4$$
$$2(8)+4$$
$$20$$

Check.

$$\begin{aligned} 8(0.02)+20(0.41) &\stackrel{?}{=} 8.36 \\ 0.16+8.20 &\stackrel{?}{=} 8.36 \\ 8.36 &= 8.36\checkmark \end{aligned} \qquad (13.6.3.3)$$

Monica bought eight two-cent stamps and 20 41-cent stamps

Try It 13.6.3.11

Eric paid $13.36 for stamps. The number of 41-cent stamps was eight more than twice the number of two-cent stamps. How many 41-cent stamps and how many two-cent stamps did Eric buy?

Answer

32 at $0.41, 12 at $0.02

Try It 13.6.3.12

Kailee paid $12.66 for stamps. The number of 41-cent stamps was four less than three times the number of 20-cent stamps. How many 41-cent stamps and how many 20-cent stamps did Kailee buy?

Answer

26 at $0.41, 10 at $0.20

Solve Mixture Word Problems

Now we'll solve some more general applications of the mixture model. Grocers and bartenders use the mixture model to set a fair price for a product made from mixing two or more ingredients. Financial planners use the mixture model when they invest money in a variety of accounts and want to find the overall interest rate. Landscape designers use the mixture model when they have an assortment of plants and a fixed budget, and event coordinators do the same when choosing appetizers and entrees for a banquet.

Our first mixture word problem will be making trail mix from raisins and nuts.

Example 13.6.3.7

Henning is mixing raisins and nuts to make 10 pounds of trail mix. Raisins cost $2 a pound and nuts cost $6 a pound. If Henning wants his cost for the trail mix to be $5.20 a pound, how many pounds of raisins and how many pounds of nuts should he use?

Solution

As before, we fill in a chart to organize our information.

The 10 pounds of trail mix will come from mixing raisins and nuts.

$$\text{Let } x = \text{ number of pounds of raisins.}$$
$$10 - x = \text{ number of pounds of nuts}$$

We enter the price per pound for each item.

We multiply the number times the value to get the total value.

Type	Number of pounds	• Price per pound ($)	= Total Value ($)
Raisins	x	2	$2x$
Nuts	$10 - x$	6	$6(10 - x)$
Trail mix	10	5.20	10(5.20)

Notice that the last line in the table gives the information for the total amount of the mixture.

We know the value of the raisins plus the value of the nuts will be the value of the trail mix.

Write the equation from the total values.	$2x + 6(10 - x) = 10(5.20)$
Solve the equation.	$2x + 60 - 6x = 52$
	$-4x = -8$

	$x = 2$ pounds of raisins
Find the number of pounds of nuts.	$10 - x$
	$10 - 2$
	8 pounds of nuts
Check. $2(\$2) + 8(\$6) \stackrel{?}{=} 10(\$5.20)$ $\$4 + \$48 \stackrel{?}{=} \$52$ $\$52 = \$52\checkmark$	
	Henning mixed two pounds of raisins with eight pounds of nuts.

Try It 13.6.3.13

Orlando is mixing nuts and cereal squares to make a party mix. Nuts sell for $7 a pound and cereal squares sell for $4 a pound. Orlando wants to make 30 pounds of party mix at a cost of $6.50 a pound, how many pounds of nuts and how many pounds of cereal squares should he use?

Answer

5 pounds cereal squares, 25 pounds nuts

Try It 13.6.3.14

Becca wants to mix fruit juice and soda to make a punch. She can buy fruit juice for $3 a gallon and soda for $4 a gallon. If she wants to make 28 gallons of punch at a cost of $3.25 a gallon, how many gallons of fruit juice and how many gallons of soda should she buy?

Answer

21 gallons of fruit punch, 7 gallons of soda

We can also use the mixture model to solve investment problems using **simple interest**. We have used the simple interest formula, $I = Prt$, where t represented the number of years. When we just need to find the interest for one year, $t = 1$, so then $I = Pr$.

Example 13.6.3.8

Stacey has $20,000 to invest in two different bank accounts. One account pays interest at 3% per year and the other account pays interest at 5% per year. How much should she invest in each account if she wants to earn 4.5% interest per year on the total amount?

Solution

We will fill in a chart to organize our information. We will use the simple interest formula to find the interest earned in the different accounts.

The interest on the mixed investment will come from adding the interest from the account earning 3% and the interest from the account earning 5% to get the total interest on the $20,000.

$$\begin{aligned} \text{Let } x &= \text{amount invested at } 3\% \\ 20,000 - x &= \text{amount invested at } 5\% \end{aligned}$$

The amount invested is the *principal* for each account.

We enter the interest rate for each account.

We multiply the amount invested times the rate to get the interest.

Type	Amount Invested	• Rate	= Interest
3%	x	0.03	$0.03x$
5%	$20{,}000 - x$	0.05	$0.05(20{,}000 - x)$
4.5%	20,000	0.045	$0.045(20{,}000)$

Notice that the total amount invested, 20,000, is the sum of the amount invested at 3% and the amount invested at 5%. And the total interest, $0.045(20,000)$ is the sum of the interest earned in the 3% account and the interest earned in the 5% account.

As with the other mixture applications, the last column in the table gives us the equation to solve.

Write the equation from the interest earned. Solve the equation.	$0.03x + 0.05(20000 - x) = 0.045(20000)$ $0.03x + 1000 - 0.05x = 900$ $-0.02x = -100$ $x = 5000$ $\text{amount invested at } 3\%$
Find the amount invested at 5%.	$20{,}000 - x$ $20{,}000 - 5{,}000$ 15,000 = amount invested at 5%
Check. $0.03x + 0.05(15000 + x) \stackrel{?}{=} 0.045(20000)$ $150 + 750 \stackrel{?}{=} 900$ $900 = 900\checkmark$	
	Stacey should invest \$5,000 in the account that earns 3% and \$15,000 in the account that earns 5%.

? Try It 13.6.3.15

Remy has \$14,000 to invest in two mutual funds. One fund pays interest at 4% per year and the other fund pays interest at 7% per year. How much should she invest in each fund if she wants to earn 6.1% interest on the total amount?

Answer

\$4,200 at 4%, \$9,800 at 7%

? Try It 13.6.3.16

Marco has \$8,000 to save for his daughter's college education. He wants to divide it between one account that pays 3.2% interest per year and another account that pays 8% interest per year. How much should he invest in each account if he wants the interest on the total investment to be 6.5%?

Answer

\$2,500 at 3.2%, \$5,500 at 8%

Key Concepts

- **Total Value of Coins** For the same type of coin, the total value of a number of coins is found by using the model. number·value=total value where *number* is the number of coins and *value* is the value of each coin; *total value* is the total value of all the coins
- **Problem-Solving Strategy—Coin Word Problems**

1. **Read** the problem. Make all the words and ideas are understood. Determine the types of coins involved.
   - Create a table to organize the information.
   - Label the columns type, number, value, total value.
   - List the types of coins.
   - Write in the value of each type of coin.
   - Write in the total value of all the coins.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
   Use variable expressions to represent the number of each type of coin and write them in the table.
   Multiply the number times the value to get the total value of each type of coin.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information.
   Then, translate the sentence into an equation.
   Write the equation by adding the total values of all the types of coins.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Glossary

mixture problems

Mixture problems combine two or more items with different values together.

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13.6.3E: Exercises

Practice Makes Perfect

Solve Coin Word Problems

In the following exercises, solve each coin word problem.

Exercise 13.6.3E.1

Jaime has $2.60 in dimes and nickels. The number of dimes is 14 more than the number of nickels. How many of each coin does he have?

Answer

8 nickels, 22 dimes

Exercise 13.6.3E.2

Lee has $1.75 in dimes and nickels. The number of nickels is 11 more than the number of dimes. How many of each coin does he have?

Exercise 13.6.3E.3

Ngo has a collection of dimes and quarters with a total value of $3.50. The number of dimes is seven more than the number of quarters. How many of each coin does he have?

Answer

15 dimes, 8 quarters

Exercise 13.6.3E.4

Connor has a collection of dimes and quarters with a total value of $6.30. The number of dimes is 14 more than the number of quarters. How many of each coin does he have?

Exercise 13.6.3E.5

A cash box of $1 and $5 bills is worth $45. The number of $1 bills is three more than the number of $5 bills. How many of each bill does it contain?

Answer

10 at $1, 7 at $5

Exercise 13.6.3E.6

Joe’s wallet contains $1 and $5 bills worth $47. The number of $1 bills is five more than the number of $5 bills. How many of each bill does he have?

Exercise 13.6.3E.7

Rachelle has $6.30 in nickels and quarters in her coin purse. The number of nickels is twice the number of quarters. How many coins of each type does she have?

Answer

18 quarters, 36 nickels

Exercise 13.6.3E.8

Deloise has $1.20 in pennies and nickels in a jar on her desk. The number of pennies is three times the number of nickels. How many coins of each type does she have?

Exercise 13.6.3E.9

Harrison has $9.30 in his coin collection, all in pennies and dimes. The number of dimes is three times the number of pennies. How many coins of each type does he have?

Answer

30 pennies, 90 dimes

Exercise 13.6.3E.10

Ivan has $8.75 in nickels and quarters in his desk drawer. The number of nickels is twice the number of quarters. How many coins of each type does he have?

Exercise 13.6.3E.11

In a cash drawer there is $125 in $5 and $10 bills. The number of $10 bills is twice the number of $5 bills. How many of each are in the drawer?

Answer

10 at $10, 5 at $5

Exercise 13.6.3E.12

John has $175 in $5 and $10 bills in his drawer. The number of $5 bills is three times the number of $10 bills. How many of each are in the drawer?

Exercise 13.6.3E.13

Carolyn has $2.55 in her purse in nickels and dimes. The number of nickels is nine less than three times the number of dimes. Find the number of each type of coin.

Answer

12 dimes and 27 nickels

Exercise 13.6.3E.14

Julio has $2.75 in his pocket in nickels and dimes. The number of dimes is 10 less than twice the number of nickels. Find the number of each type of coin.

Exercise 13.6.3E.15

Chi has $11.30 in dimes and quarters. The number of dimes is three more than three times the number of quarters. How many of each are there?

Answer

63 dimes, 20 quarters

Exercise 13.6.3E.16

Tyler has $9.70 in dimes and quarters. The number of quarters is eight more than four times the number of dimes. How many of each coin does he have?

Exercise 13.6.3E.17

Mukul has $3.75 in quarters, dimes and nickels in his pocket. He has five more dimes than quarters and nine more nickels than quarters. How many of each coin are in his pocket?

Answer

16 nickels, 12 dimes, 7 quarters

Exercise 13.6.3E.18

Vina has $4.70 in quarters, dimes and nickels in her purse. She has eight more dimes than quarters and six more nickels than quarters. How many of each coin are in her purse?

Solve Ticket and Stamp Word Problems

In the following exercises, solve each ticket or stamp word problem.

Exercise 13.6.3E.19

The school play sold $550 in tickets one night. The number of $8 adult tickets was 10 less than twice the number of $5 child tickets. How many of each ticket were sold?

Answer

30 child tickets, 50 adult tickets

Exercise 13.6.3E.20

If the number of $8 child tickets is seventeen less than three times the number of $12 adult tickets and the theater took in $584, how many of each ticket were sold?

Exercise 13.6.3E.21

The movie theater took in $1,220 one Monday night. The number of $7 child tickets was ten more than twice the number of $9 adult tickets. How many of each were sold?

Answer

110 child tickets, 50 adult tickets

Exercise 13.6.3E.22

The ball game sold $1,340 in tickets one Saturday. The number of $12 adult tickets was 15 more than twice the number of $5 child tickets. How many of each were sold?

Exercise 13.6.3E.23

The ice rink sold 95 tickets for the afternoon skating session, for a total of $828. General admission tickets cost $10 each and youth tickets cost $8 each. How many general admission tickets and how many youth tickets were sold?

Answer

34 general, 61 youth

Exercise 13.6.3E.24

For the 7:30 show time, 140 movie tickets were sold. Receipts from the $13 adult tickets and the $10 senior tickets totaled $1,664. How many adult tickets and how many senior tickets were sold?

Exercise 13.6.3E.25

The box office sold 360 tickets to a concert at the college. The total receipts were $4170. General admission tickets cost $15 and student tickets cost $10. How many of each kind of ticket was sold?

Answer

114 general, 246 student

Exercise 13.6.3E.26

Last Saturday, the museum box office sold 281 tickets for a total of $3954. Adult tickets cost $15 and student tickets cost $12. How many of each kind of ticket was sold?

Exercise 13.6.3E.27

Julie went to the post office and bought both $0.41 stamps and $0.26 postcards. She spent $51.40. The number of stamps was 20 more than twice the number of postcards. How many of each did she buy?

Answer

40 postcards, 100 stamps

Exercise 13.6.3E.28

Jason went to the post office and bought both $0.41 stamps and $0.26 postcards and spent $10.28. The number of stamps was four more than twice the number of postcards. How many of each did he buy?

Exercise 13.6.3E.29

Maria spent $12.50 at the post office. She bought three times as many $0.41 stamps as $0.02 stamps. How many of each did she buy?

Answer

30 at $0.41, 10 at $0.02

Exercise 13.6.3E.30

Hector spent $33.20 at the post office. He bought four times as many $0.41 stamps as $0.02 stamps. How many of each did he buy?

Exercise 13.6.3E.31

Hilda has $210 worth of $10 and $12 stock shares. The numbers of $10 shares is five more than twice the number of $12 shares. How many of each does she have?

Answer

15 $10 shares, 5 $12 shares

Exercise 13.6.3E.32

Mario invested $475 in $45 and $25 stock shares. The number of $25 shares was five less than three times the number of $45 shares. How many of each type of share did he buy?

Solve Mixture Word Problems

In the following exercises, solve each mixture word problem.

Exercise 13.6.3E.33

Lauren in making 15 liters of mimosas for a brunch banquet. Orange juice costs her $1.50 per liter and champagne costs her $12 per liter. How many liters of orange juice and how many liters of champagne should she use for the mimosas to cost Lauren $5 per liter?

Answer

5 liters champagne, 10 liters orange juice

Exercise 13.6.3E.34

Macario is making 12 pounds of nut mixture with macadamia nuts and almonds. Macadamia nuts cost $9 per pound and almonds cost $5.25 per pound. How many pounds of macadamia nuts and how many pounds of almonds should Macario use for the mixture to cost $6.50 per pound to make?

Exercise 13.6.3E.35

Kaapo is mixing Kona beans and Maui beans to make 25 pounds of coffee blend. Kona beans cost Kaapo $15 per pound and Maui beans cost $24 per pound. How many pounds of each coffee bean should Kaapo use for his blend to cost him $17.70 per pound?

Answer

7.5 lbs Maui beans, 17.5 Kona beans

Exercise 13.6.3E.36

Estelle is making 30 pounds of fruit salad from strawberries and blueberries. Strawberries cost $1.80 per pound and blueberries cost $4.50 per pound. If Estelle wants the fruit salad to cost her $2.52 per pound, how many pounds of each berry should she use?

Exercise 13.6.3E.37

Carmen wants to tile the floor of his house. He will need 1000 square feet of tile. He will do most of the floor with a tile that costs $1.50 per square foot, but also wants to use an accent tile that costs $9.00 per square foot. How many square feet of each tile should he plan to use if he wants the overall cost to be $3 per square foot?

Answer

800 at $1.50, 200 at $9.00

Exercise 13.6.3E.38

Riley is planning to plant a lawn in his yard. He will need nine pounds of grass seed. He wants to mix Bermuda seed that costs $4.80 per pound with Fescue seed that costs $3.50 per pound. How much of each seed should he buy so that the overall cost will be $4.02 per pound?

Exercise 13.6.3E.39

Vartan was paid $25,000 for a cell phone app that he wrote and wants to invest it to save for his son's education. He wants to put some of the money into a bond that pays 4% annual interest and the rest into stocks that pay 9% annual interest. If he wants to earn 7.4% annual interest on the total amount, how much money should he invest in each account?

Answer

$8000 at 4%, $17,000 at 9%

Exercise 13.6.3E.40

Vern sold his 1964 Ford Mustang for $55,000 and wants to invest the money to earn him 5.8% interest per year. He will put some of the money into Fund A that earns 3% per year and the rest in Fund B that earns 10% per year. How much should he invest into each fund if he wants to earn 5.8% interest per year on the total amount?

Exercise 13.6.3E.41

Stephanie inherited $40,000. She wants to put some of the money in a certificate of deposit that pays 2.1% interest per year and the rest in a mutual fund account that pays 6.5% per year. How much should she invest in each account if she wants to earn 5.4% interest per year on the total amount?

Answer

$10,000 in CD, $30,000 in mutual fund

Exercise 13.6.3E.42

Avery and Caden have saved $27,000 towards a down payment on a house. They want to keep some of the money in a bank account that pays 2.4% annual interest and the rest in a stock fund that pays 7.2% annual interest. How much should they put into each account so that they earn 6% interest per year?

Exercise 13.6.3E.43

Dominic pays 7% interest on his $15,000 college loan and 12% interest on his $11,000 car loan. What average interest rate does he pay on the total $26,000 he owes? (Round your answer to the nearest tenth of a percent.)

Answer

9.1%

Exercise 13.6.3E.44

Liam borrowed a total of $35,000 to pay for college. He pays his parents 3% interest on the $8,000 he borrowed from them and pays the bank 6.8% on the rest. What average interest rate does he pay on the total $35,000? (Round your answer to the nearest tenth of a percent.)

Everyday Math

Exercise 13.6.3E.45

As the treasurer of her daughter's Girl Scout troop, Laney collected money for some girls and adults to go to a 3-day camp. Each girl paid $75 and each adult paid $30. The total amount of money collected for camp was $765. If the number of girls is three times the number of adults, how many girls and how many adults paid for camp?

Answer

9 girls, 3 adults

Exercise 13.6.3E.46

Laurie was completing the treasurer's report for her son's Boy Scout troop at the end of the school year. She didn't remember how many boys had paid the $15 full-year registration fee and how many had paid the $10 partial-year fee. She knew that the number of boys who paid for a full-year was ten more than the number who paid for a partial-year. If $250 was collected for all the registrations, how many boys had paid the full-year fee and how many had paid the partial-year fee?

Writing Exercises

Exercise 13.6.3E.47

Suppose you have six quarters, nine dimes, and four pennies. Explain how you find the total value of all the coins.

Answer

Answers will vary.

Exercise 13.6.3E.48

Do you find it helpful to use a table when solving coin problems? Why or why not?

Exercise 13.6.3E.49

In the table used to solve coin problems, one column is labeled "number" and another column is labeled "value." What is the difference between the "number" and the "value?"

Answer

Answers will vary.

Exercise 13.6.3E.50

What similarities and differences did you see between solving the coin problems and the ticket and stamp problems?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve coin word problems.			
solve ticket and stamp word problems.			
solve mixture word problems.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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13.6.4: Triangles, Rectangles, and the Pythagorean Theorem

Learning Objectives

By the end of this section, you will be able to:

- Solve applications using properties of triangles
- Use the Pythagorean Theorem
- Solve applications using rectangle properties

Be prepared

Before you get started, take this readiness quiz.

1. Simplify: $12(6h)$.
   If you missed this problem, review Exercise 1.10.1.
2. The length of a rectangle is three less than the width. Let w represent the width. Write an expression for the length of the rectangle.
   If you missed this problem, review Exercise 1.3.43.
3. Solve: $A = \frac{1}{2}bh$ for b when A=260 and h=52.
   If you missed this problem, review Exercise 2.6.10.
4. Simplify: $\sqrt{144}$.
   If you missed this problem, review Exercise 1.9.10.

Solve Applications Using Properties of Triangles

In this section we will use some common geometry formulas. We will adapt our problem-solving strategy so that we can solve geometry applications. The geometry formula will name the variables and give us the equation to solve. In addition, since these applications will all involve shapes of some sort, most people find it helpful to draw a figure and label it with the given information. We will include this in the first step of the problem solving strategy for geometry applications.

SOLVE GEOMETRY APPLICATIONS

1. **Read** the problem and make sure all the words and ideas are understood. Draw the figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Label** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer by substituting it back into the equation solved in step 5 and by making sure it makes sense in the context of the problem.
7. **Answer** the question with a complete sentence.

We will start geometry applications by looking at the properties of triangles. Let's review some basic facts about triangles. Triangles have three sides and three interior angles. Usually each side is labeled with a lowercase letter to match the uppercase letter of the opposite vertex.

The plural of the word *vertex* is *vertices*. All triangles have three **vertices**. Triangles are named by their vertices: The triangle in Figure 13.6.4.1is called $\triangle ABC$.

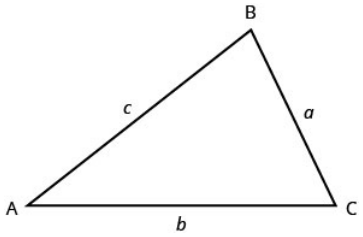


Figure 13.6.4.1: Triangle ABC has vertices A, B, and C. The lengths of the sides are a, b, and c.

The three angles of a triangle are related in a special way. The sum of their measures is 180°. Note that we read $m\angle A$ as "the measure of angle A." So in $\triangle ABC$ in Figure 13.6.4.1

$$m\angle A + m\angle B + m\angle C = 180^\circ$$

Because the **perimeter** of a figure is the length of its boundary, the perimeter of $\triangle ABC$ is the sum of the lengths of its three sides.

$$P = a + b + c$$

To find the **area** of a triangle, we need to know its base and height. The height is a line that connects the base to the opposite vertex and makes a 90° angle with the base. We will draw $\triangle ABC$ again, and now show the height, h. See Figure 13.6.4.2

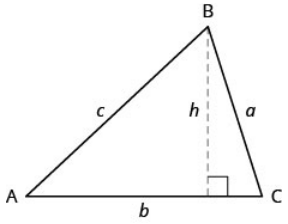
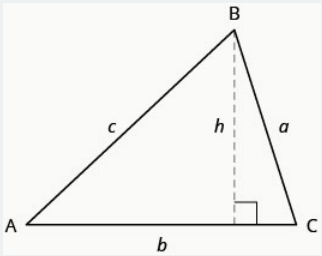


Figure 13.6.4.2: The formula for the area of $\triangle ABC$ is $A = \frac{1}{2}bh$ where b is the base and h is the height.

TRIANGLE PROPERTIES



For $\triangle ABC$

Angle measures:

$$m\angle A + m\angle B + m\angle C = 180^\circ \qquad (13.6.4.1)$$

- The sum of the measures of the angles of a triangle is 180°.

Perimeter:

$$P = a + b + c \qquad (13.6.4.2)$$

- The perimeter is the sum of the lengths of the sides of the triangle.

Area:

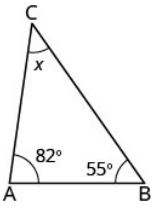
$A = \frac{1}{2}bh, b =$ base , $h =$ height

- The area of a triangle is one-half the base times the height.

Example 13.6.4.1

The measures of two angles of a triangle are 55 and 82 degrees. Find the measure of the third angle.

Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the measure of the third angle in a triangle
Step 3. Name. Choose a variable to represent it.	Let $x =$ the measure of the angle.
Step 4. Translate.	
Write the appropriate formula and substitute.	$m\angle A + m\angle B + m\angle C = 180^\circ$
Step 5. Solve the equation.	$\begin{aligned} 55 + 82 + x &= 180 \\ 137 + x &= 180 \\ x &= 43 \end{aligned}$
Step 6. Check. $\begin{aligned} 55 + 82 + 43 &\stackrel{?}{=} 180 \\ 180 &= 180\checkmark \end{aligned}$	
Step 7. Answer the question.	The measure of the third angle is 43 degrees.

Try It 13.6.4.1

The measures of two angles of a triangle are 31 and 128 degrees. Find the measure of the third angle.

Answer

21 degrees

Try It 13.6.4.2

The measures of two angles of a triangle are 49 and 75 degrees. Find the measure of the third angle.

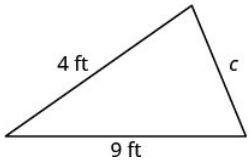
Answer

56 degrees

Example 13.6.4.2

The perimeter of a triangular garden is 24 feet. The lengths of two sides are four feet and nine feet. How long is the third side?

Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	 P = 24 ft
Step 2. Identify what you are looking for.	length of the third side of a triangle
Step 3. Name. Choose a variable to represent it.	Let $c =$ the third side.
Step 4. Translate.	
Write the appropriate formula and substitute.	$\underbrace{P}\ = \underbrace{a}\ + \underbrace{b}\ + \underbrace{c}$
Substitute in the given information.	$24 \text{ ft} = 4 \text{ ft} + 9 \text{ ft} + c$
Step 5. Solve the equation.	$24 = 13 + c$ $11 = c$
Step 6. Check. $P = a+b+c$ $24 \stackrel{?}{=} 4+9+11$ $24 = 24\checkmark$	
Step 7. Answer the question.	The third side is 11 feet long.

Try It 13.6.4.3

The perimeter of a triangular garden is 48 feet. The lengths of two sides are 18 feet and 22 feet. How long is the third side?

Answer

8 feet

Try It 13.6.4.4

The lengths of two sides of a triangular window are seven feet and five feet. The perimeter is 18 feet. How long is the third side?

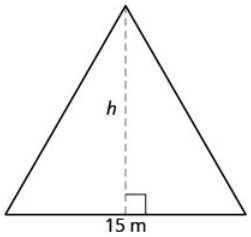
Answer

6 feet

Example 13.6.4.3

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window's height?

Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	 Area $= 90m^2$
Step 2. Identify what you are looking for.	height of a triangle
Step 3. Name. Choose a variable to represent it.	Let $h =$ the height.
Step 4. Translate.	
Write the appropriate formula.	$A = \frac{1}{2} \cdot b \cdot h$
Substitute in the given information.	$90\ \text{m}^2 = \frac{1}{2} \cdot 15\ \text{m} \cdot h$
Step 5. Solve the equation.	$90 = \frac{15}{2}h$ $12 = h$
Step 6. Check. $A = \frac{1}{2}bh$ $90 \stackrel{?}{=} \frac{1}{2} \cdot 15 \cdot 12$ $90 = 90\checkmark$	
Step 7. Answer the question.	The height of the triangle is 12 meters.

? Try It 13.6.4.5

The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Answer

14 inches

? Try It 13.6.4.6

A triangular tent door has area 15 square feet. The height is five feet. What is the base?

Answer

6 feet

The triangle properties we used so far apply to all triangles. Now we will look at one specific type of triangle—a right triangle. A **right triangle** has one 90° angle, which we usually mark with a small square in the corner.

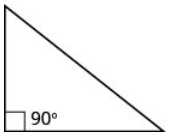


Figure 13.6.4.3

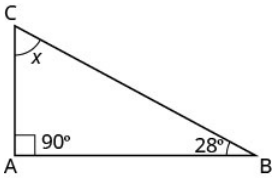
Definition: RIGHT TRIANGLE

A **right triangle** has one 90° angle, which is often marked with a square at the vertex.

Example 13.6.4.4

One angle of a right triangle measures 28°. What is the measure of the third angle?

Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the measure of an angle
Step 3. Name. Choose a variable to represent it.	Let $x =$ the measure of an angle.
Step 4. Translate.	$m\angle A + m\angle B + m\angle C = 180$
Write the appropriate formula and substitute.	$x + 90 + 28 = 180$
Step 5. Solve the equation.	$x = 62$
Step 6. Check. $180 \stackrel{?}{=} 90 + 28 + 62$ $180 = 180\checkmark$	
Step 7. Answer the question.	The measure of the third angle is 62°.

Try It 13.6.4.7

One angle of a right triangle measures 56°. What is the measure of the other small angle?

Answer

34°

Try It 13.6.4.8

One angle of a right triangle measures 45°. What is the measure of the other small angle?

Answer

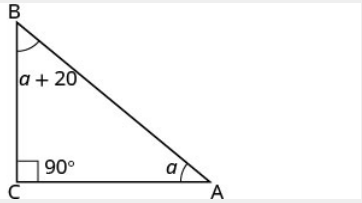
45°

In the examples we have seen so far, we could draw a figure and label it directly after reading the problem. In the next example, we will have to define one angle in terms of another. We will wait to draw the figure until we write expressions for all the angles we are looking for.

Example 13.6.4.5

The measure of one angle of a right triangle is 20 degrees more than the measure of the smallest angle. Find the measures of all three angles.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the measures of all three angles
Step 3. Name. Choose a variable to represent it.	Let $a = 1^{st}$ angle. $a + 20 = 2^{nd}$ angle $90 = 3^{rd}$ angle (the right angle)
Draw the figure and label it with the given information	
Step 4. Translate	$m\angle A + m\angle B + m\angle C = 180$
Write the appropriate formula. Substitute into the formula.	$a + (a + 20) + 90 = 180$
Step 5. Solve the equation.	$2a + 110 = 180$ $2a = 70$ $a = 35$ first angle $a + 20$ second angle $35 + 20 = 55$ And the third angle is 90.
Step 6. Check. $35 + 55 + 90 \stackrel{?}{=} 180$ $180 = 180$✓	
Step 7. Answer the question.	The three angles measure 35°, 55°, and 90°.

? Try It 13.6.4.9

The measure of one angle of a right triangle is 50° more than the measure of the smallest angle. Find the measures of all three angles.

Answer

20°,70°,90°

? Try It 13.6.4.10

The measure of one angle of a right triangle is 30° more than the measure of the smallest angle. Find the measures of all three angles.

Answer

30°,60°,90°

Use the Pythagorean Theorem

We have learned how the measures of the angles of a triangle relate to each other. Now, we will learn how the lengths of the sides relate to each other. An important property that describes the relationship among the lengths of the three sides of a right triangle is

called the **Pythagorean Theorem**. This theorem has been used around the world since ancient times. It is named after the Greek philosopher and mathematician, Pythagoras, who lived around 500 BC.

Before we state the Pythagorean Theorem, we need to introduce some terms for the sides of a triangle. Remember that a right triangle has a 90° angle, marked with a small square in the corner. The side of the triangle opposite the 90°90° angle is called the *hypotenuse* and each of the other sides are called *legs*.

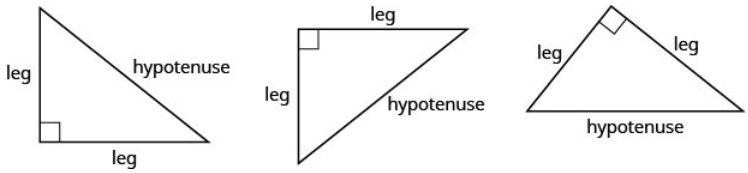


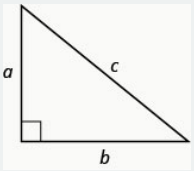
Figure 13.6.4.4

The Pythagorean Theorem tells how the lengths of the three sides of a right triangle relate to each other. It states that in any right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse. In symbols we say: in any right triangle, $a^2+b^2=c^2$, where a and b are the lengths of the legs and cc is the length of the hypotenuse.

Writing the formula in every exercise and saying it aloud as you write it, may help you remember the Pythagorean Theorem.

THE PYTHAGOREAN THEOREM

In any right triangle, where a and b are the lengths of the legs, c is the length of the hypotenuse.



Then

$$a^2+b^2=c^2 \tag{13.6.4.3}$$

To solve exercises that use the Pythagorean Theorem (Equation 13.6.4.3), we will need to find square roots. We have used the notation $\sqrt{m}$ and the definition:

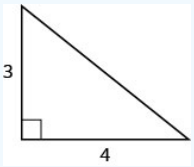
If $m=n^2$, then $\sqrt{m}=n$, for $n\geq 0$.

For example, we found that $\sqrt{25}$ is 5 because $25=5^2$.

Because the Pythagorean Theorem contains variables that are squared, to solve for the length of a side in a right triangle, we will have to use square roots.

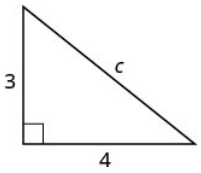
Example 13.6.4.6

Use the Pythagorean Theorem to find the length of the hypotenuse shown below.



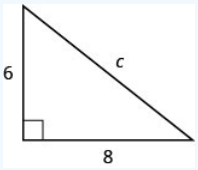
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length of the hypotenuse of the triangle

Step 3. Name. Choose a variable to represent it. Label side c on the figure.	Let c = the length of the hypotenuse. 
Step 4. Translate.	
Write the appropriate formula.	$a^2+b^2=c^2$
Substitute.	$3^2+4^2=c^2$
Step 5. Solve the equation.	$9+16=c^2$
Simplify.	$25=c^2$
Use the definition of square root.	$\sqrt{25}=c$
Simplify.	$5=c$
Step 6. Check. $3^2+4^2 \stackrel{?}{=} 5^2$ $9+16 \stackrel{?}{=} 25$ $25=25$ ✓	
Step 7. Answer the question.	The length of the hypotenuse is 5.

Try It 13.6.4.11

Use the Pythagorean Theorem to find the length of the hypotenuse in the triangle shown below.

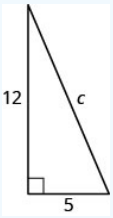


Answer

c=10

Try It 13.6.4.12

Use the Pythagorean Theorem to find the length of the hypotenuse in the triangle shown below.

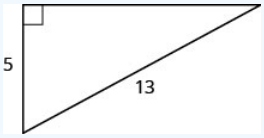


Answer

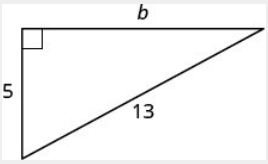
c=13

Example 13.6.4.7

Use the Pythagorean Theorem to find the length of the leg shown below.

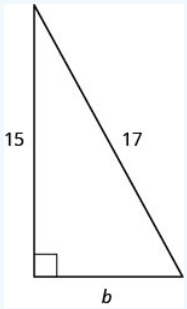


Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length of the leg of the triangle
Step 3. Name. Choose a variable to represent it.	Let $b =$ the leg of the triangle.
Label side b.	
Step 4. Translate	
Write the appropriate formula.	$a^2 + b^2 = c^2$
Substitute.	$5^2 + b^2 = 13^2$
Step 5. Solve the equation.	$25 + b^2 = 169$
Isolate the variable term.	$b^2 = 144$
Use the definition of square root.	$b = \sqrt{144}$
Simplify.	$b = 12$
Step 6. Check. $5^2 + 12^2 \stackrel{?}{=} 13^2$ $25 + 144 \stackrel{?}{=} 169$ $169 = 169$ ✓	
Step 7. Answer the question.	The length of the leg is 12.

Try It 13.6.4.13

Use the Pythagorean Theorem to find the length of the leg in the triangle shown below.

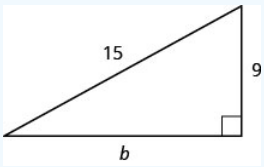


Answer

8

Try It 13.6.4.14

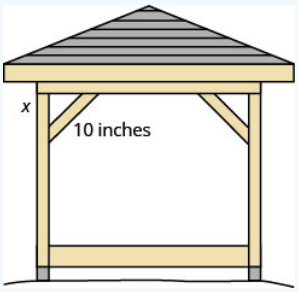
Use the Pythagorean Theorem to find the length of the leg in the triangle shown below.



Answer

12

Example 13.6.4.8



Kelvin is building a gazebo and wants to brace each corner by placing a 10″ piece of wood diagonally as shown above.

If he fastens the wood so that the ends of the brace are the same distance from the corner, what is the length of the legs of the right triangle formed? Approximate to the nearest tenth of an inch.

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for.	the distance from the corner that the bracket should be attached
Step 3. Name. Choose a variable to represent it.	Let x = distance from the corner.
Step 4. Translate Write the appropriate formula and substitute.	$a^2+b^2=c^2$ $x^2+x^2=10^2$
Step 5. Solve the equation.	
	$2x^2=100$
Isolate the variable.	$x^2=50$
Simplify. Approximate to the nearest tenth.	$x\approx 7.1$

Step 6. Check.
$a^2+b^2=c^2$
$(7.1)^2+(7.1)^2\approx 10^2$ Yes.

Step 7. Answer the question.	Kelven should fasten each piece of wood approximately 7.1'' from the corner.

? Try It 13.6.4.15

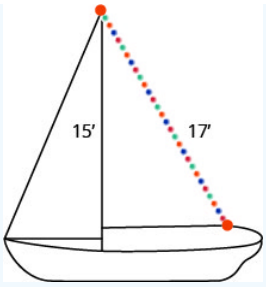
John puts the base of a 13-foot ladder five feet from the wall of his house as shown below. How far up the wall does the ladder reach?

Answer

12 feet

? Try It 13.6.4.16

Randy wants to attach a 17 foot string of lights to the top of the 15 foot mast of his sailboat, as shown below. How far from the base of the mast should he attach the end of the light string?

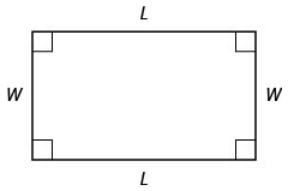


Answer

8 feet

Solve Applications Using Rectangle Properties

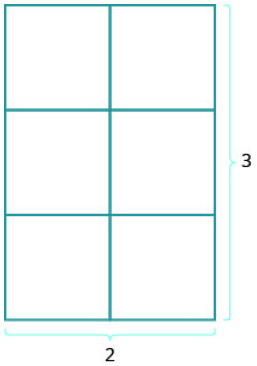
You may already be familiar with the properties of rectangles. Rectangles have four sides and four right (90°) angles. The opposite sides of a **rectangle** are the same length. We refer to one side of the rectangle as the length, L, and its adjacent side as the width, W.



The distance around this rectangle is $L+W+L+W$, or $2L+2W$. This is the **perimeter**, P, of the rectangle.

$$P=2L+2W \tag{13.6.4.4}$$

What about the **area** of a rectangle? Imagine a rectangular rug that is 2-feet long by 3-feet wide. Its area is 6 square feet. There are six squares in the figure.



$$\begin{aligned} A&=6\\ A&=2\cdot 3\\ A&=L\cdot W \end{aligned} \tag{13.6.4.5}$$

The area is the length times the width. The formula for the area of a rectangle is

$$A=LW. \tag{13.6.4.6}$$

PROPERTIES OF RECTANGLES

Rectangles have four sides and four right (90°) angles.

The lengths of opposite sides are equal.

The perimeter of a rectangle is the sum of twice the length and twice the width.

$$P = 2L + 2W \tag{13.6.4.7}$$

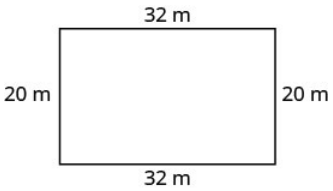
The area of a rectangle is the product of the length and the width.

$$A = L \cdot W \tag{13.6.4.8}$$

Example 13.6.4.9

The length of a rectangle is 32 meters and the width is 20 meters. What is the perimeter?

Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the perimeter of a rectangle
Step 3. Name. Choose a variable to represent it.	Let $P =$ the perimeter.
Step 4. Translate.	
Write the appropriate formula.	$P = 2L + 2W$
Substitute.	$P = 2(32\text{ m}) + 2(20\text{ m})$
Step 5. Solve the equation.	$P = 64 + 40$ $P = 104$
Step 6. Check. $P \stackrel{?}{=} 104$ $20 + 32 + 20 + 32 \stackrel{?}{=} 104$ $104 = 104\checkmark$	
Step 7. Answer the question.	The perimeter of the rectangle is 104 meters.

Try It 13.6.4.17

The length of a rectangle is 120 yards and the width is 50 yards. What is the perimeter?

Answer

340 yards

Try It 13.6.4.18

The length of a rectangle is 62 feet and the width is 48 feet. What is the perimeter?

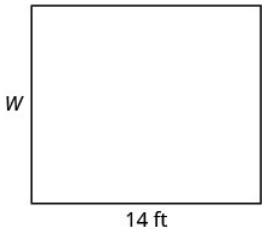
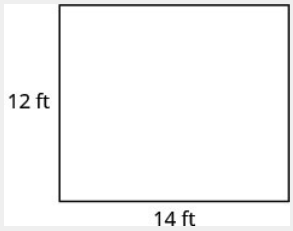
Answer

220 feet

Example 13.6.4.10

The area of a rectangular room is 168 square feet. The length is 14 feet. What is the width?

Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the width of a rectangular room
Step 3. Name. Choose a variable to represent it.	Let $W =$ the width.
Step 4. Translate.	
Write the appropriate formula.	$A = LW$
Substitute.	$168 = 14W$
Step 5. Solve the equation.	$\frac{168}{14} = \frac{14W}{14}$ $12 = W$
Step 6. Check.  $A = LW$ $168 \stackrel{?}{=} 14 \cdot 12$ $168 = 168$✓	
Step 7. Answer the question.	The width of the room is 12 feet.

Try It 13.6.4.19

The area of a rectangle is 598 square feet. The length is 23 feet. What is the width?

Answer

26 feet

Try It 13.6.4.20

The width of a rectangle is 21 meters. The area is 609 square meters. What is the length?

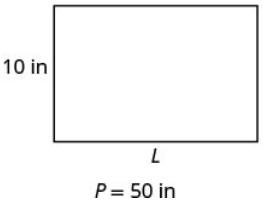
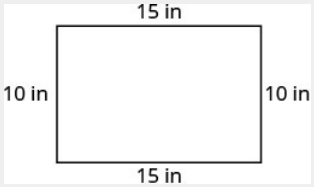
Answer

29 meters

Example 13.6.4.11

Find the length of a rectangle with perimeter 50 inches and width 10 inches.

Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the length of the rectangle
Step 3. Name. Choose a variable to represent it.	Let $L =$ the length.
Step 4. Translate.	
Write the appropriate formula.	$P = 2L + 2W$
Substitute.	$50 = 2L + 2(10)$
Step 5. Solve the equation.	$50 - 20 = 2L + 20 - 20$ $30 = 2L$ $\frac{30}{2} = \frac{2L}{2}$ $15 = L$
Step 6. Check.  $P = 50$ $15 + 10 + 15 + 10 \stackrel{?}{=} 50$ $50 = 50\checkmark$	
Step 7. Answer the question.	The length is 15 inches.

Try It 13.6.4.21

Find the length of a rectangle with: perimeter 80 and width 25.

Answer

15

Try It 13.6.4.22

Find the length of a rectangle with: perimeter 30 and width 6.

Answer

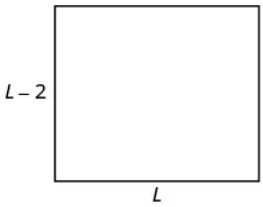
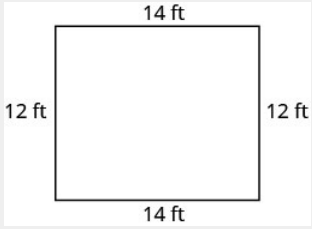
9

We have solved problems where either the length or width was given, along with the perimeter or area; now we will learn how to solve problems in which the width is defined in terms of the length. We will wait to draw the figure until we write an expression for the width so that we can label one side with that expression.

Example 13.6.4.12

The width of a rectangle is two feet less than the length. The perimeter is 52 feet. Find the length and width.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length and width of a rectangle
Step 3. Name. Choose a variable to represent it. Since the width is defined in terms of the length, we let $L =$ length. The width is two feet less than the length, so we let $L-2$ width.	 $P = 52$ ft
Step 4. Translate.	
Write the appropriate formula. The formula for the perimeter of a rectangle relates all the information.	$P = 2L + 2W$
Substitute in the given information.	$52 = 2L + 2(L-2)$
Step 5. Solve the equation.	$52 = 2L + 2L - 4$
Combine like terms.	$52 = 4L - 4$
Add 4 to each side.	$56 = 4L$
Divide by 4.	$\frac{56}{4} = \frac{4L}{4}$ $14 = L$ The length is 14 feet.
Now we need to find the width.	The width is $L-2$. $L-2$ $14-2$ 12 The width is 12 feet.
Step 6. Check. Since $14+12+14+12=52$, this works! 	
Step 7. Answer the question.	The length is 14 feet and the width is 12 feet.

Try It 13.6.4.23

The width of a rectangle is seven meters less than the length. The perimeter is 58 meters. Find the length and width.

Answer

18 meters, 11 meters

Try It 13.6.4.24

The length of a rectangle is eight feet more than the width. The perimeter is 60 feet. Find the length and width.

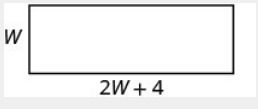
Answer

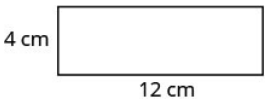
19 feet, 11 feet

Example 13.6.4.13

The length of a rectangle is four centimeters more than twice the width. The perimeter is 32 centimeters. Find the length and width.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length and the width
Step 3. Name. Choose a variable to represent the width.	Let W = width
The length is four more than twice the width.	$2W + 4$ = length  P = 32 cm
Step 4. Translate	
Write the appropriate formula.	$P = 2L + 2W$
Substitute in the given information.	$32 = 2(2W + 4) + 2W$
Step 5. Solve the equation.	$32 = 4W + 8 + 2W$ $32 = 6W + 8$ $24 = 6W$ $4 = W$ (width) $2W + 4$ (length) $2(4) + 4$ 12 The length is 12 cm.

Step 6. Check.  $P = 2L + 2W$ $32 \stackrel{?}{=} 2 \cdot 12 + 2 \cdot 4$ $32 = 32\checkmark$	
Step 7. Answer the question.	The length is 12 cm and the width is 4 cm.

Try It 13.6.4.25

The length of a rectangle is eight more than twice the width. The perimeter is 64. Find the length and width.

Answer

24, 8

Try It 13.6.4.26

The width of a rectangle is six less than twice the length. The perimeter is 18. Find the length and width.

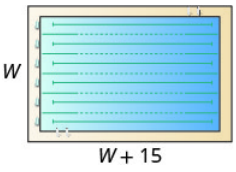
Answer

5, 4

Example 13.6.4.14

The perimeter of a rectangular swimming pool is 150 feet. The length is 15 feet more than the width. Find the length and width.

Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	 $P = 150$ ft
Step 2. Identify what you are looking for.	the length and the width of the pool
Step 3. Name. Choose a variable to represent the width. The length is 15 feet more than the width.	Let W = width $W + 15$ = length
Step 4. Translate	
Write the appropriate formula.	$P = 2L + 2W$
Substitute.	$150 = 2(W + 15) + 2W$

Step 5. Solve the equation.	$150 = 2W + 30 + 2W$ $150 = 4W + 30$ $120 = 4W$ $30 = W$ (the width of the pool) $W + 15$ (the length of the pool) $30 + 15$ 45
Step 6. Check. $P = 2L + 2W$ $150 \stackrel{?}{=} 2(45) + 2(30)$ $150 = 150\checkmark$	
Step 7. Answer the question.	The length of the pool is 45 feet and the width is 30 feet.

? Try It 13.6.4.27

The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.

Answer

70 feet, 30 feet

? Try It 13.6.4.28

The length of a rectangular garden is 30 yards more than the width. The perimeter is 300 yards. Find the length and width.

Answer

90 yards, 60 yards

Key Concepts

- **Problem-Solving Strategy for Geometry Applications**
  1. **Read** the problem and make all the words and ideas are understood. Draw the figure and label it with the given information.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for by choosing a variable to represent it.
  4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- **Triangle Properties For △ABC**
  Angle measures:
  - $m\angle A + m\angle B + m\angle C = 180$

  Perimeter:
  - $P = a + b + c$

  Area:
  - $A = \frac{1}{2}bh$, b=base,h=height

  A right triangle has one 90° angle.
- **The Pythagorean Theorem** In any right triangle, $a^2 + b^2 = c^2$ where c is the length of the hypotenuse and a and b are the lengths of the legs.

- **Properties of Rectangles**
  - Rectangles have four sides and four right (90°) angles.
  - The lengths of opposite sides are equal.
  - The perimeter of a rectangle is the sum of twice the length and twice the width: $P = 2L + 2W$.
  - The area of a rectangle is the length times the width: $A = LW$.

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13.6.4E: Exercises

Practice Makes Perfect

Solving Applications Using Triangle Properties

In the following exercises, solve using triangle properties.

Exercise 13.6.4E.1

The measures of two angles of a triangle are 26 and 98 degrees. Find the measure of the third angle.

Answer

56 degrees

Exercise 13.6.4E.2

The measures of two angles of a triangle are 61 and 84 degrees. Find the measure of the third angle.

Exercise 13.6.4E.3

The measures of two angles of a triangle are 105 and 31 degrees. Find the measure of the third angle.

Answer

44 degrees

Exercise 13.6.4E.4

The measures of two angles of a triangle are 47 and 72 degrees. Find the measure of the third angle.

Exercise 13.6.4E.5

The perimeter of a triangular pool is 36 yards. The lengths of two sides are 10 yards and 15 yards. How long is the third side?

Answer

11 feet

Exercise 13.6.4E.6

A triangular courtyard has perimeter 120 meters. The lengths of two sides are 30 meters and 50 meters. How long is the third side?

Exercise 13.6.4E.7

If a triangle has sides 6 feet and 9 feet and the perimeter is 23 feet, how long is the third side?

Answer

8 feet

Exercise 13.6.4E.8

If a triangle has sides 14 centimeters and 18 centimeters and the perimeter is 49 centimeters, how long is the third side?

Exercise 13.6.4E.9

A triangular flag has base one foot and height 1.5 foot. What is its area?

Answer

0.75 sq. ft.

Exercise 13.6.4E.10

A triangular window has base eight feet and height six feet. What is its area?

Exercise 13.6.4E.11

What is the base of a triangle with area 207 square inches and height 18 inches?

Answer

23 inches

Exercise 13.6.4E.12

What is the height of a triangle with area 893 square inches and base 38 inches?

Exercise 13.6.4E.13

One angle of a right triangle measures 33 degrees. What is the measure of the other small angle?

Answer

57

Exercise 13.6.4E.14

One angle of a right triangle measures 51 degrees. What is the measure of the other small angle?

Exercise 13.6.4E.15

One angle of a right triangle measures 22.5 degrees. What is the measure of the other small angle?

Answer

67.5

Exercise 13.6.4E.16

One angle of a right triangle measures 36.5 degrees. What is the measure of the other small angle?

Exercise 13.6.4E.17

The perimeter of a triangle is 39 feet. One side of the triangle is one foot longer than the second side. The third side is two feet longer than the second side. Find the length of each side.

Answer

13 ft., 12 ft., 14 ft.

Exercise 13.6.4E.18

The perimeter of a triangle is 35 feet. One side of the triangle is five feet longer than the second side. The third side is three feet longer than the second side. Find the length of each side.

Exercise 13.6.4E.19

One side of a triangle is twice the shortest side. The third side is five feet more than the shortest side. The perimeter is 17 feet. Find the lengths of all three sides.

Answer

3 ft., 6 ft., 8 ft.

Exercise 13.6.4E.20

One side of a triangle is three times the shortest side. The third side is three feet more than the shortest side. The perimeter is 13 feet. Find the lengths of all three sides.

Exercise 13.6.4E.21

The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.

Answer

$45^\circ, 45^\circ, 90^\circ$

Exercise 13.6.4E.22

The measure of the smallest angle of a right triangle is 20° less than the measure of the next larger angle. Find the measures of all three angles.

Exercise 13.6.4E.23

The angles in a triangle are such that one angle is twice the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

Answer

$30^\circ, 60^\circ, 90^\circ$

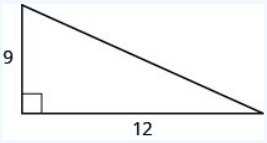
Exercise 13.6.4E.24

The angles in a triangle are such that one angle is 20° more than the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

Use the Pythagorean Theorem

In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.

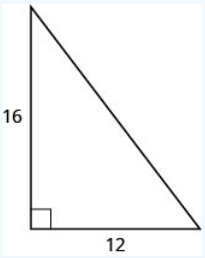
Exercise 13.6.4E.25



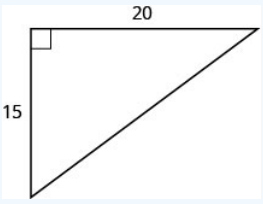
Answer

15

Exercise 13.6.4E.26



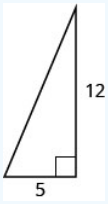
Exercise 13.6.4E.27



Answer

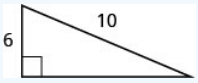
25

Exercise 13.6.4E.28



In the following exercises, use the Pythagorean Theorem to find the length of the leg. Round to the nearest tenth, if necessary.

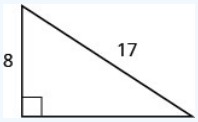
Exercise 13.6.4E.29



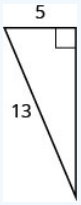
Answer

8

Exercise 13.6.4E.30



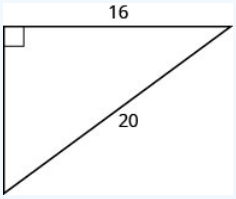
Exercise 13.6.4E.31



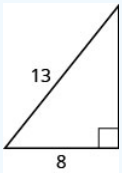
Answer

12

Exercise 13.6.4E.32



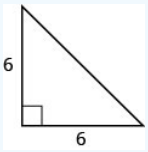
Exercise 13.6.4E.33



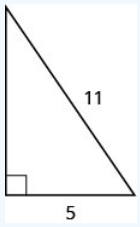
Answer

10.2

Exercise 13.6.4E.34



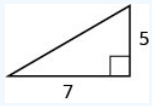
Exercise 13.6.4E.35



Answer

9.8

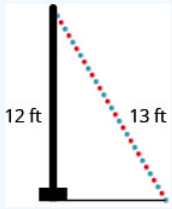
Exercise 13.6.4E.36



In the following exercises, solve using the Pythagorean Theorem. Approximate to the nearest tenth, if necessary.

Exercise 13.6.4E.37

A 13-foot string of lights will be attached to the top of a 12-foot pole for a holiday display, as shown below. How far from the base of the pole should the end of the string of lights be anchored?



Answer

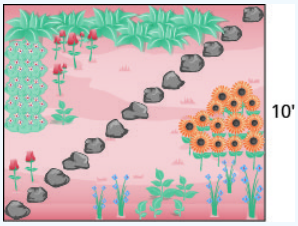
5 feet

Exercise 13.6.4E.38

Pam wants to put a banner across her garage door, as shown below, to congratulate her son for his college graduation. The garage door is 12 feet high and 16 feet wide. How long should the banner be to fit the garage door?

Exercise 13.6.4E. 39

Chi is planning to put a path of paving stones through her flower garden, as shown below. The flower garden is a square with side 10 feet. What will the length of the path be?



Answer

14.1 feet

Exercise 13.6.4E. 40

Brian borrowed a 20 foot extension ladder to use when he paints his house. If he sets the base of the ladder 6 feet from the house, as shown below, how far up will the top of the ladder reach?

Solve Applications Using Rectangle Properties

In the following exercises, solve using rectangle properties.

Exercise 13.6.4E. 41

The length of a rectangle is 85 feet and the width is 45 feet. What is the perimeter?

Answer

260 feet

Exercise 13.6.4E. 42

The length of a rectangle is 26 inches and the width is 58 inches. What is the perimeter?

Exercise 13.6.4E. 43

A rectangular room is 15 feet wide by 14 feet long. What is its perimeter?

Answer

58 feet

Exercise 13.6.4E.44

A driveway is in the shape of a rectangle 20 feet wide by 35 feet long. What is its perimeter?

Exercise 13.6.4E.45

The area of a rectangle is 414 square meters. The length is 18 meters. What is the width?

Answer

23 meters

Exercise 13.6.4E.46

The area of a rectangle is 782 square centimeters. The width is 17 centimeters. What is the length?

Exercise 13.6.4E.47

The width of a rectangular window is 24 inches. The area is 624 square inches. What is the length?

Answer

26 inches

Exercise 13.6.4E.48

The length of a rectangular poster is 28 inches. The area is 1316 square inches. What is the width?

Exercise 13.6.4E.49

Find the length of a rectangle with perimeter 124 and width 38.

Answer

24

Exercise 13.6.4E.50

Find the width of a rectangle with perimeter 92 and length 19.

Exercise 13.6.4E.51

Find the width of a rectangle with perimeter 16.2 and length 3.2.

Answer

4.9

Exercise 13.6.4E.52

Find the length of a rectangle with perimeter 20.2 and width 7.8.

Exercise 13.6.4E.53

The length of a rectangle is nine inches more than the width. The perimeter is 46 inches. Find the length and the width.

Answer

16 in., 7 in.

Exercise 13.6.4E.54

The width of a rectangle is eight inches more than the length. The perimeter is 52 inches. Find the length and the width.

Exercise 13.6.4E.55

The perimeter of a rectangle is 58 meters. The width of the rectangle is five meters less than the length. Find the length and the width of the rectangle.

Answer

17 m, 12 m

Exercise 13.6.4E.56

The perimeter of a rectangle is 62 feet. The width is seven feet less than the length. Find the length and the width.

Exercise 13.6.4E.57

The width of the rectangle is 0.7 meters less than the length. The perimeter of a rectangle is 52.6 meters. Find the dimensions of the rectangle.

Answer

13.5 m length, 12.8 m width

Exercise 13.6.4E.58

The length of the rectangle is 1.1 meters less than the width. The perimeter of a rectangle is 49.4 meters. Find the dimensions of the rectangle.

Exercise 13.6.4E.59

The perimeter of a rectangle is 150 feet. The length of the rectangle is twice the width. Find the length and width of the rectangle.

Answer

50 ft., 25 ft.

Exercise 13.6.4E.60

The length of a rectangle is three times the width. The perimeter of the rectangle is 72 feet. Find the length and width of the rectangle.

Exercise 13.6.4E.61

The length of a rectangle is three meters less than twice the width. The perimeter of the rectangle is 36 meters. Find the dimensions of the rectangle.

Answer

7 m width, 11 m length

Exercise 13.6.4E.62

The length of a rectangle is five inches more than twice the width. The perimeter is 34 inches. Find the length and width.

Exercise 13.6.4E.63

The perimeter of a rectangular field is 560 yards. The length is 40 yards more than the width. Find the length and width of the field.

Answer

160 yd., 120 yd.

Exercise 13.6.4E.64

The perimeter of a rectangular atrium is 160 feet. The length is 16 feet more than the width. Find the length and width of the atrium.

Exercise 13.6.4E.65

A rectangular parking lot has perimeter 250 feet. The length is five feet more than twice the width. Find the length and width of the parking lot.

Answer

85 ft., 40 ft.

Exercise 13.6.4E.66

A rectangular rug has perimeter 240 inches. The length is 12 inches more than twice the width. Find the length and width of the rug.

Everyday Math

Exercise 13.6.4E.67

Christa wants to put a fence around her triangular flowerbed. The sides of the flowerbed are six feet, eight feet and 10 feet. How many feet of fencing will she need to enclose her flowerbed?

Answer

24 feet

Exercise 13.6.4E.68

Jose just removed the children's playset from his back yard to make room for a rectangular garden. He wants to put a fence around the garden to keep out the dog. He has a 50 foot roll of fence in his garage that he plans to use. To fit in the backyard, the width of the garden must be 10 feet. How long can he make the other length?

Writing Exercises

Exercise 13.6.4E.69

If you need to put tile on your kitchen floor, do you need to know the perimeter or the area of the kitchen? Explain your reasoning.

Answer

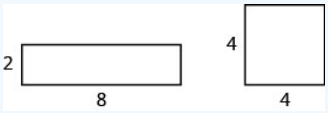
area; answers will vary

Exercise 13.6.4E.70

If you need to put a fence around your backyard, do you need to know the perimeter or the area of the backyard? Explain your reasoning.

Exercise 13.6.4E.71

Look at the two figures below.



1. Which figure looks like it has the larger area?
2. Which looks like it has the larger perimeter?
3. Now calculate the area and perimeter of each figure.
4. Which has the larger area?
5. Which has the larger perimeter?

Answer

1. Answers will vary.
2. Answers will vary.
3. Answers will vary.
4. The areas are the same.
5. The 2x8 rectangle has a larger perimeter than the 4x4 square.

Exercise 13.6.4E.72

Write a geometry word problem that relates to your life experience, then solve it and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve applications using triangle properties.			
use the Pythagorean Theorem.			
solve applications using rectangle properties.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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13.6.5: Solve Uniform Motion Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve uniform motion applications

Note

Before you get started, take this readiness quiz.

1. Find the distance traveled by a car going 70 miles per hour for 3 hours.
   If you missed this problem, review Exercise 2.6.1.
2. Solve $x+1.2(x-10)=98$.
   If you missed this problem, review Exercise 2.4.7.
3. Convert 90 minutes to hours.
   If you missed this problem, review Exercise 1.11.1.

Solve Uniform Motion Applications

When planning a road trip, it often helps to know how long it will take to reach the destination or how far to travel each day. We would use the distance, rate, and time formula, D=rt, which we have already seen.

In this section, we will use this formula in situations that require a little more algebra to solve than the ones we saw earlier. Generally, we will be looking at comparing two scenarios, such as two vehicles traveling at different rates or in opposite directions. When the speed of each vehicle is constant, we call applications like this *uniform motion problems*.

Our problem-solving strategies will still apply here, but we will add to the first step. The first step will include drawing a diagram that shows what is happening in the example. Drawing the diagram helps us understand what is happening so that we will write an appropriate equation. Then we will make a table to organize the information, like we did for the money applications.

The steps are listed here for easy reference:

USE A PROBLEM-SOLVING STRATEGY IN DISTANCE, RATE, AND TIME APPLICATIONS.

1. **Read** the problem. Make sure all the words and ideas are understood.
   - Draw a diagram to illustrate what it happening.
   - Create a table to organize the information.
   - Label the columns rate, time, distance.
   - List the two scenarios.
   - Write in the information you know.

	Rate	•	Time	=	Distance

2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
   - Complete the chart.
   - Use variable expressions to represent that quantity in each row.
   - Multiply the rate times the time to get the distance.
4. **Translate** into an equation.
   - Restate the problem in one sentence with all the important information.
   - Then, translate the sentence into an equation.
5. **Solve** the equation using good algebra techniques.

6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

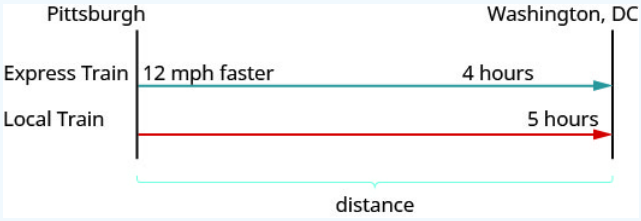
Example 13.6.5.1

An express train and a local train leave Pittsburgh to travel to Washington, D.C. The express train can make the trip in 4 hours and the local train takes 5 hours for the trip. The speed of the express train is 12 miles per hour faster than the speed of the local train. Find the speed of both trains.

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.

Draw a diagram to illustrate what it happening. Shown below is a sketch of what is happening in the example.



	Rate (mph)	•	Time (hrs)	=	Distance (miles)
Express			4		
Local			5		

Create a table to organize the information. Label the columns “Rate,” “Time,” and “Distance.” List the two scenarios. Write in the information you know.

Step 2. Identify what we are looking for.

We are asked to find the speed of both trains. Notice that the distance formula uses the word “rate,” but it is more common to use “speed” when we talk about vehicles in everyday English.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

Complete the chart Use variable expressions to represent that quantity in each row. We are looking for the speed of the trains. Let’s let r represent the speed of the local train. Since the speed of the express train is 12 mph faster, we represent that as r+12.

$$\begin{aligned} r &= \text{speed of the local train} \\ r+12 &= \text{speed of the express train} \end{aligned}$$

Fill in the speeds into the chart.

	Rate (mph)	•	Time (hrs)	=	Distance (miles)
Express	$r+12$		4		
Local	r		5		

Multiply the rate times the time to get the distance.

	Rate (mph)	•	Time (hrs)	=	Distance (miles)
Express	$r+12$		4		$4(r+12)$
Local	r		5		$5r$

Step 4. Translate into an equation.

Restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.

	distance traveled by express train	=	distance traveled by local train
Translate to an equation.	$4(r+12)$	=	$5r$

- The equation to model this situation will come from the relation between the distances. Look at the diagram we drew above. How is the distance traveled by the express train related to the distance traveled by the local train?
- Since both trains leave from Pittsburgh and travel to Washington, D.C. they travel the same distance. So we write:

Step 5. Solve the equation using good algebra techniques.

Now solve this equation.	$4(r+12)=5r$ $4r+48=5r$ $48=r$ So the speed of the local train is 48 mph.
Find the speed of the express train.	$r+12$ $48+12$ 60 he speed of the express train is 60 mph.

Step 6. Check the answer in the problem and make sure it makes sense. \[\begin{array}{ll}{\text { express train }} & {60 \mathrm{mph}(4 \text { hours })=240 \mathrm{miles}} \\ {\text { local train }} & {48 \mathrm{mph}(5 \text { hours })=240 \mathrm{miles}} \checkmark \end{array}\]

Step 7. Answer the question with a complete sentence.

The speed of the local train is 48 mph and the speed of the express train is 60 mph.

Try It 13.6.5.2

Wayne and Dennis like to ride the bike path from Riverside Park to the beach. Dennis's speed is seven miles per hour faster than Wayne's speed, so it takes Wayne 2 hours to ride to the beach while it takes Dennis 1.5 hours for the ride. Find the speed of both bikers.

Answer

Wayne 21 mph, Dennis 28 mph

Try It 13.6.5.3

Jeromy can drive from his house in Cleveland to his college in Chicago in 4.5 hours. It takes his mother 6 hours to make the same drive. Jeromy drives 20 miles per hour faster than his mother. Find Jeromy's speed and his mother's speed.

Answer

Jeromy 80 mph, mother 60 mph

In Exercise 13.6.5.4 the last example, we had two trains traveling the same distance. The diagram and the chart helped us write the equation we solved. Let's see how this works in another case.

Example 13.6.5.4

Christopher and his parents live 115 miles apart. They met at a restaurant between their homes to celebrate his mother's birthday. Christopher drove 1.5 hours while his parents drove 1 hour to get to the restaurant. Christopher's average speed was 10 miles per hour faster than his parents' average speed. What were the average speeds of Christopher and of his parents as they drove to the restaurant?

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.

Draw a diagram to illustrate what it happening. Below shows a sketch of what is happening in the example.

Create a table to organize the information.

Label the columns rate, time, distance.

List the two scenarios.

Write in the information you know.

	Rate (mph)	•	Time (hrs)	=	Distance (miles)
Christopher			1.5		
Parents			1		
					115

Step 2. Identify what we are looking for.

We are asked to find the average speeds of Christopher and his parents.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

Complete the chart.

Use variable expressions to represent that quantity in each row.

We are looking for their average speeds. Let's let r represent the average speed of the parents. Since the Christopher's speed is 10 mph faster, we represent that as r+10.

Fill in the speeds into the chart.

	Rate (mph)	•	Time (hrs)	=	Distance (miles)
Christopher	$r+10$		1.5		$1.5(r+10)$
Parents	r		1		r
					115

Multiply the rate times the time to get the distance.

Step 4. Translate into an equation.

Restate the problem in one sentence with all the important information. Then, translate the sentence into an equation. Again, we need to identify a relationship between the distances in order to write an equation. Look at the diagram we created above and notice the relationship between the distance Christopher traveled and the distance his parents traveled.

The distance Christopher travelled plus the distance his parents travel must add up to 115 miles. So we write:

	distance traveled by Christopher	+	distance traveled by his parents	= 115
Translate to an equation.	$1.5(r+10)$	+	r	= 115

Step 5. Solve the equation using good algebra techniques.

	$1.5(r+10)+r=115$
	$1.5r+15+r=115$
Now solve this equation.	$2.5r+15=115$
	$2.5r=100$
	$r=40$
	so the parents' speed was 40 mph.
	$r+10$
Christopher's speed is r + 10	$40+10$
	50
	Christopher's speed was 50 mph.

Step 6. Check the answer in the problem and make sure it makes sense.

Christopher drove	50 mph (1.5 hours)	=	75 miles
His parents drove	40 mph (1 hour)	=	40 miles
			115 miles

Step 7. Answer the question with a complete sentence.

Christopher's speed was 50 mph.
His parents' speed was 40 mph.

Try It 13.6.5.5

Carina is driving from her home in Anaheim to Berkeley on the same day her brother is driving from Berkeley to Anaheim, so they decide to meet for lunch along the way in Buttonwillow. The distance from Anaheim to Berkeley is 410 miles. It takes Carina 3 hours to get to Buttonwillow, while her brother drives 4 hours to get there. The average speed Carina's brother drove was 15 miles per hour faster than Carina's average speed. Find Carina's and her brother's average speeds.

Answer

Carina 50 mph, brother 65 mph

Try It 13.6.5.6

Ashley goes to college in Minneapolis, 234 miles from her home in Sioux Falls. She wants her parents to bring her more winter clothes, so they decide to meet at a restaurant on the road between Minneapolis and Sioux Falls. Ashley and her parents both drove 2 hours to the restaurant. Ashley's average speed was seven miles per hour faster than her parents' average speed. Find Ashley's and her parents' average speed.

Answer

parents 55 mph, Ashley 62 mph

As you read the next example, think about the relationship of the distances traveled. Which of the previous two examples is more similar to this situation?

Example 13.6.5.7

Two truck drivers leave a rest area on the interstate at the same time. One truck travels east and the other one travels west. The truck traveling west travels at 70 mph and the truck traveling east has an average speed of 60 mph. How long will they travel before they are 325 miles apart?

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.

Draw a diagram to illustrate what it happening.

Create a table to organize the information.

	Rate (mph)	• Time (hrs)	= Distance (miles)
West	70		
East	60		
			325

Step 2. Identify what we are looking for.

We are asked to find the amount of time the trucks will travel until they are 325 miles apart.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

We are looking for the time traveled. Both trucks will travel the same amount of time. Let's call the time t. Since their speeds are different, they will travel different distances. Complete the chart.

	Rate (mph)	• Time (hrs)	= Distance (miles)
West	70	t	$70t$
East	60	t	$60t$
			325

Step 4. Translate into an equation.

We need to find a relation between the distances in order to write an equation. Looking at the diagram, what is the relationship between the distance each of the trucks will travel? The distance traveled by the truck going west plus the distance traveled by the truck going east must add up to 325 miles. So we write:

	distance traveled by westbound truck	+	distance traveled by eastbound truck	= 32
Translate to an equation.	70t	+	60t	= 32

Step 5. Solve the equation using good algebra techniques.

$$\begin{aligned} \text{Now solve this equation.} \quad 70t+60t &= 325 \\ 130t &= 325 \\ t &= 2.5 \end{aligned} \tag{13.6.5.1}$$

Step 6. Check the answer in the problem and make sure it makes sense.

$$\begin{aligned} \text{Truck going West} \quad 70 \text{ mph } (2.5 \text{ hours}) &= 175 \text{ miles} \\ \text{Truck going East} \quad 60 \text{ mph } (2.5 \text{ hour}) &= \underline{150 \text{ miles}} \\ & \quad 325 \text{ miles} \end{aligned}$$

Step 7. Answer the question with a complete sentence. It will take the truck 2.5 hours to be 325 miles apart.

? Try It 13.6.5.8

Pierre and Monique leave their home in Portland at the same time. Pierre drives north on the turnpike at a speed of 75 miles per hour while Monique drives south at a speed of 68 miles per hour. How long will it take them to be 429 miles apart?

Answer

3 hours

Try It 13.6.5.9

Thanh and Nhat leave their office in Sacramento at the same time. Thanh drives north on I-5 at a speed of 72 miles per hour. Nhat drives south on I-5 at a speed of 76 miles per hour. How long will it take them to be 330 miles apart?

Answer

2.2 hours

MATCHING UNITS IN PROBLEMS

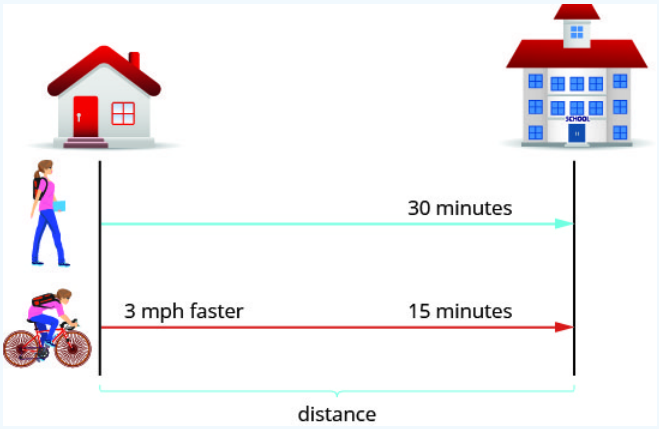
It is important to make sure the units match when we use the distance rate and time formula. For instance, if the rate is in miles per hour, then the time must be in hours.

Example 13.6.5.10

When Katie Mae walks to school, it takes her 30 minutes. If she rides her bike, it takes her 15 minutes. Her speed is three miles per hour faster when she rides her bike than when she walks. What are her walking speed and her speed riding her bike?

Solution

First, we draw a diagram that represents the situation to help us see what is happening.



We are asked to find her speed walking and riding her bike. Let's call her walking speed r. Since her biking speed is three miles per hour faster, we will call that speed r+3. We write the speeds in the chart.

The speed is in miles per hour, so we need to express the times in hours, too, in order for the units to be the same. Remember, one hour is 60 minutes. So:

$$\begin{aligned} 30 \text{ minutes is } \tfrac{30}{60} \text{ or } \tfrac{1}{2} \text{ hour} \\ 15 \text{ minutes is } \tfrac{15}{60} \text{ or } \tfrac{1}{4} \text{ hour} \end{aligned} \qquad (13.6.5.2)$$

Next, we multiply rate times time to fill in the distance column.

	Rate (mph)	• Time (hrs)	= Distance (miles)
Walk	r	$\frac{1}{2}$	$\frac{1}{2}r$
Bike	$r+3$	$\frac{1}{4}$	$\frac{1}{4}(r+3)$

The equation will come from the fact that the distance from Katie Mae's home to her school is the same whether she is walking or riding her bike.

So we say:

	distance walked = distance covered by bike
Translate into an equation.	$\frac{1}{2}r = \frac{1}{4}(r+3)$
Solve this equation.	$\frac{1}{2}r = \frac{1}{4}(r+3)$
Clear the fractions by multiplying by the LCD of all the fractions in the equation.	$8 \cdot \frac{1}{2}r = 8 \cdot \frac{1}{4}(r+3)$
Simplify.	$4r = 2(r+3)$ $4r = 2r + 6$ $2r = 6$ $r = 3$ mph (Katie Mae's walking speed) $r + 3$ biking speed $3 + 3$ 6 mph (Katie Mae's biking speed)
Let's check if this works. Walk 3 mph (0.5 hour) = 1.5 miles Bike 6 mph (0.25 hour) = 1.5 miles	
Yes, either way Katie Mae travels 1.5 miles to school.	Katie Mae's walking speed is 3 mph. Her speed riding her bike is 6 mph.

Try It 13.6.5.11

Suzy takes 50 minutes to hike uphill from the parking lot to the lookout tower. It takes her 30 minutes to hike back down to the parking lot. Her speed going downhill is 1.2 miles per hour faster than her speed going uphill. Find Suzy's uphill and downhill speeds.

Answer

uphill 1.8 mph, downhill three mph

Try It 13.6.5.12

Llewyn takes 45 minutes to drive his boat upstream from the dock to his favorite fishing spot. It takes him 30 minutes to drive the boat back downstream to the dock. The boat's speed going downstream is four miles per hour faster than its speed going upstream. Find the boat's upstream and downstream speeds.

Answer

upstream 8 mph, downstream 12 mph

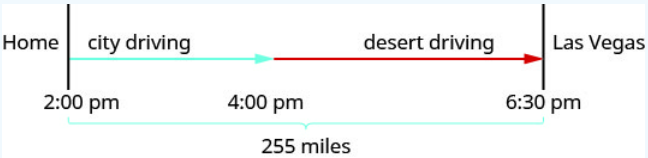
In the distance, rate, and time formula, time represents the actual amount of **elapsed time** (in hours, minutes, etc.). If a problem gives us starting and ending times as clock times, we must find the elapsed time in order to use the formula.

Example 13.6.5.13

Hamilton loves to travel to Las Vegas, 255 miles from his home in Orange County. On his last trip, he left his house at 2:00 pm. The first part of his trip was on congested city freeways. At 4:00 pm, the traffic cleared and he was able to drive through the desert at a speed 1.75 times as fast as when he drove in the congested area. He arrived in Las Vegas at 6:30 pm. How fast was he driving during each part of his trip?

Solution

A diagram will help us model this trip.



Next, we create a table to organize the information.

We know the total distance is 255 miles. We are looking for the rate of speed for each part of the trip. The rate in the desert is 1.75 times the rate in the city. If we let r= the rate in the city, then the rate in the desert is 1.75r.

The times here are given as clock times. Hamilton started from home at 2:00 pm and entered the desert at 4:30 pm. So he spent two hours driving the congested freeways in the city. Then he drove faster from 4:00 pm until 6:30 pm in the desert. So he drove 2.5 hours in the desert.

Now, we multiply the rates by the times.

	Rate (mph)	• Time (hrs)	= Distance (miles)
City	r	2	$2r$
Desert	$1.75r$	2.5	$2.5(1.75r)$
			255

By looking at the diagram below, we can see that the sum of the distance driven in the city and the distance driven in desert is 255 miles.

	distance driven in the city + distance driven in desert = 255
Translate into an equation.	$2r + 2.5(1.75r) = 255$
Solve this equation.	$2r + 2.5(1.75r) = 255$ $2r + 4.375r = 255$ $6.375r = 255$ $r = 40$ mph city $1.75r$ desert speed $1.75(40)$ 70 mph
Check. City 40 mph (2 hours) = 80 miles Desert 70 mph (2.5 hours) = 175 miles 255 miles	
	Hamilton drove 40 mph in the city and 70 mph in the desert.

? Try It 13.6.5.14

Cruz is training to compete in a triathlon. He left his house at 6:00 and ran until 7:30. Then he rode his bike until 9:45. He covered a total distance of 51 miles. His speed when biking was 1.6 times his speed when running. Find Cruz's biking and running speeds.

Answer

biking 16 mph, running 10 mph

? Try It 13.6.5.15

Phuong left home on his bicycle at 10:00. He rode on the flat street until 11:15, then rode uphill until 11:45. He rode a total of 31 miles. His speed riding uphill was 0.6 times his speed on the flat street. Find his speed biking uphill and on the flat street.

Answer

uphill 12 mph, flat street 20 mph

Key Concepts

- **Distance, Rate, and Time**
  - $D = rt$ where D = distance, r = rate, t = time
- **Problem-Solving Strategy—Distance, Rate, and Time Applications**
  1. **Read** the problem. Make sure all the words and ideas are understood.
     Draw a diagram to illustrate what it happening.
     Create a table to organize the information: Label the columns rate, time, distance. List the two scenarios. Write in the information you know.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
     Complete the chart.
     Use variable expressions to represent that quantity in each row.
     Multiply the rate times the time to get the distance.
  4. **Translate** into an equation.
     Restate the problem in one sentence with all the important information.
     Then, translate the sentence into an equation.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

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13.6.5E: Exercises

Practice Makes Perfect

Solve Uniform Motion Applications

In the following exercises, solve.

Exercise 13.6.5E.1

Lilah is moving from Portland to Seattle. It takes her three hours to go by train. Mason leaves the train station in Portland and drives to the train station in Seattle with all Lilah's boxes in his car. It takes him 2.4 hours to get to Seattle, driving at 15 miles per hour faster than the speed of the train. Find Mason's speed and the speed of the train.

Answer

Mason 75 mph, train 60 mph

Exercise 13.6.5E.2

Kathy and Cheryl are walking in a fundraiser. Kathy completes the course in 4.8 hours and Cheryl completes the course in 8 hours. Kathy walks two miles per hour faster than Cheryl. Find Kathy's speed and Cheryl's speed.

Exercise 13.6.5E.3

Two busses go from Sacramento for San Diego. The express bus makes the trip in 6.8 hours and the local bus takes 10.2 hours for the trip. The speed of the express bus is 25 mph faster than the speed of the local bus. Find the speed of both busses.

Answer

express bus 75mph, local 50 mph

Exercise 13.6.5E.4

A commercial jet and a private airplane fly from Denver to Phoenix. It takes the commercial jet 1.1 hours for the flight, and it takes the private airplane 1.8 hours. The speed of the commercial jet is 210 miles per hour faster than the speed of the private airplane. Find the speed of both airplanes.

Exercise 13.6.5E.5

Saul drove his truck 3 hours from Dallas towards Kansas City and stopped at a truck stop to get dinner. At the truck stop he met Erwin, who had driven 4 hours from Kansas City towards Dallas. The distance between Dallas and Kansas City is 542 miles, and Erwin's speed was eight miles per hour slower than Saul's speed. Find the speed of the two truckers.

Answer

Saul 82 mph, Erwin 74 mph

Exercise 13.6.5E.6

Charlie and Violet met for lunch at a restaurant between Memphis and New Orleans. Charlie had left Memphis and drove 4.8 hours towards New Orleans. Violet had left New Orleans and drove 2 hours towards Memphis, at a speed 10 miles per hour faster than Charlie's speed. The distance between Memphis and New Orleans is 394 miles. Find the speed of the two drivers.

Exercise 13.6.5E.7

Sisters Helen and Anne live 332 miles apart. For Thanksgiving, they met at their other sister's house partway between their homes. Helen drove 3.2 hours and Anne drove 2.8 hours. Helen's average speed was four miles per hour faster than Anne's. Find Helen's average speed and Anne's average speed.

Answer

Helen 60 mph, Anne 56 mph

Exercise 13.6.5E.8

Ethan and Leo start riding their bikes at the opposite ends of a 65-mile bike path. After Ethan has ridden 1.5 hours and Leo has ridden 2 hours, they meet on the path. Ethan's speed is six miles per hour faster than Leo's speed. Find the speed of the two bikers.

Exercise 13.6.5E.9

Elvira and Aletheia live 3.1 miles apart on the same street. They are in a study group that meets at a coffee shop between their houses. It took Elvira half an hour and Aletheia two-thirds of an hour to walk to the coffee shop. Aletheia's speed is 0.6 miles per hour slower than Elvira's speed. Find both women's walking speeds.

Answer

Aletheia 2.4 mph, Elvira 3 mph

Exercise 13.6.5E.10

DaMarcus and Fabian live 23 miles apart and play soccer at a park between their homes. DaMarcus rode his bike for three-quarters of an hour and Fabian rode his bike for half an hour to get to the park. Fabian's speed was six miles per hour faster than DaMarcus' speed. Find the speed of both soccer players.

Exercise 13.6.5E.11

Cindy and Richard leave their dorm in Charleston at the same time. Cindy rides her bicycle north at a speed of 18 miles per hour. Richard rides his bicycle south at a speed of 14 miles per hour. How long will it take them to be 96 miles apart?

Answer

3 hours

Exercise 13.6.5E.12

Matt and Chris leave their uncle's house in Phoenix at the same time. Matt drives west on I-60 at a speed of 76 miles per hour. Chris drives east on I-60 at a speed of 82 miles per hour. How many hours will it take them to be 632 miles apart?

Exercise 13.6.5E.13

Two busses leave Billings at the same time. The Seattle bus heads west on I-90 at a speed of 73 miles per hour while the Chicago bus heads east at a speed of 79 miles an hour. How many hours will it take them to be 532 miles apart?

Answer

3.5 hours

Exercise 13.6.5E.14

Two boats leave the same dock in Cairo at the same time. One heads north on the Mississippi River while the other heads south. The northbound boat travels four miles per hour. The southbound boat goes eight miles per hour. How long will it take them to be 54 miles apart?

Exercise 13.6.5E.15

Lorena walks the path around the park in 30 minutes. If she jogs, it takes her 20 minutes. Her jogging speed is 1.5 miles per hour faster than her walking speed. Find Lorena's walking speed and jogging speed.

Answer

walking 3 mph, jogging 4.5 mph

Exercise 13.6.5E.16

Julian rides his bike uphill for 45 minutes, then turns around and rides back downhill. It takes him 15 minutes to get back to where he started. His uphill speed is 3.2 miles per hour slower than his downhill speed. Find Julian's uphill and downhill speed.

Exercise 13.6.5E.17

Cassius drives his boat upstream for 45 minutes. It takes him 30 minutes to return downstream. His speed going upstream is three miles per hour slower than his speed going downstream. Find his upstream and downstream speeds.

Answer

upstream 6 mph, downstream 9 mph

Exercise 13.6.5E.18

It takes Darline 20 minutes to drive to work in light traffic. To come home, when there is heavy traffic, it takes her 36 minutes. Her speed in light traffic is 24 miles per hour faster than her speed in heavy traffic. Find her speed in light traffic and in heavy traffic.

Exercise 13.6.5E.19

At 1:30 Marlon left his house to go to the beach, a distance of 7.6 miles. He rode his skateboard until 2:15, then walked the rest of the way. He arrived at the beach at 3:00. Marlon's speed on his skateboard is 2.5 times his walking speed. Find his speed when skateboarding and when walking.

Answer

skateboarding 8 mph, walking 3.2 mph

Exercise 13.6.5E.20

Aaron left at 9:15 to drive to his mountain cabin 108 miles away. He drove on the freeway until 10:45, and then he drove on the mountain road. He arrived at 11:05. His speed on the freeway was three times his speed on the mountain road. Find Aaron's speed on the freeway and on the mountain road.

Exercise 13.6.5E.21

Marisol left Los Angeles at 2:30 to drive to Santa Barbara, a distance of 95 miles. The traffic was heavy until 3:20. She drove the rest of the way in very light traffic and arrived at 4:20. Her speed in heavy traffic was 40 miles per hour slower than her speed in light traffic. Find her speed in heavy traffic and in light traffic.

Answer

heavy traffic 30 mph, light traffic 70 mph

Exercise 13.6.5E.22

Lizette is training for a marathon. At 7:00 she left her house and ran until 8:15, then she walked until 11:15. She covered a total distance of 19 miles. Her running speed was five miles per hour faster than her walking speed. Find her running and walking speeds.

Everyday Math

Exercise 13.6.5E.23

John left his house in Irvine at 8:35 am to drive to a meeting in Los Angeles, 45 miles away. He arrived at the meeting at 9:50. At 3:30 pm, he left the meeting and drove home. He arrived home at 5:18.

1. What was his average speed on the drive from Irvine to Los Angeles?
2. What was his average speed on the drive from Los Angeles to Irvine?
3. What was the total time he spent driving to and from this meeting?
4. John drove a total of 90 miles roundtrip. Find his average speed. (Round to the nearest tenth.)

Answer

1. 36 mph
2. 25 mph
3. 3.05 hours
4. 29.5 mph

Exercise 13.6.5E.24

Sarah wants to arrive at her friend's wedding at 3:00. The distance from Sarah's house to the wedding is 95 miles. Based on usual traffic patterns, Sarah predicts she can drive the first 15 miles at 60 miles per hour, the next 10 miles at 30 miles per hour, and the remainder of the drive at 70 miles per hour.

1. How long will it take Sarah to drive the first 15 miles?
2. How long will it take Sarah to drive the next 10 miles?
3. How long will it take Sarah to drive the rest of the trip?
4. What time should Sarah leave her house?

Writing Exercises

Exercise 13.6.5E.25

When solving a uniform motion problem, how does drawing a diagram of the situation help you?

Answer

Answers will vary.

Exercise 13.6.5E. 26

When solving a uniform motion problem, how does creating a table help you?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve uniform motion applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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13.6.6: Solve Applications with Linear Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve applications with linear inequalities

Note

Before you get started, take this readiness quiz.

1. Write as an inequality: x is at least 30.
   If you missed this problem, review Exercise 2.7.34.
2. Solve $8-3y<41$.
   If you missed this problem, review Exercise 2.7.22.

Solve Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. In fact, inequality applications are so common that we often do not even realize we are doing algebra. For example, how many gallons of gas can be put in the car for $20? Is the rent on an apartment affordable? Is there enough time before class to go get lunch, eat it, and return? How much money should each family member's holiday gift cost without going over budget?

The method we will use to solve applications with linear inequalities is very much like the one we used when we solved applications with equations. We will read the problem and make sure all the words are understood. Next, we will identify what we are looking for and assign a variable to represent it. We will restate the problem in one sentence to make it easy to translate into an inequality. Then, we will solve the inequality.

Example 13.6.6.1

Emma got a new job and will have to move. Her monthly income will be $5,265. To qualify to rent an apartment, Emma's monthly income must be at least three times as much as the rent. What is the highest rent Emma will qualify for?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the highest rent Emma will qualify for
Step 3. Name what we are looking for.	
Choose a variable to represent that quantity.	Let r = rent
Step 4. Translate into an inequality.	
First write a sentence that gives the information to find it.	Emma's monthly income must be at least three times the rent.
Step 5. Solve the inequality.	$5265 \geq 3r$
Remember, $a>x$ has the same meaning as $x<a$	$1755 \geq r$ $r \leq 1755$
Step 6. Check the answer in the problem and make sure it makes sense.	
A maximum rent of $1,755 seems reasonable for an income of $5,265.	
Step 7. Answer the question with a complete sentence.	The maximum rent is $1,755.

Try It 13.6.6.2

Alan is loading a pallet with boxes that each weighs 45 pounds. The pallet can safely support no more than 900 pounds. How many boxes can he safely load onto the pallet?

Answer

There can be no more than 20 boxes.

Try It 13.6.6.3

The elevator in Yehire's apartment building has a sign that says the maximum weight is 2,100 pounds. If the average weight of one person is 150 pounds, how many people can safely ride the elevator?

Answer

A maximum of 14 people can safely ride in the elevator.

Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round the algebraic solution to a whole number. The context of the application will determine whether we round up or down. To check applications like this, we will round our answer to a number that is easy to compute with and make sure that number makes the inequality true.

Example 13.6.6.4

Dawn won a mini-grant of $4,000 to buy tablet computers for her classroom. The tablets she would like to buy cost $254.12 each, including tax and delivery. What is the maximum number of tablets Dawn can buy?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the maximum number of tablets Dawn can buy
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let n = the number of tablets.
Step 4. Translate. write a sentence that gives the information to find it.	$254.12 times the number of tablets is no more than $4000.
Translate into an inequality.	$254.12n \leq 4000$
Step 5. Solve the inequality.	$n \leq 15.74$
But n must be a whole number of tablets, so round to 15.	$n \leq 15$
Step 6. Check the answer in the problem and make sure it makes sense. Rounding down the price to $250, 15 tablets would cost $3750, while 16 tablets would be $4000. So a maximum of 15 tablets at $254.12 seems reasonable.	
Step 7. Answer the answer in the problem complete sentence.	the question with a Dawn can buy a maximum of 15 tablets.

Try It 13.6.6.5

Angie has $20 to spend on juice boxes for her son's preschool picnic. Each pack of juice boxes costs $2.63. What is the maximum number of packs she can buy?

Answer

seven packs

Try It 13.6.6.6

Daniel wants to surprise his girlfriend with a birthday party at her favorite restaurant. It will cost $42.75 per person for dinner, including tip and tax. His budget for the party is $500. What is the maximum number of people Daniel can have at the party?

Answer

11 people

Example 13.6.6.7

Pete works at a computer store. His weekly pay will be either a fixed amount, $925, or $500 plus 12% of his total sales. How much should his total sales be for his variable pay option to exceed the fixed amount of $925?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the total sales needed for his variable pay option to exceed the fixed amount of $925
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let s = the total sales.
Step 4. Translate. write a sentence that gives the information to find it. Translate into an inequality. Remember to convert the percent to a decimal.	$500 plus 12% of total sales is more than $925. $500 + 0.12s > 925$
Step 5. Solve the inequality.	$0.12s > 425$ $s > 3541.\overline{66}$
Step 6. Check the answer in the problem and make sure it makes sense. Rounding down the price to $250, 15 tablets would cost $3750, while If we round the total sales up to $4000, we see that $500+0.12(4000) = 980$, which is more than $925.	
Step 7. Answer the the question with a complete sentence.	The total sales must be more than $3541.67

Try It 13.6.6.8

Tiffany just graduated from college and her new job will pay her $20000 per year plus 2% of all sales. She wants to earn at least $100000 per year. For what total sales will she be able to achieve her goal?

Answer

at least $4000000

Try It 13.6.6.9

Christian has been offered a new job that pays $24000 a year plus 3% of sales. For what total sales would this new job pay more than his current job which pays $60000?

Answer

at least $1200000

Example 13.6.6.10

Sergio and Lizeth have a very tight vacation budget. They plan to rent a car from a company that charges $75 a week plus $0.25 a mile. How many miles can they travel and still keep within their $200 budget?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the number of miles Sergio and Lizeth can travel
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let m = the number of miles.
Step 4. Translate. write a sentence that gives the information to find it. Translate into an inequality.	$75 plus 0.25 times the number of miles is less than or equal to $200. $75 + 25m \leq 200$
Step 5. Solve the inequality.	$0.25m \leq 125$ $m \leq 500$ miles
Step 6. Check the answer in the problem and make sure it makes sense. Yes, $75 + 0.25(500) = 200$.	
Step 7. Answer the the question with a complete sentence.	Sergio and Lizeth can travel 500 miles and still stay on budget.

Try It 13.6.6.11

Taleisha's phone plan costs her $28.80 a month plus $0.20 per text message. How many text messages can she use and keep her monthly phone bill no more than $50?

Answer

no more than 106 text messages

Try It 13.6.6.12

Rameen's heating bill is $5.42 per month plus $1.08 per therm. How many therms can Rameen use if he wants his heating bill to be a maximum of $87.50?

Answer

no more than 76 therms

A common goal of most businesses is to make a profit. *Profit* is the money that remains when the expenses have been subtracted from the money earned. In the next example, we will find the number of jobs a small businessman needs to do every month in order to make a certain amount of **profit**.

Example 13.6.6.13

Elliot has a landscape maintenance business. His monthly expenses are $1,100. If he charges $60 per job, how many jobs must he do to earn a profit of at least $4,000 a month?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the number of jobs Elliot needs
Step 3. Name what we are looking for.	
Choose a variable to represent it	Let j = the number of jobs.
Step 4. Translate. write a sentence that	$60 times the number of jobs minus $1,100 is at least $4,000.
gives the information to find it.	less than or equal to $200.
Translate into an inequality.	$60j - 1100 \geq 4000$
Step 5. Solve the inequality.	$60j \geq 5100$
	$j \geq 85$ jobs
Step 6. Check the answer in the problem and make sure it makes sense.	If Elliot did 90 jobs, his profit would be $60(90) - 1{,}100$, or $4,300. This is more than $4,000.
Step 7. Answer the the question with a complete sentence.	Elliot must work at least 85 jobs.

Try It 13.6.6.14

Caleb has a pet sitting business. He charges $32 per hour. His monthly expenses are $2272. How many hours must he work in order to earn a profit of at least $800 per month?

Answer

at least 96 hours

Try It 13.6.6.15

Felicity has a calligraphy business. She charges $2.50 per wedding invitation. Her monthly expenses are $650. How many invitations must she write to earn a profit of at least $2800 per month?

Answer

at least 1380 invitations

Sometimes life gets complicated! There are many situations in which several quantities contribute to the total expense. We must make sure to account for all the individual expenses when we solve problems like this.

Example 13.6.6.16

Brenda's best friend is having a destination wedding and the event will last 3 days. Brenda has $500 in savings and can earn $15 an hour babysitting. She expects to pay $350 airfare, $375 for food and entertainment and $60 a night for her share of a hotel room. How many hours must she babysit to have enough money to pay for the trip?

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the number of hours Brenda must babysit
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let h = the number of hours.
Step 4. Translate. write a sentence that gives the information to find it.	The expenses must be less than or equal to the income. The cost of airfare plus the cost of food and entertainment and the hotel bill must be less than or equal to the savings plus the amount earned babysitting.
Translate into an inequality.	$\$350 + \$375 + \$60(3) \leq \$500 + \$15h$
Step 5. Solve the inequality.	$905 \leq 500 + 15h$ $405 \leq 15h$ $27 \leq h$ $h \geq 27$
Step 6. Check the answer in the problem and make sure it makes sense.	
We substitute 27 into the inequality.	
$905 \leq 500 + 15h$	
$905 \leq 500 + 15(27)$	
$905 \leq 905$	
Step 7. Answer the the question with a complete sentence.	Brenda must babysit at least 27 hours.

? Try It 13.6.6.17

Malik is planning a 6-day summer vacation trip. He has $840 in savings, and he earns $45 per hour for tutoring. The trip will cost him $525 for airfare, $780 for food and sightseeing, and $95 per night for the hotel. How many hours must he tutor to have enough money to pay for the trip?

Answer

at least 23 hours

? Try It 13.6.6.18

Josue wants to go on a 10-day road trip next spring. It will cost him $180 for gas, $450 for food, and $49 per night for a motel. He has $520 in savings and can earn $30 per driveway shoveling snow. How many driveways must he shovel to have enough money to pay for the trip?

Answer

at least 20 driveways

Key Concepts

- **Solving inequalities**
  1. **Read** the problem.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate.** Write a sentence that gives the information to find it. Translate into an inequality.

5. **Solve** the inequality.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

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13.6.6E: Exercises

Practice Makes Perfect

Solve Applications with Linear Inequalities

In the following exercises, solve.

Exercise 13.6.6E.1

Mona is planning her son's birthday party and has a budget of $285. The Fun Zone charges $19 per child. How many children can she have at the party and stay within her budget?

Answer

15 children

Exercise 13.6.6E.2

Carlos is looking at apartments with three of his friends. They want the monthly rent to be no more than $2360. If the roommates split the rent evenly among the four of them, what is the maximum rent each will pay?

Exercise 13.6.6E.3

A water taxi has a maximum load of 1,800 pounds. If the average weight of one person is 150 pounds, how many people can safely ride in the water taxi?

Answer

12 people

Exercise 13.6.6E.4

Marcela is registering for her college classes, which cost $105 per unit. How many units can she take to have a maximum cost of $1,365?

Exercise 13.6.6E.5

Arleen got a $20 gift card for the coffee shop. Her favorite iced drink costs $3.79. What is the maximum number of drinks she can buy with the gift card?

Answer

five drinks

Exercise 13.6.6E.6

Teegan likes to play golf. He has budgeted $60 next month for the driving range. It costs him $10.55 for a bucket of balls each time he goes. What is the maximum number of times he can go to the driving range next month?

Exercise 13.6.6E.7

Joni sells kitchen aprons online for $32.50 each. How many aprons must she sell next month if she wants to earn at least $1,000?

Answer

31 aprons

Exercise 13.6.6E.8

Ryan charges his neighbors $17.50 to wash their car. How many cars must he wash next summer if his goal is to earn at least $1,500?

Exercise 13.6.6E.9

Keshad gets paid $2,400 per month plus 6% of his sales. His brother earns $3,300 per month. For what amount of total sales will Keshad's monthly pay be higher than his brother's monthly pay?

Answer

$15,000

Exercise 13.6.6E.10

Kimuyen needs to earn $4,150 per month in order to pay all her expenses. Her job pays her $3,475 per month plus 4% of her total sales. What is the minimum Kimuyen's total sales must be in order for her to pay all her expenses?

Exercise 13.6.6E.11

Andre has been offered an entry-level job. The company offered him $48,000 per year plus 3.5% of his total sales. Andre knows that the average pay for this job is $62,000. What would Andre's total sales need to be for his pay to be at least as high as the average pay for this job?

Answer

$400,000

Exercise 13.6.6E.12

Nataly is considering two job offers. The first job would pay her $83,000 per year. The second would pay her $66,500 plus 15% of her total sales. What would her total sales need to be for her salary on the second offer be higher than the first?

Exercise 13.6.6E.13

Jake's water bill is $24.80 per month plus $2.20 per ccf (hundred cubic feet) of water. What is the maximum number of ccf Jake can use if he wants his bill to be no more than $60?

Answer

16 ccf

Exercise 13.6.6E.14

Kiyoshi's phone plan costs $17.50 per month plus $0.15 per text message. What is the maximum number of text messages Kiyoshi can use so the phone bill is no more than $56.50?

Exercise 13.6.6E.15

Marlon's TV plan costs $49.99 per month plus $5.49 per first-run movie. How many first-run movies can he watch if he wants to keep his monthly bill to be a maximum of $100?

Answer

nine movies

Exercise 13.6.6E.16

Kellen wants to rent a banquet room in a restaurant for her cousin's baby shower. The restaurant charges $350 for the banquet room plus $32.50 per person for lunch. How many people can Kellen have at the shower if she wants the maximum cost to be $1,500?

Exercise 13.6.6E.17

Moshde runs a hairstyling business from her house. She charges $45 for a haircut and style. Her monthly expenses are $960. She wants to be able to put at least $1,200 per month into her savings account order to open her own salon. How many "cut & styles" must she do to save at least $1,200 per month?

Answer

48 cut & styles

Exercise 13.6.6E.18

Noe installs and configures software on home computers. He charges $125 per job. His monthly expenses are $1,600. How many jobs must he work in order to make a profit of at least $2,400?

Exercise 13.6.6E.19

Katherine is a personal chef. She charges $115 per four-person meal. Her monthly expenses are $3,150. How many four-person meals must she sell in order to make a profit of at least $1,900?

Answer

44 meals

Exercise 13.6.6E.20

Melissa makes necklaces and sells them online. She charges $88 per necklace. Her monthly expenses are $3745. How many necklaces must she sell if she wants to make a profit of at least $1,650?

Exercise 13.6.6E.21

Five student government officers want to go to the state convention. It will cost them $110 for registration, $375 for transportation and food, and $42 per person for the hotel. There is $450 budgeted for the convention in the student government savings account. They can earn the rest of the money they need by having a car wash. If they charge $5 per car, how many cars must they wash in order to have enough money to pay for the trip?

Answer

49 cars

Exercise 13.6.6E.22

Cesar is planning a 4-day trip to visit his friend at a college in another state. It will cost him $198 for airfare, $56 for local transportation, and $45 per day for food. He has $189 in savings and can earn $35 for each lawn he mows. How many lawns must he mow to have enough money to pay for the trip?

Exercise 13.6.6E.23

Alonzo works as a car detailer. He charges $175 per car. He is planning to move out of his parents' house and rent his first apartment. He will need to pay $120 for application fees, $950 for security deposit, and first and last months' rent at $1,140 per month. He has $1,810 in savings. How many cars must he detail to have enough money to rent the apartment?

Answer

9 cars

Exercise 13.6.6E.24

Eun-Kyung works as a tutor and earns $60 per hour. She has $792 in savings. She is planning an anniversary party for her parents. She would like to invite 40 guests. The party will cost her $1,520 for food and drinks and $150 for the photographer. She will also have a favor for each of the guests, and each favor will cost $7.50. How many hours must she tutor to have enough money for the party?

Everyday Math

Exercise 13.6.6E.25

Maximum Load on a Stage In 2014, a high school stage collapsed in Fullerton, California, when 250 students got on stage for the finale of a musical production. Two dozen students were injured. The stage could support a maximum of 12,750 pounds. If the average weight of a student is assumed to be 140 pounds, what is the maximum number of students who could safely be on the stage?

Answer

91 students

Exercise 13.6.6E.26

Maximum Weight on a Boat In 2004, a water taxi sank in Baltimore harbor and five people drowned. The water taxi had a maximum capacity of 3,500 pounds (25 people with average weight 140 pounds). The average weight of the 25 people on the water taxi when it sank was 168 pounds per person. What should the maximum number of people of this weight have been?

Exercise 13.6.6E.27

Wedding Budget Adele and Walter found the perfect venue for their wedding reception. The cost is $9,850 for up to 100 guests, plus $38 for each additional guest. How many guests can attend if Adele and Walter want the total cost to be no more than $12,500?

Answer

169 guests

Exercise 13.6.6E.28

Shower Budget Penny is planning a baby shower for her daughter-in-law. The restaurant charges $950 for up to 25 guests, plus $31.95 for each additional guest. How many guests can attend if Penny wants the total cost to be no more than $1,500?

Writing Exercises

Exercise 13.6.6E. 29

Find your last month's phone bill and the hourly salary you are paid at your job. (If you do not have a job, use the hourly salary you would realistically be paid if you had a job.) Calculate the number of hours of work it would take you to earn at least enough money to pay your phone bill by writing an appropriate inequality and then solving it.

Answer

Answers will vary.

Exercise 13.6.6E. 30

Find out how many units you have left, after this term, to achieve your college goal and estimate the number of units you can take each term in college. Calculate the number of terms it will take you to achieve your college goal by writing an appropriate inequality and then solving it.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve applications with linear inequalities.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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13.6.7: Chapter 3 Review Exercises

Chapter 3 Review Exercises

3.1 Using a Problem Solving Strategy

Approach Word Problems with a Positive Attitude

In the following exercises, reflect on your approach to word problems.

Exercise 13.6.7.1

How has your attitude towards solving word problems changed as a result of working through this chapter? Explain.

Answer

answers will vary

Exercise 13.6.7.2

Did the problem-solving strategy help you solve word problems in this chapter? Explain.

Use a Problem-Solving Strategy for Word Problems

In the following exercises, solve using the problem-solving strategy for word problems. Remember to write a complete sentence to answer each question.

Exercise 13.6.7.3

Three-fourths of the people at a concert are children. If there are 87 children, what is the total number of people at the concert?

Answer

116

Exercise 13.6.7.4

There are nine saxophone players in the band. The number of saxophone players is one less than twice the number of tuba players. Find the number of tuba players.

Solve Number Problems

In the following exercises, solve each number word problem.

Exercise 13.6.7.5

The sum of a number and three is forty-one. Find the number.

Answer

38

Exercise 13.6.7.6

Twice the difference of a number and ten is fifty-four. Find the number.

Exercise 13.6.7.7

One number is nine less than another. Their sum is negative twenty-seven. Find the numbers.

Answer

−18,−9

Exercise 13.6.7.8

One number is eleven more than another. If their sum is increased by seventeen, the result is 90. Find the numbers.

Exercise 13.6.7.9

One number is two more than four times another. Their sum is −13. Find the numbers.

Answer

−3,−10

Exercise 13.6.7.10

The sum of two consecutive integers is −135. Find the numbers.

Exercise 13.6.7.11

Find three consecutive integers whose sum is −141.

Answer

−48,−47,−46

Exercise 13.6.7.12

Find three consecutive even integers whose sum is 234.

Exercise 13.6.7.13

Find three consecutive odd integers whose sum is 51.

Answer

15, 17, 19

Exercise 13.6.7.14

Koji has $5,502 in his savings account. This is $30 less than six times the amount in his checking account. How much money does Koji have in his checking account?

3.2 Solve Percent Applications

Translate and Solve Basic Percent Equations

In the following exercises, translate and solve.

Exercise 13.6.7.15

What number is 67% of 250?

Answer

167.5

Exercise 13.6.7.16

300% of 82 is what number?

Exercise 13.6.7.17

12.5% of what number is 20?

Answer

160

Exercise 13.6.7.18

72 is 30% of what number?

Exercise 13.6.7.19

What percent of 125 is 150?

Answer

120%

Exercise 13.6.7.20

127.5 is what percent of 850?

Solve Percent Applications

In the following exercises, solve.

Exercise 13.6.7.21

The bill for Dino's lunch was $19.45. He wanted to leave 20% of the total bill as a tip. How much should the tip be?

Answer

$3.89

Exercise 13.6.7.22

Reza was very sick and lost 15% of his original weight. He lost 27 pounds. What was his original weight?

Exercise 13.6.7.23

Dolores bought a crib on sale for $350. The sale price was 40% of the original price. What was the original price of the crib?

Answer

$875

Exercise 13.6.7.24

Jaden earns $2,680 per month. He pays $938 a month for rent. What percent of his monthly pay goes to rent?

Find Percent Increase and Percent Decrease

In the following exercises, solve.

Exercise 13.6.7.25

Angel's got a raise in his annual salary from $55,400 to $56,785. Find the percent increase.

Answer

2.5%

Exercise 13.6.7.26

Rowena's monthly gasoline bill dropped from $83.75 last month to $56.95 this month. Find the percent decrease.

Solve Simple Interest Applications

In the following exercises, solve.

Exercise 13.6.7.27

Winston deposited $3,294 in a bank account with interest rate 2.6%. How much interest was earned in 5 years?

Answer

$428.22

Exercise 13.6.7.28

Moira borrowed $4,500 from her grandfather to pay for her first year of college. Three years later, she repaid the $4,500 plus $243 interest. What was the rate of interest?

Exercise 13.6.7.29

Jaime's refrigerator loan statement said he would pay $1,026 in interest for a 4-year loan at 13.5%. How much did Jaime borrow to buy the refrigerator?

Answer

$1,900

Exercise 13.6.7.30

In 12 years, a bond that paid 6.35% interest earned $7,620 interest. What was the principal of the bond?

Solve Applications with Discount or Mark-up

In the following exercises, find the sale price.

Exercise 13.6.7.31

The original price of a handbag was $84. Carole bought it on sale for $21 off.

Answer

$63

Exercise 13.6.7.32

Marian wants to buy a coffee table that costs $495. Next week the coffee table will be on sale for $149 off.

In the following exercises, find

1. the amount of discount and
2. the sale price.

Exercise 13.6.7.33

Emmett bought a pair of shoes on sale at 40% off from an original price of $138.

Answer

1. $55.20
2. $82.80

Exercise 13.6.7.34

Anastasia bought a dress on sale at 75% off from an original price of $280.

In the following exercises, find ⓐ the amount of discount and ⓑ the discount rate. (Round to the nearest tenth of a percent, if needed.)

Exercise 13.6.7.35

Zack bought a printer for his office that was on sale for $380. The original price of the printer was $450.

Answer

1. $70
2. 15.6%

Exercise 13.6.7.36

Lacey bought a pair of boots on sale for $95. The original price of the boots was $200.

In the following exercises, find

1. the amount of the mark-up and
2. the list price.

Exercise 13.6.7.37

Nga and Lauren bought a chest at a flea market for $50. They re-finished it and then added a 350% mark-up.

Answer

1. $175
2. $225

Exercise 13.6.7.38

Carly bought bottled water for $0.24 per bottle at the discount store. She added a 75% mark-up before selling them at the football game.

3.3 Solve Mixture Applications

Solve Coin Word Problems

In the following exercises, solve each coin word problem.

Exercise 13.6.7.39

Francie has $4.35 in dimes and quarters. The number of dimes is five more than the number of quarters. How many of each coin does she have?

Answer

16 dimes, 11 quarters

Exercise 13.6.7.40

Scott has $0.39 in pennies and nickels. The number of pennies is eight times the number of nickels. How many of each coin does he have?

Exercise 13.6.7.41

Paulette has $140 in $5 and $10 bills. The number of $10 bills is one less than twice the number of $5 bills. How many of each does she have?

Answer

six $5 bills, 11 $10 bills

Exercise 13.6.7.42

Lenny has $3.69 in pennies, dimes, and quarters. The number of pennies is three more than the number of dimes. The number of quarters is twice the number of dimes. How many of each coin does he have?

Solve Ticket and Stamp Word Problems

In the following exercises, solve each ticket or stamp word problem.

Exercise 13.6.7.43

A church luncheon made $842. Adult tickets cost $10 each and children's tickets cost $6 each. The number of children was 12 more than twice the number of adults. How many of each ticket were sold?

Answer

35 adults, 82 children

Exercise 13.6.7.44

Tickets for a basketball game cost $2 for students and $5 for adults. The number of students was three less than 10 times the number of adults. The total amount of money from ticket sales was $619. How many of each ticket were sold?

Exercise 13.6.7.45

125 tickets were sold for the jazz band concert for a total of $1,022. Student tickets cost $6 each and general admission tickets cost $10 each. How many of each kind of ticket were sold?

Answer

57 students, 68 adults

Exercise 13.6.7.46

One afternoon the water park sold 525 tickets for a total of $13,545. Child tickets cost $19 each and adult tickets cost $40 each. How many of each kind of ticket were sold?

Exercise 13.6.7.47

Ana spent $4.06 buying stamps. The number of $0.41 stamps she bought was five more than the number of $0.26 stamps. How many of each did she buy?

Answer

three $0.26 stamps, eight $0.41 stamps

Exercise 13.6.7.48

Yumi spent $34.15 buying stamps. The number of $0.56 stamps she bought was 10 less than four times the number of $0.41 stamps. How many of each did she buy?

Solve Mixture Word Problems

In the following exercises, solve each mixture word problem.

Exercise 13.6.7.49

Marquese is making 10 pounds of trail mix from raisins and nuts. Raisins cost $3.45 per pound and nuts cost $7.95 per pound. How many pounds of raisins and how many pounds of nuts should Marquese use for the trail mix to cost him $6.96 per pound?

Answer

2.2 lb. of raisins, 7.8 lb. of nuts

Exercise 13.6.7.50

Amber wants to put tiles on the backsplash of her kitchen counters. She will need 36 square feet of tile. She will use basic tiles that cost $8 per square foot and decorator tiles that cost $20 per square foot. How many square feet of each tile should she use so that the overall cost of the backsplash will be $10 per square foot?

Exercise 13.6.7.51

Shawn has $15,000 to invest. She will put some of it into a fund that pays 4.5% annual interest and the rest in a certificate of deposit that pays 1.8% annual interest. How much should she invest in each account if she wants to earn 4.05% annual interest on the total amount?

Answer

$12,500 at 4.5%, $2,500 at 1.8%

Exercise 13.6.7.52

Enrique borrowed $23,500 to buy a car. He pays his uncle 2% interest on the $4,500 he borrowed from him, and he pays the bank 11.5% interest on the rest. What average interest rate does he pay on the total $23,500? (Round your answer to the nearest tenth of a percent.)

3.4 Solve Geometry Applications: Triangles, Rectangles and the Pythagorean Theorem

Solve Applications Using Triangle Properties

In the following exercises, solve using triangle properties.

Exercise 13.6.7.53

The measures of two angles of a triangle are 22 and 85 degrees. Find the measure of the third angle.

Answer

73°

Exercise 13.6.7.54

The playground at a shopping mall is a triangle with perimeter 48 feet. The lengths of two sides are 19 feet and 14 feet. How long is the third side?

Exercise 13.6.7.55

A triangular road sign has base 30 inches and height 40 inches. What is its area?

Answer

600 square inches

Exercise 13.6.7.56

What is the height of a triangle with area 67.5 square meters and base 9 meters?

Exercise 13.6.7.57

One angle of a triangle is 30° more than the smallest angle. The largest angle is the sum of the other angles. Find the measures of all three angles.

Answer

30°,60°,90°

Exercise 13.6.7.58

One angle of a right triangle measures 58°. What is the measure of the other angles of the triangle?

Exercise 13.6.7.59

The measure of the smallest angle in a right triangle is 45° less than the measure of the next larger angle. Find the measures of all three angles.

Answer

22.5°,67.5°,90°

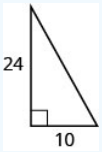
Exercise 13.6.7.60

The perimeter of a triangle is 97 feet. One side of the triangle is eleven feet more than the smallest side. The third side is six feet more than twice the smallest side. Find the lengths of all sides.

Use the Pythagorean Theorem

In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.

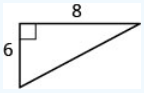
Exercise 13.6.7.61



Answer

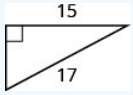
26

Exercise 13.6.7.62



In the following exercises, use the Pythagorean Theorem to find the length of the missing side. Round to the nearest tenth, if necessary.

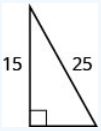
Exercise 13.6.7.63



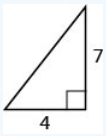
Answer

8

Exercise 13.6.7.64



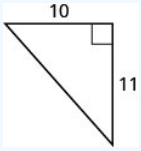
Exercise 13.6.7.65



Answer

8.1

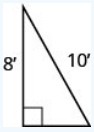
Exercise 13.6.7.66



In the following exercises, solve. Approximate to the nearest tenth, if necessary.

Exercise 13.6.7.67

Sergio needs to attach a wire to hold the antenna to the roof of his house, as shown in the figure. The antenna is 8 feet tall and Sergio has 10 feet of wire. How far from the base of the antenna can he attach the wire?

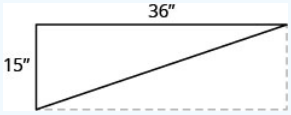


Answer

$6'$

Exercise 13.6.7.68

Seong is building shelving in his garage. The shelves are 36 inches wide and 15 inches tall. He wants to put a diagonal brace across the back to stabilize the shelves, as shown. How long should the brace be?



Solve Applications Using Rectangle Properties

In the following exercises, solve using rectangle properties.

Exercise 13.6.7.69

The length of a rectangle is 36 feet and the width is 19 feet. Find the

1. perimeter
2. area.

Answer

1. 110 ft.
2. 684 sq. ft.

Exercise 13.6.7.70

A sidewalk in front of Kathy's house is in the shape of a rectangle four feet wide by 45 feet long. Find the

1. perimeter
2. area.

Exercise 13.6.7.71

The area of a rectangle is 2356 square meters. The length is 38 meters. What is the width?

Answer

62 m

Exercise 13.6.7.72

The width of a rectangle is 45 centimeters. The area is 2,700 square centimeters. What is the length?

Exercise 13.6.7.73

The length of a rectangle is 12 cm more than the width. The perimeter is 74 cm. Find the length and the width.

Answer

24.5 cm, 12.5 cm

Exercise 13.6.7.74

The width of a rectangle is three more than twice the length. The perimeter is 96 inches. Find the length and the width.

3.5 Solve Uniform Motion Applications

Solve Uniform Motion Applications

In the following exercises, solve.

Exercise 13.6.7.75

When Gabe drives from Sacramento to Redding it takes him 2.2 hours. It takes Elsa 2 hours to drive the same distance. Elsa's speed is seven miles per hour faster than Gabe's speed. Find Gabe's speed and Elsa's speed.

Answer

Gabe 70 mph, Elsa 77 mph

Exercise 13.6.7.76

Louellen and Tracy met at a restaurant on the road between Chicago and Nashville. Louellen had left Chicago and drove 3.2 hours towards Nashville. Tracy had left Nashville and drove 4 hours towards Chicago, at a speed one mile per hour faster than Louellen's speed. The distance between Chicago and Nashville is 472 miles. Find Louellen's speed and Tracy's speed.

Exercise 13.6.7.77

Two busses leave Amarillo at the same time. The Albuquerque bus heads west on the I-40 at a speed of 72 miles per hour, and the Oklahoma City bus heads east on the I-40 at a speed of 78 miles per hour. How many hours will it take them to be 375 miles apart?

Answer

2.5 hours

Exercise 13.6.7.78

Kyle rowed his boat upstream for 50 minutes. It took him 30 minutes to row back downstream. His speed going upstream is two miles per hour slower than his speed going downstream. Find Kyle's upstream and downstream speeds.

Exercise 13.6.7.79

At 6:30, Devon left her house and rode her bike on the flat road until 7:30. Then she started riding uphill and rode until 8:00. She rode a total of 15 miles. Her speed on the flat road was three miles per hour faster than her speed going uphill. Find Devon's speed on the flat road and riding uphill.

Answer

flat road 11 mph, uphill 8 mph

Exercise 13.6.7.80

Anthony drove from New York City to Baltimore, a distance of 192 miles. He left at 3:45 and had heavy traffic until 5:30. Traffic was light for the rest of the drive, and he arrived at 7:30. His speed in light traffic was four miles per hour more than twice his speed in heavy traffic. Find Anthony's driving speed in heavy traffic and light traffic.

3.6 Solve Applications with Linear Inequalities

Solve Applications with Linear Inequalities

In the following exercises, solve.

Exercise 13.6.7.81

Julianne has a weekly food budget of $231 for her family. If she plans to budget the same amount for each of the seven days of the week, what is the maximum amount she can spend on food each day?

Answer

$33 per day

Exercise 13.6.7.82

Rogelio paints watercolors. He got a $100 gift card to the art supply store and wants to use it to buy $12'' \times 16''$ canvases. Each canvas costs $10.99. What is the maximum number of canvases he can buy with his gift card?

Exercise 13.6.7.83

Briana has been offered a sales job in another city. The offer was for $42,500 plus 8% of her total sales. In order to make it worth the move, Briana needs to have an annual salary of at least $66,500. What would her total sales need to be for her to move?

Answer

at least $300,000

Exercise 13.6.7.84

Renee's car costs her $195 per month plus $0.09 per mile. How many miles can Renee drive so that her monthly car expenses are no more than $250?

Exercise 13.6.7.85

Costa is an accountant. During tax season, he charges $125 to do a simple tax return. His expenses for buying software, renting an office, and advertising are $6,000. How many tax returns must he do if he wants to make a profit of at least $8,000?

Answer

at least 112 jobs

Exercise 13.6.7.86

Jenna is planning a 5-day resort vacation with three of her friends. It will cost her $279 for airfare, $300 for food and entertainment, and $65 per day for her share of the hotel. She has $550 saved towards her vacation and can earn $25 per hour as an assistant in her uncle's photography studio. How many hours must she work in order to have enough money for her vacation?

Practice Test

Exercise 13.6.7.87

Four-fifths of the people on a hike are children. If there are 12 children, what is the total number of people on the hike?

Answer

15

Exercise 13.6.7.88

One number is three more than twice another. Their sum is −63. Find the numbers.

Exercise 13.6.7.89

The sum of two consecutive odd integers is −96. Find the numbers.

Answer

−49,−47

Exercise 13.6.7.90

Marla's breakfast was 525 calories. This was 35% of her total calories for the day. How many calories did she have that day?

Exercise 13.6.7.91

Humberto's hourly pay increased from $16.25 to $17.55. Find the percent increase.

Answer

8%

Exercise 13.6.7.92

Melinda deposited $5,985 in a bank account with an interest rate of 1.9%. How much interest was earned in 2 years?

Exercise 13.6.7.93

Dotty bought a freezer on sale for $486.50. The original price of the freezer was $695. Find

1. the amount of discount and
2. the discount rate.

Answer

1. $208.50
2. 30%

Exercise 13.6.7.94

Bonita has $2.95 in dimes and quarters in her pocket. If she has five more dimes than quarters, how many of each coin does she have?

Exercise 13.6.7.95

At a concert, $1,600 in tickets were sold. Adult tickets were $9 each and children's tickets were $4 each. If the number of adult tickets was 30 less than twice the number of children's tickets, how many of each kind were sold?

Answer

140 adult, 85 children

Exercise 13.6.7.96

Kim is making eight gallons of punch from fruit juice and soda. The fruit juice costs $6.04 per gallon and the soda costs $4.28 per gallon. How much fruit juice and how much soda should she use so that the punch costs $5.71 per gallon?

Exercise 13.6.7.97

The measure of one angle of a triangle is twice the measure of the smallest angle. The measure of the third angle is 14 more than the measure of the smallest angle. Find the measures of all three angles.

Answer

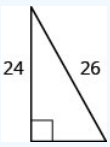
41.5°,55.5°,83°

Exercise 13.6.7.98

What is the height of a triangle with area 277.2 square inches and base 44 inches?

In the following exercises, use the Pythagorean Theorem to find the length of the missing side. Round to the nearest tenth, if necessary.

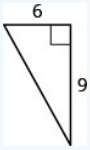
Exercise 13.6.7.99



Answer

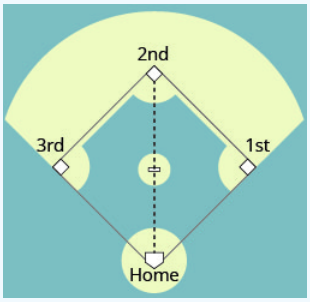
10

Exercise 13.6.7.100



Exercise 13.6.7.101

A baseball diamond is really a square with sides of 90 feet. How far is it from home plate to second base, as shown?



Answer

127.3 ft.

Exercise 13.6.7.102

The length of a rectangle is two feet more than five times the width. The perimeter is 40 feet. Find the dimensions of the rectangle.

Exercise 13.6.7.103

Two planes leave Dallas at the same time. One heads east at a speed of 428 miles per hour. The other plane heads west at a speed of 382 miles per hour. How many hours will it take them to be 2,025 miles apart?

Answer

2.5 hours

Exercise 13.6.7.104

Leon drove from his house in Cincinnati to his sister's house in Cleveland, a distance of 252 miles. It took him 412412 hours. For the first half hour he had heavy traffic, and the rest of the time his speed was five miles per hour less than twice his speed in heavy traffic. What was his speed in heavy traffic?

Exercise 13.6.7.105

Chloe has a budget of $800 for costumes for the 18 members of her musical theater group. What is the maximum she can spend for each costume?

Answer

at most $44.44 per costume

Exercise 13.6.7.106

Frank found a rental car deal online for $49 per week plus $0.24 per mile. How many miles could he drive if he wants the total cost for one week to be no more than $150?

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SECTION OVERVIEW

13.7: Foundations

In this chapter, you will review the language of algebra and take your first steps toward working with algebraic concepts.

13.7.1: Prelude to the Foundations of Algebra

13.7.2: Use the Language of Algebra

13.7.2E: Exercises

13.7.3: Integers

13.7.3E: Exercises

13.7.4: Fractions

13.7.4E: Exercises

13.7.5: Decimals

13.7.5E: Exercises

13.7.6: Properties of Real Numbers

13.7.6E: Exercises

13.7.7: Chapter 1 Review Exercises

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13.7.1: Prelude to the Foundations of Algebra

For years, doctors and engineers have worked to make artificial limbs, such as this hand for people who need them. This particular product is different, however, because it was developed using a 3D printer. As a result, it can be printed much like you print words on a sheet of paper. This makes producing the limb less expensive and faster than conventional methods.

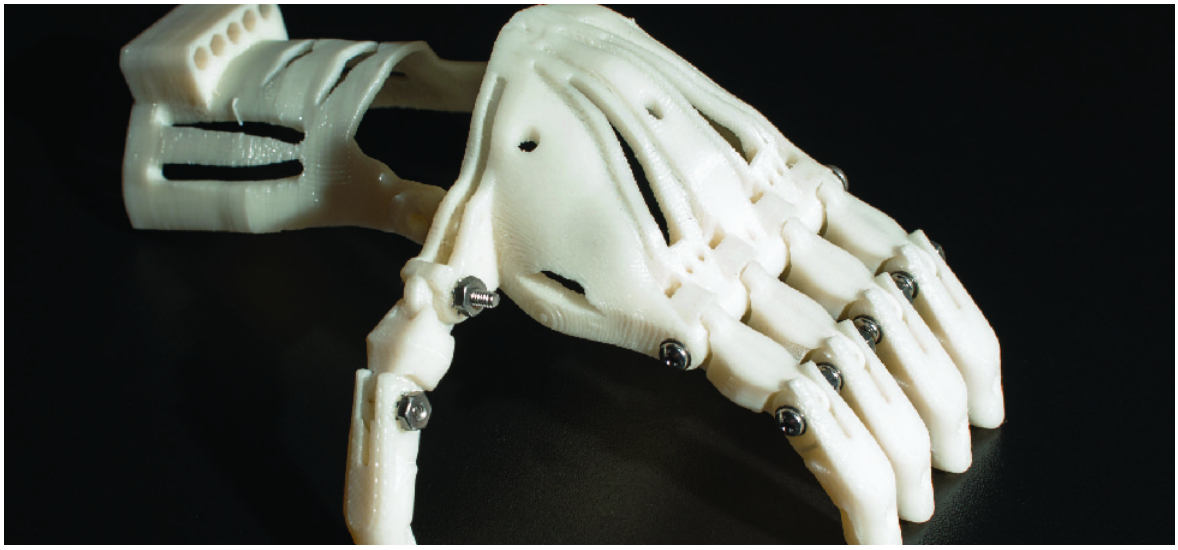

Figure 13.7.1.1: This hand may change someone's life. Amazingly, it was created using a special kind of printer known as a 3D printer. (credit: U.S. Food and Drug Administration/Wikimedia Commons)

Biomedical engineers are working to develop organs that may one day save lives. Scientists at NASA are designing ways to use 3D printers to build on the moon or Mars. Already, animals are benefiting from 3D-printed parts, including a tortoise shell and a dog leg. Builders have even constructed entire buildings using a 3D printer. The technology and use of 3D printers depends on the ability to understand the language of algebra. Engineers must be able to translate observations and needs in the natural world to complex mathematical commands that can provide directions to a printer. In this chapter, you will review the language of algebra and take your first steps toward working with algebraic concepts.

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13.7.2: Use the Language of Algebra

Learning Objectives

By the end of this section, you will be able to:

- Find factors, prime factorizations, and least common multiples
- Use variables and algebraic symbols
- Simplify expressions using the order of operations
- Evaluate an expression
- Identify and combine like terms
- Translate an English phrase to an algebraic expression

This chapter is intended to be a brief review of concepts that will be needed in an Intermediate Algebra course. A more thorough introduction to the topics covered in this chapter can be found in the *Elementary Algebra* chapter, Foundations.

Find Factors, Prime Factorizations, and Least Common Multiples

The numbers 2, 4, 6, 8, 10, 12 are called multiples of 2. A **multiple** of 2 can be written as the product of a counting number and 2.

Multiples of 2:

2,	4,	6,	8,	10,	12, ...
$2 \cdot 1$	$2 \cdot 2$	$2 \cdot 3$	$2 \cdot 4$	$2 \cdot 5$	$2 \cdot 6$

Similarly, a multiple of 3 would be the product of a counting number and 3.

Multiples of 3:

3,	6,	9,	12,	15,	18, ...
$3 \cdot 1$	$3 \cdot 2$	$3 \cdot 3$	$3 \cdot 4$	$3 \cdot 5$	$3 \cdot 6$

We could find the multiples of any number by continuing this process.

Counting Number	1	2	3	4	5	6	7	8	9	10	11	12
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108

MULTIPLE OF A NUMBER

A number is a **multiple** of n if it is the product of a counting number and n.

Another way to say that 15 is a multiple of 3 is to say that 15 is **divisible** by 3. That means that when we divide 3 into 15, we get a counting number. In fact, $15 \div 3$ is 5, so 15 is $5 \cdot 3$.

DIVISIBLE BY A NUMBER

If a number m is a multiple of n, then m is **divisible** by n.

If we were to look for patterns in the multiples of the numbers 2 through 9, we would discover the following divisibility tests:

DIVISIBILITY TESTS

A number is divisible by:

- 2 if the last digit is 0, 2, 4, 6, or 8.
- 3 if the sum of the digits is divisible by 3.
- 5 if the last digit is 5 or 0.
- 6 if it is divisible by both 2 and 3.
- 10 if it ends with 0.

Example 13.7.2.1

Is 5,625 divisible by

a. 2?
b. 3?
c. 5 or 10?
d. 6?

Answer

a.

Is 5,625 divisible by 2?

Does it end in 0, 2, 4, 6 or 8?	No. 5,625 is not divisible by 2.

b.

5,625 divisible by 3?

What is the sum of the digits?	$5+6+2+5=18$
Is the sum divisible by 3?	Yes. 5,625 is divisible by 3.

c.

Is 5,625 divisible by 5 or 10?

What is the last digit? It is 5.	5,625 is divisible by 5 but not by 10.

d.

Is 5,625 divisible by 6?

Is it divisible by both 2 and 3?	No, 5,625 is not divisible by 2, so 5,625 is not divisible by 6.

Example 13.7.2.2

Is 4,962 divisible by a. 2? b. 3? c. 5? d. 6? e. 10?

Answer

a. yes b. yes c. no d. yes e. no

Example 13.7.2.3

Is 3,765 divisible by a. 2? b. 3? c. 5? d. 6? e. 10?

Answer

a. no b. yes c. yes d. no e. no

In mathematics, there are often several ways to talk about the same ideas. So far, we've seen that if m is a multiple of n, we can say that m is divisible by n. For example, since 72 is a multiple of 8, we say 72 is divisible by 8. Since 72 is a multiple of 9, we say 72 is divisible by 9. We can express this still another way.

Since $8 \cdot 9 = 72$, we say that 8 and 9 are factors of 72. When we write $72 = 8 \cdot 9$, we say we have factored 72.

$$\underbrace{8 \cdot 9}_{\text{factors}} = \underbrace{72}_{\text{product}}$$

Other ways to factor 72 are $1 \cdot 72$, $2 \cdot 36$, $3 \cdot 24$, $4 \cdot 18$, and $6 \cdot 12$. The number 72 has many factors: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, and 72.

FACTORS

If a and b are counting numbers, and $a \cdot b = m$, then a and b are **factors** of m.

Some numbers, such as 72, have many factors. Other numbers have only two factors. A **prime number** is a counting number greater than 1 whose only factors are 1 and itself.

PRIME NUMBER AND COMPOSITE NUMBER

A **prime number** is a counting number greater than 1 whose only factors are 1 and the number itself.

A **composite number** is a counting number that is not prime. A composite number has factors other than 1 and the number itself.

The counting numbers from 2 to 20 are listed in the table with their factors. Make sure to agree with the "prime" or "composite" label for each!

Number	Factors	Prime or Composite?
2	1,2	Prime
3	1,3	Prime
4	1,2,4	Composite
5	1,5	Prime
6	1,2,3,6	Composite
7	1,7	Prime
8	1,2,4,8	Composite
9	1,3,9	Composite
10	1,2,5,10	Composite
11	1,11	Prime

Number	Factors	Prime or Composite?
12	1,2,3,4,6,12	Composite
13	1,13	Prime
14	1,2,7,14	Composite
15	1,3,5,15	Composite
16	1,2,4,8,16	Composite
17	1,17	Prime
18	1,2,3,6,9,18	Composite
19	1,19	Prime
20	1,2,4,5,10,20	Composite

The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. Notice that the only even prime number is 2.

A composite number can be written as a unique product of primes. This is called the **prime factorization** of the number. Finding the prime factorization of a composite number will be useful in many topics in this course.

PRIME FACTORIZATION

The **prime factorization** of a number is the product of prime numbers that equals the number.

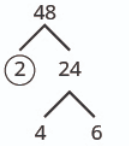
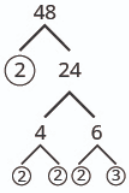
To find the prime factorization of a composite number, find any two factors of the number and use them to create two branches. If a factor is prime, that branch is complete. Circle that prime. Otherwise it is easy to lose track of the prime numbers.

If the factor is not prime, find two factors of the number and continue the process. Once all the branches have circled primes at the end, the factorization is complete. The composite number can now be written as a product of prime numbers.

Example 13.7.2.4: How to Find the Prime Factorization of a Composite Number

Factor 48.

Answer

Step 1. Find two factors whose product is the given number. Use these numbers to create two branches.	$48 = 2 \cdot 24$	
Step 2. If a factor is prime, that branch is complete. Circle the prime.	2 is prime. Circle the prime.	
Step 3. If a factor is not prime, write it as the product of two factors and continue the process.	24 is not prime. Break it into 2 more factors. 4 and 6 are not prime. Break them each into two factors. 2 and 3 are prime, so circle them.	 

We say $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ is the prime factorization of 48. We generally write the primes in ascending order. Be sure to multiply the factors to verify your answer.$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ is the prime factorization of 48. We generally write the primes in ascending order. Be sure to multiply the factors to verify your answer.

If we first factored 48 in a different way, for example as $6 \cdot 8$, the result would still be the same. Finish the prime factorization and verify this for yourself.

Exercise 13.7.2.4

Find the prime factorization of 80.

Answer

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

Example 13.7.2.6

Find the prime factorization of 60.

Answer

$2 \cdot 2 \cdot 3 \cdot 5$

FIND THE PRIME FACTORIZATION OF A COMPOSITE NUMBER

1. Find two factors whose product is the given number, and use these numbers to create two branches.
2. If a factor is prime, that branch is complete. Circle the prime, like a leaf on the tree.
3. If a factor is not prime, write it as the product of two factors and continue the process.
4. Write the composite number as the product of all the circled primes.

One of the reasons we look at primes is to use these techniques to find the **least common multiple** of two numbers. This will be useful when we add and subtract fractions with different denominators.

LEAST COMMON MULTIPLE

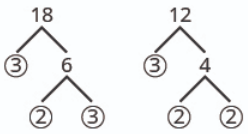
The **least common multiple (LCM)** of two numbers is the smallest number that is a multiple of both numbers.

To find the least common multiple of two numbers we will use the Prime Factors Method. Let's find the LCM of 12 and 18 using their prime factors.

Example 13.7.2.7: How to Find the Least Common Multiple Using the Prime Factors Method

Find the least common multiple (LCM) of 12 and 18 using the prime factors method.

Answer

Step 1. Write each number as a product of primes.		
Step 2. List the primes of each number. Match primes vertically when possible.	List the primes of 12. List the primes of 18. Line up with the primes of 12 when possible. If not create a new column.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$
Step 3. Bring down the number from each column.		$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 3 \cdot 3$
Step 4. Multiply the factors.		LCM = 36

Notice that the prime factors of 12 $(2 \cdot 2 \cdot 3)$ and the prime factors of 18 $(2 \cdot 3 \cdot 3)$ are included in the LCM $(2 \cdot 2 \cdot 3 \cdot 3)$. So 36 is the least common multiple of 12 and 18.

By matching up the common primes, each common prime factor is used only once. This way you are sure that 36 is the *least* common multiple.

Find the LCM of 9 and 12 using the Prime Factors Method.

Answer

36

Example 13.7.2.9

Find the LCM of 18 and 24 using the Prime Factors Method.

Answer

72

FIND THE LEAST COMMON MULTIPLE USING THE PRIME FACTORS METHOD

1. Write each number as a product of primes.
2. List the primes of each number. Match primes vertically when possible.
3. Bring down the columns.
4. Multiply the factors.

Use Variables and Algebraic Symbols

In algebra, we use a letter of the alphabet to represent a number whose value may change. We call this a **variable** and letters commonly used for variables are x, y, a, b, and c.

VARIABLE

A **variable** is a letter that represents a number whose value may change.

A number whose value always remains the same is called a **constant**.

CONSTANT

A **constant** is a number whose value always stays the same.

To write algebraically, we need some operation symbols as well as numbers and variables. There are several types of symbols we will be using. There are four basic arithmetic operations: addition, subtraction, multiplication, and division. We'll list the symbols used to indicate these operations below.

OPERATION SYMBOLS

Operation	Notation	Say:	The result is…
Addition	$a+b$	a plus b	the sum of a and b
Subtraction	$a-b$	a minus b	the difference of a and b
Multiplication	$a \cdot b$, ab, $(a)(b)$, $(a)b$, $a(b)$	a times b	the product of a and b
Division	$a \div b$, a/b, $\frac{a}{b}$, $b\overline{)a}$	a divided by b	the quotient of a and b; a is called the dividend, and b is called the divisor

When two quantities have the same value, we say they are equal and connect them with an equal sign.

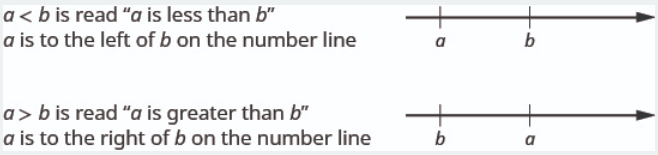
EQUALITY SYMBOL

$a = b$ is read "a is equal to b."

The symbol "$=$" is called the equal sign.

On the number line, the numbers get larger as they go from left to right. The number line can be used to explain the symbols "<" and ">".

INEQUALITY



The expressions $a < b$ or $a > b$ can be read from left to right or right to left, though in English we usually read from left to right. In general,

$$a < b \text{ is equivalent to } b > a. \text{ For example, } 7 < 11 \text{ is equivalent to } 11 > 7. \tag{13.7.2.1}$$

$$a > b \text{ is equivalent to } b < a. \text{ For example, } 17 > 4 \text{ is equivalent to } 4 < 17. \tag{13.7.2.2}$$

INEQUALITY SYMBOLS

Inequality Symbols	Words
$a \neq b$	a is *not equal to* b.
$a < b$	a is *less than* b.
$a \leq b$	a is *less than or equal to* b.
$a > b$	a is *greater than* b.
$a \geq b$	a is *greater than or equal to* b.

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in English. They help identify an expression, which can be made up of number, a variable, or a combination of numbers and variables using operation symbols. We will introduce three types of grouping symbols now.

GROUPING SYMBOLS

$$\begin{array}{ll} \text{Parentheses} & () \\ \text{Brackets} & [] \\ \text{Braces} & \{\} \end{array} \tag{13.7.2.3}$$

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

$$8(14-8) \qquad 21-3[2+4(9-8)] \qquad 24 \div \{13-2[1(6-5)+4]\} \tag{13.7.2.4}$$

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. A sentence has a subject and a verb. In algebra, we have *expressions* and *equations*.

EXPRESSION

An **expression** is a number, a variable, or a combination of numbers and variables using operation symbols.

Expression	**Words**	**English Phrase**
$3+5$	3 plus 5	the sum of three and five
$n-1$	n minus one	the difference of n and one
$6 \cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

(13.7.2.5)

Notice that the English phrases do not form a complete sentence because the phrase does not have a verb.

An equation is two expressions linked by an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb.

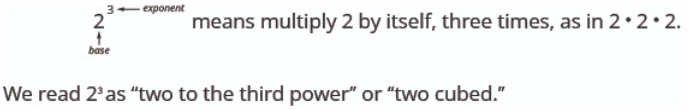
EQUATION

An **equation** is two expressions connected by an equal sign.

Equation	English Sentence
$3+5=8$	The sum of three and five is equal to eight.
$n-1=14$	n minus one equals fourteen.
$6\cdot 7=42$	The product of six and seven is equal to forty-two.
$x=53$	x is equal to fifty-three.
$y+9=2y-3$	y plus nine is equal to two y minus three.

(13.7.2.6)

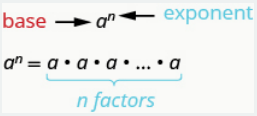
Suppose we need to multiply 2 nine times. We could write this as $2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2$. This is tedious and it can be hard to keep track of all those 2s, so we use exponents. We write $2\cdot 2\cdot 2$ as 2^3 and $2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2$ as 2^9. In expressions such as 2^3, the 2 is called the *base* and the 3 is called the *exponent*. The **exponent** tells us how many times we need to multiply the **base**.



EXPONENTIAL NOTATION

We say 2^3 is in *exponential notation* and $2\cdot 2\cdot 2$ is in *expanded notation*.

a^n means multiply n factors of the number a.



The expression a^n is read a to the nth power.

While we read a^n as "a to the nth power", we usually read:

a^2 "a squared"
a^3 "a cubed"

(13.7.2.7)

We'll see later why a^2 and a^3 have special names.

Table shows how we read some expressions with exponents.

Expression	In Words	
7^2	7 to the second power or	7 squared
5^3	5 to the third power or	5 cubed
9^4	9 to the fourth power	
12^5	12 to the fifth power	

Simplify Expressions Using the Order of Operations

To **simplify an expression** means to do all the math possible. For example, to simplify $4\cdot 2+1$ we would first multiply $4\cdot 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:

$$\begin{array}{l} 4 \cdot 2+1 \\ 8+1 \\ 9 \end{array} \tag{13.7.2.8}$$

By not using an equal sign when you simplify an expression, you may avoid confusing expressions with equations.

SIMPLIFY AN EXPRESSION

To **simplify an expression**, do all operations in the expression.

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the **order of operations**. Otherwise, expressions may have different meanings, and they may result in different values.

For example, consider the expression $4+3 \cdot 7$. Some students simplify this getting 49, by adding $4+3$ and then multiplying that result by 7. Others get 25, by multiplying $3 \cdot 7$ first and then adding 4.

The same expression should give the same result. So mathematicians established some guidelines that are called the order of operations.

USE THE ORDER OF OPERATIONS.

1. Parentheses and Other Grouping Symbols
   - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
2. Exponents
   - Simplify all expressions with exponents.
3. Multiplication and Division
   - Perform all multiplication and division in order from left to right. These operations have equal priority.
4. Addition and Subtraction
   - Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, "How will I remember the order?" Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase "Please Excuse My Dear Aunt Sally".

$$\begin{array}{ll} \text{Parentheses} & \text{Please} \\ \text{Exponents} & \text{Excuse} \\ \text{Multiplication Division} & \text{My Dear} \\ \text{Addition Subtraction} & \text{Aunt Sally} \end{array} \tag{13.7.2.9}$$

It's good that "**M**y **D**ear" goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, "**A**unt **S**ally" goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

Example 13.7.2.10

Simplify: $18 \div 6+4(5-2)$.

Answer

	$18 \div 6+4(5-2)$
Parentheses? Yes, subtract first.	$18 \div 6+4(3)$
Exponents? No.	
Multiplication or division? Yes.	
Divide first because we multiply and divide left to right.	$3+4(3)$

Any other multiplication or division? Yes.	
Multiply.	$3 + 12$
Any other multiplication of division? No.	
Any addition or subtraction? Yes.	
Add.	15

Example 13.7.2.11

Simplify: $30 \div 5 + 10(3 - 2)$.

Answer

16

Example 13.7.2.12

Simplify: $70 \div 10 + 4(6 - 2)$.

Answer

23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

Example 13.7.2.13

Simplify: $5 + 2^3 + 3[6 - 3(4 - 2)]$.

Answer

	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Are there any parentheses (or other grouping symbols)? Yes.	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Focus on the parentheses that are inside the brackets. Subtract.	$5 + 2^3 + 3[6 - 3(2)]$
Continue inside the brackets and multiply.	$5 + 2^3 + 3[6 - 6]$
Continue inside the brackets and subtract.	$5 + 2^3 + 3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes. Simplify exponents.	$5 + 8 + 3[0]$
Is there any multiplication or division? Yes.	
Multiply.	$5 + 8 + 0$
Is there any addition of subtraction? Yes.	
Add.	$13 + 0$
Add.	13

Example 13.7.2.14

Simplify: $9+5^3-[4(9+3)]$.

Answer

86

Example 13.7.2.15

Simplify: $7^2-2[4(5+1)]$.

Answer

1

Evaluate an Expression

In the last few examples, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

EVALUATE AN EXPRESSION

To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression, substitute that number for the variable in the expression and then simplify the expression.

Evaluate when $x=4$: a. x^2 b. 3^x c. $2x^2+3x+8$.

Answer

a.

	x^2
Replace x with 4.	4^2
Use definition of exponent.	$4 \cdot 4$
Simplify.	16

b.

	3^x
Replace x with 4.	3^4
Use definition of exponent.	$3 \cdot 3 \cdot 3 \cdot 3$
Simplify.	81

c.

	$2x^2+3x+8$
Replace x with 4.	$2(4)^2+3(4)+8$
Follow the order of operations.	$2(16)+3(4)+8$
	$32+12+8$

52

Evaluate when $x = 3$, a. x^2 b. 4^x c. $3x^2 + 4x + 1$.

Answer

a. 9
b. 64
c. 40

Example 13.7.2.18

Evaluate when $x = 6$, a. x^3 b. 2^x c. $6x^2 - 4x - 7$.

Answer

a. 216
b. 64
c. 185

Identify and Combine Like Terms

Algebraic expressions are made up of terms. A **term** is a constant, or the product of a constant and one or more variables.

TERM

A **term** is a constant or the product of a constant and one or more variables.

Examples of terms are 7, y, $5x^2$, $9a$, and b^5.

The constant that multiplies the variable is called the **coefficient**.

COEFFICIENT

The **coefficient** of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1 \cdot x$.

Some terms share common traits. When two terms are constants or have the same variable and exponent, we say they are **like terms**.

Look at the following 6 terms. Which ones seem to have traits in common?

$$5x \quad 7 \quad n^2 \quad 4 \quad 3x \quad 9n^2 \tag{13.7.2.10}$$

We say,

7 and 4 are like terms.

$5x$ and $3x$ are like terms.

n^2 and $9n^2$ are like terms.

LIKE TERMS

Terms that are either constants or have the same variables raised to the same powers are called **like terms.**

If there are like terms in an expression, you can simplify the expression by combining the like terms. We add the coefficients and keep the same variable.

$$\begin{array}{ll} \text{Simplify.} & 4x+7x+x \\ \text{Add the coefficients.} & 12x \end{array} \qquad (13.7.2.11)$$

ExAMPLE 13.7.2.19: How To Combine Like Terms

Simplify: $2x^2+3x+7+x^2+4x+5$.

Answer

Step 1. Identify the like terms.	$2x^2+3x+7+x^2+4x+5$ $2x^2+3x+7+x^2+4x+5$ $2x^2+3x+7+x^2+4x+5$
Step 2. Rearrange the expression so the like terms are together.	$2x^2+x^2+3x+4x+7+5$ $2x^2+x^2+3x+4x+7+5$
Step 3. Combine like terms.	$3x^2+7x+12$

Simplify: $3x^2+7x+9+7x^2+9x+8$.

Answer

$10x^2+16x+17$

ExAMPLE 13.7.2.21

Simplify: $4y^2+5y+2+8y^2+4y+5$.

Answer

$12y^2+9y+7$

COMBINE LIKE TERMS.

1. Identify like terms.
2. Rearrange the expression so like terms are together.
3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Translate an English Phrase to an Algebraic Expression

We listed many operation symbols that are used in algebra. Now, we will use them to translate English phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. Table summarizes them.

Operation	Phrase	Expression

Operation	Phrase	Expression
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a+b$
Subtraction	a minus b the difference of a and b a decreased by b b less than a b subtracted from a	$a-b$
Multiplication	a times b the product of a and b twice a	$a \cdot b,\ ab,\ a(b),\ (a)(b)$ $2a$
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a \div b,\ a/b,\ \frac{a}{b},\ b\overline{)a}$

Look closely at these phrases using the four operations:

the **sum** *of a and b*

the **difference** *of a and b*

the **product** *of a and b*

the **quotient** *of a and b*

Each phrase tells us to operate on two numbers. Look for the words *of* and *and* to find the numbers.

? Example 13.7.2.22

Each phrase tells us to operate on two numbers. Look for the words *of* and *and* to find the numbers.

Translate each English phrase into an algebraic expression:

a. the difference of $14x$ and 9

b. the quotient of $8y^2$ and 3

c. twelve more than y

d. seven less than $49x^2$

Answer

a. The key word is *difference*, which tells us the operation is subtraction. Look for the words *of* and *and* to find the numbers to subtract.

the *difference of* 14*x* *and* 9

14*x* minus 9

14*x* – 9

b. The key word is *quotient*, which tells us the operation is division.

the *quotient of* $8y^2$ *and* 3
divide $8y^2$ by 3
$8y^2 \div 3$
This can also be written $8y^2/3$ or $\frac{8y^2}{3}$.

c. The key words are *more than*. They tell us the operation is addition. *More than* means "added to."

$$\begin{gathered} \text{twelve more than } y \\ \text{twelve added to } y \\ y+12 \end{gathered} \tag{13.7.2.12}$$

d. The key words are *less than*. They tell us to subtract. *Less than* means "subtracted from."

$$\begin{gathered} \text{seven less than } 49x^2 \\ \text{seven subtracted from } 49x^2 \\ 49x^2-7 \end{gathered} \tag{13.7.2.13}$$

? Exercise 13.7.2.23

Translate the English phrase into an algebraic expression:

a. the difference of $14x^2$ and 13

b. the quotient of $12x$ and 2

c. 13 more than z

d. 18 less than $8x$

Answer

a. $14x^2-13$ b. $12x \div 2$

c. $z+13$ d. $8x-18$

? Example 13.7.2.24

Translate the English phrase into an algebraic expression:

a. the sum of $17y^2$ and 19

b. the product of 7 and y

c. Eleven more than x

d. Fourteen less than $11a$

Answer

a. $17y^2+19$ b. $7y$

c. $x+11$ d. $11a-14$

We look carefully at the words to help us distinguish between multiplying a sum and adding a product.

? Example 13.7.2.25

Translate the English phrase into an algebraic expression:

a. eight times the sum of x and y

b. the sum of eight times x and y

Answer

There are two operation words—*times* tells us to multiply and *sum* tells us to add.

a. Because we are multiplying 8 times the sum, we need parentheses around the sum of x and y, $(x+y)$. This forces us to determine the sum first. (Remember the order of operations.)

$$\begin{array}{c}\text{eight times the sum of } x \text{ and } y \\ 8(x+y)\end{array} \tag{13.7.2.14}$$

b. To take a sum, we look for the words *of* and *and* to see what is being added. Here we are taking the sum *of* eight times x and y.

the sum *of* eight times *x and y*

$8x + y$

? Example 13.7.2.26

Translate the English phrase into an algebraic expression:

a. four times the sum of p and q

b. the sum of four times p and q

Answer

a. $4(p+q)$ b. $4p+q$

? Example 13.7.2.27

Translate the English phrase into an algebraic expression:

a. the difference of two times x and 8

b. two times the difference of x and 8

Answer

a. $2x-8$ b. $2(x-8)$

Later in this course, we'll apply our skills in algebra to solving applications. The first step will be to translate an English phrase to an algebraic expression. We'll see how to do this in the next two examples.

? Example 13.7.2.28

The length of a rectangle is 14 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Answer

$$\begin{array}{ll}\text{Write a phrase about the length of the rectangle.} & 14 \text{ less than the width} \\ \text{Substitute } w \text{ for "the width."} & w \\ \text{Rewrite less than as subtracted from.} & 14 \text{ subtracted from } w \\ \text{Translate the phrase into algebra.} & w-14\end{array} \tag{13.7.2.15}$$

Example 13.7.2.29

The length of a rectangle is 7 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Answer

$w-7$

Example 13.7.2.30

The width of a rectangle is 6 less than the length. Let l represent the length of the rectangle. Write an expression for the width of the rectangle.

Answer

$l-6$

The expressions in the next example will be used in the typical coin mixture problems we will see soon.

Example 13.7.2.31

June has dimes and quarters in her purse. The number of dimes is seven less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Answer

Write a phrase about the number of dimes.	7 less than 4 times q	
Translate 4 times q.	7 less than $4q$	(13.7.2.16)
Translate the phrase into algebra.	$4q-7$	

Example 13.7.2.32

Geoffrey has dimes and quarters in his pocket. The number of dimes is eight less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Answer

$4q-8$

Example 13.7.2.33

Lauren has dimes and nickels in her purse. The number of dimes is three more than seven times the number of nickels. Let n represent the number of nickels. Write an expression for the number of dimes.

Answer

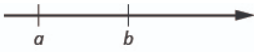
$7n+3$

Key Concepts

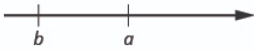
- **Divisibility Tests**
  A number is divisible by:
  2 if the last digit is 0, 2, 4, 6, or 8.
  3 if the sum of the digits is divisible by 3.
  5 if the last digit is 5 or 0.

6 if it is divisible by both 2 and 3.
10 if it ends with 0.

- **How to find the prime factorization of a composite number.**
  1. Find two factors whose product is the given number, and use these numbers to create two branches.
  2. If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
  3. If a factor is not prime, write it as the product of two factors and continue the process.
  4. Write the composite number as the product of all the circled primes.
- **How To Find the least common multiple using the prime factors method.**
  1. Write each number as a product of primes.
  2. List the primes of each number. Match primes vertically when possible.
  3. Bring down the columns.
  4. Multiply the factors.
- **Equality Symbol**
  $a = b$ is read " a is equal to b." The symbol "=" is called the equal sign.
- $a < b$ is read "a is less than b"
  a is to the left of b on the number line



- $a > b$ is read "a is greater than b"
  a is to the right of b on the number line



- **Inequality Symbols**

Inequality Symbols	Words
$a \neq b$	a is *not equal to* b.
$a < b$	a is *less than* b.
$a \leq b$	a is *less than or equal to* b.
$a > b$	a is *greater than* b.
$a \geq b$	a is *greater than or equal to* b.

- **Grouping Symbols**

Parentheses	$()$
Brackets	$[]$
Braces	$\{\}$

- **Exponential Notation** a^n means multiply a by itself, n times. The expression an is read a to the nth power.
- **Simplify an Expression**
  To simplify an expression, do all operations in the expression.
- **How to use the order of operations.**
  1. Parentheses and Other Grouping Symbols
     - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
  2. Exponents
     - Simplify all expressions with exponents.
  3. Multiplication and Division
     - Perform all multiplication and division in order from left to right. These operations have equal priority.
  4. Addition and Subtraction
     - Perform all addition and subtraction in order from left to right. These operations have equal priority.
- **How to combine like terms.**
  1. Identify like terms.
  2. Rearrange the expression so like terms are together.
  3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Operation	Phrase	Expression
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a+b$
Subtraction	a minus b the difference of a and b a decreased by b b less than a b subtracted from a	$a-b$
Multiplication	a times b the product of a and b twice a	$a \cdot b,\ ab,\ a(b),\ (a)(b)$ $2a$
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a \div b,\ a/b,\ \frac{a}{b},\ b\overline{)a}$

Glossary

coefficient

The coefficient of a term is the constant that multiplies the variable in a term.

composite number

A composite number is a counting number that is not prime. It has factors other than 1 and the number itself.

constant

A constant is a number whose value always stays the same.

divisible by a number

If a number m is a multiple of n, then m is divisible by n.

equation

An equation is two expressions connected by an equal sign.

evaluate an expression

To evaluate an expression means to find the value of the expression when the variables are replaced by a given number.

expression

An expression is a number, a variable, or a combination of numbers and variables using operation symbols.

factors

If $a \cdot b = m$, then a and b are factors of m.

least common multiple

The least common multiple (LCM) of two numbers is the smallest number that is a multiple of both numbers.

like terms

Terms that are either constants or have the same variables raised to the same powers are called like terms.

multiple of a number

A number is a multiple of n if it is the product of a counting number and n.

order of operations

The order of operations are established guidelines for simplifying an expression.

prime factorization

The prime factorization of a number is the product of prime numbers that equals the number.

prime number

A prime number is a counting number greater than 1 whose only factors are 1 and the number itself.

simplify an expression

To simplify an expression means to do all the math possible.

term

A term is a constant, or the product of a constant and one or more variables.

variable

A variable is a letter that represents a number whose value may change.

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13.7.2E: Exercises

Practice Makes Perfect

Identify Multiples and Factors

In the following exercises, use the divisibility tests to determine whether each number is divisible by 2, by 3, by 5, by 6, and by 10.

1. 84

Answer

Divisible by 2, 3, 6

2. 96

3. 896

Answer

Divisible by 2

4. 942

5. $22,335$

Answer

Divisible by 3, 5

6. $39,075$

Find Prime Factorizations and Least Common Multiples

In the following exercises, find the prime factorization.

7. 86

Answer

$2 \cdot 43$

8. 78

9. 455

Answer

$5 \cdot 7 \cdot 13$

10. 400

11. 432

Answer

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

12. 627

In the following exercises, find the least common multiple of each pair of numbers using the prime factors method.

13. $8,\ 12$

Answer

24

14. $12,\ 16$

15. $28,\ 40$

Answer

280

16. $84,\ 90$

17. $55,\ 88$

Answer

440

18. $60,\ 72$

Simplify Expressions Using the Order of Operations

In the following exercises, simplify each expression.

19. $2^3 - 12 \div (9 - 5)$

Answer

5

20. $3^2 - 18 \div (11 - 5)$

21. $2 + 8(6 + 1)$

Answer

58

22. $4 + 6(3 + 6)$

23. $20 \div 4 + 6(5 - 1)$

Answer

29

24. $33 \div 3 + 4(7 - 2)$

25. $3(1 + 9 \cdot 6) - 4^2$

Answer

149

26. $5(2+8\cdot 4)-7^2$

27. $2[1+3(10-2)]$

Answer

50

28. $5[2+4(3-2)]$

29. $8+2[7-2(5-3)]-3^2$

Answer

5

30. $10+3[6-2(4-2)]-2^4$

Evaluate an Expression

In the following exercises, evaluate the following expressions.

31. When $x=2$,

a. x^6

b. 4^x

c. $2x^2+3x-7$

Answer

a. 64
b. 16
c. 7

32. When $x=3$,

a. x^5

b. $5x$

c. $3x^2-4x-8$

33. When $x=4$ and $y=1$

$x^2+3xy-7y^2$

Answer

21

34. When $x=3$ and $y=2$

$6x^2+3xy-9y^2$

35. When $x=10$ and $y=7$

$(x-y)^2$

Answer

9

36. When $a=3$ and $b=8$

a^2+b^2

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the following expressions by combining like terms.

37. $7x+2+3x+4$

Answer

$10x+6$

38. $8y+5+2y-4$

39. $10a+7+5a-2+7a-4$

Answer

$22a+1$

40. $7c+4+6c-3+9c-1$

41. $3x^2+12x+11+14x^2+8x+5$

Answer

$17x^2+20x+16$

42. $5b^2+9b+10+2b^2+3b-4$

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the phrases into algebraic expressions.

43. a. the difference of $5x^2$ and $6xy$

b. the quotient of $6y^2$ and $5x$

c. Twenty-one more than y^2

d. $6x$ less than $81x^2$

Answer

a. $5x^2-6xy$ b. $\frac{6y^2}{5x}$

c. y^2+21 d. $81x^2-6x$

44. a. the difference of $17x^2$ and $17x^2$ and $5xy$

b. the quotient of $8y^3$ and $3x$

c. Eighteen more than a^2;

d. $11b$ less than $100b^2$

45. a. the sum of $4ab^2$ and $3a^2b$

b. the product of $4y^2$ and $5x$

c. Fifteen more than m

d. $9x$ less than $121x^2$

Answer

a. $4ab^2+3a^2b$ b. $20xy^2$

c. $m+15$ d. $121x^2-9x$

46. a. the sum of $3x^2y$ and $7xy^2$

b. The product of $6xy^2$ and $4z$

c. Twelve more than $3x^2$

d. $7x^2$ less than $63x^3$

47. a. eight times the difference of y and nine

b. the difference of eight times y and 9

Answer

a. $8(y-9)$
b. $8y-9$

48. a. seven times the difference of y and one

b. the difference of seven times y and 1

49. a. five times the sum of $3x$ and y

b. the sum of five times $3x$ and y

Answer

a. $5(3x+y)$
b. $15x+y$

50. a. eleven times the sum of $4x^2$ and $5x$

b. the sum of eleven times $4x^2$ and $5x$

51. Eric has rock and country songs on his playlist. The number of rock songs is 14 more than twice the number of country songs. Let c represent the number of country songs. Write an expression for the number of rock songs.

Answer

$2c-14$

52. The number of women in a Statistics class is 8 more than twice the number of men. Let m represent the number of men. Write an expression for the number of women.

53. Greg has nickels and pennies in his pocket. The number of pennies is seven less than three times the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

Answer

$3n-7$

54. Jeannette has $\$5$ and $\$10$ bills in her wallet. The number of fives is three more than six times the number of tens. Let t represent the number of tens. Write an expression for the number of fives.

Writing Exercises

55. Explain in your own words how to find the prime factorization of a composite number.

Answer

Answers will vary.

56. Why is it important to use the order of operations to simplify an expression?

57. Explain how you identify the like terms in the expression $8a^2+4a+9-a^2-1$.

Answer

Answers will vary.

58. Explain the difference between the phrases "4 times the sum of x and y" and "the sum of 4 times x and y".

Self Check

a. Use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify multiples and apply divisibility tests.			
find prime factorizations and least common multiples.			
use variables and algebraic symbols.			
simplify expressions using the order of operations.			
evaluate an expression.			
identify and combine like terms.			
translate English phrases to algebraic expressions.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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13.7.3: Integers

summary

By the end of this section, you will be able to:

- Simplify expressions with absolute value
- Add and subtract integers
- Multiply and divide integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate phrases to expressions with integers
- Use integers in applications

A more thorough introduction to the topics covered in this section can be found in the *Elementary Algebra* chapter, Foundations.

Simplify Expressions with Absolute Value

A negative number is a number less than 0. The negative numbers are to the left of zero on the number line (Figure 13.7.3.1).

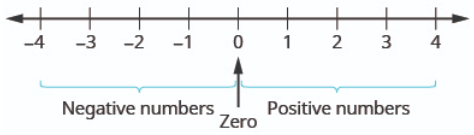


***Figure* 13.7.3.1.** *The number line shows the location of positive and negative numbers.*

You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, each one is called the **opposite** of the other. The opposite of 2 is -2, and the opposite of -2 is 2.

OPPOSITE

The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

Figure 13.7.3.2illustrates the definition.

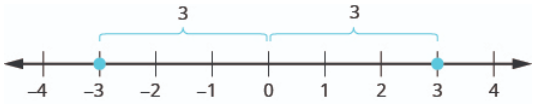


***Figure* 13.7.3.2.** *The opposite of 3 is* -3.

OPPOSITE NOTATION

$$-a \text{ means the opposite of the number } a \qquad (13.7.3.1)$$

$$\text{The notation } -a \text{ is read as "the opposite of } a\text{."} \qquad (13.7.3.2)$$

We saw that numbers such as 3 and -3 are opposites because they are the same distance from 0 on the number line. They are both three units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

Definition: ABSOLUTE VALUE

The **absolute value** of a number is its distance from 0 on the number line.

The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.

> Absolute values are always greater than or equal to zero.

For example,

$$-5 \text{ is } 5 \text{ units away from } 0, \text{ so } |-5| = 5. \tag{13.7.3.3}$$
$$5 \text{ is } 5 \text{ units away from } 0, \text{ so } |5| = 5. \tag{13.7.3.4}$$

Figure 13.7.3.3illustrates this idea.

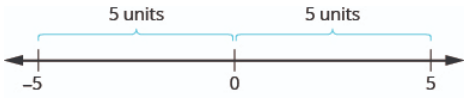


Figure 13.7.3.3: The numbers 5 and −5 are 5 units away from 0.

The absolute value of a number is never negative because distance cannot be negative. The only number with absolute value equal to zero is the number zero itself because the distance from 0 to 0 on the number line is zero units.

In the next example, we'll order expressions with absolute values.

? EXAMPLE 13.7.3.1

Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

a. $|-5|\_\_-|-5|\_\_-|5|$
b. $8\_\_-|-8|$
c. $-9\_\_-|-9|$
d. (\text{−(−16)__|−16|}\).

Answer

a.

	$\lvert-5\rvert$	__	$-\lvert-5\rvert$
Simplify.	5	__	-5
Order.	5	$>$	-5
	$\lvert-5\rvert$	$>$	$-\lvert-5\rvert$

b.

	8	__	$-\lvert-8\rvert$
Simplify.	8	__	-8
Order.	8	$>$	-8
	8	$>$	$-\lvert-8\rvert$

c.

	-9	__	$-\lvert-9\rvert$
Simplify.	-9	__	-9
Order.	-9	$=$	-9
	-9	$=$	$-\lvert-9\rvert$

d.

	$-(-16)$	__	$-\lvert-16\rvert$
Simplify.	16	__	16
Order.	16	$=$	16
	$-(-16)$	$=$	$\lvert-16\rvert$

EXAMPLE 13.7.3.2

Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

ⓐ $-9\_\_-|-9|$ ⓑ $2\_\_-|-2|$ ⓒ $-8\_\_|-8|$ ⓓ $-(-9)\_\_|-9|$.

Answer

ⓐ $>$ ⓑ $>$ ⓒ $<$

ⓓ $=$

EXAMPLE 13.7.3.3

Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

a. $7\_\_-|-7|$
b. $-(-10)\_\_|-10|$
c. $|-4|\_\_-|-4|$
d. $-1\_\_|-1|$.

Answer

ⓐ $>$ ⓑ $=$ ⓒ $>$

ⓓ $<$

We now add absolute value bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the absolute value of the resulting number.

GROUPING SYMBOLS

Parentheses	()	Braces	{}
Brackets	[]	Absolute value	||

(13.7.3.5)

In the next example, we simplify the expressions inside absolute value bars first just like we do with parentheses.

EXAMPLE 13.7.3.4

Simplify: $24-|19-3(6-2)|$.

Answer

$$24-|19-3(6-2)|$$

Work inside parentheses first:	
subtract 2 from 6.	$24-\|19-3(4)\|$
Multiply $3(4)$.	$24-\|19-12\|$
Subtract inside the absolute value bars.	$24-\|7\|$
Take the absolute value.	$24-7$
Subtract.	17

EXAMPLE 13.7.3.5

Simplify: $19-|11-4(3-1)|$.

Answer

16

? EXAMPLE 13.7.3.6

Simplify: $9-|8-4(7-5)|$.

Answer

9

Add and Subtract Integers

So far in our examples, we have only used the counting numbers and the whole numbers.

$$\begin{array}{ll}\text{Counting numbers} & 1,2,3\ldots \\ \text{Whole numbers} & 0,1,2,3\ldots .\end{array} \tag{13.7.3.6}$$

Our work with opposites gives us a way to define the **integers**. The whole numbers and their opposites are called the integers. The integers are the numbers $\ldots -3,-2,-1,0,1,2,3\ldots$

Definition: INTEGERS

The whole numbers and their opposites are called the **integers**.

The integers are the numbers

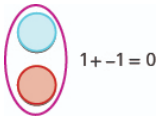
$$\ldots -3,-2,-1,0,1,2,3\ldots , \tag{13.7.3.7}$$

Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more challenging.

We will use two color counters to model addition and subtraction of negatives so that you can visualize the procedures instead of memorizing the rules.

We let one color (blue) represent positive. The other color (red) will represent the negatives.

If we have one positive counter and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero.



We will use the counters to show how to add:

$$5+3 \qquad -5+(-3) \qquad -5+3 \qquad 5+(-3) \tag{13.7.3.8}$$

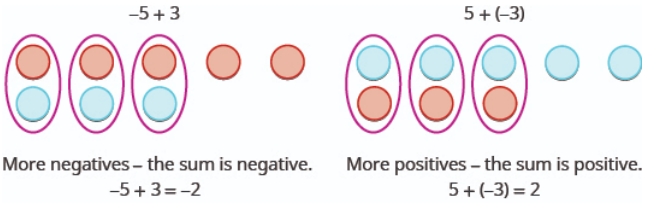
The first example, $5+3,$ adds 5 positives and 3 positives—both positives.

The second example, $-5+(-3),$ adds 5 negatives and 3 negatives—both negatives.

When the signs are the same, the counters are all the same color, and so we add them. In each case we get 8—either 8 positives or 8 negatives.

So what happens when the signs are different? Let's add $-5+3$ and $5+(-3)$.

When we use counters to model addition of positive and negative integers, it is easy to see whether there are more positive or more negative counters. So we know whether the sum will be positive or negative.



EXAMPLE 13.7.3.7

Add: ⓐ $-1+(-4)$ ⓑ $-1+5$ ⓒ $1+(-5)$.

Answer

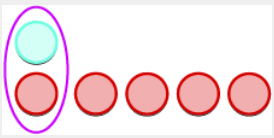
ⓐ

	–1 + (–4)
1 negative plus 4 negatives is 5 negatives	–5

ⓑ

	–1 + 5
There are more positives, so the sum is positive.	4

ⓒ

	1 + (–5)
	
There are more negatives, so the sum is negative.	–4

EXAMPLE 13.7.3.8

Add: ⓐ $-2+(-4)$ ⓑ $-2+4$ ⓒ $2+(-4)$.

Answer

ⓐ -6 ⓑ 2 ⓒ -2

EXAMPLE 13.7.3.9

Add: ⓐ $-2+(-5)$ ⓑ $-2+5$ ⓒ $2+(-5)$.

Answer

ⓐ -7 ⓑ 3 ⓒ -3

We will continue to use counters to model the subtraction. Perhaps when you were younger, you read "$5-3$" as "5 take away 3." When you use counters, you can think of subtraction the same way!

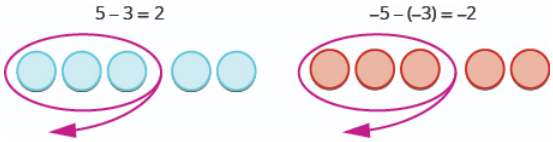
We will use the counters to show to subtract:

$$5-3 \qquad -5-(-3) \qquad -5-3 \qquad 5-(-3) \tag{13.7.3.9}$$

The first example, $5-3$, we subtract 3 positives from 5 positives and end up with 2 positives.

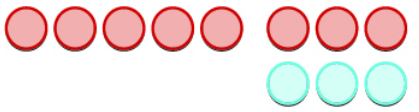
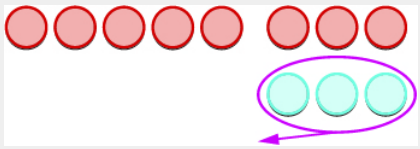
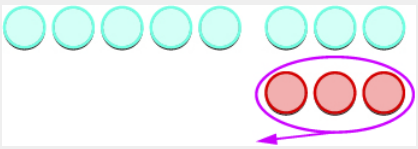
In the second example, $-5-(-3)$, we subtract 3 negatives from 5 negatives and end up with 2 negatives.

Each example used counters of only one color, and the "take away" model of subtraction was easy to apply.



What happens when we have to subtract one positive and one negative number? We'll need to use both blue and red counters as well as some neutral pairs. If we don't have the number of counters needed to take away, we add neutral pairs. Adding a neutral pair does not change the value. It is like changing quarters to nickels—the value is the same, but it looks different.

Let's look at $-5-3$ and $5-(-3)$.

	–5 – 3	5 – (–3)
Model the first number.		
We now add the needed neutral pairs.		
We remove the number of counters modeled by the second number.		
Count what is left.		
	–5 – 3 = –8	5 – (–3) = –8
	–8	8

EXAMPLE 13.7.3.10

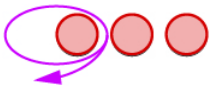
Subtract: ⓐ $3-1$ ⓑ $-3-(-1)$ ⓒ $-3-1$ ⓓ $3-(-1)$.

Answer

ⓐ

		3 – 1
Take 1 positive from 3 positives and get 2 positives.		2

ⓑ

		–3 – (–1)
Take 1 positive from 3 negatives and get 2 negatives.		–2

ⓒ

		–3 – 1
Take 1 positive from the one added neutral pair.		–4

ⓓ

		3 – (–1)
Take 1 negative from the one added neutral pair.		4

? EXAMPLE 13.7.3.11

Subtract: ⓐ $6-4$ ⓑ $-6-(-4)$ ⓒ $-6-4$ ⓓ $6-(-4)$.

Answer

ⓐ 2 ⓑ -2 ⓒ -10 ⓓ 10

? EXAMPLE 13.7.3.12

Subtract: ⓐ $7-4$ ⓑ $-7-(-4)$ ⓒ $-7-4$ ⓓ $7-(-4)$.

Answer

ⓐ 3 ⓑ -3 ⓒ -11 ⓓ 11

Have you noticed that *subtraction of signed numbers can be done by adding the opposite*? In the last example, $-3-1$ is the same as $-3+(-1)$ and $3-(-1)$ is the same as $3+1$. You will often see this idea, the Subtraction Property, written as follows:

Definition: SUBTRACTION PROPERTY

$$a-b=a+(-b) \tag{13.7.3.10}$$

Subtracting a number is the same as adding its opposite.

EXAMPLE 13.7.3.13

Simplify: ⓐ $13-8$ and $13+(-8)$ ⓑ $-17-9$ and $-17+(-9)$ ⓒ $9-(-15)$ and $9+15$ ⓓ $-7-(-4)$ and $-7+4$.

Answer

ⓐ

	$13-8$	and	$13+(-8)$
Subtract.	5		5

ⓑ

	$-17-9$	and	$-17+(-9)$
Subtract.	-26		-26

ⓒ

	$9-(-15)$	and	$9+15$
Subtract.	24		24

ⓓ

	$-7-(-4)$	and	$-7+4$
Subtract.	-3		-3

EXAMPLE 13.7.3.14

Simplify: ⓐ $21-13$ and $21+(-13)$ ⓑ $-11-7$ and $-11+(-7)$ ⓒ $6-(-13)$ and $6+13$ ⓓ $-5-(-1)$ and $-5+1$.

Answer

ⓐ $8, 8$ ⓑ $-18, -18$

ⓒ $19, 19$ ⓓ $-4, -4$

EXAMPLE 13.7.3.15

Simplify: ⓐ $15-7$ and $15+(-7)$ ⓑ $-14-8$ and $-14+(-8)$ ⓒ $4-(-19)$ and $4+19$ ⓓ $-4-(-7)$ and $-4+7$.

Answer

ⓐ $8, 8$ ⓑ $-22, -22$

ⓒ $23, 23$ ⓓ $3, 3$

What happens when there are more than three integers? We just use the order of operations as usual.

EXAMPLE 13.7.3.16

Simplify: $7-(-4-3)-9$.

Answer

	$7-(-4-3)-9$
Simplify inside the parentheses first.	$7-(-7)-9$
Subtract left to right.	$14-9$
Subtract.	5

Simplify: $8-(-3-1)-9$.

Answer

3

EXAMPLE 13.7.3.18

Simplify: $12-(-9-6)-14$.

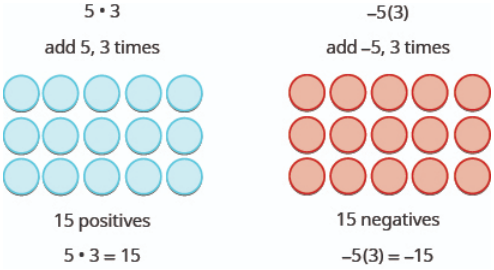
Answer

13

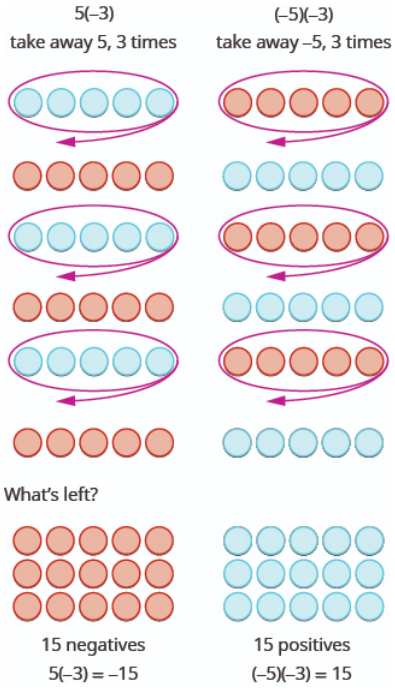
Multiply and Divide Integers

Since multiplication is mathematical shorthand for repeated addition, our model can easily be applied to show multiplication of integers. Let's look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction. Here, we are using the model just to help us discover the pattern.

We remember that a·ba·b means **add** *a*, *b* **times**.



The next two examples are more interesting. What does it mean to multiply 5 by −3? It means subtract 5,3 times. Looking at subtraction as "taking away", it means to take away 5, 3 times. But there is nothing to take away, so we start by adding neutral pairs on the workspace.



In summary:

$$\begin{aligned} 5 \cdot 3 &= 15 \qquad -5(3) = -15 \\ 5(-3) &= -15 \quad (-5)(-3) = 15 \end{aligned} \tag{13.7.3.11}$$

Notice that for multiplication of two signed numbers, when the

$$\begin{aligned} &\text{signs are the } \textbf{same}\text{, the product is } \textbf{positive.} \\ &\text{signs are } \textbf{different}\text{, the product is } \textbf{negative.} \end{aligned} \tag{13.7.3.12}$$

What about division? Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $15 \cdot 3 = 15$. In words, this expression says that 15 can be divided into 3 groups of 5 each because adding five three times gives 15. If you look at some examples of multiplying integers, you might figure out the rules for dividing integers.

$$\begin{aligned} 5 \cdot 3 &= 15 \quad \text{so} \quad 15 \div 3 = 5 \qquad -5(3) = -15 \quad \text{so} \quad -15 \div 3 = -5 \\ (-5)(-3) &= 15 \quad \text{so} \quad 15 \div (-3) = -5 \qquad 5(-3) = -15 \quad \text{so} \quad -15 \div (-3) = 5 \end{aligned} \tag{13.7.3.13}$$

Division follows the same rules as multiplication with regard to signs.

MULTIPLICATION AND DIVISION OF SIGNED NUMBERS

For multiplication and division of two signed numbers:

Same signs	Result
• Two positives	Positive
• Two negatives	Positive

If the signs are the same, the result is positive.

Different signs	Result
• Positive and negative	Negative
• Negative and positive	Negative

If the signs are different, the result is negative.

EXAMPLE 13.7.3.19

Multiply or divide: ⓐ $-100 \div (-4)$ ⓑ $7 \cdot 6$ ⓒ $4(-8)$ ⓓ $-27 \div 3$.

Answer

ⓐ

$$-100 \div (-4)$$

Divide, with signs that are the same the quotient is positive. 25

ⓑ

$$7 \cdot 6$$

Multiply, with same signs. 42

ⓒ

$$4(-8)$$

Multiply, with different signs. -32

ⓓ

$$-27 \div 3$$

Divide, with different signs, the quotient is negative. -9

EXAMPLE 13.7.3.20

Multiply or divide: ⓐ $-115 \div (-5)$ ⓑ $5 \cdot 12$ ⓒ $9(-7)$ ⓓ $-63 \div 7$.

Answer

ⓐ 23 ⓑ 60 ⓒ −63 ⓓ −9

?

Multiply or divide: ⓐ $-117 \div (-3)$ ⓑ $3 \cdot 13$ ⓒ $7(-4)$ ⓓ $-42 \div 6$.

Answer

ⓐ 39 ⓑ 39 ⓒ −28 ⓓ −7

When we multiply a number by 1, the result is the same number. Each time we multiply a number by -1, we get its opposite!

MULTIPLICATION BY −1

$$-1a = -a \tag{13.7.3.14}$$

Multiplying a number by -1 gives its opposite.

Simplify Expressions with Integers

What happens when there are more than two numbers in an expression? The order of operations still applies when negatives are included. Remember Please Excuse My Dear Aunt Sally?

Let's try some examples. We'll simplify expressions that use all four operations with integers—addition, subtraction, multiplication, and division. Remember to follow the order of operations.

EXAMPLE 13.7.3.22

Simplify: ⓐ $(-2)^4$ ⓑ -2^4.

Answer

Notice the difference in parts (a) and (b). In part (a), the exponent means to raise what is in the parentheses, the −2 to the 4^{th}power. In part (b), the exponent means to raise just the 2 to the 4th power and then take the opposite.

ⓐ

	$(-2)^4$
Write in expanded form.	$(-2)(-2)(-2)(-2)$
Multiply.	$4(-2)(-2)$
Multiply.	$-8(-2)$
Multiply.	16

ⓑ

	-2^4
Write in expanded form.	$-(2\cdot 2\cdot 2\cdot 2)$
We are asked to find the opposite of 24.	
Multiply.	$-(4\cdot 2\cdot 2)$
Multiply.	$-(8\cdot 2)$
Multiply.	-16

?

Simplify: ⓐ $(-3)^4$ ⓑ -3^4.

Answer

ⓐ 81 ⓑ −81

EXAMPLE 13.7.3.24

Simplify: ⓐ $(-7)^2$ ⓑ -7^2.

Answer

ⓐ 49 ⓑ −49

The last example showed us the difference between $(-2)^4$ and -2^4. This distinction is important to prevent future errors. The next example reminds us to multiply and divide in order left to right.

EXAMPLE 13.7.3.25

Simplify: ⓐ $8(-9)\div(-2)^3$ ⓑ $-30\div 2+(-3)(-7)$.

Answer

ⓐ

	$8(-9)\div(-2)^3$
Exponents first.	$8(-9)\div(-8)$
Multiply.	$-72\div(-8)$
Divide.	9

ⓑ

	$-30 \div 2 + (-3)(-7)$
Multiply and divide left to right, so divide first.	$-15 + (-3)(-7)$
Multiply.	$-15 + 21$
Add.	6

?

Simplify: ⓐ $12(-9) \div (-3)^3$ ⓑ $-27 \div 3 + (-5)(-6)$.

Answer

ⓐ 4 ⓑ 21

? EXAMPLE 13.7.3.27

Simplify: ⓐ $18(-4) \div (-2)^3$ ⓑ $-32 \div 4 + (-2)(-7)$.

Answer

ⓐ 9 ⓑ 6

Evaluate Variable Expressions with Integers

Remember that to evaluate an expression means to substitute a number for the variable in the expression. Now we can use negative numbers as well as positive numbers.

? EXAMPLE 13.7.3.28

Evaluate $4x^2 - 2xy + 3y^2$ when $x = 2, y = -1$.

Answer

	$4x^2 - 2xy + 3y^2$
Substitute $x = 2$, $y = -1$. Use parentheses to show multiplication.	$4(2)^2 - 2(2)(-1) + 3(-1)^2$
Simplify exponents.	$4 \cdot 4 - (-4) + 3 \cdot 1$
Multiply.	$16 - (-4) + 3$
Subtract.	$20 + 3$
Add.	23

? EXAMPLE 13.7.3.29

Evaluate: $3x^2 - 2xy + 6y^2$ when $x = 1, y = -2$.

Answer

31

? EXAMPLE 13.7.3.30

Evaluate: $4x^2 - xy + 5y^2$ when $x = -2, y = 3$.

Answer

67

Translate Phrases to Expressions with Integers

Our earlier work translating English to algebra also applies to phrases that include both positive and negative numbers.

? EXAMPLE 13.7.3.31

Translate and simplify: the sum of 8 and −12, increased by 3.

Answer

	the **sum** of -8 and -12 increased by 3
Translate.	$[8 + (-12)] + 3$
Simplify. Be careful not to confuse the brackets with an absolute value sign.	$(-4) + 3$
Add.	-1

? EXAMPLE 13.7.3.32

Translate and simplify the sum of 9 and −16, increased by 4.

Answer

$(9 + (-16)) + 4; -3$

? EXAMPLE 13.7.3.33

Translate and simplify the sum of −8 and −12, increased by 7.

Answer

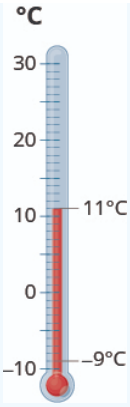
$(-8 + (-12)) + 7; -13$

Use Integers in Applications

We'll outline a plan to solve applications. It's hard to find something if we don't know what we're looking for or what to call it! So when we solve an application, we first need to determine what the problem is asking us to find. Then we'll write a phrase that gives the information to find it. We'll translate the phrase into an expression and then simplify the expression to get the answer. Finally, we summarize the answer in a sentence to make sure it makes sense.

? EXAMPLE 13.7.3.34: How to Solve Application Problems Using Integers

The temperature in Kendallville, Indiana one morning was 11 degrees. By mid-afternoon, the temperature had dropped to −9−9degrees. What was the difference in the morning and afternoon temperatures?



Answer

Step 1. Read the problem. Make sure all the words and ideas are understood.	
Step 2. Identify what we are asked to find.	The difference of the morning and afternoon temperatures
Step 3. Write a phrase that gives the information to find it.	the difference *of* 11 *and* –9
Step 4. Translate the phrase to an expression.	11 – (–9)
Step 5. Simplify the expression.	20
Step 6. Answer the question with a complete sentence.	The difference in temperatures was 20 degrees.

EXAMPLE 13.7.3.35

The temperature in Anchorage, Alaska one morning was 15 degrees. By mid-afternoon the temperature had dropped to 30 degrees below zero. What was the difference in the morning and afternoon temperatures?

Answer

The difference in temperatures was 45 degrees Fahrenheit.

EXAMPLE 13.7.3.36

The temperature in Denver was −6 degrees at lunchtime. By sunset the temperature had dropped to −15 degrees. What was the difference in the lunchtime and sunset temperatures?

Answer

The difference in temperatures was 9 degrees.

USE INTEGERS IN APPLICATIONS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are asked to find.
3. **Write a phrase** that gives the information to find it.

4. **Translate** the phrase to an expression.
5. **Simplify** the expression.
6. **Answer** the question with a complete sentence.

Access this online resource for additional instruction and practice with integers.

- Subtracting Integers with Counters

Key Concepts

- $$-a \text{ means the opposite of the number } a \quad (13.7.3.15)$$
  $$\text{The notation } -a \text{ is read as "the opposite of } a." \quad (13.7.3.16)$$
- The absolute value of a number is its distance from 0 on the number line.

  The absolute value of a number *n* is written as $|n|$ and $|n| \geq 0$ for all numbers.

  Absolute values are always greater than or equal to zero.
- $$\begin{array}{llll} \text{Parentheses} & () & \text{Braces} & \{\} \\ \text{Brackets} & [] & \text{Absolute value} & || \end{array} \quad (13.7.3.17)$$
- **Subtraction Property**
  $a-b=a+(-b)$
  Subtracting a number is the same as adding its opposite.
- For multiplication and division of two signed numbers:

Same signs	Result
• Two positives	Positive
• Two negatives	Positive

If the signs are the same, the result is positive.

Different signs	Result
• Positive and negative	Negative
• Negative and positive	Negative

If the signs are different, the result is negative.

- **Multiplication by** -1

  $-1a=-a$

  Multiplying a number by -1 gives its opposite.
- **How to Use Integers in Applications.**
  1. **Read** the problem. Make sure all the words and ideas are understood
  2. **Identify** what we are asked to find.
  3. **Write a phrase** that gives the information to find it.
  4. **Translate** the phrase to an expression.
  5. **Simplify** the expression.
  6. **Answer** the question with a complete sentence.

Glossary

absolute value

The absolute value of a number is its distance from 0 on the number line.

integers

The whole numbers and their opposites are called the integers.

negative numbers

Numbers less than 0 are negative numbers.

opposite

The opposite of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

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13.7.3E: Exercises

Practice Makes Perfect

Simplify Expressions with Absolute Value

In the following exercises, fill in $<, >$, or $=$ for each of the following pairs of numbers.

1. ⓐ $|-7|$ ___ $-|-7|$
   ⓑ 6 ___ $-|-6|$
   ⓒ $|-11|$ ___ -11
   ⓓ $-(-13)$ ___ $-|-13|$

   Answer

   ⓐ $>$ ⓑ $>$ ⓒ $>$ ⓓ $>$

2. ⓐ $-|-9|$ ___ $|-9|$
   ⓑ -8 ___ $|-8|$
   ⓒ $|-1|$ ___ -1
   ⓓ $-(-14)$ ___ $-|-14|$

3. ⓐ $-|2|$ ___ $-|-2|$
   ⓑ -12 ___ $-|-12|$
   ⓒ $|-3|$ ___ -3
   ⓓ $|-19|$ ___ $-(-19)$

   Answer

   ⓐ $=$ ⓑ $=$ ⓒ $>$ ⓓ $=$

4. ⓐ $-|-4|$ ___ $-|4|$
   ⓑ 5 ___ $-|-5|$
   ⓒ $-|-10|$ ___ -10
   ⓓ $-|-0|$ ___ $-(-0)$

In the following exercises, simplify.

5. $|15-7|-|14-6|$

   Answer

   0

6. $|17-8|-|13-4|$

7. $18-|2(8-3)|$

   Answer

   8

8. $15-|3(8-5)|$

9. $18-|12-4(4-1)+3|$

Answer

15

10. $27-|19+4(3-1)-7|$

11. $10-3|9-3(3-1)|$

Answer

1

12. $13-2|11-2(5-2)|$

Add and Subtract Integers

In the following exercises, simplify each expression.

13. ⓐ $-7+(-4)$
ⓑ $-7+4$
ⓒ $7+(-4)$.

Answer

ⓐ -11 ⓑ -3 ⓒ 3

14. ⓐ $-5+(-9)$
ⓑ $-5+9$
ⓒ $5+(-9)$

15. $48+(-16)$

Answer

32

16. $34+(-19)$

17. $-14+(-12)+4$

Answer

-22

18. $-17+(-18)+6$

19. $19+2(-3+8)$

Answer

29

20. $24+3(-5+9)$

21. ⓐ $13-7$
ⓑ $-13-(-7)$
ⓒ $-13-7$
ⓓ $13-(-7)$

Answer

ⓐ 6 ⓑ −6 ⓒ −20 ⓓ 20

22. ⓐ $15-8$
ⓑ $-15-(-8)$
ⓒ $-15-8$
ⓓ $15-(-8)$

23. $-17-42$

Answer

-59

24. $-58-(-67)$

25. $-14-(-27)+9$

Answer

22

26. $64+(-17)-9$

27. ⓐ $44-28$ ⓑ $44+(-28)$

Answer

ⓐ 16 ⓑ 16

28. ⓐ $35-16$ ⓑ $35+(-16)$

29. ⓐ $27-(-18)$ ⓑ $27+18$

Answer

ⓐ 45 ⓑ 45

30. ⓐ $46-(-37)$ ⓑ $46+37$

31. $(2-7)-(3-8)$

Answer

0

32. $(1-8)-(2-9)$

33. $-(6-8)-(2-4)$

Answer

4

34. $-(4-5)-(7-8)$

35. $25-[10-(3-12)]$

Answer

6

36. $32-[5-(15-20)]$

Multiply and Divide Integers

In the following exercises, multiply or divide.

37. ⓐ $-4\cdot 8$
ⓑ $13(-5)$
ⓒ $-24\div 6$
ⓓ $-52\div(-4)$

Answer

ⓐ -32 ⓑ -65 ⓒ -4 ⓓ 13

38. ⓐ $-3\cdot 9$
ⓑ $9(-7)$
ⓒ $35\div(-7)$
ⓓ $-84\div(-6)$

39. ⓐ $-28\div 7$
ⓑ $-180\div 15$
ⓒ $3(-13)$
ⓓ $-1(-14)$

Answer

ⓐ -4 ⓑ -12 ⓒ -39 ⓓ 14

40. ⓐ $-36\div 4$
ⓑ $-192\div 12$
ⓒ $9(-7)$
ⓓ $-1(-19)$

Simplify and Evaluate Expressions with Integers

In the following exercises , simplify each expression.

41. ⓐ $(-2)^6$ ⓑ -2^6

Answer

ⓐ 64 ⓑ -64

42. ⓐ $(-3)^5$ ⓑ -3^5

43. $5(-6)+7(-2)-3$

Answer

-47

44. $8(-4)+5(-4)-6$

45. $-3(-5)(6)$

Answer

90

46. $-4(-6)(3)$

47. $(8-11)(9-12)$

Answer

9

48. $(6-11)(8-13)$

49. $26-3(2-7)$

Answer

41

50. $23-2(4-6)$

51. $65\div(-5)+(-28)\div(-7)$

Answer

-9

52. $52\div(-4)+(-32)\div(-8)$

53. $9-2[3-8(-2)]$

Answer

-29

54. $11-3[7-4(-2)]$

55. $8-|2-4(4-1)+3|$

Answer

1

56. $7-|5-3(4-1)-6|$

57. $9-3|2(2-6)-(3-7)|$

Answer

-3

58. $5-2|2(1-4)-(2-5)|$

59. $(-3)^2-24\div(8-2)$

Answer

5

60. $(-4)^2-32\div(12-4)$

In the following exercises , evaluate each expression.

61. $y+(-14)$ when ⓐ $y=-33$ ⓑ $y=30$

Answer

ⓐ -47 ⓑ 16

62. $x+(-21)$ when ⓐ $x=-27$ ⓑ $x=44$

63. $(x+y)^2$ when $x=-3$ and $y=14$

Answer

121

64. $(y+z)^2$ when $y=-3$ and $z=15$

65. $9a-2b-8$ when $a=-6$ and $b=-3$

Answer

-56

66. $7m-4n-2$ when $m=-4$ and $n=-9$

67. $3x^2-4xy+2y^2$ when $x=-2$ and $y=-3$

Answer

6

68. $4x^2 - xy + 3y^2$ when $x = -3$ and $y = -2$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

69. the sum of 3 and −15, increased by 7

Answer

$(3 + (-15)) + 7; -5$

70. the sum of -8 and -9, increased by 23

71. ⓐ the difference of 10 and -18

ⓑ subtract 11 from -25

Answer

ⓐ $10 - (-18); 28$

ⓑ $-25 - 11; -36$

72. ⓐ the difference of -5 and -30

ⓑ subtract -6 from -13

73. the quotient of -6 and the sum of a and b

Answer

$$\frac{-6}{a+b}$$

74. the product of -13 and the difference of c and d

Use Integers in Applications

In the following exercises, solve.

75. Temperature On January 15, the high temperature in Anaheim, California, was 84°. That same day, the high temperature in Embarrass, Minnesota, was -12°. What was the difference between the temperature in Anaheim and the temperature in Embarrass?

Answer

96°

76. Temperature On January 21, the high temperature in Palm Springs, California, was 89°, and the high temperature in Whitefield, New Hampshire, was -31°. What was the difference between the temperature in Palm Springs and the temperature in Whitefield?

77. Football On the first down, the Chargers had the ball on their 25-yard line. On the next three downs, they lost 6 yards, gained 10 yards, and lost 8 yards. What was the yard line at the end of the fourth down?

Answer

21

78. Football On the first down, the Steelers had the ball on their 30-yard line. On the next three downs, they gained 9 yards, lost 14 yards, and lost 2 yards. What was the yard line at the end of the fourth down?

79. Checking Account Mayra has $124 in her checking account. She writes a check for $152. What is the new balance in her checking account?

Answer

$-\$28$

80. Checking Account Reymonte has a balance of $-\$49$ in his checking account. He deposits $281 to the account. What is the new balance?

Writing Exercises

81. Explain why the sum of -8 and 2 is negative, but the sum of 8 and -2 is positive.

Answer

Answers will vary.

82. Give an example from your life experience of adding two negative numbers.

83. In your own words, state the rules for multiplying and dividing integers.

Answer

Answers will vary.

84. Why is $-4^3 = (-4)^3$?

Self Check

ⓐ After completing the EXAMPLEs, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with absolute value.			
add and subtract integers.			
multiply and divide integers.			
simplify and evaluate expressions with integers.			
translate English phrases to algebraic expressions.			
use integers in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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13.7.4: Fractions

Learning Objectives

By the end of this section, you will be able to:

- Simplify fractions
- Multiply and divide fractions
- Add and subtract fractions
- Use the order of operations to simplify fractions
- Evaluate variable expressions with fractions

A more thorough introduction to the topics covered in this section can be found in the *Elementary Algebra* chapter, Foundations.

Simplify Fractions

A fraction is a way to represent parts of a whole. The fraction $\frac{2}{3}$ represents two of three equal parts (Figure 13.7.4.1). In the fraction $\frac{2}{3}$, the 2 is called the **numerator** and the 3 is called the **denominator**. The line is called the fraction bar.

Figure $\boldsymbol{13.7.4.1}$ *In the circle, $\frac{2}{3}$ of the circle is shaded—2 of the 3 equal parts.*

FRACTION

A **fraction** is written $\frac{a}{b}$, where $b \neq 0$ and

a is the **numerator** and b is the **denominator**.

A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

Fractions that have the same value are **equivalent fractions**. The Equivalent Fractions

Property allows us to find equivalent fractions and also simplify fractions.

If a, b, and c are numbers where $b \neq 0, c \neq 0$,

then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator.

For example,

$\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.

$\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. If an expression has fractions, it is not completely simplified until the fractions are simplified.

Sometimes it may not be easy to find common factors of the numerator and denominator. When this happens, a good idea is to factor the numerator and the denominator into prime numbers. Then divide out the common factors using the Equivalent Fractions Property.

Simplify $\frac{-315}{770}$.

Answer

Step 1. Rewrite the numerator and denominator to show the common factors. If needed, use a factor tree.	Rewrite 315 and 770 as the product of the primes.	$-\frac{315}{770}$ $-\frac{3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 5 \cdot 7 \cdot 11}$
Step 2. Simplify using the equivalent fractions property by dividing out common factors.	Mark the common factors 5 and 7. Divide out the common factors	$-\frac{3 \cdot 3 \cdot \cancel{5} \cdot \cancel{7}}{2 \cdot \cancel{5} \cdot \cancel{7} \cdot 11}$ $-\frac{3 \cdot 3}{2 \cdot 11}$
Step 3. Multiply the remaining factors, if necessary.		$-\frac{9}{22}$

Simplify $-\frac{69}{120}$.

Answer

$-\frac{23}{40}$

EXAMPLE 13.7.4.3

Simplify $-\frac{120}{192}$.

Answer

$-\frac{5}{8}$

We now summarize the steps you should follow to simplify fractions.

SIMPLIFY A FRACTION.

1. Rewrite the numerator and denominator to show the common factors.
   If needed, factor the numerator and denominator into prime numbers first.
2. Simplify using the Equivalent Fractions Property by dividing out common factors.
3. Multiply any remaining factors.

Multiply and Divide Fractions

Many people find multiplying and dividing fractions easier than adding and subtracting fractions.

To multiply fractions, we multiply the numerators and multiply the denominators.

FRACTION MULTIPLICATION

If a, b, c, and d are numbers where $b \neq 0$, and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \tag{13.7.4.1}$$

To multiply fractions, multiply the numerators and multiply the denominators.

When multiplying fractions, the properties of positive and negative numbers still apply, of course. It is a good idea to determine the sign of the product as the first step. In Example, we will multiply negative and a positive, so the product will be negative.

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $\frac{a}{1}$. So, for example, $3 = \frac{3}{1}$.

Exercise 13.7.4.4

Multiply: $-\frac{12}{5}(-20x)$.

Answer

The first step is to find the sign of the product. Since the signs are the same, the product is positive.

	$-\frac{12}{5}(-20x)$
Determine the sign of the product. The signs are the same, so the product is positive.	$\frac{12}{5}(20x)$
Write $20x$ as a fraction.	$\frac{12}{5}\left(\frac{20x}{1}\right)$
Multiply.	$\frac{12 \cdot 20x}{5 \cdot 1}$
Rewrite 20 to show the common factor 5 and divide it out.	$\frac{12 \cdot 4 \cdot \cancel{5} \cdot x}{\cancel{5} \cdot 1}$
Simplify.	$48x$

Exercise 13.7.4.5

Multiply: $\frac{1}{13}(-9a)$.

Answer

$-33a$

Exercise 13.7.4.6

Multiply: $\frac{13}{7}(-14b)$.

Answer

$-26b$

Now that we know how to multiply fractions, we are almost ready to divide. Before we can do that, we need some vocabulary. The **reciprocal** of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the

numerator. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. Since 4 is written in fraction form as $\frac{4}{1}$, the reciprocal of 4 is $\frac{1}{4}$.

To divide fractions, we multiply the first fraction by the reciprocal of the second.

FRACTION DIVISION

If a, b, c, and d are numbers where $b \neq 0, c \neq 0$, and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \tag{13.7.4.2}$$

To divide fractions, we multiply the first fraction by the **reciprocal** of the second.

We need to say $b \neq 0$, $c \neq 0$, and $d \neq 0$, to be sure we don't divide by zero!

Exercise 13.7.4.7

Find the quotient: $-\frac{7}{18} \div (-\frac{14}{27})$.

Answer

	$-\frac{7}{18} \div (-\frac{14}{27})$
To divide, multiply the first fraction by the reciprocal of the second.	$-\frac{7}{18}\left(-\frac{27}{14}\right)$
Determine the sign of the product, and then multiply.	$\frac{7 \cdot 27}{18 \cdot 14}$
Rewrite showing common factors.	$\frac{\cancel{7} \cdot \cancel{9} \cdot 3}{\cancel{9} \cdot 2 \cdot \cancel{7} \cdot 2}$
Remove common factors.	$\frac{3}{2 \cdot 2}$
Simplify.	$\frac{3}{4}$

?

Divide: $-\frac{7}{27} \div (-\frac{35}{36})$.

Answer

$\frac{4}{15}$

Exercise 13.7.4.9

Divide: $-\frac{5}{14} \div (-\frac{15}{28})$.

Answer

$\frac{2}{3}$

The numerators or denominators of some fractions contain fractions themselves. A fraction in which the numerator or the denominator is a fraction is called a **complex fraction**.

Definition: COMPLEX FRACTION

A **complex fraction** is a fraction in which the numerator or the denominator contains a fraction.

Some examples of complex fractions are:

$$\frac{\frac{6}{7}}{3} \quad \frac{\frac{3}{4}}{\frac{5}{8}} \quad \frac{\frac{x}{2}}{\frac{5}{6}} \tag{13.7.4.3}$$

To simplify a complex fraction, remember that the fraction bar means division. For example, the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ means $\frac{3}{4} \div \frac{5}{8}$.

Exercise 13.7.4.10

Simplify: $\frac{\frac{x}{2}}{\frac{xy}{6}}$.

Answer

$\frac{\frac{x}{2}}{\frac{xy}{6}}$

Rewrite as division. $\frac{x}{2} \div \frac{xy}{6}$

Multiply the first fraction by the reciprocal of the second. $\frac{x}{2} \cdot \frac{6}{xy}$

Multiply. $\frac{x \cdot 6}{2 \cdot xy}$

Look for common factors. $\frac{\cancel{x} \cdot 3 \cdot \cancel{2}}{\cancel{2} \cdot \cancel{x} \cdot y}$

Divide common factors and simplify. $\frac{3}{y}$

Simplify: $\frac{\frac{a}{8}}{\frac{ab}{6}}$.

Answer

$\frac{3}{4b}$

Simplify: $\dfrac{\frac{p}{2}}{\frac{pq}{8}}$.

Answer

$\dfrac{4}{q}$

Add and Subtract Fractions

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators right straight across. To add or subtract fractions, they must have a common denominator.

FRACTION ADDITION AND SUBTRACTION

If a, b, and c are numbers where $c \neq 0$, then

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \text{ and } \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c} \tag{13.7.4.4}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

The **least common denominator** (LCD) of two fractions is the smallest number that can be used as a common denominator of the fractions. The LCD of the two fractions is the least common multiple (LCM) of their denominators.

LEAST COMMON DENOMINATOR

The **least common denominator** (LCD) of two fractions is the least common multiple (LCM) of their denominators.

After we find the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD. Putting these steps together allows us to add and subtract fractions because their denominators will be the same!

EXAMPLE 13.7.4.13: How to Add or Subtract Fractions

Add: $\frac{7}{12}+\frac{5}{18}$.

Answer

Step 1. Do they have a common denominator? • No—rewrite each fraction with the LCD (least common denominator).	No. Find the LCD of 12, 18. We multiply the numerator and denominator of each fraction by the factor needed to get the denominator to be 36. Do not simplify the equivalent fractions! If you do, you'll get back to the original fractions and lose the common denominator!	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $LCD = 2 \cdot 2 \cdot 3 \cdot 3$ $LCD = 36$ LCD is 36. $\frac{7}{12} + \frac{5}{18}$ $\frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$ $\frac{21}{36} + \frac{10}{36}$
Step 2. Add or subtract the fractions.	Add.	$\frac{31}{36}$
Step 3. Simplify, if possible.	Since 31 is prime, its only factors are 1 and 31. Since 31 does not go into 36, the answer is simplified.	

? EXAMPLE 13.7.4.14

Add: $\frac{7}{12} + \frac{11}{15}$.

Answer

$\frac{79}{60}$

? EXAMPLE 13.7.4.15

Add: $\frac{13}{15} + \frac{17}{20}$.

Answer

$\frac{103}{60}$

ADD OR SUBTRACT FRACTIONS.

1. Do they have a common denominator?
   - Yes—go to step 2.
   - No—rewrite each fraction with the LCD (least common denominator).
     - Find the LCD.
     - Change each fraction into an equivalent fraction with the LCD as its denominator.
2. Add or subtract the fractions.
3. Simplify, if possible.

We now have all four operations for fractions. Table summarizes fraction operations.

Fraction Multiplication	Fraction Division
$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$	$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$
Multiply the numerators and multiply the denominators	Multiply the first fraction by the reciprocal of the second.
Fraction Addition	**Fraction Subtraction**
$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$	$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$
Add the numerators and place the sum over the common denominator.	Subtract the numerators and place the difference over the common denominator.
To multiply or divide fractions, an LCD is NOT needed. To add or subtract fractions, an LCD is needed.	

When starting an exercise, always identify the operation and then recall the methods needed for that operation.

?

Simplify: ⓐ $\frac{5x}{6}-\frac{3}{10}$ ⓑ $\frac{5x}{6}\cdot\frac{3}{10}$.

Answer

First ask, "What is the operation?" Identifying the operation will determine whether or not we need a common denominator. Remember, we need a common denominator to add or subtract, but not to multiply or divide.

ⓐ

What is the operation? The operation is subtraction.

Do the fractions have a common denominator? No. $\frac{5x}{6}-\frac{3}{10}$

Find the LCD of 6 and 10 The LCD is 30.

$6=2\cdot 3$

$10=2\cdot 5$

$\text{LCD}=2\cdot 3\cdot 5$

$\text{LCD}=30$

Rewrite each fraction as an equivalent fraction with the LCD. $\frac{5x\cdot 5}{6\cdot 5}-\frac{3\cdot 3}{10\cdot 3}$

$\frac{25x}{30}-\frac{9}{30}$

Subtract the numerators and place the

difference over the common denominators. $\frac{25x-9}{30}$

Simplify,if possible.There are no common factors.

The fraction is simplified.

ⓑ

What is the operation? Multiplication. $\frac{25x}{6} \cdot \frac{3}{10}$

To multiply fractions,multiply the numerators and multiply the denominators. $\frac{25x \cdot 3}{6 \cdot 10}$

Rewrite, showing common factors.

Remove common factors. $\frac{\cancel{5}x \cdot \cancel{3}}{2 \cdot \cancel{3} \cdot 2 \cdot \cancel{5}}$

Simplify. $\frac{x}{4}$

Notice, we needed an LCD to add $\frac{25x}{6} - \frac{3}{10}$, but not to multiply $\frac{25x}{6} \cdot \frac{3}{10}$.

? EXAMPLE 13.7.4.17

Simplify: ⓐ $\frac{3a}{4} - \frac{8}{9}$ ⓑ $\frac{3a}{4} \cdot \frac{8}{9}$.

Answer

ⓐ $\frac{27a-32}{36}$ ⓑ $\frac{2a}{3}$

? EXAMPLE 13.7.4.18

Simplify: ⓐ $\frac{4k}{5} - \frac{1}{6}$ ⓑ $\frac{4k}{5} \cdot \frac{1}{6}$.

Answer

ⓐ $\frac{24k-5}{30}$ ⓑ $\frac{2k}{15}$

Use the Order of Operations to Simplify Fractions

The fraction bar in a fraction acts as grouping symbol. The order of operations then tells us to simplify the numerator and then the denominator. Then we divide.

SIMPLIFY AN EXPRESSION WITH A FRACTION BAR.

1. Simplify the expression in the numerator. Simplify the expression in the denominator.
2. Simplify the fraction.

Where does the negative sign go in a fraction? Usually the negative sign is in front of the fraction, but you will sometimes see a fraction with a negative numerator, or sometimes with a negative denominator. Remember that fractions represent division. When the numerator and denominator have different signs, the quotient is negative.

$$\frac{-1}{3} = -\frac{1}{3} \quad \frac{\text{negative}}{\text{positive}} = \text{negative} \tag{13.7.4.5}$$

$$\frac{1}{-3} = -\frac{1}{3} \quad \frac{\text{positive}}{\text{negative}} = \text{negative} \tag{13.7.4.6}$$

PLACEMENT OF NEGATIVE SIGN IN A FRACTION

For any positive numbers a and b,

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b} \tag{13.7.4.7}$$

EXAMPLE 13.7.4.19

Simplify: $\frac{4(-3)+6(-2)}{-3(2)-2}$.

Answer

The fraction bar acts like a grouping symbol. So completely simplify the numerator and the denominator separately.

$$\frac{4(-3)+6(-2)}{-3(2)-2}$$

Multiply. $\frac{-12+(-12)}{-6-2}$

Simplify. $\frac{-24}{-8}$

Divide. 3

EXAMPLE 13.7.4.20

Simplify: $\frac{8(-2)+4(-3)}{-5(2)+3}$.

Answer

4

EXAMPLE 13.7.4.21

Simplify: $\frac{7(-1)+9(-3)}{-5(3)-2}$.

Answer

2

Now we'll look at complex fractions where the numerator or denominator contains an expression that can be simplified. So we first must completely simplify the numerator and denominator separately using the order of operations. Then we divide the numerator by the denominator as the fraction bar means division.

EXAMPLE 13.7.4.22: How to Simplify Complex Fractions

Simplify: $\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$.

Answer

Step 1. Simplify the numerator.	$\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$ $\frac{\frac{1}{4}}{4+3^2}$
Step 2. Simplify the denominator.	$\frac{\frac{1}{4}}{4+9}$ $\frac{\frac{1}{4}}{13}$
Step 3. Divide the numerator by the denominator. Simplify if possible	$\frac{1}{4} \div \frac{13}{1}$ $\frac{1}{4} \cdot \frac{1}{13}$ $\frac{1}{52}$

? EXAMPLE 13.7.4.23

Simplify: $\frac{\left(\frac{1}{3}\right)^2}{2^3+2}$.

Answer

$\frac{1}{90}$

? EXAMPLE 13.7.4.24

Simplify: $\frac{1+4^2}{\left(\frac{1}{4}\right)^2}$.

Answer

272

SIMPLIFY COMPLEX FRACTIONS.

1. Simplify the numerator.
2. Simplify the denominator.
3. Divide the numerator by the denominator. Simplify if possible.

? EXAMPLE 13.7.4.25

Simplify: $\frac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$.

Answer

It may help to put parentheses around the numerator and the denominator.

$$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$$

Simplify the numerator ($LCD = 6$) and simplify the denominator ($LCD = 12$). $$\frac{\left(\frac{3}{6}+\frac{4}{6}\right)}{\left(\frac{9}{12}-\frac{2}{12}\right)}$$

Simplify. $$\left(\frac{7}{6}\right)\left(\frac{7}{12}\right)$$

Divide the numerator by the denominator. $$\frac{7}{6} \div \frac{7}{12}$$

Simplify. $$\frac{7}{6} \cdot \frac{12}{7}$$

Divide out common factors. $$\frac{\cancel{7} \cdot \cancel{6} \cdot 2}{\cancel{6} \cdot \cancel{7} \cdot 1}$$

Simplify. $$2$$

? EXAMPLE 13.7.4.26

Simplify: $\dfrac{\frac{1}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$.

Answer

2

? EXAMPLE 13.7.4.27

Simplify: $\dfrac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{4}+\frac{1}{3}}$.

Answer

$\frac{2}{7}$

Evaluate Variable Expressions with Fractions

We have evaluated expressions before, but now we can evaluate expressions with fractions. Remember, to evaluate an expression, we substitute the value of the variable into the expression and then simplify.

? EXAMPLE 13.7.4.28

Evaluate $2x^2y$ when $x = \frac{1}{4}$ and $y = -\frac{2}{3}$.

Answer

Substitute the values into the expression.

	$2x^2y$
Substitute $\frac{1}{4}$ for x and $-\frac{2}{3}$ for y.	$2\left(\frac{1}{4}\right)^2\left(-\frac{2}{3}\right)$
Simplify exponents first.	$2\left(\frac{1}{16}\right)\left(-\frac{2}{3}\right)$
Multiply; divide out the common factors. Notice we write 16 as $2\cdot2\cdot4\cdot2\cdot4$ to make it easy to remove common factors.	$-\frac{\cancel{2}\cdot1\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot4\cdot3}$
Simplify.	$-\frac{1}{12}$

? EXAMPLE 13.7.4.29

Evaluate $3ab^2$ when $a=-\frac{2}{3}$ and $b=-\frac{1}{2}$.

Answer

$-\frac{1}{2}$

? EXAMPLE 13.7.4.30

Evaluate $4c^3d$ when $c=-\frac{1}{2}$ and $d=-\frac{4}{3}$.

Answer

$\frac{2}{3}$

Access this online resource for additional instruction and practice with fractions.

- Adding Fractions with Unlike Denominators

Key Concepts

- If a, b, and c are numbers where $b\neq0, c\neq0$, then
  $\frac{a}{b}=\frac{a\cdot c}{b\cdot c}$ and $\frac{a\cdot c}{b\cdot c}=\frac{a}{b}$.
- **How to simplify a fraction.**
  1. Rewrite the numerator and denominator to show the common factors.
     If needed, factor the numerator and denominator into prime numbers first.
  2. Simplify using the Equivalent Fractions Property by dividing out common factors.
  3. Multiply any remaining factors.
- If a, b, c, and d are numbers where $b\neq0$, and $d\neq0$, then
  $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$
  To multiply fractions, multiply the numerators and multiply the denominators.
- If a, b, c, and d are numbers where $b\neq0, c\neq0$, and $d\neq0$, then
  $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$
  To divide fractions, we multiply the first fraction by the reciprocal of the second.
- If a, b, and c are numbers where $c\neq0$, then

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \text{ and } \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

- **How to add or subtract fractions.**
  1. Do they have a common denominator?
     - Yes—go to step 2.
     - No—rewrite each fraction with the LCD (least common denominator).
       - Find the LCD.
       - Change each fraction into an equivalent fraction with the LCD as its denominator.
  2. Add or subtract the fractions.
  3. Simplify, if possible.
- **How to simplify an expression with a fraction bar.**
  1. Simplify the expression in the numerator. Simplify the expression in the denominator.
  2. Simplify the fraction.
- For any positive numbers a and b,

  $$\frac{-a}{b}=\frac{a}{-b}=-\frac{a}{b}$$
- **How to simplify complex fractions.**
  1. Simplify the numerator.
  2. Simplify the denominator.
  3. Divide the numerator by the denominator. Simplify if possible.

Glossary

complex fraction

A fraction in which the numerator or the denominator is a fraction is called a complex fraction.

denominator

In a fraction, written $\frac{a}{b}$, where $b \neq 0$, the denominator b is the number of equal parts the whole has been divided into.

equivalent fractions

Equivalent fractions are fractions that have the same value.

fraction

A fraction is written $\frac{a}{b}$, where $b \neq 0$, and a is the numerator and b is the denominator. A fraction represents parts of a whole.

least common denominator

The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

numerator

In a fraction, written $\frac{a}{b}$, where $b \neq 0$, the numerator a indicates how many parts are included.

reciprocal

The reciprocal of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator.

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13.7.4E: Exercises

Practice Makes Perfect

Simplify Fractions

In the following exercises, simplify.

1. $-\frac{108}{63}$

Answer

$-\frac{12}{7}$

2. $-\frac{104}{48}$

3. $\frac{120}{252}$

Answer

$\frac{10}{21}$

4. $\frac{182}{294}$

5. $\frac{14x^2}{21y}$

Answer

$\frac{2x^2}{3y}$

6. $\frac{24a}{32b^2}$

7. $-\frac{210a^2}{110b^2}$

Answer

$-\frac{21a^2}{11b^2}$

8. $-\frac{30x^2}{105y^2}$

Multiply and Divide Fractions

In the following exercises, perform the indicated operation.

9. $-\frac{3}{4}\left(-\frac{4}{9}\right)$

Answer

$\frac{1}{3}$

10. $-\frac{3}{8} \cdot \frac{4}{15}$

11. $\left(-\frac{14}{15}\right)\left(\frac{9}{20}\right)$

Answer

$-\frac{21}{50}$

12. $\left(-\frac{9}{10}\right)\left(\frac{25}{33}\right)$

13. $\left(-\frac{63}{84}\right)\left(-\frac{44}{90}\right)$

Answer

$\frac{11}{30}$

14. $\left(-\frac{33}{60}\right)\left(-\frac{40}{88}\right)$

15. $\frac{3}{7} \cdot 21n$

Answer

$9n$

16. $\frac{5}{6} \cdot 30m$

17. $\frac{3}{4} \div \frac{x}{11}$

Answer

$\frac{33}{4x}$

18. $\frac{2}{5} \div \frac{y}{9}$

19. $\frac{5}{18} \div \left(-\frac{15}{24}\right)$

Answer

$-\frac{4}{9}$

20. $\frac{7}{18} \div \left(-\frac{14}{27}\right)$

21. $\frac{8u}{15} \div \frac{12v}{25}$

Answer

$\frac{10u}{9v}$

22. $\frac{12r}{25} \div \frac{18s}{35}$

23. $\frac{3}{4} \div (-12)$

Answer

$-\frac{1}{16}$

24. $-15 \div \left(-\frac{5}{3}\right)$

In the following exercises, simplify.

25. $-\frac{\frac{8}{21}}{\frac{12}{35}}$

Answer

$-\frac{10}{9}$

26. $-\frac{\frac{9}{16}}{\frac{33}{40}}$

27. $-\frac{\frac{4}{5}}{2}$

Answer

$-\frac{2}{5}$

28. $\frac{\frac{5}{3}}{10}$

29. $\frac{\frac{m}{3}}{\frac{n}{2}}$

Answer

$\frac{2m}{3n}$

30. $\frac{-\frac{3}{8}}{-\frac{y}{12}}$

Add and Subtract Fractions

In the following exercises, add or subtract.

31. $\frac{7}{12}+\frac{5}{8}$

Answer

$\frac{29}{24}$

32. $\frac{5}{12}+\frac{3}{8}$

33. $\frac{7}{12}-\frac{9}{16}$

Answer

$\frac{1}{48}$

34. $\frac{7}{16}-\frac{5}{12}$

35. $-\frac{13}{30}+\frac{25}{42}$

Answer

$\frac{17}{105}$

36. $-\frac{23}{30}+\frac{5}{48}$

37. $-\frac{39}{56}-\frac{22}{35}$

Answer

$-\frac{53}{40}$

38. $-\frac{33}{49}-\frac{18}{35}$

39. $-\frac{2}{3}-\left(-\frac{3}{4}\right)$

Answer

$\frac{1}{12}$

40. $-\frac{3}{4}-\left(-\frac{4}{5}\right)$

41. $\frac{x}{3}+\frac{1}{4}$

Answer

$\frac{4x+3}{12}$

42. $\frac{x}{5}-\frac{1}{4}$

43. ⓐ $\frac{2}{3}+\frac{1}{6}$
ⓑ $\frac{2}{3}\div\frac{1}{6}$

Answer

ⓐ $\frac{5}{6}$ ⓑ 4

44. ⓐ $-\frac{2}{5}-\frac{1}{8}$
ⓑ $-\frac{2}{5}\cdot\frac{1}{8}$

45. ⓐ $\frac{5n}{6}\div\frac{8}{15}$
ⓑ $\frac{5n}{6}-\frac{8}{15}$

Answer

ⓐ $\frac{25n}{16}$ ⓑ $\frac{25n-16}{30}$

46. ⓐ $\frac{3a}{8}\div\frac{7}{12}$
ⓑ $\frac{3a}{8}-\frac{7}{12}$

47. ⓐ $-\frac{4x}{9}-\frac{5}{6}$
ⓑ $-\frac{4k}{9}\cdot\frac{5}{6}$

Answer

ⓐ $\frac{-8x-15}{18}$ ⓑ $-\frac{10k}{27}$

48. ⓐ $-\frac{3y}{8}-\frac{4}{3}$

ⓑ $-\frac{3y}{8}\cdot\frac{4}{3}$

49. ⓐ $-\frac{5a}{3}+\left(-\frac{10}{6}\right)$

ⓑ $-\frac{5a}{3}\div\left(-\frac{10}{6}\right)$

Answer

ⓐ $\frac{-5(a+1)}{3}$ ⓑ a

50. ⓐ $\frac{2b}{5}+\frac{8}{15}$

ⓑ $\frac{2b}{5}\div\frac{8}{15}$

Use the Order of Operations to Simplify Fractions

In the following exercises, simplify.

51. $\frac{5\cdot 6-3\cdot 4}{4\cdot 5-2\cdot 3}$

Answer

$\frac{9}{7}$

52. $\frac{8\cdot 9-7\cdot 6}{5\cdot 6-9\cdot 2}$

53. $\frac{5^2-3^2}{3-5}$

Answer

-8

54. $\frac{6^2-4^2}{4-6}$

55. $\frac{7\cdot 4-2(8-5)}{9\cdot 3-3\cdot 5}$

Answer

$\frac{11}{6}$

56. $\frac{9\cdot 7-3(12-8)}{8\cdot 7-6\cdot 6}$

57. $\dfrac{9(8-2)-3(15-7)}{6(7-1)-3(17-9)}$

Answer

$\dfrac{5}{2}$

58. $\dfrac{8(9-2)-4(14-9)}{7(8-3)-3(16-9)}$

59. $\dfrac{2^3+4^2}{\left(\frac{2}{3}\right)^2}$

Answer

54

60. $\dfrac{3^3-3^2}{\left(\frac{3}{4}\right)^2}$

61. $\dfrac{\left(\frac{3}{5}\right)^2}{\left(\frac{3}{7}\right)^2}$

Answer

$\dfrac{49}{25}$

62. $\dfrac{\left(\frac{3}{4}\right)^2}{\left(\frac{5}{8}\right)^2}$

63. $\dfrac{2}{\frac{1}{3}+\frac{1}{5}}$

Answer

$\dfrac{15}{4}$

64. $\dfrac{5}{\frac{1}{4}+\frac{1}{3}}$

65. $\dfrac{\frac{7}{8}-\frac{2}{3}}{\frac{1}{2}+\frac{3}{8}}$

Answer

$\frac{5}{21}$

66. $\dfrac{\frac{3}{4}-\frac{3}{5}}{\frac{1}{4}+\frac{2}{5}}$

Mixed Practice

In the following exercises, simplify.

67. $-\frac{3}{8} \div \left(-\frac{3}{10}\right)$

Answer

$\frac{5}{4}$

68. $-\frac{3}{12} \div \left(-\frac{5}{9}\right)$

69. $-\frac{3}{8}+\frac{5}{12}$

Answer

$\frac{1}{24}$

70. $-\frac{1}{8}+\frac{7}{12}$

71. $-\frac{7}{15}-\frac{y}{4}$

Answer

$\frac{-28-15y}{60}$

72. $-\frac{3}{8}-\frac{x}{11}$

73. $\frac{11}{12a} \cdot \frac{9a}{16}$

Answer

$\frac{33}{64}$

74. $\frac{10y}{13} \cdot \frac{8}{15y}$

75. $\frac{1}{2}+\frac{2}{3} \cdot \frac{5}{12}$

Answer

$\frac{7}{9}$

76. $\frac{1}{3}+\frac{2}{5}\cdot\frac{3}{4}$

77. $1-\frac{3}{5}\div\frac{1}{10}$

Answer

-5

78. $1-\frac{5}{6}\div\frac{1}{12}$

79. $\frac{3}{8}-\frac{1}{6}+\frac{3}{4}$

Answer

$\frac{23}{24}$

80. $\frac{2}{5}+\frac{5}{8}-\frac{3}{4}$

81. $12\left(\frac{9}{20}-\frac{4}{15}\right)$

Answer

$\frac{11}{5}$

82. $8\left(\frac{15}{16}-\frac{5}{6}\right)$

83. $\frac{\frac{5}{8}+\frac{1}{6}}{\frac{19}{24}}$

Answer

1

84. $\frac{\frac{1}{6}+\frac{3}{10}}{\frac{14}{30}}$

85. $\left(\frac{5}{9}+\frac{1}{6}\right)\div\left(\frac{2}{3}-\frac{1}{2}\right)$

Answer

$\frac{13}{3}$

86. $\left(\frac{3}{4}+\frac{1}{6}\right) \div \left(\frac{5}{8}-\frac{1}{3}\right)$

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

87. $\frac{7}{10}-w$ when ⓐ $w=\frac{1}{2}$ ⓑ $w=-\frac{1}{2}$

Answer

ⓐ $\frac{1}{5}$ ⓑ $\frac{6}{5}$

88. $512-w$ when ⓐ $w=\frac{1}{4}$ ⓑ $w=-\frac{1}{4}$

89. $2x^2y^3$ when $x=-\frac{2}{3}$ and $y=-\frac{1}{2}$

Answer

$-\frac{1}{9}$

90. $8u^2v^3$ when $u=-\frac{3}{4}$ and $v=-\frac{1}{2}$

91. $\frac{a+b}{a-b}$ when $a=-3$ and $b=8$

Answer

$-\frac{5}{11}$

92. $\frac{r-s}{r+s}$ when $r=10$ and $s=-5$

Writing Exercises

93. Why do you need a common denominator to add or subtract fractions? Explain.

Answer

Answers will vary.

94. How do you find the LCD of 2 fractions?

95. Explain how you find the reciprocal of a fraction.

Answer

Answers will vary.

96. Explain how you find the reciprocal of a negative number.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify fractions.			
multiply and divide fractions.			
add and subtract fractions.			
use the order of operations to simplify fractions.			
evaluate variable expressions with fractions.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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13.7.5: Decimals

Learning Objectives

By the end of this section, you will be able to:

- Round decimals
- Add and subtract decimals
- Multiply and divide decimals
- Convert decimals, fractions, and percents
- Simplify expressions with square roots
- Identify integers, rational numbers, irrational numbers, and real numbers
- Locate fractions and decimals on the number line

A more thorough introduction to the topics covered in this section can be found in the *Elementary Algebra* chapter, Foundations.

Round Decimals

Decimals are another way of writing fractions whose denominators are powers of ten.

$$\begin{aligned} 0.1 &= \frac{1}{10} && \text{is “one tenth”} \\ 0.01 &= \frac{1}{100} && \text{is “one hundredth”} \\ 0.001 &= \frac{1}{1000} && \text{is “one thousandth”} \\ 0.0001 &= \frac{1}{10,000} && \text{is “one ten-thousandth”} \end{aligned} \tag{13.7.5.1}$$

Just as in whole numbers, each digit of a decimal corresponds to the place value based on the powers of ten. Figure shows the names of the place values to the left and right of the decimal point.

Place Value											
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
						.					

Figure 1.

When we work with decimals, it is often necessary to round the number to the nearest required place value. We summarize the steps for rounding a decimal here.

ROUND DECIMALS.

1. Locate the given place value and mark it with an arrow.
2. Underline the digit to the right of the place value.
3. Is the underlined digit greater than or equal to 5?
   - Yes: add 1 to the digit in the given place value.
   - No: do not change the digit in the given place value
4. Rewrite the number, deleting all digits to the right of the rounding digit.

EXAMPLE 13.7.5.1

Round 18.379 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Answer

Round 18.379.

ⓐ to the nearest hundredth

Locate the hundredths place with an arrow.	hundredths place ↓ 18.379
Underline the digit to the right of the given place value.	hundredths place ↓ 18.379
Because 9 is greater than or equal to 5, add 1 to the 7.	18.379 add 1 ↗ delete
Rewrite the number, deleting all digits to the right of the rounding digit.	18.38
Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest hundredth is 18.38.

ⓑ to the nearest tenth

Locate the tenths place with an arrow.	tenths place ↓ 18.379
Underline the digit to the right of the given place value.	tenths place ↓ 18.379
Because 7 is greater than or equal to 5, add 1 to the 3.	18.379 add 1 ↗ delete
Rewrite the number, deleting all digits to the right of the rounding digit.	18.4
Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest tenth is 18.4.

ⓒ to the nearest whole number

Locate the ones place with an arrow.	ones place ↓ 18.379
Underline the digit to the right of the given place value.	ones place ↓ 18.379
Since 3 is not greater than or equal to 5, do not add 1 to the 8.	18.379 do not add 1 ↗ delete
Rewrite the number, deleting all digits to the right of the rounding digit.	18

So, 18.379 rounded to the nearest whole number is 18.

EXAMPLE 13.7.5.2

Round 6.582 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Answer

ⓐ 6.58 ⓑ 6.6 ⓒ 7

EXAMPLE 13.7.5.3

Round 15.2175 to the nearest ⓐ thousandth ⓑ hundredth ⓒ tenth.

Answer

ⓐ 15.218 ⓑ 15.22

ⓒ 15.2

Add and Subtract Decimals

To add or subtract decimals, we line up the decimal points. By lining up the decimal points this way, we can add or subtract the corresponding place values. We then add or subtract the numbers as if they were whole numbers and then place the decimal point in the sum.

ADD OR SUBTRACT DECIMALS.

1. Determine the sign of the sum or difference.
2. Write the numbers so the decimal points line up vertically.
3. Use zeros as placeholders, as needed.
4. Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.
5. Write the sum or difference with the appropriate sign.

EXAMPLE 13.7.5.4

Add or subtract: ⓐ $-23.5-41.38$ ⓑ $14.65-20$.

Answer

ⓐ

$-23.5-41.38$

The difference will be negative. To subtract, we add the numerals. Write the numbers so the decimal points line up vertically.

$$\begin{array}{r} 23.5 \\ +41.38 \\ \hline \end{array}$$

Put 0 as a placeholder after the 5 in 23.5.
Remember, $\frac{5}{10}=\frac{50}{100}$ so $0.5=0.50$.

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline \end{array}$$

Add the numbers as if they were whole numbers. Then place the decimal point in the sum.

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline 64.88 \end{array}$$

Write the result with the correct sign. $64.88-23.5-41.38=-64.88$

ⓑ

$14.65-20$

The difference will be negative. To subtract, we subtract 14.65 from 20.

Write the numbers so the decimal points line up vertically.

$$\begin{array}{r} 20 \\ -14.65 \\ \hline \end{array}$$

Remember, 20 is a whole number, so place the decimal point after the 0.

Put in zeros to the right as placeholders.

$$\begin{array}{r} 20.00 \\ -14.65 \\ \hline \end{array}$$

Subtract and place the decimal point in the answer.

$$\begin{array}{rrrrr} & 9 & & 9 & \\ 1 & \cancel{0} & & \cancel{0} & 10 \\ 2 & 0 & . & 0 & 0 \\ -1 & 4 & . & 6 & 5 \\ \hline & 5 & . & 3 & 5 \end{array}$$

Write the result with the correct sign. $14.65-20=-5.35$

? EXAMPLE 13.7.5.5

Add or subtract: ⓐ $-4.8-11.69$ ⓑ $9.58-10$.

Answer

ⓐ -16.49 ⓑ -0.42

? EXAMPLE 13.7.5.6

Add or subtract: ⓐ $-5.123-18.47$ ⓑ $37.42-50$.

Answer

ⓐ -23.593 ⓑ -12.58

Multiply and Divide Decimals

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. We multiply the numbers temporarily ignoring the decimal point and then count the number of decimal points in the factors and that sum tells us the number of decimal places in the product. Finally, we write the product with the appropriate sign.

MULTIPLY DECIMALS.

1. Determine the sign of the product.
2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
3. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
4. Write the product with the appropriate sign.

?

Multiply: $(-3.9)(4.075)$.

Answer

	$(-3.9)(4.075)$
The signs are different. The product will be negative.	The product will be negative.
Write in vertical format, lining up the numbers on the right.	4.075 × 3.9
Multiply.	4.075 × 3.9 36675 12225 158925
Add the number of decimal places in the factors (1 + 3). Place the decimal point 4 places from the right. (–3.9) (4.075) 1 place 3 places	4.075 × 3.9 36675 12225 15.8925 4 places
The signs are the different, so the product is negative.	$(-3.9)(4.075)=-15.8925$

Multiply: $-4.5(6.107)$

Answer

-27.4815

EXAMPLE 13.7.5.9

Multiply: $-10.79(8.12)$

Answer

-87.6148

Often, especially in the sciences, you will multiply decimals by powers of 10 (10, 100, 1000, etc). If you multiply a few products on paper, you may notice a pattern relating the number of zeros in the power of 10 to number of decimal places we move the decimal point to the right to get the product.

MULTIPLY A DECIMAL BY A POWER OF TEN.

1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
2. Add zeros at the end of the number as needed.

EXAMPLE 13.7.5.10

Multiply: 5.63 by ⓐ 10 ⓑ 100 ⓒ 1000.

Answer

By looking at the number of zeros in the multiple of ten, we see the number of places we need to move the decimal to the right.

ⓐ

	5.63 (10)
There is 1 zero in 10, so move the decimal point 1 place to the right.	5.63
	5.63

ⓑ

	5.63(100)
There are 2 zeroes in 100, so move the decimal point 2 places to the right.	5.63
	563

ⓒ

	5.63(1,000)
There are 3 zeroes in 1,000, so move the decimal point 3 place to the right.	5.63

A zero must be added to the end.	5,630

? EXAMPLE 13.7.5.11

Multiply 2.58 by ⓐ 10 ⓑ 100 ⓒ 1000.

Answer

ⓐ 25.8 ⓑ 258 ⓒ 2,580

? EXAMPLE 13.7.5.12

Multiply 14.2 by ⓐ 10 ⓑ 100 ⓒ 1000.

Answer

ⓐ 142 ⓑ 1,420 ⓒ 14,200

Just as with multiplication, division of signed decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed and the sign of the quotient. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

We review the notation and vocabulary for division:

$$\underset{\text{dividend}}{a} \div \underset{\text{divisor}}{b} = \underset{\text{quotient}}{c} \qquad \begin{array}{r} \overset{\text{quotient}}{c} \\ \text{divisor } b \overline{)\, a} \\ \text{dividend} \end{array}$$

We'll write the steps to take when dividing decimals for easy reference.

DIVIDE DECIMALS.

1. Determine the sign of the quotient.
2. Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places—adding zeros as needed.
3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
4. Write the quotient with the appropriate sign.

? EXAMPLE 13.7.5.13

Divide: $-25.65 \div (-0.06)$.

Answer

Remember, you can "move" the decimals in the divisor and dividend because of the Equivalent Fractions Property.

	–25.65 ÷ (–0.06)
The signs are the same.	The quotient is positive.
Make the divisor a whole number by "moving" the decimal point all the way to the right.	
"Move" the decimal point in the dividend the same number of places.	$0.06\overline{)25.65}$

Divide. Place the decimal point in the quotient above the decimal point in the dividend.	427.5 006.)2565.0 −24 16 −12 45 −42 30 30
Write the quotient with the appropriate sign.	−25.65 ÷ (−0.06) = 427.5

EXAMPLE 13.7.5.14

Divide: $-23.492 \div (-0.04)$.

Answer

587.3

EXAMPLE 13.7.5.15

Divide: $-4.11 \div (-0.12)$.

Answer

34.25

Convert Decimals, Fractions, and Percents

In our work, it is often necessary to change the form of a number. We may have to change fractions to decimals or decimals to percent.

We convert decimals into fractions by identifying the place value of the last (farthest right) digit. In the decimal 0.03. the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03.

$$0.03 = \frac{3}{100} \tag{13.7.5.2}$$

The steps to take to convert a decimal to a fraction are summarized in the procedure box.

CONVERT A DECIMAL TO A PROPER FRACTION AND A FRACTION TO A DECIMAL.

1. To convert a decimal to a proper fraction, determine the place value of the final digit.
2. Write the fraction.
   - numerator—the "numbers" to the right of the decimal point
   - denominator—the place value corresponding to the final digit
3. To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

EXAMPLE 13.7.5.16

Write: ⓐ 0.374 as a fraction ⓑ $-\frac{5}{8}$ as a decimal.

Answer

ⓐ

	0.374
Determine the place value of the final digit.	0.3 tenths, 7 hundredths, 4 thousandths

Write the fraction for 0.374: The numerator is 374. The denominator is 1,000.	$\frac{374}{1000}$
Simplify the fraction.	$\frac{2 \cdot 187}{2 \cdot 500}$
Divide out the common factors.	$\frac{187}{500}$
	so, $0.374 = \frac{187}{500}$

ⓑ Since a fraction bar means division, we begin by writing the fraction $\frac{5}{8}$ as $8\overline{)5}$. Now divide.

$$\begin{array}{r} 0.625 \\ 8\overline{)5.000} \\ \underline{48} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ \end{array}$$

so, $-\frac{5}{8} = -0.625$

? EXAMPLE 13.7.5.17

Write: ⓐ 0.234 as a fraction ⓑ $-\frac{7}{8}$ as a decimal.

Answer

ⓐ $\frac{117}{500}$ ⓑ -0.875

? EXAMPLE 13.7.5.18

Write: ⓐ 0.024 as a fraction ⓑ $-\frac{3}{8}$ as a decimal.

Answer

ⓐ $\frac{3}{125}$ ⓑ -0.375

A **percent** is a ratio whose denominator is 100. Percent means per hundred. We use the percent symbol, %, to show percent. Since a percent is a ratio, it can easily be expressed as a fraction. Percent means per 100, so the denominator of the fraction is 100. We then change the fraction to a decimal by dividing the numerator by the denominator. After doing this many times, you may see the pattern.

To convert a percent number to a decimal number, we move the decimal point two places to the left.

6%	78%	2.7%	135%
0.06	0.78	0.027	1.35

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent. After many conversions, you may recognize the pattern.

To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.

0.05	0.83	1.05	0.075	0.3
5%	83%	105%	7.5%	30%

CONVERT A PERCENT TO A DECIMAL AND A DECIMAL TO A PERCENT.

1. To convert a percent to a decimal, move the decimal point two places to the left after removing the percent sign.
2. To convert a decimal to a percent, move the decimal point two places to the right and then add the percent sign.

EXAMPLE 13.7.5.19

Convert each:

ⓐ percent to a decimal: 62%, 135%, and 13.7%.

ⓑ decimal to a percent: 0.51, 1.25, and 0.093.

Answer

ⓐ

	62%	135%	35.7%
Move the decimal point two places to the left.	0.62	1.35	0.357

ⓑ

	0.51	1.25	0.093
Move the decimal point two places to the right.	51%	125%	9.3%

EXAMPLE 13.7.5.20

Convert each:

ⓐ percent to a decimal: 9%, 87%, and 3.9%.

ⓑ decimal to a percent: 0.17, 1.75, and 0.0825.

Answer

ⓐ 0.09, 0.87, 0.039 ⓑ 17%, 175%, 8.25%

EXAMPLE 13.7.5.21

Convert each:

ⓐ percent to a decimal: 3%, 91%, and 8.3%.

ⓑ decimal to a percent: 0.41, 2.25, and 0.0925.

Answer

ⓐ 0.03, 0.91, 0.083 ⓑ 41%, 225%, 9.25%

Simplify Expressions with Square Roots

Remember that when a number n is multiplied by itself, we write n^2 and read it "n squared." The result is called the **square of a number** n. For example, $\frac{8}{2}$ is read "8 squared" and 64 is called the *square* of 8. Similarly, 121 is the square of 11 because 11^2 is 121. It will be helpful to learn to recognize the perfect square numbers.

SQUARE OF A NUMBER

If $n^2 = m$, then m is the **square** of n.

What about the squares of negative numbers? We know that when the signs of two numbers are the same, their product is positive. So the square of any negative number is also positive.

$$(-3)^2 = 9 \qquad (-8)^2 = 64 \qquad (-11)^2 = 121 \qquad (-15)^2 = 225 \tag{13.7.5.3}$$

Because $10^2 = 100$, we say 100 is the square of 10. We also say that 10 is a *square root* of 100. A number whose square is m is called a **square root of a number** m.

SQUARE ROOT OF A NUMBER

If $n^2 = m$, then n is a **square root** of m.

Notice $(-10)^2 = 100$ also, so -10 is also a square root of 100. Therefore, both 10 and -10 are square roots of 100. So, every positive number has two square roots—one positive and one negative. The radical sign, $\sqrt{m}$, denotes the positive square root. The positive square root is called the **principal square root**. When we use the radical sign that always means we want the principal square root.

SQUARE ROOT NOTATION

$\sqrt{m}$ is read "the square root of mm."

radical sign → $\sqrt{m}$ ← radicand

If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

The square root of m, $\sqrt{m}$, is the positive number whose square is m.

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{100} = 10$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{100} = -10$. We read $-\sqrt{100}$ as "the opposite of the principal square root of 10."

Exercise 13.7.5.22

Simplify: ⓐ $\sqrt{25}$ ⓑ $\sqrt{121}$ ⓒ $-\sqrt{144}$.

Answer

ⓐ

	$\sqrt{25}$
Since $5^2 = 25$	5

ⓑ

	$\sqrt{121}$
Since $11^2 = 121$	11

ⓒ

	$-\sqrt{144}$
The negative is in front of the radical sign.	-12

Exercise 13.7.5.23

Simplify: ⓐ $\sqrt{36}$ ⓑ $\sqrt{169}$ ⓒ $-\sqrt{225}$

Answer

ⓐ 6 ⓑ 13 ⓒ −15

Exercise 13.7.5.24

Simplify: ⓐ $\sqrt{16}$ ⓑ $\sqrt{196}$ ⓒ $-\sqrt{100}$

Answer

ⓐ 4 ⓑ 14 ⓒ −10

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

We have already described numbers as *counting numbers*, *whole numbers*, and *integers*. What is the difference between these types of numbers? Difference could be confused with subtraction. How about asking how we distinguish between these types of numbers?

$$\begin{array}{ll} \text{Counting numbers} & 1, 2, 3, 4, \ldots . \\ \text{Whole numbers} & 0, 1, 2, 3, 4, \ldots . \\ \text{Integers} & \ldots . -3, -2, -1, 0, 1, 2, 3, \ldots . \end{array} \tag{13.7.5.4}$$

What type of numbers would we get if we started with all the integers and then included all the fractions? The numbers we would have form the set of rational numbers. A **rational number** is a number that can be written as a ratio of two integers.

In general, any decimal that ends after a number of digits (such as 7.3 or −1.2684) is a rational number. We can use the place value of the last digit as the denominator when writing the decimal as a fraction. The decimal for $\frac{1}{3}$ is the number $0.\overline{3}$. The bar over the 3 indicates that the number 3 repeats infinitely. Continuously has an important meaning in calculus. The number(s) under the bar is called the repeating block and it repeats continuously.

Since all integers can be written as a fraction whose denominator is 1, the integers (and so also the counting and whole numbers. are rational numbers.

Every rational number can be written both as a ratio of integers $\frac{p}{q}$, where p and q are integers and $q \neq 0$, and as a decimal that stops or repeats.

RATIONAL NUMBER

A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Its decimal form stops or repeats.

Are there any decimals that do not stop or repeat? Yes! The number $\pi\pi$ (the Greek letter *pi*, pronounced “pie”), which is very important in describing circles, has a decimal form that does not stop or repeat. We use three dots (…) to indicate the decimal does not stop or repeat.

$$\pi = 3.141592654... \tag{13.7.5.5}$$

The square root of a number that is not a perfect square is a decimal that does not stop or repeat.

A numbers whose decimal form does not stop or repeat cannot be written as a fraction of integers. We call this an **irrational number**.

IRRATIONAL NUMBER

An **irrational number** is a number that cannot be written as the ratio of two integers.

Its decimal form does not stop and does not repeat.

Let's summarize a method we can use to determine whether a number is rational or irrational.

RATIONAL OR IRRATIONAL

If the decimal form of a number

- *repeats or stops*, the number is a **rational number**.
- *does not repeat and does not stop*, the number is an **irrational number.**

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. The irrational numbers are numbers whose decimal form does not stop and does not repeat. When we put together the rational numbers and the irrational numbers, we get the set of **real numbers**.

REAL NUMBER

A **real number** is a number that is either rational or irrational.

Later in this course we will introduce numbers beyond the real numbers. Figure illustrates how the number sets we've used so far fit together.

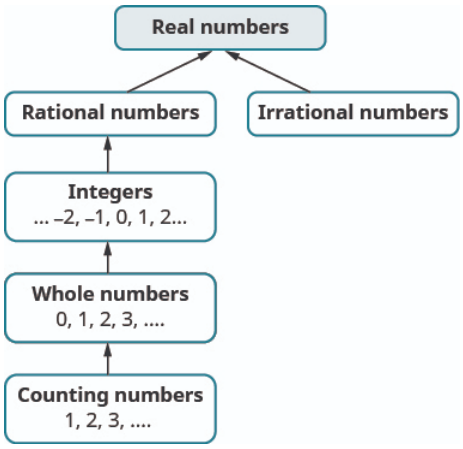


Figure 2. This chart shows the number sets that make up the set of real numbers.

Does the term "real numbers" seem strange to you? Are there any numbers that are not "real," and, if so, what could they be? Can we simplify $-\sqrt{25}$? Is there a number whose square is -25?

$$()^2 = -25? \tag{13.7.5.6}$$

None of the numbers that we have dealt with so far has a square that is -25. Why? Any positive number squared is positive. Any negative number squared is positive. So we say there is no real number equal to $\sqrt{-25}$. The square root of a negative number is not a real number.

Exercise 13.7.5.25

Given the numbers $-7, \frac{14}{5}, 8, \sqrt{5}, 5.9, -\sqrt{64}$ list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔreal numbers.

Answer

ⓐ Remember, the whole numbers are $0, 1, 2, 3, \ldots$,so 8 is the only whole number given.

ⓑ The integers are the whole numbers and their opposites (which includes 0). So the whole number 8 is an integer, and -7 is the opposite of a whole number so it is an integer, too. Also, notice that 64 is the square of 8 so $-\sqrt{64}=-8$. So the integers are $-7, 8$, and $-\sqrt{64}$.

ⓒ Since all integers are rational, then $-7, 8$, and $-\sqrt{64}$ are rational. Rational numbers also include fractions and decimals that repeat or stop, so $\frac{14}{5}$ and 5.9 are rational. So the list of rational numbers is -7, $\frac{14}{5}, 8, 5.9$,and $-\sqrt{64}$.

ⓓ Remember that 5 is not a perfect square, so $\sqrt{5}$ is irrational.

ⓔ All the numbers listed are real numbers.

? Exercise 13.7.5.26

Given the numbers $-3, -\sqrt{2}, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers.

Answer

ⓐ $4, \sqrt{49}$ ⓑ $-3, 4, \sqrt{49}$

ⓒ $-3, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$ⓓ $-\sqrt{2}$

ⓔ $-3, -\sqrt{2}, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$

? Exercise 13.7.5.27

Given numbers $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975\ldots$, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers.

Answer

ⓐ $6, \sqrt{121}$

ⓑ $-\sqrt{25}, -1, 6, \sqrt{121}$

ⓒ $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}$

ⓓ $2.041975\ldots$

ⓔ $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975\ldots$

Locate Fractions and Decimals on the Number Line

We now want to include fractions and decimals on the number line. Let's start with fractions and locate $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5$ and $\frac{8}{3}$ on the number line.

We'll start with the whole numbers 3 and -5 because they are the easiest to plot. See Figure.

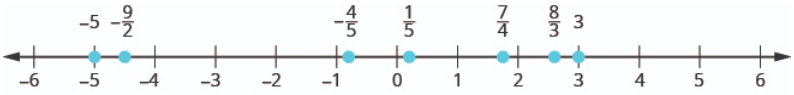
The proper fractions listed are $\frac{1}{5}$ and $-\frac{4}{5}$. We know the proper fraction $\frac{1}{5}$ has value less than one and so would be located between 0 and 1. The denominator is 5, so we divide the unit from 0 to 1 into 5 equal parts $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. We plot $\frac{1}{5}$.

Similarly, $-\frac{4}{5}$ is between 0 and -1. After dividing the unit into 5 equal parts we plot $-\frac{4}{5}$.

Finally, look at the improper fractions $\frac{7}{4}, \frac{9}{2}, \frac{8}{3}$. Locating these points may be easier if you change each of them to a mixed number.

$$\frac{7}{4}=1\frac{3}{4} \qquad -\frac{9}{2}=-4\frac{1}{2} \qquad \frac{8}{3}=2\frac{2}{3} \tag{13.7.5.7}$$

Figure shows the number line with all the points plotted.



Exercise 13.7.5.28

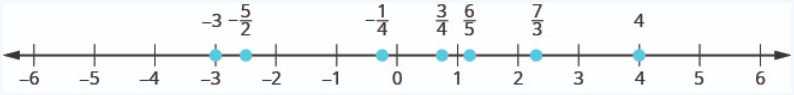
Locate and label the following on a number line: $4, \frac{3}{4}, -\frac{1}{4}, -3, \frac{6}{5}, -\frac{5}{2}$, and $\frac{7}{3}$.

Answer

Locate and plot the integers, $4, -3$.

Locate the proper fraction $\frac{3}{4}$ first. The fraction $\frac{3}{4}$ is between 0 and 1. Divide the distance between 0 and 1 into four equal parts, then we plot $\frac{3}{4}$. Similarly plot $-\frac{1}{4}$.

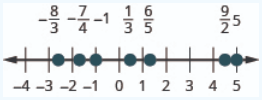
Now locate the improper fractions $\frac{6}{5}, -\frac{5}{2}$, and $\frac{7}{3}$. It is easier to plot them if we convert them to mixed numbers and then plot them as described above: $\frac{6}{5} = 1\frac{1}{5}, -\frac{5}{2} = -2\frac{1}{2}, \frac{7}{3} = 2\frac{1}{3}$.



Exercise 13.7.5.29

Locate and label the following on a number line: $-1, \frac{1}{3}, \frac{6}{5}, -\frac{7}{4}, \frac{9}{2}, 5, -\frac{8}{3}$.

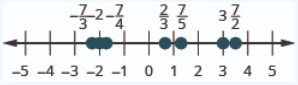
Answer



Exercise 13.7.5.30

Locate and label the following on a number line: $-2, \frac{2}{3}, \frac{7}{5}, -\frac{7}{4}, \frac{7}{2}, 3, -\frac{7}{3}$.

Answer



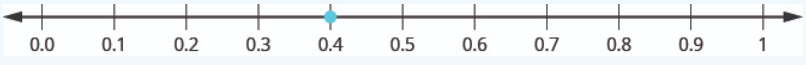
Exercise 13.7.5.31

Since decimals are forms of fractions, locating decimals on the number line is similar to locating fractions on the number line.

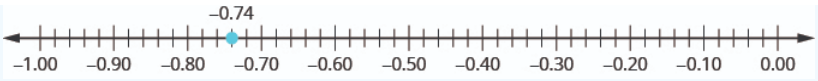
Locate on the number line: ⓐ 0.4 ⓑ −0.74.

Answer

ⓐ The decimal number 0.4 is equivalent to $\frac{4}{10}$, a proper fraction, so 0.4 is located between 0 and 1. On a number line, divide the interval between 0 and 1 into 10 equal parts. Now label the parts 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. We write 0 as 0.0 and 1 as 1.0, so that the numbers are consistently in tenths. Finally, mark 0.4 on the number line.



ⓑ The decimal -0.74 is equivalent to $-\frac{74}{100}$, so it is located between 0 and .−1. On a number line, mark off and label the hundredths in the interval between 0 and −1.



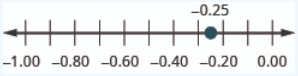
Exercise 13.7.5.32

Locate on the number line: ⓐ 0.6 ⓑ -0.25.

Answer

ⓐ

ⓑ

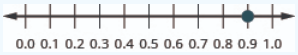


Exercise 13.7.5.33

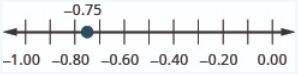
Locate on the number line: ⓐ $0.90.9$ ⓑ $-0.75.-0.75$.

Answer

ⓐ



ⓑ



Access this online resource for additional instruction and practice with decimals.

- Arithmetic Basics: Dividing Decimals

Key Concepts

- **How to round decimals.**
  1. Locate the given place value and mark it with an arrow.
  2. Underline the digit to the right of the place value.
  3. Is the underlined digit greater than or equal to 5?
     - Yes: add 1 to the digit in the given place value.
     - No: do not change the digit in the given place value
  4. Rewrite the number, deleting all digits to the right of the rounding digit.
- **How to add or subtract decimals.**
  1. Determine the sign of the sum or difference.
  2. Write the numbers so the decimal points line up vertically.
  3. Use zeros as placeholders, as needed.
  4. Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.
  5. Write the sum or difference with the appropriate sign
- **How to multiply decimals.**
  1. Determine the sign of the product.

2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
3. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
4. Write the product with the appropriate sign.

- **How to multiply a decimal by a power of ten.**
  1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
  2. Add zeros at the end of the number as needed.
- **How to divide decimals.**
  1. Determine the sign of the quotient.
  2. Make the divisor a whole number by "moving" the decimal point all the way to the right. "Move" the decimal point in the dividend the same number of places—adding zeros as needed.
  3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
  4. Write the quotient with the appropriate sign.
- **How to convert a decimal to a proper fraction and a fraction to a decimal.**
  1. To convert a decimal to a proper fraction, determine the place value of the final digit.
  2. Write the fraction.
     - numerator—the "numbers" to the right of the decimal point
     - denominator—the place value corresponding to the final digit
  3. To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.
- **How to convert a percent to a decimal and a decimal to a percent.**
  1. To convert a percent to a decimal, move the decimal point two places to the left after removing the percent sign.
  2. To convert a decimal to a percent, move the decimal point two places to the right and then add the percent sign.
- **Square Root Notation** $\sqrt{m}$ is read "the square root of *m*." If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$. The square root of *m*, $\sqrt{m}$, is the positive number whose square is *m*.
- **Rational or Irrational** If the decimal form of a number
  - *repeats or stops*, the number is a rational number.
  - *does not repeat and does not stop*, the number is an irrational number.
- **Real Numbers**

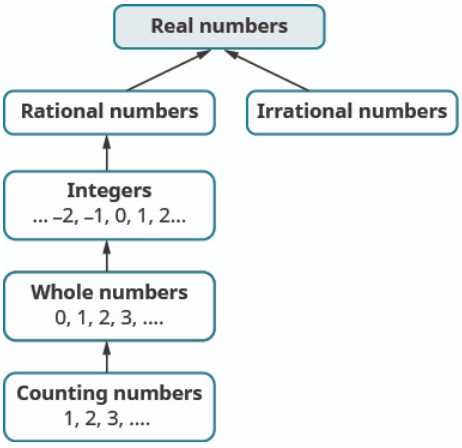


Figure 4.

Glossary

irrational number

An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

percent

A percent is a ratio whose denominator is 100.

principal square root

The positive square root is called the principal square root.

rational number

A rational number is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Its decimal form stops or repeats.

real number

A real number is a number that is either rational or irrational.

square of a number

If $n^2 = m$, then m is the square of n.

square root of a number

If $n^2 = m$, then n is a square root of m.

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13.7.5E: Exercises

Practice Makes Perfect

Round Decimals

In the following exercises, round each number to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

1. 5.781

Answer

ⓐ 5.78 ⓑ 5.8 ⓒ 6

2. 1.638

3. 0.299

Answer

ⓐ 0.30 ⓑ 0.3 ⓒ 0

4. 0.697

5. 63.479

Answer

ⓐ 63.48 ⓑ 63.5 ⓒ 63

6. 84.281

Add and Subtract Decimals

In the following exercises, add or subtract.

7. $-16.53-24.38$

Answer

-40.91

8. $-19.47-32.58$

9. $-38.69+31.47$

Answer

-7.22

10. $-29.83+19.76$

11. $72.5-100$

Answer

-27.5

12. $86.2 - 100$

13. $91.75 - (-10.462)$

Answer

02.212

14. $94.69 - (-12.678)$

15. $55.01 - 3.7$

Answer

51.31

16. $59.08 - 4.6$

17. $2.51 - 7.4$

Answer

-4.89

18. $3.84 - 6.1$

Multiply and Divide Decimals

In the following exercises, multiply.

19. $94.69 - (-12.678)$

Answer

-11.653

20. $(-8.5)(1.69)$

21. $(-5.18)(-65.23)$

Answer

337.8914

22. $(-9.16)(-68.34)$

23. $(0.06)(21.75)$

Answer

1.305

24. $(0.08)(52.45)$

25. $(9.24)(10)$

Answer

92.4

26. $(6.531)(10)$

27. $(0.025)(100)$

Answer

2.5

28. $(0.037)(100)$

29. $(55.2)(1000)$

Answer

55200

30. $(99.4)(1000)$

In the following exercises, divide. Round money monetary answers to the nearest cent.

31. $\$117.25 \div 48$

Answer

$\$2.44$

32. $\$109.24 \div 36$

33. $1.44 \div (-0.3)$

Answer

-4.8

34. $-1.15 \div (-0.05)$

35. $5.2 \div 2.5$

Answer

2.08

36. $14 \div 0.35$

Convert Decimals, Fractions and Percents

In the following exercises, write each decimal as a fraction.

37. 0.04

Answer

$\frac{1}{25}$

38. 1.464

39. 0.095

Answer

$\frac{19}{200}$

40. -0.375

In the following exercises, convert each fraction to a decimal.

41. $\frac{17}{20}$

Answer

0.85

42. $\frac{17}{4}$

43. $-\frac{310}{25}$

Answer

-12.4

44. $-\frac{18}{11}$

In the following exercises, convert each percent to a decimal.

45. 71%

Answer

0.71

46. 150%

47. 39.3%

Answer

0.393

48. 7.8%

In the following exercises, convert each decimal to a percent.

49. 1.56

Answer

156%

50. 3

51. 0.0625

Answer

6.25%

52. 2.254

Simplify Expressions with Square Roots

In the following exercises, simplify.

53. $\sqrt{64}$

Answer

8

54. $\sqrt{169}$

55. $\sqrt{144}$

Answer

12

56. $-\sqrt{4}$

57. $-\sqrt{100}$

Answer

-10

58. $-\sqrt{121}$

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercises, list the ⓐ whole numbers, ⓑ integers, ⓒ rational numbers, ⓓ irrational numbers, ⓔ real numbers for each set of numbers.

59. $-8, 0, 1.95286\ldots, \frac{12}{5}, \sqrt{36}, 9$

Answer

ⓐ Whole numbers: $0, \sqrt{36}, 9$
ⓑ Integers: $-8, 0, \sqrt{36}, 9$
ⓒ Rational numbers: $-8, 0, \sqrt{36}, 9$
ⓓ Irrational numbers: $1.95286\ldots,$
ⓔ Real numbers: $-8, 0, 1.95286\ldots, \frac{12}{5}, \sqrt{36}, 9$

60. $-9, -3\frac{4}{9}, -\sqrt{9}, 0.40\overline{9}, \frac{11}{6}, 7$

61. $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$

Answer

ⓐ Whole numbers: none
ⓑ Integers: $-\sqrt{100}, -7, -1$
ⓒ Rational numbers: $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$
ⓓ Irrational numbers: none
ⓔ Real numbers: $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$

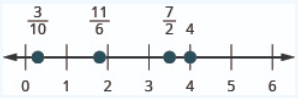
62. $-6, -\frac{5}{2}, 0, 0.\overline{714285}, 2\frac{1}{5}, \sqrt{14}$

Locate Fractions and Decimals on the Number Line

In the following exercises, locate the numbers on a number line.

63. $\frac{3}{10}, \frac{7}{2}, \frac{11}{6}, 4$

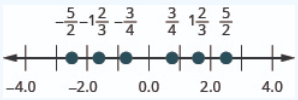
Answer



64. $\frac{7}{10}, \frac{5}{2}, \frac{13}{8}, 3$

65. $\frac{3}{4}, -\frac{3}{4}, 1\frac{2}{3}, -1\frac{2}{3}, \frac{5}{2}, -\frac{5}{2}$

Answer



66. $\frac{2}{5}, -\frac{2}{5}, 1\frac{3}{4}, -1\frac{3}{4}, \frac{8}{3}, -\frac{8}{3}$

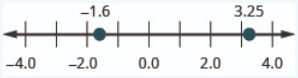
67. ⓐ 0.8 ⓑ -1.25

Answer

68. ⓐ -0.9 ⓑ -2.75

69. ⓐ -1.6 ⓑ 3.25

Answer



70. ⓐ 3.1 ⓑ -3.65

Writing Exercises

71. How does knowing about U.S. money help you learn about decimals?

Answer

Answers will vary.

72. When the Szetos sold their home, the selling price was 500% of what they had paid for the house 30 years ago. Explain what 500% means in this context.Szetos sold their home, the selling price was 500% of what they had paid for the house 30 years ago. Explain what 500% means in this context.

73. In your own words, explain the difference between a rational number and an irrational number.

Answer

Answers will vary.

74. Explain how the sets of numbers (counting, whole, integer, rational, irrationals, reals) are related to each other.

Self Check

ⓐ Use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
round decimals.			
add and subtract decimals.			
multiply and divide decimals.			
convert decimals, fractions and percents.			
simplify expressions with square roots.			
identify integers, rational numbers, irrational numbers and real numbers.			
locate fractions and decimals on the number line.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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13.7.6: Properties of Real Numbers

Learning Objectives

By the end of this section, you will be able to:

- Use the commutative and associative properties
- Use the properties of identity, inverse, and zero
- Simplify expressions using the Distributive Property

Use the Commutative and Associative Properties

The order we add two numbers doesn't affect the result. If we add $8+9$ or $9+8$, the results are the same—they both equal 17. So, $8+9=9+8$. The order in which we add does not matter!

Similarly, when multiplying two numbers, the order does not affect the result. If we multiply $9 \cdot 8$ or $8 \cdot 9$ the results are the same —they both equal 72. So, $9 \cdot 8=8 \cdot 9$. The order in which we multiply does not matter! These examples illustrate the **Commutative Property**.

COMMUTATIVE PROPERTY

of Addition	If a and $b$are real numbers, then	$a+b=b+a$.	(13.7.6.1)
of Multiplication	If a and $b$are real numbers, then	$a \cdot b=b \cdot a$.	

When adding or multiplying, changing the *order* gives the same result.

The Commutative Property has to do with order. We subtract $9-8$ and $8-9$, and see that $9-8 \neq 8-9$. Since changing the order of the subtraction does not give the same result, we know that *subtraction is not commutative*.

Division is not commutative either. Since $12 \div 3 \neq 3 \div 12$, changing the order of the division did not give the same result. The commutative properties apply only to addition and multiplication!

- Addition and multiplication *are* commutative.
- Subtraction and division are *not* commutative.

When adding three numbers, changing the grouping of the numbers gives the same result. For example, $(7+8)+2=7+(8+2)$, since each side of the equation equals 17.

This is true for multiplication, too. For example, $\left(5 \cdot \frac{1}{3}\right) \cdot 3=5 \cdot\left(\frac{1}{3} \cdot 3\right)$, since each side of the equation equals 5.

These examples illustrate the **Associative Property**.

ASSOCIATIVE PROPERTY

of Addition	If $a, b,$ and c are real numbers, then	$(a+b)+c=a+(b+c)$.	(13.7.6.2)
of Multiplication	If $a, b,$ and c are real numbers, then	$(a \cdot b) \cdot c=a \cdot(b \cdot c)$.	

When adding or multiplying, changing the *grouping* gives the same result.

The Associative Property has to do with grouping. If we change how the numbers are grouped, the result will be the same. Notice it is the same three numbers in the same order—the only difference is the grouping.

We saw that subtraction and division were not commutative. They are not associative either.

$$\begin{array}{ll} (10-3)-2 \neq 10-(3-2) & (24 \div 4) \div 2 \neq 24 \div(4 \div 2) \\ 7-2 \neq 10-1 & 6 \div 2 \neq 24 \div 2 \\ 5 \neq 9 & 3 \neq 12 \end{array} \quad (13.7.6.3)$$

When simplifying an expression, it is always a good idea to plan what the steps will be. In order to combine like terms in the next example, we will use the Commutative Property of addition to write the like terms together.

Example 13.7.6.1

Simplify: $18p+6q+15p+5q$.

Answer

	$18p+6q+15p+5q$	
Use the Commutative Property of addition to reorder so that like terms are together.	$18p+15p+6q+5q$	(13.7.6.4)
Add like terms.	$33p+11q$	

Example 13.7.6.2

Simplify: $23r+14s+9r+15s$.

Answer

$32r+29s$

Example 13.7.6.3

Simplify: $37m+21n+4m-15n$.

Answer

$41m+6n$

When we have to simplify algebraic expressions, we can often make the work easier by applying the Commutative Property or Associative Property first.

EXAMPLE 13.7.6.4

Simplify: $(\frac{5}{13}+\frac{3}{4})+\frac{1}{4}$.

Answer

	$(\frac{5}{13}+\frac{3}{4})+\frac{1}{4}$
Notice that the last 2 terms have a common denominator, so change the grouping.	$\frac{5}{13}+(\frac{3}{4}+\frac{1}{4})$
Add in parentheses first.	$\frac{5}{13}+(\frac{4}{4})$
Simplify the fraction.	$\frac{5}{13}+1$
Add.	$1\frac{5}{13}$
Convert to an improper fraction.	$\frac{18}{13}$

EXAMPLE 13.7.6.5

Simplify: $(\frac{7}{15}+\frac{5}{8})+\frac{3}{8}$.

Answer

$1\frac{7}{15}$

EXAMPLE 13.7.6.6

Simplify: $(\frac{2}{9}+\frac{7}{12})+\frac{5}{12}$.

Answer

$1\frac{2}{9}$

Use the Properties of Identity, Inverse, and Zero

What happens when we add 0 to any number? Adding 0 doesn't change the value. For this reason, we call 0 the **additive identity**. The **Identity Property of Addition** that states that for any real number $a, a+0=a$ and $0+a=a$.

What happens when we multiply any number by one? Multiplying by 1 doesn't change the value. So we call 1 the multiplicative identity. The **Identity Property of Multiplication** that states that for any real number $a, a\cdot 1=a$ and $1\cdot a=a$.

We summarize the Identity Properties here.

IDENTITY PROPERTY

of Addition For any real number a : $a+0=a$ $\quad 0+a=a$

0 is the **additive identity**

of Multiplication For any real number a : $a\cdot 1=a$ $\quad 1\cdot a=a$ (13.7.6.5)

1 is the **multiplicative identity**

What number added to 5 gives the additive identity, 0? We know

$$5+(-5)=0$$

The missing number was the opposite of the number!

We call $-a$ the additive inverse of a. *The opposite of a number is its additive inverse.* A number and its opposite add to zero, which is the additive identity. This leads to the **Inverse Property of Addition** that states for any real number $a, a+(-a)=0$.

What number multiplied by $\frac{2}{3}$ gives the multiplicative identity, 1? In other words, $\frac{2}{3}$ times what results in 1? We know

$$\frac{2}{3}\cdot\frac{3}{2}=1$$

The missing number was the reciprocal of the number!

We call $\frac{1}{a}$ the **multiplicative inverse** of a. *The reciprocal of a number is its multiplicative inverse.* This leads to the **Inverse Property of Multiplication** that states that for any real number $a, a\neq 0, a\cdot\frac{1}{a}=1$.

We'll formally state the inverse properties here.

INVERSE PROPERTY

of addition For any real number a, $a+(-a)=0$

$-a$ is the **additive inverse** of a

A number and its *opposite* add to zero.

of multiplication For any real number $a, a \neq 0$ $a \cdot \frac{1}{a}=1$

$\frac{1}{a}$ is the **multiplicative inverse** of a

A number and its *reciprocal* multiply to one. (13.7.6.6)

The Identity Property of addition says that when we add 0 to any number, the result is that same number. What happens when we multiply a number by 0? Multiplying by 0 makes the product equal zero.

What about division involving zero? What is $0 \div 3$? Think about a real example: If there are no cookies in the cookie jar and 3 people are to share them, how many cookies does each person get? There are no cookies to share, so each person gets 0 cookies. So, $0 \div 3 = 0$.

We can check division with the related multiplication fact. So we know $0 \div 3 = 0$ because $0 \cdot 3 = 0$.

Now think about dividing *by* zero. What is the result of dividing 4 by 0? Think about the related multiplication fact:

$4 \div 0 = \square$ means $\square \cdot 0 = 4$

Is there a number that multiplied by 0 gives 4? Since any real number multiplied by 0 gives 0, there is no real number that can be multiplied by 0 to obtain 4. We conclude that there is no answer to $4 \div 0$ and so we say that division by 0 is **undefined**.

We summarize the properties of zero here.

PROPERTIES OF ZERO

Multiplication by Zero: For any real number a,

$a \cdot 0 = 0 \quad 0 \cdot a = 0$ The product of any number and 0 is 0. (13.7.6.7)

Division by Zero: For any real number a, $a \neq 0$

$\frac{0}{a} = 0$ Zero divided by any real number, except itself, is zero.

$\frac{a}{0}$ is undefined Division by zero is undefined. (13.7.6.8)

We will now practice using the properties of identities, inverses, and zero to simplify expressions.

EXAMPLE 13.7.6.7

Simplify: $-84n+(-73n)+84n$.

Answer

	$-84n+(-73n)+84n$
Notice that the first and third terms are opposites; use the Commutative Property of addition to re-order the terms.	$-84n+84n+(-73n)$
Add left to right.	$0+(-73n)$
Add.	$-73n$

EXAMPLE 13.7.6.8

Simplify: $-27a+(-48a)+27a$.

Answer

$-48a$

EXAMPLE 13.7.6.9

Simplify: $39x+(-92x)+(-39x)$.

Answer

$-92x$

Now we will see how recognizing reciprocals is helpful. Before multiplying left to right, look for reciprocals—their product is 1.

EXAMPLE 13.7.6.10

Simplify: $\frac{7}{15}\cdot\frac{8}{23}\cdot\frac{15}{7}$.

Answer

	$\frac{7}{15}\cdot\frac{8}{23}\cdot\frac{15}{7}$
Notice the first and third terms are reciprocals, so use the Commutative Property of multiplication to re-order the factors.	$\frac{7}{15}\cdot\frac{15}{7}\cdot\frac{8}{23}$
Multiply left to right.	$1\cdot\frac{8}{23}$
Multiply.	$\frac{8}{23}$

EXAMPLE 13.7.6.11

Simplify: $\frac{9}{16}\cdot\frac{5}{49}\cdot\frac{16}{9}$.

Answer

$\frac{5}{49}$

Simplify: $\frac{6}{17}\cdot\frac{11}{25}\cdot\frac{17}{6}$.

Answer

$\frac{11}{25}$

The next example makes us aware of the distinction between dividing 0 by some number or some number being divided by 0.

Simplify: a. $\frac{0}{n+5}$, where $n\neq -5$ b. $\frac{10-3p}{0}$ where $10-3p\neq 0$.

Answer

a.

	$\frac{0}{n+5}$
Zero divided by any real number except itself is 0.	0

b.

	$\frac{10-3p}{0}$
Division by 0 is undefined.	undefined

EXAMPLE 13.7.6.14

Simplify: a. $\frac{0}{m+7}$, where $m \neq -7$ b. $\frac{18-6c}{0}$, where $18-6c \neq 0$.

Answer

a. 0
b. undefined

EXAMPLE 13.7.6.15

Simplify: a. $\frac{0}{d-4}$, where $d \neq 4$ b. $\frac{15-4q}{0}$, where $15-4q \neq 0$.

Answer

a. 0
b. undefined

Simplify Expressions Using the Distributive Property

Suppose that three friends are going to the movies. They each need $9.25—that's 9 dollars and 1 quarter—to pay for their tickets. How much money do they need all together?

You can think about the dollars separately from the quarters. They need 3 times $9 so $27 and 3 times 1 quarter, so 75 cents. In total, they need $27.75. If you think about doing the math in this way, you are using the Distributive Property.

DISTRIBUTIVE PROPERTY

If $a, b,$ and c are real numbers, then

$$a(b+c) = ab+ac$$
$$(b+c)a = ba+ca$$
$$a(b-c) = ab-ac$$
$$(b-c)a = ba-ca$$

In algebra, we use the Distributive Property to remove parentheses as we simplify expressions.

EXAMPLE 13.7.6.16

Simplify: $3(x+4)$.

Answer

	$3(x+4)$
Distribute.	$3 \cdot x+3 \cdot 4$
Multiply.	$3x+12$

Simplify: $4(x+2)$.

Answer

$4x8$

EXAMPLE 13.7.6.18

Simplify: $6(x+7)$.

Answer

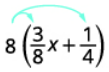
$6x42$

Some students find it helpful to draw in arrows to remind them how to use the Distributive Property. Then the first step in Example would look like this:

EXAMPLE 13.7.6.19

Simplify: $8(\frac{3}{8}x+\frac{1}{4})$.

Answer

	
Distribute.	$8\cdot\frac{3}{8}x+8\cdot\frac{1}{4}$
Multiply.	$3x+2$

EXAMPLE 13.7.6.20

Simplify: $6(\frac{5}{6}y+\frac{1}{2})$.

Answer

$5y+3$

EXAMPLE 13.7.6.21

Simplify: $12(\frac{1}{3}n+\frac{3}{4})$

Answer

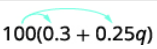
$4n+9$

Using the Distributive Property as shown in the next example will be very useful when we solve money applications in later chapters.

EXAMPLE 13.7.6.22

Simplify: $100(0.3+0.25q)$.

Answer

Distribute.	$100(0.3) + 100(0.25q)$
Multiply.	$30 + 25q$

EXAMPLE 13.7.6.23

Simplify: $100(0.7+0.15p)$.

Answer

$70+15p$

EXAMPLE 13.7.6.24

Simplify: $100(0.04+0.35d)$.

Answer

$4+35d$

When we distribute a negative number, we need to be extra careful to get the signs correct!

EXAMPLE 13.7.6.25

Simplify: $-11(4-3a)$.

Answer

	$-11(4-3a)$
Distribute.	$-11\cdot 4-(-11)\cdot 3a$
Multiply.	$-44-(-33a)$
Simplify.	$-44+33a$

Notice that you could also write the result as $33a-44$. Do you know why?

Simplify: $-5(2-3a)$.

Answer

$-10+15a$

EXAMPLE 13.7.6.27

Simplify: $-7(8-15y)$.

Answer

$-56+105y$

In the next example, we will show how to use the Distributive Property to find the opposite of an expression.

Simplify: $-(y+5)$.

Answer

	$-(y+5)$
Multiplying by -1 results in the opposite.	$-1(y+5)$
Distribute.	$-1\cdot y+(-1)\cdot 5$
Simplify.	$-y+(-5)$
Simplify.	$-y-5$

? EXAMPLE 13.7.6.29

Simplify: $-(z-11)$.

Answer

$-z+11$

? EXAMPLE 13.7.6.30

Simplify: $-(x-4)$.

Answer

$-x+4$

There will be times when we'll need to use the Distributive Property as part of the order of operations. Start by looking at the parentheses. If the expression inside the parentheses cannot be simplified, the next step would be multiply using the Distributive Property, which removes the parentheses. The next two examples will illustrate this.

? EXAMPLE 13.7.6.31

Simplify: $8-2(x+3)$

Answer

We follow the order of operations. Multiplication comes before subtraction, so we will distribute the 2 first and then subtract.

	$8-2(x+3)$
Distribute.	$8-2\cdot x-2\cdot 3$
Multiply.	$8-2x-6$
Combine like terms.	$-2x+2$

? EXAMPLE 13.7.6.32

Simplify: $9-3(x+2)$.

Answer

$3-3x$

? EXAMPLE 13.7.6.33

Simplify: $7x-5(x+4)$.

Answer

$2x-20$

? EXAMPLE 13.7.6.34

Simplify: $4(x-8)-(x+3)$.

Answer

	$4(x-8)-(x+3)$
Distribute.	$4x-32-x-3$
Combine like terms.	$3x-35$

? EXAMPLE 13.7.6.35

Simplify: $6(x-9)-(x+12)$.

Answer

$5x-66$

? EXAMPLE 13.7.6.36

Simplify: $8(x-1)-(x+5)$.

Answer

$7x-13$

All the properties of real numbers we have used in this chapter are summarized here.

Commutative Property

When adding or multiplying, changing the *order* gives the same result

of Addition	If a and $b$are real numbers, then	$a+b=b+a$.	(13.7.6.9)
of Multiplication	If a and $b$are real numbers, then	$a\cdot b=b\cdot a$.	

Associative Property

When adding or multiplying, changing the *grouping* gives the same result.

of Addition	If $a, b,$ and c are real numbers, then	$(a+b)+c=a+(b+c)$.	(13.7.6.10)
of Multiplication	If $a, b,$ and c are real numbers, then	$(a\cdot b)\cdot c=a\cdot(b\cdot c)$.	

Distributive Property

If $a, b,$and $c$are real numbers, then

$$a(b+c)=ab+ac$$
$$(b+c)a=ba+ca$$
$$a(b-c)=ab-ac$$
$$(b-c)a=ba-ca$$

(13.7.6.11)

Identity Property

of Addition For any real number a : $a+0=a$	$0+a=a$	(13.7.6.12)
0 is the **additive identity**		
of Multiplication For any real number a : $a\cdot 1=a$	$1\cdot a=a$	
1 is the **multiplicative identity**		

Inverse Property

of addition For any real number a, $a+(-a)=0$

$-a$ is the **additive inverse** of a

A number and its *opposite* add to zero.

of multiplication For any real number $a, a \neq 0$ $a \cdot \frac{1}{a} = 1$

$\frac{1}{a}$ is the **multiplicative inverse** of a

A number and its *reciprocal* multiply to one.

(13.7.6.13)

Properties of Zero

For any real number a, $a \cdot 0 = 0$

$0 \cdot a = 0$

For any real number $a, a \neq 0$, $\frac{0}{a} = 0$

For any real number a, $\frac{a}{0}$ is undefined

(13.7.6.14)

Key Concepts

Commutative Property

When adding or multiplying, changing the *order* gives the same result

of Addition If a and $b$are real numbers, then $a+b=b+a$.

of Multiplication If a and $b$are real numbers, then $a \cdot b = b \cdot a$.

(13.7.6.15)

Associative Property When adding or multiplying, changing the *grouping* gives the same result.

of Addition If $a, b,$ and c are real numbers, then $(a+b)+c=a+(b+c)$.

of Multiplication If $a, b,$ and c are real numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(13.7.6.16)

Distributive Property

If $a, b,$and $c$are real numbers, then

$a(b+c)=ab+ac$

$(b+c)a=ba+ca$

$a(b-c)=ab-ac$

$(b-c)a=ba-ca$

(13.7.6.17)

Identity Property

of Addition For any real number $a: a+0=a$ $0+a=a$

0 is the **additive identity**

of Multiplication For any real number $a: a \cdot 1 = a$ $1 \cdot a = a$

1 is the **multiplicative identity**

(13.7.6.18)

Inverse Property

of addition For any real number a, $-a$ is the **additive inverse** of a A number and its *opposite* add to zero.	$a+(-a)=0$	(13.7.6.19)
of multiplication For any real number a, $a \neq 0$ $\frac{1}{a}$ is the **multiplicative inverse** of a A number and its *reciprocal* multiply to one.	$a \cdot \frac{1}{a}=1$	

Properties of Zero

For any real number a,	$a \cdot 0=0$ $0 \cdot a=0$	(13.7.6.20)
For any real number a, $a \neq 0$,	$\frac{0}{a}=0$	
For any real number a,	$\frac{a}{0}$ is undefined	

Glossary

additive identity

The number 0 is the additive identity because adding 0 to any number does not change its value.

additive inverse

The opposite of a number is its additive inverse.

multiplicative identity

The number 1 is the multiplicative identity because multiplying 1 by any number does not change its value.

multiplicative inverse

The reciprocal of a number is its multiplicative inverse.

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13.7.6E: Exercises

Practice Makes Perfect

Use the Commutative and Associative Properties

In the following exercises, simplify.

1. $43m+(-12n)+(-16m)+(-9n)$

Answer

$27m+(-21n)$

2. $-22p+17q+(-35p)+(-27q)$

3. $\frac{3}{8}g+\frac{1}{12}h+\frac{7}{8}g+\frac{5}{12}h$

Answer

$\frac{5}{4}g+\frac{1}{2}h$

4. $\frac{5}{6}a+\frac{3}{10}b+\frac{1}{6}a+\frac{9}{10}b$

5. $6.8p+9.14q+(-4.37p)+(-0.88q)$

Answer

$2.43p+8.26q$

6. $9.6m+7.22n+(-2.19m)+(-0.65n)$

7. $-24\cdot 7\cdot \frac{3}{8}$

Answer

-63

8. $-36\cdot 11\cdot \frac{4}{9}$

9. $\left(\frac{5}{6}+\frac{8}{15}\right)+\frac{7}{15}$

Answer

$1\frac{5}{6}$

10. $\left(\frac{11}{12}+\frac{4}{9}\right)+\frac{5}{9}$

11. $17(0.25)(4)$

Answer

17

12. $36(0.2)(5)$

13. $[2.48(12)](0.5)$

Answer

14.88

14. $[9.731(4)](0.75)$

15. $12\left(\frac{5}{6}p\right)$

Answer

$10p$

16. $20\left(\frac{3}{5}q\right)$

Use the Properties of Identity, Inverse and Zero

In the following exercises, simplify.

17. $19a+44-19a$

Answer

44

18. $27c+16-27c$

19. $\frac{1}{2}+\frac{7}{8}+\left(-\frac{1}{2}\right)$

Answer

$\frac{7}{8}$

20. $\frac{2}{5}+\frac{5}{12}+\left(-\frac{2}{5}\right)$

21. $10(0.1d)$

Answer

d

22. $100(0.01p)$

23. $\frac{3}{20}\cdot\frac{49}{11}\cdot\frac{20}{3}$

Answer

$\frac{49}{11}$

24. $\frac{13}{18}\cdot\frac{25}{7}\cdot\frac{18}{13}$

25. $\frac{0}{u-4.99}$, where $u\neq 4.99$

Answer

0

26. $0 \div (y - \frac{1}{6})$, where $x \neq 16$

27. $\frac{32-5a}{0}$, where $32 - 5a \neq 0$

Answer

undefined

28. $\frac{28-9b}{0}$, where $28 - 9b \neq 0$

29. $(\frac{3}{4} + \frac{9}{10}m) \div 0$, where $\frac{3}{4} + \frac{9}{10}m \neq 0$

Answer

undefined

30. $(\frac{5}{16}n - \frac{3}{7}) \div 0$, where $\frac{5}{16}n - \frac{3}{7} \neq 0$

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the Distributive Property.

31. $8(4y + 9)$

Answer

$32y + 72$

32. $9(3w + 7)$

33. $6(c - 13)$

Answer

$6c - 78$

34. $7(y - 13)$

35. $\frac{1}{4}(3q + 12)$

Answer

$\frac{3}{4}q + 3$

36. $\frac{1}{5}(4m + 20)$

37. $9(\frac{5}{9}y - \frac{1}{3})$

Answer

$5y - 3$

38. $10(\frac{3}{10}x - \frac{2}{5})$

39. $12\left(\frac{1}{4}+\frac{2}{3}r\right)$

Answer

$3+8r$

40. $12\left(\frac{1}{6}+\frac{3}{4}s\right)$

41. $15\cdot\frac{3}{5}(4d+10)$

Answer

$36d+90$

42. $18\cdot\frac{5}{6}(15h+24)$

43. $r(s-18)$

Answer

$rs-18r$

44. $u(v-10)$

45. $(y+4)p$

Answer

$yp+4p$

46. $(a+7)x$

47. $-7(4p+1)$

Answer

$-28p-7$

48. $-9(9a+4)$

49. $-3(x-6)$

Answer

$-3x+18$

50. $-4(q-7)$

51. $-(3x-7)$

Answer

$-3x+7$

52. $-(5p-4)$

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53. $16-3(y+8)$

Answer

$-3y-8$

54. $18-4(x+2)$

55. $4-11(3c-2)$

Answer

$-33c+26$

56. $9-6(7n-5)$

57. $22-(a+3)$

Answer

$-a+19$

58. $8-(r-7)$

59. $(5m-3)-(m+7)$

Answer

$4m-10$

60. $(4y-1)-(y-2)$

61. $9(8x-3)-(-2)$

Answer

$72x-25$

62. $4(6x-1)-(-8)$

63. $5(2n+9)+12(n-3)$

Answer

$22n+9$

64. $9(5u+8)+2(u-6)$

65. $14(c-1)-8(c-6)$

Answer

$6c+34$

66. $11(n-7)-5(n-1)$

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67. $6(7y+8)-(30y-15)$

Answer

$12y+63$

68. $7(3n+9)-(4n-13)$

Writing Exercises

69. In your own words, state the Associative Property of addition.

Answer

Answers will vary.

70. What is the difference between the additive inverse and the multiplicative inverse of a number

71. Simplify $8(x-\frac{1}{4})$ using the Distributive Property and explain each step.

Answer

Answers will vary.

72. Explain how you can multiply $4(\$5.97)$ without paper or calculator by thinking of $\$5.97$ as $6-0.03$ and then using the Distributive Property.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the commutative and associative properties.			
use the properties of identity, inverse and zero.			
simplify expressions using the distributive property.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

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13.7.7: Chapter 1 Review Exercises

Chapter Review Exercises

Use the Language of Algebra

Identify Multiples and Factors

1. Use the divisibility tests to determine whether 180 is divisible by 2, by 3, by 5, by 6, and by 10.

Answer

Divisible by $2, 3, 5, 6$

2. Find the prime factorization of 252.

3. Find the least common multiple of 24 and 40.

Answer

120

In the following exercises, simplify each expression.

4. $24 \div 3 + 4(5 - 2)$

5. $7 + 3[6 - 4(5 - 4)] - 3^2$

Answer

4

Evaluate an Expression

In the following exercises, evaluate the following expressions.

6. When $x = 4$, ⓐ x^3 ⓑ $5x$ ⓒ $2x^2 - 5x + 3$

7. $2x^2 - 4xy - 3y^2$ when $x = 3$ and $y = 1$

Answer

3

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the following expressions by combining like terms.

8. $12y + 7 + 2y - 5$

9. $14x^2 - 9x + 11 - 8x^2 + 8x - 6$

Answer

$6x^2 - x + 5$

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the phrases into algebraic expressions.

10. ⓐ the sum of $4ab^2$ and $7a3b24ab^2$ and $7a^3b^2$

ⓑ the product of $6y^2$ and $3y$

ⓒ twelve more than $5x$

ⓓ $5y$ less than $8y^2$

11. ⓐ eleven times the difference of y and two

ⓑ the difference of eleven times y and two

Answer

ⓐ $11(y-2)$
ⓑ $11y-2$

12. Dushko has nickels and pennies in his pocket. The number of pennies is four less than five the number of nickels. Let nn represent the number of nickels. Write an expression for the number of pennies.

Integers

Simplify Expressions with Absolute Value

In the following exercise, fill in $<, >$, or $=$ for each of the following pairs of numbers.

13. ⓐ $-|7|\_\_\_-|-7|$

ⓑ $-8\_\_\_-|-8|$

ⓒ $|-13|\_\_\_-13$

ⓓ $|-12|\_\_\_-(-12)$

Answer

ⓐ $=$
ⓑ $=$
ⓒ $>$
ⓓ $=$

In the following exercises, simplify.

14. $9-|3(4-8)|$

15. $12-3|1-4(4-2)|$

Answer

-9

Add and Subtract Integers

In the following exercises, simplify each expression.

16. $-12+(-8)+7$

17. ⓐ $15-7$

ⓑ $-15-(-7)$

ⓒ $-15-7$

ⓓ $15-(-7)$

Answer

ⓐ 8
ⓑ -8
ⓒ -22
ⓓ 22

18. $-11-(-12)+5$

19. ⓐ $23-(-17)$ ⓑ $23+17$

Answer

ⓐ 40 ⓑ 40

20. $-(7-11)-(3-5)$

Multiply and Divide Integers

In the following exercise, multiply or divide.

21. ⓐ $-27\div 9$ ⓑ $120\div(-8)$ ⓒ $4(-14)$ ⓓ $-1(-17)$

Answer

ⓐ -3 ⓑ -15 ⓒ -56 ⓓ 17

Simplify and Evaluate Expressions with Integers

In the following exercises, simplify each expression.

22. ⓐ $(-7)^3$ ⓑ -7^3

23. $(7-11)(6-13)$

Answer

16

24. $63\div(-9)+(-36)\div(-4)$

25. $6-3|4(1-2)-(7-5)|$

Answer

-12

26. $(-2)^4-24\div(13-5)$

For the following exercises, evaluate each expression.

27. $(y+z)^2$ when $y=-4$ and $z=7$

Answer

9

28. $3x^2 - 2xy + 4y^2$ when $x = -2$ and $y = -3$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

29. the sum of -4 and -9, increased by 23

Answer

$(-4 + (-9)) + 23; 10$

30. ⓐ the difference of 17 and −8 ⓑ subtract 17 from −25

Use Integers in Applications

In the following exercise, solve.

31. Temperature On July 10, the high temperature in Phoenix, Arizona, was 109°, and the high temperature in Juneau, Alaska, was 63°. What was the difference between the temperature in Palm Springs and the temperature in Whitefield?

Answer

46°

Fractions

Simplify Fractions

In the following exercises, simplify.

32. $\frac{204}{228}$

33. $-\frac{270x^3}{198y^2}$

Answer

$-\frac{15x^3}{11y^2}$

Multiply and Divide Fractions

In the following exercises, perform the indicated operation.

34. $\left(-\frac{14}{15}\right)\left(\frac{10}{21}\right)$

35. $\frac{6x}{25} \div \frac{9y}{20}$

Answer

$\frac{8x}{15y}$

36. $\frac{-\frac{4}{9}}{\frac{8}{21}}$

Add and Subtract Fractions

In the following exercises, perform the indicated operation.

37. $\frac{5}{18}+\frac{7}{12}$

Answer

$\frac{31}{36}$

38. $\frac{11}{36}-\frac{15}{48}$

39. ⓐ $\frac{5}{8}+\frac{3}{4}$ ⓑ $\frac{5}{8}\div\frac{3}{4}$

Answer

ⓐ $\frac{11}{8}$ ⓑ $\frac{5}{6}$

40. ⓐ $-\frac{3y}{10}-\frac{5}{6}$ ⓑ $-\frac{3y}{10}\cdot\frac{5}{6}$

Use the Order of Operations to Simplify Fractions

In the following exercises, simplify.

41. $\frac{4\cdot 3-2\cdot 5}{-6\cdot 3+2\cdot 3}$

Answer

$-\frac{1}{6}$

42. $\frac{4(7-3)-2(4-9)}{-3(4+2)+7(3-6)}$

43. $\frac{4^3-4^2}{(\frac{4}{5})^2}$

Answer

75

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

44. $4x^2y^2$ when $x=\frac{2}{3}$ and $y=-\frac{3}{4}$

45. $\frac{a+b}{a-b}$ when $a=-4$ and $b=6$

Answer

-15

Decimals

Round Decimals

46. Round 6.738 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Add and Subtract Decimals

In the following exercises, perform the indicated operation.

47. $-23.67+29.84$

Answer

6.17

48. $54.3-100$

49. $79.38-(-17.598)$

Answer

96.978

Multiply and Divide Decimals

In the following exercises, perform the indicated operation.

50. $(-2.8)(3.97)$

51. $(-8.43)(-57.91)$

Answer

488.1813

52. $(53.48)(10)$

53. $(0.563)(100)$

Answer

56.3

54. $\$118.35 \div 2.6$

55. $1.84 \div (-0.8)$

Answer

-23

Convert Decimals, Fractions and Percents

In the following exercises, write each decimal as a fraction.

56. $\frac{13}{20}$

57. $-\frac{240}{25}$

Answer

-9.6

In the following exercises, convert each fraction to a decimal.

58. $-\frac{5}{8}$

59. $\frac{14}{11}$

Answer

$1.\overline{27}$

In the following exercises, convert each decimal to a percent.

60. 2.43

61. 0.0475

Answer

4.75%

Simplify Expressions with Square Roots

In the following exercises, simplify.

62. $\sqrt{289}$

63. $\sqrt{-121}$

Answer

no real number

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercise, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers for each set of numbers

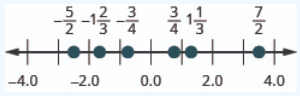
64. $-8, 0, 1.95286\ldots, \frac{12}{5}, \sqrt{36}, 9$

Locate Fractions and Decimals on the Number Line

In the following exercises, locate the numbers on a number line.

65. $\frac{3}{4}, -\frac{3}{4}, 1\frac{1}{3}, -1\frac{2}{3}, \frac{7}{2}, -\frac{5}{2}$

Answer



66. ⓐ 3.2 ⓑ -1.35

Properties of Real Numbers

Use the Commutative and Associative Properties

In the following exercises, simplify.

67. $\frac{5}{8}x+\frac{5}{12}y+\frac{1}{8}x+\frac{7}{12}y$

Answer

$\frac{3}{4}x+y$

68. $-32\cdot 9\cdot \frac{5}{8}$

69. $\left(\frac{11}{15}+\frac{3}{8}\right)+\frac{5}{8}$

Answer

$1\frac{11}{15}$

Use the Properties of Identity, Inverse and Zero

In the following exercises, simplify.

70. $\frac{4}{7}+\frac{8}{15}+\left(-\frac{4}{7}\right)$

71. $\frac{13}{15}\cdot\frac{9}{17}\cdot\frac{15}{13}$

Answer

$\frac{9}{17}$

72. $\frac{0}{x-3}, x\neq 3$

73. $\frac{5x-7}{0}, 5x-7\neq 0$

Answer

undefined

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the Distributive Property.

74. $8(a-4)$

75. $12\left(\frac{2}{3}b+\frac{5}{6}\right)$

Answer

$8b+10$

76. $18 \cdot \frac{5}{6}(2x-5)$

77. $(x-5)p$

Answer

$xp-5p$

78. $-4(y-3)$

79. $12-6(x+3)$

Answer

$-6x-6$

80. $6(3x-4)-(-5)$

81. $5(2y+3)-(4y-1)$

Answer

$y+16$

Practice Test

1. Find the prime factorization of 756.

2. Combine like terms: $5n+8+2n-1$

Answer

$7n+7$

3. Evaluate when $x=-2$ and $y=3$: $\frac{|3x-4y|}{6}$

4. Translate to an algebraic expression and simplify:

ⓐ eleven less than negative eight

ⓑ the difference of -8 and -3, increased by 5

Answer

$-8-11=-19$
$(-8-(-3))+5=0$

5. Dushko has nickels and pennies in his pocket. The number of pennies is seven less than four times the number of nickels. Let nn represent the number of nickels. Write an expression for the number of pennies.

6. Round $28.1458$to the nearest

ⓐ hundredth ⓑ thousandth

Answer

ⓐ 28.15 ⓑ 28.146

7. Convert

ⓐ $\frac{5}{11}$ to a decimal ⓑ 1.15 to a percent

8. Locate $\frac{3}{5}, 2.8, and -\frac{5}{2}$ on a number line.

Answer

In the following exercises, simplify each expression.

9. $8+3[6-3(5-2)]-4^2$

10. $-(4-9)-(9-5)$

Answer

1

11. $56 \div (-8)+(-27) \div (-3)$

12. $16-2|3(1-4)-(8-5)|$

Answer

-8

13. $-5+2(-3)^2-9$

14. $\frac{180}{204}$

Answer

$\frac{15}{17}$

15. $-\frac{7}{18}+\frac{5}{12}$

16. $\frac{4}{5} \div (-\frac{12}{25})$

Answer

$-\frac{5}{3}$

17. $\dfrac{9-3\cdot 9}{15-9}$

18. $\dfrac{4(-3+2(3-6))}{3(11-3(2+3))}$

Answer

3

19. $\dfrac{5}{13}\cdot\dfrac{4}{7}\cdot\dfrac{13}{5}$

20. $\dfrac{-\dfrac{5}{9}}{\dfrac{10}{21}}$

Answer

$-\dfrac{7}{6}$

21. $-4.8+(-6.7)$

22. $34.6-100$

Answer

-65.4

23. $-12.04\cdot(4.2)$

24. $-8\div 0.05$

Answer

160

25. $-\sqrt{121}$

26. $(\dfrac{8}{13}+\dfrac{5}{7})+\dfrac{2}{7}$

Answer

$1\dfrac{8}{13}$

27. $5x+(-8y)-6x+3y$

28. ⓐ $\dfrac{0}{9}$ ⓑ $\dfrac{11}{0}$

Answer

ⓐ 0 ⓑ undefined

29. $-3(8x-5)$

30. $6(3y-1)-(5y-3)$

Answer

$13y-3$

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SECTION OVERVIEW

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13.8.1: Simplify and Use Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with square roots
- Estimate square roots
- Approximate square roots
- Simplify variable expressions with square roots

BE PREPARED

Before you get started, take this readiness quiz.

1. Simplify: ⓐ 9^2 ⓑ $(-9)^2$ ⓒ -9^2.
   If you missed this problem, review [link].
2. Round 3.846 to the nearest hundredth.
   If you missed this problem, review [link].
3. For each number, identify whether it is a real number or not a real number:
   ⓐ $-\sqrt{100}$ ⓑ $\sqrt{-100}$.
   If you missed this problem, review [link].

Simplify Expressions with Square Roots

Remember that when a number n is multiplied by itself, we write n^2 and read it "n squared." For example, 15^2 reads as "15 squared," and 225 is called the square of 15, since $15^2 = 225$.

Definition: SQUARE OF A NUMBER

If $n^2 = m$, then m is the square of n.

Sometimes we will need to look at the relationship between numbers and their squares in reverse. Because 225 is the square of 15, we can also say that 15 is a square root of 225. A number whose square is m is called a *square root* of m.

Definition: SQUARE ROOT OF A NUMBER

If $n^2 = m$, then n is a square root of m.

Notice $(-15)^2 = 225$ also, so −15 is also a square root of 225. Therefore, both 15 and −15 are square roots of 225.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? The *radical sign*, $\sqrt{m}$, denotes the positive square root. The positive square root is also called the principal square root.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Definition: SQUARE ROOT NOTATION

radical sign → $\sqrt{m}$ ← radicand

$\sqrt{m}$ is read as "the square root of m."

If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

The square root of m, $\sqrt{m}$, is the positive number whose square is m.

Since 15 is the positive square root of 225, we write $\sqrt{225}=15$. Fill in Figure to make a table of square roots you can refer to as you work this chapter.

$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$
														15

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{225}=15$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{225}=-15$.

? Example 13.8.1.1

Simplify:

1. $\sqrt{36}$
2. $\sqrt{196}$
3. $-\sqrt{81}$
4. $-\sqrt{289}$.

Answer

1.

$$\begin{array}{ll} & \sqrt{36} \\ \text{Since} 6^2=36 & 6 \end{array} \tag{13.8.1.1}$$

2.

$$\begin{array}{ll} & \sqrt{196} \\ \text{Since} 14^2=196 & 14 \end{array} \tag{13.8.1.2}$$

3.

$$\begin{array}{ll} & -\sqrt{81} \\ \text{The negative is in front of the radical sign} & -9 \end{array} \tag{13.8.1.3}$$

4.

$$\begin{array}{ll} & -\sqrt{289} \\ \text{The negative is in front of the radical sign} & -17 \end{array} \tag{13.8.1.4}$$

? Example 13.8.1.2

Simplify:

1. $-\sqrt{49}$
2. $\sqrt{225}$.

Answer

1. -7
2. 15

Example 13.8.1.3

implify:

1. $\sqrt{64}$
2. $-\sqrt{121}$.

Answer

1. 8
2. −11

Example 13.8.1.4

Simplify:

1. $\sqrt{-169}$
2. $-\sqrt{64}$

Answer

1.

$$\begin{array}{ll} \text{There is no real number whose square is} s-169 & \sqrt{-169} \\ & \sqrt{-169}\text{is not a real number.} \end{array} \tag{13.8.1.5}$$

2.

$$\begin{array}{ll} \text{The negative is in front of the radical sign} & -\sqrt{64} \\ & -8 \end{array} \tag{13.8.1.6}$$

Example 13.8.1.5

Simplify:

1. $\sqrt{-196}$
2. $-\sqrt{81}$.

Answer

1. not a real number
2. −9

Example 13.8.1.6

Simplify:

1. $-\sqrt{49}$
2. $\sqrt{-121}$.

Answer

1. −7
2. not a real number

When using the order of operations to simplify an expression that has square roots, we treat the radical as a grouping symbol.

Example 13.8.1.7

Simplify:

1. $\sqrt{25}+\sqrt{144}$
2. $\sqrt{25+144}$.

Answer

1.

$$\begin{array}{ll} & \sqrt{25}+\sqrt{144} \\ \text{Use the order of operations} & 5+12 \\ \text{Simplify.} & 17 \end{array} \tag{13.8.1.7}$$

2.

$$\begin{array}{ll} & \sqrt{25+144} \\ \text{Simplify under the radical sign.} & \sqrt{169} \\ \text{Simplify.} & 13 \end{array} \tag{13.8.1.8}$$

Notice the different answers in parts 1 and 2!

Example 13.8.1.8

Simplify:

1. $\sqrt{9}+\sqrt{16}$
2. $\sqrt{9+16}$.

Answer

1. 7
2. 5

Example 13.8.1.9

Simplify:

1. $\sqrt{64+225}$
2. $\sqrt{64}+\sqrt{225}$.

Answer

1. 17
2. 23

Estimate Square Roots

So far we have only considered square roots of perfect square numbers. The square roots of other numbers are not whole numbers. Look at Table below.

Number	Square Root
4	$\sqrt{4}=2$
5	$\sqrt{5}$

Number	Square Root
6	$\sqrt{6}$
7	$\sqrt{7}$
8	$\sqrt{8}$
9	$\sqrt{9} = 3$

The square roots of numbers between 4 and 9 must be between the two consecutive whole numbers 2 and 3, and they are not whole numbers. Based on the pattern in the table above, we could say that $\sqrt{5}$ must be between 2 and 3. Using inequality symbols, we write:

$2 < \sqrt{5} < 3$

Example 13.8.1.10

Estimate $\sqrt{60}$ between two consecutive whole numbers.

Answer

Think of the perfect square numbers closest to 60. Make a small table of these perfect squares and their squares roots.

Number	Square root
36	6
49	7
64	8
81	9

60 $\sqrt{60}$

Locate 60 between two consecutive perfect squares.	$49 < 60 < 64$
$\sqrt{60}$ is between their square roots.	$7 < \sqrt{60} < 8$

Example 13.8.1.11

Estimate the square root $\sqrt{38}$ between two consecutive whole numbers.

Answer

$6 < \sqrt{38} < 7$

Example 13.8.1.12

Estimate the square root $\sqrt{84}$ between two consecutive whole numbers.

Answer

$9 < \sqrt{84} < 10$

Approximate Square Roots

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find them. Find the $\sqrt{x}$ key on your calculator. You will use this key to approximate square roots.

When you use your calculator to find the square root of a number that is not a perfect square, the answer that you see is not the exact square root. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately.'

Suppose your calculator has a 10-digit display. You would see that

$\sqrt{5} \approx 2.236067978$

If we wanted to round $\sqrt{5}$ to two decimal places, we would say

$\sqrt{5} \approx 2.24$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$\begin{aligned} (2.236067978)^2 &= 5.000000002 \\ (2.24)^2 &= 5.0176 \end{aligned} \tag{13.8.1.9}$$

Their squares are close to 5, but are not exactly equal to 5.

Using the square root key on a calculator and then rounding to two decimal places, we can find:

$$\begin{aligned} \sqrt{4} &= 2 \\ \sqrt{5} &\approx 2.24 \\ \sqrt{6} &\approx 2.45 \\ \sqrt{7} &\approx 2.65 \\ \sqrt{8} &\approx 2.83 \\ \sqrt{9} &= 3 \end{aligned} \tag{13.8.1.10}$$

Example 13.8.1.13

Round $\sqrt{17}$ to two decimal places.

Answer

	$\sqrt{17}$
Use the calculator square root key.	$4.123105626\ldots$
Round to two decimal places.	4.12
	$\sqrt{17} \approx 4.12$

(13.8.1.11)

Example 13.8.1.14

Round $\sqrt{11}$ to two decimal places.

Answer

≈ 3.32

Example 13.8.1.15

Round $\sqrt{13}$ to two decimal places.

Answer

≈ 3.61

Simplify Variable Expressions with Square Roots

What if we have to find a square root of an expression with a variable? Consider $\sqrt{9x^2}$. Can you think of an expression whose square is $9x^2$?

$$\begin{aligned} (?)^2 &= 9x^2 \\ (3x)^2 &= 9x^2 \quad \text{so} \sqrt{9x^2} = 3x \end{aligned} \tag{13.8.1.12}$$

When we use the radical sign to take the square root of a variable expression, we should specify that x≥0x≥0 to make sure we get the *principal square root*.

However, in this chapter we will assume that each variable in a square-root expression represents a non-negative number and so we will not write $x \geq 0$ next to every radical.

What about square roots of higher powers of variables? Think about the Power Property of Exponents we used in Chapter 6.

$(a^m)^n = a^{m \cdot n}$

If we square a^m, the exponent will become 2m.

$(a^m)^2 = a^{2m}$

How does this help us take square roots? Let's look at a few:

$$\begin{aligned} \sqrt{25u^8} &= 5u^4 \quad \text{Because} (5u^4)^2 = 25u^8 \\ \sqrt{16r^{20}} &= 4r^{10} \quad \text{Because} (4r^{10})^2 = 16r^{20} \\ \sqrt{196q^{36}} &= 14q^{18} \quad \text{Because} (14r^{18})^2 = 196q^{36} \end{aligned} \tag{13.8.1.13}$$

? Example 13.8.1.16

Simplify:

1. $\sqrt{x^6}$
2. $\sqrt{y^{16}}$

Answer

1.

$$\begin{aligned} & \sqrt{x^6} \\ \text{Since} (x^3)^2 = x^6 \quad & x^3 \end{aligned} \tag{13.8.1.14}$$

2.

$$\begin{aligned} & \sqrt{y^{16}} \\ \text{Since} (y^8)^2 = y^{16} \quad & y^8 \end{aligned} \tag{13.8.1.15}$$

? Example 13.8.1.17

Simplify:

1. $\sqrt{y^8}$
2. $\sqrt{z^{12}}$.

Answer

1. y^4
2. z^6

? Example 13.8.1.18

Simplify:

1. $\sqrt{m^4}$
2. $\sqrt{b^{10}}$.

Answer

1. m^2
2. b^5

Example 13.8.1.19

Simplify: $\sqrt{16n^2}$

Answer

$$\begin{array}{cc} & \sqrt{16n^2} \\ \text{Since} (4n)^2 = 16n^2 & 4n \end{array} \quad (13.8.1.16)$$

Example 13.8.1.20

Simplify: $\sqrt{64x^2}$.

Answer

$8x$

Example 13.8.1.21

Simplify: $\sqrt{169y^2}$.

Answer

$13y$

Example 13.8.1.22

Simplify: $-\sqrt{81c^2}$.

Answer

$$\begin{array}{cc} & -\sqrt{81c^2} \\ \text{Since} (9c)^2 = 81c^2 & -9c \end{array} \quad (13.8.1.17)$$

Example 13.8.1.23

Simplify: $-\sqrt{121y^2}$.

Answer

$-11y$

Example 13.8.1.24

Simplify: $-\sqrt{100p^2}$.

Answer

$-10p$

Example 13.8.1.25

Simplify: $\sqrt{36x^2y^2}$.

Answer

$$\text{Since} (6xy)^2 = \sqrt{36x^2y^2} \quad \begin{matrix} \sqrt{36x^2y^2} \\ 6xy \end{matrix} \tag{13.8.1.18}$$

Example 13.8.1.26

Simplify: $\sqrt{100a^2b^2}$.

Answer

10ab

Example 13.8.1.27

Simplify: $\sqrt{225m^2n^2}$.

Answer

15mn

Example 13.8.1.28

Simplify: $\sqrt{64p^{64}}$.

Answer

$$\text{Since} (8p^8)^2 = \sqrt{64p^{64}} \quad \begin{matrix} \sqrt{64p^{64}} \\ 8p^8 \end{matrix} \tag{13.8.1.19}$$

Example 13.8.1.29

Simplify: $\sqrt{49x^{30}}$.

Answer

$7x^{15}$

Example 13.8.1.30

Simplify: $\sqrt{81w^{36}}$

Answer

$9w^{18}$

Example 13.8.1.31

Simplify: $\sqrt{121a^6b^8}$

Answer

$$\text{Since}(11a^3b^4)^2 = \sqrt{121a^6b^8} \quad \begin{matrix}\sqrt{121a^6b^8} \\ 11a^3b^4\end{matrix} \tag{13.8.1.20}$$

? Example 13.8.1.32

Simplify: $\sqrt{169x^{10}y^{14}}$

Answer

$13x^5y^7$

? Example 13.8.1.33

Simplify: $\sqrt{144p^{12}q^{20}}$

Answer

$\sqrt{12p^6q^{10}}$

Access this online resource for additional instruction and practice with square roots.

- Square Roots

Key Concepts

- Note that the square root of a negative number is not a real number.
- Every positive number has two square roots, one positive and one negative. The positive square root of a positive number is the principal square root.
- We can estimate square roots using nearby perfect squares.
- We can approximate square roots using a calculator.
- When we use the radical sign to take the square root of a variable expression, we should specify that $x \geq 0$ to make sure we get the principal square root.

Glossary

square of a number

- If $n^2 = m$, then m is the square of n

square root of a number

- If $n^2 = m$, then n is a square root of m

square root notation

- If $m = n^2$, then $\sqrt{m} = n$. We read $\sqrt{m}$ as 'the square root of m.'

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13.8.1E: Exercises

Practice Makes Perfect

Simplify Expressions with Square Roots

In the following exercises, simplify.

Example 13.8.1E.34

$\sqrt{36}$

Answer

6

Example 13.8.1E.35

$\sqrt{4}$

Example 13.8.1E.36

$\sqrt{64}$

Answer

8

Example 13.8.1E.37

$\sqrt{169}$

Example 13.8.1E.38

$\sqrt{9}$

Answer

3

Example 13.8.1E.39

$\sqrt{16}$

Example 13.8.1E.40

$\sqrt{100}$

Answer

10

Example 13.8.1E.41

$\sqrt{144}$

Example 13.8.1E.42

$-\sqrt{4}$

Answer

-2

Example 13.8.1E.43

$-\sqrt{100}$

Example 13.8.1E.44

$-\sqrt{1}$

Answer

-1

Example 13.8.1E.45

$-\sqrt{121}$

Example 13.8.1E.46

$\sqrt{-121}$

Answer

not a real number

Example 13.8.1E.47

$\sqrt{-36}$

Example 13.8.1E.48

$\sqrt{-9}$

Answer

not a real number

Example 13.8.1E.49

$\sqrt{-49}$

Example 13.8.1E.50

$\sqrt{9+16}$

Answer

5

Example 13.8.1E.51

$\sqrt{25+144}$

Example 13.8.1E.52

$\sqrt{9}+\sqrt{16}$

Answer

7

Example 13.8.1E.53

$\sqrt{25}+\sqrt{144}$

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

Example 13.8.1E.54

$\sqrt{70}$

Answer

$8<\sqrt{70}<9$

Example 13.8.1E.55

$\sqrt{55}$

Example 13.8.1E.56

$\sqrt{200}$

Answer

$14<\sqrt{200}<15$

Example 13.8.1E.57

$\sqrt{172}$

Approximate Square Roots

In the following exercises, approximate each square root and round to two decimal places.

Example 13.8.1E.58

$\sqrt{19}$

Answer

4.36

Example 13.8.1E.59

$\sqrt{21}$

Example 13.8.1E.60

$\sqrt{53}$

Answer

7.28

Example 13.8.1E.61

$\sqrt{47}$

Simplify Variable Expressions with Square Roots

In the following exercises, simplify.

Example 13.8.1E.62

$\sqrt{y^2}$

Answer

y

Example 13.8.1E.63

$\sqrt{b^2}$

Example 13.8.1E.64

$\sqrt{a^{14}}$

Answer

a^7

Example 13.8.1E.65

$\sqrt{w^{24}}$

Example 13.8.1E.66

$\sqrt{49x^2}$

Answer

$7x$

Example 13.8.1E.67

$\sqrt{100y^2}$

Example 13.8.1E.68

$\sqrt{121m^{20}}$

Answer

$11m^{10}$

Example 13.8.1E.69

$25h^{44}$

Example 13.8.1E.70

$\sqrt{81x^{36}}$

Answer

$9x^{18}$

Example 13.8.1E.71

$\sqrt{144z^{84}}$

Example 13.8.1E.72

$-\sqrt{81x^{18}}$

Answer

$-9x^{9}$

Example 13.8.1E.73

$-\sqrt{100m^{32}}$

Example 13.8.1E.74

$-\sqrt{64a^2}$

Answer

$-8a$

Example 13.8.1E.75

$-\sqrt{25x^2}$

Example 13.8.1E.76

$\sqrt{144x^2y^2}$

Answer

$12xy$

https://math.libretexts.org/@go/page/46317

Example 13.8.1E.77

$\sqrt{196a^2b^2}$

Example 13.8.1E.78

$\sqrt{169w^8y^{10}}$

Answer

$13w^4y^5$

Example 13.8.1E.79

$\sqrt{81p^{24}q^6}$

Example 13.8.1E.80

$\sqrt{9c^8d^{12}}$

Answer

$3c^4d^6$

Examlpe 13.8.1E.81

$\sqrt{36r^6s^{20}}$

Everyday Math

Example 13.8.1E.82

Decorating Denise wants to have a square accent of designer tiles in her new shower. She can afford to buy 625 square centimeters of the designer tiles. How long can a side of the accent be?

Answer

25 centimeters

Example 13.8.1E.83

Decorating Morris wants to have a square mosaic inlaid in his new patio. His budget allows for 2025 square inch tiles. How long can a side of the mosaic be?

Writing Exercises

Example 13.8.1E.84

Why is there no real number equal to $\sqrt{-64}$?

Answer

Answers will vary.

? Example 13.8.1E. 85

What is the difference between 9^2 and $\sqrt{9}$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with square roots.			
estimate square roots.			
approximate square roots.			
simplify variable expressions with square roots.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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13.8.2: Simplify Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify square roots
- Use the Quotient Property to simplify square roots

BE PREPARED

Before you get started take this readiness quiz.

1. Simplify: $\frac{80}{176}$.
   If you missed this problem, review [link].
2. Simplify: $\frac{n^9}{n^3}$.
   If you missed this problem, review [link].
3. Simplify: $\frac{q^4}{q^{12}}$.
   If you missed this problem, review [link].

In the last section, we estimated the square root of a number between two consecutive whole numbers. We can say that $\sqrt{50}$ is between 7 and 8. This is fairly easy to do when the numbers are small enough that we can use [link].

But what if we want to estimate $\sqrt{500}$? If we simplify the square root first, we'll be able to estimate it easily. There are other reasons, too, to simplify square roots as you'll see later in this chapter.

A square root is considered *simplified* if its radicand contains no perfect square factors.

Definition: SIMPLIFIED SQUARE ROOT

$\sqrt{a}$ is considered simplified if a has no perfect square factors.

So $\sqrt{31}$ is simplified. But $\sqrt{32}$ is not simplified, because 16 is a perfect square factor of 32.

Use the Product Property to Simplify Square Roots

The properties we will use to simplify expressions with square roots are similar to the properties of exponents. We know that $(ab)^m = a^m b^m$. The corresponding property of square roots says that $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

Definition: PRODUCT PROPERTY OF SQUARE ROOTS

If *a*, *b* are non-negative real numbers, then $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

We use the Product Property of Square Roots to remove all perfect square factors from a radical. We will show how to do this in Example.

How To Use the Product Property to Simplify a Square Root

Example 13.8.2.1

Simplify: $\sqrt{50}$.

Answer

Step 1. Find the largest perfect square factor of the radicand.	25 is the largest perfect square factor of 50.	$\sqrt{50}$
Rewrite the radicand as a product using the perfect square factor.	$50 = 25 \cdot 2$	
	Always write the perfect square factor first.	$\sqrt{25 \cdot 2}$
Step 2. Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{25} \cdot \sqrt{2}$
Step 3. Simplify the square root of the perfect square.		$5\sqrt{2}$

Example 13.8.2.2

Simplify: $\sqrt{48}$.

Answer

$4\sqrt{3}$

Example 13.8.2.3

Simplify: $\sqrt{45}$.

Answer

$3\sqrt{5}$

Notice in the previous example that the simplified form of $\sqrt{50}$ is $5\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Definition: SIMPLIFY A SQUARE ROOT USING THE PRODUCT PROPERTY.

1. Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect-square factor.
2. Use the product rule to rewrite the radical as the product of two radicals.
3. Simplify the square root of the perfect square.

Example 13.8.2.4

Simplify: $\sqrt{500}$.

Answer

	$\sqrt{500}$	
Rewrite the radicand as a product using the largest perfect square factor	$\sqrt{100 \cdot 5}$	
Rewrite the radical as the product of two radicals	$\sqrt{100} \cdot \sqrt{5}$	(13.8.2.1)
Simplify	$10\sqrt{5}$	

Example 13.8.2.5

Simplify: $\sqrt{288}$.

Answer

$12\sqrt{2}$

Example 13.8.2.6

Simplify: $\sqrt{432}$.

Answer

$12\sqrt{3}$

We could use the simplified form $10\sqrt{5}$ to estimate $\sqrt{500}$. We know $\sqrt{5}$ is between 2 and 3, and $\sqrt{500}$ is $10\sqrt{5}$. So $\sqrt{500}$ is between 20 and 30.

The next example is much like the previous examples, but with variables.

Example 13.8.2.7

Simplify: $\sqrt{x^3}$.

Answer

	$\sqrt{x^3}$	
Rewrite the radicand as a product using the largest perfect square factor	$\sqrt{x^2 \cdot x}$	
Rewrite the radical as the product of two radicals	$\sqrt{x^2} \cdot \sqrt{x}$	(13.8.2.2)
Simplify	$x\sqrt{x}$	

Example 13.8.2.8

Simplify: $\sqrt{b^5}$.

Answer

$b^2\sqrt{b}$

Example 13.8.2.9

Simplify: $\sqrt{p^9}$.

Answer

$p^4\sqrt{p}$

We follow the same procedure when there is a coefficient in the radical, too.

Example 13.8.2.10

Simplify: $\sqrt{25y^5}$.

Answer

	$\sqrt{25y^5}$	
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{25y^4 \cdot y}$	
Rewrite the radical as the product of two radicals.	$\sqrt{25y^4} \cdot \sqrt{y}$	(13.8.2.3)
Simplify.	$5y^2\sqrt{y}$	

Example 13.8.2.11

Simplify: $\sqrt{16x^7}$.

Answer

$4x^3\sqrt{x}$

Example 13.8.2.12

Simplify: $\sqrt{49v^9}$.

Answer

$7v^4\sqrt{v}$

In the next example both the constant and the variable have perfect square factors.

Example 13.8.2.13

Simplify: $\sqrt{72n^7}$.

Answer

	$\sqrt{72n^7}$	
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{36n^6 \cdot 2n}$	
Rewrite the radical as the product of two radicals.	$\sqrt{36n^6} \cdot \sqrt{2n}$	(13.8.2.4)
Simplify.	$6n^3\sqrt{2n}$	

Example 13.8.2.14

Simplify: $\sqrt{32y^5}$.

Answer

$4y^2\sqrt{2y}$

? Example 13.8.2.15

Simplify: $\sqrt{75a^9}$.

Answer

$5a^4\sqrt{3a}$

? Example 13.8.2.16

Simplify: $\sqrt{63u^3v^5}$.

Answer

	$\sqrt{63u^3v^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9u^2v^4 \cdot 7uv}$
Rewrite the radical as the product of two radicals.	$\sqrt{9u^2v^4} \cdot \sqrt{7uv}$
Simplify.	$3uv^2\sqrt{7uv}$

(13.8.2.5)

? Example 13.8.2.17

Simplify: $\sqrt{98a^7b^5}$.

Answer

$7a^3b^2\sqrt{2ab}$

? Example 13.8.2.18

Simplify: $\sqrt{180m^9n^{11}}$.

Answer

$6m^4n^5\sqrt{5mn}$

We have seen how to use the Order of Operations to simplify some expressions with radicals. To simplify $\sqrt{25}+\sqrt{144}$ we must simplify each square root separately first, then add to get the sum of 17.

The expression $\sqrt{17}+\sqrt{7}$ cannot be simplified—to begin we'd need to simplify each square root, but neither 17 nor 7 contains a perfect square factor.

In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer.

? Example 13.8.2.19

Simplify: $3+\sqrt{32}$.

Answer

	$3+\sqrt{32}$
Rewrite the radicand as a product using the largest perfect square factor.	$3+\sqrt{16 \cdot 2}$
Rewrite the radical as the product of two radicals.	$3+\sqrt{16} \cdot \sqrt{2}$
Simplify.	$3+4\sqrt{2}$

(13.8.2.6)

The terms are not like and so we cannot add them. Trying to add an integer and a radical is like trying to add an integer and a variable—they are not like terms!

? Example 13.8.2.20

Simplify: $5+\sqrt{75}$.

Answer

$5+5\sqrt{3}$

Example 13.8.2.21

Simplify: $2+\sqrt{98}$.

Answer

$2+7\sqrt{2}$

The next example includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

Example 13.8.2.22

Simplify: $\frac{4-\sqrt{48}}{2}$.

Answer

	$\frac{4-\sqrt{48}}{2}$	
Rewrite the radicand as a product using thelargest perfect square factor.	$\frac{4-\sqrt{16\cdot 3}}{2}$	
Rewrite the radical as the product of two radicals.	$\frac{4-\sqrt{16}\cdot\sqrt{3}}{2}$	(13.8.2.7)
Simplify.	$\frac{4-4\sqrt{3}}{2}$	
Factor the common factor from thenumerator.	$\frac{4(1-\sqrt{3})}{2}$	
Remove the common factor, 2, from thenumerator and denominator.	$2(1-\sqrt{3})$	

Example 13.8.2.23

Simplify: $\frac{10-\sqrt{75}}{5}$.

Answer

$2-\sqrt{3}$

Example 13.8.2.24

Simplify: $\frac{6-\sqrt{45}}{3}$.

Answer

$2-\sqrt{5}$

Use the Quotient Property to Simplify Square Roots

Whenever you have to simplify a square root, the first step you should take is to determine whether the radicand is a perfect square. A *perfect square fraction* is a fraction in which both the numerator and the denominator are perfect squares.

Example 13.8.2.25

Simplify: $\sqrt{\frac{9}{64}}$.

Answer

	$\sqrt{\frac{9}{64}}$	
Since$(\frac{3}{8})^2$	$\frac{3}{8}$	(13.8.2.8)

Example 13.8.2.26

Simplify: $\sqrt{\frac{25}{16}}$.

Answer

$\frac{5}{4}$

? Example 13.8.2.27

Simplify: $\sqrt{\frac{49}{81}}$.

Answer

$\frac{7}{9}$

If the numerator and denominator have any common factors, remove them. You may find a perfect square fraction!

? Example 13.8.2.28

Simplify: $\sqrt{\frac{45}{80}}$.

Answer

	$\sqrt{\frac{45}{80}}$
Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.	$\sqrt{\frac{5 \cdot 9}{5 \cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt{\frac{9}{16}}$
Simplify. $(\frac{3}{4})^2 = \frac{9}{16}$	$\frac{3}{4}$

(13.8.2.9)

? Example 13.8.2.29

Simplify: $\sqrt{\frac{75}{48}}$.

Answer

$\frac{5}{4}$

? Example 13.8.2.30

Simplify: $\sqrt{\frac{98}{162}}$.

Answer

$\frac{7}{9}$

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents, $\frac{a^m}{a^n} = a^{m-n}$, $a \neq 0$.

? Example 13.8.2.31

Simplify: $\sqrt{\frac{m^6}{m^4}}$.

Answer

	$\sqrt{\frac{m^6}{m^4}}$
Simplify the fraction inside the radical first	$\sqrt{m^2}$
Divide the like bases by subtracting the exponents.	
Simplify.	m

(13.8.2.10)

? Example 13.8.2.32

Simplify: $\sqrt{\frac{a^8}{a^6}}$.

Answer

a

Example 13.8.2.33

Simplify: $\sqrt{\frac{x^{14}}{x^{10}}}$.

Answer

x^2

Example 13.8.2.34

Simplify: $\sqrt{\frac{48p^7}{3p^3}}$.

Answer

$$\begin{array}{ll} & \sqrt{\frac{48p^7}{3p^3}} \\ \text{Simplify the fraction inside the radical first.} & \sqrt{16p^4} \\ \text{Simplify.} & 4p^2 \end{array} \qquad (13.8.2.11)$$

Example 13.8.2.35

Simplify: $\sqrt{\frac{75x^5}{3x}}$.

Answer

$5x^2$

Example 13.8.2.36

Simplify: $\sqrt{\frac{72z^{12}}{2z^{10}}}$.

Answer

6z

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$(\frac{a}{b})^m = \frac{a^m}{b^m}$, $b \neq 0$

We can use a similar property to simplify a square root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect square we simplify the numerator and denominator separately.

Definition: QUOTIENT PROPERTY OF SQUARE ROOTS

If a, b are non-negative real numbers and $b \neq 0$, then

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Example 13.8.2.37

Simplify: $\sqrt{\frac{21}{64}}$.

Answer

$$\begin{array}{ll} & \sqrt{\frac{21}{64}} \\ \text{We cannot simplify the fraction inside the radical. Rewrite using the quotient property.} & \frac{\sqrt{21}}{\sqrt{64}} \\ \text{Simplify the square root of 64. The numerator cannot be simplified.} & \frac{\sqrt{21}}{8} \end{array} \qquad (13.8.2.12)$$

Example 13.8.2.38

Simplify: $\sqrt{\frac{19}{49}}$.

Answer

$\frac{\sqrt{19}}{7}$

Example 13.8.2.39

Simplify: $\sqrt{\frac{28}{81}}$

Answer

$\frac{2\sqrt{7}}{9}$

How to Use the Quotient Property to Simplify a Square Root

Example 13.8.2.40

Simplify: $\sqrt{\frac{27m^3}{196}}$.

Answer

Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$
Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$
Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2}\cdot\sqrt{3m}}{\sqrt{196}}$ $\frac{3m\sqrt{3m}}{14}$

Example 13.8.2.41

Simplify: $\sqrt{\frac{24p^3}{49}}$

Answer

$\frac{2p\sqrt{6p}}{7}$

Example 13.8.2.42

Simplify: $\sqrt{\frac{48x^5}{100}}$

Answer

$\frac{2x^2\sqrt{3x}}{5}$

Definition: SIMPLIFY A SQUARE ROOT USING THE QUOTIENT PROPERTY.

1. Simplify the fraction in the radicand, if possible.
2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
3. Simplify the radicals in the numerator and the denominator.

Example 13.8.2.43

Simplify: $\sqrt{\frac{45x^5}{y^4}}$.

Answer

We cannot simplify the fraction inside the radical. Rewrite using the quotient property. $\frac{\sqrt{45x^5}}{\sqrt{y^4}}$

Simplify the radicals in the numerator and the denominator. $\frac{\sqrt{9x^4}\sqrt{5x}}{y^2}$

Simplify. $\frac{3x^2\sqrt{5x}}{y^2}$

(13.8.2.13)

Example 13.8.2.44

Simplify: $\sqrt{\frac{80m^3}{n^6}}$

Answer

$\frac{4m\sqrt{5m}}{n^3}$

Example 13.8.2.45

Simplify: $\sqrt{\frac{54u^7}{v^8}}$.

Answer

$\frac{3u^3\sqrt{6u}}{v^4}$

Be sure to simplify the fraction in the radicand first, if possible.

Example 13.8.2.46

Simplify: $\sqrt{\frac{81d^9}{25d^4}}$.

Answer

Simplify the fraction in the radicand. $\sqrt{\frac{81d^5}{25}}$

Rewrite using the quotient property. $\frac{\sqrt{81d^5}}{\sqrt{25}}$

Simplify the radicals in the numerator and the denominator. $\frac{\sqrt{81d^4}\sqrt{d}}{5}$

Simplify. $\frac{9d^2\sqrt{d}}{5}$

(13.8.2.14)

Example 13.8.2.47

Simplify: $\sqrt{\frac{64x^7}{9x^3}}$.

Answer

$\frac{8x^2}{3}$

Example 13.8.2.48

Simplify: $\sqrt{\frac{16a^9}{100a^5}}$.

Answer

$\frac{2a^2}{5}$

? Example 13.8.2.49

Simplify: $\sqrt{\frac{18p^5q^7}{32pq^2}}$.

Answer

	$\sqrt{\frac{18p^5q^7}{32pq^2}}$
Simplify the fraction in the radicand.	$\sqrt{\frac{9p^4q^5}{16}}$
Rewrite using the quotient property.	$\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{9p^4q^4}\sqrt{q}}{4}$
Simplify.	$\frac{3p^2q^2\sqrt{q}}{4}$

(13.8.2.15)

? ExAMPLe 13.8.2.50

Simplify: $\sqrt{\frac{50x^5y^3}{72x^4y}}$.

Answer

$\frac{5y\sqrt{x}}{6}$

? Example 13.8.2.51

Simplify: $\sqrt{\frac{48m^7n^2}{125m^5n^9}}$.

Answer

$\frac{4m\sqrt{3}}{5n^3\sqrt{5n}}$

Key Concepts

- **Simplified Square Root** $\sqrt{a}$ is considered simplified if a has no perfect-square factors.
- **Product Property of Square Roots** If a, b are non-negative real numbers, then

  $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$
- **Simplify a Square Root Using the Product Property** To simplify a square root using the Product Property:
  1. Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect square factor.
  2. Use the product rule to rewrite the radical as the product of two radicals.
  3. Simplify the square root of the perfect square.
- **Quotient Property of Square Roots** If a, b are non-negative real numbers and $b \neq 0$, then

  $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- **Simplify a Square Root Using the Quotient Property** To simplify a square root using the Quotient Property:
  1. Simplify the fraction in the radicand, if possible.
  2. Use the Quotient Rule to rewrite the radical as the quotient of two radicals.
  3. Simplify the radicals in the numerator and the denominator.

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13.8.2E: Exercises

Practice Makes Perfect

Use the Product Property to Simplify Square Roots

In the following exercises, simplify.

Example 13.8.2E.52

$\sqrt{27}$

Answer

$3\sqrt{3}$

Example 13.8.2E.53

$\sqrt{80}$

Example 13.8.2E.54

$\sqrt{125}$

Answer

$5\sqrt{5}$

Example 13.8.2E.55

$\sqrt{96}$

Example 13.8.2E.56

$\sqrt{200}$

Answer

$10\sqrt{2}$

Example 13.8.2E.57

$\sqrt{147}$

Example 13.8.2E.58

$\sqrt{450}$

Answer

$15\sqrt{2}$

Example 13.8.2E.59

$\sqrt{252}$

Example 13.8.2E.60

$\sqrt{800}$

Answer

$20\sqrt{2}$

Example 13.8.2E.61

$\sqrt{288}$

Example 13.8.2E.62

$\sqrt{675}$

Answer

$15\sqrt{3}$

Example 13.8.2E.63

$\sqrt{1250}$

Example 13.8.2E.64

$\sqrt{x^7}$

Answer

$x^3\sqrt{x}$

Example 13.8.2E.65

$\sqrt{y^{11}}$

Example 13.8.2E.66

$\sqrt{p^3}$

Answer

$p\sqrt{p}$

Example 13.8.2E.67

$\sqrt{q^5}$

Example 13.8.2E.68

$\sqrt{m^{13}}$

Answer

$m^6\sqrt{m}$

Example 13.8.2E.69

$\sqrt{n^{21}}$

Example 13.8.2E.70

$\sqrt{r^{25}}$

Answer

$r^{12}\sqrt{r}$

Example 13.8.2E.71

$\sqrt{s^{33}}$

Example 13.8.2E.72

$\sqrt{49n^{17}}$

Answer

$7n^{8}\sqrt{n}$

Example 13.8.2E.73

$\sqrt{25m^{9}}$

Example 13.8.2E.74

$\sqrt{81r^{15}}$

Answer

$9r^{7}\sqrt{r}$

Example 13.8.2E.75

$\sqrt{100s^{19}}$

Example 13.8.2E.76

$\sqrt{98m^{5}}$

Answer

$7m^{2}\sqrt{2m}$

Example 13.8.2E.77

$\sqrt{32n^{11}}$

Example 13.8.2E.78

$\sqrt{125r^{13}}$

Answer

$5r^6\sqrt{5r}$

Example 13.8.2E.79

$\sqrt{80s^{15}}$

Example 13.8.2E.80

$\sqrt{200p^{13}}$

Answer

$10p^6\sqrt{2p}$

Example 13.8.2E.81

$\sqrt{128q^3}$

Example 13.8.2E.82

$\sqrt{242m^{23}}$

Answer

$11m^{11}\sqrt{2m}$

Example 13.8.2E.83

$\sqrt{175n^{13}}$

Exercise 13.8.2E.84

$\sqrt{147m^7n^{11}}$

Answer

$7m^3n^5\sqrt{3mn}$

Example 13.8.2E.85

$\sqrt{48m^7n^5}$

Example 13.8.2E.86

$\sqrt{75r^{13}s^9}$

Answer

$5r^6s^4\sqrt{3rs}$

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Example 13.8.2E.87

$\sqrt{96r^3s^3}$

Example 13.8.2E.88

$\sqrt{300p^9q^{11}}$

Answer

$10p^4q^5\sqrt{3pq}$

Example 13.8.2E.89

$\sqrt{192q^3r^7}$

Example 13.8.2E.90

$\sqrt{242m^{13}n^{21}}$

Answer

$11m^6n^{10}\sqrt{2mn}$

Example 13.8.2E.91

$\sqrt{150m^9n^3}$

Example 13.8.2E.92

$5+\sqrt{12}$

Answer

$5+2\sqrt{3}$

Example 13.8.2E.93

$8+\sqrt{96}$

Example 13.8.2E.94

$1+\sqrt{45}$

Answer

$1+3\sqrt{5}$

Example 13.8.2E.95

$3+\sqrt{125}$

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Example 13.8.2E.96

$\frac{10-\sqrt{24}}{2}$

Answer

$5-\sqrt{6}$

Example 13.8.2E.97

$\frac{8-\sqrt{80}}{4}$

Example 13.8.2E.98

$\frac{3+\sqrt{90}}{3}$

Answer

$1+\sqrt{10}$

Example 13.8.2E.99

$\frac{15+\sqrt{75}}{5}$

Use the Quotient Property to Simplify Square Roots

In the following exercises, simplify.

Example 13.8.2E.100

$\sqrt{\frac{49}{64}}$

Answer

$\frac{7}{8}$

Example 13.8.2E.101

$\sqrt{\frac{100}{36}}$

Example 13.8.2E.102

$\sqrt{\frac{121}{16}}$

Answer

$\frac{11}{4}$

Example 13.8.2E.103

$\sqrt{\frac{144}{169}}$

Example 13.8.2E.104

$\sqrt{\frac{72}{98}}$

Answer

$\frac{6}{7}$

Example 13.8.2E.105

$\sqrt{\frac{75}{12}}$

Example 13.8.2E.106

$\sqrt{\frac{45}{125}}$

Answer

$\frac{3}{5}$

Example 13.8.2E.107

$\sqrt{\frac{300}{243}}$

Example 13.8.2E.108

$\sqrt{\frac{x^{10}}{x^6}}$

Answer

x^2

Example 13.8.2E.109

$\sqrt{\frac{p^{20}}{p^{10}}}$

Example 13.8.2E.110

$\sqrt{\frac{y^4}{y^8}}$

Answer

$\frac{1}{y^2}$

Example 13.8.2E.111

$\sqrt{\frac{q^8}{q^{14}}}$

Example 13.8.2E.112

$\sqrt{\frac{200x^7}{2x^3}}$

Answer

$10x^2$

Example 13.8.2E.113

$\sqrt{\frac{98y^{11}}{2y^5}}$

Example 13.8.2E.114

$\sqrt{\frac{96p^9}{6p}}$

Answer

$4p^4$

Example 13.8.2E.115

$\sqrt{\frac{108q^{10}}{3q^2}}$

Example 13.8.2E.116

$\sqrt{\frac{36}{35}}$

Answer

$\frac{6}{\sqrt{35}}$

Example 13.8.2E.117

$\sqrt{\frac{144}{65}}$

Example 13.8.2E.118

$\sqrt{\frac{20}{81}}$

Answer

$\frac{2\sqrt{5}}{9}$

Example 13.8.2E.119

$\sqrt{\frac{211}{96}}$

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Example 13.8.2E.120

$\sqrt{\frac{96x^7}{121}}$

Answer

$\frac{4x^3\sqrt{6x}}{11}$

Example 13.8.2E.121

$\sqrt{\frac{108y^4}{49}}$

Example 13.8.2E.122

$\sqrt{\frac{300m^5}{64}}$

Answer

$\frac{5m^2\sqrt{3m}}{4}$

Example 13.8.2E.123

$\sqrt{\frac{125n^7}{169}}$

Example 13.8.2E.124

$\sqrt{\frac{98r^5}{100}}$

Answer

$\frac{7r^2\sqrt{2r}}{10}$

Example 13.8.2E.125

$\sqrt{\frac{180s^{10}}{144}}$

Example 13.8.2E.126

$\sqrt{\frac{28q^6}{225}}$

Answer

$\frac{2q^3\sqrt{7}}{15}$

Example 13.8.2E.127

$\sqrt{\frac{150r^3}{256}}$

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Example 13.8.2E.128

$\sqrt{\frac{75r^9}{s^8}}$

Answer

$\frac{5r^4\sqrt{3r}}{s^4}$

Example 13.8.2E.129

$\sqrt{\frac{72x^5}{y^6}}$

Example 13.8.2E.130

$\sqrt{\frac{28p^7}{q^2}}$

Answer

$\frac{4p^3\sqrt{7p}}{q}$

Example 13.8.2E.131

$\sqrt{\frac{45r^3}{s^{10}}}$

Example 13.8.2E.132

$\sqrt{\frac{100x^5}{36x^3}}$

Answer

$\frac{5x}{3}$

Example 13.8.2E.133

$\sqrt{\frac{49r^{12}}{16r^6}}$

Example 13.8.2E.134

$\sqrt{\frac{121p^5}{81p^2}}$

Answer

$\frac{11p\sqrt{p}}{9}$

Example 13.8.2E.135

$\sqrt{\frac{25r^8}{64r}}$

Example 13.8.2E.136

$\sqrt{\frac{32x^5y^3}{18x^3y}}$.

Answer

$\frac{4xy}{3}$

Example 13.8.2E.137

$\sqrt{\frac{75r^6s^8}{48rs^4}}$

Example 13.8.2E.138

$\sqrt{\frac{27p^2q^{10}}{8p^5q^3}}$

Answer

$\frac{1}{2pq\sqrt{p}}$

Example 13.8.2E.139

$\sqrt{\frac{50r^5s^2}{128r^2s^5}}$

Everyday Math

Example 13.8.2E.140

1. Elliott decides to construct a square garden that will take up 288 square feet of his yard. Simplify $\sqrt{288}$ to determine the length and the width of his garden. Round to the nearest tenth of a foot.
2. Suppose Elliott decides to reduce the size of his square garden so that he can create a 5-foot-wide walking path on the north and east sides of the garden. Simplify $\sqrt{288}-5$ to determine the length and width of the new garden. Round to the nearest tenth of a foot.

Answer

1. 17.0 feet
2. 15.0 feet

Example 13.8.2E.141

1. Melissa accidentally drops a pair of sunglasses from the top of a roller coaster, 64 feet above the ground. Simplify $\sqrt{\frac{64}{16}}$ to determine the number of seconds it takes for the sunglasses to reach the ground.
2. Suppose the sunglasses in the previous example were dropped from a height of 144 feet. Simplify $\sqrt{\frac{144}{16}}$ to determine the number of seconds it takes for the sunglasses to reach the ground.

Writing Exercises

? Example $13.8.2E.142$

Explain why $\sqrt{x^4} = x^2$. Then explain why $\sqrt{x^{16}} = x^8$.

Answer

Answers will vary.

? Example $13.8.2E.143$

Explain why $7 + \sqrt{9}$ is not equal to $\sqrt{7+9}$.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify square roots.			
use the Quotient Property to simplify square roots.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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13.8.3: Add and Subtract Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract like square roots
- Add and subtract square roots that need simplification

BE PREPARED

Before you get started, take this readiness quiz.

1. Add: ⓐ $3x+9x$ ⓑ $5m+5n$.
   If you missed this problem, review [link].
2. Simplify: $\sqrt{50x^3}$.
   If you missed this problem, review [link].

We know that we must follow the order of operations to simplify expressions with square roots. The radical is a grouping symbol, so we work inside the radical first. We simplify $\sqrt{2+7}$ in this way:

$$\begin{array}{ll} & \sqrt{2+7} \\ \text{Add inside the radical.} & \sqrt{9} \\ \text{Simplify.} & 3 \end{array} \tag{13.8.3.1}$$

So if we have to add $\sqrt{2}+\sqrt{7}$, we must not combine them into one radical.

$\sqrt{2}+\sqrt{7} \neq \sqrt{2+7}$

Trying to add square roots with different radicands is like trying to add unlike terms.

$$\begin{array}{llll} \text{But, just like we can} & x+x & \text{we can add} & \sqrt{3}+\sqrt{3} \\ & x+x=2x & & \sqrt{3}+\sqrt{3}=2\sqrt{3} \end{array} \tag{13.8.3.2}$$

Adding square roots with the same radicand is just like adding like terms. We call square roots with the same radicand like square roots to remind us they work the same as like terms.

Definition: LIKE SQUARE ROOTS

Square roots with the same radicand are called **like square roots**.

We add and subtract like square roots in the same way we add and subtract like terms. We know that 3x+8x is 11x. Similarly we add $3\sqrt{x}+8\sqrt{x}$ and the result is $11\sqrt{x}$.

Add and Subtract Like Square Roots

Think about adding like terms with variables as you do the next few examples. When you have like radicands, you just add or subtract the coefficients. When the radicands are not like, you cannot combine the terms.

Example 13.8.3.1

Simplify: $2\sqrt{2}-7\sqrt{2}$.

Answer

$$\begin{array}{ll} & 2\sqrt{2}-7\sqrt{2} \\ \text{Since the radicals are like, we subtract the coefficients.} & -5\sqrt{2} \end{array} \tag{13.8.3.3}$$

Example 13.8.3.2

Simplify: $8\sqrt{2}-9\sqrt{2}$.

Answer

$-\sqrt{2}$

Example 13.8.3.3

Simplify: $5\sqrt{3}-9\sqrt{3}$.

Answer

$-4\sqrt{3}$

Example 13.8.3.4

Simplify: $3\sqrt{y}+4\sqrt{y}$.

Answer

$$\begin{array}{ll} & 3\sqrt{y}+4\sqrt{y} \\ \text{Since the radicals are like, we add the coefficients.} & 7\sqrt{y} \end{array} \tag{13.8.3.4}$$

Example 13.8.3.5

Simplify: $2\sqrt{x}+7\sqrt{x}$.

Answer

$9\sqrt{x}$

Example 13.8.3.6

Simplify: $5\sqrt{u}+3\sqrt{u}$.

Answer

$8\sqrt{u}$

Example 13.8.3.7

Simplify: $4\sqrt{x}-2\sqrt{y}$

Answer

$$\begin{array}{ll} & 4\sqrt{x}-2\sqrt{y} \\ \text{Since the radicals are not like, we cannot subtract them. We leave the expression as is.} & 4\sqrt{x}-2\sqrt{y} \end{array} \tag{13.8.3.5}$$

Example 13.8.3.8

Simplify: $7\sqrt{p}-6\sqrt{q}$.

Answer

$7\sqrt{p}-6\sqrt{q}$

Example 13.8.3.9

Simplify: $6\sqrt{a}-3\sqrt{b}$.

Answer

$6\sqrt{a}-3\sqrt{b}$

Example 13.8.3.10

Simplify: $5\sqrt{13}+4\sqrt{13}+2\sqrt{13}$.

Answer

$$\begin{array}{ll} & 5\sqrt{13}+4\sqrt{13}+2\sqrt{13} \\ \text{Since the radicals are like, we add the coefficients.} & 11\sqrt{13} \end{array} \tag{13.8.3.6}$$

Example 13.8.3.11

Simplify: $4\sqrt{11}+2\sqrt{11}+3\sqrt{11}$.

Answer

$9\sqrt{11}$

Example 13.8.3.12

Simplify: $6\sqrt{10}+2\sqrt{10}+3\sqrt{10}$.

Answer

$11\sqrt{10}$

Example 13.8.3.13

Simplify: $2\sqrt{6}-6\sqrt{6}+3\sqrt{3}$.

Answer

$$\begin{array}{ll} & 2\sqrt{6}-6\sqrt{6}+3\sqrt{3} \\ \text{Since the first two radicals are like, we subtract their coefficients.} & -4\sqrt{6}+3\sqrt{3} \end{array} \tag{13.8.3.7}$$

Example 13.8.3.14

Simplify: $5\sqrt{5}-4\sqrt{5}+2\sqrt{6}$.

Answer

$\sqrt{5}+2\sqrt{6}$

Example 13.8.3.15

Simplify: $3\sqrt{7}-8\sqrt{7}+2\sqrt{5}$.

Answer

$-5\sqrt{7}+2\sqrt{5}$

Example 13.8.3.16

Simplify: $2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$.

Answer

$$\begin{array}{ll} & 2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n} \\ \text{Since the radicals are like, we combine them.} & -0\sqrt{5n} \\ \text{Simplify.} & 0 \end{array} \tag{13.8.3.8}$$

Example 13.8.3.17

Simplify: $\sqrt{7x}-7\sqrt{7x}+4\sqrt{7x}$.

Answer

$-2\sqrt{7x}$

Example 13.8.3.18

Simplify: $4\sqrt{3y}-7\sqrt{3y}+2\sqrt{3y}$.

Answer

$-3\sqrt{y}$

When radicals contain more than one variable, as long as all the variables and their exponents are identical, the radicals are like.

Example 13.8.3.19

Simplify: $\sqrt{3xy}+5\sqrt{3xy}-4\sqrt{3xy}$.

Answer

$$\begin{array}{ll} & \sqrt{3xy}+5\sqrt{3xy}-4\sqrt{3xy} \\ \text{Since the radicals are like, we combine them.} & 2\sqrt{3xy} \end{array} \tag{13.8.3.9}$$

Example 13.8.3.20

Simplify: $\sqrt{5xy}+4\sqrt{5xy}-7\sqrt{5xy}$.

Answer

$-2\sqrt{5xy}$

Example 13.8.3.21

Simplify: $3\sqrt{7mn}+\sqrt{7mn}-4\sqrt{7mn}$.

Answer

0

Add and Subtract Square Roots that Need Simplification

Remember that we always simplify square roots by removing the largest perfect-square factor. Sometimes when we have to add or subtract square roots that do not appear to have **like radicals**, we find like radicals after simplifying the square roots.

Example 13.8.3.22

Simplify: $\sqrt{20}+3\sqrt{5}$.

Answer

$$\begin{array}{ll} & \sqrt{20}+3\sqrt{5} \\ \text{Simplify the radicals, when possible.} & \sqrt{4}\cdot\sqrt{5}+3\sqrt{5} \\ & 2\sqrt{5}+3\sqrt{5} \\ \text{Combine the like radicals.} & 5\sqrt{5} \end{array} \tag{13.8.3.10}$$

Example 13.8.3.23

Simplify: $\sqrt{18}+6\sqrt{2}$.

Answer

$9\sqrt{2}$

Example 13.8.3.24

Simplify: $\sqrt{27}+4\sqrt{3}$.

Answer

$7\sqrt{3}$

Example 13.8.3.25

Simplify: $\sqrt{48}-\sqrt{75}$

Answer

$$\begin{aligned} & & \sqrt{48}-\sqrt{75} \\ & \text{Simplify the radicals.} & \sqrt{16}\cdot\sqrt{3}-\sqrt{25}\cdot\sqrt{3} \\ & & 4\sqrt{3}-5\sqrt{3} \\ & \text{Combine the like radicals.} & -\sqrt{3} \end{aligned} \tag{13.8.3.11}$$

Example 13.8.3.26

Simplify: $\sqrt{32}-\sqrt{18}$.

Answer

$\sqrt{2}$

Example 13.8.3.27

Simplify: $\sqrt{20}-\sqrt{45}$.

Answer

$-\sqrt{5}$

Just like we use the Associative Property of Multiplication to simplify 5(3x) and get 15x, we can simplify $5(3\sqrt{x})$ and get $15\sqrt{x}$. We will use the Associative Property to do this in the next example.

Example 13.8.3.28

Simplify: $5\sqrt{18}-2\sqrt{8}$.

Answer

$$\begin{aligned} & & 5\sqrt{18}-2\sqrt{8} \\ & \text{Simplify the radicals.} & 5\cdot\sqrt{9}\cdot\sqrt{2}-2\cdot\sqrt{4}\cdot\sqrt{2} \\ & & 5\cdot 3\cdot\sqrt{2}-2\cdot 2\cdot\sqrt{2} \\ & & 15\sqrt{2}-4\sqrt{2} \\ & \text{Combine the like radicals.} & 11\sqrt{2} \end{aligned} \tag{13.8.3.12}$$

Example 13.8.3.29

Simplify: $4\sqrt{27}-3\sqrt{12}$.

Answer

$6\sqrt{3}$

Example 13.8.3.30

Simplify: $3\sqrt{20}-7\sqrt{45}$.

Answer

$-15\sqrt{5}$

Example 13.8.3.31

Simplify: $\frac{3}{4}\sqrt{192}-\frac{5}{6}\sqrt{108}$.

Answer

$$\begin{aligned} & \frac{3}{4}\sqrt{192}-\frac{5}{6}\sqrt{108} \\ \text{Simplify the radicals.} \quad & \frac{3}{4}\sqrt{64}\cdot\sqrt{3}-\frac{5}{6}\sqrt{36}\cdot\sqrt{3} \\ & \frac{3}{4}\cdot 8\cdot\sqrt{3}-\frac{5}{6}\cdot 6\cdot\sqrt{3} \\ & 6\sqrt{3}-5\sqrt{3} \\ \text{Combine the like radicals.} \quad & \sqrt{3} \end{aligned} \tag{13.8.3.13}$$

Example 13.8.3.32

Simplify: $\frac{2}{3}\sqrt{108}-\frac{5}{7}\sqrt{147}$.

Answer

$-\sqrt{3}$

Example 13.8.3.33

Simplify: $\frac{3}{5}\sqrt{200}-\frac{3}{4}\sqrt{128}$.

Answer

0

Example 13.8.3.34

Simplify: $\frac{2}{3}\sqrt{48}-\frac{3}{4}\sqrt{12}$.

Answer

$$\begin{aligned} & \frac{2}{3}\sqrt{48}-\frac{3}{4}\sqrt{12} \\ \text{Simplify the radicals.} \quad & \frac{2}{3}\sqrt{16}\cdot\sqrt{3}-\frac{3}{4}\sqrt{4}\cdot\sqrt{3} \\ & \frac{2}{3}\cdot 4\cdot\sqrt{3}-\frac{3}{4}\cdot 2\cdot\sqrt{3} \\ & \frac{8}{3}\sqrt{3}-\frac{3}{2}\sqrt{3} \\ \text{Find a common denominator to subtract the coefficients of the like radicals.} \quad & \frac{16}{6}\sqrt{3}-\frac{9}{6}\sqrt{3} \\ \text{Simplify.} \quad & \frac{7}{6}\sqrt{3} \end{aligned} \tag{13.8.3.14}$$

Example 13.8.3.35

Simplify: $\frac{2}{5}\sqrt{32}-\frac{1}{3}\sqrt{8}$

Answer

$\frac{14}{15}\sqrt{2}$

Example 13.8.3.36

Simplify: $\frac{1}{3}\sqrt{80}-\frac{1}{4}\sqrt{125}$

Answer

$\frac{1}{12}[\sqrt{5}$

In the next example, we will remove constant and variable factors from the square roots.

Example 13.8.3.37

Simplify: $\sqrt{18n^5}-\sqrt{32n^5}$

Answer

$$\begin{array}{ll} & \sqrt{18n^5}-\sqrt{32n^5} \\ \text{Simplify the radicals.} & \sqrt{9n^4}\cdot\sqrt{2n}-\sqrt{16n^4}\cdot\sqrt{2n} \\ & 3n^2\sqrt{2n}-4n^2\sqrt{2n} \\ \text{Combine the like radicals.} & -n^2\sqrt{2n} \end{array} \tag{13.8.3.15}$$

Example 13.8.3.38

Simplify: $\sqrt{32m^7}-\sqrt{50m^7}$.

Answer

$-m^3\sqrt{2m}$

Example 13.8.3.39

Simplify: $\sqrt{27p^3}-\sqrt{48p^3}$

Answer

$-p^3\sqrt{p}$

Example 13.8.3.40

Simplify: $9\sqrt{50m^2}-6\sqrt{48m^2}$.

Answer

\[\begin{array}{ll} {}&{9\sqrt{50m^{2}}-6\sqrt{48m^{2}}}\\ {\text{Simplify the radicals.}}&{9\sqrt{25m^{2}}\cdot\sqrt{2}-6\cdot\sqrt{16m^{2}}\cdot\sqrt{3}}\\ {}&{9\cdot 5m\cdot\sqrt{2}-6\cdot 4m\cdot\sqrt{3}}\\ {}&{45m\sqrt{2}-24m\sqrt{3}}\\ \end{array}\]

Example 13.8.3.41

Simplify: $5\sqrt{32x^2}-3\sqrt{48x^2}$.

Answer

$20x\sqrt{2}-12x\sqrt{3}$

Example 13.8.3.42

Simplify: $7\sqrt{48y^2}-4\sqrt{72y^2}$.

Answer

$28y\sqrt{3}-24y\sqrt{2}$

Example 13.8.3.43

Simplify: $2\sqrt{8x^2}-5x\sqrt{32}+5\sqrt{18x^2}$.

Answer

$$\begin{array}{ll} & 2\sqrt{8x^2}-5x\sqrt{32}+5\sqrt{18x^2} \\ \text{Simplify the radicals.} & 2\sqrt{4x^2}\cdot\sqrt{2}-5x\sqrt{16}\cdot\sqrt{2}+5\sqrt{9x^2}\cdot\sqrt{2} \\ & 2\cdot 2x\cdot\sqrt{2}-5x\cdot 4\cdot\sqrt{2}+5\cdot 3x\cdot\sqrt{2} \\ & 4x\sqrt{2}-20x\sqrt{2}+15x\sqrt{2} \\ \text{Combine the like radicals.} & -x\sqrt{2} \end{array} \qquad (13.8.3.16)$$

Example 13.8.3.44

Simplify: $3\sqrt{12x^2}-2x\sqrt{48}+4\sqrt{27x^2}$

Answer

$10x\sqrt{3}$

Example 13.8.3.45

Simplify: $3\sqrt{18x^2}-6x\sqrt{32}+2\sqrt{50x^2}$.

Answer

$-5x\sqrt{2}$

Access this online resource for additional instruction and practice with the adding and subtracting square roots.

- Adding/Subtracting Square Roots

Glossary

like square roots

Square roots with the same radicand are called like square roots.

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13.8.3E: Exercises

Practice Makes Perfect

Add and Subtract Like Square Roots

In the following exercises, simplify.

Example 13.8.3E.46

$8\sqrt{2}-5\sqrt{2}$

Answer

$3\sqrt{2}$

Example 13.8.3E.47

$7\sqrt{2}-3\sqrt{2}$

Example 13.8.3E.48

$3\sqrt{5}+6\sqrt{5}$

Answer

$9\sqrt{5}$

Example 13.8.3E.49

$4\sqrt{5}+8\sqrt{5}$

Example 13.8.3E.50

$9\sqrt{7}-10\sqrt{7}$

Answer

$-\sqrt{7}$

Example 13.8.3E.51

$11\sqrt{7}-12\sqrt{7}$

Example 13.8.3E.52

$7\sqrt{y}+2\sqrt{y}$

Answer

$9\sqrt{y}$

Example 13.8.3E.53

$9\sqrt{n}+3\sqrt{n}$

Example 13.8.3E.54

$\sqrt{a}-4\sqrt{a}$

Answer

$-3\sqrt{a}$

Example 13.8.3E.55

$\sqrt{b}-6\sqrt{b}$

Example 13.8.3E.56

$5\sqrt{c}+2\sqrt{c}$

Answer

$7\sqrt{c}$

Example 13.8.3E.57

$7\sqrt{d}+2\sqrt{d}$

Example 13.8.3E.58

$8\sqrt{a}-2\sqrt{b}$

Answer

$8\sqrt{a}-2\sqrt{b}$

Example 13.8.3E.59

$5\sqrt{c}-3\sqrt{d}$

Example 13.8.3E.60

$5\sqrt{m}+\sqrt{n}$

Answer

$5\sqrt{m}+\sqrt{n}$

Example 13.8.3E.61

$\sqrt{n}+3\sqrt{p}$

Example 13.8.3E.62

$8\sqrt{7}+2\sqrt{7}+3\sqrt{7}$

Answer

$13\sqrt{7}$

Example 13.8.3E.63

$6\sqrt{5}+3\sqrt{5}+\sqrt{5}$

Example 13.8.3E.64

$3\sqrt{11}+2\sqrt{11}-8\sqrt{11}$

Answer

$-3\sqrt{11}$

Example 13.8.3E.65

$2\sqrt{15}+5\sqrt{15}-9\sqrt{15}$

Example 13.8.3E.66

$3\sqrt{3}-8\sqrt{3}+7\sqrt{5}$

Answer

$-5\sqrt{3}+7\sqrt{5}$

Example 13.8.3E.67

$5\sqrt{7}-8\sqrt{7}+6\sqrt{3}$

Example 13.8.3E.68

$6\sqrt{2}+2\sqrt{2}-3\sqrt{5}$

Answer

$8\sqrt{2}-3\sqrt{5}$

Example 13.8.3E.69

$7\sqrt{5}+\sqrt{5}-8\sqrt{10}$

Example 13.8.3E.70

$3\sqrt{2a}-4\sqrt{2a}+5\sqrt{2a}$

Answer

$4\sqrt{2a}$

Example 13.8.3E.71

$\sqrt{11b}-5\sqrt{11b}+3\sqrt{11b}$

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Example 13.8.3E.72

$8\sqrt{3c}+2\sqrt{3c}-9\sqrt{3c}$

Answer

$\sqrt{3c}$

Example 13.8.3E.73

$3\sqrt{5d}+8\sqrt{5d}-11\sqrt{5d}$

Example 13.8.3E.74

$5\sqrt{3ab}+\sqrt{3ab}-2\sqrt{3ab}$

Answer

\(4\sqrt{3ab}\

Example 13.8.3E.75

$8\sqrt{11cd}+5\sqrt{11cd}-9\sqrt{11cd}$

Example 13.8.3E.76

$2\sqrt{pq}-5\sqrt{pq}+4\sqrt{pq}$

Answer

$\sqrt{pq}$

Example 13.8.3E.77

$11\sqrt{2rs}-9\sqrt{2rs}+3\sqrt{2rs}$

Add and Subtract Square Roots that Need Simplification

In the following exercises, simplify.

Example 13.8.3E.78

$\sqrt{50}+4\sqrt{2}$

Answer

$9\sqrt{2}$

Example 13.8.3E.79

$\sqrt{48}+2\sqrt{3}$

Example 13.8.3E.80

$\sqrt{80}-3\sqrt{5}$

Answer

$\sqrt{5}$

Example 13.8.3E.81

$\sqrt{28}-4\sqrt{7}$

Example 13.8.3E.82

$\sqrt{27}-\sqrt{75}$

Answer

$-2\sqrt{3}$

Example 13.8.3E.83

$\sqrt{72}-\sqrt{98}$

Example 13.8.3E.84

$\sqrt{48}+\sqrt{27}$

Answer

$7\sqrt{3}$

Example 13.8.3E.85

$\sqrt{45}+\sqrt{80}$

Example 13.8.3E.86

$2\sqrt{50}-3\sqrt{72}$

Answer

$-8\sqrt{2}$

Example 13.8.3E.87

$3\sqrt{98}-\sqrt{128}$

Example 13.8.3E.88

$2\sqrt{12}+3\sqrt{48}$

Answer

$16\sqrt{3}$

Example 13.8.3E.89

$4\sqrt{75}+2\sqrt{108}$

Example 13.8.3E.90

$\frac{2}{3}\sqrt{72}+\frac{1}{5}\sqrt{50}$

Answer

$5\sqrt{2}$

Example 13.8.3E.91

$\frac{2}{5}\sqrt{75}+\frac{3}{4}\sqrt{48}$

Example 13.8.3E.92

$\frac{1}{2}\sqrt{20}-\frac{2}{3}\sqrt{45}$

Answer

$-\sqrt{5}$

Example 13.8.3E.93

$\frac{2}{3}\sqrt{54}-\frac{3}{4}\sqrt{96}$

Example 13.8.3E.94

$\frac{1}{6}\sqrt{27}-\frac{3}{8}\sqrt{48}$

Answer

$-\sqrt{3}$

Example 13.8.3E.95

$\frac{1}{8}\sqrt{32}-\frac{1}{10}\sqrt{50}$

Example 13.8.3E.96

$\frac{1}{4}\sqrt{98}-\frac{1}{3}\sqrt{128}$

Answer

$-\frac{3}{4}\sqrt{2}$

Example 13.8.3E.97

$\frac{1}{3}\sqrt{24}+\frac{1}{4}\sqrt{54}$

Example 13.8.3E.98

$\sqrt{72a^5}-\sqrt{50a^5}$

Answer

$a^2\sqrt{2a}$

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Example 13.8.3E.99

$\sqrt{48b^5} - \sqrt{75b^5}$

Example 13.8.3E.100

$\sqrt{80c^7} - \sqrt{20c^7}$

Answer

$2c^3\sqrt{5c}$

Example 13.8.3E.101

$\sqrt{96d^9} - \sqrt{24d^9}$

Example 13.8.3E.102

$9\sqrt{80p^4} - 6\sqrt{98p^4}$

Answer

$36p^2\sqrt{5} - 42p^2\sqrt{2}$

Example 13.8.3E.103

$8\sqrt{72q^6} - 3\sqrt{75q^6}$

Example 13.8.3E.104

$2\sqrt{50r^8} + 4\sqrt{54r^8}$

Answer

$10r^4\sqrt{2} + 12r^4\sqrt{6}$

Example 13.8.3E.105

$5\sqrt{27s^6} + 2\sqrt{20s^6}$

Example 13.8.3E.106

$3\sqrt{20x^2} - 4\sqrt{45x^2} + 5x\sqrt{80}$

Answer

$14x\sqrt{5}$

Example 13.8.3E.107

$2\sqrt{28x^2} - 6\sqrt{3x^2} + 6x\sqrt{7}$

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Example 13.8.3E.108

$3\sqrt{128y^2}+4y\sqrt{162}-8\sqrt{98y^2}$

Answer

$-12y\sqrt{2}$

Example 13.8.3E.109

$3\sqrt{75y^2}+8y\sqrt{48}-\sqrt{300y^2}$

Mixed Practice

Example 13.8.3E.110

$2\sqrt{8}+6\sqrt{8}-5\sqrt{8}$

Answer

$3\sqrt{8}$

Example 13.8.3E.111

$\frac{2}{3}\sqrt{27}+\frac{3}{4}\sqrt{48}$

Example 13.8.3E.112

$\sqrt{175k^4}-\sqrt{63k^4}$

Answer

$2k^2\sqrt{7}$

Example 13.8.3E.113

$\frac{5}{6}\sqrt{162}+\frac{3}{16}\sqrt{128}$

Example 13.8.3E.114

$2\sqrt{363}-2\sqrt{300}$

Answer

$2\sqrt{3}$

Example 13.8.3E.115

$\sqrt{150}+4\sqrt{6}$

Example 13.8.3E.116

$9\sqrt{2}-8\sqrt{2}$

Answer

$\sqrt{2}$

Example 13.8.3E.117

$5\sqrt{x}-8\sqrt{y}$

Example 13.8.3E.118

$8\sqrt{13}-4\sqrt{13}-3\sqrt{13}$

Answer

$\sqrt{13}$

Example 13.8.3E.119

$5\sqrt{12c^4}-3\sqrt{27c^6}$

Example 13.8.3E.120

$\sqrt{80a^5}-\sqrt{45a^5}$

Answer

$a^2\sqrt{5a}$

Example 13.8.3E.121

$\frac{3}{5}\sqrt{75}-\frac{1}{4}\sqrt{48}$

Example 13.8.3E.122

$21\sqrt{19}-2\sqrt{19}$

Answer

$19\sqrt{19}$

Example 13.8.3E.123

$\sqrt{500}+\sqrt{405}$

Example 13.8.3E.124

$\frac{5}{6}\sqrt{27}+\frac{5}{8}\sqrt{48}$

Answer

$5\sqrt{3}$

Example 13.8.3E.125

$11\sqrt{11}-10\sqrt{11}$

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Example 13.8.3E.126

$\sqrt{75}-\sqrt{108}$

Answer

$-\sqrt{3}$

Example 13.8.3E.127

$2\sqrt{98}-4\sqrt{72}$

Example 13.8.3E.128

$4\sqrt{24x^2}-\sqrt{54x^2}+3x\sqrt{6}$

Answer

$8x\sqrt{6}$

Example 13.8.3E.129

$8\sqrt{80y^6}-6\sqrt{48y^6}$

Everyday Math

Example 13.8.3E.130

A decorator decides to use square tiles as an accent strip in the design of a new shower, but she wants to rotate the tiles to look like diamonds. She will use 9 large tiles that measure 8 inches on a side and 8 small tiles that measure 2 inches on a side. Determine the width of the accent strip by simplifying the expression $9(8\sqrt{2})+8(2\sqrt{2})$. (Round to the nearest tenth of an inch.)

Answer

124.5 inches

Example 13.8.3E.131

Suzy wants to use square tiles on the border of a spa she is installing in her backyard. She will use large tiles that have area of 12 square inches, medium tiles that have area of 8 square inches, and small tiles that have area of 4 square inches. Once section of the border will require 4 large tiles, 8 medium tiles, and 10 small tiles to cover the width of the wall. Simplify the expression $4\sqrt{12}+8\sqrt{8}+10\sqrt{4}$ to determine the width of the wall.

Writing Exercises

Example 13.8.3E.132

Explain the difference between like radicals and unlike radicals. Make sure your answer makes sense for radicals containing both numbers and variables.

Answer

Answers will vary.

? Example 13.8.3E.133

Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract like square roots.			
add and subtract square roots that need simplification.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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13.8.4: Multiply Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Multiply square roots
- Use polynomial multiplication to multiply square roots

Note

Before you get started, take this readiness quiz.

1. Simplify: (3u)(8v).
   If you missed this problem, review Example 6.2.31.
2. Simplify: 6(12−7n).
   If you missed this problem, review Example 6.3.1.
3. Simplify: (2+a)(4−a).
   If you missed this problem, review Example 6.3.34.

Multiply Square Roots

We have used the Product Property of Square Roots to simplify square roots by removing the perfect square factors. The Product Property of Square Roots says

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

We can use the Product Property of Square Roots 'in reverse' to multiply square roots.

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Square Roots so we see both ways together.

Definition: PRODUCT PROPERTY OF SQUARE ROOTS

If a, b are nonnegative real numbers, then

$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ and $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

So we can multiply $\sqrt{3} \cdot \sqrt{5}$ in this way:

$$\sqrt{3} \cdot \sqrt{5}$$
$$\sqrt{3 \cdot 5}$$
$$\sqrt{15}$$

Sometimes the product gives us a perfect square:

$$\sqrt{2} \cdot \sqrt{8}$$
$$\sqrt{2 \cdot 8}$$
$$\sqrt{16}$$
$$4$$

Even when the product is not a perfect square, we must look for perfect-square factors and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply 4x·3y we multiply the coefficients together and then the variables. The result is 12xy. Keep this in mind as you do these examples.

Example 13.8.4.1

Simplify:

1. $\sqrt{2} \cdot \sqrt{6}$
2. $(4\sqrt{3})(2\sqrt{12})$.

Answer

1.	$\sqrt{2} \cdot \sqrt{6}$
Multiply using the Product Property.	$\sqrt{12}$
Simplify the radical.	$\sqrt{4} \cdot \sqrt{3}$
Simplify.	$2\sqrt{3}$
2.	$(4\sqrt{3})(2\sqrt{12})$
Multiply using the Product Property.	$8\sqrt{36}$
Simplify the radical.	$8 \cdot 6$
Simplify.	48

Example 13.8.4.2

Simplify:

1. $\sqrt{3} \cdot \sqrt{6}$
2. $(2\sqrt{6})(3\sqrt{12})$.

Answer

1. $3\sqrt{2}$
2. $36\sqrt{2}$

Example 13.8.4.3

Simplify:

1. $\sqrt{5} \cdot \sqrt{10}$
2. $(6\sqrt{3})(5\sqrt{6})$

Answer

1. $5\sqrt{2}$
2. $90\sqrt{2}$

Example 13.8.4.4

Simplify: $(6\sqrt{2})(3\sqrt{10})$

Answer

	$(6\sqrt{2})(3\sqrt{10})$
Multiply using the Product Property.	$18\sqrt{20}$
Simplify the radical.	$18\sqrt{4} \cdot \sqrt{5}$
Simplify.	$18 \cdot 2 \cdot \sqrt{5}$

	$36\sqrt{5}$

Example 13.8.4.5

Simplify: $(3\sqrt{2})(2\sqrt{30})$

Answer

$12\sqrt{15}$

Example 13.8.4.6

Simplify: $(3\sqrt{3})(3\sqrt{6})$.

Answer

$27\sqrt{2}$

When we have to multiply square roots, we first find the product and then remove any perfect square factors.

Example 13.8.4.7

Simplify:

1. $(\sqrt{8x^3})(\sqrt{3x})$
2. $(\sqrt{20y^2})(\sqrt{5y^3})$

Answer

1.	$(\sqrt{8x^3})(\sqrt{3x})$
Multiply using the Product Property.	$\sqrt{24x^4}$
Simplify the radical.	$\sqrt{4x^4}\cdot\sqrt{6}$
Simplify.	$2x^2\sqrt{6}$
2.	$(\sqrt{20y^2})(\sqrt{5y^3})$
Multiply using the Product Property.	$\sqrt{100y^5}$
Simplify the radical.	$10y^2\sqrt{y}$

Example 13.8.4.8

Simplify:

1. $(\sqrt{6x^3})(\sqrt{3x})$
2. $(\sqrt{2y^3})(\sqrt{50y^2})$.

Answer

1. $3x^2\sqrt{2}$
2. $10y^2\sqrt{y}$

Example 13.8.4.9

Simplify:

1. $(\sqrt{6x^5})(\sqrt{2x})$
2. $(\sqrt{12y^2})(\sqrt{3y^5})$

Answer

1. $2x^3\sqrt{3}$
2. $6y^2\sqrt{y}$

Example 13.8.4.10

Simplify: $(10\sqrt{6p^3})(3\sqrt{18p})$

Answer

	$(10\sqrt{6p^3})(3\sqrt{18p})$
Multiply.	$30\sqrt{108p^4}$
Simplify the radical.	$30\sqrt{36p^4}\cdot\sqrt{3}$
	$30\cdot 6p^2\cdot\sqrt{3}$
	$180p^2\sqrt{3}$

Example 13.8.4.11

Simplify: $(\sqrt{62x^2})(8\sqrt{45x^4})$

Answer

$144x^3\sqrt{10}$

Example 13.8.4.12

Simplify: $(2\sqrt{6y^4})(12\sqrt{30y})$.

Answer

$144y^2\sqrt{5y}$

Example 13.8.4.13

Simplify:

1. $(\sqrt{2})^2$
2. $(-\sqrt{11})^2$.

Answer

1.	$(\sqrt{2})^2$
Rewrite as a product.	$(\sqrt{2})(\sqrt{2})$
Multiply.	$\sqrt{4}$
Simplify.	2

2.	$(-\sqrt{11})^2$
Rewrite as a product.	$(-\sqrt{11})(-\sqrt{11})$
Multiply.	$\sqrt{121}$
Simplify.	11

? Example 13.8.4.14

Simplify:

1. $(\sqrt{12})^2$
2. $(-\sqrt{15})^2$.

Answer

1. 12
2. 15

? Example 13.8.4.15

Simplify:

1. $(\sqrt{16})^2$
2. $(-\sqrt{20})^2$.

Answer

1. 16
2. 20

The results of the previous example lead us to this property.

Definition: SQUARING A SQUARE ROOT

If a is a nonnegative real number, then

$(\sqrt{a})^2 = a$

By realizing that squaring and taking a square root are 'opposite' operations, we can simplify $(\sqrt{2})^2$ and get 2 right away. When we multiply the two **like square roots** in part (a) of the next example, it is the same as squaring.

? Example 13.8.4.16

Simplify:

1. $(2\sqrt{3})(8\sqrt{3})$
2. $(3\sqrt{6})^2$.

Answer

1.	$(2\sqrt{3})(8\sqrt{3})$
Multiply. Remember, $(\sqrt{3^2})$	16·3
Simplify.	48
2.	$(3\sqrt{6})^2$
Multiply.	9·6

Simplify.	54

Example 13.8.4.17

Simplify:

1. $(6\sqrt{11})(5\sqrt{11})$
2. $(5\sqrt{8})^2$.

Answer

1. 330
2. 200

Example 13.8.4.18

Simplify:

1. $(3\sqrt{7})(10\sqrt{7})$
2. $(-4\sqrt{6})^2$.

Answer

1. 210
2. 96

Use Polynomial Multiplication to Multiply Square Roots

In the next few examples, we will use the Distributive Property to multiply expressions with square roots.

We will first distribute and then simplify the square roots when possible.

Example 13.8.4.19

Simplify:

1. $3(5-\sqrt{2})$
2. $\sqrt{2}(4-\sqrt{10})$.

Answer

1.	$3(5-\sqrt{2})$
Distribute.	$15-3\sqrt{2})$
2.	$\sqrt{2}(4-\sqrt{10})$
Distribute.	$4\sqrt{2}-\sqrt{20}$
Simplify.	$4\sqrt{2}-2\sqrt{5}$

Example 13.8.4.20

Simplify:

1. $2(3-\sqrt{5})$
2. $\sqrt{3}(2-\sqrt{18})$.

Answer

1. $6-2\sqrt{5}$
2. $2\sqrt{3}-3\sqrt{6}$

Example 13.8.4.21

Simplify:

1. $6(2+\sqrt{6})$
2. $\sqrt{7}(1+\sqrt{14})$.

Answer

1. $12+\sqrt{6}$
2. $\sqrt{7}+7\sqrt{2}$

Example 13.8.4.22

Simplify:

1. $\sqrt{5}(7+2\sqrt{5})$
2. $\sqrt{6}(\sqrt{2}+\sqrt{18})$.

Answer

1.	$\sqrt{5}(7+2\sqrt{5})$
Distribute.	$7\sqrt{5}+2\cdot 5$
Simplify.	$7\sqrt{5}+10$
	$10+7\sqrt{5}$
2.	$\sqrt{6}(\sqrt{2}+\sqrt{18})$
Multiply.	$\sqrt{12}+\sqrt{108}$
Simplify.	$(\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3})$
	$2\sqrt{3}+6\sqrt{3}$
Combine like radicals.	$8\sqrt{3}$

Example 13.8.4.23

Simplify:

1. $\sqrt{6}(1+3\sqrt{6})$
2. $\sqrt{12}(\sqrt{3}+\sqrt{24})$

Answer

1. $18+\sqrt{6}$
2. $6+12\sqrt{2}$

Example 13.8.4.24

Simplify:

1. $\sqrt{8}(2-5\sqrt{8})$
2. $\sqrt{14}(\sqrt{2}+\sqrt{42})$

Answer

1. $-40+4\sqrt{2}$
2. $2\sqrt{7}+14\sqrt{3}$

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

Example 13.8.4.25

Simplify: $(2+\sqrt{3})(4-\sqrt{3})$

Answer

	$(2+\sqrt{3})(4-\sqrt{3})$
Multiply.	$8-2\sqrt{3}+4\sqrt{3}-3$
Combine like terms.	$5+2\sqrt{3}$

Example 13.8.4.26

Simplify: $(1+\sqrt{6})(3-\sqrt{6})$.

Answer

$-3+2\sqrt{6}$

Example 13.8.4.27

Simplify: $(4-\sqrt{10})(2+\sqrt{10})$.

Answer

$-2+2\sqrt{10}$

Example 13.8.4.28

Simplify: $(3-2\sqrt{7})(4-2\sqrt{7})$.

Answer

	$(3-2\sqrt{7})(4-2\sqrt{7})$
Multiply.	$12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$
Simplify.	$12-6\sqrt{7}-8\sqrt{7}+28$
Combine like terms.	$40-14\sqrt{7}$

Example 13.8.4.29

Simplify: $(6-3\sqrt{7})(3+4\sqrt{7})$.

Answer

$-66+15\sqrt{7}$

Example 13.8.4.30

Simplify: $(2-3\sqrt{11})(4-\sqrt{11})$

Answer

$41+14\sqrt{11}$

Example 13.8.4.31

Simplify: $(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$.

Answer

	$3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$
Multiply.	$3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$
Simplify.	$6+12\sqrt{10}-\sqrt{10}-20$
Combine like terms.	$-14+11\sqrt{10}$

Example 13.8.4.32

Simplify: $(5\sqrt{3}-\sqrt{7})(\sqrt{3}+2\sqrt{7})$

Answer

$1+9\sqrt{21}$

Example 13.8.4.33

Simplify: $(\sqrt{6}-3\sqrt{8})(2\sqrt{6}+\sqrt{8})$

Answer

$-12-20\sqrt{3}$

Example 13.8.4.34

Simplify: $(4-2\sqrt{x})(1+3\sqrt{x})$.

Answer

	$(4-2\sqrt{x})(1+3\sqrt{x})$.
Multiply.	$4+12\sqrt{x}-2\sqrt{x}-6x$
Combine like terms.	$4+10\sqrt{x}-6x$

Example 13.8.4.35

Simplify: $(6-5\sqrt{m})(2+3\sqrt{m})$.

Answer

$12+8\sqrt{m}-15m$

Example 13.8.4.36

Simplify: $(10+3\sqrt{n})(1-5\sqrt{n})$

Answer

$10-47\sqrt{n}-15n$

Note that some special products made our work easier when we multiplied binomials earlier. This is true when we multiply square roots, too. The special product formulas we used are shown below.

Definition: SPECIAL PRODUCT FORMULAS

Binomial Squares	Product of Conjugates
$(a+b)^2=a^2+2ab+b^2$	$(a-b)(a+b)=a^2-b^2$
$(a-b)^2=a^2-2ab+b^2$	

We will use the special product formulas in the next few examples. We will start with the Binomial Squares formula.

Example 13.8.4.37

Simplify:

1. $(2+\sqrt{3})^2$
2. $(4-2\sqrt{5})^2$.

Answer

Be sure to include the 2ab term when squaring a binomial.

1.

	$\overset{(a+b)^2}{(2+\sqrt{3})^2}$
Multiply using the binomial square pattern.	$\overset{a^2+2ab+b^2}{2^2+2\cdot 2\cdot\sqrt{3}+(\sqrt{3})^2}$
Simplify.	$4+4\sqrt{3}+3$
Combine like terms.	$7+4\sqrt{3}$

2.

	$\overset{(a-b)^2}{(4-2\sqrt{5})^2}$
Multiply using the binomial square pattern.	$\overset{a^2-2ab+b^2}{4^2-2\cdot 4\cdot 2\sqrt{5}+(2\sqrt{5})^2}$
Simplify.	$16-16\sqrt{5}+4\cdot 5$ $16-16\sqrt{5}+20$
Combine like terms.	$36-16\sqrt{5}$

Example 13.8.4.38

Simplify:

1. $(10+\sqrt{2})^2$
2. $(1+3\sqrt{6})^2$.

Answer

1. $102+20\sqrt{2}$
2. $55+6\sqrt{6}$

Exercise 13.8.4.39

Simplify:

1. $(6-\sqrt{5})^2$
2. $(9-2\sqrt{10})^2$.

Answer

1. $41-12\sqrt{5}$
2. $121-36\sqrt{10}$

Example 13.8.4.40

Simplify: $(1+3\sqrt{x})^2$.

Answer

	$\overset{(a+b)^2}{(1+3\sqrt{x})^2}$
Multiply using the binomial square pattern.	$\overset{a^2+2ab+b^2}{1^2+2\cdot 1\cdot 3\sqrt{x}+(3\sqrt{x})^2}$
Simplify.	$1+6\sqrt{x}+9x$

Example 13.8.4.41

Simplify: $(2+5\sqrt{m})^2$.

Answer

$4+20\sqrt{m}+25m$

Example 13.8.4.42

Simplify: $(3-4\sqrt{n})^2$.

Answer

$9-24\sqrt{n}+16n$

In the next two examples, we will find the product of conjugates.

Example 13.8.4.43

Simplify: $(4-\sqrt{2})(4+\sqrt{2})$.

Answer

	$\overset{(a-b)(a+b)}{(4-\sqrt{2})(4+\sqrt{2})}$
Multiply using the binomial square pattern.	$\overset{a^2-b^2}{4^2-(\sqrt{2})^2}$

Simplify.	$16-2$ 14

Example 13.8.4.44

Simplify: $(2-\sqrt{3})(2+\sqrt{3})$

Answer

1

ExAMPLe 13.8.4.45

Simplify: $(1+\sqrt{5})(1-\sqrt{5})$

Answer

-4

Example 13.8.4.46

Simplify: $(5-2\sqrt{3})(5+2\sqrt{3})$

Answer

	$(a-b)(a+b)$ $(5-2\sqrt{3})(5+2\sqrt{3})$
Multiply using the binomial square pattern.	a^2-b^2 $5^2-(2\sqrt{3})^2$
Simplify.	$25-4\cdot 3$ 13

Example 13.8.4.47

Simplify: $(3-2\sqrt{5})(3+2\sqrt{5})$.

Answer

-11

Example 13.8.4.48

Simplify: $(4+5\sqrt{7})(4-5\sqrt{7})$.

Answer

-159

Access these online resources for additional instruction and practice with multiplying square roots.

- Product Property
- Multiply Binomials with Square Roots

Key Concepts

- **Product Property of Square Roots** If a, b are nonnegative real numbers, then

  $\sqrt{ab}=\sqrt{a\cdot b}$ and $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$

- **Special formulas** for multiplying binomials and conjugates:

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (a-b)(a+b) = a^2 - b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

- The FOIL method can be used to multiply binomials containing radicals.

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13.8.4E: Exercises

Practice Makes Perfect

Multiply Square Roots

In the following exercises, simplify.

Example 13.8.4E.48

1. $\sqrt{2} \cdot \sqrt{8}$
2. $(3\sqrt{3})(2\sqrt{18})$

Answer

1. 44
2. $18\sqrt{6}$

Example 13.8.4E.49

1. $\sqrt{6} \cdot \sqrt{6}$
2. $(3\sqrt{2})(2\sqrt{32})$

Example 13.8.4E.50

1. $\sqrt{7} \cdot \sqrt{14}$
2. $(4\sqrt{8})(5\sqrt{8})$

Answer

1. $7\sqrt{2}$
2. 160

Example 13.8.4E.51

1. $\sqrt{6} \cdot \sqrt{12}$
2. $(2\sqrt{5})(2\sqrt{10})$

Example 13.8.4E.52

$(5\sqrt{2})(3\sqrt{6})$

Answer

$30\sqrt{3}$

Example 13.8.4E.53

$(2\sqrt{3})(4\sqrt{6})$

Example 13.8.4E.54

$(-2\sqrt{3})(3\sqrt{18})$

Answer

$-18\sqrt{6}$

Example 13.8.4E.55

$(-4\sqrt{5})(5\sqrt{10})$

Example 13.8.4E.56

$(5\sqrt{6})(-\sqrt{12})$

Answer

$-30\sqrt{2}$

Example 13.8.4E.57

$(6\sqrt{2})(-\sqrt{10})$

Example 13.8.4E.58

$(-2\sqrt{7})(-2\sqrt{14})$

Answer

$28\sqrt{2}$

Example 13.8.4E.59

$(-2\sqrt{11})(-4\sqrt{22})$

Example 13.8.4E.60

1. $(\sqrt{15y})(\sqrt{5y^3})$
2. $(\sqrt{2n^2})(\sqrt{18n^3})$

Answer

1. $5y^2\sqrt{3}$
2. $6n^2\sqrt{n}$

Example 13.8.4E.61

1. $(\sqrt{14x^3})(\sqrt{7x^3})$
2. $(\sqrt{3q^2})(\sqrt{48q^3})$

Example 13.8.4E.62

1. $(\sqrt{16y^2})(\sqrt{8y^4})$
2. $(\sqrt{11s^6})(\sqrt{11s})$

Answer

1. $8y^3\sqrt{2}$
2. $11s^3\sqrt{s}$

Example 13.8.4E.63

ⓐ $(\sqrt{8x^3})(\sqrt{3x})$
ⓑ $(\sqrt{7r})(\sqrt{7r^8})$

Example 13.8.4E.64

$(2\sqrt{5b^3})(4\sqrt{15b})$

Answer

$40b^2\sqrt{3}$

Example 13.8.4E.65

$(\sqrt{38c^5})(\sqrt{26c^3})$

Example 13.8.4E.66

$(6\sqrt{3d^3})(4\sqrt{12d^5})$

Answer

$144d^4$

Example 13.8.4E.67

$(2\sqrt{5b^3})(4\sqrt{15b})$

Example 13.8.4E.68

$(2\sqrt{5d^6})(3\sqrt{20d^2})$

Answer

$60d^4$

Example 13.8.4E.69

$(-2\sqrt{7z^3})(3\sqrt{14z^8})$

Example 13.8.4E.70

$(4\sqrt{2k^5})(-3\sqrt{32k^6})$

Answer

$-96k^5\sqrt{k}$

Example 13.8.4E.71

1. $(\sqrt{7})^2$
2. $(-\sqrt{15})^2$

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Example 13.8.4E.72

1. $(\sqrt{11})^2$
2. $(-\sqrt{21})^2$

Answer

1. 11
2. 21

Example 13.8.4E.73

1. $(\sqrt{19})^2$
2. $(-\sqrt{5})^2$

Exercise 13.8.4E.74

1. $(\sqrt{23})^2$
2. $(-\sqrt{3})^2$

Answer

1. 23
2. 3

Example 13.8.4E.75

1. $(4\sqrt{11})(-3\sqrt{11})$
2. $(5\sqrt{3})^2$

Example 13.8.4E.76

1. $(2\sqrt{13})(-9\sqrt{13})$
2. $(6\sqrt{5})^2$

Answer

1. -234
2. 180

Example 13.8.4E.77

1. $(-3\sqrt{12})(-2\sqrt{6})$
2. $(-4\sqrt{10})^2$

Example 13.8.4E.78

1. $(-7\sqrt{5})(-3\sqrt{10})$
2. $(-2\sqrt{14})^2$

Answer

1. $105\sqrt{2}$
2. 56

Use Polynomial Multiplication to Multiply Square Roots

In the following exercises, simplify.

Example 13.8.4E.79

1. $3(4-\sqrt{3})$
2. $\sqrt{2}(4-\sqrt{6})$

Example 13.8.4E.80

1. $4(6-\sqrt{11})$
2. $\sqrt{2}(5-\sqrt{12})$

Answer

1. $24-4\sqrt{11}$
2. $5\sqrt{2}-2\sqrt{6}$

Example 13.8.4E.81

1. $5(3-\sqrt{7})$
2. $\sqrt{3}(4-\sqrt{15})$

Example 13.8.4E.82

1. $7(-2-\sqrt{11})$
2. $\sqrt{7}(6-\sqrt{14})$

Answer

1. $-14-7\sqrt{11}$
2. $6\sqrt{7}-7\sqrt{2}$

Example 13.8.4E.83

1. $\sqrt{7}(5+2\sqrt{7})$
2. $\sqrt{5}(\sqrt{10}+\sqrt{18})$

Example 13.8.4E.84

1. $\sqrt{11}(8+4\sqrt{11})$
2. $\sqrt{3}(\sqrt{12}+\sqrt{27})$

Answer

1. $44+8\sqrt{11}$
2. 15

Example 13.8.4E.85

1. $\sqrt{11}(-3+4\sqrt{1})$
2. $\sqrt{3}(\sqrt{15}-\sqrt{18})$

Example 13.8.4E.86

1. $\sqrt{2}(-5+9\sqrt{2})$
2. $\sqrt{7}(\sqrt{3}-\sqrt{21})$

Answer

1. $18-5\sqrt{2}$

2. $\sqrt{21}-7\sqrt{3}$

Example 13.8.4E.87

$(8+\sqrt{3})(2-\sqrt{3})$

Example 13.8.4E.88

$(7+\sqrt{3})(9-\sqrt{3})$

Answer

$60+2\sqrt{3}$

Example 13.8.4E.89

$(8-\sqrt{2})(3+\sqrt{2})$

Example 13.8.4E.90

$(9-\sqrt{2})(6+\sqrt{2})$

Answer

$52+3\sqrt{2}$

Example 13.8.4E.91

$(3-\sqrt{7})(5-\sqrt{7})$

Example 13.8.4E.92

$(5-\sqrt{7})(4-\sqrt{7})$

Answer

$27-9\sqrt{7}$

Example 13.8.4E.93

$(1+3\sqrt{10})(5-2\sqrt{10})$

Exercise 13.8.4E.94

$(7-2\sqrt{5})(4+9\sqrt{5})$

Answer

$-62+55\sqrt{5}$

Example 13.8.4E.95

$(\sqrt{3}+\sqrt{10})(\sqrt{3}+2\sqrt{10})$

Example 13.8.4E.96

$(\sqrt{11}+\sqrt{5})(\sqrt{11}+6\sqrt{5})$

Answer

$41+7\sqrt{55}$

Example 13.8.4E.97

$(2\sqrt{7}-5\sqrt{11})(4\sqrt{7}+9\sqrt{11})$

Example 13.8.4E.98

$(4\sqrt{6}+7\sqrt{13})(8\sqrt{6}-3\sqrt{13})$

Answer

$-81+44\sqrt{78}$

Example 13.8.4E.99

$(5-\sqrt{u})(3+\sqrt{u})$

Example 13.8.4E.100

$(9-\sqrt{w})(2+\sqrt{w})$

Answer

$18+7\sqrt{w}$

Example 13.8.4E.101

$(7+2\sqrt{m})(4+9\sqrt{m})$

Example 13.8.4E.102

$(6+5\sqrt{n})(11+3\sqrt{n})$

Answer

$66+73\sqrt{n}+15n$

Example 13.8.4E.103

1. $(3+\sqrt{5})^2$
2. $(2-5\sqrt{3})^2$

Example 13.8.4E.104

1. $(4+\sqrt{11})^2$
2. $(3-2\sqrt{5})^2$

Answer

1. $27+8\sqrt{11}$

2. $29-12\sqrt{5}$

Example 13.8.4E.105

1. $(9-\sqrt{6})^2$
2. $(10+3\sqrt{7})^2$

Example 13.8.4E.106

1. $(5-\sqrt{10})^2$
2. $(8+3\sqrt{2})^2$

Answer

1. $35-10\sqrt{10}$
2. $82+48\sqrt{2}$

Example 13.8.4E.107

$(3-\sqrt{5})(3+\sqrt{5})$

Example 13.8.4E.108

$(10-\sqrt{3})(10+\sqrt{3})$

Answer

97

Example 13.8.4E.109

$(4+\sqrt{2})(4-\sqrt{2})$

Example 13.8.4E.110

$(7+\sqrt{10})(7-\sqrt{10})$

Answer

39

Example 13.8.4E.111

$(4+9\sqrt{3})(4-9\sqrt{3})$

Example 13.8.4E.112

$(1+8\sqrt{2})(1-8\sqrt{2})$

Answer

-127

Example 13.8.4E.113

$(12-5\sqrt{5})(12+5\sqrt{5})$

Example 13.8.4E.114

$(9-4\sqrt{3})(9+4\sqrt{3})$

Answer

33

Mixed Practice

In the following exercises, simplify.

Example 13.8.4E.115

$\sqrt{3}\cdot\sqrt{21}$

Example 13.8.4E.116

$(4\sqrt{6})(-\sqrt{18})$

Answer

$-24\sqrt{3}$

Example 13.8.4E.117

$(-5+\sqrt{7})(6+\sqrt{21})$

Example 13.8.4E.118

$(-5\sqrt{7})(6\sqrt{21})$

Answer

$-210\sqrt{3}$

Example 13.8.4E.119

$(-4\sqrt{2})(2\sqrt{18})$

Example 13.8.4E.120

$(\sqrt{35y^3})(\sqrt{7y^3})$

Answer

$7y^3\sqrt{5}$

Example 13.8.4E.121

$(4\sqrt{12x^5})(2\sqrt{6x^3})$

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Example 13.8.4E.122

$(\sqrt{29})^2$

Answer

29

Example 13.8.4E.123

$(-4\sqrt{17})(-3\sqrt{17})$

Example 13.8.4E.124

$(-4+\sqrt{17})(-3+\sqrt{17})$

Answer

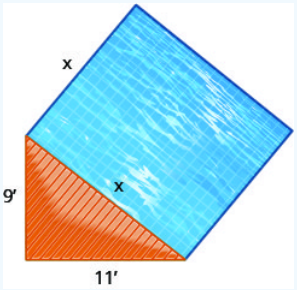
$29-7\sqrt{17}$

Everyday Math

Example 13.8.4E.125

A landscaper wants to put a square reflecting pool next to a triangular deck, as shown below. The triangular deck is a right triangle, with legs of length 9 feet and 11 feet, and the pool will be adjacent to the hypotenuse.

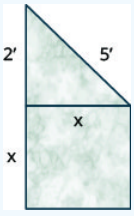
1. Use the Pythagorean Theorem to find the length of a side of the pool. Round your answer to the nearest tenth of a foot.
2. Find the exact area of the pool.



Example 13.8.4E.126

An artist wants to make a small monument in the shape of a square base topped by a right triangle, as shown below. The square base will be adjacent to one leg of the triangle. The other leg of the triangle will measure 2 feet and the hypotenuse will be 5 feet.

1. Use the Pythagorean Theorem to find the length of a side of the square base. Round your answer to the nearest tenth of a foot.



2. Find the exact area of the face of the square base.

Answer

1. 4.6feet
2. 21 sq. feet

Example 13.8.4E.127

A square garden will be made with a stone border on one edge. If only $3+\sqrt{10}$ feet of stone are available, simplify $(3+\sqrt{10})^2$ to determine the area of the largest such garden.

Example 13.8.4E.128

A garden will be made so as to contain two square sections, one section with side length $\sqrt{5}+\sqrt{6}$ yards and one section with side length $\sqrt{2}+\sqrt{3}$ yards. Simplify $(\sqrt{5}+\sqrt{6})(\sqrt{2}+\sqrt{3})$ to determine the total area of the garden.

Example 13.8.4E.129

Suppose a third section will be added to the garden in the previous exercise. The third section is to have a width of $\sqrt{432}$ feet. Write an expression that gives the total area of the garden.

Writing Exercises

Example 13.8.4E.130

1. Explain why $(-\sqrt{n})^2$ is always positive, for $n \geq 0$.
2. Explain why $-(\sqrt{n})^2$ is always negative, for $n \geq 0$.

Answer

1. when squaring a negative, it becomes a positive
2. since the negative is not included in the parenthesis, it is not squared, and remains negative

Example 13.8.4E.131

Use the binomial square pattern to simplify $(3+\sqrt{2})^2$. Explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
multiply square roots.			
use polynomial multiplication to multiply square roots.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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13.8.5: Divide Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Divide square roots
- Rationalize a one-term denominator
- Rationalize a two-term denominator

Note

Before you get started, take this readiness quiz.

1. Find a fraction equivalent to $\frac{5}{8}$ with denominator 48.
   If you missed this problem, review Exercise 1.6.1.
2. Simplify: $(\sqrt{5})^2$.
   If you missed this problem, review Example 9.4.13.
3. Multiply: (7+3x)(7−3x).
   If you missed this problem, review Exercise 6.4.22.

Divide Square Roots

We know that we simplify fractions by removing factors common to the numerator and the denominator. When we have a fraction with a square root in the numerator, we first simplify the square root. Then we can look for common factors.

Common Factors	No common factors
$\frac{3\sqrt{2}}{3 \cdot 5}$	$\frac{2\sqrt{3}}{3 \cdot 5}$

Example 13.8.5.1

Simplify: $\frac{\sqrt{54}}{6}$

Answer

	$\frac{\sqrt{54}}{6}$
Simplify the radical.	$\frac{\sqrt{9} \cdot \sqrt{6}}{6}$
Simplify.	$\frac{3\sqrt{6}}{6}$
Remove the common factors.	$\frac{3\sqrt{6}}{3 \cdot 2}$
Simplify.	$\frac{\sqrt{6}}{2}$

Example 13.8.5.2

Simplify: $\frac{\sqrt{32}}{8}$.

Answer

$\frac{\sqrt{2}}{2}$

Example 13.8.5.3

Simplify: $\frac{\sqrt{75}}{15}$.

Answer

$\frac{\sqrt{3}}{3}$

Example 13.8.5.4

Simplify: $\frac{6-\sqrt{24}}{12}$.

Answer

	$\frac{6-\sqrt{24}}{12}$
Simplify the radical.	$\frac{6-\sqrt{4}\cdot\sqrt{6}}{12}$
Simplify.	$\frac{6-2\sqrt{6}}{12}$
Factor the common factor from the numerator.	$\frac{2(3-\sqrt{6})}{12}$
Remove the common factors.	$\frac{2(3-\sqrt{6})}{2\cdot 6}$
Simplify.	$\frac{3-\sqrt{6}}{6}$

Example 13.8.5.5

Simplify: $\frac{8-\sqrt{40}}{10}$.

Answer

$\frac{4-\sqrt{10}}{5}$

Example 13.8.5.6

Simplify: $\frac{10-\sqrt{75}}{20}$.

Answer

$\frac{5-\sqrt{3}}{4}$

We have used the Quotient Property of Square Roots to simplify square roots of fractions. The Quotient Property of Square Roots says

$$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}, b\neq 0.$$

Sometimes we will need to use the Quotient Property of Square Roots 'in reverse' to simplify a fraction with square roots.

$$\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}, b\neq 0.$$

We will rewrite the Quotient Property of Square Roots so we see both ways together. Remember: we assume all variables are greater than or equal to zero so that their square roots are real numbers.

Definition: QUOTIENT PROPERTY OF SQUARE ROOTS

If a, b are non-negative real numbers and $b \neq 0$, then

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ and $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

We will use the Quotient Property of Square Roots 'in reverse' when the fraction we start with is the quotient of two square roots, and neither radicand is a perfect square. When we write the fraction in a single square root, we may find common factors in the numerator and denominator.

Example 13.8.5.7

Simplify: $\frac{\sqrt{27}}{\sqrt{75}}$

Answer

	$\frac{\sqrt{27}}{\sqrt{75}}$
Neither radicand is a perfect square, so rewrite using the quotient property of square root.	$\sqrt{\frac{27}{75}}$
Remove common factors in the numerator and denominator.	$\sqrt{\frac{9}{25}}$
Simplify.	$\frac{3}{5}$

Example 13.8.5.8

Simplify: $\frac{\sqrt{48}}{\sqrt{108}}$

Answer

$\frac{2}{3}$

Example 13.8.5.9

Simplify: $\frac{\sqrt{96}}{\sqrt{54}}$

Answer

$\frac{4}{3}$

We will use the Quotient Property for Exponents, $\frac{a^m}{a^n} = a^{m-n}$, when we have variables with exponents in the radicands.

Example 13.8.5.10

Simplify: $\frac{\sqrt{6y^5}}{\sqrt{2y}}$

Answer

	$\frac{\sqrt{6y^5}}{\sqrt{2y}}$
Neither radicand is a perfect square, so rewrite using the quotient property of square root.	$\sqrt{\frac{6y^5}{2y}}$

Remove common factors in the numerator and denominator.	$\sqrt{3y^4}$
Simplify.	$y^2\sqrt{3}$

Example 13.8.5.11

Simplify: $\frac{\sqrt{12r^3}}{\sqrt{6r}}$.

Answer

$r\sqrt{2}$

Example 13.8.5.12

Simplify: $\frac{\sqrt{14p^9}}{\sqrt{2p^5}}$

Answer

$p^2\sqrt{7}$

Example 13.8.5.13

Simplify: $\frac{\sqrt{72x^3}}{\sqrt{162x}}$

Answer

	$\frac{\sqrt{72x^3}}{\sqrt{162x}}$
Rewrite using the quotient property of square roots.	$\sqrt{\frac{72x^3}{162x}}$
Remove common factors.	$\sqrt{\frac{18\cdot4\cdot x^2\cdot x}{18\cdot9\cdot x}}$
Simplify.	$\sqrt{\frac{4x^2}{9}}$
Simplify the radical.	$\frac{2x}{3}$

ExAMPLe 13.8.5.14

Simplify: $\frac{\sqrt{50s^3}}{\sqrt{128s}}$.

Answer

$\frac{5s}{8}$

Example 13.8.5.15

Simplify: $\frac{\sqrt{75q^5}}{\sqrt{108q}}$.

Answer

$\frac{5q^2}{6}$

Example 13.8.5.16

Simplify: $\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$.

Answer

	$\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$
Rewrite using the quotient property of square roots.	$\sqrt{\frac{147ab^8}{3a^3b^4}}$
Remove common factors.	\(\sqrt{\frac{49b^4}{a^2}}\
Simplify the radical.	$\frac{7b^2}{a}$

Example 13.8.5.17

Simplify: $\frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$.

Answer

$\frac{9x^2}{y^2}$

Example 13.8.5.18

Simplify: $\frac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$.

Answer

$\frac{10n^3}{m}$

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, tables of square roots were used to find approximate values of square roots. Figure shows a portion of a table of squares and square roots. Square roots are approximated to five decimal places in this table.

n	n^2	$\sqrt{n}$
200	40,000	14.14214
201	40,401	14.17745
202	40,804	14.21267
203	41,209	14.24781
204	41,616	14.28286
205	42,025	14.31782
206	42,436	14.35270
207	42,849	14.38749
208	43,264	14.42221
209	43,681	14.45683
210	44,100	14.49138

A table of square roots was used to find approximate values of square roots before there were calculators.

If someone needed to approximate a fraction with a square root in the denominator, it meant doing long division with a five decimal-place divisor. This was a very cumbersome process.

For this reason, a process called rationalizing the denominator was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. This process is still used today and is useful in other areas of mathematics, too.

Definition: RATIONALIZING THE DENOMINATOR

The process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer is called **rationalizing the denominator**.

Square roots of numbers that are not perfect squares are irrational numbers. When we **rationalize the denominator**, we write an equivalent fraction with a rational number in the denominator.

Let's look at a numerical example.

Suppose we need an approximate value for the fraction.	$\frac{1}{\sqrt{2}}$
A five decimal place approximation to $\sqrt{2}$ is 1.41421	$\frac{1}{1.41421}$
Without a calculator, would you want to do this division?	$1.41421\overline{)1.0}$

But we can find a fraction equivalent to $\frac{1}{\sqrt{2}}$ by multiplying the numerator and denominator by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}}$$
$$\frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$$
$$\frac{\sqrt{2}}{2}$$

Now if we need an approximate value, we divide $2\overline{)1.41421}$. This is much easier.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a square root.

Similarly, a square root is not considered simplified if the radicand contains a fraction.

Definition: SIMPLIFIED SQUARE ROOTS

A square root is considered simplified if there are

- no perfect-square factors in the radicand
- no fractions in the radicand
- no square roots in the denominator of a fraction

To rationalize a denominator, we use the property that $(\sqrt{a})^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

Example 13.8.5.19

Simplify: $\frac{4}{\sqrt{3}}$.

Answer

To remove the square root from the denominator, we multiply it by itself. To keep the fractions equivalent, we multiply both the numerator and denominator by the same factor.

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$	$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	$\frac{4\sqrt{3}}{3}$

Example 13.8.5.20

Simplify: $\frac{5}{\sqrt{3}}$.

Answer

$\frac{5\sqrt{3}}{3}$

Example 13.8.5.21

Simplify: $\frac{6}{\sqrt{5}}$.

Answer

$\frac{6\sqrt{5}}{5}$

Example 13.8.5.22

Simplify: $-\frac{8}{3\sqrt{6}}$

Answer

To remove the square root from the denominator, we multiply it by itself. To keep the fractions equivalent, we multiply both the numerator and denominator by $\sqrt{6}$.

	$-\frac{8}{3\sqrt{6}}$
Multiply both the numerator and the denominator by $\sqrt{6}$.	$-\frac{8 \cdot \sqrt{6}}{3\sqrt{6} \cdot \sqrt{6}}$
Simplify.	$-\frac{8\sqrt{6}}{3 \cdot 6}$
Remove common factors.	$-\frac{4 \cdot \cancel{2}\sqrt{6}}{3 \cdot \cancel{2} \cdot 3}$
Simplify.	$-\frac{4\sqrt{6}}{9}$

Example 13.8.5.23

Simplify: $\frac{5}{2\sqrt{5}}$.

Answer

$\frac{\sqrt{5}}{2}$

Example 13.8.5.24

Simplify: $-\frac{9}{4\sqrt{3}}$.

Answer

$-\frac{3\sqrt{3}}{4}$

Always simplify the radical in the denominator first, before you rationalize it. This way the numbers stay smaller and easier to work with.

Example 13.8.5.25

Simplify: $\sqrt{\frac{5}{12}}$.

Answer

	$\sqrt{\frac{5}{12}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$\frac{\sqrt{5}}{\sqrt{12}}$
Simplify the denominator.	$\frac{\sqrt{5}}{2\sqrt{3}}$
Rationalize the denominator.	$\frac{\sqrt{5}\cdot\sqrt{3}}{2\sqrt{3}\cdot\sqrt{3}}$
Simplify.	$\frac{\sqrt{15}}{2\cdot 3}$
Simplify.	$\frac{\sqrt{15}}{6}$

Example 13.8.5.26

Simplify: $\sqrt{\frac{7}{18}}$.

Answer

$\frac{\sqrt{14}}{6}$

Example 13.8.5.27

Simplify: $\sqrt{\frac{3}{32}}$.

Answer

$\frac{\sqrt{6}}{8}$

Example 13.8.5.28

Simplify: $\sqrt{\frac{11}{28}}$

Answer

	$\sqrt{\frac{11}{28}}$
Rewrite using the Quotient Property.	$\frac{\sqrt{11}}{\sqrt{28}}$
Simplify the denominator.	$\frac{\sqrt{11}}{2\sqrt{7}}$

Rationalize the denominator.	$\frac{\sqrt{11}\cdot\sqrt{7}}{2\sqrt{7}\cdot\sqrt{7}}$
Simplify.	$\frac{\sqrt{77}}{2\cdot 7}$
Simplify.	$\frac{\sqrt{77}}{14}$

Example 13.8.5.29

Simplify: $\sqrt{\frac{3}{27}}$.

Answer

$\frac{1}{3}$

Example 13.8.5.30

Simplify: $\sqrt{\frac{10}{50}}$

Answer

$\frac{\sqrt{5}}{5}$

Rationalize a Two-Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates pattern to rationalize the denominator.

$$\begin{array}{ll} (a-b)(a+b) & (2-\sqrt{5})(2+\sqrt{5}) \\ a^2-b^2 & 2^2-(\sqrt{5})^2 \\ & 4-5 \\ & -1 \end{array}$$

When we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

Example 13.8.5.31

Simplify: $\frac{4}{4+\sqrt{2}}$.

Answer

	$\frac{4}{4+\sqrt{2}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{4(4-\sqrt{2})}{(4+\sqrt{2})(4-\sqrt{2})}$
Multiply the conjugates in the denominator.	$\frac{4(4-\sqrt{2})}{4^2-(\sqrt{2})^2}$
Simplify the denominator.	$\frac{4(4-\sqrt{2})}{16-2}$
Simplify the denominator.	$\frac{4(4-\sqrt{2})}{14}$

Remove common factors from the numerator and denominator.	$\frac{2(4-\sqrt{2})}{7}$
We leave the numerator in factored form to make it easier to look for common factors after we have simplified the denominator.	

Example 13.8.5.32

Simplify: $\frac{2}{2+\sqrt{3}}$.

Answer

$\frac{2(2-\sqrt{3})}{1}$

Example 13.8.5.33

Simplify: $\frac{5}{5+\sqrt{3}}$.

Answer

$\frac{5(5-\sqrt{3})}{22}$

Example 13.8.5.34

Simplify: $\frac{5}{2-\sqrt{3}}$.

Answer

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$
Multiply the conjugates in the denominator.	$\frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{1}$
Simplify.	$5(2+\sqrt{3})$

Example 13.8.5.35

Simplify: $\frac{3}{1-\sqrt{5}}$.

Answer

$-\frac{3(1+\sqrt{5})}{4}$

Example 13.8.5.36

Simplify: $\frac{2}{4-\sqrt{6}}$.

Answer

$\frac{4+\sqrt{6}}{5}$

Example 13.8.5.37

Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Answer

	$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u}-\sqrt{6})(\sqrt{u}+\sqrt{6})}$
Multiply the conjugates in the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u})^2-(\sqrt{6})^2}$
Simplify the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}$

Example 13.8.5.38

Simplify: $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

Answer

$\frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{x-2}$

Example 13.8.5.39

Simplify: $\frac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$.

Answer

$\frac{\sqrt{10}(\sqrt{y}+\sqrt{3})}{y-3}$

Example 13.8.5.40

Simplify: $\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Answer

	$\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$

Multiply the conjugates in the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$\frac{(\sqrt{x}+\sqrt{7})^2}{x-7}$
We do not square the numerator. In factored form, we can see there are no common factors to remove from the numerator and denominator.	

Example 13.8.5.41

Simplify: $\frac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

Answer

$\frac{(\sqrt{p}+\sqrt{2})^2}{p-2}$

Example 13.8.5.42

Simplify: $\frac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$.

Answer

$\frac{(\sqrt{q}-\sqrt{10})^2}{q-10}$

Access this online resource for additional instruction and practice with dividing and rationalizing.

- Dividing and Rationalizing

Key Concepts

- **Quotient Property of Square Roots**
  - If a, b are non-negative real numbers and $b \neq 0$, then

    $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ and $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- **Simplified Square Roots**
  A square root is considered simplified if there are
  - no perfect square factors in the radicand
  - no fractions in the radicand
  - no square roots in the denominator of a fraction

Glossary

rationalizing the denominator

The process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer is called rationalizing the denominator.

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13.8.6: Solve Equations with Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Use square roots in applications

Note

Before you get started, take this readiness quiz.

1. Simplify: ⓐ $\sqrt{9}$ ⓑ 9^2.
   If you missed this problem, review Example 9.1.1 and Exercise 1.3.22.
2. Solve: 5(x+1)−4=3(2x−7).
   If you missed this problem, review Exercise 2.4.16.
3. Solve: $n^2 - 6n + 8 = 0$.
   If you missed this problem, review Exercise 7.6.13.

Solve Radical Equations

In this section we will solve equations that have the variable in the radicand of a square root. Equations of this type are called radical equations.

Definition: RADICAL EQUATION

An equation in which the variable is in the radicand of a square root is called a **radical equation**.

As usual, in solving these equations, what we do to one side of an equation we must do to the other side as well. Since squaring a quantity and taking a square root are 'opposite' operations, we will square both sides in order to remove the radical sign and solve for the variable inside.

But remember that when we write $\sqrt{a}$ we mean the principal square root. So $\sqrt{a} \geq 0$ always. When we solve radical equations by squaring both sides we may get an algebraic solution that would make $\sqrt{a}$ negative. This algebraic solution would not be a solution to the original **radical equation**; it is an *extraneous solution.* We saw extraneous solutions when we solved rational equations, too.

Example 13.8.6.1

For the equation $\sqrt{x+2} = x$:

1. Is x=2 a solution?
2. Is x=−1 a solution?

Answer

1. Is x=2 a solution?

	$\sqrt{x+2} = x$
Let x = 2.	$\sqrt{2+2} \stackrel{?}{=} 2$
Simplify.	$\sqrt{4} \stackrel{?}{=} 2$
	$2 = 2$ ✓
	2 is a solution.

2. Is x=−1 a solution?

	$\sqrt{x+2}=x$
Let $x=-1$.	$\sqrt{-1+2}\stackrel{?}{=}-1$
Simplify.	$\sqrt{1}\stackrel{?}{=}-1$
	$1\neq -1$
	-1 is not a solution.
	-1 is an extraneous solution to the equation.

Example 13.8.6.2

For the equation $\sqrt{x+6}=x$:

1. Is x=−2 a solution?
2. Is x=3 a solution?

Answer

1. no
2. yes

Example 13.8.6.3

For the equation $\sqrt{-x+2}=x$:

1. Is x=−2 a solution?
2. Is x=1 a solution?

Answer

1. no
2. yes

Now we will see how to solve a radical equation. Our strategy is based on the relation between taking a square root and squaring.

For$a\geq 0$, $(\sqrt{a})^2=a$

How to Solve Radical Equations

Example 13.8.6.4

Solve: $\sqrt{2x-1}=7$

Answer

Step 1. Isolate the radical on one side of the equation.	$\sqrt{2x-1}$ is already isolated on the left side.	$\sqrt{2x-1}=7$
Step 2. Square both sides of the equation.	Remember, $(\sqrt{a})^2=a$	$(\sqrt{2x-1})^2=(7)^2$

Step 3. Solve the new equation.		$2x-1=49$ $2x=50$ $x=25$
Step 4. Check the answer.		Check: $\sqrt{2x-1}=7$ $\sqrt{2(25)-1}\stackrel{?}{=}7$ $\sqrt{50-1}\stackrel{?}{=}7$ $\sqrt{49}\stackrel{?}{=}7$ $7=7$ ✓ The solution is $x=25$.

Example 13.8.6.5

Solve: $\sqrt{3x-5}=5$.

Answer

10

Example 13.8.6.6

Solve: $\sqrt{4x+8}=6$.

Answer

7

Definition: SOLVE A RADICAL EQUATION.

1. Isolate the radical on one side of the equation.
2. Square both sides of the equation.
3. Solve the new equation.
4. Check the answer.

Example 13.8.6.7

Solve: $\sqrt{5n-4}-9=0$.

Answer

	$\sqrt{5n-4}-9=0$
To isolate the radical, add 9 to both sides.	$\sqrt{5n-4}-9+9=0+9$
Simplify.	$\sqrt{5n-4}=9$
Square both sides of the equation.	$(\sqrt{5n-4})^2=(9)^2$
Solve the new equation.	$5n-4=81$
	$5n=85$
	$n=17$

Check the answer. $\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9\stackrel{?}{=}0$ $0=0$ ✓	
	The solution is $n = 17$.

Example 13.8.6.8

Solve: $\sqrt{3m+2}-5=0$.

Answer

$\frac{23}{3}$

Example 13.8.6.9

Solve: $\sqrt{10z+1}-2=0$.

Answer

$\frac{3}{10}$

Example 13.8.6.10

Solve: $\sqrt{3y+5}+2=5$.

Answer

	$\sqrt{3y+5}+2=5$
To isolate the radical, subtract 2 from both sides.	$\sqrt{3y+5}+2-2=5-2$
Simplify.	$\sqrt{3y+5}=3$
Square both sides of the equation.	$(\sqrt{3y+5})^2=(3)^2$
Solve the new equation.	$3y+5=9$
	$3y=4$
	$y=\frac{4}{3}$
Check the answer. $\sqrt{3y+5}+2=5$ $\sqrt{3\left(\frac{4}{3}\right)+5}+2\stackrel{?}{=}5$ $\sqrt{4+5}+2\stackrel{?}{=}5$ $\sqrt{9}+2\stackrel{?}{=}5$ $3+2\stackrel{?}{=}5$ $5=5$ ✓	

	The solution is $y = \frac{4}{3}$

Example 13.8.6.11

Solve: $\sqrt{3p+3}+3=5$.

Answer

$\frac{1}{3}$

Example 13.8.6.12

Solve: $\sqrt{5q+1}+4=6$.

Answer

$\frac{3}{5}$

When we use a radical sign, we mean the principal or positive root. If an equation has a square root equal to a negative number, that equation will have no solution.

Example 13.8.6.13

Solve: $\sqrt{9k-2}+1=0$.

Answer

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 from both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$
Since the square root is equal to a negative number, the equation has no solution.	

Example 13.8.6.14

Solve: $\sqrt{2r-3}+5=0$

Answer

no solution

Example 13.8.6.15

Solve: $\sqrt{7s-3}+2=0$.

Answer

no solution

If one side of the equation is a binomial, we use the binomial squares formula when we square it.

Definition: BINOMIAL SQUARES

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (a-b)^2 = a^2 - 2ab + b^2$$

Don't forget the middle term!

Example 13.8.6.16

Solve: $\sqrt{p-1}+1=p$.

Answer

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$
Use the zero product property.	$0=p-1 \quad 0=p-2$
Solve each equation.	$p=1 \quad p=2$
Check the answers. $p=1$: $\sqrt{p-1}+1=p$; $\sqrt{1-1}+1\stackrel{?}{=}1$; $\sqrt{0}+1\stackrel{?}{=}1$; $1=1$ ✓ — $p=2$: $\sqrt{p-1}+1=p$; $\sqrt{2-1}+1\stackrel{?}{=}2$; $\sqrt{1}+1\stackrel{?}{=}2$; $2=2$ ✓	
	The solutions are $p = 1$, $p = 2$.

Example 13.8.6.17

Solve: $\sqrt{x-2}+2=x$.

Answer

2, 3

Example 13.8.6.18

Solve: $\sqrt{y-5}+5=y$.

Answer

5, 6

Example 13.8.6.19

Solve: $\sqrt{r+4}-r+2=0$.

Answer

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$(\sqrt{r+4})^2=(r-2)^2$
Solve the new equation.	$r+4=r^2-4r+4$
It is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the zero product property.	0=r 0=r−5
Solve the equation.	r=0 r=5
Check the answer. $r=0$ $\sqrt{r+4}-r+2=0$ $\sqrt{0+4}-0+2\stackrel{?}{=}0$ $\sqrt{4}+2\stackrel{?}{=}0$ $4\neq 0$ $r=5$ $\sqrt{r+4}-r+2=0$ $\sqrt{5+4}-5+2\stackrel{?}{=}0$ $\sqrt{9}-3\stackrel{?}{=}0$ $0=0$ ✓	r=5
	r=0 is an extraneous solution.

Example 13.8.6.20

Solve: $\sqrt{m+9}-m+3=0$.

Answer

7

Example 13.8.6.21

Solve: $\sqrt{n+1}-n+1=0$

Answer

3

When there is a coefficient in front of the radical, we must square it, too.

Example 13.8.6.22

Solve: $3\sqrt{3x-5}-8=4$.

Answer

	$3\sqrt{3x-5}-8=4$
Isolate the radical.	$3\sqrt{3x-5}=12$
Square both sides of the equation.	$(3\sqrt{3x-5})^2=(12)^2$

Simplify, then solve the new equation.	9(3x−5)=144
Distribute.	27x−45=144
Solve the equation.	27x=189
	x=7
Check the answer. $x=7 \quad 3\sqrt{3x-5}-8=4$ $3\sqrt{3(7)-5}-8\stackrel{?}{=}4$ $3\sqrt{21-5}-8\stackrel{?}{=}4$ $3\sqrt{16}-8\stackrel{?}{=}4$ $3(4)-8\stackrel{?}{=}4$ $4=4\checkmark$	The solution is x=7.

? Example 13.8.6.23

Solve: $\sqrt{24a+2}-16=16$.

Answer

$\frac{127}{2}$

? Example 13.8.6.24

Solve: $\sqrt{36b+3}-25=50$.

Answer

$\frac{311}{3}$

? Example 13.8.6.25

Solve: $\sqrt{4z-3}=\sqrt{3z+2}$.

Answer

	$\sqrt{4z-3}=\sqrt{3z+2}$
The radical terms are isolated	$\sqrt{4z-3}=\sqrt{3z+2}$
Square both sides of the equation.	$(\sqrt{4z-3})^2=(\sqrt{3z+2})^2$
Simplify, then solve the new equation	4z−3=3z+2
	z−3=2
	z=5
	x=7
Check the answer. We leave it to you to show that 5 checks!	The solution is z=5.

Example 13.8.6.26

Solve: $\sqrt{2x-5}=\sqrt{5x+3}$.

Answer

no solution

Example 13.8.6.27

Solve: $\sqrt{7y+1}=\sqrt{2y-5}$.

Answer

no solution

Sometimes after squaring both sides of an equation, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and square both sides of the equation again.

Example 13.8.6.28

Solve: $\sqrt{m}+1=\sqrt{m+9}$.

Answer

	$\sqrt{m}+1=\sqrt{m+9}$
The radical on the right side is isolated. Square both sides	$(\sqrt{m}+1)^2=(\sqrt{m+9})^2$
Simplify—be very careful as you multiply!	$m+2\sqrt{m}+1=m+9$
There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical.	$2\sqrt{m}=8$
Square both sides.	$(2\sqrt{m})^2=(8)^2$
Simplify, then solve the new equation.	4m=64
	m=16
Check the answer. We leave it to you to show that m=16 checks!	The solution is m=16.

Example 13.8.6.29

Solve: $\sqrt{x}+3=\sqrt{x+5}$.

Answer

no solution

Example 13.8.6.30

Solve: $\sqrt{m}+5=\sqrt{m+16}$.

Answer

no solution

Example 13.8.6.31

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Answer

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right side is isolated. Square both sides	$(\sqrt{q-2}+3)^2=(\sqrt{4q+1})^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides.	$(6\sqrt{q-2})^2=(3q-6)^2$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$
Distribute.	$36q-72=9q^2-36q+36$
It is a quadratic equation, so get zero on one side.	$0=9q^2-72q+108$
Factor the right side.	$0=9(q^2-8q+12)$ $0=9(q-6)(q-2)$
Use the zero product property	$q-6=0 \quad q-2=0$ $q=6 \quad q=2$
The checks are left to you. (Both solutionsshould work.)	The solutions are q=6 and q=2.

Example 13.8.6.32

Solve: $\sqrt{y-3}+2=\sqrt{4y+2}$.

Answer

no solution

Example 13.8.6.33

Solve: $\sqrt{n-4}+5=\sqrt{3n+3}$.

Answer

no solution

Use Square Roots in Applications

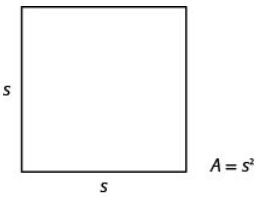
As you progress through your college courses, you'll encounter formulas that include square roots in many disciplines. We have already used formulas to solve geometry applications.

We will use our Problem Solving Strategy for Geometry Applications, with slight modifications, to give us a plan for solving applications with formulas from any discipline.

Definition: SOLVE APPLICATIONS WITH FORMULAS.

1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We used the formula A=L·W to find the area of a rectangle with length *L* and width *W*. A square is a rectangle in which the length and width are equal. If we let *s* be the length of a side of a square, the area of the square is \(s^2\).

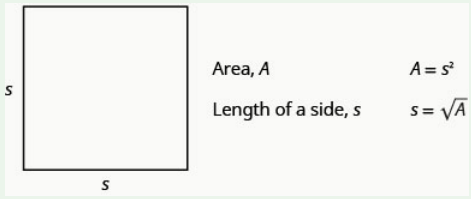


The formula $A = s^2$ gives us the area of a square if we know the length of a side. What if we want to find the length of a side for a given area? Then we need to solve the equation for *s*.

	$A = s^2$
Take the square root of both sides.	$\sqrt{A} = \sqrt{s^2}$
Simplify.	$s = \sqrt{A}$

We can use the formula $s = \sqrt{A}$ to find the length of a side of a square for a given area.

Definition: AREA OF A SQUARE

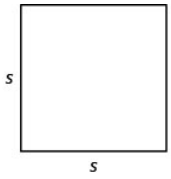


We will show an example of this in the next example.

Example 13.8.6.34

Mike and Lychelle want to make a square patio. They have enough concrete to pave an area of 200 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of the patio. Round your answer to the nearest tenth of a foot.

Answer

Step 1. Read the problem. Draw a figure and label it with the given information.	
	A = 200 square feet

Step 2. Identify what you are looking for.	The length of a side of the square patio.
Step 3. Name what you are looking for by choosing a variable to represent it.	Let s = the length of a side.
Step 4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute the given information.	$s = \sqrt{A}$, and $A = 200$ $s = \sqrt{200}$
Step 5. Solve the equation using good algebra techniques. Round to one decimal place.	$s = 14.14213...$ $s \approx 14.1$
Step 6. Check the answer in the problem and make sure it makes sense.	
$14.1^2 \stackrel{?}{\approx} 200$ $14.1^2 \approx 198.81$ ✓	
This is close enough because we rounded the square root. Is a patio with side 14.1 feet reasonable? Yes.	
Step 7. Answer the question with a complete sentence.	Each side of the patio should be 14.1 feet.

? Example 13.8.6.35

Katie wants to plant a square lawn in her front yard. She has enough sod to cover an area of 370 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of her lawn. Round your answer to the nearest tenth of a foot.

Answer

19.2 feet

? Example 13.8.6.36

Sergio wants to make a square mosaic as an inlay for a table he is building. He has enough tile to cover an area of 2704 square centimeters. Use the formula $s = \sqrt{A}$ to find the length of each side of his mosaic. Round your answer to the nearest tenth of a foot.

Answer

52.0 cm

Another application of square roots has to do with gravity.

Definition: FALLING OBJECTS

On Earth, if an object is dropped from a height of hh feet, the time in seconds it will take to reach the ground is found by using the formula,

$t = \frac{\sqrt{h}}{4}$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting h=64 into the formula.

	$t=\frac{\sqrt{h}}{4}$
	$t=\frac{\sqrt{64}}{4}$
Take the square root of 64.	$t=\frac{8}{4}$
Simplify the fraction.	$t=2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

? Example 13.8.6.37

Christy dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	The time it takes for the sunglasses to reach the river.
Step 3. Name what you are looking for by choosing a variable to represent it.	Let t = time.
Step 4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.	$t=\frac{\sqrt{h}}{4}$, and $h=400$ $t=\frac{\sqrt{400}}{4}$
Step 5. Solve the equation using good algebra techniques.	$t=\frac{20}{4}$ $t=5$
Step 6. Check the answer in the problem and make sure it makes sense.	
$5\stackrel{?}{=}\frac{\sqrt{400}}{4}$ $5\stackrel{?}{=}\frac{20}{4}$ 5=5✓	
Does 5 seconds seem reasonable? Yes.	
Step 7. Answer the question with a complete sentence.	It will take 5 seconds for the sunglasses to hit the water.

? Exercise 13.8.6.38

A helicopter dropped a rescue package from a height of 1,296 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Answer

9 seconds

Example 13.8.6.39

A window washer dropped a squeegee from a platform 196 feet above the sidewalk Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Answer

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Definition: SKID MARKS AND SPEED OF A CAR

If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula,

$s = \sqrt{24d}$

Example 13.8.6.40

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

Step 1. **Read** the problem.	
Step 2. **Identify** what we are looking for.	The speed of a car.
Step 3. **Name** what we are looking for.	Let s = the speed.
Step 4. **Translate** into an equation by writing the appropriate formula.	$s = \sqrt{24d}$, and $d = 190$
Substitute the given information.	$s = \sqrt{24(190)}$
Step 5. **Solve the equation**.	$s = \sqrt{4560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
Step 6. **Check** the answer in the problem. $67.5 \approx ?\sqrt{24(190)}$ $67.5 \approx ?\sqrt{4560}$ $67.5 \approx ?67.5277...$	
Is 67.5 mph a reasonable speed?	Yes.
Step 7. **Answer** the question with a complete sentence.	The speed of the car was approximately 67.5 miles per hour.

Example 13.8.6.41

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

42.7 feet

Example 13.8.6.42

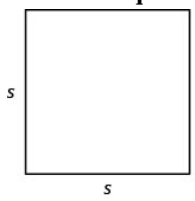
The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

54.1 feet

Key Concepts

- **To Solve a Radical Equation:**
  1. Isolate the radical on one side of the equation.
  2. Square both sides of the equation.
  3. Solve the new equation.
  4. Check the answer. Some solutions obtained may not work in the original equation.
- **Solving Applications with Formulas**
  1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for by choosing a variable to represent it.
  4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. **Solve the equation** using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- **Area of a Square**



Area, A	$A = s^2$
Length of a side, s	$s = \sqrt{A}$

- **Falling Objects**
  - On Earth, if an object is dropped from a height of hh feet, the time in seconds it will take to reach the ground is found by using the formula$t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
  - If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Glossary

radical equation

An equation in which the variable is in the radicand of a square root is called a radical equation

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13.8.6E: Exercises

Practice Makes Perfect

Solve Radical Equations

In the following exercises, check whether the given values are solutions.

Example 13.8.6E.43

For the equation $\sqrt{x+12}=x$:

1. Is x=4 a solution?
2. Is x=−3 a solution?

Answer

1. yes
2. no

Example 13.8.6E.44

For the equation $\sqrt{-y+20}=y$

1. Is y=4 a solution?
2. Is y=−5 a solution?

Example 13.8.6E.45

For the equation $\sqrt{t+6}=t$:

1. Is t=−2 a solution?
2. Is t=3 a solution?

Answer

1. no
2. yes

Example 13.8.6E.46

For the equation $\sqrt{u+42}=u$:

1. Is u=−6 a solution?
2. Is u=7 a solution?

In the following exercises, solve.

Example 13.8.6E.47

$\sqrt{5y+1}=4$

Answer

3

Example 13.8.6E.48

$\sqrt{7z+15}=6$

Example 13.8.6E.49

$\sqrt{5x-6}=8$

Answer

14

Example 13.8.6E.50

$\sqrt{4x-3}=7$

Example 13.8.6E.51

$\sqrt{2m-3}-5=0$

Answer

14

Example 13.8.6E.52

$\sqrt{2n-1}-3=0$

Example 13.8.6E.53

$\sqrt{6v-2}-10=0$

Answer

17

Example 13.8.6E.54

$\sqrt{4u+2}-6=0$

Example 13.8.6E.55

$\sqrt{5q+3}-4=0$

Answer

$\frac{13}{5}$

Example 13.8.6E.56

$\sqrt{4m+2}+2=6$

Example 13.8.6E.57

$\sqrt{6n+1}+4=8$

Answer

$\frac{5}{2}$

Example 13.8.6E.58

$\sqrt{2u-3}+2=0$

Example 13.8.6E.59

$\sqrt{5v-2}+5=0$

Answer

no solution

Example 13.8.6E.60

$\sqrt{3z-5}+2=0$

Example 13.8.6E.61

$\sqrt{2m+1}+4=0$

Answer

no solution

Example 13.8.6E.62

1. $\sqrt{u-3}+3=u$
2. $\sqrt{x+1}-x+1=0$

Example 13.8.6E.63

1. $\sqrt{v-10}+10=v$
2. $\sqrt{y+4}-y+2=0$

Answer

1. 10, 11
2. 5

Example 13.8.6E.64

1. $\sqrt{r-1}-r=-1$
2. $\sqrt{z+100}-z+10=0$

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Example 13.8.6E.65

1. $\sqrt{s-8}-s=-8$
2. $\sqrt{w+25}-w+5=0$

Answer

1. 8,9
2. 11

Example 13.8.6E.66

$3\sqrt{2x-3}-20=7$

Example 13.8.6E.67

$2\sqrt{5x+1}-8=0$

Answer

3

Example 13.8.6E.68

$2\sqrt{8r+1}-8=2$

Example 13.8.6E.69

$3\sqrt{7y+1}-10=8$

Answer

5

Example 13.8.6E.70

$\sqrt{3u-2}=\sqrt{5u+1}$

Example 13.8.6E.71

$\sqrt{4v+3}=\sqrt{v-6}$

Answer

not a real number

Example 13.8.6E.72

$\sqrt{8+2r}=\sqrt{3r+10}$

Example 13.8.6E.73

$\sqrt{12c+6}=\sqrt{10-4c}$

Answer

$\frac{1}{4}$

Example 13.8.6E.74

1. $\sqrt{a}+2=\sqrt{a+4}$
2. $\sqrt{b-2}+1=\sqrt{3b+2}$

Example 13.8.6E.75

1. $\sqrt{r}+6=\sqrt{r+8}$
2. $\sqrt{s-3}+2=\sqrt{s+4}$

Answer

1. no solution
2. $\frac{57}{16}$

Example 13.8.6E.76

1. $\sqrt{u}+1=\sqrt{u+4}$
2. $\sqrt{n-5}+4=\sqrt{3n+7}$

Example 13.8.6E.77

1. $\sqrt{x}+10=\sqrt{x+2}$
2. $\sqrt{y-2}+2=\sqrt{2y+4}$

Answer

1. no solution
2. 6

Example 13.8.6E.78

$\sqrt{2y+4}+6=0$

Example 13.8.6E.79

$\sqrt{8u+1}+9=0$

Answer

no solution

Example 13.8.6E.80

$\sqrt{a}+1=\sqrt{a+5}$

Example 13.8.6E.81

$\sqrt{d}-2=\sqrt{d-20}$

Answer

36

Example 13.8.6E.82

$\sqrt{6s+4} = \sqrt{8s-28}$

Example 13.8.6E.83

$\sqrt{9p+9} = \sqrt{10p-6}$

Answer

15

Use Square Roots in Applications

In the following exercises, solve. Round approximations to one decimal place.

Example 13.8.6E.84

Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

Example 13.8.6E.85

Landscaping Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.

Answer

11.4 feet

Example 13.8.6E.86

Gravity While putting up holiday decorations, Renee dropped a light bulb from the top of a 64 foot tall tree. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the light bulb to reach the ground.

Example 13.8.6E.87

Gravity An airplane dropped a flare from a height of 1024 feet above a lake. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the flare to reach the water.

Answer

8 seconds

Example 13.8.6E.88

Gravity A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.

Example 13.8.6E.89

Gravity A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.

Answer

15.8 seconds

Example 13.8.6E.90

Accident investigation The skid marks for a car involved in an accident measured 54 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Example 13.8.6E.91

Accident investigation The skid marks for a car involved in an accident measured 216 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

72 feet

Example 13.8.6E.92

Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Example 13.8.6E.93

Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 117 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

53.0 feet

Writing Exercises

Example 13.8.6E.94

Explain why an equation of the form $\sqrt{x} + 1 = 0$ has no solution.

Example 13.8.6E.95

1. ⓐ Solve the equation $\sqrt{r+4} - r + 2 = 0$.
2. ⓑ Explain why one of the "solutions" that was found was not actually a solution to the equation.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use square roots in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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13.8.7: Higher Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with higher roots
- Use the Product Property to simplify expressions with higher roots
- Use the Quotient Property to simplify expressions with higher roots
- Add and subtract higher roots

Note

1. Simplify: $y^5 y^4$.
   If you missed this problem, review Example 6.2.7.
2. Simplify: $(n^2)^6$.
   If you missed this problem, review Example 6.2.19.
3. Simplify: $\frac{x^8}{x^3}$.
   If you missed this problem, review Example 6.5.1.

Simplify Expressions with Higher Roots

Up to now, in this chapter we have worked with squares and square roots. We will now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

We write:	We say:
n^2	n squared
n^3	n cubed
n^4	n to the fourth
n^5	n to the fifth

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from -5to5. See Figure \PageIdnex1.

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

Figure 13.8.7.1: First through fifth powers of integers from −5 to 5.

Notice the signs in Figure 13.8.7.1. All powers of positive numbers are positive, of course. But when we have a negative number, the even powers are positive and the odd powers are negative. We'll copy the row with the powers of −2 below to help you see this.

n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

Earlier in this chapter we defined the square root of a number.

If $n^2 = m$,then n is a square root of m.

And we have used the notation $\sqrt{m}$ to denote the **principal square root**. So $\sqrt{m} \geq 0$ always.

We will now extend the definition to higher roots.

Definition: *N*TH ROOT OF A NUMBER

If $b^n = a$, then b is an ***n*th root of a number** a.

The principal *n*th root of a is written $\sqrt[n]{a} = b$

n is called the **index** of the radical.

We do not write the index for a square root. Just like we use the word 'cubed' for b^3, we use the term 'cube root' for $\sqrt[3]{a}$.

We refer to Figure 13.8.7.1to help us find higher roots.

$$\begin{aligned} 4^3 &= 64 & \sqrt[3]{64} &= 4 \\ 3^4 &= 81 & \sqrt[4]{81} &= 3 \\ (-2)^5 &= -32 & \sqrt[5]{-32} &= -2 \end{aligned}$$

Could we have an even root of a negative number? No. We know that the square root of a negative number is not a real number. The same is true for any even root. Even roots of negative numbers are not real numbers. Odd roots of negative numbers are real numbers.

Definition: PROPERTIES OF $\sqrt[n]{a}$

When n is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number
- $a < 0$, then $\sqrt[n]{a}$ is not a real number

When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

Example 13.8.7.1

Simplify:

1. $\sqrt[3]{8}$
2. $\sqrt[4]{81}$
3. $\sqrt[5]{32}$.

Answer

1.	$\sqrt[3]{8}$
Since $(2)^3 = 8$.	2
2.	$\sqrt[4]{81}$
Since $(3)^4 = 81$.	3
3.	$\sqrt[5]{32}$
Since $(2)^5 = 32$.	2

Example 13.8.7.2

Simplify:

1. $\sqrt[3]{27}$
2. $\sqrt[4]{256}$
3. $\sqrt[5]{243}$.

Answer

1. 3

2. 4
3. 3

Example 13.8.7.3

Simplify:

1. $\sqrt[3]{1000}$
2. $\sqrt[4]{16}$
3. $\sqrt[5]{32}$.

Answer

1. 10
2. 2
3. 2

Example 13.8.7.4

Simplify:

1. $\sqrt[3]{-64}$
2. $\sqrt[4]{-16}$
3. $\sqrt[5]{-243}$.

Answer

1.	$\sqrt[3]{-64}$
Since $(-4)^3 = -64$.	-4
2.	$\sqrt[4]{-16}$
Think, $(?)^4 = -16$.No real number raised to the fourth power is positive.	Not a real number.
3.	$\sqrt[5]{-243}$
Since $(-3)^5 = -243$.	-3

Example 13.8.7.5

Simplify:

1. $\sqrt[3]{-125}$
2. $\sqrt[4]{-16}$
3. $\sqrt[5]{-32}$.

Answer

1. -5
2. not real
3. -2

Example 13.8.7.6

Simplify:

1. $\sqrt[3]{-216}$

2. $\sqrt[4]{-81}$
3. $\sqrt[5]{-1024}$.

Answer

1. −6
2. not real
3. −4

When we worked with square roots that had variables in the radicand, we restricted the variables to non-negative values. Now we will remove this restriction.

The odd root of a number can be either positive or negative. We have seen that $\sqrt[3]{-64} = -4$.

But the even root of a non-negative number is always non-negative, because we take the **principal *n*th root**.

Suppose we start with a=−5.

$$(-5)^4 = 625 \quad \sqrt[4]{625} = 5$$

How can we make sure the fourth root of −5 raised to the fourth power, $(-5)^4$ is 5? We will see in the following property.

Definition: SIMPLIFYING ODD AND EVEN ROOTS

For any integer $n \geq 2$,

$$\begin{array}{ll} \text{when n is odd} & \sqrt[n]{a^n} = a \\ \text{when n is even} & \sqrt[n]{a^n} = |a| \end{array}$$

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Example 13.8.7.7

Simplify:

1. $\sqrt{x^2}$
2. $\sqrt[3]{n^3}$
3. $\sqrt[4]{p^4}$
4. $\sqrt[5]{y^5}$.

Answer

We use the absolute value to be sure to get the positive root.

1.	$\sqrt{x^2}$
Since $(x)^2 = x^2$ and we want the positive root.	|x|
2.	$\sqrt[3]{n^3}$
Since $(n)^3 = n^3$. It is an odd root so there is no need for an absolute value sign.	n
3.	$\sqrt[4]{p^4}$
Since $(p)^4 = p^4$ and we want the positive root.	|p|
4.	$\sqrt[5]{y^5}$
Since $(y)^5 = y^5$. It is an odd root so there is no need for an absolute value sign.	y

Example 13.8.7.8

Simplify:

1. $\sqrt{b^2}$
2. $\sqrt[3]{w^3}$
3. $\sqrt[4]{m^4}$
4. $\sqrt[5]{q^5}$.

Answer

1. |b|
2. w
3. |m|
4. q

Example 13.8.7.9

Simplify:

1. $\sqrt{y^2}$
2. $\sqrt[3]{p^3}$
3. $\sqrt[4]{z^4}$
4. $\sqrt[5]{q^5}$

Answer

1. |y|
2. p
3. |z|
4. q

Example 13.8.7.10

Simplify:

1. $\sqrt[3]{y^{18}}$
2. $\sqrt[4]{z^8}$.

Answer

1.	$\sqrt[3]{y^{18}}$
Since $(y^6)^3 = y^18$.	$\sqrt[3]{(y^6)^3}$
	y^6
2.	$\sqrt[4]{z^8}$
Since $(z^2)^4 = z^8$.	$\sqrt[4]{(z^2)^4}$
Since z^2 is positive, we do not need an absolute value sign.	z^2

Example 13.8.7.11

Simplify:

1. $\sqrt[4]{u^{12}}$
2. $\sqrt[3]{v^{15}}$.

Answer

1. u^3
2. v^5

Example 13.8.7.12

Simplify:

1. $\sqrt[5]{c^{20}}$
2. $\sqrt[6]{d^{24}}$.

Answer

1. c^4
2. d^4

Example 13.8.7.13

Simplify:

1. $\sqrt[3]{64p^6}$
2. $\sqrt[4]{16q^{12}}$.

Answer

1.	$\sqrt[3]{64p^6}$
Rewrite $64p^6$ as $(4p^2)^3$.	$\sqrt[3]{(4p^2)^3}$
Take the cube root.	$4p^2$
2.	$\sqrt[4]{16q^{12}}$
Rewrite the radicand as a fourth power.	$\sqrt[4]{(2q^3)^4}$
Take the fourth root.	$2\lvert q^3\rvert$

Example 13.8.7.14

Simplify:

1. $\sqrt[3]{27x^{27}}$
2. $\sqrt[4]{81q^{28}}$.

Answer

1. $3x^9$
2. $3 \mid q^7\rvert$

Example 13.8.7.15

Simplify:

1. $\sqrt[3]{125p^9}$
2. $\sqrt[5]{243q^{25}}$

Answer

1. $5p^3$
2. $3q^5$

Use the Product Property to Simplify Expressions with Higher Roots

We will simplify expressions with higher roots in much the same way as we simplified expressions with square roots. An *n*th root is considered simplified if it has no factors of m^n.

Definition: SIMPLIFIED *N*TH ROOT

$\sqrt[n]{a}$ is considered simplified if a has no factors of m^n.

We will generalize the Product Property of Square Roots to include any integer root $n \geq 2$.

Definition: PRODUCT PROPERTY OF *N*TH ROOTS

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

when $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and for any integer $n \geq 2$

Example 13.8.7.16

Simplify:

1. $\sqrt[3]{x^4}$
2. $\sqrt[4]{x^7}$.

Answer

1.	$\sqrt[3]{x^4}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{x^3 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$
Simplify.	$x\sqrt[3]{x}$
2.	$\sqrt[4]{x^7}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{x^4 \cdot x^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$
Simplify.	$\lvert x \rvert \sqrt[4]{x^3}$

Example 13.8.7.17

Simplify:

1. $\sqrt[4]{y^6}$
2. $\sqrt[3]{z^5}$.

Answer

1. $\lvert y \rvert \sqrt[4]{y^2}$
2. $z\sqrt[3]{z^2}$

Example 13.8.7.18

Simplify:

1. $\sqrt[5]{p^8}$
2. $\sqrt[6]{q^{13}}$.

Answer

1. $p\sqrt[5]{p^3}$
2. $q^2\sqrt[6]{q}$

Example 13.8.7.19

Simplify:

1. $\sqrt[3]{16}$
2. $\sqrt[4]{243}$.

Answer

1.	$\sqrt[3]{16}$
	$\sqrt[3]{2^4}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{2^3 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{2^3} \cdot \sqrt[3]{2}$
Simplify.	$2\sqrt[3]{2}$
2.	$\sqrt[4]{243}$
	$\sqrt[4]{3^5}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{3^4 \cdot 3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{3^4} \cdot \sqrt[4]{3}$
Simplify.	$3\sqrt[4]{3}$

Example 13.8.7.20

Simplify:

1. $\sqrt[3]{81}$
2. $\sqrt[4]{64}$.

Answer

1. $3\sqrt[3]{3}$
2. $2\sqrt[4]{4}$

Example 13.8.7.21

Simplify:

1. $\sqrt[3]{625}$
2. $\sqrt[4]{729}$.

Answer

1. $5\sqrt[3]{5}$
2. $3\sqrt[4]{9}$

Don't forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

? Example 13.8.7.22

Simplify:

1. $\sqrt[3]{24x^7}$
2. $\sqrt[4]{80y^{14}}$.

Answer

1.	$\sqrt[3]{24x^7}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{2^3x^6\cdot 3x}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{2^3x^6}\cdot\sqrt[3]{3x}$
Rewrite the first radicand as $(2x^2)^3$	$\sqrt[3]{(2x^2)^3}\cdot\sqrt[3]{3x}$
Simplify.	$2x^2\sqrt[3]{3x}$
2.	$\sqrt[4]{80y^{14}}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{2^4y^{12}\cdot 5y^2}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{2^4y^{12}}\cdot\sqrt[4]{5y^2}$
Rewrite the first radicand as $(2y^3)^4$	$\sqrt[4]{(2y^3)^4}\cdot\sqrt[4]{5y^2}$
Simplify.	$2\lvert y^3\rvert\sqrt[4]{5y^2}$

? Example 13.8.7.23

Simplify:

1. $\sqrt[3]{54p^{10}}$
2. $\sqrt[4]{64q^{10}}$.

Answer

1. $3p^3\sqrt[3]{2p}$
2. $2q^2\sqrt[4]{4q^2}$

? Example 13.8.7.24

Simplify:

1. $\sqrt[3]{128m^{11}}$
2. $\sqrt[4]{162n^7}$.

Answer

1. $4m^3\sqrt[3]{2m^2}$
2. $3\lvert n\rvert\sqrt[4]{2n^3}$

Example 13.8.7.25

Simplify:

1. $\sqrt[3]{-27}$
2. $\sqrt[4]{-16}$.

Answer

1.	$\sqrt[3]{-27}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3}$
Take the cube root.	-3
2.	$\sqrt[4]{-16}$
There is no real number n where $n^4 = -16$.	Not a real number.

Example 13.8.7.26

Simplify:

1. $\sqrt[3]{-108}$
2. $\sqrt[4]{-48}$.

Answer

1. $-3\sqrt[3]{4}$
2. not real

Example 13.8.7.27

Simplify:

1. $\sqrt[3]{-625}$
2. $\sqrt[4]{-324}$.

Answer

1. $-5\sqrt[3]{5}$
2. not real

Use the Quotient Property to Simplify Expressions with Higher Roots

We can simplify higher roots with quotients in the same way we simplified square roots. First we simplify any fractions inside the radical.

Example 13.8.7.28

Simplify:

1. $\sqrt[3]{\frac{a^8}{a^5}}$
2. $\sqrt[4]{\frac{a^{10}}{a^2}}$.

Answer

1.	$\sqrt[3]{\frac{a^8}{a^5}}$
Simplify the fraction under the radical first.	$\sqrt[3]{a^3}$

Simplify.	a
2.	$\sqrt[4]{\frac{a^{10}}{a^2}}$
Simplify the fraction under the radical first.	$\sqrt[4]{a^8}$
Rewrite the radicand using perfect fourth power factors.	$\sqrt[4]{(a^2)^4}$
Simplify.	a^2

Example 13.8.7.29

Simplify:

1. $\sqrt[4]{\frac{x^7}{x^3}}$
2. $\sqrt[4]{\frac{y^{17}}{y^5}}$.

Answer

1. |x|
2. y^3

Example 13.8.7.30

Simplify:

1. $\sqrt[3]{\frac{m^{13}}{m^7}}$
2. $\sqrt[5]{\frac{n^{12}}{n^2}}$.

Answer

1. m^2
2. n^2

Previously, we used the Quotient Property 'in reverse' to simplify square roots. Now we will generalize the formula to include higher roots.

Definition: QUOTIENT PROPERTY OF *N*TH ROOTS

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

when $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$

Exercise 13.8.7.31

Simplify:

1. $\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$
2. $\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$

Answer

1.	$\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$

Neither radicand is a perfect cube, so use the Quotient Property to write as one radical.	$\sqrt[3]{\frac{-108}{2}}$
Simplify the fraction under the radical.	$\sqrt[3]{-54}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{(-3)^3} \cdot \sqrt[3]{2}$
Simplify.	$-3\sqrt[3]{2}$
2.	$\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$
Neither radicand is a perfect fourth power, so use the Quotient Property to write as one radical	$\sqrt[4]{\frac{96x^7}{3x^2}}$
Simplify the fraction under the radical.	$\sqrt[4]{32x^5}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{2^4x^4 \cdot 2x}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$
Simplify.	$2\lvert x\rvert\sqrt[4]{2x}$

? Example 13.8.7.32

Simplify:

1. $\frac{\sqrt[3]{-532}}{\sqrt[3]{2}}$
2. $\frac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$

Answer

1. not real
2. $3\lvert m\rvert\sqrt[4]{2m^2}$

? Example 13.8.7.33

Simplify:

1. $\frac{\sqrt[3]{-192}}{\sqrt[3]{3}}$
2. $\frac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$.

Answer

1. -4
2. $3\lvert n\rvert\sqrt[4]{2}$

If the fraction inside the radical cannot be simplified, we use the first form of the Quotient Property to rewrite the expression as the quotient of two radicals.

? Example 13.8.7.34

Simplify:

1. $\sqrt[3]{\frac{24x^7}{y^3}}$

2. $\sqrt[4]{\frac{48x^{10}}{y^8}}$.

Answer

1.	$\sqrt[3]{\frac{24x^7}{y^3}}$
The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.	$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$
Rewrite each radicand as a product using perfect cube factors.	$\frac{\sqrt[3]{8x^6 \cdot 3x}}{\sqrt[3]{y^3}}$
Rewrite the numerator as the product of two radicals.	$\frac{\sqrt[3]{(2x^2)^3} \cdot \sqrt[3]{3x}}{\sqrt[3]{y^3}}$
Simplify.	$\frac{2x^2\sqrt[3]{3x}}{y}$
2.	$\sqrt[4]{\frac{48x^{10}}{y^8}}$
The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.	$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$
Rewrite each radicand as a product using perfect cube factors.	$\frac{\sqrt[4]{16x^8 \cdot 3x^2}}{\sqrt[4]{y^8}}$
Rewrite the numerator as the product of two radicals.	$\frac{\sqrt[4]{(2x^2)^4} \cdot \sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$
Simplify.	$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$

Example 13.8.7.35

Simplify:

1. $\sqrt[3]{\frac{108c^{10}}{d^6}}$
2. $\sqrt[4]{\frac{80x^{10}}{y^5}}$.

Answer

1. $\frac{3c^3\sqrt[3]{4c}}{d^2}$
2. $\frac{x^2}{|y|}\sqrt[4]{\frac{80x^2}{y}}$

Example 13.8.7.36

Simplify:

1. $\sqrt[3]{\frac{40r^3}{s}}$
2. $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$

Answer

1. $r\sqrt[3]{\frac{40}{s}}$
2. $\frac{3m^3\sqrt[4]{2m^2}}{|n^3|}$

Add and Subtract Higher Roots

We can add and subtract higher roots like we added and subtracted square roots. First we provide a formal definition of **like radicals**.

Definition: LIKE RADICALS

Radicals with the same index and same radicand are called **like radicals**.

Like radicals have the same index and the same radicand.

- $9\sqrt[4]{42x}$ and $-2\sqrt[4]{42x}$ are like radicals.
- $5\sqrt[3]{125x}$ and $6\sqrt[3]{125y}$ are not like radicals. The radicands are different.
- $2\sqrt[5]{1000q}$ and $-4\sqrt[4]{1000q}$ are not like radicals. The indices are different.

We add and subtract like radicals in the same way we add and subtract like terms. We can add $9\sqrt[4]{42x}+(-2\sqrt[4]{42x})$ and the result is $7\sqrt[4]{42x}$.

Example 13.8.7.37

Simplify:

1. $\sqrt[3]{4x}+\sqrt[3]{4x}$
2. $4\sqrt[4]{8}-2\sqrt[4]{8}$

Answer

1.	$\sqrt[3]{4x}+\sqrt[3]{4x}$
The radicals are like, so we add the coefficients	$2\sqrt[3]{4x}$
2.	$4\sqrt[4]{8}-2\sqrt[4]{8}$
The radicals are like, so we subtract the coefficients.	$2\sqrt[4]{8}$

Example 13.8.7.38

Simplify:

1. $\sqrt[5]{3x}+\sqrt[5]{3x}$
2. $3\sqrt[3]{9}-\sqrt[3]{9}$

Answer

1. $2\sqrt[5]{3x}$
2. $2\sqrt[3]{9}$

Example 13.8.7.39

Simplify:

1. $\sqrt[4]{10y}+\sqrt[4]{10y}$
2. $5\sqrt[6]{32}-3\sqrt[6]{32}$.

Answer

1. $2\sqrt[4]{10y}$
2. $2\sqrt[6]{32}$

When an expression does not appear to have like radicals, we will simplify each radical first. Sometimes this leads to an expression with like radicals.

Example 13.8.7.40

Simplify:

1. $\sqrt[3]{54}-\sqrt[3]{16}$
2. $\sqrt[4]{48}+\sqrt[4]{243}$.

Answer

1.	$\sqrt[3]{54}-\sqrt[3]{16}$
Rewrite each radicand using perfect cube factors.	$\sqrt[3]{27}\cdot\sqrt[3]{2}-\sqrt[3]{8}\cdot\sqrt[3]{2}$
Rewrite the perfect cubes.	$\sqrt[3]{(3)^3}\cdot\sqrt[3]{2}-\sqrt[3]{(2)^3}\cdot\sqrt[3]{2}$
Simplify the radicals where possible.	$3\sqrt[3]{2}-2\sqrt[3]{2}$
Combine like radicals.	$\sqrt[3]{2}$
2.	$\sqrt[4]{48}+\sqrt[4]{243}$
Rewrite using perfect fourth power factors.	$\sqrt[4]{16}\cdot\sqrt[4]{3}+\sqrt[4]{81}\cdot\sqrt[4]{3}$
Rewrite each radicand as a product using perfect cube factors.	$\sqrt[4]{(2)^4}\cdot\sqrt[4]{3}+\sqrt[4]{(3)^4}\cdot\sqrt[4]{3}$
Rewrite the numerator as the product of two radicals.	$2\sqrt[4]{3}+3\sqrt[4]{3}$
Simplify.	$5\sqrt[4]{3}$

Example 13.8.7.41

Simplify:

1. $\sqrt[3]{192}-\sqrt[3]{81}$
2. $\sqrt[4]{32}+\sqrt[4]{512}$.

Answer

1. $\sqrt[3]{3}$
2. $6\sqrt[4]{2}$

Example 13.8.7.42

Simplify:

1. $\sqrt[3]{108}-\sqrt[3]{250}$
2. $\sqrt[5]{64}+\sqrt[5]{486}$.

Answer

1. $-\sqrt[3]{2}$
2. $5\sqrt[5]{2}$

Example 13.8.7.43

Simplify:

1. $\sqrt[3]{24x^4}-\sqrt[3]{-81x^7}$
2. $\sqrt[4]{162y^9}+\sqrt[4]{512y^5}$.

Answer

1.	$\sqrt[3]{24x^4}-\sqrt[3]{-81x^7}$
Rewrite each radicand using perfect cube factors.	$\sqrt[3]{8x^3}\cdot\sqrt[3]{3x}-\sqrt[3]{-27x^6}\cdot\sqrt[3]{3x}$
Rewrite the perfect cubes.	$\sqrt[3]{(2x)^3}\cdot\sqrt[3]{3x}-\sqrt[3]{(-3x^2)^3}\cdot\sqrt[3]{3x}$
Simplify the radicals where possible.	$2x\sqrt[3]{3x}-(-3x^2\sqrt[3]{3x})$
2.	$\sqrt[4]{162y^9}+\sqrt[4]{516y^5}$
Rewrite using perfect fourth power factors.	$\sqrt[4]{81y^8}\cdot\sqrt[4]{2y}+\sqrt[4]{256y^4}\cdot\sqrt[4]{2y}$
Rewrite each radicand as a product using perfect cube factors.	$\sqrt[4]{(3y^2)^4}\cdot\sqrt[4]{2y}+\sqrt[4]{(4y)^4}\cdot\sqrt[4]{2y}$
Rewrite the numerator as the product of two radicals.	$3y^2\sqrt[4]{2y}+4\lvert y\rvert\sqrt[4]{2y}$

? Example 13.8.7.44

Simplify:

1. $\sqrt[3]{32y^5}-\sqrt[3]{-108y^8}$
2. $\sqrt[4]{243r^{11}}+\sqrt[4]{768r^{10}}$.

Answer

1. $2y\sqrt[3]{4y^2}+3y^2\sqrt[3]{4y^2}$
2. $3r^2\sqrt[4]{3r^3}+4r^2\sqrt[4]{3r^2}$

? Example 13.8.7.45

Simplify:

1. $\sqrt[3]{40z^7}-\sqrt[3]{-135z^4}$
2. $\sqrt[4]{80s^{13}}+\sqrt[4]{1280s^6}$.

Answer

1. $2z^2\sqrt[3]{5z}+3z^5\sqrt[3]{5z}$
2. $2\mid s^3\mid\sqrt[4]{5s}+4\lvert s\rvert\sqrt[4]{5s}$

Access these online resources for additional instruction and practice with simplifying higher roots.

- Simplifying Higher Roots
- Add/Subtract Roots with Higher Indices

Key Concepts

- **Properties of**
- $\sqrt[n]{a}$ when n is an even number and
  - $a\geq 0$, then $\sqrt[n]{a}$ is a real number
  - $a<0$, then $\sqrt[n]{a}$ is not a real number
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
  - For any integer $n\geq 2$, when n is odd $\sqrt[n]{a^n}=a$
  - For any integer $n\geq 2$, when n is even $\sqrt[n]{a^n}=|a|$
- $\sqrt[n]{a}$ is considered simplified if a has no factors of m^n.
- $\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}$ and $\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}$
- $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$

- To combine like radicals, simply add or subtract the coefficients while keeping the radical the same.

Glossary

_n_th root of a number

If $b^n = a$, then b is an *n*th root of a.

principal _n_th root

The principal *n*th root of a is written $\sqrt[n]{a}$.

index

$\sqrt[n]{a}$ *n* is called the *index* of the radical.

like radicals

Radicals with the same index and same radicand are called like radicals.

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13.8.8: Rational Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the Laws of Exponents to simply expressions with rational exponents

Be Prepared

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{15}+\frac{5}{12}$.
   If you missed this problem, review [link].
2. Simplify: $(4x^2y^5)^3$.
   If you missed this problem, review [link].
3. Simplify: 5^{-3}.
   If you missed this problem, review [link].

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use **rational exponents**, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n=a^{m\cdot n}$ when *m* and *n* are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number *p* such that $(8^p)^3=8$. We will use the Power Property of Exponents to find the value of *p*.

	$(8^p)^3=8$
Multiply the exponents on the left.	$8^{3p}=8$
Write the exponent 1 on the right.	$8^{3p}=8^1$
The exponents must be equal.	$3p=1$
Solve for p.	$p=\frac{1}{3}$

But we know also $(\sqrt[3]{8})^3=8$. Then it must be that $8^{\frac{1}{3}}=\sqrt[3]{8}$

This same logic can be used for any positive integer exponent *n* to show that $a^{\frac{1}{n}}=\sqrt[n]{a}$.

Definition: RATIONAL EXPONENT $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number and $n\geq 2$, $a^{\frac{1}{n}}=\sqrt[n]{a}$.

There will be times when working with expressions will be easier if you use **rational exponents** and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

Example 13.8.8.1

Write as a radical expression:

1. $x^{\frac{1}{2}}$
2. $y^{\frac{1}{3}}$
3. $z^{\frac{1}{4}}$.

Answer

We want to write each expression in the form $\sqrt[n]{a}$.

1.	$x^{\frac{1}{2}}$
The denominator of the exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.	$\sqrt{x}$
2.	$y^{\frac{1}{3}}$
The denominator of the exponent is 3, so the index is 3.	$\sqrt[3]{y}$
3.	z^\frac{1}{4}}
The denominator of the exponent is 4, sothe index is 4.	$\sqrt[4]{z}$

Example 13.8.8.2

Write as a radical expression:

1. $t^{\frac{1}{2}}$
2. $m^{\frac{1}{3}}$
3. $r^{\frac{1}{4}}$.

Answer

1. $\sqrt{t}$
2. $\sqrt[3]{m}$
3. $\sqrt[4]{r}$

Example 13.8.8.3

Write as a radical expression:

1. $b^{\frac{1}{2}}$
2. $z^{\frac{1}{3}}$
3. $p^{\frac{1}{4}}$.

Answer

1. $\sqrt{b}$
2. $\sqrt[3]{z}$
3. $\sqrt[4]{p}$

Example 13.8.8.4

Write with a rational exponent:

1. $\sqrt{x}$
2. $\sqrt[3]{y}$
3. $\sqrt[4]{z}$.

Answer

We want to write each radical in the form $a^{\frac{1}{n}}$.

1.	$\sqrt{x}$

No index is shown, so it is 2. The denominator of the exponent will be 2.	$x^{\frac{1}{2}}$
2.	$\sqrt[3]{y}$
The index is 3, so the denominator of the exponent is 3.	$y^{\frac{1}{3}}$
3.	$\sqrt[4]{z}$
The index is 4, so the denominator of the exponent is 4.	$z^{\frac{1}{4}}$

Example 13.8.8.5

Write with a rational exponent:

1. $\sqrt{s}$
2. $\sqrt[3]{x}$
3. $\sqrt[4]{b}$.

Answer

1. $s^{\frac{1}{2}}$
2. $x^{\frac{1}{3}}$
3. \(b^{\frac{1}{4}}\

Example 13.8.8.6

Write with a rational exponent:

1. $\sqrt{v}$
2. $\sqrt[3]{p}$
3. $\sqrt[4]{p}$.

Answer

1. $v^{\frac{1}{2}}$
2. $p^{\frac{1}{3}}$
3. $p^{\frac{1}{4}}$

Example 13.8.8.7

Write with a rational exponent:

1. $\sqrt{5y}$
2. $\sqrt[3]{4x}$
3. $3\sqrt[4]{5z}$.

Answer

1.	$\sqrt{5y}$
No index is shown, so it is 2. The denominator of the exponent will be 2.	$(5y)^{\frac{1}{2}}$
2.	$\sqrt[3]{4x}$
The index is 3, so the denominator of the exponent is 3.	$(4x)^{\frac{1}{3}}$

3.	$3\sqrt[4]{5z}$
The index is 4, so the denominator of the exponent is 4.	$3(5z)^{\frac{1}{4}}$

Example 13.8.8.8

Write with a rational exponent:

1. $\sqrt{10m}$
2. $\sqrt[5]{3n}$
3. $3\sqrt[4]{6y}$.

Answer

1. $(10^m)^{\frac{1}{2}}$
2. $(3n)^{\frac{1}{5}}$
3. $(486y)^{\frac{1}{4}}$

Example 13.8.8.9

Write with a rational exponent:

1. $\sqrt[7]{3k}$
2. $\sqrt[4]{5j}$
3. $\sqrt[3]{82a}$.

Answer

1. $(3k)^{\frac{1}{7}}$
2. $(5j)^{\frac{1}{4}}$
3. $(1024a)^{\frac{1}{3}}$

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

Example 13.8.8.10

Simplify:

1. $25^{\frac{1}{2}}$
2. $64^{\frac{1}{3}}$
3. $256^{\frac{1}{4}}$.

Answer

1.	$25^{\frac{1}{2}}$
Rewrite as a square root.	$\sqrt{25}$
Simplify.	5
2.	$64^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{64}$
Recognize 64 is a perfect cube.	$\sqrt[3]{4^3}$
Simplify.	4

3.	$256^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{256}$
Recognize 256 is a perfect fourth power.	$\sqrt[4]{4^4}$
Simplify.	4

Example 13.8.8.11

Simplify:

1. $36^{\frac{1}{2}}$
2. $8^{\frac{1}{3}}$
3. $16^{\frac{1}{4}}$.

Answer

1. 6
2. 2
3. 2

Example 13.8.8.12

Simplify:

1. $100^{\frac{1}{2}}$
2. $27^{\frac{1}{3}}$
3. $81^{\frac{1}{4}}$.

Answer

1. 10
2. 3
3. 3

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n} = \frac{1}{a^n}$ in one case.

Example 13.8.8.13

Simplify:

1. $(-64)^{\frac{1}{3}}$
2. $-64^{\frac{1}{3}}$
3. $(64)^{-\frac{1}{3}}$.

Answer

1.	$(-64)^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{-64}$
Rewrite-64 as a perfect cube.	$\sqrt[3]{(-4)^3}$
Simplify.	-4
2.	$-64^{\frac{1}{3}}$

The exponent applies only to the 64.	$-(64^{\frac{1}{3}})$
Rewrite as a cube root.	$-\sqrt[3]{64}$
Rewrite 64 as 4^3.	$-\sqrt[3]{4^3}$
Simplify.	-4
3.	$(64)^{-\frac{1}{3}}$
Rewrite as a fraction with a positive exponent, using the property, $a^{-n}=\frac{1}{a^n}$. Write as a cube root.	$\frac{1}{\sqrt[3]{64}}$
Rewrite 64 as 4^3.	$\frac{1}{\sqrt[3]{4^3}}$
Simplify.	$\frac{1}{4}$

Example 13.8.8.14

Simplify:

1. $(-125)^{\frac{1}{3}}$
2. $-125^{\frac{1}{3}}$
3. $(125)^{-\frac{1}{3}}$.

Answer

1. -5
2. -5
3. $\frac{1}{5}$

Example 13.8.8.15

Simplify:

1. $(-32)^{\frac{1}{5}}$
2. $-32^{\frac{1}{5}}$
3. $(32)^{-\frac{1}{5}}$.

Answer

1. -2
2. -2
3. $\frac{1}{2}$

Example 13.8.8.16

Simplify:

1. $(-16)^{\frac{1}{4}}$
2. $-16^{\frac{1}{4}}$
3. $(16)^{-\frac{1}{4}}$.

Answer

1.	$(-16)^{\frac{1}{4}}$

Rewrite as a fourth root.	$\sqrt[4]{-16}$
There is no real number whose fourth power is −16.	
2.	$-16^{\frac{1}{4}}$
The exponent applies only to the 16.	$-(16^{\frac{1}{4}})$
Rewrite as a fourth root.	$-\sqrt[4]{16}$
Rewrite 16 as 2^4	$-\sqrt[4]{2^4}$
Simplify.	-2
3.	$(16)^{-\frac{1}{4}}$
Rewrite as a fraction with a positive exponent, using the property, $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{(16)^{\frac{1}{4}}}$
Rewrite as a fourth root.	$\frac{1}{\sqrt[4]{16}}$
Rewrite 16 as 2^4.	$\frac{1}{\sqrt[4]{2^4}}$
Simplify.	$\frac{1}{2}$

? Example 13.8.8.17

Simplify:

1. $(-64)^{\frac{1}{2}}$
2. $-64^{\frac{1}{2}}$
3. $(64)^{-\frac{1}{2}}$.

Answer

1. −8
2. −8
3. $\frac{1}{8}$

? Example 13.8.8.18

Simplify:

1. $(-256)^{\frac{1}{4}}$
2. $-256^{\frac{1}{4}}$
3. $(256)^{-\frac{1}{4}}$.

Answer

1. −4
2. −4
3. $\frac{1}{4}$

Simplify Expressions with $a^{\frac{m}{n}}$

Let's work with the Power Property for Exponents some more.

Suppose we raise $a^{\frac{1}{n}}$ to the power *m*.

	$(a^{\frac{1}{n}})^m$
Multiply the exponents.	$a^{\frac{1}{n}\cdot m}$
Simplify.	$a^{\frac{m}{n}}$
So $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ also.	

Now suppose we take a^m to the $\frac{1}{n}$ power.

	$(a^m)^{\frac{1}{n}}$
Multiply the exponents.	$a^{m\cdot\frac{1}{n}}$
Simplify.	$a^{\frac{m}{n}}$
So $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ also.	

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller.

Definition: RATIONAL EXPONENT $a^{\frac{m}{n}}$

For any positive integers *m* and *n*,

$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Example 13.8.8.19

Write with a rational exponent:

1. $\sqrt{y^3}$
2. $\sqrt[3]{x^2}$
3. $\sqrt[4]{z^3}$

Answer

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$.

1. The numerator of the exponent is the exponent of *y*, 3. $\sqrt{y^3}$
   The denominator of the exponent is the index of the radical, 2. $y^{\frac{3}{2}}$
2. The numerator of the exponent is the exponent of *x*, 2. $\sqrt[3]{x^2}$
   The denominator of the exponent is the index of the radical, 3. $x^{\frac{2}{3}}$
3. The numerator of the exponent is the exponent of *z*, 3. $\sqrt[4]{z^3}$
   The denominator of the exponent is the index of the radical, 4. $z^{\frac{3}{4}}$

Example 13.8.8.20

Write with a rational exponent:

1. $\sqrt{x^5}$
2. $\sqrt[4]{z^3}$
3. $\sqrt[5]{y^2}$.

Answer

1. $x^{\frac{5}{2}}$
2. $z^{\frac{3}{4}}$

3. $y^{\frac{2}{5}}$

Example 13.8.8.21

Write with a rational exponent:

1. $\sqrt[5]{a^2}$
2. $\sqrt[3]{b^7}$
3. $\sqrt[4]{m^5}$.

Answer

1. $a^{\frac{2}{5}}$
2. $b^{\frac{7}{3}}$
3. $m^{\frac{5}{4}}$

Example 13.8.8.22

Simplify:

1. $9^{\frac{3}{2}}$
2. $125^{\frac{2}{3}}$
3. $81^{\frac{3}{4}}$.

Answer

We will rewrite each expression as a radical first using the property, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

1.	$9^{\frac{3}{2}}$
The power of the radical is the numerator of the exponent, 3. Since the denominator of the exponent is 2, this is a square root.	$(\sqrt{9})^3$
Simplify.	3^3
	27
2.	$125^{\frac{2}{3}}$
The power of the radical is the numerator of the exponent, 2. Since the denominator of the exponent is 3, this is a square root.	$(\sqrt[3]{125})^2$
Simplify.	5^2
	25
3.	$81^{\frac{3}{4}}$
The power of the radical is the numerator of the exponent, 2. Since the denominator of the exponent is 3, this is a square root.	$(\sqrt[4]{81})^3$
Simplify.	3^3
	27

Example 13.8.8.23

Simplify:

1. $4^{\frac{3}{2}}$
2. $27^{\frac{2}{3}}$
3. $625^{\frac{3}{4}}$.

Answer

1. 8
2. 9
3. 125

Example 13.8.8.24

Simplify:

1. $8^{\frac{5}{3}}$
2. $81^{\frac{3}{2}}$
3. $16^{\frac{3}{4}}$.

Answer

1. 32
2. 729
3. 8

Remember that $b^{-p}=\frac{1}{b^p}$. The negative sign in the exponent does not change the sign of the expression.

Example 13.8.8.25

Simplify:

1. $16^{-\frac{3}{2}}$
2. $32^{-\frac{2}{5}}$
3. $4^{-\frac{5}{2}}$

Answer

We will rewrite each expression first using $b^{-p}=\frac{1}{b^p}$ and then change to radical form.

1.	$16^{-\frac{3}{2}}$
Rewrite using $b^{-p}=\frac{1}{b^p}$.	$\frac{1}{16^{\frac{3}{2}}}$
Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.	$\frac{1}{(\sqrt{16})^3}$
Simplify.	$\frac{1}{4^3}$
	$\frac{1}{64}$
2.	$32^{-\frac{2}{5}}$
Rewrite using $b^{-p}=\frac{1}{b^p}$.	$\frac{1}{32^{\frac{2}{5}}}$

Change to radical form.	$\frac{1}{(\sqrt[5]{32})^2}$
Rewrite the radicand as a power.	$\frac{1}{(\sqrt[5]{2^5})^2}$
Simplify.	$\frac{1}{2^2}$
	$\frac{1}{4}$
3.	$4^{-\frac{5}{2}}$
Rewrite using $b^{-p}=\frac{1}{b^p}$.	$\frac{1}{4^{\frac{5}{2}}}$
Change to radical form.	$\frac{1}{(\sqrt{4})^5}$
Simplify.	$\frac{1}{2^5}$
	$\frac{1}{32}$

Example 13.8.8.26

Simplify:

1. $8^{-\frac{5}{3}}8$
2. $81^{-\frac{3}{2}}$
3. $16^{-\frac{3}{4}}$.

Answer

1. $\frac{1}{32}$
2. $\frac{1}{729}$
3. $\frac{1}{8}$

Example 13.8.8.27

Simplify:

1. $4^{-\frac{3}{2}}$
2. $27^{-\frac{2}{3}}$
3. $625^{-\frac{3}{4}}$.

Answer

1. $\frac{1}{8}$
2. $\frac{1}{9}$
3. $\frac{1}{125}$

Example 13.8.8.28

Simplify:

1. $-25^{\frac{3}{2}}$
2. $-25^{-\frac{3}{2}}$
3. $(-25)^{\frac{3}{2}}$.

Answer

1.	$-25^{\frac{3}{2}}$
Rewrite in radical form.	$-(\sqrt{25})^3$
Simplify the radical	-5^3
Simplify.	-125
2.	$-25^{-\frac{3}{2}}$
Rewrite using $b^{-p}=\frac{1}{b^p}$.	$-(\frac{1}{25^{\frac{3}{2}}})$
Rewrite in radical form.	$-(\frac{1}{(\sqrt{25})^3})$
Simplify the radical.	$-(\frac{1}{5^3})$
Simplify.	$-\frac{1}{125}$
3.	$(-25)^{\frac{3}{2}}$.
Rewrite in radical form.	$(\sqrt{-25})^3$
There is no real number whose square root is -25.	Not a real number.

? Example 13.8.8.29

Simplify:

1. $-16^{\frac{3}{2}}$
2. $-16^{-\frac{3}{2}}$
3. $(-16)^{-\frac{3}{2}}$.

Answer

1. -64
2. $-\frac{1}{64}$
3. not a real number

? Example 13.8.8.30

Simplify:

1. $-81^{\frac{3}{2}}$
2. $-81^{-\frac{3}{2}}$
3. $(-81)^{-\frac{3}{2}}$.

Answer

1. -729
2. $-\frac{1}{729}$
3. not a real number

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

The same laws of exponents that we already used apply to rational exponents, too. We will list the Exponent Properties here to have them for reference as we simplify expressions.

SUMMARY OF EXPONENT PROPERTIES

If a,b are real numbers and m,n are rational numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0, m > n$
	$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, a \neq 0, n > m$
Zero Exponent Definition	$a^0 = 1, a \neq 0$
Quotient to a Power Property	$(\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0$

When we multiply the same base, we add the exponents.

Example 13.8.8.31

Simplify:

1. $2^{\frac{1}{2}} \cdot 2^{\frac{5}{2}}$
2. $x^{\frac{2}{3}} \cdot x^{\frac{4}{3}}$
3. $z^{\frac{3}{4}} \cdot z^{\frac{5}{4}}$.

Answer

1.	$2^{\frac{1}{2}} \cdot 2^{\frac{5}{2}}$
The bases are the same, so we add the exponents.	$2^{\frac{1}{2}+\frac{5}{2}}$
Add the fractions.	$2^{\frac{6}{2}}$
Simplify the exponent.	2^3
Simplify.	8
2.	$x^{\frac{2}{3}} \cdot x^{\frac{4}{3}}$
The bases are the same, so we add the exponents.	$x^{\frac{2}{3}+\frac{4}{3}}$
Add the fractions.	$x^{\frac{6}{3}}$
Simplify.	x^2
3.	$z^{\frac{3}{4}} \cdot z^{\frac{5}{4}}$
The bases are the same, so we add the exponents.	$z^{\frac{3}{4}+\frac{5}{4}}$
Add the fractions.	$z^{\frac{8}{4}}$
Simplify.	z^2

Example 13.8.8.32

Simplify:

1. $3^{\frac{2}{3}} \cdot 3^{\frac{4}{3}}$
2. $y^{\frac{1}{3}} \cdot y^{\frac{8}{3}}$
3. $m^{\frac{1}{4}} \cdot m^{\frac{3}{4}}$.

Answer

1. 9
2. y^3
3. m

Example 13.8.8.33

Simplify:

1. $5^{\frac{3}{5}} \cdot 5^{\frac{7}{5}}$
2. $z^{\frac{1}{8}} \cdot z^{\frac{7}{8}}$
3. $n^{\frac{2}{7}} \cdot n^{\frac{5}{7}}$.

Answer

1. 25
2. z
3. n

We will use the Power Property in the next example.

Example 13.8.8.34

Simplify:

1. $(x^4)^{\frac{1}{2}}$
2. $(y^6)^{\frac{1}{3}}$
3. $(z^9)^{\frac{2}{3}}$.

Answer

1.	$(x^4)^{\frac{1}{2}}$
To raise a power to a power, we multiply the exponents.	$x^{4 \cdot \frac{1}{2}}$
Simplify.	x^2
2.	$(y^6)^{\frac{1}{3}}$
To raise a power to a power, we multiply the exponents.	$y^{6 \cdot \frac{1}{3}}$
Simplify.	y^2
3.	$(z^9)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$z^{9 \cdot \frac{2}{3}}$
Simplify.	z^6

Example 13.8.8.35

Simplify:

1. $(p^{10})^{\frac{1}{5}}$
2. $(q^8)^{\frac{3}{4}}$
3. $(x^6)^{\frac{4}{3}}$

Answer

1. p^
2. q^6
3. x^8

Example 13.8.8.36

Simplify:

1. $(r^6)^{\frac{5}{3}}$
2. $(s^{12})^{\frac{3}{4}}$
3. $(m^9)^{\frac{2}{9}}$

Answer

1. r^{10}
2. s^9
3. m^2

The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

Example 13.8.8.37

Simplify:

1. $\frac{x^{\frac{4}{3}}}{x^{\frac{1}{3}}}$
2. $\frac{y^{\frac{3}{4}}}{y^{\frac{1}{4}}}$
3. $\frac{z^{\frac{2}{3}}}{z^{\frac{5}{3}}}$.

Answer

1.	$\frac{x^{\frac{4}{3}}}{x^{\frac{1}{3}}}$
To divide with the same base, we subtract the exponents.	$x^{\frac{4}{3}-\frac{1}{3}}$
Simplify.	x
2.	$\frac{y^{\frac{3}{4}}}{y^{\frac{1}{4}}}$
To divide with the same base, we subtract the exponents.	$y^{\frac{3}{4}-\frac{1}{4}}$
Simplify.	$y^{\frac{1}{2}}$
3.	$\frac{z^{\frac{2}{3}}}{z^{\frac{5}{3}}}$
To divide with the same base, we subtract the exponents.	$z^{\frac{2}{3}-\frac{5}{3}}$
Rewrite without a negative exponent.	$\frac{1}{z}$

Example 13.8.8.38

Simplify:

1. $\frac{u^{\frac{5}{4}}}{u^{\frac{1}{4}}}$
2. $\frac{v^{\frac{3}{5}}}{v^{\frac{2}{5}}}$
3. $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$.

Answer

1. u
2. $v^{\frac{1}{5}}$
3. $\frac{1}{x}$

Example 13.8.8.39

Simplify:

1. $\frac{c^{\frac{12}{5}}}{c^{\frac{2}{5}}}$
2. $\frac{m^{\frac{5}{4}}}{m^{\frac{9}{4}}}$
3. $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$.

Answer

1. c^2
2. $\frac{1}{m}$
3. $\frac{1}{d}$

Sometimes we need to use more than one property. In the next two examples, we will use both the Product to a Power Property and then the Power Property.

Example 13.8.8.40

Simplify:

1. $(27u^{\frac{1}{2}})^{\frac{2}{3}}$
2. $(8v^{\frac{1}{4}})^{\frac{2}{3}}$.

Answer

1.	$(27u^{\frac{1}{2}})^{\frac{2}{3}}$
First we use the Product to a Power Property.	$(27)^{\frac{2}{3}}(u^{\frac{1}{2}})^{\frac{2}{3}}$
Rewrite 27 as a power of 3.	$(3^3)^{\frac{2}{3}}(u^{\frac{1}{2}})^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$(3^2)(u^{\frac{1}{3}})$
Simplify.	$9u^{\frac{1}{3}}$
2.	$(8v^{\frac{1}{4}})^{\frac{2}{3}}$.

First we use the Product to a Power Property.	$(8)^{\frac{2}{3}}(v^{\frac{1}{4}})^{\frac{2}{3}}$
Rewrite 8 as a power of 2.	$(2^3)^{\frac{2}{3}}(v^{\frac{1}{4}})^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$(2^2)(v^{\frac{1}{6}})$
Simplify.	$4v^{\frac{1}{6}}$

Example 13.8.8.41

Simplify:

1. $32x^{\frac{1}{3}})^{\frac{3}{5}}$
2. $(64y^{\frac{2}{3}})^{\frac{1}{3}}$.

Answer

1. $8x^{\frac{1}{5}}$
2. $4y^{\frac{2}{9}}$

Example 13.8.8.42

Simplify:

1. $(16m^{\frac{1}{3}})^{\frac{3}{2}}$
2. $(81n^{\frac{2}{5}})^{\frac{3}{2}}$.

Answer

1. $64m^{\frac{1}{2}}$
2. $729n^{\frac{3}{5}}$

Example 13.8.8.43

Simplify:

1. $(m^3n^9)^{\frac{1}{3}}$
2. $(p^4q^8)^{\frac{1}{4}}$.

Answer

1.	$(m^3n^9)^{\frac{1}{3}}$
First we use the Product to a Power Property.	$(m^3)^{\frac{1}{3}}(n^9)^{\frac{1}{3}}$
To raise a power to a power, we multiply the exponents.	mn^3
2.	$(p^4q^8)^{\frac{1}{4}}$
First we use the Product to a Power Property.	$(p^4)^{\frac{1}{4}}(q^8)^{\frac{1}{4}}$
To raise a power to a power, we multiply the exponents.	pq^2

We will use both the Product and Quotient Properties in the next example.

Exercise 13.8.8.44

Simplify:

1. $\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$
2. $\frac{y^{\frac{4}{3}} \cdot y}{y^{-\frac{2}{3}}}$.

Answer

1.	$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$
Use the Product Property in the numerator, add the exponents.	$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$
Use the Quotient Property, subtract the exponents.	$x^{\frac{8}{4}}$
Simplify.	x^2
2.	$\frac{y^{\frac{4}{3}} \cdot y}{y^{-\frac{2}{3}}}$
Use the Product Property in the numerator, add the exponents.	$\frac{y^{\frac{7}{3}}}{y^{-\frac{2}{3}}}$
Use the Quotient Property, subtract the exponents.	$y^{\frac{9}{3}}$
Simplify.	y^3

Example 13.8.8.45

Simplify:

1. $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$
2. $\frac{n^{\frac{1}{6}} \cdot n}{n^{-\frac{11}{6}}}$.

Answer

1. m^2
2. n^3

Example 13.8.8.46

Simplify:

1. $\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$
2. $\frac{v^{\frac{1}{2}} \cdot v}{v^{-\frac{7}{2}}}$.

Answer

1. u^3
2. v^5

Key Concepts

- **Summary of Exponent Properties**
- If a,b are real numbers and m,n are rational numbers, then
  - **Product Property** $a^m \cdot a^n = a^{m+n}$
  - **Power Property** $(a^m)^n = a^{m \cdot n}$
  - **Product to a Power** $(ab)^m = a^m b^m$
  - **Quotient Property**:

    $\frac{a^m}{a^n} = a^{m-n}, a \neq 0, m > n$

    $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, a \neq 0, n > m$
  - **Zero Exponent Definition** $a^0 = 1, a \neq 0$
  - **Quotient to a Power Property** $(\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0$

Glossary

rational exponents

- If $\sqrt[n]{a}$ is a real number and $n \geq 2$, $a^{\frac{1}{n}} = \sqrt[n]{a}$
- For any positive integers *m* and *n*, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

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13.8.8E: Exercises

Practice Makes Perfect

Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

Example 13.8.8E.47

1. $x^{\frac{1}{2}}$
2. $y^{\frac{1}{3}}$
3. $z^{\frac{1}{4}}$

Example 13.8.8E.48

1. $r^{\frac{1}{2}}$
2. $s^{\frac{1}{3}}$
3. $t^{\frac{1}{4}}$

Answer

1. $\sqrt{r}$
2. $\sqrt[3]{s}$
3. $\sqrt[4]{t}$

Example 13.8.8E.49

1. $u^{\frac{1}{5}}$
2. $v^{\frac{1}{9}}$
3. $w^{\frac{1}{20}}$

Example 13.8.8E.50

1. $g^{\frac{1}{7}}$
2. $h^{\frac{1}{5}}$
3. $j^{\frac{1}{25}}$

Answer

1. $\sqrt[7]{g}$
2. $\sqrt[5]{h}$
3. $\sqrt[25]{j}$

In the following exercises, write with a rational exponent.

Example 13.8.8E.51

1. $-\sqrt[7]{x}$
2. $\sqrt[9]{y}$
3. $\sqrt[5]{f}$

Example 13.8.8E.52

1. $\sqrt[8]{r}$
2. $\sqrt[10]{s}$
3. \(\sqrt[4]{t}\)

Answer

1. $r^{\frac{1}{8}}$
2. $s^{\frac{1}{10}}$
3. $t^{\frac{1}{4}}$

Example 13.8.8E.53

1. \(\sqrt[3]{a}\
2. $\sqrt[12]{b}$
3. $\sqrt{c}$

Example 13.8.8E.54

1. $\sqrt[5]{u}$
2. $\sqrt{v}$
3. $\sqrt[16]{w}$

Answer

1. $u^{\frac{1}{5}}$
2. $v^{\frac{1}{2}}$
3. $w^{\frac{1}{16}}$

Example 13.8.8E.55

1. $\sqrt[3]{7c}$
2. $\sqrt[7]{12d}$
3. $3\sqrt[4]{5f}$

Example 13.8.8E.56

1. $\sqrt[4]{5x}$
2. $\sqrt[8]{9y}$
3. $7\sqrt[5]{3z}$

Answer

1. $(5x)^{\frac{1}{4}}$
2. $(9y)^{\frac{1}{8}}$
3. $7(3z)^{\frac{1}{5}}$

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Example 13.8.8E.57

1. $\sqrt{21p}$
2. $\sqrt[4]{8q}$
3. $\sqrt[6]{436r}$

Example 13.8.8E.58

1. $\sqrt[3]{25a}$
2. $\sqrt{3b}$
3. $\sqrt[10]{40c}$

Answer

1. $(25a)^{\frac{1}{3}}$
2. $(3b)^{\frac{1}{2}}$
3. $(40c)^{\frac{1}{10}}$

In the following exercises, simplify.

Example 13.8.8E.59

1. $81^{\frac{1}{2}}$
2. $125^{\frac{1}{3}}$
3. $64^{\frac{1}{2}}$

Example 13.8.8E.60

1. $625^{\frac{1}{4}}$
2. $243^{\frac{1}{5}}$
3. $32^{\frac{1}{5}}$

Answer

1. 5
2. 3
3. 2

Example 13.8.8E.61

1. $16^{\frac{1}{4}}$
2. $16^{\frac{1}{2}}$
3. $3125^{\frac{1}{5}}$

Example 13.8.8E.62

1. $216^{\frac{1}{3}}$
2. $32^{\frac{1}{5}}$
3. $81^{\frac{1}{4}}$

Answer

1. 6
2. 2
3. 3

Example 13.8.8E.63

1. $(-216)^{\frac{1}{3}}$
2. $-216^{\frac{1}{3}}$
3. $(216)^{-\frac{1}{3}}$

Example 13.8.8E.64

1. $(-243)^{\frac{1}{5}}$
2. $-243^{\frac{1}{5}}$
3. $(243)^{-\frac{1}{5}}$

Answer

1. -3
2. -3
3. $\frac{1}{3}$

Example 13.8.8E.65

1. $(-1)^{\frac{1}{3}}$
2. $-1^{\frac{1}{3}}$
3. $(1)^{-\frac{1}{3}}$

Example 13.8.8E.66

1. $(-1000)^{\frac{1}{3}}$
2. $-1000^{\frac{1}{3}}$
3. $(1000)^{-\frac{1}{3}}$

Answer

1. -10
2. -10
3. \(\frac{1}{10}

Example 13.8.8E.67

1. $(-81)^{\frac{1}{4}}$
2. $-81^{\frac{1}{4}}$
3. $(81)^{-\frac{1}{4}}$

Example 13.8.8E.68

1. $(-49)^{\frac{1}{2}}$
2. $-49^{\frac{1}{2}}$
3. $(49)^{-\frac{1}{2}}$

Answer

1. not a real number
2. -7
3. $\frac{1}{7}$

Example 13.8.8E.69

1. $(-36)^{\frac{1}{2}}$
2. $-36^{\frac{1}{2}}$
3. $(36)^{-\frac{1}{2}}$

Example 13.8.8E.70

1. $(-1)^{\frac{1}{4}}$
2. $(1)^{-\frac{1}{4}}$
3. $-1^{\frac{1}{4}}$

Answer

1. not a real number
2. 1
3. -1

Example 13.8.8E.71

1. $(-100)^{\frac{1}{2}}$
2. $-100^{\frac{1}{2}}$
3. $(100)^{-\frac{1}{2}}$

Example 13.8.8E.72

1. $(-32)^{\frac{1}{5}}$
2. $(243)^{-\frac{1}{5}}$
3. $-125^{\frac{1}{3}}$

Answer

1. -2
2. $\frac{1}{3}$
3. -5

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

Example 13.8.8E.73

1. $\sqrt{m^5}$
2. $\sqrt[3]{n^2}$
3. $\sqrt[4]{p^3}$

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Example 13.8.8E.74

1. $\sqrt[4]{r^7}$
2. $\sqrt[5]{s^3}$
3. $\sqrt[3]{t^7}$

Answer

1. $r^{\frac{7}{4}}$
2. $s^{\frac{3}{5}}$
3. $t^{\frac{7}{3}}$

Exercise 13.8.8E.75

1. $\sqrt[5]{u^2}$
2. $\sqrt[5]{v^8}$
3. $\sqrt[9]{w^4}$

Example 13.8.8E.76

1. $\sqrt[3]{a}$
2. $\sqrt[5]{b}$
3. $\sqrt[3]{c^5}$

Answer

1. $a^{\frac{1}{3}}$
2. $b^{\frac{1}{5}}$
3. $c^{\frac{5}{3}}$

In the following exercises, simplify.

Example 13.8.8E.77

1. $16^{\frac{3}{2}}$
2. $8^{\frac{2}{3}}$
3. $10,000^{\frac{3}{4}}$

Example 13.8.8E.78

1. $1000^{\frac{2}{3}}$
2. $25^{\frac{3}{2}}$
3. $32^{\frac{3}{5}}$

Answer

1. 100
2. 125
3. 8

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Example 13.8.8E.79

1. $275^{\frac{3}{2}}$
2. $16^{\frac{5}{4}}$
3. $32^{\frac{2}{5}}$

Example 13.8.8E.80

1. $16^{\frac{3}{2}}$
2. $125^{\frac{5}{3}}$
3. $64^{\frac{4}{3}}$

Answer

1. 64
2. 3125
3. 256

Example 13.8.8E.81

1. $32^{\frac{2}{5}}$
2. $27^{-\frac{2}{3}}$
3. $25^{-\frac{3}{2}}$

Example 13.8.8E.82

1. $64^{\frac{5}{2}}$
2. $81^{-\frac{3}{2}}$
3. $27^{-\frac{4}{3}}$

Answer

1. 32,768
2. $\frac{1}{729}$
3. $\frac{1}{81}$

Example 13.8.8E.83

1. $25^{\frac{3}{2}}$
2. $9^{-\frac{3}{2}}$
3. $(-64)^{\frac{2}{3}}$

Example 13.8.8E.84

1. $100^{\frac{3}{2}}$
2. $49^{-\frac{5}{2}}$
3. $(-100)^{\frac{3}{2}}$

Answer

1. 1000
2. $\frac{1}{16,807}$

3. not a real numbe

Example 13.8.8E.85

1. $-9^{\frac{3}{2}}$
2. $-9^{-\frac{3}{2}}$
3. $(-9)^{\frac{3}{2}}$

Example 13.8.8E.86

1. $-64^{\frac{3}{2}}$
2. $-64^{-\frac{3}{2}}$
3. $(-64)^{\frac{3}{2}}$

Answer

1. -512
2. $-\frac{1}{512}$
3. not a real number

Example 13.8.8E.87

1. $-100^{\frac{3}{2}}$
2. $-100^{-\frac{3}{2}}$
3. $(-100)^{\frac{3}{2}}$

Example 13.8.8E.88

1. $-49^{\frac{3}{2}}$
2. $-49^{-\frac{3}{2}}$
3. $(-49)^{\frac{3}{2}}$

Answer

1. -343
2. $-\frac{1}{343}$
3. not a real number

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

Example 13.8.8E.89

1. $4^{\frac{5}{8}} \cdot 4^{\frac{11}{8}}$
2. $m^{\frac{7}{12}} \cdot m^{\frac{17}{12}}$
3. $p^{\frac{3}{7}} \cdot p^{\frac{18}{7}}$

Example 13.8.8E.90

1. $6^{\frac{5}{2}} \cdot 6^{\frac{12}{6}}$
2. $n^{\frac{2}{10}} \cdot n^{\frac{8}{10}}$

3. $q^{\frac{2}{5}} \cdot q^{\frac{13}{5}}$

Answer

1. 216
2. n
3. q^3

Example 13.8.8E.91

1. $5^{\frac{1}{2}} \cdot 5^{\frac{7}{2}}$
2. $c^{\frac{3}{4}} \cdot c^{\frac{9}{4}}$
3. $d^{\frac{3}{5}} \cdot d^{\frac{2}{5}}$

Example 13.8.8E.92

1. $10^{\frac{1}{3}} \cdot 10^{\frac{5}{3}}$
2. $x^{\frac{5}{6}} \cdot x^{\frac{7}{6}}$
3. $y^{\frac{11}{8}} \cdot y^{\frac{21}{8}}$

Answer

1. 100
2. x^2
3. y^4

Example 13.8.8E.93

1. $(m^6)^{\frac{5}{2}}$
2. $(n^9)^{\frac{4}{3}}$
3. $(p^{12})^{\frac{3}{4}}$

Example 13.8.8E.94

1. $(a^{12})^{\frac{1}{6}}$
2. $(b^{15})^{\frac{3}{5}}$
3. $(c^{11})^{\frac{1}{11}}$

Answer

1. a^2
2. b^9
3. c

Example 13.8.8E.95

1. $(x^{12})^{\frac{2}{3}}$
2. $(y^{20})^{\frac{2}{5}}$
3. $(z^{16})^{\frac{1}{16}}$

Example 13.8.8E.96

1. $(h^6)^{\frac{4}{3}}$
2. $(k^{12})^{\frac{3}{4}}$
3. $(j^{10})^{\frac{7}{5}}$

Answer

1. h^8
2. k^9
3. j^{14}

Example 13.8.8E.97

1. $\frac{x^{\frac{7}{2}}}{x^{\frac{5}{2}}}$
2. $\frac{y^{\frac{5}{2}}}{y^{\frac{1}{2}}}$
3. $\frac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$

Example 13.8.8E.98

1. $\frac{s^{\frac{11}{5}}}{s^{\frac{6}{5}}}$
2. $\frac{z^{\frac{7}{3}}}{z^{\frac{1}{3}}}$
3. $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

Answer

1. s
2. z^2
3. $\frac{1}{w}$

Example 13.8.8E.99

1. $\frac{t^{\frac{12}{5}}}{t^{\frac{7}{5}}}$
2. $\frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$
3. $\frac{m^{\frac{13}{8}}}{m^{\frac{5}{8}}}$

Example 13.8.8E.100

1. $\frac{u^{\frac{13}{9}}}{u^{\frac{4}{9}}}$
2. $\frac{r^{\frac{15}{7}}}{r^{\frac{8}{7}}}$
3. $\frac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

Answer

1. u
2. r
3. $\frac{1}{n}$

Example 13.8.8E.101

1. $(9p^{\frac{2}{3}})^{\frac{5}{2}}$
2. $(27q^{\frac{3}{2}})^{\frac{4}{3}}$

Example 13.8.8E.102

1. $(81r^{\frac{4}{5}})^{\frac{1}{4}}$
2. $(64s^{\frac{3}{7}})^{\frac{1}{6}}$

Answer

1. $3r^{\frac{1}{5}}$
2. $2s^{\frac{1}{14}}$

Example 13.8.8E.103

1. $(16u^{\frac{1}{3}})^{\frac{3}{4}}$
2. $(100v^{\frac{2}{5}})^{\frac{3}{2}}$

Example 13.8.8E.104

1. $(27m^{\frac{3}{4}})^{\frac{2}{3}}$
2. $(625n^{\frac{8}{3}})^{\frac{3}{4}}$

Answer

1. $9m^{\frac{1}{2}}$
2. $125n^2$

Example 13.8.8E.105

1. $(x^8y^{10})^{\frac{1}{2}}$
2. $(a^9by^{12})^{\frac{1}{3}}$

Example 13.8.8E.106

1. $(r^8s^4)^{\frac{1}{4}}$
2. $(u^{15}v^{20})^{\frac{1}{5}}$

Answer

1. r^2s
2. u^3v^4

Example 13.8.8E.107

1. $(a^6 b^{16})^{\frac{1}{2}}$
2. $(j^9 k^6)^{\frac{2}{3}}$

Example 13.8.8E.108

1. $(r^{16} s^{10})^{\frac{1}{2}}$
2. $(u^{10} v^5)^{\frac{4}{5}}$

Answer

1. $r^8 s^5$
2. $u^8 v^4$

Example 13.8.8E.109

1. $\frac{r^{\frac{5}{2}} \cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$
2. $\frac{s^{\frac{1}{5}} \cdot s}{s^{-\frac{9}{5}}}$

Example 13.8.8E.110

1. $\frac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$
2. $\frac{b^{\frac{2}{3}} \cdot b}{b^{-\frac{7}{3}}}$

Answer

1. a^3
2. b^4

Example 13.8.8E.111

1. $\frac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$
2. $\frac{d^{\frac{3}{5}} \cdot d}{d^{-\frac{2}{5}}}$

Example 13.8.8E.112

1. $\frac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$
2. $\frac{n^{\frac{3}{7}} \cdot n}{n^{-\frac{4}{7}}}$

Answer

1. m
2. n^2

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Example 13.8.8E.113

$4^{\frac{5}{2}} \cdot 4^{\frac{1}{2}}$

Example 13.8.8E.114

$n^{\frac{2}{6}} \cdot n^{\frac{4}{6}}$

Answer

n

Example 13.8.8E.115

$(a^2 4)^{\frac{1}{6}}$

Example 13.8.8E.116

$(b^{10})^{\frac{3}{5}}$

Answer

b^6

Example 13.8.8E.117

$\frac{w^{\frac{2}{5}}}{w^{\frac{7}{5}}}$

Example 13.8.8E.118

$\frac{z^{\frac{2}{3}}}{z^{\frac{8}{3}}}$

Answer

$\frac{1}{z^2}$

Example 13.8.8E.119

$(27r^{\frac{3}{5}})^{\frac{1}{3}}$

Example 13.8.8E.120

$(64s^{\frac{3}{5}})^{\frac{1}{6}}$

Answer

$2s^{\frac{1}{10}}$

Example 13.8.8E.121

$(r^9 s^{12})^{\frac{1}{3}}$

Example 13.8.8E.122

$(u^{12} v^{18})^{\frac{1}{6}}$

Answer

$u^2 v^3$

Everyday Math

Example 13.8.8E.123

Landscaping Joe wants to have a square garden plot in his backyard. He has enough compost to cover an area of 144 square feet. Simplify $144^{\frac{1}{2}}$ to find the length of each side of his garden.

Example 13.8.8E.124

Landscaping Elliott wants to make a square patio in his yard. He has enough concrete to pave an area of 242 square feet. Simplify $242^{\frac{1}{2}}$ to find the length of each side of his patio.Round to the nearest tenth of a foot.

Answer

15.6 feet

Example 13.8.8E.125

Gravity While putting up holiday decorations, Bob dropped a decoration from the top of a tree that is 12 feet tall. Simplify $\frac{12^{\frac{1}{2}}}{16^{\frac{1}{2}}}$ to find how many seconds it took for the decoration to reach the ground. Round to the nearest tenth of a second.

Example 13.8.8E.126

Gravity An airplane dropped a flare from a height of 1024 feet above a lake. Simplify $\frac{1024^{\frac{1}{2}}}{16^{\frac{1}{2}}}$ to find how many seconds it took for the flare to reach the water.

Answer

8 seconds

Writing Exercises

Example 13.8.8E.127

Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.

Example 13.8.8E.128

Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

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13.8.9: Chapter 9 Review Exercises

Chapter 9 Review Exercises

Simplify and Use Square Roots

Simplify Expressions with Square Roots

In the following exercises, simplify.

Exercise 13.8.9.1

$\sqrt{64}$

Exercise 13.8.9.2

$\sqrt{144}$

Answer

12

Exercise 13.8.9.3

$-\sqrt{25}$

Exercise 13.8.9.4

$-\sqrt{81}$

Answer

-9

Exercise 13.8.9.5

$\sqrt{-9}$

Exercise 13.8.9.6

$\sqrt{-36}$

Answer

not a real number

Exercise 13.8.9.7

$\sqrt{64}+\sqrt{225}$

Exercise 13.8.9.8

$\sqrt{64+225}$

Answer

17

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

Exercise 13.8.9.9

$\sqrt{28}$

Exercise 13.8.9.10

$\sqrt{155}$

Answer

$12 < \sqrt{155} < 13$

Approximate Square Roots

In the following exercises, approximate each square root and round to two decimal places.

Exercise 13.8.9.11

$\sqrt{15}$

Example 13.8.9.12

$\sqrt{57}$

Answer

7.55

Simplify Variable Expressions with Square Roots

In the following exercises, simplify.

Exercise 13.8.9.13

$\sqrt{q^2}$

Exercise 13.8.9.14

$\sqrt{64b^2}$

Answer

8b

Exercis\e 13.8.9.15

$-\sqrt{121a^2}$

Exercise 13.8.9.16

$\sqrt{225m^2n^2}$

Answer

15mn

Exercise 13.8.9.17

$-\sqrt{100q^2}$

Exercise 13.8.9.18

$\sqrt{49y^2}$

Answer

7y

Exercise 13.8.9.19

$\sqrt{4a^2b^2}$

Exercise 13.8.9.20

$\sqrt{121c^2d^2}$

Answer

11cd

Simplify Square Roots

Use the Product Property to Simplify Square Roots

In the following exercises, simplify.

Exercise 13.8.9.21

$\sqrt{300}$

Exercise 13.8.9.22

$\sqrt{98}$

Answer

$7\sqrt{2}$

Exercise 13.8.9.23

$\sqrt{x^{13}}$

Exercise 13.8.9.24

$\sqrt{y^{19}}$

Answer

$y^9\sqrt{y}$

Exercise 13.8.9.25

$\sqrt{16m^4}$

Exercise 13.8.9.26

$\sqrt{36n^{13}}$

Answer

$6n^6\sqrt{n}$

Exercise 13.8.9.27

$\sqrt{288m^{21}}$

Exercise 13.8.9.28

$\sqrt{150n^7}$

Answer

$5n^3\sqrt{6n}$

Exercise 13.8.9.29

$\sqrt{48r^5s^4}$

Exercise 13.8.9.30

$\sqrt{108r^5s^3}$

Answer

$6r^2s\sqrt{3rs}$

Exercise 13.8.9.31

$\frac{10-\sqrt{50}}{5}$

Exercise 13.8.9.32

$\frac{6+\sqrt{72}}{6}$

Answer

$1+\sqrt{2}$

Use the Quotient Property to Simplify Square Roots

In the following exercises, simplify.

Exercise 13.8.9.33

$\sqrt{\frac{16}{25}}$

Exercise 13.8.9.34

$\sqrt{\frac{81}{36}}$

Answer

$\frac{3}{2}$

Exercise 13.8.9.35

$\sqrt{\frac{x^8}{x^4}}$

Exercise 13.8.9.36

$\sqrt{\frac{y^6}{y^2}}$

Answer

y^2

Exercise 13.8.9.37

$\sqrt{\frac{98p^6}{2p^2}}$

Exercise 13.8.9.38

$\sqrt{\frac{72q^8}{2q^4}}$

Answer

$6q^2$

Exercise 13.8.9.39

$\sqrt{\frac{65}{121}}$

Exercise 13.8.9.40

$\sqrt{\frac{26}{169}}$

Answer

$\frac{\sqrt{26}}{13}$

Exercise 13.8.9.41

$\sqrt{\frac{64x^4}{25x^2}}$

Exercise 13.8.9.42

$\sqrt{\frac{36r^{10}}{16r^5}}$

Answer

$\frac{3r^2\sqrt{r}}{2}$

Exercise 13.8.9.43

$\sqrt{\frac{48p^3q^5}{27pq}}$

Exercise 13.8.9.44

$\sqrt{\frac{12r^5s^7}{75r^2s}}$

Answer

$\frac{2rs^3\sqrt{r}}{5}$

Add and Subtract Square Roots

Add and Subtract Like Square Roots

In the following exercises, simplify.

Exercise 13.8.9.45

$3\sqrt{2}+\sqrt{2}$

Exercise 13.8.9.46

$5\sqrt{5}+7\sqrt{5}$

Answer

$12\sqrt{5}$

Exercise 13.8.9.47

$4\sqrt{y}+4\sqrt{y}$

Exercise 13.8.9.48

$6\sqrt{m}-2\sqrt{m}$

Answer

$4\sqrt{m}$

Exercise 13.8.9.49

$-3\sqrt{7}+2\sqrt{7}-\sqrt{7}$

Exercise 13.8.9.50

$8\sqrt{13}+2\sqrt{3}+3\sqrt{13}$

Answer

$11\sqrt{13}+2\sqrt{3}$

Exercise 13.8.9.51

$3\sqrt{5xy}-\sqrt{5xy}+3\sqrt{5xy}$

Exercise 13.8.9.52

$2\sqrt{3rs}+\sqrt{3rs}-5\sqrt{rs}$

Answer

$3\sqrt{3rs}-5\sqrt{rs}$

Add and Subtract Square Roots that Need Simplification

In the following exercises, simplify.

Exercise 13.8.9.53

$\sqrt{32}+3\sqrt{2}$

Exercise 13.8.9.54

$\sqrt{8}+\sqrt{32}$

Answer

$5\sqrt{2}$

Exercise 13.8.9.55

$\sqrt{72}+\sqrt{50}$

Exercise 13.8.9.56

$\sqrt{48}+\sqrt{75}$

Answer

$9\sqrt{3}$

Exercise 13.8.9.57

$3\sqrt{32}+\sqrt{98}$

Exercise 13.8.9.58

$\frac{1}{3}\sqrt{27}-\frac{1}{8}\sqrt{192}$

Answer

0

Exercise 13.8.9.59

$\sqrt{50y^5}-\sqrt{72y^5}$

Exercise 13.8.9.60

Add exercises text here.

Answer

$17n^2\sqrt{2}$

Multiply Square Roots

Multiply Square Roots

In the following exercises, simplify.

Exercise 13.8.9.61

$\sqrt{2}\cdot\sqrt{20}$

Exercise 13.8.9.62

$2\sqrt{2}\cdot 6\sqrt{14}$

Answer

$24\sqrt{7}$

Exercise 13.8.9.63

$\sqrt{2m^2}\cdot\sqrt{20m^4}$

Exercise 13.8.9.64

$(\sqrt{62y})(\sqrt{350y^3})$

Answer

$180y^2$

Exercise 13.8.9.65

$(6\sqrt{3v^4})(5\sqrt{30v})$

Exercise 13.8.9.66

$(\sqrt{8})^2$

Answer

8

Exercise 13.8.9.67

$(-\sqrt{10})^2$

Exercise 13.8.9.68

$(2\sqrt{5})(5\sqrt{5})$

Answer

50

Exercise 13.8.9.69

$(-3\sqrt{3})(5\sqrt{18})$

Use Polynomial Multiplication to Multiply Square Roots

In the following exercises, simplify.

Exercise 13.8.9.70

$10(2-\sqrt{7})$

Answer

$20-10\sqrt{7}$

Exercise 13.8.9.71

$\sqrt{3}(4+\sqrt{12})$

Exercise 13.8.9.72

$(5+\sqrt{2})(3-\sqrt{2})$

Answer

$13-2\sqrt{2}$

Exercise 13.8.9.73

$(5-3\sqrt{7})(1-2\sqrt{7})$

Exercise 13.8.9.74

$(1-3\sqrt{x})(5+2\sqrt{x})$

Answer

$5-13\sqrt{x}-6x$

Exercise 13.8.9.75

$(3+4\sqrt{y})(10-\sqrt{y})$

Exercise 13.8.9.76

$(1+6\sqrt{p})^2$

Answer

$1+12\sqrt{p}+36p$

Exercise 13.8.9.77

$(2-6\sqrt{5})^2$

Exercise 13.8.9.78

$(3+2\sqrt{7})(3-2\sqrt{7})$

Answer

-19

Exercise 13.8.9.79

$(6-\sqrt{11})(6+\sqrt{11})$

Divide Square Roots

Divide Square Roots

In the following exercises, simplify.

Exercise 13.8.9.80

$\frac{\sqrt{75}}{10}$

Answer

$\frac{\sqrt{3}}{2}$

Exercise 13.8.9.81

$\frac{2-\sqrt{12}}{6}$

Exercise 13.8.9.82

$\frac{\sqrt{48}}{\sqrt{27}}$

Answer

$\frac{4}{3}$

Exercise 13.8.9.83

$\frac{\sqrt{75x^7}}{\sqrt{3x^3}}$

Exercise 13.8.9.84

$\frac{\sqrt{20y^5}}{\sqrt{2y}}$

Answer

$y^2\sqrt{10}$

Exercise 13.8.9.85

$\frac{\sqrt{98p^6q^4}}{\sqrt{2p^4q^8}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

Exercise 13.8.9.86

$\frac{10}{\sqrt{15}}$

Answer

$\frac{2\sqrt{15}}{3}$

Exercise 13.8.9.87

$\frac{6}{\sqrt{6}}$

Exercise 13.8.9.88

$\frac{5}{3\sqrt{5}}$

Answer

$\frac{\sqrt{5}}{3}$

Exercise 13.8.9.89

$\frac{10}{2\sqrt{6}}$

Exercise 13.8.9.90

$\sqrt{\frac{3}{28}}$

Answer

$\frac{\sqrt{21}}{14}$

Exercise 13.8.9.91

$\sqrt{\frac{9}{75}}$

Rationalize a Two Term Denominator

In the following exercises, rationalize the denominator.

Exercise 13.8.9.92

$\frac{4}{4+\sqrt{27}}$

Answer

$\frac{16-12\sqrt{3}}{-11}$

Exercise 13.8.9.93

$\frac{5}{2-\sqrt{10}}$

Exercise 13.8.9.94

$\frac{4}{2-\sqrt{5}}$

Answer

$-8-4\sqrt{5}$

Exercise 13.8.9.95

$\frac{5}{4-\sqrt{8}}$

Exercise 13.8.9.96

$\frac{\sqrt{2}}{\sqrt{p}+\sqrt{3}}$

Answer

$\frac{\sqrt{2p}-\sqrt{6}}{p-3}$

Exercise 13.8.9.97

$\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}}$

Solve Equations with Square Roots

Solve Radical Equations

In the following exercises, solve the equation.

Exercise 13.8.9.98

$\sqrt{7z+1}=6$

Answer

5

Exercise 13.8.9.99

$\sqrt{4u-2}-4=0$

Exercise 13.8.9.100

$\sqrt{6m+4}-5=0$

Answer

$\frac{7}{2}$

Exercise 13.8.9.101

$\sqrt{2u-3}+2=0$

Exercise 13.8.9.102

$\sqrt{u-4}+4=u$

Answer

no solution

Exercise 13.8.9.103

$\sqrt{v-9}+9=0$

Exercise 13.8.9.104

$\sqrt{r-4}-r=-10$

Answer

13

Exercise 13.8.9.105

$\sqrt{s-9}-s=-9$

Exercise 13.8.9.106

$2\sqrt{2x-7}-4=8$

Answer

$\frac{43}{2}$

Exercise 13.8.9.107

$\sqrt{2-x} = \sqrt{2x-7}$

Exercise 13.8.9.108

$\sqrt{a}+3 = \sqrt{a+9}$

Answer

0

Exercise 13.8.9.109

$\sqrt{r}+3 = \sqrt{r+4}$

Exercise 13.8.9.110

$\sqrt{u}+2 = \sqrt{u+5}$

Answer

$\frac{11}{6}$

Exercise 13.8.9.111

$\sqrt{n+11}-1 = \sqrt{n+4}$

Exercise 13.8.9.112

$\sqrt{y+5}+1 = \sqrt{2y+3}$

Answer

11

Use Square Roots in Applications

In the following exercises, solve. Round approximations to one decimal place.

Exercise 13.8.9.113

A pallet of sod will cover an area of about 600 square feet. Trinh wants to order a pallet of sod to make a square lawn in his backyard. Use the formula $s = \sqrt{A}$ to find the length of each side of his lawn.

Exercise 13.8.9.114

A helicopter dropped a package from a height of 900 feet above a stranded hiker. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the hiker.

Answer

7.5 seconds

Exercise 13.8.9.115

Officer Morales measured the skid marks of one of the cars involved in an accident. The length of the skid marks was 245 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied.

Higher Roots

Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise 13.8.9.116

1. $\sqrt[6]{64}$
2. $\sqrt[3]{64}$

Answer

1. 2
2. 4

Exercise 13.8.9.117

1. $\sqrt[3]{-27}$
2. $\sqrt[4]{-64}$

Exercise 13.8.9.118

1. $\sqrt[9]{d^9}$
2. $\sqrt[8]{v^8}$

Answer

1. d
2. |v|

Exercise 13.8.9.119

1. $\sqrt[5]{a^{10}}$
2. $\sqrt[3]{b^{27}}$

Exercise 13.8.9.120

1. $\sqrt[4]{16x^8}$
2. $\sqrt[6]{64y^{12}}$

Answer

1. $2x^2$
2. $2y^2$

Exercise 13.8.9.121

1. $\sqrt[7]{128r^{14}}$
2. $\sqrt[4]{81s^{24}}$

Use the Product Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise 13.8.9.122

1. $\sqrt[9]{d^9}$
2. $\sqrt[11]{m^{17}}$

Answer

1. d
2. $m\sqrt[11]{m^6}$

Exercise 13.8.9.123

1. $\sqrt[3]{54}$
2. $\sqrt[4]{128}$

Exercise 13.8.9.124

1. $\sqrt[5]{64c^8}$
2. $\sqrt[4]{48d^7}$

Answer

1. $2c\sqrt[5]{2c^3}$
2. $2d\sqrt[4]{3d^3}$

Exercise 13.8.9.125

1. $\sqrt[3]{343q^7}$
2. $\sqrt[6]{192r^9}$

Exercise 13.8.9.126

1. $\sqrt[3]{-500}$
2. $\sqrt[4]{-16}$

Answer

1. $-5\sqrt[3]{4}$
2. not a real number

Use the Quotient Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise 13.8.9.127

$\sqrt[5]{\frac{r^{10}}{r^5}}$

Exercise 13.8.9.128

$\sqrt[3]{\frac{w^{12}}{w^2}}$

Answer

$w^3\sqrt[3]{w}$

Exercise 13.8.9.129

$\sqrt[4]{\frac{64y^8}{4y^5}}$

Exercise 13.8.9.130

$\sqrt[3]{\frac{54z^9}{2z^3}}$

Answer

$3z^2$

Exercise 13.8.9.131

$\sqrt[6]{\frac{64a^7}{b^2}}$

Add and Subtract Higher Roots

In the following exercises, simplify.

Exercise 13.8.9.132

$4\sqrt[5]{20}-2\sqrt[5]{20}$

Answer

$2\sqrt[5]{20}$

Exercise 13.8.9.133

$4\sqrt[3]{18}+3\sqrt[3]{18}$

Exercise 13.8.9.134

$\sqrt[4]{1250}-\sqrt[4]{162}$

Answer

$2\sqrt[4]{2}$

Exercise 13.8.9.135

$\sqrt[3]{640c^5}-\sqrt[3]{-80c^3}$

Exercise 13.8.9.136

$\sqrt[5]{96t^8} + \sqrt[5]{486t^4}$

Answer

$2t\sqrt[5]{3t^3} + 3\sqrt[5]{2t^4}$

Rational Exponents

Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

Exercise 13.8.9.137

$r^{\frac{1}{8}}$

Exercise 13.8.9.138

$s^{\frac{1}{10}}$

Answer

$\sqrt[10]{s}$

In the following exercises, write with a rational exponent.

Exercise 13.8.9.139

$\sqrt[5]{u}$

Exercise 13.8.9.140

$\sqrt[6]{v}$

Answer

$v^{\frac{1}{6}}$

Exercise 13.8.9.141

$\sqrt[3]{9m}$

Exercise 13.8.9.142

$\sqrt[6]{10z}$

Answer

$(10z)^{\frac{1}{6}}$

In the following exercises, simplify.

Exercise 13.8.9.143

$16^{\frac{1}{4}}$

Exercise 13.8.9.144

$32^{\frac{1}{5}}$

Answer

2

Exercise 13.8.9.145

$(-125)^{\frac{1}{3}}$

Exercise 13.8.9.146

$(125)^{-\frac{1}{3}}$

Answer

$\frac{1}{5}$

Exercise 13.8.9.147

$(-9)^{\frac{1}{2}}$

Exercise 13.8.9.148

$(36)^{-\frac{1}{2}}$

Answer

$\frac{1}{6}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

Exercise 13.8.9.149

$\sqrt[3]{q^5}$

Exercise 13.8.9.150

$\sqrt[5]{n^8}$

Answer

$n^{\frac{8}{5}}$

In the following exercises, simplify.

Exercise 13.8.9.151

$27^{-\frac{2}{3}}$

Exercise 13.8.9.152

$64^{\frac{5}{2}}$

Answer

32,768

Exercise 13.8.9.153

$36^{\frac{3}{2}}$

Exercise 13.8.9.154

$81^{-\frac{5}{2}}$

Answer

$\frac{1}{59,049}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

Exercise 13.8.9.155

$3^{\frac{4}{5}} \cdot 3^{\frac{6}{5}}$

Exercise 13.8.9.156

$(x^6)^{\frac{4}{3}}$

Answer

x^8

Exercise 13.8.9.157

$\frac{z^{\frac{5}{2}}}{z^{\frac{7}{5}}}$

Exercise 13.8.9.158

$(16s^{\frac{9}{4}})^{\frac{1}{4}}$

Answer

$2s^{\frac{9}{16}}$

Exercise 13.8.9.159

$(m^8 n^{12})^{\frac{1}{4}}$

Exercise 13.8.9.160

$\frac{z^{\frac{2}{3}} \cdot z^{-\frac{1}{3}}}{z^{-\frac{5}{3}}}$

Answer

z^2

Practice Test

In the following exercises, simplify.

Exercise 13.8.9.161

$\sqrt{81+144}$

Exercise 13.8.9.162

$\sqrt{169m^4n^2}$

Answer

$13m^2|n|$

Exercise 13.8.9.163

$\sqrt{36n^{13}}$

Exercise 13.8.9.164

$3\sqrt{13}+5\sqrt{2}+\sqrt{13}$

Answer

$4\sqrt{13}+5\sqrt{2}$

Exercise 13.8.9.165

$5\sqrt{20}+2\sqrt{125}$

Exercise 13.8.9.166

$(3\sqrt{6y})(\sqrt{250y^3})$

Answer

$180y^2\sqrt{3}$

Exercise 13.8.9.167

$(2-5\sqrt{x})(3+\sqrt{x})$

Exercise 13.8.9.168

$(1-2\sqrt{q})^2$

Answer

$1-4\sqrt{q}+4q$

Exercise 13.8.9.169

1. $\sqrt{a^{12}}$
2. $\sqrt[3]{b^{21}}$

Exercise 13.8.9.170

1. $\sqrt[4]{81x^{12}}$
2. $\sqrt[6]{64y^{18}}$

Answer

1. $3x^3$
2. $2y^3$

Exercise 13.8.9.171

$\sqrt[6]{\frac{64r^{12}}{25r^6}}$

Exercise 13.8.9.172

$\sqrt{\frac{14y^3}{7y}}$

Answer

$y\sqrt{2}$

Exercise 13.8.9.173

$\frac{\sqrt{256x^7}}{\sqrt{54x^2}}$

Exercise 13.8.9.174

$\sqrt[4]{512}-2\sqrt[4]{32}$

Answer

0

Exercise 13.8.9.175

1. $256^{\frac{1}{4}}$
2. $243^{\frac{1}{5}}$

Exercise 13.8.9.176

$49^{\frac{3}{2}}$

Answer

343

Exercise 13.8.9.177

$25^{-\frac{5}{2}}$

Exercise 13.8.9.178

$\frac{w^{\frac{3}{4}}}{w^{\frac{7}{4}}}$

Answer

$\frac{1}{w}$

Exercise 13.8.9.179

$(27s^{\frac{3}{5}})^{\frac{1}{3}}$

In the following exercises, rationalize the denominator.

Exercise 13.8.9.180

$\frac{3}{2\sqrt{6}}$

Answer

$\frac{\sqrt{6}}{4}$

Exercise 13.8.9.181

$\frac{\sqrt{3}}{\sqrt{x}+\sqrt{5}}$

In the following exercises, solve.

Exercise 13.8.9.182

$3\sqrt{2x-3}-20=7$

Answer

42

Exercise 13.8.9.183

$\sqrt{3u-2}=\sqrt{5u+1}$

In the following exercise, solve.

Exercise 13.8.9.184

A helicopter flying at an altitude of 600 feet dropped a package to a lifeboat. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the hiker. Round your answer to the nearest tenth of a second.

Answer

6.1 seconds

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SECTION OVERVIEW

13.9: Rational Expressions and Equations

13.9.1: Simplify Rational Expressions

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13.9.2: Multiply and Divide Rational Expressions

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13.9.1: Simplify Rational Expressions

Summary

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Evaluate rational expressions
- Simplify rational expressions
- Simplify rational expressions with opposite factors

Are you Ready?

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Simplify: $\frac{90y}{15y^2}$.
   If you missed this problem, review [link].
2. Factor: $6x^2-7x+2$.
   If you missed this problem, review [link].
3. Factor: n^3+8.
   If you missed this problem, review [link].

In Chapter 1, we reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers, and the denominator is not zero.

In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call these rational expressions.

Definition: RATIONAL EXPRESSION

A **rational expression** is an expression of the form $\frac{p(x)}{q(x)}$, where p and q are polynomials and $q \neq 0$.

Remember, division by 0 is undefined.

Here are some examples of rational expressions:

$$-\frac{13}{42} \quad \frac{7y}{8z} \quad \frac{5x+2}{x^2-7} \quad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{13}{42}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will perform same operations with rational expressions that we do with fractions. We will simplify, add, subtract, multiply, divide, and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

When we work with a numerical fraction, it is easy to avoid dividing by zero, because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

Definition: DETERMINE THE VALUES FOR WHICH A RATIONAL EXPRESSION IS UNDEFINED.

1. Set the denominator equal to zero.
2. Solve the equation in the set of reals, if possible.

Example 13.9.1.1

Determine the values for which the rational expression is undefined:

1. $\frac{9y}{x}$
2. $\frac{4b-3}{2b+5}$
3. $\frac{x+4}{x^2+5x+6x}$

Solution

The expression will be undefined when the denominator is zero.

1.	$\frac{9y}{x}$
Set the denominator equal to zero. Solve for the variable.	x=0
	$\frac{9y}{x}$ is undefined for x=0.
2.	$\frac{4b-3}{2b+5}$
Set the denominator equal to zero. Solve for the variable.	2b+5=0
	2b=−5
	$b=-\frac{5}{2}$
	$\frac{4b-3}{2b+5}$ is undefined for $b=-\frac{5}{2}$.
3.	$\frac{x+4}{x^2+5x+6x}$
Set the denominator equal to zero. Solve for the variable.	$x^2+5x+6x=0$
	$(x+2)(x+3)=0$
	x+2=0 or x+3=0
	x=−2 or x=−3
	$\frac{x+4}{x^2+5x+6x}$ is undefined for x=−2 or x=−3.

Saying that the rational expression $\frac{x+4}{x^2+5x+6x}$ is undefined for x=−2orx=−3 is similar to writing the phrase “void where prohibited” in contest rules.

Try It 13.9.1.2

Determine the values for which the rational expression is undefined:

1. $\frac{3y}{x}$
2. $\frac{8n-5}{3n+1}$
3. $\frac{a+10}{a^2+4a+3a}$

Answer

1. x=0
2. $n=-\frac{1}{3}$
3. a=−1, a=−3

Try It 13.9.1.3

Determine the values for which the rational expression is undefined:

1. $\frac{4p}{5q}$
2. $\frac{y-1}{3y+2}$
3. $\frac{m-5}{m^2+m-6}$

Answer

1. q=0
2. $y=-\frac{2}{3}$
3. m=2, m=−3

Evaluate Rational Expressions

To evaluate a rational expression, we substitute values of the variables into the expression and simplify, just as we have for many other expressions in this book.

Example 13.9.1.4

Evaluate $\frac{2x+3}{3x-5}$ for each value:

1. x=0
2. x=2
3. x=−3

Solution

1.	$\frac{2x+3}{3x-5}$
Substitute 0 for x.	$\frac{2(0)+3}{3(0)-5}$
Simplify.	$-\frac{3}{5}$
2.	$\frac{2x+3}{3x-5}$
Substitute 2 for x.	$\frac{2(2)+3}{3(2)-5}$
Simplify.	$\frac{4+3}{6-5}$
	$\frac{7}{1}$
	7
3.	$\frac{2x+3}{3x-5}$
Substitute –3 for x.	$\frac{2(-3)+3}{3(-3)-5}$
Simplify.	$\frac{-6+3}{-9-5}$
	$\frac{-3}{-14}$
	$\frac{3}{14}$

Try It 13.9.1.5

Evaluate $\frac{y+1}{2y-3}$ for each value:

1. y=1
2. y=−3
3. y=0

Answer

1. −2
2. $\frac{2}{9}$
3. $-\frac{1}{3}$

Try It 13.9.1.6

Evaluate $\frac{5x-1}{2x+1}$ for each value:

1. x=1
2. x=−1
3. x=0

Answer

1. $\frac{4}{3}$
2. 6
3. −1

Example 13.9.1.7

Evaluate $\frac{x^2+8x+7}{x^2-4}$ for each value:

1. x=0
2. x=2
3. x=−1

Solution

1.	$\frac{x^2+8x+7}{x^2-4}$
Substitute 0 for x.	$\frac{(0)^2+8(0)+7}{(0)^2-4}$
Simplify.	$\frac{7}{-4}$
	$-\frac{7}{4}$
2.	$\frac{x^2+8x+7}{x^2-4}$
Substitute 2 for x.	$\frac{(2)^2+8(2)+7}{(2)^2-4}$
Simplify.	$\frac{4+16+7}{4-4}$
	$\frac{27}{0}$
This rational expression is undefined for x = 2.	

3.	$\frac{x^2+8x+7}{x^2-4}$
Substitute -1 for x.	$\frac{(-1)^2+8(-1)+7}{(-1)^2-4}$
Simplify.	$\frac{1-8+7}{1-4}$
	$\frac{-7+7}{-3}$
	$\frac{0}{-3}$
	0

Try It 13.9.1.8

Evaluate $\frac{x^2+1}{x^2-3x+2}$ for each value:

1. x=0
2. x=−1
3. x=3

Answer

1. $\frac{1}{2}$
2. $\frac{1}{3}$
3. 2

Try It 13.9.1.9

Evaluate $\frac{x^2+x-6}{x^2-9}$ for each value.

1. x=0
2. x=−2
3. x=1

Answer

1. $\frac{2}{3}$
2. $\frac{4}{5}$
3. $\frac{1}{2}$

Remember that a fraction is simplified when it has no common factors, other than 1, in its numerator and denominator. When we evaluate a rational expression, we make sure to simplify the resulting fraction.

Example 13.9.1.10

Evaluate $\frac{a^2+2ab+b^2}{3ab}$ for each value.

1. a=1, b=2
2. a=−2, b=−1
3. $a=\frac{1}{3}$, b=0

Solution

1.	$\frac{a^2+2ab+b^2}{3ab}$ when a=1, b=2

Substitute 1 for a and 2 for b.	$\frac{(1)^2+2(1)(2)+(2)^2}{3(1)(2)^2}$
Simplify.	$\frac{1+4+4}{3(4)}$
	$\frac{9}{12}$
	$\frac{3}{4}$
2.	$\frac{a^2+2ab+b^2}{3ab}$ when a=−2, b=−1
Substitute −2 for a and −1 for b.	$\frac{(-2)^2+2(-2)(-1)+(-1)^2}{3(-2)(-1)^2}$
Simplify.	$\frac{4+4+1}{-6}$
	$-\frac{9}{6}$
	$-\frac{3}{2}$
3.	$\frac{a^2+2ab+b^2}{3ab}$ when $a=\frac{1}{3}$, b=0
Substitute $\frac{1}{3}$ for a and 0 for b.	$\frac{\left(\frac{1}{3}\right)^2+2\left(\frac{1}{3}\right)(0)+(0)^2}{3\left(\frac{1}{3}\right)(0)^2}$
Simplify.	$\frac{\frac{1}{9}+0+0}{0}$
	$\frac{\frac{1}{9}}{0}$

Try It 13.9.1.11

Evaluate $\frac{2a^3b}{a^2+2ab+b^2}$ for each value.

1. a=−1, b=2
2. a=0, b=−1
3. a=1, $b=\frac{1}{2}$

Answer

1. −4
2. 0
3. $\frac{4}{9}$

Try It 13.9.1.12

Evaluate $\frac{a^2-b^2}{8ab^3}$ for each value:

1. a=1, b=−1
2. $a=\frac{1}{2}$, b=−1
3. a=−2, b=1

Answer

1. 0
2. $\frac{3}{16}$

3. $\frac{3}{16}$

Simplify Rational Expressions

Just like a fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator, a rational expression is *simplified* if it has no common factors, other than 1, in its numerator and denominator.

Definition: SIMPLIFIED RATIONAL EXPRESSION

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example:

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{2x}{3x}$ is not simplified because x is a common factor of $2x$ and $3x$.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify **rational expression**s.

Definition: EQUIVALENT FRACTIONS PROPERTY

If a, b, and c are numbers where $b \neq 0$, $c \neq 0$, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0$, $c \neq 0$ clearly stated. Every time we write a rational expression, we should make a similar statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Let's start by reviewing how we simplify numerical fractions.

Example 13.9.1.13

Simplify: $-\frac{36}{63}$.

Solution

	$-\frac{36}{63}$
Rewrite the numerator and denominator showing the common factors.	$-\frac{4 \cdot 9}{7 \cdot 9}$
Simplify using the Equivalent Fractions Property.	$-\frac{4}{7}$

Notice that the fraction $-\frac{4}{7}$ is simplified because there are no more common factors.

Try It 13.9.1.14

Simplify: $-\frac{45}{81}$.

Answer

$-\frac{5}{9}$

Try It 13.9.1.15

Simplify: $-\frac{42}{54}$.

Answer

$-\frac{7}{9}$

Throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$ and $y \neq 0$.

Example 13.9.1.16

Simplify: $\frac{3xy}{18x^2y^2}$.

Solution

	$\frac{3xy}{18x^2y^2}$
Rewrite the numerator and denominator showing the common factors.	$\frac{1 \cdot 3xy}{6xy \cdot 3xy}$
Simplify using the Equivalent Fractions Property.	$\frac{1}{6xy}$

Did you notice that these are the same steps we took when we divided monomials in Polynomials?

Try It 13.9.1.17

Simplify: $\frac{4x^2y}{12xy^2}$.

Answer

$\frac{x}{3y}$

Try It $PageIndex18

Simplify: $\frac{16x^2y}{2xy^2}$.

Answer

$\frac{8x}{y}$

To simplify rational expressions we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$	$\frac{3x(x-9)}{5(x-9)}$	$\frac{x+5}{x}$
$\frac{2}{5}$	$\frac{3x}{5}$	NO COMMON FACTORS
We removed the common factors 3 and 7. They are factors of the product.	We removed the common factor $(x-9)$. It is a factor of the product.	While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Note that removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

Example 13.9.1.19

Simplify: $\frac{2x+8}{5x+20}$.

Solution

Step 1. Factor the numerator and denominator completely.	Factor $2x + 8$ and $5x - 20$.	$\frac{2x+8}{5x+20}$ $\frac{2(x+4)}{5(x+4)}$
Step 2. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{2(\cancel{x+4})}{5(\cancel{x+4})}$ $\frac{2}{5}$

Try It 13.9.1.20

Simplify: $\frac{3x-6}{2x-4}$.

Answer

$\frac{3}{2}$

Try It 13.9.1.21

Simplify: $\frac{7y+35}{5y+25}$.

Answer

$\frac{7}{5}$

We now summarize the steps you should follow to simplify rational expressions.

Definition: SIMPLIFY A RATIONAL EXPRESSION.

1. Factor the numerator and denominator completely.
2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way it is easy to check that we have removed all the common factors!

We'll use the methods we covered in Factoring to factor the polynomials in the numerators and denominators in the following examples.

Example 13.9.1.22

Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$.

Solution

	$\frac{x^2+5x+6}{x^2+8x+12}$
Factor the numerator and denominator.	$\frac{(x+2)(x+3)}{(x+2)(x+6)}$

Remove the common factor x+2 from the numerator and the denominator.	$\frac{x+3}{x+6}$

Can you tell which values of *x* must be excluded in this example?

Try It 13.9.1.23

Simplify: $\frac{x^2-x-2}{x^2-3x+2}$.

Answer

$\frac{x+1}{x-1}$

Try It 13.9.1.24

Simplify: $\frac{x^2-3x-10}{x^2+x-2}$.

Answer

$\frac{x-5}{x-1}$

Example 13.9.1.25

Simplify: $\frac{y^2+y-42}{y^2-36}$.

Solution

	$\frac{y^2+y-42}{y^2-36}$.
Factor the numerator and denominator.	$\frac{(y+7)(y-6)}{(y+6)(y-6)}$
Remove the common factor y−6 from the numerator and the denominator.	$\frac{y+7}{y+6}$

Try It 13.9.1.26

Simplify: $\frac{x^2+x-6}{x^2-4}$.

Answer

$\frac{x+3}{x+2}$

Try It 13.9.1.27

Simplify: $\frac{x^2+8x+7}{x^2-49}$.

Answer

$\frac{x+1}{x-7}$

Example 13.9.1.28

Simplify: $\frac{p^3-2p^2+2p-4}{p^2-7p+10}$.

Solution

	$\frac{p^3-2p^2+2p-4}{p^2-7p+10}$
Factor the numerator and denominator, using grouping to factor the numerator.	$\frac{p^2(p-2)+2(p-2)}{(p-5)(p-2)}$
	$\frac{(p^2+2)(p-2)}{(p-5)(p-2)}$
Remove the common factor p−2 from the numerator and the denominator.	$\frac{p^2+2}{p-5}$

Try It 13.9.1.29

Simplify: $\frac{y^3-3y^2+y-3}{y^2-y-6}$.

Answer

$\frac{y^2+1}{y+2}$

Try It 13.9.1.30

Simplify: $\frac{p^3-p^2+2p-2}{p^2+4p-5}$.

Answer

$\frac{p^2+2}{p+5}$

Example 13.9.1.31

Simplify: $\frac{2n^2-14n}{4n^2-16n-48}$.

Solution

	$\frac{2n^2-14n}{4n^2-16n-48}$
Factor the numerator and denominator, first factoring out the GCF.	$\frac{2n(n-7)}{4(n^2-4n-12)}$
	$\frac{2n(n-7)}{4(n-6)(n+2)}$
Remove the common factor, 2.	$\frac{n(n-7)}{2(n-6)(n+2)}$

Try It 13.9.1.32

Simplify: $\frac{2n^2-10n}{4n^2-16n-20}$.

Answer

$\frac{n}{2(n+1)}$

Try It 13.9.1.33

Simplify: $\frac{4x^2-16x}{8x^2-16x-64}$.

Answer

$\frac{x}{2(x+2)}$

Example 13.9.1.34

Simplify: $\frac{3b^2-12b+12}{6b^2-24}$.

Solution

	$\frac{3b^2-12b+12}{6b^2-24}$
Factor the numerator and denominator, first factoring out the GCF.	$\frac{3(b^2-4b+4)}{6(b^2-4)}$
	$\frac{3(b-2)(b-2)}{6(b-2)(b+2)}$
Remove the common factors of b−2 and 3.	$\frac{3(b-2)}{2(b+2)}$

Try It 13.9.1.35

Simplify: $\frac{2x^2-12x+18}{3x^2-27}$.

Answer

$\frac{2(x-3)}{3(x+3)}$

Try It $PageIndex36$

Simplify: $\frac{5y^2-30y+25}{2y^2-50}$.

Answer

$\frac{5(x-1)}{2(x+5)}$

Example 13.9.1.37

Simplify: $\frac{m^3+8}{m^2-4}$.

Solution

	$\frac{m^3+8}{m^2-4}$
Factor the numerator and denominator, using the formulas for sum of cubes and difference of squares.	$\frac{(m+2)(m^2-2m+4)}{(m+2)(m-2)}$

Remove the common factors of m+2.	$\frac{m^2-2m+4}{m-2}$

Try It 13.9.1.38

Simplify: $\frac{p^3-64}{p^2-16}$.

Answer

$\frac{p^2+4p+16}{p+4}$

Try It 13.9.1.39

Simplify: $\frac{x^3+8}{x^2-4}$.

Answer

$\frac{x^2-2x+4}{x-2}$

Simplify Rational Expressions with Opposite Factors

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. Let's start with a numerical fraction, say $\frac{7}{-7}$.

We know this fraction simplifies to −1. We also recognize that the numerator and denominator are opposites.

In Foundations, we introduced opposite notation: the opposite of a is −a. We remember, too, that −a=−1·a

We simplify the fraction $\frac{a}{-a}$

	$\frac{a}{-a}$
We could rewrite this.	$\frac{1\cdot a}{-1\cdot a}$
Remove the common factors.	$\frac{1}{-1}$
Simplify.	-1

So, in the same way, we can simplify the fraction $\frac{x-3}{-(x-3)}$

	$\frac{x-3}{-(x-3)}$
We could rewrite this.	$\frac{1\cdot(x-3)}{-1\cdot(x-3)}$
Remove the common factors.	$\frac{1}{-1}$
Simplify.	-1

But the opposite of x−3 could be written differently:

	$-(x-3)$
Distribute.	$-x+3$
Rewrite.	$3-x$

This means the fraction $\frac{x-3}{3-x}$ simplifies to −1.

In general, we could write the opposite of a−b as b−a. So the rational expression $\frac{a-b}{b-a}$ simplifies to −1.

Definition: OPPOSITES IN A RATIONAL EXPRESSION

The opposite of a−b is b−a

$\frac{a-b}{b-a}=-1, a \neq b$

An expression and its opposite divide to −1

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators.

Example 13.9.1.40

Simplify: $\frac{x-8}{8-x}$.

Solution

	$\frac{x-8}{8-x}$.
Recognize that x−8 and 8−x are opposites	−1

Try It 13.9.1.41

Simplify: $\frac{y-2}{2-y}$.

Answer

−1

Try It 13.9.1.42

Simplify: $\frac{n-9}{9-n}$.

Answer

−1

Remember, the first step in simplifying a rational expression is to factor the numerator and denominator completely.

Example 13.9.1.43

Simplify: $\frac{14-2x}{x^2-49}$.

Solution

	$\frac{14-2x}{x^2-49}$
Factor the numerator and denominator.	$\frac{2(7-x)}{(x+7)(x-7)}$
Recognize 7−x and x−7 are opposites.	$\frac{2(7-x)}{(x+7)(x-7)}(-1)$
Simplify.	$-\frac{2}{x+7}$

Try It 13.9.1.44

Simplify: $\frac{10-2y}{y^2-25}$.

Answer

$-\frac{2}{y+5}$

Try It 13.9.1.45

Simplify: $\frac{3y-27}{81-y^2}$.

Answer

$-\frac{3}{9+y}$

Example 13.9.1.46

Simplify: $\frac{x^2-4x-32}{64-x^2}$.

Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Simplify.	$-\frac{x+4}{x+8}$

Try It 13.9.1.47

Simplify: $\frac{x^2-4x-5}{25-x^2}$.

Answer

$-\frac{x+1}{x+5}$

Try It 13.9.1.48

Simplify: $\frac{x^2+x-2}{1-x^2}$.

Answer

$-\frac{x+2}{x+1}$

Key Concepts

- **Determine the Values for Which a Rational Expression is Undefined**
  1. Set the denominator equal to zero.
  2. Solve the equation, if possible.
- **Simplified Rational Expression**

- A rational expression is considered simplified if there are no common factors in its numerator and denominator.

- **Simplify a Rational Expression**
  1. Factor the numerator and denominator completely.
  2. Simplify by dividing out common factors.

- **Opposites in a Rational Expression**
  - The opposite of a−b is b−a
    $\frac{a-b}{b-a} = -1 \; a \neq b, b \neq 0, a \neq b$

Practice Makes Perfect

In the following exercises, determine the values for which the rational expression is undefined.

Example 13.9.1.49

1. $\frac{2x}{z}$
2. $\frac{4p-1}{6p-5}$
3. $\frac{n-3}{n^2+2n-8}$

Answer

1. z=0
2. $p = \frac{5}{6}$
3. n=−4, n=2

Example 13.9.1.50

1. $\frac{10m}{11n}$
2. $\frac{6y+13}{4y-9}$
3. $\frac{b-8}{b^2-36}$

Example 13.9.1.51

1. $\frac{4x^2y}{3y}$
2. $\frac{3x-2}{2x+1}$
3. $\frac{u-1}{u^2-3u-28}$

Answer

1. y=0
2. $x = -\frac{1}{2}$
3. u=−4, u=7

Example 13.9.1.52

1. $\frac{5pq^2}{9q}$
2. $\frac{7a-4}{3a+5}$
3. $\frac{1}{x^2-4}$

Evaluate Rational Expressions

In the following exercises, evaluate the rational expression for the given values.

Example 13.9.1.53

$\frac{2x}{x-1}$

1. x=0
2. x=2
3. x=−1

Answer

1. 0
2. 4
3. 1

Example 13.9.1.54

$\frac{4y-1}{5y-3}$

1. y=0
2. y=2
3. y=−1

Example 13.9.1.55

$\frac{2p+3}{p^2+1}$

1. p=0
2. p=1
3. p=−2

Answer

1. 3
2. $\frac{5}{2}$
3. $-\frac{1}{5}$

Example 13.9.1.56

$\frac{x+3}{2-3x}$

1. x=0
2. x=1
3. x=−2

Example 13.9.1.57

$\frac{y^2+5y+6}{y^2-1}$

1. y=0
2. y=2
3. y=−2

Answer

1. −6
2. $\frac{20}{3}$
3. 0

Example 13.9.1.58

$\frac{z^2+3z-10}{z^2-1}$

1. z=0
2. z=2
3. z=−2

Example 13.9.1.59

$\frac{a^2-4}{a^2+5a+4}$

1. a=0
2. a=1
3. a=−2

Answer

1. −1
2. $-\frac{3}{10}$
3. 0

Example 13.9.1.60

$\frac{b^2+2}{b^2-3b-4}$

1. b=0
2. b=2
3. b=−2

Example 13.9.1.61

$\frac{x^2+3xy+2y^2}{2x^3y}$

1. x=1, y=−1
2. x=2, y=1
3. x=−1, y=−2

Answer

1. 0
2. $\frac{3}{4}$
3. $\frac{15}{4}$

Example 13.9.1.62

$\frac{c^2+cd-2d^2}{cd^3}$

1. c=2, d=−1
2. c=1, d=−1
3. c=−1, d=2

Example 13.9.1.63

$\frac{m^2-4n^2}{5mn^3}$

1. m=2, n=1
2. m=−1, n=−1
3. m=3, n=2

Answer

1. 0
2. $-\frac{3}{5}$
3. $-\frac{7}{20}$

Example 13.9.1.64

$\frac{2s^2t}{s^2-9t^2}$

1. s=4, t=1
2. s=−1, t=−1
3. s=0, t=2

Simplify Rational Expressions

In the following exercises, simplify.

Example 13.9.1.65

$-\frac{4}{52}$

Answer

$-\frac{1}{13}$

Example 13.9.1.66

$-\frac{44}{55}$

Example 13.9.1.67

$\frac{56}{63}$

Answer

$\frac{8}{9}$

Example 13.9.1.68

$\frac{65}{104}$

Example 13.9.1.69

$\frac{6ab^2}{12a^2b}$

Answer

$\frac{b}{2ab}$

Example 13.9.1.70

$\frac{15xy^3}{x^3y^3}$

Example 13.9.1.71

$\frac{8m^3n}{12mn^2}$

Answer

$\frac{2m^2}{3n}$

Example 13.9.1.72

$\frac{36v^3w^2}{27vw^3}$

Example 13.9.1.73

$\frac{3a+6}{4a+8}$

Answer

$\frac{3}{4}$

Example 13.9.1.74

$\frac{5b+5}{6b+6}$

Example 13.9.1.75

$\frac{3c-9}{5c-15}$

Answer

$\frac{3}{5}$

Example 13.9.1.76

$\frac{4d+8}{9d+18}$

Example 13.9.1.77

$\frac{7m+63}{5m+45}$

Answer

$\frac{7}{5}$

Example 13.9.1.78

$\frac{8n-96}{3n-36}$

Exercise 13.9.1.79

$\frac{12p-240}{5p-100}$

Answer

$\frac{12}{5}$

Example 13.9.1.80

$\frac{6q+210}{5q+175}$

Example 13.9.1.81

$\frac{a^2-a-12}{a^2-8a+16}$

Answer

$\frac{a+3}{a-4}$

Example 13.9.1.82

$\frac{x^2+4x-5}{x^2-2x+1}$

Example 13.9.1.83

$\frac{y^2+3y-4}{y^2-6y+5}$

Answer

$\frac{y+4}{y-5}$

Example 13.9.1.84

$\frac{v^2+8v+15}{v^2-v-12}$

Example 13.9.1.85

$\frac{x^2-25}{x^2+2x-15}$

Answer

$\frac{x-5}{x-3}$

Example 13.9.1.86

$\frac{a^2-4}{a^2+6a-16}$

Example 13.9.1.87

$\frac{y^2-2y-3}{y^2-9}$

Answer

$\frac{y+1}{y+3}$

Example 13.9.1.88

$\frac{b^2+9b+18}{b^2-36}$

Example 13.9.1.89

$\frac{y^3+y^2+y+1}{y^2+2y+1}$

Answer

$\frac{y^2+1}{y+1}$

Example 13.9.1.90

$\frac{p^3+3p^2+4p+12}{p^2+p-6}$

Example 13.9.1.91

$\frac{x^3-2x^2-25x+50}{x^2-25}$

Answer

x−2

Example 13.9.1.92

$\frac{q^3+3q^2-4q-12}{q^2-4}$

Example 13.9.1.93

$\frac{3a^2+15a}{6a^2+6a-36}$

Answer

$\frac{a(a+5)}{2(a+3)(a-2)}$

Example 13.9.1.94

$\frac{8b^2-32b}{2b^2-6b-80}$

Example 13.9.1.95

$\frac{-5c^2-10c}{-10c^2+30c+100}$

Answer

$\frac{c}{2(c-5)}$

Example 13.9.1.96

$\frac{4d^2-24d}{2d^2-4d-48}$

Example 13.9.1.97

$\frac{3m^2+30m+75}{4m^2-100}$

Answer

$\frac{3(m+5)}{4(m-5)}$

Example 13.9.1.98

$\frac{5n^2+30n+45}{2n^2-18}$

Example 13.9.1.99

$\frac{5r^2+30r-35}{r^2-49}$

Answer

$\frac{5(r-1)}{r+7}$

Example 13.9.1.100

$\frac{3s^2+30s+24}{3s^2-48}$

Example 13.9.1.101

$\frac{t^3-27}{t^2-9}$

Answer

$\frac{t^2+3t+9}{t+3}$

Example 13.9.1.102

$\frac{v^3-1}{v^2-1}$

Example 13.9.1.103

$\frac{w^3+216}{w^2-36}$

Answer

$\frac{w^2-6w+36}{w-6}$

Example 13.9.1.104

$\frac{v^3+125}{v^2-25}$

Simplify Rational Expressions with Opposite Factors

In the following exercises, simplify each rational expression.

Example 13.9.1.105

$\frac{a-5}{5-a}$

Answer

-1

Example 13.9.1.106

$\frac{b-12}{12-b}$

Example 13.9.1.107

$\frac{11-c}{c-11}$

Answer

-1

Example 13.9.1.108

$\frac{5-d}{d-5}$

Example 13.9.1.109

$\frac{12-2x}{x^2-36}$

Answer

$-\frac{2}{x+6}$

Example 13.9.1.110

$\frac{20-5y}{y^2-16}$

Example 13.9.1.111

$\frac{4v-32}{64-v^2}$

Answer

$-\frac{4}{8+v}$

Example 13.9.1.112

$\frac{7w-21}{9-w^2}$

Example 13.9.1.113

$\frac{y^2-11y+24}{9-y^2}$

Answer

$-\frac{y-8}{3+y}$

Example 13.9.1.114

$\frac{z^2-9z+20}{16-z^2}$

Example 13.9.1.115

$\frac{a^2-5a-36}{81-a^2}$

Answer

$-\frac{a+4}{9+a}$

Example 13.9.1.116

$\frac{b^2+b-42}{36-b^2}$

Everyday Math

Example 13.9.1.117

Tax Rates For the tax year 2015, the amount of tax owed by a single person earning between $37,450 and $90,750, can be found by evaluating the formula 0.25x−4206.25, where x is income. The average tax rate for this income can be found by evaluating the formula $\frac{0.25x-4206.25}{x}$. What would be the average tax rate for a single person earning $50,000?

Answer

16.5%

Example 13.9.1.118

Work The length of time it takes for two people for perform the same task if they work together can be found by evaluating the formula $\frac{xy}{x+y}$. If Tom can paint the den in x=45 minutes and his brother Bobby can paint it in y=60 minutes, how many minutes will it take them if they work together?

Writing Exercises

Example 13.9.1.119

Explain how you find the values of x for which the rational expression $\frac{x^2-x-20}{x^2-4}$ is undefined.

Example 13.9.1.120

Explain all the steps you take to simplify the rational expression $\frac{p^2+4p-21}{9-p^2}$.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
evaluate rational expressions.			
simplify rational expressions.			
simplify rational expressions with opposite factors.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

rational expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

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13.9.1E: Exercises

Practice Makes Perfect

In the following exercises, determine the values for which the rational expression is undefined.

? Example 13.9.1E.49

1. $\frac{2x}{z}$
2. $\frac{4p-1}{6p-5}$
3. $\frac{n-3}{n^2+2n-8}$

Answer

1. z=0
2. $p=\frac{5}{6}$
3. n=−4, n=2

? Example 13.9.1E.50

1. $\frac{10m}{11n}$
2. $\frac{6y+13}{4y-9}$
3. $\frac{b-8}{b^2-36}$

? Example 13.9.1E.51

1. $\frac{4x^2y}{3y}$
2. $\frac{3x-2}{2x+1}$
3. $\frac{u-1}{u^2-3u-28}$

Answer

1. y=0
2. $x=-\frac{1}{2}$
3. u=−4, u=7

? Example 13.9.1E.52

1. $\frac{5pq^2}{9q}$
2. $\frac{7a-4}{3a+5}$
3. $\frac{1}{x^2-4}$

Evaluate Rational Expressions

In the following exercises, evaluate the rational expression for the given values.

Example 13.9.1E.53

$$\frac{2x}{x-1}$$

1. x=0
2. x=2
3. x=−1

Answer

1. 0
2. 4
3. 1

Example 13.9.1E.54

$$\frac{4y-1}{5y-3}$$

1. y=0
2. y=2
3. y=−1

Example 13.9.1E.55

$$\frac{2p+3}{p^2+1}$$

1. p=0
2. p=1
3. p=−2

Answer

1. 3
2. $\frac{5}{2}$
3. $-\frac{1}{5}$

Example 13.9.1E.56

$$\frac{x+3}{2-3x}$$

1. x=0
2. x=1
3. x=−2

Example 13.9.1E.57

$$\frac{y^2+5y+6}{y^2-1}$$

1. y=0
2. y=2
3. y=−2

Answer

1. -6
2. $\frac{20}{3}$
3. 0

Example 13.9.1E.58

$$\frac{z^2+3z-10}{z^2-1}$$

1. z=0
2. z=2
3. z=−2

Example 13.9.1E.59

$$\frac{a^2-4}{a^2+5a+4}$$

1. a=0
2. a=1
3. a=−2

Answer

1. -1
2. $-\frac{3}{10}$
3. 0

Example 13.9.1E.60

$$\frac{b^2+2}{b^2-3b-4}$$

1. b=0
2. b=2
3. b=−2

Example 13.9.1E.61

$$\frac{x^2+3xy+2y^2}{2x^3y}$$

1. x=1, y=−1
2. x=2, y=1
3. x=−1, y=−2

Answer

1. 0
2. $\frac{3}{4}$
3. $\frac{15}{4}$

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Example 13.9.1E.62

$$\frac{c^2+cd-2d^2}{cd^3}$$

1. c=2, d=−1
2. c=1, d=−1
3. c=−1, d=2

Example 13.9.1E.63

$$\frac{m^2-4n^2}{5mn^3}$$

1. m=2, n=1
2. m=−1, n=−1
3. m=3, n=2

Answer

1. 0
2. $-\frac{3}{5}$
3. $-\frac{7}{20}$

Example 13.9.1E.64

$$\frac{2s^2t}{s^2-9t^2}$$

1. s=4, t=1
2. s=−1, t=−1
3. s=0, t=2

Simplify Rational Expressions

In the following exercises, simplify.

Example 13.9.1E.65

$$-\frac{4}{52}$$

Answer

$$-\frac{1}{13}$$

Example 13.9.1E.66

$$-\frac{44}{55}$$

Example 13.9.1E.67

$$\frac{56}{63}$$

Answer

$\frac{8}{9}$

Example 13.9.1E.68

$\frac{65}{104}$

Example 13.9.1E.69

$\frac{6ab^2}{12a^2b}$

Answer

$\frac{b}{2a}$

Example 13.9.1E.70

$\frac{15xy^3}{x^3y^3}$

Example 13.9.1E.71

$\frac{8m^3n}{12mn^2}$

Answer

$\frac{2m^2}{3n}$

Example 13.9.1E.72

$\frac{36v^3w^2}{27vw^3}$

Example 13.9.1E.73

$\frac{3a+6}{4a+8}$

Answer

$\frac{3}{4}$

Example 13.9.1E.74

$\frac{5b+5}{6b+6}$

Example 13.9.1E.75

$$\frac{3c-9}{5c-15}$$

Answer

$$\frac{3}{5}$$

Example 13.9.1E.76

$$\frac{4d+8}{9d+18}$$

Example 13.9.1E.77

$$\frac{7m+63}{5m+45}$$

Answer

$$\frac{7}{5}$$

Example 13.9.1E.78

$$\frac{8n-96}{3n-36}$$

Exercise 13.9.1E.79

$$\frac{12p-240}{5p-100}$$

Answer

$$\frac{12}{5}$$

Example 13.9.1E.80

$$\frac{6q+210}{5q+175}$$

Example 13.9.1E.81

$$\frac{a^2-a-12}{a^2-8a+16}$$

Answer

$$\frac{a+3}{a-4}$$

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Example 13.9.1E.82

$$\frac{x^2+4x-5}{x^2-2x+1}$$

Example 13.9.1E.83

$$\frac{y^2+3y-4}{y^2-6y+5}$$

Answer

$$\frac{y+4}{y-5}$$

Example 13.9.1E.84

$$\frac{v^2+8v+15}{v^2-v-12}$$

Example 13.9.1E.85

$$\frac{x^2-25}{x^2+2x-15}$$

Answer

$$\frac{x-5}{x-3}$$

Example 13.9.1E.86

$$\frac{a^2-4}{a^2+6a-16}$$

Example 13.9.1E.87

$$\frac{y^2-2y-3}{y^2-9}$$

Answer

$$\frac{y+1}{y+3}$$

Example 13.9.1E.88

$$\frac{b^2+9b+18}{b^2-36}$$

Example 13.9.1E.89

$$\frac{y^3+y^2+y+1}{y^2+2y+1}$$

Answer

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$$\frac{y^2+1}{y+1}$$

Example 13.9.1E.90

$$\frac{p^3+3p^2+4p+12}{p^2+p-6}$$

Example 13.9.1E.91

$$\frac{x^3-2x^2-25x+50}{x^2-25}$$

Answer

x−2

Example 13.9.1E.92

$$\frac{q^3+3q^2-4q-12}{q^2-4}$$

Example 13.9.1E.93

$$\frac{3a^2+15a}{6a^2+6a-36}$$

Answer

$$\frac{a(a+5)}{2(a+3)(a-2)}$$

Example 13.9.1E.94

$$\frac{8b^2-32b}{2b^2-6b-80}$$

Example 13.9.1E.95

$$\frac{-5c^2-10c}{-10c^2+30c+100}$$

Answer

$$\frac{c}{2(c-5)}$$

Example 13.9.1E.96

$$\frac{4d^2-24d}{2d^2-4d-48}$$

Example 13.9.1E.97

$$\frac{3m^2+30m+75}{4m^2-100}$$

Answer

$$\frac{3(m+5)}{4(m-5)}$$

Example 13.9.1E.98

$$\frac{5n^2+30n+45}{2n^2-18}$$

Example 13.9.1E.99

$$\frac{5r^2+30r-35}{r^2-49}$$

Answer

$$\frac{5(r-1)}{r+7}$$

Example 13.9.1E.100

$$\frac{3s^2+30s+72}{3s^2-48}$$

Example 13.9.1E.101

$$\frac{t^3-27}{t^2-9}$$

Answer

$$\frac{t^2+3t+9}{t+3}$$

Example 13.9.1E.102

$$\frac{v^3-1}{v^2-1}$$

Example 13.9.1E.103

$$\frac{w^3+216}{w^2-36}$$

Answer

$$\frac{w^2-6w+36}{w-6}$$

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Example 13.9.1E.104

$$\frac{v^3+125}{v^2-25}$$

Simplify Rational Expressions with Opposite Factors

In the following exercises, simplify each rational expression.

Example 13.9.1E.105

$$\frac{a-5}{5-a}$$

Answer

-1

Example 13.9.1E.106

$$\frac{b-12}{12-b}$$

Example 13.9.1E.107

$$\frac{11-c}{c-11}$$

Answer

-1

Example 13.9.1E.108

$$\frac{5-d}{d-5}$$

Example 13.9.1E.109

$$\frac{12-2x}{x^2-36}$$

Answer

$$-\frac{2}{x+6}$$

Example 13.9.1E.110

$$\frac{20-5y}{y^2-16}$$

Example 13.9.1E.111

$$\frac{4v-32}{64-v^2}$$

Answer

$$-\frac{4}{8+v}$$

Example 13.9.1E.112

$$\frac{7w-21}{9-w^2}$$

Example 13.9.1E.113

$$\frac{y^2-11y+24}{9-y^2}$$

Answer

$$-\frac{y-8}{3+y}$$

Example 13.9.1E.114

$$\frac{z^2-9z+20}{16-z^2}$$

Example 13.9.1E.115

$$\frac{a^2-5a-36}{81-a^2}$$

Answer

$$-\frac{a+4}{9+a}$$

Example 13.9.1E.116

$$\frac{b^2+b-42}{36-b^2}$$

Everyday Math

Example 13.9.1E.117

Tax Rates For the tax year 2015, the amount of tax owed by a single person earning between $37,450 and $90,750, can be found by evaluating the formula 0.25x−4206.25, where x is income. The average tax rate for this income can be found by evaluating the formula $\frac{0.25x-4206.25}{x}$. What would be the average tax rate for a single person earning $50,000?

Answer

16.5%

Example 13.9.1E.118

Work The length of time it takes for two people for perform the same task if they work together can be found by evaluating the formula $\frac{xy}{x+y}$. If Tom can paint the den in x=45 minutes and his brother Bobby can paint it in y=60 minutes, how many

minutes will it take them if they work together?

Writing Exercises

Example 13.9.1E.119

Explain how you find the values of x for which the rational expression $\frac{x^2-x-20}{x^2-4}$ is undefined.

Answer

Answers will vary, but all should reference setting the denominator function to zero.

Example 13.9.1E.120

Explain all the steps you take to simplify the rational expression $\frac{p^2+4p-21}{9-p^2}$.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
evaluate rational expressions.			
simplify rational expressions.			
simplify rational expressions with opposite factors.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

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13.9.2: Multiply and Divide Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Multiply rational expressions
- Divide rational expressions

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.
   If you missed this problem, review Exercise 1.6.13.
2. Divide: $\frac{14}{15} \div \frac{6}{35}$.
   If you missed this problem, review Exercise 1.6.22.
3. Factor completely: $2x^2 - 98$.
   If you missed this problem, review Exercise 7.5.10.
4. Factor completely: $10n^3 + 10$.
   If you missed this problem, review Exercise 7.5.19.
5. Factor completely: $10p^2 - 25pq - 15q^2$.
   If you missed this problem, review Exercise 7.5.28.

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Definition: MULTIPLICATION OF RATIONAL EXPRESSIONS

If p,q,r,s are polynomials where $q \neq 0$ and $s \neq 0$

$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$

To multiply rational expressions, multiply the numerators and multiply the denominators.

We'll do the first example with numerical fractions to remind us of how we multiplied fractions without variables.

Example 13.9.2.1

Multiply: $\frac{10}{28} \cdot \frac{8}{15}$.

Answer

	$\frac{10}{28} \cdot \frac{8}{15}$
Multiply the numerators and denominators.	$\frac{10 \cdot 8}{28 \cdot 15}$
Look for common factors, and then remove them.	$\frac{2 \cdot \cancel{5} \cdot 2 \cdot \cancel{4}}{7 \cdot \cancel{4} \cdot 3 \cdot \cancel{5}}$
Simplify.	$\frac{4}{21}$

Example 13.9.2.2

Mulitply: $\frac{6}{10} \cdot \frac{15}{12}$.

Answer

$\frac{3}{4}$

Example 13.9.2.3

Mulitply: $\frac{20}{15} \cdot \frac{6}{8}$.

Answer

1

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$ and $y \neq 0$.

Example 13.9.2.4

Mulitply: $\frac{2x}{3y^2} \cdot \frac{6xy^3}{x^2y}$.

Answer

	$\frac{2x}{3y^2} \cdot \frac{6xy^3}{x^2y}$
Multiply.	$\frac{2x \cdot 6xy^3}{3y^2 \cdot x^2y}$
Factor the numerator and denominator completely, and then remove common factors.	$\frac{2 \cdot x \cdot 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y}{3 \cdot y \cdot y \cdot x \cdot x \cdot y}$
Simplify.	4

Example 13.9.2.5

Mulitply: $\frac{3pq}{q^2} \cdot \frac{5p^2q}{6pq}$.

Answer

$\frac{5p^2}{2q}$

Example 13.9.2.6

Mulitply: $\frac{6x^3y}{7x^2} \cdot \frac{2xy}{3x^2y}$.

Answer

$\frac{12y^3}{7}$

How to Multiply Rational Expressions

Example 13.9.2.7

Mulitply: $\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$.

Answer

Step 1. Factor the numerator and denominator completely.	Factor $x^2 - 9$ and $x^2 + x + 12$.	$\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$ $\frac{2x}{(x-3)(x-4)} \cdot \frac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\frac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\frac{\cancel{2}x\cancel{(x-3)}(x+3)}{\cancel{2}\cdot 3\cdot x\cdot \cancel{(x-3)}(x-4)}$ $\frac{(x+3)}{3x(x-4)}$

Example 13.9.2.8

Mulitply: $\frac{5x}{x^2+5x+6} \cdot \frac{x^2-4}{10x}$.

Answer

$\frac{x-2}{2(x+3)}$

Example 13.9.2.9

Mulitply: $\frac{9x^2}{x^2+11x+30} \cdot \frac{x^2-36}{3x^2}$.

Answer

$\frac{3(x-6)}{x+5}$

Definition: MULTIPLY A RATIONAL EXPRESSION.

1. Factor each numerator and denominator completely.
2. Multiply the numerators and denominators.
3. Simplify by dividing out common factors.

Example 13.9.2.10

Multiply: $\frac{n^2-7n}{n^2+2n+1} \cdot \frac{n+1}{2n}$.

Answer

	$\frac{n^2-7n}{n^2+2n+1} \cdot \frac{n+1}{2n}$
Factor each numerator and denominator.	$\frac{n(n-7)}{(n+1)(n+1)} \cdot \frac{n+1}{2n}$
Multiply the numerators and the denominators.	$\frac{n(n-7)(n+1)}{(n+1)(n+1)2n}$
Simplify.	$\frac{n-7}{2(n+1)}$

Example 13.9.2.11

Multiply: $\frac{x^2-25}{x^2-3x-10} \cdot \frac{x+2}{x}$.

Answer

$\frac{x+5}{x}$

Example 13.9.2.12

Multiply: $\frac{x^2-4x}{x^2+5x+6} \cdot \frac{x+2}{x}$.

Answer

$\frac{x-4}{x+3}$

Example 13.9.2.13

Multiply: $\frac{16-4x}{2x-12} \cdot \frac{x^2-5x-6}{x^2-16}$.

Answer

	$\frac{16-4x}{2x-12} \cdot \frac{x^2-5x-6}{x^2-16}$
Factor each numerator and denominator.	$\frac{4(4-x)}{2(x-6)} \cdot \frac{(x-6)(x+1)}{(x-4)(x+4)}$
Multiply the numerators and the denominators.	$\frac{4(4-x)(x-6)(x+1)}{2(x-6)(x-4)(x+4)}$
Simplify.	$-\frac{2(x+1)}{(x+4)}$

Example 13.9.2.14

Multiply: $\frac{12x-6x^2}{x^2+8x} \cdot \frac{x^2+11x+24}{x^2-4}$.

Answer

$-\frac{6(x+3)}{x+2}$

Example 13.9.2.15

Multiply: $\frac{9v-3v^2}{9v+36} \cdot \frac{v^2+7v+12}{v^2-9}$.

Answer

$-\frac{v}{3}$

Example 13.9.2.16

Multiply: $\frac{2x-6}{x^2-8x+15} \cdot \frac{x^2-25}{2x+10}$.

Answer

	$\frac{2x-6}{x^2-8x+15} \cdot \frac{x^2-25}{2x+10}$
Factor each numerator and denominator.	$\frac{2(x-3)}{(x-3)(x-5)} \cdot \frac{(x-5)(x+5)}{2(x+5)}$
Multiply the numerators and denominators.	$\frac{2(x-3)(x-5)(x+5)}{2(x-3)(x-5)(x+5)}$
Remove common factors.	$\frac{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}$
Simplify.	1

Example 13.9.2.17

Multiply: $\frac{3a-21}{a^2-9a+14} \cdot \frac{a^2-4}{3a+6}$.

Answer

1

Example 13.9.2.18

Multiply: $\frac{b^2-b}{b^2+9b-10} \cdot \frac{b^2-100}{b^2-10b}$.

Answer

1

Divide Rational Expressions

To divide rational expressions we multiply the first fraction by the reciprocal of the second, just like we did for numerical fractions.

Remember, the **reciprocal** of $\frac{a}{b}$ is $\frac{b}{a}$. To find the reciprocal we simply put the numerator in the denominator and the denominator in the numerator. We "flip" the fraction.

Definition: DIVISION OF RATIONAL EXPRESSIONS

If p,q,r,s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$

$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$

To divide rational expressions multiply the first fraction by the reciprocal of the second.

How to Divide Rational Expressions

Example 13.9.2.19

Divide: $\frac{x+9}{6-x} \div \frac{x^2-81}{x-6}$.

Answer

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{x+9}{6-x} \div \frac{x^2-81}{x-6}$ $\frac{x+9}{6-x} \cdot \frac{x-6}{x^2-81}$
Step 2. Factor the numerators and denominators completely.	Factor $x^2 - 81$.	$\frac{x+9}{6-x} \cdot \frac{x-6}{(x-9)(x+9)}$
Step 3. Multiply the numerators and denominators.		$\frac{(x+9)(x-6)}{(6-x)(x-9)(x+9)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors. Remember opposites divide to –1.	$(-1)\frac{\cancel{(x+9)}\cancel{(x-6)}}{\cancel{(6-x)}(x-9)\cancel{(x+9)}}$ $-\frac{1}{(x-9)}$

Example 13.9.2.20

Divide: $\frac{c+3}{5-c} \div \frac{c^2-9}{c-5}$.

Answer

$-\frac{1}{c-3}$

Example 13.9.2.21

Divide: $\frac{2-d}{d-4} \div \frac{4-d^2}{4-d}$.

Answer

$-\frac{1}{2+d}$

Definition: DIVIDE RATIONAL EXPRESSIONS.

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

Example 13.9.2.22

Divide: $\frac{3n^2}{n^2-4n} \div \frac{9n^2-45n}{n^2-7n+10}$.

Answer

	$\frac{3n^2}{n^2-4n} \div \frac{9n^2-45n}{n^2-7n+10}$
Rewrite the division as the product of the first rational expression and the reciprocal of the second.	$\frac{3n^2}{n^2-4n} \cdot \frac{n^2-7n+10}{9n^2-45n}$
Factor the numerators and denominators and then multiply.	$\frac{3 \cdot n \cdot n \cdot (n-5)(n-2)}{n(n-4) \cdot 3 \cdot 3 \cdot n \cdot (n-5)}$
Simplify by dividing out common factors.	$\frac{\cancel{3} \cdot \cancel{n} \cdot \cancel{n(n-5)}(n-2)}{\cancel{n}(n-4)\cancel{3} \cdot 3 \cdot \cancel{n(n-5)}}$

	$\frac{n-2}{3(n-4)}$

Example 13.9.2.23

Divide: $\frac{2m^2}{m^2-8m} \div \frac{8m^2+24m}{m^2+m-6}$.

Answer

$\frac{(m-2)}{4(m-8)}$

Example 13.9.2.24

Divide: $\frac{15n^2}{3n^2+33n} \div \frac{5n-5}{n^2+9n-22}$.

Answer

$\frac{n(n-2)}{n-1}$

Remember, first rewrite the division as multiplication of the first expression by the reciprocal of the second. Then factor everything and look for common factors.

Example 13.9.2.25

Divide: $\frac{2x^2+5x-12}{x^2-16} \div \frac{2x^2-13x+15}{x^2-8x+16}$.

Answer

	$\frac{2x^2+5x-12}{x^2-16} \div \frac{2x^2-13x+15}{x^2-8x+16}$
Rewrite the division as the product of the first rational expression and the reciprocal of the second.	$\frac{2x^2+5x-12}{x^2-16} \cdot \frac{x^2-8x+16}{2x^2-13x+15}$
Factor the numerators and denominators and then multiply.	$\frac{(2x-3)(x+4)(x-4)(x-4)}{(x-4)(x+4)(2x-3)(x-5)}$
Simplify.	$\frac{(x-4)}{(x-5)}$

Example 13.9.2.26

Divide: $\frac{3a^2-8a-3}{a^2-25} \div \frac{3a^2-14a-5}{a^2+10a+25}$.

Answer

$\frac{(a-3)(a+5)}{(a-5)(a-5)}$

Exercise 13.9.2.27

Divide: $\frac{4b^2+7b-2}{1-b^2} \div \frac{4b^2+15b-4}{b^2-2b+1}$.

Answer

$-\frac{(b+2)(b-1)}{(1+b)(b+4)}$

Example 13.9.2.28

Divide: $\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$.

Answer

	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$
Rewrite the division as the product of the first rational expression and the reciprocal of the second.	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$
Factor the numerators and denominators and then multiply.	$\frac{(p+q)(p^2-pq+q^2)6}{2(p^2+pq+q^2)(p-q)(p+q)}$
Simplify.	$\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Example 13.9.2.29

Divide: $\frac{x^3-8}{3x^2-6x+12} \div \frac{x^2-4}{6}$.

Answer

$$\frac{2(x^2+2x+4)}{(x+2)(x^2-2x+4)}$$

Example 13.9.2.30

Divide: $\frac{2z^2}{z^2-1} \div \frac{z^3-z^2+z}{z^3-1}$.

Answer

$$\frac{2z(z^2+z+1)}{(z+1)(z^2-z+1)}$$

Before doing the next example, let's look at how we divide a fraction by a whole number. When we divide $\frac{3}{5} \div 4$

$$\frac{3}{5} \div 4$$
$$\frac{3}{5} \div \frac{4}{1}$$
$$\frac{3}{5} \cdot \frac{1}{4}$$

We do the same thing when we divide rational expressions.

Example 13.9.2.31

$\frac{a^2-b^2}{3ab} \div (a^2+2ab+b^2)$.

Answer

	$\frac{a^2-b^2}{3ab} \div (a^2+2ab+b^2)$
Write the second expression as a fraction.	$\frac{a^2-b^2}{3ab} \div \frac{a^2+2ab+b^2}{1}$
Rewrite the division as the first expression times the reciprocal of the second expression.	$\frac{a^2-b^2}{3ab} \cdot \frac{1}{a^2+2ab+b^2}$
Factor the numerators and the denominators, and then multiply.	$\frac{(a-b)(a+b)1}{3ab \cdot (a+b)(a+b)}$

Simplify.	$\frac{a-b}{3ab(a+b)}$

Example 13.9.2.32

$\frac{2x^2-14x-16}{4} \div (x2+2x+1)$.

Answer

$\frac{x-8}{2(x+1)}$

Example 13.9.2.33

$\frac{y^2-6y+8}{y^2-4y} \div (3y2-12y)$.

Answer

$\frac{y-2}{3y(y-4)}$

Example 13.9.2.34

$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$.

Answer

	$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$
Rewrite with a division sign.	$\frac{6x^2-7x+2}{4x-8} \div \frac{2x^2-7x+3}{x^2-5x+6}$
Rewrite as product of first times reciprocal of second.	$\frac{6x^2-7x+2}{4x-8} \cdot \frac{x^2-5x+6}{2x^2-7x+3}$
Factor the numerators and the denominators, and then multiply	$\frac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$
Simplify.	$\frac{3x-2}{4}$

Example 13.9.2.35

$\frac{\frac{3x^2+7x+2}{4x+24}}{\frac{3x^2-14x-5}{x^2+x-30}}$.

Answer

$\frac{x+2}{4}$

Example 13.9.2.36

$\frac{\frac{y^2-36}{2y^2+11y-6}}{\frac{2y^2-2y-60}{8y-4}}$.

Answer

$\frac{2}{y+5}$

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then we factor and multiply.

Example 13.9.2.37

$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$.

Answer

	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \cdot \frac{8x+16}{2x+12}$
Factor the numerators and the denominators, and then multiply.	$\frac{3 \cdot 8(x-2)(x+3)(x-1)(x+2)}{4 \cdot 2(x-1)(x+2)(x-5)(x+6)}$
Simplify by dividing out common factors.	$\frac{3 \cdot \cancel{8}(x-2)(x+3)\cancel{(x-1)(x+2)}}{\cancel{4} \cdot \cancel{2}\cancel{(x-1)(x+2)}(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

Example 13.9.2.38

$\frac{4m+4}{3m-15} \cdot \frac{m^2-3m-10}{m^2-4m-32} \div \frac{12m-36}{6m-48}$.

Answer

$\frac{2(m+1)(m+2)}{3(m+4)(m-3)}$

Example 13.9.2.39

$\frac{2n^2+10n}{n-1} \div \frac{n^2+10n+24}{n^2+8n-9} \cdot \frac{n+4}{8n^2+12n}$.

Answer

$\frac{(n+5)(n+9)}{2(n+6)(2n+3)}$

Key Concepts

- **Multiplication of Rational Expressions**
  - If p,q,r,s are polynomials where $q \neq 0$ and $s \neq 0$, then $\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$
  - To multiply rational expressions, multiply the numerators and multiply the denominators
- **Multiply a Rational Expression**
  1. Factor each numerator and denominator completely.
  2. Multiply the numerators and denominators.
  3. Simplify by dividing out common factors.
- **Division of Rational Expressions**
  - If p,q,r,s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then $\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$
  - To divide rational expressions multiply the first fraction by the reciprocal of the second.
- **Divide Rational Expressions**

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

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13.9.2E: Exersices

Practice Makes Perfect

Multiply Rational Expressions

In the following exercises, multiply.

Example 13.9.2E.40

$\frac{12}{16} \cdot \frac{4}{10}$

Answer

$\frac{3}{10}$

Example 13.9.2E.41

$\frac{32}{5} \cdot \frac{16}{24}$

Example 13.9.2E.42

$\frac{18}{10} \cdot \frac{4}{30}$

Answer

$\frac{6}{25}$

Example 13.9.2E.43

$\frac{21}{36} \cdot \frac{45}{24}$

Example 13.9.2E.44

$\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2}$

Answer

$\frac{x^3}{8y}$

Example 13.9.2E.45

$\frac{8w^3y^9}{y^2} \cdot \frac{3y}{4w^4}$

Example 13.9.2E.46

$\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3}$

Answer

$\frac{8a^4}{3b^2}$

Example 13.9.2E.47

$\frac{4mn^2}{5n^3} \cdot \frac{mn^3}{8m^2}$

Example 13.9.2E.48

$\frac{5p^2}{p^2-5p-36} \cdot \frac{p^2-16}{10p}$

Answer

$\frac{p(p-4)}{2(p-9)}$

Example 13.9.2E.49

$\frac{3q^2}{q^2+q-6} \cdot \frac{q^2-9}{9q}$

Example 13.9.2E.50

$\frac{4r}{r^2-3r-10} \cdot \frac{r^2-25}{8r^2}$

Answer

$\frac{r+5}{2r(r+2)}$

Example 13.9.2E.51

$\frac{s}{s^2-9s+14} \cdot \frac{s^2-49}{7s^2}$

Example 13.9.2E.52

$\frac{x^2-7x}{x^2+6x+9} \cdot \frac{x+3}{4x}$

Answer

$\frac{x-7}{4(x+3)}$

Example 13.9.2E.53

$\frac{2y^2-10y}{y^2+10y+25} \cdot \frac{y+5}{6y}$

Example 13.9.2E.54

$\frac{z^2+3z}{z^2-3z-4} \cdot \frac{z-4}{z^2}$

Answer

$\frac{z+3}{z(z+1)}$

Example 13.9.2E.55

$\frac{2a^2+8a}{a^2-9a+20} \cdot \frac{a-5}{a^2}$

Example 13.9.2E.56

$\frac{28-4b}{3b-3} \cdot \frac{b^2+8b-9}{b^2-49}$

Answer

$-\frac{4(b+9)}{3(b+7)}$

Example 13.9.2E.57

$\frac{18c-2c^2}{6c+30} \cdot \frac{c^2+7c+10}{c^2-81}$

Example 13.9.2E.58

$\frac{35d-7d^2}{d^2+7d} \cdot \frac{d^2+12d+35}{d^2-25}$

Answer

-7

Example 13.9.2E.59

$\frac{72m-12m^2}{8m+32} \cdot \frac{m^2+10m+24}{m^2-36}$

Example 13.9.2E.60

$\frac{4n+20}{n^2+n-20} \cdot \frac{n^2-16}{4n+16}$

Answer

1

Example 13.9.2E.61

$\frac{6p^2-6p}{p^2+7p-18} \cdot \frac{p^2-81}{3p^2-27p}$

Example 13.9.2E.62

$\frac{q^2-2q}{q^2+6q-16} \cdot \frac{q^2-64}{q^2-8q}$

Answer

1

Example 13.9.2E.63

$\frac{2r^2-2r}{r^2+4r-5} \cdot \frac{r^2-25}{2r^2-10r}$

Divide Rational Expressions

In the following exercises, divide.

Example 13.9.2E.64

$\frac{t-6}{3-t} \div \frac{t^2-9}{t-5}$

Answer

$-\frac{2t}{t^3-5t-9}$

Example 13.9.2E.65

$\frac{v-5}{11-v} \div \frac{v^2-25}{v-11}$

Example 13.9.2E.66

$\frac{10+w}{w-8} \div \frac{100-w^2}{8-w}$

Answer

$-\frac{1}{10-w}$

Example 13.9.2E.67

$\frac{7+x}{x-6} \div \frac{49-x^2}{x+6}$

Example 13.9.2E.68

$\frac{27y^2}{3y-21} \div \frac{3y^2+18}{y^2+13y+42}$

Answer

$\frac{3y^2(y+6)(y+7)}{(y-7)(y2+6)}$

Example 13.9.2E.69

$\frac{24z^2}{2z-8} \div \frac{4z-28}{z^2-11z+28}$

Example 13.9.2E.70

$\frac{16a^2}{4a+36} \div \frac{4a^2-24a}{a^2+4a-45}$

Answer

$\frac{a(a-5)}{a-6}$

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Example 13.9.2E.71

$\frac{24b^2}{2b-4} \div \frac{12b^2+36b}{b^2-11b+18}$

Example 13.9.2E.72

$\frac{3c^2-16c+5}{c^2-25} \div \frac{3c^2-14c-5}{c^2+10c+25}$

Answer

$\frac{(3c-1)(c+5)}{(3c+1)(c-5)}$

Example 13.9.2E.73

$\frac{2d^2+d-3}{d^2-16} \div \frac{2d^2-9d-18}{d^2-8d+16}$

Example 13.9.2E.74

$\frac{6m^2-13m+2}{9-m^2} \div \frac{6m^2+23m-4}{m^2-6m+9}$

Answer

$-\frac{(m-2)(m-3)}{(3+m)(m+4)}$

Example 13.9.2E.75

$\frac{2n^2-3n-14}{25-n^2} \div \frac{2n^2-13n+21}{n^2-10n+25}$

Example 13.9.2E.76

$\frac{3s^2}{s^2-16} \div \frac{s^3+4s^2+16s}{s^3-64}$

Answer

$\frac{3s}{s+4}$

Example 13.9.2E.77

$\frac{r^2-9}{15} \div \frac{r^3-27}{5r^2+15r+45}$

Example 13.9.2E.78

$\frac{p^3+q^3}{3p^2+3pq+3q^2} \div \frac{p^2-q^2}{12}$

Answer

$\frac{4(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Example 13.9.2E.79

$\frac{v^3-8w^3}{2v^2+4vw+8w^2} \div \frac{v^2-4w^2}{4}$

Example 13.9.2E.80

$\frac{t^2-9}{2t} \div (t^2-6t+9)$

Answer

$\frac{t+3}{2t(t-3)}$

Example 13.9.2E.81

$\frac{x^2+3x-10}{4x} \div (2x^2+20x+50)$

Example 13.9.2E.82

$\frac{2y^2-10yz-48z^2}{2y-1} \div (4y^2-32yz)$

Answer

$\frac{y+3z}{2y(2y-1)}$

Example 13.9.2E.83

$\frac{2m^2-98n^2}{2m+6} \div (m^2-7mn)$

Example 13.9.2E.84

$\frac{\frac{2a^2-a-21}{5a+20}}{\frac{a^2+7a+12}{a^2+8a+16}}$

Answer

$\frac{2a-7}{5}$

Example 13.9.2E.85

$\frac{\frac{3b^2+2b-8}{12b+18}}{\frac{3b^2+2b-8}{2b^2-7b-15}}$

Example 13.9.2E.86

$\frac{\frac{12c^2-12}{2c^2-3c+14}}{\frac{c+4}{6c^2-13c+5}}$

Answer

3(3c−5)

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Example 13.9.2E.87

$$\frac{\frac{4d^2+7d-2}{35d+10}}{\frac{d^2-4}{7d^2-12d-4}}$$

Example 13.9.2E.88

$$\frac{10m^2+80m}{3m-9} \cdot \frac{m^2+4m-21}{m^2-9m+20} \div \frac{5m^2+10m}{2m-10}$$

Answer

$$\frac{4(m+8)(m+7)}{3(m-4)(m+2)}$$

Example 13.9.2E.89

$$\frac{4n^2+32n}{3n+2} \cdot \frac{3n^2-n-2}{n^2+n-30} \div \frac{108n^2-24n}{n+6}$$

Example 13.9.2E.90

$$\frac{12p^2+3p}{p+3} \div \frac{p^2+2p-63}{p^2-p-12} \cdot \frac{p-7}{9p^3-9p^2}$$

Answer

$$\frac{(4p+1)(p-7)}{3p(p+9)(p-1)}$$

Example 13.9.2E.91

$$\frac{6q+3}{9q^2-9q} \div \frac{q^2+14q+33}{q^2+4q-5} \cdot \frac{4q^2+12q}{12q+6}$$

Everyday Math

Example 13.9.2E.92

Probability The director of large company is interviewing applicants for two identical jobs. If w= the number of women applicants and m= the number of men applicants, then the probability that two women are selected for the jobs is $\frac{w}{w+m} \cdot \frac{w-1}{w+m-1}$.

1. Simplify the probability by multiplying the two rational expressions.
2. Find the probability that two women are selected when w=5 and m=10.

Answer

1. $\frac{w(w-1)}{(w+m)(w+m-1)}$
2. $\frac{2}{21}$

Example 13.9.2E.93

Area of a triangle The area of a triangle with base b and height h is $\frac{bh}{2}$. If the triangle is stretched to make a new triangle with base and height three times as much as in the original triangle, the area is $\frac{9bh}{2}$. Calculate how the area of the new triangle compares to the area of the original triangle by dividing $\frac{9bh}{2}$ by $\frac{bh}{2}$.

Writing Exercises

Example 13.9.2E.94

1. Multiply $\frac{7}{4} \cdot \frac{9}{10}$ and explain all your steps.
2. Multiply $\frac{n}{n-3} \cdot \frac{9n+3}{n}$ and explain all your steps.
3. Evaluate your answer to part (b) when n=7. Did you get the same answer you got in part (a)? Why or why not?

Answer

Answers will vary.

Example 13.9.2E.95

1. Divide $\frac{24}{5} \div 6$ and explain all your steps.
2. Divide $\frac{x^2-1}{x} \div (x+1)$ and explain all your steps.
3. Evaluate your answer to part (b) when x=5. Did you get the same answer you got in part (a)? Why or why not?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
multiply rational expressions.			
divide rational expressions.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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13.9.3: Add and Subtract Rational Expressions with a Common Denominator

Learning Objectives

By the end of this section, you will be able to:

- Add rational expressions with a common denominator
- Subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Add: $\frac{y}{3}+\frac{9}{3}$.
   If you missed this problem, review Exercise 1.7.1.
2. Subtract: $\frac{10}{x}-\frac{2}{x}$.
   If you missed this problem, review Exercise 1.7.7.
3. Factor completely: $8n^5-20n^3$.
   If you missed this problem, review Exercise 7.5.1.
4. Factor completely: $45a^3-5ab^2$.
   If you missed this problem, review Exercise 7.5.10.

Add Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

Definition: RATIONAL EXPRESSION ADDITION

If p, q, and r are polynomials where $r\neq 0$, then

$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r}$

To add rational expressions with a common denominator, add the numerators and place the sum over the common denominator.

We will add two numerical fractions first, to remind us of how this is done.

Example 13.9.3.1

Add: $\frac{5}{18}+\frac{7}{18}$.

Answer

	$\frac{5}{18}+\frac{7}{18}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{5+7}{18}$
Add in the numerator.	$\frac{12}{18}$
Factor the numerator and denominator to show the common factors.	$\frac{6\cdot 2}{6\cdot 3}$

Simplify.	$\frac{2}{3}$

Example 13.9.3.2

Add: $\frac{7}{16}+\frac{5}{16}$.

Answer

$\frac{3}{4}$

Example 13.9.3.3

Add: $\frac{3}{10}+\frac{1}{10}$.

Answer

$\frac{2}{5}$

Remember, we do not allow values that would make the denominator zero. What value of yy should be excluded in the next example?

Example 13.9.3.4

Add: $\frac{3y}{4y-3}+\frac{7}{4y-3}$.

Answer

	$\frac{3y}{4y-3}+\frac{7}{4y-3}$.
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{3y+7}{4y-3}$
The numerator and denominator cannot be factored. The fraction is simplified.	

Example 13.9.3.5

Add: $\frac{5x}{2x+3}+\frac{2}{2x+3}$.

Answer

$\frac{5x+2}{2x+3}$.

Example 13.9.3.6

Add: $\frac{x}{x-2}+\frac{1}{x-2}$.

Answer

$\frac{x+1}{x-2}$

Example 13.9.3.7

Add: $\frac{7x+12}{x+3}+\frac{x^2}{x+3}$.

Answer

	$\frac{7x+12}{x+3}+\frac{x^2}{x+3}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{7x+12+x^2}{x+3}$
Write the degrees in descending order.	$\frac{x^2+7x+12}{x+3}$
Factor the numerator.	$\frac{(x+3)(x+4)}{x+3}$
Simplify.	x+4

Example 13.9.3.8

Add: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

Answer

x+2

Example 13.9.3.9

Add: $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Answer

x+3

Subtract Rational Expressions with a Common Denominator

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator.

Definition: RATIONAL EXPRESSION SUBTRACTION

If p, q, and r are polynomials where $r \neq 0$

$\frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$

To subtract rational expressions, subtract the numerators and place the difference over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Example 13.9.3.10

Subtract: $\frac{n^2}{n-10}-\frac{100}{n-10}$.

Answer

	$\frac{n^2}{n-10}-\frac{100}{n-10}$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{n^2-100}{n-10}$
Factor the numerator.	$\frac{(n-10)(n+10)}{n-10}$
Simplify.	n+10

Example 13.9.3.11

Subtract: $\frac{x^2}{x+3}-\frac{9}{x+3}$.

Answer

x−3

Example 13.9.3.12

Subtract: $\frac{4x^2}{2x-5}-\frac{25}{2x-5}$.

Answer

2x+5

Be careful of the signs when you subtract a binomial!

Example 13.9.3.13

Subtract: $\frac{y^2}{y-6}-\frac{2y+24}{y-6}$.

Answer

	$\frac{y^2}{y-6}-\frac{2y+24}{y-6}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{y^2-(2y+24)}{y-6}$
Distribute the sign in the numerator.	$\frac{y^2-2y-24}{y-6}$
Factor the numerator.	$\frac{(y-6)(y+4)}{y-6}$
Simplify.	y+4

Example 13.9.3.14

Subtract: $\frac{n^2}{n-4}-\frac{n+12}{n-4}$.

Answer

n+3

Example 13.9.3.15

Subtract: $\frac{y^2}{y-1}-\frac{9y-8}{y-1}$.

Answer

y−8

Example 13.9.3.16

Subtract: $\frac{5x^2-7x+3}{x^2-3x-18} - \frac{4x^2+x-9}{x^2-3x-18}$.

Answer

	$\frac{5x^2-7x+3}{x^2-3x+18} - \frac{4x^2+x-9}{x^2-3x+18}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x+18}$
Combine like terms.	$\frac{x^2-8x+12}{x^2-3x+18}$
Factor the numerator and the denominator.	$\frac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify.	$\frac{x-2}{x+3}$

Example 13.9.3.17

Subtract: $\frac{4x^2-11x+8}{x^2-3x+2} - \frac{3x^2+x-3}{x^2-3x+2}$.

Answer

$\frac{x-11}{x-2}$

Example 13.9.3.18

Subtract: $\frac{6x^2-x+20}{x^2-81} - \frac{5x^2+11x-7}{x^2-81}$.

Answer

$\frac{x-3}{x+9}$

Add and Subtract Rational Expressions whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$

Let's see how this works.

	$\frac{7}{d} + \frac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{7}{d} + \frac{(-1)5}{(-1)(-d)}$
The denominators are the same.	$\frac{7}{d} + \frac{-5}{d}$
Simplify.	$\frac{2}{d}$

Example 13.9.3.19

Add: $\frac{4u-1}{3u-1}+\frac{u}{1-3u}$.

Answer

	$\frac{4u-1}{3u-1}+\frac{u}{1-3u}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{4u-1}{3u-1}+\frac{(-1)u}{(-1)(1-3u)}$
Simplify the second fraction.	$\frac{4u-1}{3u-1}+\frac{-u}{3u-1}$
The denominators are the same. Add the numerators.	$\frac{4u-1-u}{3u-1}$
Simplify.	$\frac{3u-1}{3u-1}$
Simplify.	1

Example 13.9.3.20

Add: $\frac{8x-15}{2x-5}+\frac{2x}{5-2x}$.

Answer

3

Example 13.9.3.21

Add: $\frac{6y^2+7y-10}{4y-7}+\frac{2y^2+2y+11}{7-4y}$.

Answer

y+3

Example 13.9.3.22

Subtract: $\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$.

Answer

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	m2−6m+3m+2m2−1
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify by removing common factors.	$\frac{\cancel{(m-1)}(m-2)}{\cancel{(m-1)}(m+1)}$

Simplify.	$\frac{m-2}{m+1}$

Example 13.9.3.23

Subtract: $\frac{y^2-5y}{y^2-4} - \frac{6y-6}{4-y^2}$.

Answer

$\frac{y+3}{y+2}$

Example 13.9.3.24

Subtract: $\frac{2n^2+8n-1}{n^2-1} - \frac{n^2-7n-1}{1-n^2}$.

Answer

$\frac{3n-2}{n-1}$

Key Concepts

- **Rational Expression Addition**
  - If p, q, and r are polynomials where $r \neq 0$, then

    $\frac{p}{r} + \frac{q}{r} = \frac{p+q}{r}$
  - To add rational expressions with a common denominator, add the numerators and place the sum over the common denominator.
- **Rational Expression Subtraction**
  - If p, q, and r are polynomials where $r \neq 0$

    $\frac{p}{r} - \frac{q}{r} = \frac{p-q}{r}$
  - To subtract rational expressions, subtract the numerators and place the difference over the common denominator.

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13.9.3E: Exersices

Practice Makes Perfect

Add Rational Expressions with a Common Denominator

In the following exercises, add.

Example 13.9.3E.25

$\frac{2}{15}+\frac{7}{15}$

Answer

$\frac{3}{5}$

Example 13.9.3E.26

$\frac{4}{21}+\frac{3}{21}$

Example 13.9.3E.27

$\frac{7}{24}+\frac{11}{24}$

Answer

$\frac{3}{4}$

Example 13.9.3E.28

$\frac{7}{36}+\frac{13}{36}$

Example 13.9.3E.29

$\frac{3a}{a-b}+\frac{1}{a-b}$

Answer

$\frac{3a+1}{a-b}$

Example 13.9.3E.30

$\frac{3c}{4c-5}+\frac{5}{4c-5}$

Example 13.9.3E.31

$\frac{d}{d+8}+\frac{5}{d+8}$

Answer

$\frac{d+5}{d+8}$

Example 13.9.3E.32

$\frac{7m}{2m+n}+\frac{4}{2m+n}$

Example 13.9.3E.33

$\frac{p^2+10p}{p+2}+\frac{16}{p+2}$

Answer

p+8

Example 13.9.3E.34

$\frac{q^2+12q}{q+3}+\frac{27}{q+3}$

Example 13.9.3E.35

$\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

Answer

r+8

Example 13.9.3E.36

$\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

Example 13.9.3E.37

$\frac{8t^2}{t+4}+\frac{32t}{t+4}$

Answer

8t

Example 13.9.3E.38

$\frac{6v^2}{v+5}+\frac{30v}{v+5}$

Example 13.9.3E.39

$\frac{2w^2}{w^2-16}+\frac{8w}{w^2-16}$

Answer

$\frac{2w}{w-4}$

Example 13.9.3E.40

$\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

Subtract Rational Expressions with a Common Denominator

In the following exercises, subtract.

Example 13.9.3E.41

$\frac{y^2}{y+8} - \frac{64}{y+8}$

Answer

y−8

Example 13.9.3E.42

$\frac{z^2}{z+2} - \frac{4}{z+2}$

Example 13.9.3E.43

$\frac{9a^2}{3a-7} - \frac{49}{3a-7}$

Answer

3a+7

Example 13.9.3E.44

$\frac{25b^2}{5b-6} - \frac{36}{5b-6}$

Example 13.9.3E.45

$\frac{c^2}{c-8} - \frac{6c+16}{c-8}$

Answer

c+2

Example 13.9.3E.46

$\frac{d^2}{d-9} - \frac{6d+27}{d-9}$

Example 13.9.3E.47

$\frac{3m^2}{6m-30} - \frac{21m-30}{6m-30}$

Answer

$\frac{m-2}{3}$

Example 13.9.3E.48

$\frac{2n^2}{4n-32} - \frac{30n-16}{4n-32}$

Example 13.9.3E.49

$\frac{6p^2+3p+4}{p^2+4p-5} - \frac{5p^2+p+7}{p^2+4p-5}$

Answer

$\frac{p+3}{p+5}$

Example 13.9.3E.50

$\frac{5q^2+3q-9}{q^2+6q+8} - \frac{4q^2+9q+7}{q^2+6q+8}$

Example 13.9.3E.51

$\frac{5r^2+7r-33}{r^2-49} - \frac{4r^2-5r-30}{r^2-49}$

Answer

$\frac{r+9}{r+7}$

Example 13.9.3E.52

$\frac{7t^2-t-4}{t^2-25} - \frac{6t^2+2t-1}{t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add.

Example 13.9.3E.53

$\frac{10v^2}{v-1} + \frac{2v+4}{1-2v}$

Answer

4

Example 13.9.3E.54

$\frac{20w}{5w-2} + \frac{5w+6}{2-5w}$

Example 13.9.3E.55

$\frac{10x^2+16x-7}{8x-3} + \frac{2x^2+3x-1}{3-8x}$

Answer

x+2

Example 13.9.3E.56

$\frac{6y^2+2y-11}{3y-7} + \frac{3y^2-3y+17}{7-3y}$

In the following exercises, subtract.

Example 13.9.3E.57

$\frac{z^2+6z}{z^2-25} - \frac{3z+20}{25-z^2}$

Answer

$\frac{z+4}{z-5}$

Example 13.9.3E.58

$\frac{a^2+3a}{a^2-9} - \frac{3a-27}{9-a^2}$

Example 13.9.3E.59

$\frac{2b^2+30b-13}{b^2-49} - \frac{2b^2-5b-8}{49-b^2}$

Answer

$\frac{4b-3}{b-7}$

Example 13.9.3E.60

$\frac{c^2+5c-10}{c^2-16} - \frac{c^2-8c-10}{16-c^2}$

Everyday Math

Example 13.9.3E.61

Sarah ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. If rr represents Sarah's speed when she ran, then her running time is modeled by the expression $\frac{8}{r}$ and her biking time is modeled by the expression $\frac{24}{r+4}$. Add the rational expressions $\frac{8}{r} + \frac{24}{r+4}$ to get an expression for the total amount of time Sarah ran and biked.

Answer

$\frac{32r+32}{r(r+4)}$

Example 13.9.3E.62

If Pete can paint a wall in pp hours, then in one hour he can paint $\frac{1}{p}$ of the wall. It would take Penelope 3 hours longer than Pete to paint the wall, so in one hour she can paint $\frac{1}{p+3}$ of the wall. Add the rational expressions $\frac{1}{p} + \frac{1}{p+3}$ to get an expression for the part of the wall Pete and Penelope would paint in one hour if they worked together.

Writing Exercises

Example 13.9.3E.63

Donald thinks that $\frac{3}{x} + \frac{4}{x}$ is $\frac{7}{2x}$. Is Donald correct? Explain.

Example 13.9.3E.64

Explain how you find the Least Common Denominator of x^2+5x+4 and x^2-16.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add rational expressions with a common denominator.			
subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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13.9.4: Add and Subtract Rational Expressions with Unlike Denominators

Learning Objectives

By the end of this section, you will be able to:

- Find the least common denominator of rational expressions
- Find equivalent rational expressions
- Add rational expressions with different denominators
- Subtract rational expressions with different denominators

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Add: $\frac{7}{10}+\frac{8}{15}$.
   If you missed this problem, review Exercise 1.7.13.
2. Subtract: $6(2x+1)-4(x-5)$.
   If you missed this problem, review Exercise 1.10.52.
3. Find the Greatest Common Factor of $9x^2y^3$ and $12xy^5$
   If you missed this problem, review Exercise 7.1.7.
4. Factor completely −48n−12
   If you missed this problem, review Exercise 7.1.31.

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at the example $\frac{7}{12}+\frac{5}{18}$ from Foundations. Since the denominators are not the same, the first step was to find the least common denominator (LCD). Remember, the LCD is the least common multiple of the denominators. It is the smallest number we can use as a common denominator.

To find the LCD of 12 and 18, we factored each number into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied the factors to find the LCD.

12=2·2·3

18=2·3·3

LCD=2·2·3·3

LCD=36

We do the same thing for rational expressions. However, we leave the LCD in factored form.

Definition: FIND THE LEAST COMMON DENOMINATOR OF RATIONAL EXPRESSIONS.

1. Factor each expression completely.
2. List the factors of each expression. Match factors vertically when possible.
3. Bring down the columns.
4. Multiply the factors.

Remember, we always exclude values that would make the denominator zero. What values of *xv*should we exclude in this next example?

Example 13.9.4.1

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$

Answer

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each expression completely, lining up common factors. Bring down the columns.	$x^2-2x-3=(x-3)(x+1)$
	$x^2+4x+3=(x+1)(x+3)$
Multiply the factors.	LCD=(x+1)(x−3)(x+3)
	The LCD is (x+1)(x−3)(x+3).

Example 13.9.4.2

Find the LCD for $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$

Answer

(x−4)(x+4)(x+3)

Example 13.9.4.3

Find the LCD for $\frac{x}{x^2+8x+15}, \frac{5}{x^2+9x+18}$

Answer

(x+3)(x+6)(x+5)

Find Equivalent Rational Expressions

When we add numerical fractions, once we find the LCD, we rewrite each fraction as an equivalent fraction with the LCD.

$$\frac{7}{12}+\frac{5}{18}$$

$$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$$

$$\frac{21}{36}+\frac{10}{36}$$

$$12=2\cdot 2\cdot 3$$
$$18=2\cdot \quad 3\cdot 3$$
$$\text{LCD}=2\cdot 2\cdot 3\cdot 3$$
$$\text{LCD}=36$$

We will do the same thing for rational expressions.

Example 13.9.4.4

Rewrite as equivalent rational expressions with denominator (x+1)(x−3)(x+3): $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.

Answer

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Find the LCD. $x^2-2x-3=(x+1)(x-3)$, $x^2+4x+3=(x+1)(x+3)$, LCD $=(x+1)(x-3)(x+3)$	

Multiply each denominator by the 'missing' factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

Example 13.9.4.5

Rewrite as equivalent rational expressions with denominator (x+3)(x−4)(x+4):
$\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$.

Answer

$\frac{2x+8}{(x-4)(x+3)(x+4)}, \frac{x+3}{(x-4)(x+3)(x+4)}$

Example 13.9.4.6

Rewrite as equivalent rational expressions with denominator (x+3)(x+5)(x+6)
$\frac{x}{x^2+8x+15}, \frac{5}{x^2+9x+18}$.

Answer

$\frac{x^2+6x}{(x+3)(x+5)(x+6)}, \frac{x+3}{(x+3)(x+5)(x+6)}$

Add Rational Expressions with Different Denominators

Now we have all the steps we need to add rational expressions with different denominators. As we have done previously, we will do one example of adding numerical fractions first.

Example 13.9.4.7

Add: $\frac{7}{12}+\frac{5}{18}$.

Answer

	$\frac{7}{12}+\frac{5}{18}$
Find the LCD of 12 and 18. $\begin{array}{l} 12=2\cdot 2\cdot 3 \\ 18=2\cdot \quad 3\cdot 3 \\ \hline \text{LCD}=2\cdot 2\cdot 3\cdot 3 \\ \text{LCD}=36 \end{array}$	
Rewrite each fraction as an equivalent fraction with the LCD.	$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$
Add the fractions.	$\frac{21}{36}+\frac{10}{36}$
The fraction cannot be simplified.	$\frac{31}{36}$

Example 13.9.4.8

Add: $\frac{11}{30}+\frac{7}{12}$.

Answer

$\frac{19}{20}$

Example 13.9.4.9

Add: $\frac{3}{8}+\frac{9}{20}$.

Answer

$\frac{33}{40}$

Now we will add rational expressions whose denominators are monomials.

Example 13.9.4.10

Add: $\frac{5}{12x^2y}+\frac{4}{21xy^2}$.

Answer

	$\frac{5}{12x^2y}+\frac{4}{21xy^2}$
Find the LCD of $12x^2y$ and $21xy^2$ $12x^2y = 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot y$ $21xy^2 = 3 \cdot 7 \cdot x \cdot y \cdot y$ $\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 7 \cdot x \cdot x \cdot y \cdot y$ $\text{LCD} = 84x^2y^2$	
	$\frac{5}{12x^2y}+\frac{4}{21xy^2}$
Rewrite each rational expression as an equivalent fraction with the LCD.	$\frac{5 \cdot 7y}{12x^2y \cdot 7y}+\frac{4 \cdot 4x}{21xy^2 \cdot 4x}$
Simplify.	$\frac{35y}{84x^2y^2}+\frac{16x}{84x^2y^2}$
Add the rational expressions.	$\frac{16x+35y}{84x^2y^2}$
There are no factors common to the numerator and denominator. The fraction cannot be simplified.	

Example 13.9.4.11

Add: $\frac{2}{15a^2b}+\frac{5}{6ab^2}$.

Answer

$\frac{4b+25a}{30a^2b^2}$

Example 13.9.4.12

Add: $\frac{5}{16c}+\frac{3}{8cd^2}$.

Answer

$\frac{5d^2+6}{16cd^2}$

Now we are ready to tackle polynomial denominators.

How to Add Rational Expressions with Different Denominators

Example 13.9.4.13

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Answer

Step 1. Determine if the expressions have a common denominator. • Yes – Go to step 2. • No – Rewrite each rational expression with the LCD. ○ Find the LCD. ○ Rewrite each rational expression as an equivalent rational expression with the LCD.	No Find the LCD of $(x-3), (x-2)$ Change into equivalent rational expressions with the LCD, $(x-3)(x-2)$. Keep the denominators factored!	$x-3 : (x-3)$ $x-2 : (x-2)$ LCD $: (x-3)(x-2)$ $\frac{3}{x-3}+\frac{2}{x-2}$ $\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$ $\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$
Step 2. Add the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$ $\frac{5x-12}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	

Example 13.9.4.14

Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

Answer

$\frac{7x-4}{(x+3)(x-2)}$

Example 13.9.4.15

Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

Answer

$\frac{7m+25}{(m+3)(m+4)}$

The steps to use to add rational expressions are summarized in the following procedure box.

Definition: ADD RATIONAL EXPRESSIONS.

1. Determine if the expressions have a common denominator.
   Yes – go to step 2.
   No – Rewrite each rational expression with the LCD.

Find the LCD.
Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add the rational expressions.
3. Simplify, if possible.

ExAMPLe 13.9.4.16

Add: $\frac{2a}{2ab+b^2}+\frac{3a}{4a^2-b^2}$.

Answer

	$\frac{2a}{2ab+b^2}+\frac{3a}{4a^2-b^2}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{array}{rl} 2ab+b^2 &= b(2a+b) \\ 4a^2-b^2 &= (2a+b)(2a-b) \\ \text{LCD} &= b(2a+b)(2a-b) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2a(2a-b)}{b(2a+b)(2a-b)}+\frac{3a\cdot b}{(2a+b)(2a-b)\cdot b}$
Simplify the numerators.	$\frac{4a^2-2ab}{b(2a+b)(2a-b)}+\frac{3ab}{b(2a+b)(2a-b)}$
Add the rational expressions.	$\frac{4a^2-2ab+3ab}{b(2a+b)(2a-b)}$
Simplify the numerator.	$\frac{4a^2+ab}{b(2a+b)(2a-b)}$
Factor the numerator.	$\frac{a(4a+b)}{b(2a+b)(2a-b)}$
There are no factors common to the numerator and denominator. The fraction cannot be simplified.	

Example 13.9.4.17

Add: $\frac{5x}{xy-y^2}+\frac{2x}{x^2+y^2}$.

Answer

$\frac{x(5x+7y)}{y(x-y)(x+y)}$

Example 13.9.4.18

Add: $\frac{7}{2m+6}+\frac{4}{m^2+4m+3}$.

Answer

$\frac{7m+15}{2(m+3)(m+1)}$

Avoid the temptation to simplify too soon! In the example above, we must leave the first rational expression as $\frac{2a(2a-b)}{b(2a+b)(2a-b)}$ to be able to add it to $\frac{3a\cdot b}{(2a+b)(2a-b)\cdot b}$.

Example 13.9.4.19

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Answer

	$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{array}{l} x^2-2x-3=(x+1)(x-3) \\ x^2+4x+3=(x+1)\quad\quad(x+3) \\ \text{LCD}\quad\quad=(x+1)(x-3)(x+3) \end{array}$	
Rewrite each rational expression as an equivalent fraction with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2+9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x^2+24}{(x+1)(x-3)(x+3)}$
The numerator is prime, so there are no common factors.	

Example 13.9.4.20

Add: $\frac{1}{m^2-m-2}+\frac{5m}{m^2+3m+2}$.

Answer

$\frac{5m^2-9m+2}{(m-2)(m+1)(m+2)}$

Example 13.9.4.21

Add: $\frac{2n}{n^2-3n-10}+\frac{6}{n^2+5n+6}$.

Answer

$\frac{2(n2+6n-15)}{(n+2)(n-5)(n+3)}$

Subtract Rational Expressions with Different Denominators

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

How to Subtract Rational Expressions with Different Denominators

Example 13.9.4.22

Subtract: $\frac{x}{x-3}-\frac{x-2}{x+3}$.

Answer

Step 1. Determine if the expressions have a common denominator. • Yes – Go to step 2. • No – Rewrite each rational expression with the LCD. ○ Find the LCD. ○ Rewrite each rational expression as an equivalent rational expression with the LCD.	No Find the LCD of $(x-3), (x+3)$. Change into equivalent fractions with the LCD, $(x-3)(x+3)$. Keep the denominators factored!	$x-3$: $(x-3)$ $x+3$: $(x+3)$ LCD: $(x-3)(x+3)$ $\frac{x}{x-3} - \frac{x-2}{x+3}$ $\frac{x(x+3)}{(x-3)(x+3)} - \frac{(x-2)(x-3)}{(x+3)(x-3)}$ $\frac{x^2+3x}{(x-3)(x+3)} - \frac{x^2-5x+6}{(x-3)(x+3)}$
Step 2. Subtract the rational expressions.	Subtract the numerators and place the difference over the common denominator. Be careful with the signs!	$\frac{x^2+3x-(x^2-5x+6)}{(x-3)(x+3)}$ $\frac{x^2+3x-x^2+5x-6}{(x-3)(x+3)}$ $\frac{8x-6}{(x-3)(x+3)}$
Step 3. Simplify, if possible.	The numerator and denominator have no factors in common. The answer is simplified.	$\frac{2(4x-3)}{(x-3)(x+3)}$

? Example 13.9.4.23

Subtract: $\frac{y}{y+4} - \frac{y-2}{y-5}$.

Answer

$\frac{-7y+8}{(y+4)(y-5)}$

? Example 13.9.4.24

Subtract: $\frac{z+3}{z+2} - \frac{z}{z+3}$.

Answer

$\frac{4z+9}{(z+2)(z+3)}$

The steps to take to subtract rational expressions are listed below.

Definition: SUBTRACT RATIONAL EXPRESSIONS.

1. Determine if they have a common denominator.
   Yes – go to step 2.
   No – Rewrite each rational expression with the LCD.
   Find the LCD.
   Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Subtract the rational expressions.
3. Simplify, if possible.

Example 13.9.4.25

Subtract: $\frac{8y}{y^2-16} - \frac{4}{y-4}$.

Answer

	$\frac{8y}{y^2-16} - \frac{4}{y-4}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{array}{l} y^2-16=(y-4)(y+4) \\ y-4 \quad =y-4 \\ \hline \text{LCD} \quad =(y-4)(y+4) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)} - \frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)} - \frac{4y+16}{(y-4)(y+4)}$
Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors.	$\frac{4\cancel{(y-4)}}{\cancel{(y-4)}(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

Example 13.9.4.26

Subtract: $\frac{2x}{x^2-4} - \frac{1}{x+2}$.

Answer

$\frac{1}{x-2}$

Example 13.9.4.27

Subtract: $\frac{3}{z+3} - \frac{6z}{z^2-9}$.

Answer

$\frac{-3}{z-3}$

There are lots of negative signs in the next example. Be extra careful!

Example 13.9.4.28

Subtract: $\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$.

Answer

	$\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$

Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{n+3}{2-n}$
Since n−2 and 2−n are opposites, we will multiply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(-1)(2-n)}$
Simplify.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)}{(n-2)}$
Do the expressions have a common denominator? No.	
Find the LCD. $\begin{array}{r} n^2+n-6=(n-2)(n+3) \\ n-2=(n-2) \\ \hline \text{LCD}=(n-2)(n+3) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)(n+3)}{(n-2)(n+3)}$
Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{n^2+6n+9}{(n-2)(n+3)}$
Simplify the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Simplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n(n+3)}{(n-2)(n+3)}$
Simplify.	$\frac{n}{(n-2)}$

Example 13.9.4.29

Subtract: $\frac{3x-1}{x^2-5x-6} - \frac{2}{6-x}$.

Answer

$\frac{1}{x-6}$

Example 13.9.4.30

Subtract: $\frac{-2y-2}{y^2+2y-8} - \frac{y-1}{2-y}$.

Answer

$\frac{y+3}{y+4}$

When one expression is not in fraction form, we can write it as a fraction with denominator 1.

Example 13.9.4.31

Subtract: $\frac{5c+4}{c-2} - 3$.

Answer

	$\frac{5c+4}{c-2} - 3$
Write 3 as $\frac{3}{1}$ to have 2 rational expressions.	$\frac{5c+4}{c-2} - \frac{3}{1}$
Do the rational expressions have a common denominator? No.	

Find the LCD of c−2 and 1. LCD = c−2.	
Rewrite $\frac{3}{1}$ as an equivalent rational expression with the LCD.	$\frac{5c+4}{c-2}-\frac{3(c-2)}{1(c-2)}$
Simplify.	$\frac{5c+4}{c-2}-\frac{3c-6}{(c-2)}$
Subtract the rational expressions.	$\frac{5c+4-(3c-6)}{c-2}$
Simplify.	$\frac{2c+10}{c-2}$
Factor to check for common factors.	$\frac{2(c+5)}{c-2}$
There are no common factors; the rational expression is simplified.	

Example 13.9.4.32

Subtract: $\frac{2x+1}{x-7}-3$.

Answer

$\frac{-x+22}{x-7}$

Example 13.9.4.33

Subtract: $\frac{4y+3}{2y-1}-5$.

Answer

$\frac{-2(3y-4)}{2y-1}$

Definition: ADD OR SUBTRACT RATIONAL EXPRESSIONS.

1. Determine if the expressions have a common denominator.
   Yes – go to step 2.
   No – Rewrite each rational expression with the LCD.
   Find the LCD.
   Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add or subtract the rational expressions.
3. Simplify, if possible.

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example we will start by factoring all three denominators to find their LCD.

Example 13.9.4.34

Simplify: $\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$.

Answer

	$\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$
Do the rational expressions have a common denominator? No.	

$u-1=\quad u-1$ $u\quad\ =u$ $u^2-u=u(u-1)$ Find the LCD. $\text{LCD}=u(u-1)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2u\cdot u}{(u-1)u}+\frac{1\cdot(u-1)}{u\cdot(u-1)}-\frac{2u-1}{u(u-1)}$
	$\frac{2u^2}{(u-1)u}+\frac{u-1}{u\cdot(u-1)}-\frac{2u-1}{u(u-1)}$
Write as one rational expression.	$\frac{2u^2+u-1-2u+1}{u(u-1)}$
Simplify.	$\frac{2u^2-u}{u(u-1)}$
Factor the numerator, and remove common factors.	$\frac{u(2u-1)}{u(u-1)}$
Simplify.	$\frac{2u-1}{u-1}$

? Example 13.9.4.35

Simplify: $\frac{v}{v+1}+\frac{3}{v-1}-\frac{6}{v^2-1}$.

Answer

$\frac{v+3}{v+1}$

? Example 13.9.4.36

Add exercises text here.Simplify: $\frac{3w}{w+2}+\frac{2}{w+7}-\frac{17w+4}{w^2+9w+14}$.

Answer

$\frac{3w}{w+7}$

Key Concepts

- **Find the Least Common Denominator of Rational Expressions**
  1. Factor each expression completely.
  2. List the factors of each expression. Match factors vertically when possible.
  3. Bring down the columns.
  4. Multiply the factors.
- **Add or Subtract Rational Expressions**
  1. Determine if the expressions have a common denominator.
     Yes – go to step 2.
     No – Rewrite each rational expression with the LCD.
     - Find the LCD.
     - Rewrite each rational expression as an equivalent rational expression with the LCD.
  2. Add or subtract the rational expressions.
  3. Simplify, if possible.

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13.9.4E: Exercises

Practice Makes Perfect

In the following exercises, find the LCD.

Example 13.9.4E.37

$\frac{5}{x^2-2x-8}, \frac{2x}{x^2-x-12}$

Answer

(x−4)(x+2)(x+3)

Example 13.9.4E.38

$\frac{8}{y^2+12y+35}, \frac{3y}{y^2+y-42}$

Example 13.9.4E.39

$\frac{9}{z^2+2z-8}, \frac{4z}{z^2-49}$

Answer

(z−2)(z+4)(z+2)

Example 13.9.4E.39

$\frac{6}{a^2+14a+45}, \frac{5a}{a^2-81}$

Example 13.9.4E.40

$\frac{4}{b^2+6b+9}, \frac{2b}{b^2-2b-15}$

Answer

(b+3)(b+3)(b−5)

Example 13.9.4E.41

$\frac{5}{c^2-4c+4}, \frac{3c}{c^2-10c+16}$

Example 13.9.4E.42

$\frac{2}{3d^2+14d-5}, \frac{5d}{3d^2-19d+6}$

Answer

(3d−1)(d+5)(d−6)

Example 13.9.4E.44

$\frac{3}{5m^2-3m-2}, \frac{6m}{5m^2+17m+6}$

In the following exercises, write as equivalent rational expressions with the given LCD.

Example 13.9.4E.45

$\frac{5}{x^2-2x-8}, \frac{2x}{x^2-x-12}$
LCD (x−4)(x+2)(x+3)

Answer

$\frac{5x+15}{(x-4)(x+2)(x+3)}, \frac{2x^2+4x}{(x-4)(x+2)(x+3)}$

Example 13.9.4E.46

$\frac{8}{y^2+12y+35}, \frac{3y}{y^2+y-42}$
LCD (y+7)(y+5)(y−6)

Example 13.9.4E.47

$\frac{9}{z^2+2z-8}, \frac{4z}{z^2-49}$
LCD (z−2)(z+4)(z+2)

Answer

$\frac{9z+18}{(z-2)(z+4)(z+2)}, \frac{4z^2+16}{(z-2)(z+4)(z+2)}$

Example 13.9.4E.48

$\frac{6}{a^2+14a+45}, \frac{5a}{a^2-81}$
LCD (a+9)(a+5)(a−9)

Example 13.9.4E.49

$\frac{4}{b^2+6b+9}, \frac{2b}{b^2-2b-15}$
LCD (b+3)(b+3)(b−5)

Answer

$\frac{4b-20}{(b+3)(b+3)(b-5)}, \frac{2b^2+6b}{(b+3)(b+3)(b-5)}$

Example 13.9.4E.50

$\frac{5}{c^2-4c+4}, \frac{3c}{c^2-10c+10}$
LCD (c−2)(c−2)(c−8)

Example 13.9.4E.51

$\frac{2}{3d^2+14d-5}, \frac{5d}{3d^2-19d+6}$
LCD (3d−1)(d+5)(d−6)

Answer

$\frac{2d-12}{(3d-1)(d+5)(d-6)}$, $\frac{5d^2+25d}{(3d-1)(d+5)(d-6)}$

Example 13.9.4E.52

$\frac{3}{5m^2-3m-2}$, $\frac{6m}{5m^2+17m+6}$

LCD (5m+2)(m−1)(m+3)

In the following exercises, add.

Example 13.9.4E.53

$\frac{5}{24}+\frac{11}{36}$

Answer

$\frac{37}{72}$

Example 13.9.4E.54

$\frac{7}{30}+\frac{13}{45}$

Example 13.9.4E.55

$\frac{9}{20}+\frac{11}{30}$

Answer

$\frac{49}{60}$

Example 13.9.4E.56

$\frac{8}{27}+\frac{7}{18}$

Example 13.9.4E.57

$\frac{7}{10x^2y}+\frac{4}{15xy^2}$

Answer

$\frac{21y+8x}{30x^2y^2}$

Example 13.9.4E.58

$\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3}$

Example 13.9.4E.59

$\frac{1}{2m}+\frac{7}{8m^2n}$

Answer

$\frac{mn+14}{16m^2n}$

Example 13.9.4E.60

$\frac{5}{6p^2q}+\frac{1}{4p}$

Example 13.9.4E.61

$\frac{3}{r+4}+\frac{2}{r-5}$

Answer

$\frac{5r-7}{(r+4)(r-5)}$

Example 13.9.4E.62

$\frac{4}{s-7}+\frac{5}{s+3}$

Example 13.9.4E.63

$\frac{8}{t+5}+\frac{6}{t-5}$

Answer

$\frac{14t-10}{(t+5)(t-5)}$

Example 13.9.4E.64

$\frac{7}{v+5}+\frac{9}{v-5}$

Example 13.9.4E.65

$\frac{5}{3w-2}+\frac{2}{w+1}$

Answer

$\frac{11w+1}{(3w-2)(w+1)}$

Example 13.9.4E.66

$\frac{4}{2x+5}+\frac{2}{x-14}$

Example 13.9.4E.67

$\frac{2y}{y+3}+\frac{3}{y-12}$

Answer

$\frac{2y^2+y+9}{(y+3)(y-1)}$

Example 13.9.4E.68

$\frac{3z}{z-2}+\frac{1}{z+5}$

Example 13.9.4E.69

$\frac{5b}{a^2b-2a^2}+\frac{2b}{b^2-4}$

Answer

$\frac{b(5b+10+2a2)}{a^2(b-2)(b+2)}$

Example 13.9.4E.70

$\frac{4}{cd+3c}+\frac{1}{d^2-9}$

Example 13.9.4E.71

$\frac{2m}{3m-3}+\frac{5m}{m^2+3m-4}$

Answer

$\frac{2m^2+23m}{3(m-1)(m+4)}$

Example 13.9.4E.72

$\frac{3}{4n+4}+\frac{6}{n^2-n-2}$

Example 13.9.4E.73

$\frac{3}{n^2+3n-18}+\frac{4n}{n^2+8n+12}$

Answer

$\frac{4n^2-9n+6}{(n-3)(n+6)(n+2)}$

Example 13.9.4E.74

$\frac{6}{q^2-3q-10}+\frac{5q}{q^2-8q+15}$

Example 13.9.4E.75

$\frac{3r}{r^2+7r+6}+\frac{9}{r^2+4r+3}$

Answer

$\frac{3(r^2+6r+18)}{(r+1)(r+6)(r+3)}$

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Example 13.9.4E.76

$\frac{2s}{s^2+2s-8}+\frac{4}{s^2+3s-10}$

In the following exercises, subtract.

Example 13.9.4E.77

$\frac{t}{t-6}-\frac{t-2}{t+6}$

Answer

$\frac{2(7t-6)}{(t-6)(t+6)}$

Example 13.9.4E.78

$\frac{v}{v-3}-\frac{v-6}{v+1}$

Example 13.9.4E.79

$\frac{w+2}{w+4}-\frac{w}{w-2}$

Answer

$\frac{-4(1+w)}{(w+4)(w-2)}$

Example 13.9.4E.80

$\frac{x-3}{x+6}-\frac{x}{x+3}$

Example 13.9.4E.81

$\frac{y-4}{y+1}-\frac{1}{y+7}$

Answer

$\frac{y^2+2y-29}{(y+1)(y+7)}$

Example 13.9.4E.82

$\frac{z+8}{z-3}-\frac{z}{z-2}$

Example 13.9.4E.83

$\frac{5a}{a+3}-\frac{a+2}{a+6}$

Answer

$\frac{4a^2+25a-6}{(a+3)(a+6)}$

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Example 13.9.4E.84

$\frac{3b}{b-2} - \frac{b-6}{b-8}$

Example 13.9.4E.85

$\frac{6c}{c^2-25} - \frac{3}{c+5}$

Answer

$\frac{3}{c-5}$

Example 13.9.4E.86

$\frac{4d}{d^2-81} - \frac{2}{d+9}$

Example 13.9.4E.87

$\frac{6}{m+6} - \frac{12m}{m^2-36}$

Answer

$\frac{-6}{m-6}$

Example 13.9.4E.88

$\frac{4}{n+4} - \frac{8n}{n^2-16}$

Example 13.9.4E.89

$\frac{-9p-17}{p^2-4p-21} - \frac{p+1}{7-p}$

Answer

$\frac{p+2}{p+3}$

Example 13.9.4E.90

$\frac{7q+8}{q^2-2q-24} - \frac{q+2}{4-q}$

Example 13.9.4E.91

$\frac{-2r-16}{r^2+6r-16} - \frac{5}{2-r}$

Answer

$\frac{3}{r-2}$

Example 13.9.4E.92

$\frac{2t-30}{t^2+6t-27} - \frac{2}{3-t}$

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Example 13.9.4E.93

$\frac{5v-2}{v+3}-4$

Answer

$\frac{-v-14}{v+3}$

Example 13.9.4E.94

$\frac{6w+5}{w-1}+2$

Example 13.9.4E.95

$\frac{2x+7}{10x-1}+3$

Answer

$\frac{4(8x+1)}{10x-1}$

Example 13.9.4E.96

$\frac{8y-4}{5y+2}-6$

In the following exercises, add and subtract.

Example 13.9.4E.97

$\frac{5a}{a-2}+\frac{9}{a}-\frac{2a+18}{a^2-2a}$

Answer

$\frac{5a^2+7a-36}{a(a-2)}$

Example 13.9.4E.98

$\frac{2b}{b-5}+\frac{3}{2b}-\frac{2}{b-15}$

Example 13.9.4E.99

$\frac{c}{c+2}+\frac{5}{c-2}-\frac{10c}{c^2-4}$

Answer

$\frac{c-5}{c+2}$

Example 13.9.4E.100

$\frac{6d}{d-5}+\frac{1}{d+4}-\frac{7d-5}{d^2-d-20}$

In the following exercises, simplify.

Example 13.9.4E.101

$\frac{6a}{3ab+b^2}+\frac{3a}{9a^2-b^2}$

Answer

$\frac{3a(6a-b)}{b(3a+b)(3a-b)}$

Example 13.9.4E.102

$\frac{2c}{2c+10}+\frac{7c}{c^2+9c+20}$

Example 13.9.4E.103

$\frac{6d}{d^2-64}-\frac{3}{d-8}$

Answer

$\frac{3}{d+8}$

Example 13.9.4E.104

$\frac{5}{n+7}-\frac{10n}{n^2-49}$

Example 13.9.4E.105

$\frac{4m}{m^2+6m-7}+\frac{2}{m^2+10m+21}$

Answer

$\frac{2(2m^2+7m-1)}{(m+7)(m-1)(m+3)}$

Example 13.9.4E.106

$\frac{3p}{p^2+4p-12}+\frac{1}{p^2+p-30}$

Example 13.9.4E.107

$\frac{-5n-5}{n^2+n-6}+\frac{n+1}{2-n}$

Answer

$\frac{n+1}{n+3}$

Example 13.9.4E.108

$\frac{-4b-24}{b^2+b-30}+\frac{b+7}{5-b}$

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Example 13.9.4E.109

$\frac{7}{15p}+\frac{5}{18pq}$

Answer

$\frac{42q+25}{90pq}$

Example 13.9.4E.110

$\frac{3}{20a^2}+\frac{11}{12ab^2}$

Example 13.9.4E.111

$\frac{4}{x-2}+\frac{3}{x+5}$

Answer

$\frac{7(x+2)}{(x-2)(x+5)}$

Example 13.9.4E.112

$\frac{6}{m+4}+\frac{9}{m-8}$

Example 13.9.4E.113

$\frac{2q+7}{y+4}-2$

Answer

$\frac{17q+2}{3q-1}$

Example 13.9.4E.114

$\frac{3y-1}{y+4}-2$

Example 13.9.4E.115

$\frac{z+2}{z-5}-\frac{z}{z+1}$

Answer

$\frac{8z+2}{(z-5)(z+1)}$

Example 13.9.4E.116

$\frac{t}{t-5}-\frac{t-1}{t+5}$

Example 13.9.4E.117

$\frac{3d}{d+2}+\frac{4}{d}-\frac{d+8}{d^2+2d}$

Answer

$\frac{3(d+1)}{d+2}$

Example 13.9.4E.118

$\frac{2q}{q+5}+\frac{3}{q-3}-\frac{13q+15}{q^2+2q-15}$

Everyday Math

Example 13.9.4E.119

Decorating cupcakes Victoria can decorate an order of cupcakes for a wedding in tt hours, so in 1 hour she can decorate $\frac{1}{t}$ of the cupcakes. It would take her sister 3 hours longer to decorate the same order of cupcakes, so in 1 hour she can decorate $\frac{1}{t+3}$ of the cupcakes.

1. Find the fraction of the decorating job that Victoria and her sister, working together, would complete in one hour by adding the rational expressions $\frac{1}{t}+\frac{1}{t+3}$.
2. Evaluate your answer to part (a) when t=5.

Answer

1. $\frac{2t+3}{t(t+3)}$
2. $\frac{13}{40}$

Example 13.9.4E.120

Kayaking When Trina kayaks upriver, it takes her $\frac{5}{3-c}$ hours to go 5 miles, where cc is the speed of the river current. It takes her $\frac{5}{3+c}$ hours to kayak 5 miles down the river.

1. Find an expression for the number of hours it would take Trina to kayak 5 miles up the river and then return by adding $\frac{5}{3-c}+\frac{5}{3+c}$.
2. Evaluate your answer to part (a) when c=1 to find the number of hours it would take Trina if the speed of the river current is 1 mile per hour.

Writing Exercises

Example 13.9.4E.121

Felipe thinks $\frac{1}{x}+\frac{1}{y}$ is $\frac{2}{x+y}$.

1. Choose numerical values for x and y and evaluate $\frac{1}{x}+\frac{1}{y}$.
2. Evaluate $\frac{2}{x+y}$ for the same values of x and y you used in part (a).
3. Explain why Felipe is wrong.
4. Find the correct expression for $\frac{1}{x}+\frac{1}{y}$.

Answer

Answers may vary.

Example 13.9.4E.122

Simplify the expression $\frac{4}{n^2+6n+9}-\frac{1}{n^2-9}$ and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the least common denominator of rational expressions.			
find equivalent rational expressions.			
add rational expressions with different denominators.			
subtract rational expressions with different denominators.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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13.9.5: Simplify Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Simplify: $\frac{\frac{3}{5}}{\frac{9}{10}}$.
   If you missed this problem, review Exercise 1.6.25.
2. Simplify: $\frac{1-\frac{1}{3}}{4^2+4\cdot 5}$.
   If you missed this problem, review Exercise 1.6.31.

Complex fractions are fractions in which the numerator or denominator contains a fraction. In Chapter 1 we simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \quad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section we will simplify *complex rational expressions*, which are rational expressions with rational expressions in the numerator or denominator.

Definition: COMPLEX RATIONAL EXPRESSION

A **complex rational expression** is a rational expression in which the numerator or denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$$

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

$$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

Simplify a Complex Rational Expression by Writing it as Division

We have already seen this complex rational expression earlier in this chapter.

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem

$$\left(\frac{6x^2-7x+2}{4x-8}\right) \div \left(\frac{2x^2-8x+3}{x^2-5x+6}\right)$$

Then we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify rational expressions. We write it as if we were dividing two fractions.

Example 13.9.5.1

$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$.

Answer

	$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$
Rewrite the complex fraction as division.	$\frac{4}{y-3} \div \frac{8}{y^2-9}$
Rewrite as the product of first times the reciprocal of the second.	$\frac{4}{y-3} \cdot \frac{y^2-9}{8}$
Multiply.	$\frac{4(y^2-9)}{8(y-3)}$
Factor to look for common factors.	$\frac{4(y-3)(y+3)}{8(y-3)}$
Simplify.	$\frac{y+3}{2}$

Are there any value(s) of y that should not be allowed? The simplified rational expression has just a constant in the denominator. But the original **complex rational expression** had denominators of y−3 and y^2-9. This expression would be undefined if y=3 or y=−3

Example 13.9.5.2

$\frac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$.

Answer

$\frac{2}{3(x-1)}$

Example 13.9.5.3

$\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$.

Answer

$\frac{1}{2(x-3)}$

Example 13.9.5.4

$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Answer

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
Simplify the numerator and denominator.	

Find the LCD and add the fractions in the numerator. Find the LCD and add the fractions in the denominator.	$\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}$ $\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}$
Simplify the numerator and denominator.	$\frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$
Simplify the numerator and denominator, again.	$\frac{\frac{3}{6}}{\frac{1}{6}}$
Rewrite the complex rational expression as a division problem.	$\frac{3}{6}\div\frac{1}{6}$
Multiply the first times by the reciprocal of the second.	$\frac{3}{6}\cdot\frac{6}{1}$
Simplify.	3

? Example 13.9.5.5

$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$.

Answer

$\frac{14}{11}$

? Example 13.9.5.6

$\frac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$.

Answer

$\frac{10}{23}$

How to Simplify a Complex Rational Expression by Writing it as Division

? Example 13.9.5.7

$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Answer

Step 1. Simplify the numerator and denominator.	We will simplify the sum in the numerator and difference in the denominator.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find a common denominator and add the fractions in the numerator.	$\dfrac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
	Find a common denominator and subtract the fractions in the numerator.	$\dfrac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	We now have just one rational expression in the numerator and one in the denominator.	$\dfrac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$
Step 2. Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$
Step 3. Divide the expressions.	Multiply the first by the reciprocal of the second.	$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$\dfrac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$\dfrac{\cancel{xy}\cancel{(y+x)}}{\cancel{xy}(x-y)\cancel{(x+y)}}$
	Simplify.	$\dfrac{1}{x-y}$

Example 13.9.5.8

$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$.

Answer

$\dfrac{y+x}{y-x}$

Example 13.9.5.9

$\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$.

Answer

$\dfrac{ab}{b-a}$

Definition: SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY WRITING IT AS DIVISION.

1. Simplify the numerator and denominator.
2. Rewrite the complex rational expression as a division problem.
3. Divide the expressions.

Example 13.9.5.10

$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$$

Answer

	$\dfrac{n-\dfrac{4n}{n+5}}{\dfrac{1}{n+5}+\dfrac{1}{n-5}}$
Simplify the numerator and denominator.	
Find the LCD and add the fractions in the numerator. Find the LCD and add the fractions in the denominator.	$\dfrac{\dfrac{n(n+5)}{1(n+5)}-\dfrac{4n}{n+5}}{\dfrac{1(n-5)}{(n+5)(n-5)}+\dfrac{1(n+5)}{(n-5)(n+5)}}$
Simplify the numerators.	$\dfrac{\dfrac{n^2+5n}{n+5}-\dfrac{4n}{n+5}}{\dfrac{n-5}{(n+5)(n-5)}+\dfrac{n+5}{(n-5)(n+5)}}$
Subtract the rational expressions in the numerator and add in the denominator.	$\dfrac{\dfrac{n^2+n}{n+5}}{\dfrac{2n}{(n+5)(n-5)}}$
Rewrite as fraction division.	$\dfrac{n^2+n}{n+5}\div\dfrac{2n}{(n+5)(n-5)}$
Multiply the first times the reciprocal of the second.	$\dfrac{n^2+n}{n+5}\cdot\dfrac{(n+5)(n-5)}{2n}$
Factor any expressions if possible.	$\dfrac{n(n+1)(n+5)(n-5)}{(n+5)2n}$
Remove common factors.	$\dfrac{\cancel{n}(n+1)(\cancel{n+5})(n-5)}{(\cancel{n+5})2\cancel{n}}$
Simplify.	$\dfrac{(n+1)(n-5)}{2}$

Example 13.9.5.11

$$\frac{b-\frac{3b}{b+5}}{\frac{2}{b+5}+\frac{1}{b-5}}.$$

Answer

b(b+2)

Example 13.9.5.12

$$\frac{1-\frac{3}{c+4}}{\frac{1}{c+4}+\frac{c}{3}}.$$

Answer

3c+3

Simplify a Complex Rational Expression by Using the LCD

We “cleared” the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by LCD of all the rational expressions.

Let’s look at the complex rational expression we simplified one way in Example. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{LCD}{LCD}$ we are multiplying by 1, so the value stays the same.

Example 13.9.5.13

Simplify: $\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Answer

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
The LCD of all the fractions in the whole expression is 6.	
Clear the fractions by multiplying the numerator and denominator by that LCD.	$\frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$
Distribute.	$\frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$
Simplify.	$\frac{2+1}{3-2}$
	$\frac{3}{1}$
	3

Example 13.9.5.14

Simplify: $\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}$.

Answer

$\frac{7}{3}$

Example 13.9.5.15

Simplify: $\frac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}$.

Answer

$\frac{7}{3}$

How to Simplify a Complex Rational Expression by Using the LCD

Example 13.9.5.16

Simplify: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Answer

Step 1. Find the LCD of all fractions in the complex rational expression.	The LCD of all the fractions is xy.	$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Step 2. Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by xy.	$\frac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$
Step 3. Simplify the expression.	Distribute.	$\frac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$
		$\frac{y+x}{x^2-y^2}$
	Simplify.	$\frac{(y+x)}{(x-y)(x+y)}$
	Remove common factors.	$\frac{1}{x-y}$

Example 13.9.5.17

Simplify: $\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}-\frac{b}{a}}$.

Answer

$\frac{b+a}{a^2+b^2}$

Example 13.9.5.18

Simplify: $\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}-\frac{1}{y}}$.

Answer

$\frac{y-x}{xy}$

Definition: SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY USING THE LCD.

1. Find the LCD of all fractions in the complex rational expression.
2. Multiply the numerator and denominator by the LCD.
3. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

Example 13.9.5.19

Simplify: $\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$.

Answer

	$\dfrac{\dfrac{2}{x+6}}{\dfrac{4}{x-6}-\dfrac{4}{x^2-36}}$
Find the LCD of all fractions in the complex rational expression. The LCD is (x+6)(x−6)	
Multiply the numerator and denominator by the LCD.	$\dfrac{(x+6)(x-6)\dfrac{2}{x+6}}{(x+6)(x-6)\left(\dfrac{4}{x-6}-\dfrac{4}{(x+6)(x-6)}\right)}$
Simplify the expression.	
Distribute in the denominator.	$\dfrac{(x+6)(x-6)\dfrac{2}{x+6}}{(x+6)(x-6)\left(\dfrac{4}{x-6}\right)-(x+6)(x-6)\left(\dfrac{4}{(x+6)(x-6)}\right)}$
Simplify.	$\dfrac{\cancel{(x+6)}(x-6)\dfrac{2}{\cancel{x+6}}}{\cancel{(x+6)}\cancel{(x-6)}\left(\dfrac{4}{\cancel{x-6}}\right)-\cancel{(x+6)}\cancel{(x-6)}\left(\dfrac{4}{\cancel{(x+6)}\cancel{(x-6)}}\right)}$
Simplify.	$\dfrac{2(x-6)}{4(x+6)-4}$
To simplify the denominator, distribute and combine like terms.	$\dfrac{2(x-6)}{4x+20}$
Remove common factors.	$\dfrac{\cancel{2}(x-6)}{\cancel{2}(2x+10)}$
Simplify.	$\dfrac{x-6}{2x+10}$
Notice that there are no more factors common to the numerator and denominator.	

? Example 13.9.5.20

Simplify: $\dfrac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}$.

Answer

$\dfrac{3x-6}{5x+7}$

? Example 13.9.5.21

Simplify: $\dfrac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}$.

Answer

$\dfrac{x+21}{6x+43}$

? Example 13.9.5.22

Simplify: $\dfrac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$.

Answer

	$\dfrac{\dfrac{4}{m^2-7m+12}}{\dfrac{3}{m-3}-\dfrac{2}{m-4}}$
Find the LCD of all fractions in the complex rational expression. The LCD is (m−3)(m−4)	
Multiply the numerator and denominator by the LCD.	$\dfrac{(m-3)(m-4)\dfrac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\dfrac{3}{m-3}-\dfrac{2}{m-4}\right)}$
Simplify.	$\dfrac{\cancel{(m-3)}\cancel{(m-4)}\dfrac{4}{\cancel{(m-3)}\cancel{(m-4)}}}{\cancel{(m-3)}(m-4)\left(\dfrac{3}{\cancel{m-3}}\right)-(m-3)\cancel{(m-4)}\left(\dfrac{2}{\cancel{m-4}}\right)}$
Simplify.	$\dfrac{4}{3(m-4)-2(m-3)}$
Distribute.	$\dfrac{4}{3m-12-2m+6}$
Combine like terms.	$\dfrac{4}{m-6}$

Example 13.9.5.23

Simplify: $\dfrac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}$.

Answer

$\dfrac{3}{5x+22}$

Example 13.9.5.24

Simplify: $\dfrac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$.

Answer

$\dfrac{6y+34}{3y}$

Example 13.9.5.25

Simplify: $\dfrac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$.

Answer

	$\dfrac{\dfrac{y}{y+1}}{1+\dfrac{1}{y-1}}$
Find the LCD of all fractions in the complex rational expression.	
The LCD is (y+1)(y−1)	

Multiply the numerator and denominator by the LCD.	$\frac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$
Distribute in the denominator and simplify.	$\frac{\cancel{(y+1)}(y-1)\left(\frac{y}{\cancel{y+1}}\right)}{(y+1)(y-1)(1)+(y+1)\cancel{(y-1)}\left(\frac{1}{\cancel{y-1}}\right)}$
Simplify.	$\frac{(y-1)y}{(y+1)(y-1)+(y+1)}$
Simplify the denominator, and leave the numerator factored.	$\frac{y(y-1)}{y^2-1+y+1}$
	$\frac{y(y-1)}{y^2+y}$
Factor the denominator, and remove factors common with the numerator.	$\frac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$
Simplify.	$\frac{y-1}{y+1}$

? Example 13.9.5.26

Simplify: $\frac{\frac{x}{x+3}}{1+\frac{1}{x+3}}$.

Answer

$\frac{x}{x+4}$

? Example 13.9.5.27

Simplify: $\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$.

Answer

$\frac{x(x+1)}{3(x-1)}$

Key Concepts

- **To Simplify a Rational Expression by Writing it as Division**
  1. Simplify the numerator and denominator.
  2. Rewrite the complex rational expression as a division problem.
  3. Divide the expressions.
- **To Simplify a Complex Rational Expression by Using the LCD**
  1. Find the LCD of all fractions in the complex rational expression.
  2. Multiply the numerator and denominator by the LCD.
  3. Simplify the expression.

Glossary

complex rational expression

A complex rational expression is a rational expression in which the numerator or denominator contains a rational expression.

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13.9.5E: Exercises

Practice Makes Perfect

Simplify a Complex Rational Expression by Writing It as Division

In the following exercises, simplify.

Example 13.9.5E.28

$$\frac{\frac{2a}{a+4}}{\frac{4a^2}{a^2-16}}$$

Answer

$$\frac{a-4}{2a}$$

Example 13.9.5E.29

$$\frac{\frac{3b}{b-5}}{\frac{b^2}{b^2-25}}$$

Example 13.9.5E.30

$$\frac{\frac{5}{c^2+5c-14}}{\frac{10}{c+7}}$$

Answer

$$\frac{1}{2(c-2)}$$

Example 13.9.5E.31

$$\frac{\frac{8}{d^2+9d+18}}{\frac{12}{d+6}}$$

Example 13.9.5E.32

$$\frac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$$

Answer

$$\frac{24}{26}$$

Example 13.9.5E.33

$$\frac{\frac{1}{2}+\frac{3}{4}}{\frac{3}{5}+\frac{7}{10}}$$

Example 13.9.5E.34

$$\frac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$$

Answer

$\frac{20}{57}$

Example 13.9.5E.35

$\frac{\frac{1}{2}-\frac{1}{6}}{\frac{2}{3}+\frac{3}{4}}$

Example 13.9.5E.36

$\frac{\frac{n}{m}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

Answer

$\frac{n^2+m}{m-n^2}$

Example 13.9.5E.37

$\frac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

Example 13.9.5E.38

$\frac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

Answer

$\frac{rt}{t-r}$

Example 13.9.5E.39

$\frac{\frac{2}{v}+\frac{2}{w}}{\frac{1}{v^2}-\frac{1}{w^2}}$

Example 13.9.5E.40

$\frac{x-\frac{2x}{x+3}}{\frac{1}{x+3}+\frac{1}{x-3}}$

Answer

$\frac{(x+1)(x-3)}{2}$

Example 13.9.5E.41

$\frac{y-\frac{2y}{y-4}}{\frac{2}{y-4}-\frac{2}{y+4}}$

Example 13.9.5E.42

$$\frac{2-\frac{2}{a+3}}{\frac{1}{a+3}+\frac{a}{2}}$$

Answer

$$\frac{4}{a+1}$$

Example 13.9.5E.43

$$\frac{4-\frac{4}{b-5}}{\frac{1}{b-5}+\frac{b}{4}}$$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

Example 13.9.5E.44

$$\frac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$$

Answer

$$\frac{1}{18}$$

Example 13.9.5E.45

$$\frac{\frac{1}{4}+\frac{1}{9}}{\frac{1}{6}+\frac{1}{12}}$$

Example 13.9.5E.46

$$\frac{\frac{5}{6}+\frac{2}{9}}{\frac{7}{18}-\frac{1}{3}}$$

Answer

19

Example 13.9.5E.47

$$\frac{\frac{1}{6}+\frac{4}{15}}{\frac{3}{5}-\frac{1}{2}}$$

Example 13.9.5E.48

$$\frac{\frac{c}{d}+\frac{1}{d}}{\frac{1}{d}-\frac{d}{c}}$$

Answer

$$\frac{c^2+c}{c-d^2}$$

Example 13.9.5E.49

$$\frac{\frac{1}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$$

Example 13.9.5E.50

$$\frac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$$

Answer

$$\frac{pq}{q-p}$$

Example 13.9.5E.51

$$\frac{\frac{2}{r}+\frac{2}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$$

Example 13.9.5E.52

$$\frac{\frac{2}{x+5}}{\frac{3}{x-5}+\frac{1}{x^2-25}}$$

Answer

$$\frac{2x-10}{3x+16}$$

Example 13.9.5E.53

$$\frac{\frac{5}{y-4}}{\frac{3}{y+4}+\frac{2}{y^2-16}}$$

Example 13.9.5E.54

$$\frac{\frac{5}{z^2-64}+\frac{3}{z+8}}{\frac{1}{z+8}+\frac{2}{z-8}}$$

Answer

$$\frac{3z-19}{3z+8}$$

Example 13.9.5E.55

$$\frac{\frac{3}{s+6}+\frac{5}{s-6}}{\frac{1}{s^2-36}+\frac{4}{s+6}}$$

Example 13.9.5E.56

$$\frac{\frac{4}{a^2-2a-15}}{\frac{1}{a-5}+\frac{2}{a+3}}$$

Answer

$$\frac{4}{3a-2}$$

Example 13.9.5E.57

$$\frac{\frac{5}{b^2-6b-27}}{\frac{3}{b-9}+\frac{1}{b+3}}$$

Example 13.9.5E.58

$$\frac{\frac{5}{c+2}-\frac{3}{c+7}}{\frac{5c}{c^2+9c+14}}$$

Answer

$$\frac{2c+29}{5c}$$

Example 13.9.5E.59

$$\frac{\frac{6}{d-4}-\frac{2}{d+7}}{\frac{2d}{d^2+3d-28}}$$

Example 13.9.5E.60

$$\frac{2+\frac{1}{p-3}}{\frac{5}{p-3}}$$

Answer

$$\frac{(2p-5)}{5}$$

Example 13.9.5E.61

$$\frac{\frac{n}{n-2}}{3+\frac{5}{n-2}}$$

Example 13.9.5E.62

$$\frac{\frac{m}{m+5}}{4+\frac{1}{m-5}}$$

Answer

$$\frac{m(m-5)}{4m^2+m-95}$$

Example 13.9.5E.63

$$\frac{7+\frac{2}{q-2}}{\frac{1}{q+2}}$$

Simplify

In the following exercises, use either method.

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Example 13.9.5E.64

$$\frac{\frac{3}{4}-\frac{2}{7}}{\frac{1}{2}+\frac{5}{14}}$$

Answer

$\frac{13}{24}$

Example 13.9.5E.65

$$\frac{\frac{v}{w}+\frac{1}{v}}{\frac{1}{v}-\frac{v}{w}}$$

Example 13.9.5E.66

$$\frac{\frac{2}{a+4}}{\frac{1}{a^2-16}}$$

Answer

2(a−4)

Example 13.9.5E.67

$$\frac{\frac{3}{b^2-3b-40}}{\frac{5}{b+5}-\frac{2}{b-8}}$$

Example 13.9.5E.68

$$\frac{\frac{3}{m}+\frac{3}{n}}{\frac{1}{m^2}-\frac{1}{n^2}}$$

Answer

$\frac{3mn}{n-m}$

Example 13.9.5E.69

$$\frac{\frac{2}{r-9}}{\frac{1}{r+9}+\frac{3}{r^2-81}}$$

Example 13.9.5E.70

$$\frac{x-\frac{3x}{x+2}}{\frac{3}{x+2}+\frac{3}{x-2}}$$

Answer

$\frac{(x-1)(x-2)}{6}$

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Example 13.9.5E.71

$\frac{\frac{y}{y+3}}{2+\frac{1}{y-3}}$

Everyday Math

Example 13.9.5E.72

Electronics The resistance of a circuit formed by connecting two resistors in parallel is $\frac{1}{\frac{1}{R1}+\frac{1}{R2}}$

1. Simplify the complex fraction $\frac{1}{\frac{1}{R1}+\frac{1}{R2}}$
2. Find the resistance of the circuit when R1=8 and R2=12

Answer

1. $\frac{R1R2}{R2+R1}$
2. $\frac{24}{5}$

Example 13.9.5E.73

Ironing Lenore can do the ironing for her family's business in hh hours. Her daughter would take h+2 hours to get the ironing done. If Lenore and her daughter work together, using 2 irons, the number of hours it would take them to do all the ironing is $\frac{1}{\frac{1}{h}+\frac{1}{h+2}}$

1. Simplify the complex fraction $\frac{1}{\frac{1}{h}+\frac{1}{h+2}}$
2. Find the number of hours it would take Lenore and her daughter, working together, to get the ironing done if h=4

Writing Exercises

Example 13.9.5E.74

In this section, you learned to simplify the complex fraction $\frac{\frac{3}{x+2}}{\frac{x}{x^2-4}}$ two ways:

rewriting it as a division problem

multiplying the numerator and denominator by the LCD

Which method do you prefer? Why?

Answer

Answers will vary.

Example 13.9.5E.75

Efraim wants to start simplifying the complex fraction $\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$ by cancelling the variables from the numerator and denominator. Explain what is wrong with Efraim's plan.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify a complex rational expression by writing it as division.			
simplify a complex rational expression by using the LCD.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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13.9.6: Solve Rational Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve rational equations
- Solve a rational equation for a specific variable

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Solve: $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$.
   If you missed this problem, review Exercise 2.5.1.
2. Solve: $n^2-5n-36=0$.
   If you missed this problem, review Exercise 7.6.13.
3. Solve for y in terms of x: 5x+2y=10 for y.
   If you missed this problem, review Exercise 2.6.22.

After defining the terms *expression* and *equation* early in Foundations, we have used them throughout this book. We have *simplified* many kinds of *expressions* and *solved* many kinds of *equations*. We have simplified many rational expressions so far in this chapter. Now we will solve rational equations.

The definition of a rational equation is similar to the definition of equation we used in Foundations.

Definition: RATIONAL EQUATION

A **rational equation** is two rational expressions connected by an equal sign.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

Here is an example we did when we worked with linear equations:

	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$	LCD = 8
We multiplied both sides by the LCD.	$8\left(\frac{1}{8}x+\frac{1}{2}\right)=8\left(\frac{1}{4}\right)$	
Then we distributed.	$8\cdot\frac{1}{8}x+8\cdot\frac{1}{2}=8\cdot\frac{1}{4}$	
We simplified—and then we had an equation with no fractions.	$x+4=2$	
Finally, we solved that equation.	$x+4-4=2-4$	

	$x = -2$	

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then we will have an equation that does not contain rational expressions and thus is much easier for us to solve.

But because the original equation may have a variable in a denominator we must be careful that we don't end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a **rational equation** we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an *extraneous solution*.

Definition: EXTRANEOUS Solution TO A RATIONAL EQUATION

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.

How to Solve Equations with Rational Expressions

Example 13.9.6.1

Solve: $\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$.

Answer

Step 1. Note any value of the variable that would make any denominator zero.	If $x = 0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}, x \neq 0$
Step 2. Find the least common denominator of *all* denominators in the equation.	Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$.	The LCD is $6x$.
Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.	Multiply both sides of the equation by the LCD, $6x$. Use the Distributive Property. Simplify – and notice, no more fractions!	$6x \cdot \left(\frac{1}{x} + \frac{1}{3}\right) = 6x \cdot \left(\frac{5}{6}\right)$ $6x \cdot \frac{1}{x} + 6x \cdot \frac{1}{3} = 6x \cdot \left(\frac{5}{6}\right)$ $6 + 2x = 5x$
Step 4. Solve the resulting equation.	Simplify.	$6 = 3x$ $2 = x$
Step 5. Check. • If any values found in Step 1 are algebraic solutions, discard them. • Check any remaining solutions in the original equation.	We did not get 0 as an algebraic solution. We substitute $x = 2$ into the original equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$ $\frac{1}{2} + \frac{1}{3} \stackrel{?}{=} \frac{5}{6}$ $\frac{3}{6} + \frac{2}{6} \stackrel{?}{=} \frac{5}{6}$ $\frac{5}{6} = \frac{5}{6}$ ✓

Example 13.9.6.2

Solve: $\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$.

Answer

$-\frac{15}{7}$

Example 13.9.6.3

Solve: $\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$.

Answer

$\frac{15}{13}$

The steps of this method are shown below.

Definition: SOLVE EQUATIONS WITH RATIONAL EXPRESSIONS.

1. Note any value of the variable that would make any denominator zero.
2. Find the least common denominator of *all* denominators in the equation.
3. Clear the fractions by multiplying both sides of the equation by the LCD.
4. Solve the resulting equation.
5. Check.
   - If any values found in Step 1 are algebraic solutions, discard them.
   - Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

Example 13.9.6.4

Solve: $1-\frac{5}{y}=-\frac{6}{y^2}$.

Answer

	$1-\frac{5}{y}=-\frac{6}{y^2}$
Note any value of the variable that would make any denominator zero.	$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$
Find the least common denominator of all denominators in the equation. The LCD is y^2	
Clear the fractions by multiplying both sides of the equation by the LCD.	$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Distribute.	$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Multiply.	$y^2-5y=-6$
Solve the resulting equation. First write the quadratic equation in standard form.	$y^2-5y+6=0$
Factor.	$(y-2)(y-3)=0$
Use the Zero Product Property.	$y-2=0$ or $y-3=0$

Solve.	$y = 2$ or $y = 3$
Check.	
We did not get 0 as an algebraic solution.	
Check $y = 2$ and $y = 3$ in the original equation. $1 - \frac{5}{y} = -\frac{6}{y^2}$ $1 - \frac{5}{2} \stackrel{?}{=} -\frac{6}{2^2}$ $1 - \frac{5}{2} \stackrel{?}{=} -\frac{6}{4}$ $\frac{2}{2} - \frac{5}{2} \stackrel{?}{=} -\frac{6}{4}$ $-\frac{3}{2} \stackrel{?}{=} -\frac{6}{4}$ $-\frac{3}{2} = -\frac{3}{2}$ ✓ $1 - \frac{5}{y} = -\frac{6}{y^2}$ $1 - \frac{5}{3} \stackrel{?}{=} -\frac{6}{3^2}$ $1 - \frac{5}{3} \stackrel{?}{=} -\frac{6}{9}$ $\frac{3}{3} - \frac{5}{3} \stackrel{?}{=} -\frac{6}{9}$ $-\frac{2}{3} \stackrel{?}{=} -\frac{6}{9}$ $-\frac{2}{3} = -\frac{2}{3}$ ✓	

Example 13.9.6.5

Solve: $1 - \frac{2}{a} = \frac{15}{a^2}$.

Answer

5, −3

Example 13.9.6.6

Solve: $1 - \frac{4}{b} = \frac{12}{b^2}$.

Answer

6, −2

Example 13.9.6.7

Solve: $\frac{5}{3u-2} = \frac{3}{2u}$.

Answer

	$\frac{5}{3u-2} = \frac{3}{2u}$
Note any value of the variable that would make any denominator zero.	$\frac{5}{3u-2} = \frac{3}{2u}$, $u \neq \frac{2}{3}$, $u \neq 0$
Find the least common denominator of all denominators in the equation. The LCD is 2u(3u−2).	
Clear the fractions by multiplying both sides of the equation by the LCD.	$2u(3u-2)\left(\frac{5}{3u-2}\right) = 2u(3u-2)\left(\frac{3}{2u}\right)$
Remove common factors.	$2u\cancel{(3u-2)}\left(\frac{5}{\cancel{3u-2}}\right) = \cancel{2u}(3u-2)\left(\frac{3}{\cancel{2u}}\right)$
Simplify.	$2u(5) = (3u-2)(3)$
Multiply.	$10u = 9u - 6$

Solve the resulting equation.	$u = -6$
We did not get 0 or $\frac{2}{3}$ as algebraic solutions.	
Check $u = -6$ in the original equation. $\frac{5}{3u-2} = \frac{3}{2u}$ $\frac{5}{3(-6)-2} \stackrel{?}{=} \frac{3}{2(-6)}$ $\frac{5}{-20} \stackrel{?}{=} \frac{3}{-12}$ $-\frac{1}{4} = -\frac{1}{4}$ ✓	

Example 13.9.6.8

Solve: $\frac{1}{x-1} = \frac{2}{3x}$.

Answer

−2

Example 13.9.6.9

Solve: $\frac{3}{5n+1} = \frac{2}{3n}$.

Answer

−2

When one of the denominators is a quadratic, remember to factor it first to find the LCD.

Example 13.9.6.10

Solve: $\frac{2}{p+2} + \frac{4}{p-2} = \frac{p-1}{p^2-4}$.

Answer

	$\frac{2}{p+2} + \frac{4}{p-2} = \frac{p-1}{p^2-4}$
Note any value of the variable that would make any denominator zero.	$\frac{2}{p+2} + \frac{4}{p-2} = \frac{p-1}{(p+2)(p+2)}, p \neq -2, p \neq 2$
Find the least common denominator of all denominators in the equation. The LCD is (p+2)(p−2).	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(p+2)(p-2)\left(\frac{2}{p+2} + \frac{4}{p-2}\right) = (p+2)(p-2)\left(\frac{p-1}{p^2-4}\right)$
Distribute.	$(p+2)(p-2)\frac{2}{p+2} + (p+2)(p-2)\frac{4}{p-2} = (p+2)(p-2)\left(\frac{p-1}{p^2-4}\right)$
Remove common factors.	$\cancel{(p+2)}(p-2)\frac{2}{\cancel{p+2}} + (p+2)\cancel{(p-2)}\frac{4}{\cancel{p-2}} = \cancel{(p+2)}\cancel{(p-2)}\left(\frac{p-1}{\cancel{p^2-4}}\right)$
Simplify.	$2(p-2) + 4(p+2) = p-1$
Distribute.	$2p - 4 + 4p + 8 = p - 1$
Solve.	$6p + 4 = p - 1$

	$5p = -5$
	$p = -1$
We did not get 2 or −2 as algebraic solutions.	
Check $p = -1$ in the original equation. $\frac{2}{p+2} + \frac{4}{p-2} = \frac{p-1}{p^2-4}$ $\frac{2}{(-1)+2} + \frac{4}{(-1)-2} \stackrel{?}{=} \frac{(-1)-1}{(-1)^2-4}$ $\frac{2}{1} + \frac{4}{-3} \stackrel{?}{=} \frac{-2}{-3}$ $\frac{6}{3} - \frac{4}{-3} \stackrel{?}{=} \frac{-2}{-3}$ $\frac{2}{3} = \frac{2}{3}$ ✓	

Example 13.9.6.11

Solve: $\frac{2}{x+1} + \frac{1}{x-1} = \frac{1}{x^2-1}$.

Answer

$\frac{2}{3}$

Example 13.9.6.12

Solve: $\frac{5}{y+3} + \frac{2}{y-3} = \frac{5}{y^2-9}$

Answer

2

Example 13.9.6.13

Solve: $\frac{4}{q-4} - \frac{3}{q-3} = 1$.

Answer

	$\frac{4}{q-4} - \frac{3}{q-3} = 1$
Note any value of the variable that would make any denominator zero.	$\frac{4}{q-4} + \frac{3}{q-3} = 1, \quad q \neq 4, q \neq 3$
Find the least common denominator of all denominators in the equation. The LCD is (q−4)(q−3).	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(q-4)(q-3)\left(\frac{4}{q-4} - \frac{3}{q-3}\right) = (q-4)(q-3)(1)$
Distribute.	$(q-4)(q-3)\left(\frac{4}{q-4}\right) - (q-4)(q-3)\left(\frac{3}{q-3}\right) = (q-4)(q-3)(1)$
Remove common factors.	$\cancel{(q-4)}(q-3)\left(\frac{4}{\cancel{q-4}}\right) - (q-4)\cancel{(q-3)}\left(\frac{3}{\cancel{q-3}}\right) = (q-4)(q-3)(1)$
Simplify.	$4(q-3) - 3(q-4) = (q-4)(q-3)$
Simplify.	$4q - 12 - 3q + 12 = q^2 - 7q + 12$

Combine like terms.	$q = q^2 - 7q + 12$
Solve. First write in standard form.	$0 = q^2 - 8q + 12$
Factor.	$0 = (q-2)(q-6)$
Use the Zero Product Property.	$q = 2$ or $q = 6$
We did not get 4 or 3 as algebraic solutions.	
Check $q = 2$ and $q = 6$ in the original equation.	

$$\frac{4}{q-4} - \frac{3}{q-3} = 1 \qquad \frac{4}{q-4} - \frac{3}{q-3} = 1$$

$$\frac{4}{2-4} - \frac{3}{2-3} \stackrel{?}{=} 1 \qquad \frac{4}{6-4} - \frac{3}{6-3} \stackrel{?}{=} 1$$

$$\frac{4}{-2} - \frac{3}{-1} \stackrel{?}{=} 1 \qquad \frac{4}{2} - \frac{3}{1} \stackrel{?}{=} 1$$

$$-2 - (-3) \stackrel{?}{=} 1 \qquad 2 - 1 \stackrel{?}{=} 1$$

$$1 = 1 \checkmark \qquad 1 = 1 \checkmark$$

Example 13.9.6.14

Solve: $\frac{2}{x+5} - \frac{1}{x-1} = 1$.

Answer

$-1, -2$

Example 13.9.6.15

Solve: $\frac{3}{x+8} - \frac{2}{x-2} = 1$.

Answer

$-2, -3$

Example 13.9.6.16

Solve: $\frac{m+11}{m^2-5m+4} = \frac{5}{m-4} - \frac{3}{m-1}$.

Answer

	$\frac{m+11}{m^2-5m+4} = \frac{5}{m-4} = \frac{3}{m-1}$
Factor all the denominators, so we can note any value of the variable the would make any denominator zero.	$\frac{m+11}{(m-4)(m-1)} = \frac{5}{m-4} - \frac{3}{m-1},\ m \neq 4, m \neq 1$
Find the least common denominator of all denominators in the equation. The LCD is (m−4)(m−1)	
Clear the fractions.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right) = (m-4)(m-1)\left(\frac{5}{m-4} - \frac{3}{m-1}\right)$
Distribute.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right) = (m-4)(m-1)\frac{5}{m-4} - (m-4)(m-1)\frac{3}{m-1}$
Remove common factors.	$\cancel{(m-4)(m-1)}\left(\frac{m+11}{\cancel{(m-4)(m-1)}}\right) = \cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}} - (m-4)\cancel{(m-1)}\frac{3}{\cancel{m-1}}$
Simplify.	$m + 11 = 5(m-1) - 3(m-4)$
Solve the resulting equation.	$m + 11 = 5m - 5 - 3m + 12$

	$4 = m$
Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution. There is no solution to this equation.	

Example 13.9.6.17

Solve: $\frac{x+13}{x^2-7x+10} = \frac{6}{x-5} - \frac{4}{x-2}$.

Answer

no solution

Example 13.9.6.18

Solve: $\frac{y-14}{y^2+3y-4} = \frac{2}{y+4} + \frac{7}{y-1}$.

Answer

no solution

The equation we solved in Example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. Some equations have no solution.

Example 13.9.6.19

Solve: $\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}$.

Answer

	$\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}$
Note any value of the variable that would make any denominator zero.	$\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}, n \neq 0$
Find the least common denominator of all denominators in the equation. The LCD is 12n.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$12n\left(\frac{n}{12} + \frac{n+3}{3n}\right) = 12n\left(\frac{1}{n}\right)$
Distribute.	$12n\left(\frac{n}{12}\right) + 12n\left(\frac{n+3}{3n}\right) = 12n\left(\frac{1}{n}\right)$
Remove common factors.	$\cancel{12}n\left(\frac{n}{\cancel{12}}\right) + 4 \cdot \cancel{3}n\left(\frac{n+3}{\cancel{3}n}\right) = 12\cancel{n}\left(\frac{1}{\cancel{n}}\right)$
Simplify.	$n \cdot n + 4(n+3) = 12 \cdot 1$
Solve the resulting equation.	$n^2 + 4n + 12 = 12$
	$n^2 + 4n = 0$
	$n(n+4) = 0$
	$n = 0$ or $n = -4$
Check.	

n=0 is an extraneous solution.

Check $n = -4$ in the original equation.

$$\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}$$

$$\frac{-4}{12} + \frac{-4+3}{3(-4)} \stackrel{?}{=} \frac{1}{-4}$$

$$-\frac{4}{12} + \frac{1}{12} \stackrel{?}{=} -\frac{1}{4}$$

$$-\frac{3}{12} \stackrel{?}{=} -\frac{1}{4}$$

$$-\frac{1}{4} = -\frac{1}{4} \checkmark$$

Example 13.9.6.20

Solve: $\frac{x}{18} + \frac{x+6}{9x} = \frac{2}{3x}$.

Answer

−2

Example 13.9.6.21

Solve: $\frac{y+5}{5y} + \frac{y}{15} = \frac{1}{y}$.

Answer

−3

Example 13.9.6.22

Solve: $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$.

Answer

	$\frac{y}{y+6} = \frac{72}{y^2-36} + 4$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{y}{y+6} = \frac{72}{(y-6)(y+6)} + 4, y \neq 6, y \neq -6$
Find the least common denominator. The LCD is (y−6)(y+6).	
Clear the fractions.	$(y-6)(y+6)\left(\frac{y}{y+6}\right) = (y-6)(y+6)\left(\frac{72}{(y-6)(y+6)} + 4\right)$
Simplify.	$(y-6) \cdot y = 72 + (y-6)(y+6) \cdot 4$
Simplify.	$y(y-6) = 72 + 4(y^2-36)$
Solve the resulting equation.	$y^2 - 6y = 72 + 4y^2 - 144$
	$0 = 3y^2 + 6y - 72$
	$0 = 3(y^2 + 2y - 24)$
	$0 = 3(y+6)(y-4)$
	$y = -6,\ y = 4$

Check.	
y=−6 is an extraneous solution.	
Check $y = 4$ in the original equation. $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$ $\frac{4}{4+6} \stackrel{?}{=} \frac{72}{4^2-36} + 4$ $\frac{4}{10} \stackrel{?}{=} \frac{72}{-20} + 4$ $\frac{4}{10} \stackrel{?}{=} -\frac{36}{10} + \frac{40}{10}$ $\frac{4}{10} = \frac{4}{10}$ ✓	

Example 13.9.6.23

Solve: $\frac{x}{x+4} = \frac{32}{x^2-16} + 5$.

Answer

−4, 3

Example 13.9.6.24

Solve: $\frac{y}{y+8} = \frac{128}{y^2-64} + 9$.

Answer

7

Example 13.9.6.25

Solve: $\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}$.

Answer

	$\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}$
We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}$
Note any value of the variable that would make any denominator zero.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}, x \neq 1, x \neq -1$
Find the least common denominator.The LCD is 12(x−1)(x+1)	
Clear the fractions.	$12(x-1)(x+1)\left(\frac{x}{2(x-1)} - \frac{2}{3(x+1)}\right) = 12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$
Simplify.	$6(x+1) \cdot x - 4(x-1) \cdot 2 = 5x^2 - 2x + 9$
Simplify.	$6x(x+1) - 4 \cdot 2(x-1) = 5x^2 - 2x + 9$
Solve the resulting equation.	$6x^2 + 6x - 8x + 8 = 5x^2 - 2x + 9$
	$x^2 - 1 = 0$
	$(x-1)(x+1) = 0$

	$x = 1$ or $x = -1$
Check.	
x=1 and x=−1 are extraneous solutions. The equation has no solution.	

Example 13.9.6.26

Solve: $\frac{y}{5y-10} - \frac{5}{3y+6} = \frac{2y^2-19y+54}{15y^2-60}$.

Answer

no solution

Example 13.9.6.27

Solve: $\frac{z^2}{z+8} - \frac{3}{4z-8} = \frac{3z^2-16z-68}{z^2+8z-64}$.

Answer

no solution

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

We'll start with a formula relating distance, rate, and time. We have used it many times before, but not usually in this form.

Example 13.9.6.28

Solve: $\frac{D}{T} = R$ for T.

Answer

	$\frac{D}{T} = R$ for T
Note any value of the variable that would make any denominator zero.	$\frac{D}{T} = R, T \neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, T.	$T\left(\frac{D}{T}\right) = T(R)$
Simplify.	$D = T \cdot R$
Divide both sides by R to isolate T.	$\frac{D}{R} = \frac{RT}{R}$
Simplify.	$\frac{D}{R} = T$

Example 13.9.6.29

Solve: $\frac{A}{L} = W$ for L.

Answer

$L = \frac{A}{W}$

Example 13.9.6.30

Solve: $\frac{F}{A} = M$ for A.

Answer

$A = \frac{F}{M}$

Example uses the formula for slope that we used to get the point-slope form of an equation of a line.

Example 13.9.6.31

Solve: $m = \frac{x-2}{y-3}$ for y.

Answer

	$m = \frac{x-2}{y-3}$ for y
Note any value of the variable that would make any denominator zero.	$m = \frac{x-2}{y-3},\ y \neq 3$
Clear the fractions by multiplying both sides of the equations by the LCD, y−3.	$(y-3)m = (y-3)\left(\frac{x-2}{y-3}\right)$
Simplify.	$ym - 3m = x - 2$
Isolate the term with y.	$ym = x - 2 + 3m$
Divide both sides by m to isolate y.	$\frac{ym}{m} = \frac{x-2+3m}{m}$
Simplify.	$y = \frac{x-2+3m}{m}$

Example 13.9.6.32

Solve: $\frac{y-2}{x+1} = \frac{2}{3}$ for x.

Answer

$x = \frac{3y-8}{2}$

Example 13.9.6.33

Solve: $x = \frac{y}{1-y}$ for y.

Answer

$y = \frac{x}{1+x}$

Be sure to follow all the steps in Example. It may look like a very simple formula, but we cannot solve it instantly for either denominator.

Example 13.9.6.34

Solve $\frac{1}{c}+\frac{1}{m}=1$ for c.

Answer

	$\frac{1}{c}+\frac{1}{m}=1$ for c
Note any value of the variable that would make any denominator zero.	$\frac{1}{c}+\frac{1}{m}=1, c\neq 0, m\neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, cm	$cm\left(\frac{1}{c}+\frac{1}{m}\right)=cm(1)$
Distribute.	$cm\left(\frac{1}{c}\right)+cm\,\frac{1}{m}=cm(1)$
Simplify.	$m+c=cm$
Collect the terms with c to the right.	$m=cm-c$
Factor the expression on the right.	$m=c(m-1)$
To isolate c, divide both sides by m−1.	$\frac{m}{m-1}=\frac{c(m-1)}{m-1}$
Simplify by removing common factors.	$\frac{m}{m-1}=c$

Notice that even though we excluded c=0 and m=0 from the original equation, we must also now state that $m\neq 1$.

Example 13.9.6.35

Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

Answer

$a=\frac{b}{cb-1}$

Example 13.9.6.36

Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y.

Answer

$y=\frac{3x}{6+x}$

Key Concepts

- **Strategy to Solve Equations with Rational Expressions**
  1. Note any value of the variable that would make any denominator zero.
  2. Find the least common denominator of *all* denominators in the equation.
  3. Clear the fractions by multiplying both sides of the equation by the LCD.
  4. Solve the resulting equation.
  5. Check.
     - If any values found in Step 1 are algebraic solutions, discard them.
     - Check any remaining solutions in the original equation.

Glossary

rational equation

A rational equation is two rational expressions connected by an equal sign.

extraneous solution to a rational equation

An extraneous solution to a rational equation is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

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13.9.6E: Exercises

Practice Makes Perfect

Solve Rational Equations

In the following exercises, solve.

Example 13.9.6E.37

$\frac{1}{a}+\frac{2}{5}=\frac{1}{2}$

Answer

10

Example 13.9.6E.38

$\frac{5}{6}+\frac{3}{b}=\frac{1}{3}$

Example 13.9.6E.39

$\frac{5}{2}-\frac{1}{c}=\frac{3}{4}$

Answer

$\frac{4}{7}$

Example 13.9.6E.40

$\frac{6}{3}-\frac{2}{d}=\frac{4}{9}$

Example 13.9.6E.41

$\frac{4}{5}+\frac{1}{4}=\frac{2}{v}$

Answer

$\frac{40}{21}$

Example 13.9.6E.42

$\frac{3}{7}+\frac{2}{3}=\frac{1}{w}$

Example 13.9.6E.43

$\frac{7}{9}+\frac{1}{x}=\frac{2}{3}$

Answer

−9

Example 13.9.6E.44

$\frac{3}{8}+\frac{2}{y}=\frac{1}{4}$

Example 13.9.6E.45

$1-\frac{2}{m}=\frac{8}{m^2}$

Answer

$-2, 4$

Example 13.9.6E.46

$1+\frac{4}{n}=\frac{21}{n^2}$

Example 13.9.6E.47

$1+\frac{9}{p}=-\frac{20}{p^2}$

Answer

$-5, -4$

Example 13.9.6E.48

$1-\frac{7}{q}=-\frac{6}{q^2}$

Example 13.9.6E.49

$\frac{1}{r+3}=\frac{4}{2r}$

Answer

-6

Example 13.9.6E.50

$\frac{3}{t-6}=\frac{1}{t}$

Example 13.9.6E.51

$\frac{5}{3v-2}=\frac{7}{4v}$

Answer

14

Example 13.9.6E.52

$\frac{8}{2w+1}=\frac{3}{w}$

Example 13.9.6E.53

$\frac{3}{x+4}+\frac{7}{x-4}=\frac{8}{x^2-16}$

Answer

$-\frac{4}{5}$

Example 13.9.6E.54

$\frac{5}{y-9}+\frac{1}{y+9}=\frac{18}{y^2-81}$

Example 13.9.6E.55

$\frac{8}{z-10}+\frac{7}{z+10}=\frac{5}{z^2-100}$

Answer

-13

Example 13.9.6E.56

$\frac{9}{a+11}+\frac{6}{a-11}=\frac{7}{a^2-121}$

Example 13.9.6E.57

$\frac{1}{q+4}-\frac{7}{q-2}=1$

Answer

no solution

Example 13.9.6E.58

$\frac{3}{r+10}-\frac{4}{r-4}=1$

Example 13.9.6E.59

$\frac{1}{t+7}-\frac{5}{t-5}=1$

Answer

$-5, -1$

Example 13.9.6E.60

$\frac{2}{s+7}-\frac{3}{s-3}=1$

Example 13.9.6E.61

$\frac{v-10}{v^2-5v+4}=\frac{3}{v-1}-\frac{6}{v-4}$

Answer

no solution

https://math.libretexts.org/@go/page/46307

Example 13.9.6E.62

$\frac{w+8}{w^2-11w+28} = \frac{5}{w-7} + \frac{2}{w-4}$

Example 13.9.6E.63

$\frac{x-10}{x^2+8x+12} = \frac{3}{x+2} + \frac{4}{x+6}$

Answer

no solution

Example 13.9.6E.64

$\frac{y-3}{y^2-4y-5} = \frac{1}{y+1} + \frac{8}{y-5}$

Example 13.9.6E.65

$\frac{z}{16} + \frac{z+2}{4z} = \frac{1}{2z}$

Answer

-4

Example 13.9.6E.66

$\frac{a}{9} + \frac{a+3}{3a} = \frac{1}{a}$

Example 13.9.6E.67

$\frac{b+3}{3b} + \frac{b}{24} = \frac{1}{b}$

Answer

-8

Example 13.9.6E.68

$\frac{c+3}{12c} + \frac{c}{36} = \frac{1}{4c}$

Example 13.9.6E.69

$\frac{d}{d+3} = \frac{18}{d^2-9} + 4$

Answer

2

Example 13.9.6E.70

$\frac{m}{m+5} = \frac{50}{m^2-25} + 6$

Example 13.9.6E.71

$\frac{n}{n+2}=\frac{8}{n^2-4}+3$

Answer

1

Example 13.9.6E.72

$\frac{p}{p+7}=\frac{98}{p^2-49}+8$

Example 13.9.6E.73

$\frac{q}{3q-9}-\frac{3}{4q+12}=\frac{7q^2+6q+63}{24q^2-216}$

Answer

no solution

Example 13.9.6E.74

$\frac{r}{3r-15}-\frac{1}{4r+20}=\frac{3r^2+17r+40}{12r^2-300}$

Example 13.9.6E.75

$\frac{s}{2s+6}-\frac{2}{5s+5}=\frac{5s^2-3s-7}{10s^2+40s+30}$

Answer

no solution

Example 13.9.6E.76

$\frac{t}{6t-12}-\frac{5}{2t+10}=\frac{t^2-23t+70}{12t^2+36t-120}$

Solve a Rational Equation for a Specific Variable

In the following exercises, solve.

Example 13.9.6E.77

$\frac{C}{r}=2\pi$ for r

Answer

$r=\frac{C}{2\pi}$

Example 13.9.6E.78

$\frac{I}{r}=P$ for r

Example 13.9.6E.79

$\frac{V}{h} = lw$ for h

Answer

$h = \frac{v}{lw}$

Example 13.9.6E.80

$\frac{2A}{b} = h$ for b

Example 13.9.6E.81

$\frac{v+3}{w-1} = \frac{1}{2}$ for w

Answer

w=2v+7

Example 13.9.6E.82

$\frac{x+5}{2-y} = \frac{4}{3}$ for y

Example 13.9.6E.83

$a = \frac{b+3}{c-2}$ for c

Answer

$c = \frac{b+3+2a}{a}$

Example 13.9.6E.84

$m = \frac{n}{2-n}$ for n

Example 13.9.6E.85

$\frac{1}{p} + \frac{2}{q} = 4$ for p

Answer

$p = \frac{q}{4q-2}$

Example 13.9.6E.86

$\frac{3}{s} + \frac{1}{t} = 2$ for s

Example 13.9.6E.87

$\frac{2}{v} + \frac{1}{5} = \frac{1}{2}$ for w

Answer

$w = \frac{15v}{10+v}$

Example 13.9.6E.88

$\frac{6}{x}+\frac{2}{3}=\frac{1}{y}$ for y

Example 13.9.6E.89

$\frac{m+3}{n-2}=\frac{4}{5}$ for n

Answer

$n=\frac{5m+23}{m}$

Example 13.9.6E.90

$\frac{E}{c}=m^2$ for c

Example 13.9.6E.91

$\frac{3}{x}-\frac{5}{y}=\frac{1}{4}$ for y

Answer

$y=\frac{20x}{12-x}$

Example 13.9.6E.92

$\frac{R}{T}=W$ for T

Example 13.9.6E.93

$r=\frac{s}{3-t}$ for t

Answer

$t=\frac{3r-s}{r}$

Example 13.9.6E.94

$c=\frac{2}{a}+\frac{b}{5}$ for a

Everyday Math

Example 13.9.6E.95

House Painting Alain can paint a house in 4 days. Spiro would take 7 days to paint the same house. Solve the equation $\frac{1}{4}+\frac{1}{7}=\frac{1}{t}$ for t to find the number of days it would take them to paint the house if they worked together.

Answer

$2\frac{6}{11}$ days

? Example 13.9.6E.96

Boating Ari can drive his boat 18 miles with the current in the same amount of time it takes to drive 10 miles against the current. If the speed of the boat is 7 knots, solve the equation $\frac{18}{7+c} = \frac{10}{7-c}$ for c to find the speed of the current.

Writing Exercises

? Example 13.9.6E.97

Why is there no solution to the equation $\frac{3}{x-2} = \frac{5}{x-2}$

Answer

Answers will vary.

? Example 13.9.6E.98

Pete thinks the equation $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$ has two solutions, y=−6 and y=4. Explain why Pete is wrong.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve rational equations.			
solve rational equations for a specific variable.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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13.9.7: Solve Proportion and Similar Figure Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve proportions
- Solve similar figure applications

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Solve $\frac{n}{3} = 30$.
   If you missed this problem, review Exercise 2.2.25.
2. The perimeter of a triangular window is 23 feet. The lengths of two sides are ten feet and six feet. How long is the third side?
   If you missed this problem, review Example 3.4.2

Solve Proportions

When two rational expressions are equal, the equation relating them is called a *proportion*.

Definition: PROPORTION

A **proportion** is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0$, $d \neq 0$.

The proportion is read “a is to b, as c is to d"

The equation $\frac{1}{2} = \frac{4}{8}$ is a proportion because the two fractions are equal.

The proportion $\frac{1}{2} = \frac{4}{8}$ is read “1 is to 2 as 4 is to 8.”

Proportions are used in many applications to ‘scale up’ quantities. We’ll start with a very simple example so you can see how proportions work. Even if you can figure out the answer to the example right away, make sure you also learn to solve it using proportions.

Suppose a school principal wants to have 1 teacher for 20 students. She could use proportions to find the number of teachers for 60 students. We let x be the number of teachers for 60 students and then set up the proportion:

$$\frac{1 \text{ teacher}}{20 \text{ students}} = \frac{x \text{ teachers}}{60 \text{ students}}$$

We are careful to match the units of the numerators and the units of the denominators—teachers in the numerators, students in the denominators.

Since a proportion is an equation with rational expressions, we will solve proportions the same way we solved equations in Solve Rational Equations. We’ll multiply both sides of the equation by the LCD to clear the fractions and then solve the resulting equation.

	$\frac{1}{20} = \frac{x}{60}$
Multiply both sides by the LCD, 60.	$\frac{1}{20} \cdot 60 = \frac{x}{60} \cdot 60$
Simplify.	$3 = x$

	The principal needs 3 teachers for 60 students.

Now we'll do a few examples of solving numerical proportions without any units. Then we will solve applications using proportions.

Example 13.9.7.1

$\frac{x}{63} = \frac{4}{7}$.

Solution

		$\frac{x}{63} = \frac{4}{7}$
To isolate x, multiply both sides by the LCD, 63.		$63\left(\frac{x}{63}\right) = 63\left(\frac{4}{7}\right)$
Simplify.		$x = \frac{9 \cdot 7 \cdot 4}{7}$
Divide the common factors.		$x = 36$
Check. To check our answer, we substitute into the original proportion.		
	$\frac{x}{63} = \frac{4}{7}$	
Substitute $x = 36$.	$\frac{36}{63} \stackrel{?}{=} \frac{4}{7}$	
Show common factors.	$\frac{4 \cdot 9}{7 \cdot 9} \stackrel{?}{=} \frac{4}{7}$	
Simplify.	$\frac{4}{7} = \frac{4}{7}$ ✓	

Try It 13.9.7.1

$\frac{n}{84} = \frac{11}{12}$.

Answer

77

Try It 13.9.7.2

$\frac{y}{96} = \frac{13}{12}$.

Answer

104

Example 13.9.7.2

$\frac{144}{a} = \frac{9}{4}$.

Solution

		$\frac{144}{a}=\frac{9}{4}$
Multiply both sides by the LCD.		$\frac{144}{a}\cdot 4a=\frac{9}{4}\cdot 4a$
Remove common factors on each side.		$4\cdot 144=a\cdot 9$
Simplify.		$576=9a$
Divide both sides by 9.		$\frac{576}{9}=\frac{9a}{9}$
Simplify.		$64=a$
Check.		
	$\frac{144}{a}=\frac{9}{4}$	
Substitute $a=64$.	$\frac{144}{64}\stackrel{?}{=}\frac{9}{4}$	
Show common factors.	$\frac{9\cdot 16}{4\cdot 16}\stackrel{?}{=}\frac{9}{4}$	
Simplify.	$\frac{9}{4}=\frac{9}{4}$ ✓	

Try It 13.9.7.3

$\frac{91}{b}=\frac{7}{5}$.

Answer

65

Try It 13.9.7.4

$\frac{39}{c}=\frac{13}{8}$.

Answer

24

Example 13.9.7.3

$\frac{n}{n+14}=\frac{5}{7}$.

Solution

		$\frac{n}{n+14}=\frac{5}{7}$
Multiply both sides by the LCD.		$7(n+14)\left(\frac{n}{n+14}\right)=7(n+14)\left(\frac{5}{7}\right)$
Remove common factors on each side.		$7n=5(n+14)$
Simplify.		$7n=5n+70$
Solve for n.		$2n=70$
		$n=35$

Check.		
	$\frac{n}{n+14}=\frac{5}{7}$	
Substitute $n=35$.	$\frac{35}{35+14}\stackrel{?}{=}\frac{5}{7}$	
Simplify.	$\frac{35}{49}\stackrel{?}{=}\frac{5}{7}$	
Show common factors.	$\frac{5\cdot 7}{7\cdot 7}\stackrel{?}{=}\frac{5}{7}$	
Simplify.	$\frac{5}{7}=\frac{5}{7}$ ✓	

Try It 13.9.7.5

$\frac{y}{y+55}=\frac{3}{8}$.

Answer

33

Try It 13.9.7.6

$\frac{z}{z-84}=-\frac{1}{5}$.

Answer

14

Example 13.9.7.4

$\frac{p+12}{9}=\frac{p-12}{6}$.

Solution

		$\frac{p+12}{9}=\frac{p-12}{6}$
Multiply both sides by the LCD, 18.		$18\cdot\frac{p+12}{9}=18\cdot\frac{p+12}{6}$
Simplify.		$2(p+12)=3(p-12)$
Distribute.		$2p+24=3p-36$
Solve fpr p.		$60=p$
Check.		
	$\frac{p+12}{9}=\frac{p-12}{6}$	
Substitute $p=60$.	$\frac{60+12}{9}\stackrel{?}{=}\frac{60-12}{6}$	
Simplify.	$\frac{72}{9}\stackrel{?}{=}\frac{48}{6}$	
Divide.	$8=8$ ✓	

Try It 13.9.7.7

$\frac{v+30}{8} = \frac{v+66}{12}$.

Answer

42

Try It 13.9.7.8

$\frac{2x+15}{9} = \frac{7x+3}{15}$.

Answer

6

To solve applications with proportions, we will follow our usual strategy for solving applications. But when we set up the proportion, we must make sure to have the units correct—the units in the numerators must match and the units in the denominators must match.

Example 13.9.7.5

When pediatricians prescribe acetaminophen to children, they prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how many milliliters of acetaminophen will her doctor prescribe?

Solution

Identify what we are asked to find, and choose a variable to represent it.	How many ml of acetaminophen will the doctor prescribe?
	Let a=ml of acetaminophen.
Write a sentence that gives the information to find it.	If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?
Translate into a proportion–be careful of the units. $\frac{5}{25} = \frac{a}{80}$	
Multiply both sides by the LCD, 400.	$400\left(\frac{5}{25}\right) = 400\left(\frac{a}{80}\right)$
Remove common factors on each side.	$25 \cdot 16\left(\frac{5}{25}\right) = 80 \cdot 5\left(\frac{a}{80}\right)$
Simplify, but don't multiply on the left. Notice what the next step will be.	$16 \cdot 5 = 5a$
Solve for a.	$\frac{16 \cdot 5}{5} = \frac{5a}{5}$
Check.	$16 = a$
Is the answer reasonable?	
Yes, since 80 is about 3 times 25, the medicine should be about 3 times 5. So 16 ml makes sense.	

Substitute $a = 16$ in the original proportion. $\frac{5}{25} = \frac{a}{80}$ $\frac{5}{25} \stackrel{?}{=} \frac{16}{80}$ $\frac{1}{5} = \frac{1}{5}$ ✓	
Write a complete sentence.	The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

Try It 13.9.7.9

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

Answer

12 ml

Try It 13.9.7.10

For every 1 kilogram (kg) of a child's weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

Answer

180 ml

Example 13.9.7.6

A 16-ounce iced caramel macchiato has 230 calories. How many calories are there in a 24-ounce iced caramel macchiato?

Solution

Identify what we are asked to find, and choose a variable to represent it.	How many calories are in a 24 ounce iced caramel macchiato?
	Let c=calories in 24 ounces.
Write a sentence that gives the information to find it.	If there are 230 calories in 16 ounces, then how many calories are in 24 ounces?
Translate into a proportion–be careful of the units. $\frac{230}{16} = \frac{c}{24}$	
Multiply both sides by the LCD, 48.	$48\left(\frac{230}{16}\right) = 48\left(\frac{c}{24}\right)$
Remove common factors on each side.	$\cancel{16} \cdot 3\left(\frac{230}{\cancel{16}}\right) = \cancel{24} \cdot 2\left(\frac{c}{\cancel{24}}\right)$
Simplify.	$690 = 2c$
Solve for c.	$\frac{690}{2} = \frac{2c}{2}$
	$345 = c$
Check.	

Is the answer reasonable?	
Yes, 345 calories for 24 ounces is more than 290 calories for 16 ounces, but not too much more.	
Substitute $c = 345$ in the original proportion. $\frac{230}{16} = \frac{c}{24}$ $\frac{230}{16} \stackrel{?}{=} \frac{345}{24}$ $\frac{115}{8} = \frac{115}{8}$ ✓	
Write a complete sentence.	There are 345 calories in a 24-ounce iced caramel macchiato.

Try It 13.9.7.11

At a fast-food restaurant, a 22-ounce chocolate shake has 850 calories. How many calories are in their 12-ounce chocolate shake? Round your answer to nearest whole number.

Answer

464 calories

Try It 13.9.7.12

Yaneli loves Starburst candies, but wants to keep her snacks to 100 calories. If the candies have 160 calories for 8 pieces, how many pieces can she have in her snack?

Answer

5 pieces

Example 13.9.7.7

Josiah went to Mexico for spring break and changed $325 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 12.54 Mexican pesos. How many Mexican pesos did he get for his trip?

Solution

What are you asked to find?	How many Mexican pesos did Josiah get?
Assign a variable.	Let p=the number of Mexican pesos.
Write a sentence that gives the information to find it.	If $1 US is equal to 12.54 Mexican pesos, then $325 is how many pesos?
Translate into a proportion–be careful of the units. $\frac{1}{12.54} = \frac{325}{p}$	
Multiply both sides by the LCD, 12.54p.	$12.54p\left(\frac{1}{12.54}\right) = 12.54p\left(\frac{325}{p}\right)$
Remove common factors on each side.	$\cancel{12.54p}\left(\frac{1}{\cancel{12.54}}\right) = 12.54\cancel{p}\left(\frac{325}{\cancel{p}}\right)$
Simplify.	$p = 4075.5$
Check.	
Is the answer reasonable?	

Yes, $100 would be 1,254 pesos. $325 is a little more than 3 times this amount, so our answer of 4075.5 pesos makes sense.	
Substitute $p = 4075.5$ in the original proportion. Use a calculator. $\frac{1}{12.54} = \frac{325}{p}$ $\frac{1}{12.54} \stackrel{?}{=} \frac{325}{4075.5}$ $0.07874... = 0.07874...$ ✓	
Write a complete sentence.	Josiah got 4075.5 pesos for his spring break trip.

? Try It 13.9.7.13

Yurianna is going to Europe and wants to change $800 dollars into Euros. At the current exchange rate, $1 US is equal to 0.738 Euro. How many Euros will she have for her trip?

Answer

590.4 Euros

? Try It 13.9.7.14

Corey and Nicole are traveling to Japan and need to exchange $600 into Japanese yen. If each dollar is 94.1 yen, how many yen will they get?

Answer

56,460 yen

In the example above, we related the number of pesos to the number of dollars by using a proportion. We could say the number of pesos *is proportional to* the number of dollars. If two quantities are related by a proportion, we say that they are proportional.

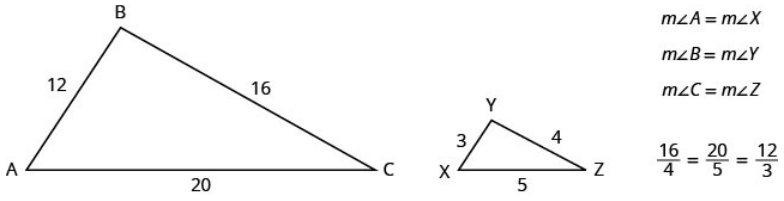
Solve Similar Figure Applications

When you shrink or enlarge a photo on a phone or tablet, figure out a distance on a map, or use a pattern to build a bookcase or sew a dress, you are working with **similar figures**. If two figures have exactly the same shape, but different sizes, they are said to be *similar.* One is a scale model of the other. All their corresponding angles have the same measures and their corresponding sides are in the same ratio.

Definition: SIMILAR FIGURES

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides are in the same ratio.

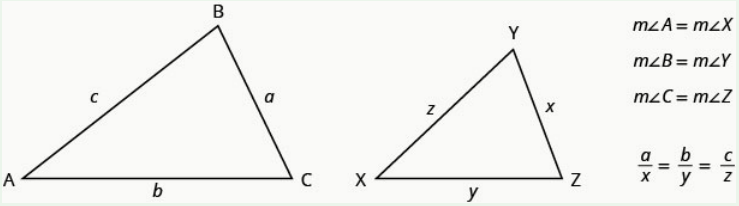
For example, the two triangles in the figure below are similar. Each side of ΔABC is 4 times the length of the corresponding side of ΔXYZ.



This is summed up in the Property of Similar Triangles.

Definition: PROPERTY OF SIMILAR TRIANGLES

If ΔABC is similar to ΔXYZ



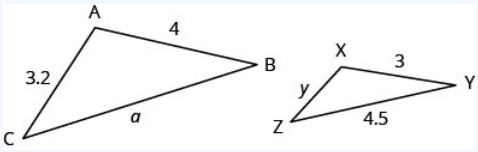
To solve applications with similar figures we will follow the Problem-Solving Strategy for Geometry Applications we used earlier.

Definition: SOLVE GEOMETRY APPLICATIONS.

1. **Read** the problem and make all the words and ideas are understood. Draw the figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Example 13.9.7.8

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure.



Find the length of the sides of the similar triangles.

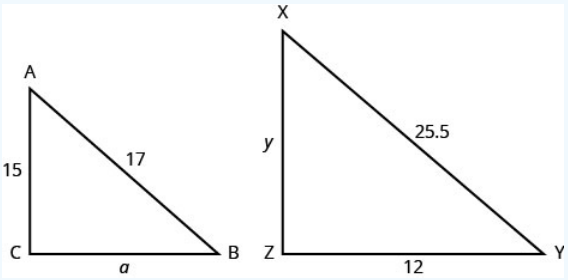
Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	Figure is given.
Step 2. Identify what we are looking for.	the length of the sides of similar triangles.
Step 3. Name the variables.	Let a= length of the third side of ΔABC. y= length of the third side of ΔXYZ.
Step 4. Translate.	Since the triangles are similar, the corresponding sides are proportional.
We need to write an equation that compares the side we are looking for to a known ratio. Since the side $AB = 4$ corresponds to the side $XY = 3$ we know $\frac{AB}{XY} = \frac{4}{3}$. So we write equations with $\frac{AB}{XY}$ to find the sides we are looking for. Be careful to match up corresponding sides correctly.	$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$. To find *a*: $\frac{AB}{XY} = \frac{BC}{YZ}$ (sides of large triangle → sides of small triangle) To find *y*: $\frac{AB}{XY} = \frac{AC}{XZ}$
Substitute.	$\frac{4}{3} = \frac{a}{4.5}$ $\frac{4}{3} = \frac{3.2}{y}$

Step 5. Solve the equation.	$3a = 4(4.5)$ Figure_08_07_030f_img_new.jpg" />
	$a = 6$ $y = 2.4$
Step 6. Check.	
Step 7. Answer the question.	The third side of ΔABC is 6 and the third side of ΔXYZ is 2.4.

? Try It 13.9.7.15

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure.



Find the length of side a

Answer

8

? Try It 13.9.7.16

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure.

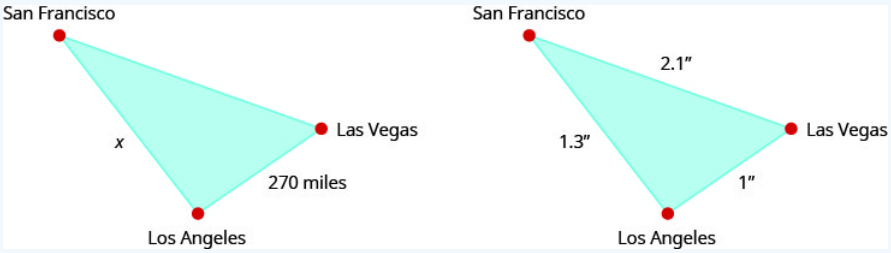
Answer

22.5

The next example shows how similar triangles are used with maps.

? Example 13.9.7.9

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle whose sides are shown in the figure below. If the actual distance from Los Angeles to Las Vegas is 270 miles find the distance from Los Angeles to San Francisco.



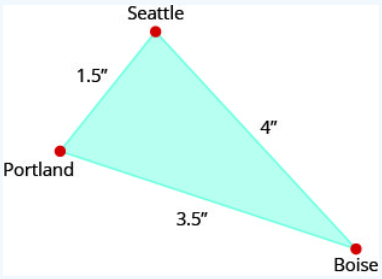
Solution

Read the problem. Draw the figures and label with the given information.	The figures are shown above.
Identify what we are looking for.	The actual distance from Los Angeles to San Francisco.

Name the variables.	Let x= distance from Los Angeles to San Francisco.
Translate into an equation. Since the triangles are similar, the corresponding sides are proportional. We'll make the numerators "miles" and the denominators "inches."	$\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$
	$1.3\left(\frac{x}{1.3}\right) = 1.3\left(\frac{270}{1}\right)$
Solve the equation.	$x = 351$
Check.	
On the map, the distance from Los Angeles to San Francisco is more than the distance from Los Angeles to Las Vegas. Since 351 is more than 270 the answer makes sense.	
Check $x = 351$ in the original proportion. Use a calculator. $\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{351 \text{ miles}}{1.3 \text{ inches}} \stackrel{?}{=} \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{270 \text{ miles}}{1 \text{ inch}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ ✓	
Answer the question.	The distance from Los Angeles to San Francisco is 351 miles.

Try It 13.9.7.17

On the map, Seattle, Portland, and Boise form a triangle whose sides are shown in the figure below. If the actual distance from Seattle to Boise is 400 miles, find the distance from Seattle to Portland.



Answer

150 miles

Try It 13.9.7.18

Using the map above, find the distance from Portland to Boise.

Answer

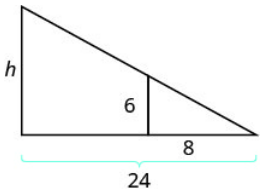
350 miles

We can use **similar figures** to find heights that we cannot directly measure.

Example 13.9.7.10

Tyler is 6 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a tree was 24 feet long. Find the height of the tree.

Solution

Read the problem and draw a figure.	
We are looking for h, the height of the tree.	
We will use similar triangles to write an equation.	
The small triangle is similar to the large triangle.	$\frac{h}{24}=\frac{6}{8}$
Solve the proportion.	$24\left(\frac{6}{8}\right)=24\left(\frac{h}{24}\right)$
Simplify.	$18=h$
Check.	
Tyler's height is less than his shadow's length so it makes sense that the tree's height is less than the length of its shadow.	
Check $h=18$ in the original proportion. $\frac{6}{8}=\frac{h}{24}$ $\frac{6}{8}\stackrel{?}{=}\frac{18}{24}$ $\frac{3}{4}=\frac{3}{4}$ ✓	

Try It 13.9.7.19

A telephone pole casts a shadow that is 50 feet long. Nearby, an 8 foot tall traffic sign casts a shadow that is 10 feet long. How tall is the telephone pole?

Answer

40 feet

Try It 13.9.7.20

A pine tree casts a shadow of 80 feet next to a 30-foot tall building which casts a 40 feet shadow. How tall is the pine tree?

Answer

60 feet

Key Concepts

- **Property of Similar Triangles**
  - If ΔABC is similar to ΔXYZ
- **Problem Solving Strategy for Geometry Applications**

1. **Read** the problem and make sure all the words and ideas are understood. Draw the figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Glossary

proportion

A proportion is an equation of the form $\frac{a}{b}=\frac{c}{d}, where \(b \neq 0$, $d \neq 0$. The proportion is read “a is to b c is to d."

similar figures

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides are in the same ratio.

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13.9.7E: Exercises

Practice Makes Perfect

Solve Proportions

In the following exercises, solve.

Example 13.9.7E.31

$\frac{x}{56}=\frac{7}{8}$

Answer

49

Example 13.9.7E.32

$\frac{n}{91}=\frac{8}{13}$

Example 13.9.7E.32

$\frac{49}{63}=\frac{z}{9}$

Answer

7

Example 13.9.7E.33

$\frac{56}{72}=\frac{y}{9}$

Example 13.9.7E.34

$\frac{5}{a}=\frac{65}{11}$

Answer

9

Example 13.9.7E.35

$\frac{4}{b}=\frac{64}{144}$

Example 13.9.7E.36

$\frac{98}{154}=-\frac{7}{p}$

Answer

-11

Example 13.9.7E.37

$$\frac{72}{156} = -\frac{6}{q}$$

Example 13.9.7E.38

$$\frac{a}{-8} = \frac{-42}{48}$$

Answer

7

Example 13.9.7E.39

$$\frac{b}{-7} = \frac{-30}{42}$$

Example 13.9.7E.40

$$\frac{2.7}{j} = \frac{0.9}{0.2}$$

Answer

0.6

Example 13.9.7E.41

$$\frac{2.8}{k} = \frac{2.1}{1.5}$$

Example 13.9.7E.42

$$\frac{a}{a+12} = \frac{4}{7}$$

Answer

16

Example 13.9.7E.43

$$\frac{b}{b-16} = \frac{11}{9}$$

Example 13.9.7E.44

$$\frac{c}{c-104} = -\frac{5}{8}$$

Answer

$-\frac{5}{8}$

https://math.libretexts.org/@go/page/46309

Example 13.9.7E.45

$$\frac{d}{d-48}=-\frac{13}{3}$$

Example 13.9.7E.46

$$\frac{m+90}{25}=\frac{m+30}{15}$$

Answer

60

Example 13.9.7E.47

$$\frac{n+10}{4}=\frac{40-n}{6}$$

Example 13.9.7E.48

$$\frac{2p+4}{8}=\frac{p+18}{6}$$

Answer

30

Example 13.9.7E.49

$$\frac{q-2}{2}=\frac{2q-7}{18}$$

Example 13.9.7E.50

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?

Answer

9 ml

Example 13.9.7E.51

Brianna, who weighs 6 kg, just received her shots and needs a pain killer. The pain killer is prescribed for children at 15 milligrams (mg) for every 1 kilogram (kg) of the child's weight. How many milligrams will the doctor prescribe?

Example 13.9.7E.52

A veterinarian prescribed Sunny, a 65 pound dog, an antibacterial medicine in case an infection emerges after her teeth were cleaned. If the dosage is 5 mg for every pound, how much medicine was Sunny given?

Answer

325 mg

Example 13.9.7E.53

Belle, a 13 pound cat, is suffering from joint pain. How much medicine should the veterinarian prescribe if the dosage is 1.8 mg per pound?

Example 13.9.7E.54

A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?

Answer

159 calories

Example 13.9.7E.55

One 12 ounce can of soda has 150 calories. If Josiah drinks the big 32 ounce size from the local mini-mart, how many calories does he get?

Example 13.9.7E.56

A new 7 ounce lemon ice drink is advertised for having only 140 calories. How many ounces could Sally drink if she wanted to drink just 100 calories?

Answer

5 oz

Example 13.9.7E.57

Reese loves to drink healthy green smoothies. A 16 ounce serving of smoothie has 170 calories. Reese drinks 24 ounces of these smoothies in one day. How many calories of smoothie is he consuming in one day?

Example 13.9.7E.58

Janice is traveling to Canada and will change $250 US dollars into Canadian dollars. At the current exchange rate, $1 US is equal to $1.01 Canadian. How many Canadian dollars will she get for her trip?

Answer

252.5 Canadian dollars

Example 13.9.7E.59

Todd is traveling to Mexico and needs to exchange $450 into Mexican pesos. If each dollar is worth 12.29 pesos, how many pesos will he get for his trip?

Example 13.9.7E.60

Steve changed $600 into 480 Euros. How many Euros did he receive for each US dollar?

Answer

0.80 Euros

Example 13.9.7E.61

Martha changed $350 US into 385 Australian dollars. How many Australian dollars did she receive for each US dollar?

Example 13.9.7E.62

When traveling to Great Britain, Bethany exchanged her $900 into 570 British pounds. How many pounds did she receive for each American dollar?

Answer

0.63 British pounds

Example 13.9.7E.63

A missionary commissioned to South Africa had to exchange his $500 for the South African Rand which is worth 12.63 for every dollar. How many Rand did he have after the exchange?

Example 13.9.7E.64

Ronald needs a morning breakfast drink that will give him at least 390 calories. Orange juice has 130 calories in one cup. How many cups does he need to drink to reach his calorie goal?

Answer

3 cups

Example 13.9.7E.65

Sarah drinks a 32-ounce energy drink containing 80 calories per 12 ounce. How many calories did she drink?

Example 13.9.7E.66

Elizabeth is returning to the United States from Canada. She changes the remaining 300 Canadian dollars she has to $230.05 in American dollars. What was $1 worth in Canadian dollars?

Answer

1.30 Canadian dollars

Example 13.9.7E.67

Ben needs to convert $1000 to the Japanese Yen. One American dollar is worth 123.3 Yen. How much Yen will he have?

Example 13.9.7E.68

A golden retriever weighing 85 pounds has diarrhea. His medicine is prescribed as 1 teaspoon per 5 pounds. How much medicine should he be given?

Answer

17 tsp

Example 13.9.7E.69

Five-year-old Lacy was stung by a bee. The dosage for the anti-itch liquid is 150 mg for her weight of 40 pounds. What is the dosage per pound?

Example 13.9.7E.70

Karen eats $\frac{1}{2}$ cup of oatmeal that counts for 2 points on her weight loss program. Her husband, Joe, can have 3 points of oatmeal for breakfast. How much oatmeal can he have?

Answer

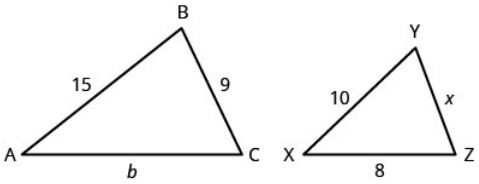
$\frac{3}{4}$ cup

Example 13.9.7E.71

An oatmeal cookie recipe calls for $\frac{1}{2}$ cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?

Solve Similar Figure Applications

In the following exercises, ΔABC is similar to ΔXYZ



Example 13.9.7E.72

side *b*

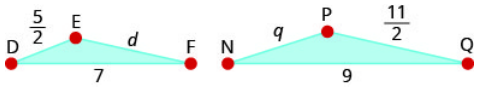
Answer

12

Example 13.9.7E.73

side *x*

In the following exercises, ΔDEF is similar to ΔNPQ



Example 13.9.7E.74

Find the length of side *d*.

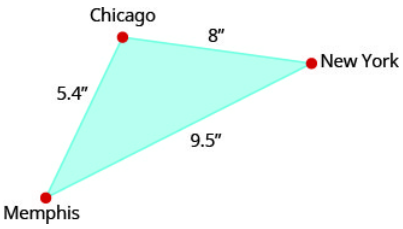
Answer

$\frac{77}{18}$

Example 13.9.7E.75

Find the length of side q.

In the following two exercises, use the map shown. On the map, New York City, Chicago, and Memphis form a triangle whose sides are shown in the figure below. The actual distance from New York to Chicago is 800 miles.



Example 13.9.7E.76

Find the actual distance from New York to Memphis.

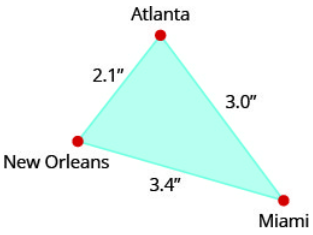
Answer

950 miles

Example 13.9.7E.77

Find the actual distance from Chicago to Memphis.

In the following two exercises, use the map shown. On the map, Atlanta, Miami, and New Orleans form a triangle whose sides are shown in the figure below. The actual distance from Atlanta to New Orleans is 420 miles.



Example 13.9.7E.78

Find the actual distance from New Orleans to Miami.

Answer

680 miles

Example 13.9.7E.79

Find the actual distance from Atlanta to Miami.

Example 13.9.7E.80

A 2 foot tall dog casts a 3 foot shadow at the same time a cat casts a one foot shadow. How tall is the cat?

Answer

$\frac{2}{3}$ foot (8in)

Example 13.9.7E.81

Larry and Tom were standing next to each other in the backyard when Tom challenged Larry to guess how tall he was. Larry knew his own height is 6.5 feet and when they measured their shadows, Larry's shadow was 8 feet and Tom's was 7.75 feet long. What is Tom's height?

Example 13.9.7E.82

The tower portion of a windmill is 212 feet tall. A six foot tall person standing next to the tower casts a seven foot shadow. How long is the windmill's shadow?

Answer

247.3 feet

Example 13.9.7E.83

The height of the Statue of Liberty is 305 feet. Nicole, who is standing next to the statue, casts a 6 foot shadow and she is 5 feet tall. How long should the shadow of the statue be?

Everyday Math

Example 13.9.7E.84

Heart Rate At the gym, Carol takes her pulse for 10 seconds and counts 19 beats.

1. How many beats per minute is this?
2. Has Carol met her target heart rate of 140 beats per minute?

Answer

1. 114 beats per minute
2. no

Example 13.9.7E.85

Heart Rate Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds.

1. How many beats per minute is this?
2. Has Kevin met his target heart rate?

Example 13.9.7E.86

Cost of a Road Trip Jesse's car gets 30 miles per gallon of gas.

1. If Las Vegas is 285 miles away, how many gallons of gas are needed to get there and then home?
2. If gas is $3.09 per gallon, what is the total cost of the gas for the trip?

Answer

1. 19 gallons
2. $58.71

Example 13.9.7E.87

Cost of a Road Trip Danny wants to drive to Phoenix to see his grandfather. Phoenix is 370 miles from Danny's home and his car gets 18.5 miles per gallon.

1. How many gallons of gas will Danny need to get to and from Phoenix?
2. If gas is $3.19 per gallon, what is the total cost for the gas to drive to see his grandfather?

Example 13.9.7E.88

Lawn Fertilizer Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil's lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?

Answer

4 bags

Example 13.9.7E.89

House Paint April wants to paint the exterior of her house. One gallon of paint covers about 350 square feet, and the exterior of the house measures approximately 2000 square feet. How many gallons of paint will she have to buy?

Example 13.9.7E.90

Cooking Natalia's pasta recipe calls for 2 pounds of pasta for 1 quart of sauce. How many pounds of pasta should Natalia cook if she has 2.5 quarts of sauce?

Answer

5

Example 13.9.7E.91

Heating Oil A 275 gallon oil tank costs $400 to fill. How much would it cost to fill a 180 gallon oil tank?

Writing Exercises

Example 13.9.7E.92

Marisol solves the proportion $\frac{144}{a}=\frac{9}{4}$ by 'cross multiplying', so her first step looks like 4·144=9·a. Explain how this differs from the method of solution shown in Example.

Answer

Answers will vary.

Example 13.9.7E.93

Find a printed map and then write and solve an application problem similar to Example.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve proportions.			
solve similar figure applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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13.9.8: Solve Uniform Motion and Work Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve uniform motion applications
- Solve work applications

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. An express train and a local bus leave Chicago to travel to Champaign. The express bus can make the trip in 2 hours and the local bus takes 5 hours for the trip. The speed of the express bus is 42 miles per hour faster than the speed of the local bus. Find the speed of the local bus.
   If you missed this problem, review [link].
2. Solve $\frac{1}{3}x+\frac{1}{4}x=\frac{5}{6}$.
   If you missed this problem, review [link].
3. Solve: $18t^2-30=-33t$.
   If you missed this problem, review [link].

Solve Uniform Motion Applications

We have solved uniform motion problems using the formula D=rt in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate •	Time =	Distance

The formula D=rt assumes we know *r* and *t* and use them to find D. If we know D and *r* and need to find *t*, we would solve the equation for *t* and get the formula $t=\frac{D}{r}$

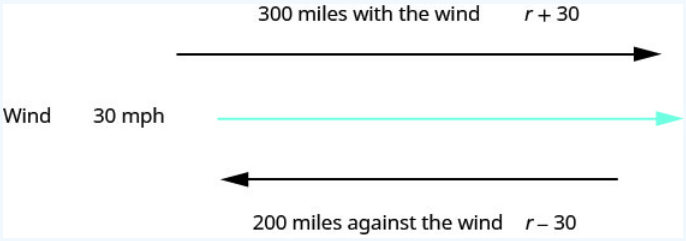
We have also explained how flying with or against a current affects the speed of a vehicle. We will revisit that idea in the next example.

Example 13.9.8.1

An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

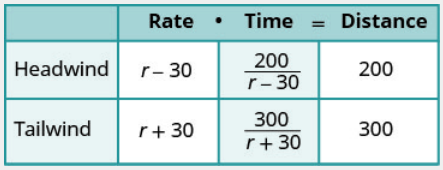
Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the airplane.	Let r= the speed of the airplane.

When the plane flies with the wind, the wind increases its speed and the rate is r+30.	
When the plane flies against the wind, the wind decreases its speed and the rate is r−30.	
Write in the rates. Write in the distances. Since D=r·t , we solve for t and get $\frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	
We know the times are equal and so we write our equation.	$\frac{200}{r-30}=\frac{300}{r+30}$
We multiply both sides by the LCD. 200(r+30)=300(r−30)	
Simplify.	(r+30)(200)=(r−30)(300)
	200r+6000=300r−9000
Solve.	15000=100r 150=r
Check.	
Is 150 mph a reasonable speed for an airplane? Yes. If the plane is traveling 150 mph and the wind is 30 mph:	
Tailwind 150+30=180 mph $\frac{300}{180}=\frac{5}{3}$ hours	
Headwind 150−30=120 mph $\frac{200}{120}=\frac{5}{3}$ hours	
The times are equal, so it checks.	The plane was traveling 150 mph.

? Try It 13.9.8.2

Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link's biking speed?

Answer

15 mph

? Try It 13.9.8.3

Judy can sail her boat 5 miles into a 7 mph headwind in the same amount of time she can sail 12 miles with a 7 mph tailwind. What is the speed of Judy's boat without a wind?

Answer

17 mph

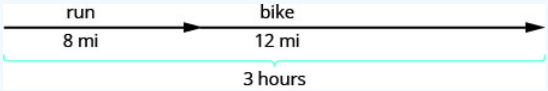
In the next example, we will know the total time resulting from travelling different distances at different speeds.

Example 13.9.8.4

Jazmine trained for 3 hours on Saturday. She ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. What is her running speed?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Jazmine's running speed.	Let r= Jazmine's running sp
Her biking speed is 4 miles faster than her running speed.	r+4= her biking speed
The distances are given, enter them into the chart.	
Since D=r·t , we solve for t and get $t = \frac{D}{r}$ We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th></th><th>Rate • Time = Distance</th></tr><tr><td>Run</td><td>r $\frac{8}{r}$ 8</td></tr><tr><td>Bike</td><td>$r+4$ $\frac{24}{r+4}$ 24</td></tr><tr><td></td><td>3</td></tr></table>
Write a word sentence.	Her time plus the time biking
Translate the sentence to get the equation.	$\frac{8}{r} + \frac{24}{r+4} = 3$
Solve.	$r(r+4)(\frac{8}{r} + \frac{24}{r+4}) = 3r(r+4)$ $8(r+4) + 24r = 3r^2 + 12r$ $8r + 32 + 24r = 3r^2 + 12r$ $32 + 32r = 3r^2 + 12r$ $0 = 3r^2 - 20r - 32$ $0 = (3r+4)(r-8)$
	(3r+4)=0, (r−8)=0
	$r = -\frac{4}{3}$, r=8
Check. r=8	
A negative speed does not make sense in this problem, so r=8 is the solution.	
Is 8 mph a reasonable running speed? Yes.	
Run 8 mph, $\frac{8miles}{8mph} = 1hour$ Bike 12 mph, $\frac{24miles}{12mph} = 2hours$ Total 3 hours. Jazmine's running speed is 8 mph.	

Try It 13.9.8.5

Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis' speed going uphill and his speed going downhill?

Answer

10 mph

Try It 13.9.8.6

Tony drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the country roads?

Answer

50 mph

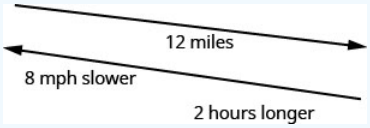
Once again, we will use the uniform motion formula solved for the variable *t*.

Example 13.9.8.7

Hamilton rode his bike downhill 12 miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was 8 miles per hour slower than his downhill speed. It took him 2 hours longer to get home than it took him to get to the ocean. Find Hamilton's downhill speed.

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Hamilton's downhill speed.	Let r= Hamilton's downhill speed.
His uphill speed is 8 miles per hour slower. Enter the rates into the chart.	h−8= Hamilton's uphill speed
The distance is the same in both directions, 12 miles. Since D=r·t , we solve for *t* and get $t = \frac{D}{r}$ We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th></th><th>Rate • Time = Distance</th></tr><tr><td>Downhill</td><td>h	$\frac{12}{h}$	12</td></tr><tr><td>Uphill</td><td>$h-8$	$\frac{12}{h-8}$	12</td></tr></table>
Write a word sentence about the time.	He took 2 hours longer uphill than downhill. The uphill time is 2 more than the downhill time.
Translate the sentence to get the equation. Solve.	$\frac{12}{h-8} = \frac{12}{h} + 2$ $h(h-8)(\frac{12}{h-8}) = h(h-8)(\frac{12}{h} + 2)$ 12h=12(h−8)+2h(h−8) $12h = 12h - 96 + 2h^2 - 16h$ $0 = 2h^2 - 16h - 96$ $0 = 2(h^2 - 8h - 48)$ 0=2(h−12)(h+4) h−12=0, h+4=0 h=12, h=−4
Check. Is 12 mph a reasonable speed for biking downhill? Yes.	
Downhill 12 mph, $\frac{12miles}{12mph} = 1hour$	

Uphill 12−8=4 mph $\frac{12miles}{4mph} = 3hours$	
	The uphill time is 2 hours more than the downhill time. Hamilton's downhill speed is 12 mph.

? Try It 13.9.8.8

Kayla rode her bike 75 miles home from college one weekend and then rode the bus back to college. It took her 2 hours less to ride back to college on the bus than it took her to ride home on her bike, and the average speed of the bus was 10 miles per hour faster than Kayla's biking speed. Find Kayla's biking speed.

Answer

15 mph

? Try It 13.9.8.9

Victoria jogs 12 miles to the park along a flat trail and then returns by jogging on an 18 mile hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes her two hours longer. Find her rate of jogging on the flat trail.

Answer

6 mph

Solve Work Applications

Suppose Pete can paint a room in 10 hours. If he works at a steady pace, in 1 hour he would paint $\frac{1}{10}$ of the room. If Alicia would take 8 hours to paint the same room, then in 1 hour she would paint $\frac{1}{8}$ of the room. How long would it take Pete and Alicia to paint the room if they worked together (and didn't interfere with each other's progress)?

This is a typical 'work' application. There are three quantities involved here – the time it would take each of the two people to do the job alone and the time it would take for them to do the job together.

Let's get back to Pete and Alicia painting the room. We will let t be the number of hours it would take them to paint the room together. So in 1 hour working together they have completed $\frac{1}{t}$ of the job.

In one hour Pete did $\frac{1}{10}$ of the job. Alicia did $\frac{1}{8}$ of the job. And together they did $\frac{1}{t}$ of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take them if they worked together, we solve for t.

	Pete's part + Alicia's part = part of total $\frac{1}{10} + \frac{1}{8} = \frac{1}{t}$
	$\frac{1}{10}+\frac{1}{8}=\frac{1}{t}$
Multiply by the LCD, 40t	$40t\left(\frac{1}{10}+\frac{1}{8}\right)=40t\left(\frac{1}{t}\right)$
Distribute.	$40t\cdot\frac{1}{10}+40t\cdot\frac{1}{8}=40t\left(\frac{1}{t}\right)$
Simplify and solve.	$4t+5t=40$

	$9t = 40$
	$t = \frac{40}{9}$
We'll write as a mixed number so that we can convert it to hours and minutes.	$t = 4\frac{4}{9}$ hours
Remember, 1 hour = 60 minutes.	$t = 4$ hours $+ \frac{4}{9}(60$ minutes$)$
Multiply, and then round to the nearest minute.	$t = 4$ hours $+ 27$ minutes
	It would take Pete and Alica about 4 hours and 27 minutes to paint the room.

Keep in mind, it should take less time for two people to complete a job working together than for either person to do it alone.

Example 13.9.8.10

The weekly gossip magazine has a big story about the Princess' baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours to do the job and Press #2 takes 12 hours to do the job. How long will it take the printer to get the magazine printed with both presses running together?

Solution

This is a work problem. A chart will help us organize the information.

Let t= the number of hours needed to complete the job together.	
Enter the hours per job for Press #1, Press #2 and when they work together. If a job on Press #1 takes 6 hours, then in 1 hour $\frac{1}{6}$ of the job is completed. Similarly find the part of the job completed/hours for Press #2 and when they both work together.	<table><tr><th></th><th>Number of hours to complete the job</th><th>Part of job completed/ hour</th></tr><tr><td>Press #1</td><td>6</td><td>$\frac{1}{6}$</td></tr><tr><td>Press #2</td><td>12</td><td>$\frac{1}{12}$</td></tr><tr><td>Together</td><td>t</td><td>$\frac{1}{t}$</td></tr></table>
Write a word sentence.	
The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.	
Translate to an equation.	Work completed by Press #1 + Press #2 = Together $\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$
Solve.	$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$
Multiply by the LCD, 12t.	$12t\left(\frac{1}{6} + \frac{1}{12}\right) = 12t\left(\frac{1}{t}\right)$
Simplify.	$2t + t = 12$
	$3t = 12$
	$t = 4$
	When both presses are running it takes 4 hours to do the job.

Try It 13.9.8.11

One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course?

Answer

2 hours and 24 minutes

Try It 13.9.8.12

Carrie can weed the garden in 7 hours, while her mother can do it in 3. How long will it take the two of them working together?

Answer

2 hours and 6 minutes

Example 13.9.8.13

Corey can shovel all the snow from the sidewalk and driveway in 4 hours. If he and his twin Casey work together, they can finish shoveling the snow in 2 hours. How many hours would it take Casey to do the job by himself?

Solution

This is a work application. A chart will help us organize the information.	
We are looking for how many hours it would take Casey to complete the job by himself.	
Let t= the number of hours needed for Casey to complete.	
Enter the hours per job for Corey, Casey, and when they work together. If Corey takes 4 hours, then in 1 hour $\frac{1}{4}$ of the job is completed. Similarly find the part of the job completed/hours for Casey and when they both work together.	<table><tr><th></th><th>Number of hours needed to complete the job</th><th>Part of job completed/ hour</th></tr><tr><td>Corey</td><td>4</td><td>$\frac{1}{4}$</td></tr><tr><td>Casey</td><td>t</td><td>$\frac{1}{t}$</td></tr><tr><td>Together</td><td>2</td><td>$\frac{1}{2}$</td></tr></table>
Write a word sentence.	
The part completed by Corey plus the part completed by Casey equals the amount completed together.	
Translate to an equation:	$\overbrace{\text{Corey} + \text{Casey}}^{} = \text{Together}$ $\frac{1}{4} + \frac{1}{t} = \frac{1}{2}$
Solve.	$\frac{1}{4}+\frac{1}{t}=\frac{1}{2}$
Multiply by the LCD, 4t.	$4t\left(\frac{1}{4}+\frac{1}{t}\right)=4t\left(\frac{1}{2}\right)$
Simplify.	$t+4=2t$
	$4=t$
	It would take Casey 4 hours to do the job alone.

Try It 13.9.8.14

Two hoses can fill a swimming pool in 10 hours. It would take one hose 26 hours to fill the pool by itself. How long would it take for the other hose, working alone, to fill the pool?

Answer

16.25 hours

Try It 13.9.8.15

Cara and Cindy, working together, can rake the yard in 4 hours. Working alone, it takes Cindy 6 hours to rake the yard. How long would it take Cara to rake the yard alone?

Answer

12 hours

Practice Makes Perfect

Solve Uniform Motion Applications

In the following exercises, solve uniform motion applications

Try It 13.9.8.16

Mary takes a sightseeing tour on a helicopter that can fly 450 miles against a 35 mph headwind in the same amount of time it can travel 702 miles with a 35 mph tailwind. Find the speed of the helicopter.

Solution

160 mph

Exercise 13.9.8.17

A private jet can fly 1210 miles against a 25 mph headwind in the same amount of time it can fly 1694 miles with a 25 mph tailwind. Find the speed of the jet.

Try It 13.9.8.18

A boat travels 140 miles downstream in the same time as it travels 92 miles upstream. The speed of the current is 6mph. What is the speed of the boat?

Answer

29 mph

Exercise 13.9.8.19

Darrin can skateboard 2 miles against a 4 mph wind in the same amount of time he skateboards 6 miles with a 4 mph wind. Find the speed Darrin skateboards with no wind.

Try It 13.9.8.20

Jane spent 2 hours exploring a mountain with a dirt bike. When she rode the 40 miles uphill, she went 5 mph slower than when she reached the peak and rode for 12 miles along the summit. What was her rate along the summit?

Answer

30 mph

Exercise 13.9.8.21

Jill wanted to lose some weight so she planned a day of exercising. She spent a total of 2 hours riding her bike and jogging. She biked for 12 miles and jogged for 6 miles. Her rate for jogging was 10 mph less than biking rate. What was her rate when jogging?

Try It 13.9.8.22

Bill wanted to try out different water craft. He went 62 miles downstream in a motor boat and 27 miles downstream on a jet ski. His speed on the jet ski was 10 mph faster than in the motor boat. Bill spent a total of 4 hours on the water. What was his rate of speed in the motor boat?

Answer

20 mph

Exercise 13.9.8.23

Nancy took a 3 hour drive. She went 50 miles before she got caught in a storm. Then she drove 68 miles at 9 mph less than she had driven when the weather was good. What was her speed driving in the storm?

Try It 13.9.8.24

Chester rode his bike uphill 24 miles and then back downhill at 2 mph faster than his uphill. If it took him 2 hours longer to ride uphill than downhill, l, what was his uphill rate?

Answer

4 mph

Exercise 13.9.8.25

Matthew jogged to his friend's house 12 miles away and then got a ride back home. It took him 2 hours longer to jog there than ride back. His jogging rate was 25 mph slower than the rate when he was riding. What was his jogging rate?

Try It 13.9.8.26

Hudson travels 1080 miles in a jet and then 240 miles by car to get to a business meeting. The jet goes 300 mph faster than the rate of the car, and the car ride takes 1 hour longer than the jet. What is the speed of the car?

Answer

60 mph

Exercise 13.9.8.27

Nathan walked on an asphalt pathway for 12 miles. He walked the 12 miles back to his car on a gravel road through the forest. On the asphalt he walked 2 miles per hour faster than on the gravel. The walk on the gravel took one hour longer than the walk on the asphalt. How fast did he walk on the gravel?

? Try It 13.9.8.28

John can fly his airplane 2800 miles with a wind speed of 50 mph in the same time he can travel 2400 miles against the wind. If the speed of the wind is 50 mph, find the speed of his airplane.

Answer

650 mph

? Exercise 13.9.8.29

Jim's speedboat can travel 20 miles upstream against a 3 mph current in the same amount of time it travels 22 miles downstream with a 3 mph current speed. Find the speed of the Jim's boat.

? Try It 13.9.8.30

Hazel needs to get to her granddaughter's house by taking an airplane and a rental car. She travels 900 miles by plane and 250 miles by car. The plane travels 250 mph faster than the car. If she drives the rental car for 2 hours more than she rode the plane, find the speed of the car.

Answer

50 mph

? Exercise 13.9.8.31

Stu trained for 3 hours yesterday. He ran 14 miles and then biked 40 miles. His biking speed is 6 mph faster than his running speed. What is his running speed?

? Try It 13.9.8.32

When driving the 9 hour trip home, Sharon drove 390 miles on the interstate and 150 miles on country roads. Her speed on the interstate was 15 more than on country roads. What was her speed on country roads?

Answer

50 mph

? Exercise 13.9.8.33

Two sisters like to compete on their bike rides. Tamara can go 4 mph faster than her sister, Samantha. If it takes Samantha 1 hours longer than Tamara to go 80 miles, how fast can Samantha ride her bike?

Solve Work Applications

In the following exercises, solve work applications.

? Try It 13.9.8.34

Mike, an experienced bricklayer, can build a wall in 3 hours, while his son, who is learning, can do the job in 6 hours. How long does it take for them to build a wall together?

Answer

2 hours

Exercise 13.9.8.35

It takes Sam 4 hours to rake the front lawn while his brother, Dave, can rake the lawn in 2 hours. How long will it take them to rake the lawn working together?

Try It 13.9.8.36

Mary can clean her apartment in 6 hours while her roommate can clean the apartment in 5 hours. If they work together, how long would it take them to clean the apartment?

Answer

2 hours and 44 minutes

Exercise 13.9.8.37

Brian can lay a slab of concrete in 6 hours, while Greg can do it in 4 hours. If Brian and Greg work together, how long will it take?

Try It 13.9.8.38

Leeson can proofread a newspaper copy in 4 hours. If Ryan helps, they can do the job in 3 hours. How long would it take for Ryan to do his job alone?

Answer

12 hours

Exercise 13.9.8.39

Paul can clean a classroom floor in 3 hours. When his assistant helps him, the job takes 2 hours. How long would it take the assistant to do it alone?

Try It 13.9.8.40

Josephine can correct her students' test papers in 5 hours, but if her teacher's assistant helps, it would take them 3 hours. How long would it take the assistant to do it alone?

Answer

7 hours and 30 minutes

Exercise 13.9.8.41

Washing his dad's car alone, eight year old Levi takes 2.5 hours. If his dad helps him, then it takes 1 hour. How long does it take the Levi's dad to wash the car by himself?

Try It 13.9.8.42

Jackson can remove the shingles off of a house in 7 hours, while Martin can remove the shingles in 5 hours. How long will it take them to remove the shingles if they work together?

Answer

2 hours and 55 minutes

Exercise 13.9.8.43

At the end of the day Dodie can clean her hair salon in 15 minutes. Ann, who works with her, can clean the salon in 30 minutes. How long would it take them to clean the shop if they work together?

Try It 13.9.8.44

Ronald can shovel the driveway in 4 hours, but if his brother Donald helps it would take 2 hours. How long would it take Donald to shovel the driveway alone?

Answer

4 hours

Exercise 13.9.8.45

It takes Tina 3 hours to frost her holiday cookies, but if Candy helps her it takes 2 hours. How long would it take Candy to frost the holiday cookies by herself?

Everyday Math

Try It 13.9.8.46

Dana enjoys taking her dog for a walk, but sometimes her dog gets away and she has to run after him. Dana walked her dog for 7 miles but then had to run for 1 mile, spending a total time of 2.5 hours with her dog. Her running speed was 3 mph faster than her walking speed. Find her walking speed.

Answer

3 mph

Exercise 13.9.8.47

Ken and Joe leave their apartment to go to a football game 45 miles away. Ken drives his car 30 mph faster Joe can ride his bike. If it takes Joe 2 hours longer than Ken to get to the game, what is Joe's speed?

Writing Exercises

Exercise 13.9.8.48

In Example, the solution $h=-4$ is crossed out. Explain why.

Exercise 13.9.8.49

Paula and Yuki are roommates. It takes Paula 3 hours to clean their apartment. It takes Yuki 4 hours to clean the apartment. The equation $\frac{1}{3}+\frac{1}{4}=\frac{1}{t}$ can be used to find t, the number of hours it would take both of them, working together, to clean their apartment. Explain how this equation models the situation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve uniform motion applications.			
solve work applications.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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13.9.8E: Exercises

Practice Makes Perfect

Solve Uniform Motion Applications

In the following exercises, solve uniform motion applications

Exercise 1

Mary takes a sightseeing tour on a helicopter that can fly 450 miles against a 35 mph headwind in the same amount of time it can travel 702 miles with a 35 mph tailwind. Find the speed of the helicopter.

Answer

160 mph

Exercise 2

A private jet can fly 1210 miles against a 25 mph headwind in the same amount of time it can fly 1694 miles with a 25 mph tailwind. Find the speed of the jet.

Exercise 3

A boat travels 140 miles downstream in the same time as it travels 92 miles upstream. The speed of the current is 6mph. What is the speed of the boat?

Answer

29 mph

Exercise 4

Darrin can skateboard 2 miles against a 4 mph wind in the same amount of time he skateboards 6 miles with a 4 mph wind. Find the speed Darrin skateboards with no wind.

Exercise 5

Jane spent 2 hours exploring a mountain with a dirt bike. When she rode the 40 miles uphill, she went 5 mph slower than when she reached the peak and rode for 12 miles along the summit. What was her rate along the summit?

Answer

30 mph

Exercise 6

Jill wanted to lose some weight so she planned a day of exercising. She spent a total of 2 hours riding her bike and jogging. She biked for 12 miles and jogged for 6 miles. Her rate for jogging was 10 mph less than biking rate. What was her rate when jogging?

Exercise 7

Bill wanted to try out different water craft. He went 62 miles downstream in a motor boat and 27 miles downstream on a jet ski. His speed on the jet ski was 10 mph faster than in the motor boat. Bill spent a total of 4 hours on the water. What was his rate of speed in the motor boat?

Answer

20 mph

Exercise 8

Nancy took a 3 hour drive. She went 50 miles before she got caught in a storm. Then she drove 68 miles at 9 mph less than she had driven when the weather was good. What was her speed driving in the storm?

Exercise 9

Chester rode his bike uphill 24 miles and then back downhill at 2 mph faster than his uphill. If it took him 2 hours longer to ride uphill than downhill, l, what was his uphill rate?

Answer

4 mph

Exercise 10

Matthew jogged to his friend's house 12 miles away and then got a ride back home. It took him 2 hours longer to jog there than ride back. His jogging rate was 25 mph slower than the rate when he was riding. What was his jogging rate?

Exercise 11

Hudson travels 1080 miles in a jet and then 240 miles by car to get to a business meeting. The jet goes 300 mph faster than the rate of the car, and the car ride takes 1 hour longer than the jet. What is the speed of the car?

Answer

60 mph

Exercise 12

Nathan walked on an asphalt pathway for 12 miles. He walked the 12 miles back to his car on a gravel road through the forest. On the asphalt he walked 2 miles per hour faster than on the gravel. The walk on the gravel took one hour longer than the walk on the asphalt. How fast did he walk on the gravel?

Exercise 13

John can fly his airplane 2800 miles with a wind speed of 50 mph in the same time he can travel 2400 miles against the wind. If the speed of the wind is 50 mph, find the speed of his airplane.

Answer

650 mph

Exercise 14

Jim's speedboat can travel 20 miles upstream against a 3 mph current in the same amount of time it travels 22 miles downstream with a 3 mph current speed. Find the speed of the Jim's boat.

Exercise 15

Hazel needs to get to her granddaughter's house by taking an airplane and a rental car. She travels 900 miles by plane and 250 miles by car. The plane travels 250 mph faster than the car. If she drives the rental car for 2 hours more than she rode the plane, find the speed of the car.

Answer

50 mph

Exercise 16

Stu trained for 3 hours yesterday. He ran 14 miles and then biked 40 miles. His biking speed is 6 mph faster than his running speed. What is his running speed?

Exercise 17

When driving the 9 hour trip home, Sharon drove 390 miles on the interstate and 150 miles on country roads. Her speed on the interstate was 15 more than on country roads. What was her speed on country roads?

Answer

50 mph

Exercise 18

Two sisters like to compete on their bike rides. Tamara can go 4 mph faster than her sister, Samantha. If it takes Samantha 1 hours longer than Tamara to go 80 miles, how fast can Samantha ride her bike?

Solve Work Applications

In the following exercises, solve work applications.

Exercise 19

Mike, an experienced bricklayer, can build a wall in 3 hours, while his son, who is learning, can do the job in 6 hours. How long does it take for them to build a wall together?

Answer

2 hours

Exercise 20

It takes Sam 4 hours to rake the front lawn while his brother, Dave, can rake the lawn in 2 hours. How long will it take them to rake the lawn working together?

Exercise 21

Mary can clean her apartment in 6 hours while her roommate can clean the apartment in 5 hours. If they work together, how long would it take them to clean the apartment?

Answer

2 hours and 44 minutes

Exercise 22

Brian can lay a slab of concrete in 6 hours, while Greg can do it in 4 hours. If Brian and Greg work together, how long will it take?

Exercise 23

Leeson can proofread a newspaper copy in 4 hours. If Ryan helps, they can do the job in 3 hours. How long would it take for Ryan to do his job alone?

Answer

12 hours

Exercise 24

Paul can clean a classroom floor in 3 hours. When his assistant helps him, the job takes 2 hours. How long would it take the assistant to do it alone?

Exercise 25

Josephine can correct her students' test papers in 5 hours, but if her teacher's assistant helps, it would take them 3 hours. How long would it take the assistant to do it alone?

Answer

7 hours and 30 minutes

Exercise 26

Washing his dad's car alone, eight year old Levi takes 2.5 hours. If his dad helps him, then it takes 1 hour. How long does it take the Levi's dad to wash the car by himself?

Exercise 27

Jackson can remove the shingles off of a house in 7 hours, while Martin can remove the shingles in 5 hours. How long will it take them to remove the shingles if they work together?

Answer

2 hours and 55 minutes

Exercise 28

At the end of the day Dodie can clean her hair salon in 15 minutes. Ann, who works with her, can clean the salon in 30 minutes. How long would it take them to clean the shop if they work together?

Exercise 29

Ronald can shovel the driveway in 4 hours, but if his brother Donald helps it would take 2 hours. How long would it take Donald to shovel the driveway alone?

Answer

4 hours

Exercise 30

It takes Tina 3 hours to frost her holiday cookies, but if Candy helps her it takes 2 hours. How long would it take Candy to frost the holiday cookies by herself?

Everyday Math

Exercise 31

Dana enjoys taking her dog for a walk, but sometimes her dog gets away and she has to run after him. Dana walked her dog for 7 miles but then had to run for 1 mile, spending a total time of 2.5 hours with her dog. Her running speed was 3 mph faster than her walking speed. Find her walking speed.

Answer

3 mph

Exercise 32

Ken and Joe leave their apartment to go to a football game 45 miles away. Ken drives his car 30 mph faster Joe can ride his bike. If it takes Joe 2 hours longer than Ken to get to the game, what is Joe's speed?

Writing Exercises

Exercise 33

In Example, the solution $h=-4$ is crossed out. Explain why.

Exercise 34

Paula and Yuki are roommates. It takes Paula 3 hours to clean their apartment. It takes Yuki 4 hours to clean the apartment. The equation $\frac{1}{3}+\frac{1}{4}=\frac{1}{t}$ can be used to find t, the number of hours it would take both of them, working together, to clean their apartment. Explain how this equation models the situation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve uniform motion applications.			
solve work applications.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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13.9.9: Use Direct and Inverse Variation

Learning Objectives

By the end of this section, you will be able to:

- Solve direct variation problems
- Solve inverse variation problems

Note

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Find the multiplicative inverse of −8
   If you missed this problem, review Exercise 1.10.13.
2. Solve for n: 45=20n
   If you missed this problem, review Exercise 2.2.1.
3. Evaluate $5x^2$ when x=10
   If you missed this problem, review Exercise 1.3.25.

When two quantities are related by a proportion, we say they are *proportional* to each other. Another way to express this relation is to talk about the *variation* of the two quantities. We will discuss direct variation and inverse variation in this section.

Solve Direct Variation Problems

Lindsay gets paid $15 per hour at her job. If we let *s* be her salary and *h* be the number of hours she has worked, we could model this situation with the equation

s=15h

Lindsay's salary is the product of a constant, 15, and the number of hours she works. We say that Lindsay's salary *varies directly* with the number of hours she works. Two variables vary directly if one is the product of a constant and the other.

Definition: DIRECT VARIATION

For any two variables *x* and *y*, *y* varies directly with *x* if

y=kx, where $n \neq 0$

In applications using direct variation, generally we will know values of one pair of the variables and will be asked to find the equation that relates *x* and *y*. Then we can use that equation to find values of *y* for other values of *x*.

How to Solve Direct Variation Problems

Example 13.9.9.1

If y varies directly with x and y=20 when x=8, find the equation that relates *x* and *y*.

Answer

Step 1. Write the formula for direct variation.	The direct variation formula is $y = kx$.	$y = kx$	**Step 2.** Substitute the given values for the variables.	We are given $y = 20$, $x = 8$.	$20 = k \cdot 8$
Step 3. Solve for the constant of variation.	Divide both sides of the equation by 8, then simplify.	$\frac{20}{8} = k$ $k = 2.5$	**Step 4.** Write the equation that relates *x* and *y*.	Rewrite the general equation with the value we found for *k*.	$y = 2.5x$

Example 13.9.9.2

If y varies directly as x and y=3, when x=10, find the equation that relates *x* and *y*.

Answer

$y = \frac{3}{10}x$

Example 13.9.9.3

If y varies directly as x and y=12 when x=4, find the equation that relates *x* and *y*.

Answer

y=3x

We'll list the steps below.

Definition: SOLVE DIRECT VARIATION PROBLEMS

1. Write the formula for direct variation.
2. Substitute the given values for the variables.
3. Solve for the constant of variation.
4. Write the equation that relates x and y.

Now we'll solve a few applications of direct variation.

Example 13.9.9.4

When Raoul runs on the treadmill at the gym, the number of calories, c, he burns varies directly with the number of minutes, m, he uses the treadmill. He burned 315 calories when he used the treadmill for 18 minutes.

1. Write the equation that relates c and m.
2. How many calories would he burn if he ran on the treadmill for 25 minutes?

Answer

1.

	The number of calories, c varies directly with the number of minutes, m, on the treadmill, and c=315 when m=18
Write the formula for direct variation.	$y = kx$
We will use c in place of y and mm in place of x.	$c = km$
Substitute the given values for the variables.	$315 = k \cdot 18$
Solve for the constant of variation.	$\frac{315}{18} = \frac{k \cdot 18}{18}$
	$17.5 = k$
Write the equation that relates c and m.	$c = km$
Substitute in the constant of variation.	$c = 17.5m$

2.

	Find c when m=25.
Write the equation that relates c and m.	$c = 17.5m$
Substitute the given value for m.	$c = 17.5(25)$
Simplify.	$c = 437.5$
	Raoul would burn 437.5 calories if he used the treadmill for 25 minutes.

Example 13.9.9.5

The number of calories, c, burned varies directly with the amount of time, t, spent exercising. Arnold burned 312 calories in 65 minutes exercising.

1. Write the equation that relates c and t.
2. How many calories would he burn if he exercises for 90 minutes?

Answer

1. c=4.8t
2. 432 calories

Example 13.9.9.6

The distance a moving body travels, d, varies directly with time, t, it moves. A train travels 100 miles in 2 hours

1. Write the equation that relates d and t.
2. How many miles would it travel in 5 hours?

Answer

1. d=50t
2. 250 miles

In the previous example, the variables c and m were named in the problem. Usually that is not the case. We will have to name the variables in the next example as part of the solution, just like we do in most applied problems.

Example 13.9.9.7

The number of gallons of gas Eunice's car uses varies directly with the number of miles she drives. Last week she drove 469.8 miles and used 14.5 gallons of gas.

1. Write the equation that relates the number of gallons of gas used to the number of miles driven.
2. How many gallons of gas would Eunice's car use if she drove 1000 miles?

Answer

1.

	The number of gallons of gas varies directly with the number of miles driven.
First we will name the variables.	Let g= number of gallons of gas. m= number of miles driven.
Write the formula for direct variation.	$y = kx$
We will use g in place of y and m in place of x.	$g = km$
Substitute the given values for the variables	$g = 14.5$ when $m = 469.8$
	$14.5 = k(469.8)$
Solve for the constant of variation.	$\frac{14.5}{469.8} = \frac{k(469.8)}{469.8}$
We will round to the nearest thousandth.	$0.031 = k$
Write the equation that relates g and m.	$g = km$
Substitute in the constant of variation.	$g = 0.031m$

2.

	Find g when m=1000.
Write the equation that relates g and m.	g=0.031m
Substitute the given value for m.	g=0.031(1000)
Simplify.	g=31
	Eunice's car would use 31 gallons of gas if she drove it 1,000 miles.

Notice that in this example, the units on the constant of variation are gallons/mile. In everyday life, we usually talk about miles/gallon.

? Example 13.9.9.8

The distance that Brad travels varies directly with the time spent traveling. Brad travelled 660 miles in 12 hours,

1. Write the equation that relates the number of miles travelled to the time.
2. How many miles could Brad travel in 4 hours?

Answer

1. m=55h
2. 220 miles

? Example 13.9.9.9

The weight of a liquid varies directly as its volume. A liquid that weighs 24 pounds has a volume of 4 gallons.

1. Write the equation that relates the weight to the volume.
2. If a liquid has volume 13 gallons, what is its weight?

Answer

1. w=6v
2. 78 pounds

In some situations, one variable varies directly with the square of the other variable. When that happens, the equation of direct variation is $y = kx^2$.

? Example 13.9.9.10

The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 4" will support a maximum load of 75 pounds.

1. Write the equation that relates the maximum load to the cross-section.
2. What is the maximum load that can be supported by a beam with diagonal 8"?

Answer

1.

	The maximum load varies directly with the square of the diagonal of the cross-section.
Name the variables.	Let L= maximum load. c= the diagonal of the cross-section
Write the formula for direct variation, where y varies directly with the square of x.	$y = kx^2$
We will use L in place of y and c in place of x.	$L = kc^2$

Substitute the given values for the variables.	$L = 75$ when $c = 4$
	$75 = k \cdot 4^2$
Solve for the constant of variation.	$\frac{75}{16} = \frac{k \cdot 16}{16}$
	$4.6875 = k$
Write the equation that relates L and c.	$L = kc^2$
Substitute in the constant of variation.	$L = 4.6875c^2$

2.

	Find L when c=8.
Write the equation that relates L and c.	$L = 4.6875c^2$
Substitute the given value for c.	$L = 4.6875(8)^2$
Simplify.	L=300
	A beam with diagonal 8” could supporta maximum load of 300 pounds.

Example 13.9.9.11

The distance an object falls is directly proportional to the square of the time it falls. A ball falls 144 feet in 3 seconds.

1. Write the equation that relates the distance to the time.
2. How far will an object fall in 4 seconds?

Answer

1. $d = 16t^2$
2. 256 feet

Example 13.9.9.12

The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.

1. Write the equation that relates the area to the radius.
2. What is the area of a pizza with a radius of 9 inches?

Answer

1. $A = 3.14r^2$
2. 254.34 square inches

Solve Inverse Variation Problems

Many applications involve two variable that *vary inversely*. As one variable increases, the other decreases. The equation that relates them is $y = \frac{k}{x}$.

Definition: INVERSE VARIATION

For any two variables x and y, y varies inversely with x if

$y=\frac{k}{x}$,where $k \neq 0$

The word ‘inverse’ in inverse variation refers to the multiplicative inverse. The multiplicative inverse of x is $\frac{1}{x}$.

We solve inverse variation problems in the same way we solved direct variation problems. Only the general form of the equation has changed. We will copy the procedure box here and just change ‘direct’ to ‘inverse’.

Definition: SOLVE INVERSE VARIATION PROBLEMS

1. Write the formula for inverse variation.
2. Substitute the given values for the variables.
3. Solve for the constant of variation.
4. Write the equation that relates x and y.

Example 13.9.9.13

If y varies inversely with x and y=20 when x=8 x and y.

Answer

Write the formula for inverse variation.	$y = \frac{k}{x}$
Substitute the given values for the variables.	$y = 20$ when $x = 8$
	$20 = \frac{k}{8}$

Solve for the constant of variation.	$8(20) = 8\left(\frac{k}{8}\right)$
	$160 = k$
Write the equation that relates x and y.	$y = \frac{k}{x}$
Substitute in the constant of variation.	$y = \frac{160}{x}$

Example 13.9.9.14

If p varies inversely with q and p=30 when q=12 find the equation that relates p and q.

Answer

$p = \frac{360}{q}$

Example 13.9.9.15

If y varies inversely with x and y=8 when x=2 find the equation that relates x and y.

Answer

$y = \frac{16}{x}$

Example 13.9.9.16

The fuel consumption (mpg) of a car varies inversely with its weight. A car that weighs 3100 pounds gets 26 mpg on the highway.

1. Write the equation of variation.
2. What would be the fuel consumption of a car that weighs 4030 pounds?

Answer

1.

	The fuel consumption varies inversely with the weight.
First we will name the variables.	Let f= fuel consumption. w= weight.
Write the formula for inverse variation.	$y = \frac{k}{x}$
We will use f in place of y and w in place of x.	$f = \frac{k}{w}$
Substitute the given values for the variables.	$f = 26$ when $w = 3100$
	$26 = \frac{k}{3100}$
Solve for the constant of variation.	$3100(26) = 3100\left(\frac{k}{3100}\right)$
	$80{,}600 = k$
Write the equation that relates f and w.	$f = \frac{k}{w}$
Substitute in the constant of variation.	$f = \frac{80{,}600}{w}$

2.

	Find f when w=4030.
Write the equation that relates f and w.	$f = \frac{80{,}600}{w}$
Substitute the given value for w.	$f = \frac{80{,}600}{4030}$
Simplify.	f=20
	A car that weighs 4030 pounds would have fuel consumption of 20 mpg.

Example 13.9.9.17

A car's value varies inversely with its age. Elena bought a two-year-old car for $20,000.

1. Write the equation of variation.
2. What will be the value of Elena's car when it is 5 years old?

Answer

1. $v = \frac{40{,}000}{a}$
2. $8,000

Example 13.9.9.18

The time required to empty a pool varies inversely as the rate of pumping. It took Lucy 2.5 hours to empty her pool using a pump that was rated at 400 gpm (gallons per minute).

1. Write the equation of variation.
2. How long will it take her to empty the pool using a pump rated at 500 gpm?

Answer

1. $t = \frac{1000}{r}$
2. 2 hours

Example 13.9.9.19

The frequency of a guitar string varies inversely with its length. A 26” long string has a frequency of 440 vibrations per second.

1. Write the equation of variation.
2. How many vibrations per second will there be if the string’s length is reduced to 20” by putting a finger on a fret?

Answer

1.

	The frequency varies inversely with the length.
Name the variables.	Let f= frequency. L=length.
Write the formula for inverse variation.	$y = \frac{k}{x}$
We will use f in place of y and L in place of x	$f = \frac{k}{L}$
Substitute the given values for the variables.	$f = 440$ when $L = 26$
	$440 = \frac{k}{26}$
Solve for the constant of variation.	$26(440) = 26\left(\frac{k}{26}\right)$
	$11{,}440 = k$
Write the equation that relates f and L.	$f = \frac{k}{L}$
Substitute in the constant of variation.	$f = \frac{11{,}440}{L}$

2.

	Find f when L=20.
Write the equation that relates f and L.	$f = \frac{11{,}440}{L}$
Substitute the given value for L.	$f = \frac{11{,}440}{20}$
Simplify.	f=572
	A 20” guitar string has frequency 572 vibrations per second.

Example 13.9.9.20

The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees.

1. Write the equation of variation.
2. How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

Answer

1. $h = \frac{130}{t}$
2. $1\frac{2}{3}$ hours

Example 13.9.9.21

The force needed to break a board varies inversely with its length. Richard uses 24 pounds of pressure to break a 2-foot long board.

1. Write the equation of variation.
2. How many pounds of pressure is needed to break a 5-foot long board?

Answer

1. $F = \frac{48}{L}$
2. 9.6 pounds

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13.9.9E: Exercises

Practice Makes Perfect

Solve Direct Variation Problems

In the following exercises, solve.

Example 13.9.9E.22

If y varies directly as x and y=14, when x=3, find the equation that relates x and y.

Answer

$y = \frac{14}{3}x$

Example 13.9.9E.23

If p varies directly as q and p=5, when q=2, find the equation that relates p and q.

Example 13.9.9E.24

If v varies directly as w and v=24, when w=8, find the equation that relates v and w.

Answer

v=3w

Example 13.9.9E.25

If a varies directly as b and a=16, when b=4, find the equation that relates a and b.

Example 13.9.9E.26

If p varies directly as q and p=9.6, when q=3, find the equation that relates p and q.

Answer

p=3.2q

Example 13.9.9E.27

If y varies directly as x and y=12.4, when x=4, find the equation that relates x and y.

Example 13.9.9E.28

If a varies directly as b and a=6, when $b = \frac{1}{3}$, find the equation that relates a and b.

Answer

a=18b

Example 13.9.9E.29

If v varies directly as w and v=8, when $w = \frac{1}{2}$, find the equation that relates v and w.

Exercise 13.9.9E.30

The amount of money Sally earns, P, varies directly with the number, n, of necklaces she sells. When Sally sells 15 necklaces she earns $150.

1. Write the equation that relates P and n.
2. How much money would she earn if she sold 4 necklaces?

Answer

1. P=10n
2. $40

Example 13.9.9E.31

The price, P, that Eric pays for gas varies directly with the number of gallons, g, he buys. It costs him $50 to buy 20 gallons of gas.

1. Write the equation that relates P and g.
2. How much would 33 gallons cost Eric?

Example 13.9.9E.32

Terri needs to make some pies for a fundraiser. The number of apples, a, varies directly with number of pies, p. It takes nine apples to make two pies.

1. Write the equation that relates a and p.
2. How many apples would Terri need for six pies?

Answer

1. a=4.5p
2. 27 apples

Example 13.9.9E.33

Joseph is traveling on a road trip. The distance, d, he travels before stopping for lunch varies directly with the speed, v, he travels. He can travel 120 miles at a speed of 60 mph.

1. Write the equation that relates d and v.
2. How far would he travel before stopping for lunch at a rate of 65 mph?

Example 13.9.9E.34

The price of gas that Jesse purchased varies directly to how many gallons he purchased. He purchased 10 gallons of gas for $39.80.

1. Write the equation that relates the price to the number of gallons.
2. How much will it cost Jesse for 15 gallons of gas?

Answer

1. p=3.98g
2. $59.70

Example 13.9.9E.35

The distance that Sarah travels varies directly to how long she drives. She travels 440 miles in 8 hours.

1. Write the equation that relates the distance to the number of hours.
2. How far can Sally travel in 6 hours?

Example 13.9.9E.36

The mass of a liquid varies directly with its volume. A liquid with mass 16 kilograms has a volume of 2 liters.

1. Write the equation that relates the mass to the volume.
2. What is the volume of this liquid if its mass is 128 kilograms?

Answer

1. m=8v
2. 16 liters

Example 13.9.9E.37

The length that a spring stretches varies directly with a weight placed at the end of the spring. When Sarah placed a 10 pound watermelon on a hanging scale, the spring stretched 5 inches.

1. Write the equation that relates the length of the spring to the weight.
2. What weight of watermelon would stretch the spring 6 inches?

Example 13.9.9E.38

The distance an object falls varies directly to the square of the time it falls. A ball falls 45 feet in 3 seconds.

1. Write the equation that relates the distance to the time.
2. How far will the ball fall in 7 seconds?

Answer

1. $d = 5t^2$
2. 245 feet

Example 13.9.9E.39

The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 6 inch will support a maximum load of 108 pounds.

1. Write the equation that relates the load to the diagonal of the cross-section.
2. What load will a beam with a 10 inch diagonal support?

Example 13.9.9E.40

The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.

1. Write the equation that relates the area to the radius.
2. What is the area of a personal pizza with a radius 4 inches?

Answer

1. $A = 3.14r^2$
2. 50.24sq. in.

Example 13.9.9E.41

The distance an object falls varies directly to the square of the time it falls. A ball falls 72 feet in 3 seconds,

1. Write the equation that relates the distance to the time.
2. How far will the ball have fallen in 8 seconds?

Solve Inverse Variation Problems

In the following exercises, solve.

Example 13.9.9E.42

If y varies inversely with x and y=5 when x=4, find the equation that relates x and y.

Answer

$y = \frac{20}{x}$

Example 13.9.9E.43

If p varies inversely with q and p=2 when q=1, find the equation that relates p and q.

Example 13.9.9E.44

If v varies inversely with w and v=6 when $w = \frac{1}{2}$, find the equation that relates v and w.

Answer

$v = \frac{3}{w}$

Example 13.9.9E.45

If a varies inversely with b and a=12 when $b = \frac{1}{3}$, find the equation that relates a and b.

Write an inverse variation equation to solve the following problems.

Example 13.9.9E.46

The fuel consumption (mpg) of a car varies inversely with its weight. A Toyota Corolla weighs 2800 pounds and gets 33 mpg on the highway.

1. Write the equation that relates the mpg to the car's weight.
2. What would the fuel consumption be for a Toyota Sequoia that weighs 5500 pounds?

Answer

1. $g = \frac{92{,}400}{w}$
2. 16.8 mpg

Example 13.9.9E.47

A car's value varies inversely with its age. Jackie bought a 10 year old car for $2,400.

1. Write the equation that relates the car's value to its age.
2. What will be the value of Jackie's car when it is 15 years old ?

Example 13.9.9E.48

The time required to empty a tank varies inversely as the rate of pumping. It took Janet 5 hours to pump her flooded basement using a pump that was rated at 200 gpm (gallons per minute),

1. Write the equation that relates the number of hours to the pump rate.
2. How long would it take Janet to pump her basement if she used a pump rated at 400 gpm?

Answer

1. $t = \frac{1000}{r}$
2. 2.5 hours

Example 13.9.9E.49

The volume of a gas in a container varies inversely as pressure on the gas. A container of helium has a volume of 370 cubic inches under a pressure of 15 psi.

1. Write the equation that relates the volume to the pressure.
2. What would be the volume of this gas if the pressure was increased to 20 psi?

Example 13.9.9E.50

On a string instrument, the length of a string varies inversely as the frequency of its vibrations. An 11-inch string on a violin has a frequency of 400 cycles per second.

1. Write the equation that relates the string length to its frequency.
2. What is the frequency of a 10-inch string?

Answer

1. $L = \frac{4,400}{f}$
2. 440 cycles per second

Example 13.9.9E.51

Paul, a dentist, determined that the number of cavities that develops in his patient's mouth each year varies inversely to the number of minutes spent brushing each night. His patient, Lori, had 4 cavities when brushing her teeth 30 seconds (0.5 minutes) each night.

1. Write the equation that relates the number of cavities to the time spent brushing.
2. How many cavities would Paul expect Lori to have if she had brushed her teeth for 2 minutes each night?

Example 13.9.9E.52

The number of tickets for a sports fundraiser varies inversely to the price of each ticket. Brianna can buy 25 tickets at $5 each.

1. Write the equation that relates the number of tickets to the price of each ticket.
2. How many tickets could Brianna buy if the price of each ticket was $2.50?

Answer

1. $t = \frac{125}{p}$
2. 50 tickets

Example 13.9.9E.53

Boyle's Law states that if the temperature of a gas stays constant, then the pressure varies inversely to the volume of the gas. Braydon, a scuba diver, has a tank that holds 6 liters of air under a pressure of 220 psi.

1. Write the equation that relates pressure to volume.
2. If the pressure increases to 330 psi, how much air can Braydon's tank hold?

Mixed practice

Example 13.9.9E.54

If y varies directly as x and y=5, when x=3, find the equation that relates x and y.

Answer

$y = \frac{5}{3}x$

Example 13.9.9E.55

If v varies directly as w and v=21, when w=8, find the equation that relates v and w.

Example 13.9.9E.56

If p varies inversely with q and p=5 when q=6, find the equation that relates q and p.

Answer

$p = \frac{30}{q}$

Example 13.9.9E.57

If y varies inversely with x and y=11 when x=3, find the equation that relates x and y.

Example 13.9.9E.58

If p varies directly as q and p=10, when q=2, find the equation that relates p and q.

Answer

p=5q

Example 13.9.9E.59

If v varies inversely with w and v=18 when $w = \frac{1}{3}$, find the equation that relates v and w.

Example 13.9.9E.60

The force needed to break a board varies inversely with its length. If Tom uses 20 pounds of pressure to break a 1.5-foot long board, how many pounds of pressure would he need to use to break a 6 foot long board?

Answer

5 pounds

Example 13.9.9E.61

The number of hours it takes for ice to melt varies inversely with the air temperature. A block of ice melts in 2.5 hours when the temperature is 54 degrees. How long would it take for the same block of ice to melt if the temperature was 45 degrees?

Example 13.9.9E.62

The length a spring stretches varies directly with a weight placed at the end of the spring. When Meredith placed a 6-pound cantaloupe on a hanging scale, the spring stretched 2 inches. How far would the spring stretch if the cantaloupe weighed 9 pounds?

Answer

3 inches

Example 13.9.9E.63

The amount that June gets paid varies directly the number of hours she works. When she worked 15 hours, she got paid $111. How much will she be paid for working 18 hours?

Example 13.9.9E.64

The fuel consumption (mpg) of a car varies inversely with its weight. A Ford Focus weighs 3000 pounds and gets 28.7 mpg on the highway. What would the fuel consumption be for a Ford Expedition that weighs 5,500 pounds? Round to the nearest tenth.

Answer

15.6 mpg

Example 13.9.9E.65

The volume of a gas in a container varies inversely as the pressure on the gas. If a container of argon has a volume of 336 cubic inches under a pressure of 2,500 psi, what will be its volume if the pressure is decreased to 2,000 psi?

Example 13.9.9E.66

The distance an object falls varies directly to the square of the time it falls. If an object falls 52.8 feet in 4 seconds, how far will it fall in 9 seconds?

Answer

267.3 feet

Example 13.9.9E.67

The area of the face of a Ferris wheel varies directly with the square of its radius. If the area of one face of a Ferris wheel with diameter 150 feet is 70,650 square feet, what is the area of one face of a Ferris wheel with diameter of 16 feet?

Everyday Math

Example 13.9.9E.68

Ride Service It costs $35 for a ride from the city center to the airport, 14 miles away.

1. Write the equation that relates the cost, *c*, with the number of miles, *m*.
2. What would it cost to travel 22 miles with this service?

Answer

1. c=2.5m
2. $55

Example 13.9.9E.69

Road Trip The number of hours it takes Jack to drive from Boston to Bangor is inversely proportional to his average driving speed. When he drives at an average speed of 40 miles per hour, it takes him 6 hours for the trip.

1. Write the equation that relates the number of hours, *h*, with the speed, *s*.
2. How long would the trip take if his average speed was 75 miles per hour?

Writing Exercises

Example 13.9.9E.70

In your own words, explain the difference between direct variation and inverse variation.

Answer

Answers will vary.

Example 13.9.9E.71

Make up an example from your life experience of inverse variation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve direct variation problems.			
solve inverse variation problems.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next chapter? Why or why not?

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13.9.10: Chapter 8 Review Exercises

Chapter Review Exercises

Simplify Rational Expressions

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

Exercise 13.9.10.1

$$\frac{2a+1}{3a-2}$$

Answer

$$a \neq \frac{2}{3}$$

Exercise 13.9.10.2

$$\frac{b-3}{b^2-16}$$

Exercise 13.9.10.3

$$\frac{3xy^2}{5y}$$

Answer

$$y \neq 0$$

Exercise 13.9.10.4

$$\frac{u-3}{u^2-u-30}$$

Evaluate Rational Expressions

In the following exercises, evaluate the rational expressions for the given values.

Exercise 13.9.10.5

$\frac{4p-1}{p^2+5}$ when $p=-1$

Answer

$$-\frac{5}{6}$$

Exercise 13.9.10.6

$\frac{q^2-5}{q+3}$ when $q=7$

Exercise 13.9.10.7

$\frac{y^2-8}{y^2-y-2}$ when $y=1$

Answer

$\frac{7}{2}$

Example 13.9.10.8

$\frac{z^2+2}{4z-z^2}$ when $z=3$

Simplify Rational Expressions

In the following exercises, simplify.

Exercise 13.9.10.9

$\frac{10}{24}$

Answer

$\frac{5}{12}$

Exercise 13.9.10.10

$\frac{8m^4}{16mn^3}$

Exercise 13.9.10.11

$\frac{14a-14}{a-1}$

Answer

14

Exercise 13.9.10.12

$\frac{b^2+7b+12}{b^2+8b+16}$

Simplify Rational Expressions with Opposite Factors

In the following exercises, simplify.

Exercise 13.9.10.13

$\frac{c^2-c-2}{4-c^2}$

Answer

$$-\frac{c+1}{c+2}$$

Exercise 13.9.10.14

$$\frac{d-16}{16-d}$$

Exercise 13.9.10.15

$$\frac{7v-35}{25-v^2}$$

Answer

$$-\frac{7}{5+v}$$

Exercise 13.9.10.16

$$\frac{w^2-3w-28}{49-w^2}$$

Multiply and Divide Rational Expressions

Multiply Rational Expressions

In the following exercises, multiply.

Exercise 13.9.10.17

$$\frac{3}{8}\cdot\frac{2}{15}$$

Answer

$$\frac{1}{20}$$

Exercise 13.9.10.18

$$\frac{2xy^2}{8y^3}\cdot\frac{16y}{24x}$$

Exercise 13.9.10.19

$$\frac{3a^2+21a}{a^2+6a-7}\cdot\frac{a-1}{ab}$$

Answer

$$\frac{3}{b}$$

Exercise 13.9.10.20

$$\frac{5z^2}{5z^2+40z+35}\cdot\frac{z^2-1}{3z}$$

Divide Rational Expressions

In the following exercises, divide.

Exercise 13.9.10.21

$$\frac{t^2-4t-12}{t^2+8t+12}\div\frac{t^2-36}{6t}$$

Answer

$$\frac{6t}{(t+6)^2}$$

Exercise 13.9.10.22

$$\frac{r^2-16}{4}\div\frac{r^3-64}{2r^2-8r+32}$$

Exercise 13.9.10.23

$$\frac{11+w}{w-9}\div\frac{121-w^2}{9-w}$$

Answer

$$\frac{1}{11+w}$$

Exercise 13.9.10.24

$$\frac{3y^2-12y-63}{4y+3}\div(6y^2-42y)$$

Exercise 13.9.10.25

$$\frac{\frac{c^2-64}{3c^2+26c+16}}{\frac{c^2-4c-32}{15c+10}}$$

Answer

$$5c+4$$

Exercise 13.9.10.26

$$\frac{8m^2-8m}{m-4}\cdot\frac{m^2+2m-24}{m^2+7m+10}\div\frac{2m^2-6m}{m+5}$$

Add and Subtract Rational Expressions with a Common Denominator

Add Rational Expressions with a Common Denominator

In the following exercises, add.

Exercise 13.9.10.27

$$\frac{3}{5}+\frac{2}{5}$$

Answer

1

Exercise 13.9.10.28

$$\frac{4a^2}{2a-1}-\frac{1}{2a-1}$$

Exercise 13.9.10.29

$$\frac{p^2+10p}{p+5}+\frac{25}{p+5}$$

Answer

$p+5$

Exercise 13.9.10.30

$$\frac{3x}{x-1}+\frac{2}{x-1}$$

Subtract Rational Expressions with a Common Denominator

In the following exercises, subtract.

Exercise 13.9.10.31

$$\frac{d^2}{d+4}-\frac{3d+28}{d+4}$$

Answer

$d-7$

Exercise 13.9.10.32

$$\frac{z^2}{z+10}-\frac{100}{z+10}$$

Exercise 13.9.10.33

$$\frac{4q^2-q+3}{q^2+6q+5}-\frac{3q^2+q+6}{q^2+6q+5}$$

Answer

$$\frac{q-3}{q+5}$$

Exercise 13.9.10.34

$$\frac{5t+4t+3}{t^2-25}-\frac{4t^2-8t-32}{t^2-25}$$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add and subtract.

Exercise 13.9.10.35

$$\frac{18w}{6w-1}+\frac{3w-2}{1-6w}$$

Answer

$$\frac{15w+2}{6w-1}$$

Exercise 13.9.10.36

$$\frac{a^2+3a}{a^2-4}-\frac{3a-8}{4-a^2}$$

Exercise 13.9.10.37

$$\frac{2b^2+3b-15}{b^2-49}-\frac{b^2+16b-1}{49-b^2}$$

Answer

$$\frac{3b-2}{b+7}$$

Exercise 13.9.10.38

$$\frac{8y^2-10y+7}{2y-5}+\frac{2y^2+7y+2}{5-2y}$$

Add and Subtract Rational Expressions With Unlike Denominators

Find the Least Common Denominator of Rational Expressions

In the following exercises, find the LCD.

Exercise 13.9.10.38

$$\frac{4}{m^2-3m-10}, \quad \frac{2m}{m^2-m-20}$$

Answer

$$(m+2)(m-5)(m+4)$$

Exercise 13.9.10.39

$$\frac{6}{n^2-4}, \quad \frac{2n}{n^2-4n+4}$$

Exercise 13.9.10.40

$$\frac{5}{3p^2+17p-6}, \quad \frac{2m}{3p^2-23p-8}$$

Answer

$$(3p+1)(p+6)(p+8)$$

Find Equivalent Rational Expressions

In the following exercises, rewrite as equivalent rational expressions with the given denominator.

Exercise 13.9.10.41

Rewrite as equivalent rational expressions with denominator $(m+2)(m-5)(m+4)$

$$\frac{4}{m^2-3m-10}, \quad \frac{2m}{m^2-m-20}.$$

Exercise 13.9.10.42

Rewrite as equivalent rational expressions with denominator $(n-2)(n-2)(n+2)$

$$\frac{6}{n^2-4n+4}, \quad \frac{2n}{n^2-4}.$$

Answer

$$\frac{6n+12}{(n-2)(n-2)(n+2)}, \quad \frac{2n^2-4n}{(n-2)(n-2)(n+2)}$$

Exercise 13.9.10.43

Rewrite as equivalent rational expressions with denominator $(3p+1)(p+6)(p+8)$

$$\frac{5}{3p^2+19p+6}, \quad \frac{7p}{3p^2+25p+8}$$

Add Rational Expressions with Different Denominators

In the following exercises, add.

Exercise 13.9.10.44

$$\frac{2}{3}+\frac{3}{5}$$

Answer

$$\frac{19}{15}$$

Exercise 13.9.10.45

$$\frac{7}{5a}+\frac{3}{2b}$$

Exercise 13.9.10.46

$$\frac{2}{c-2}+\frac{9}{c+3}$$

Answer

$$\frac{11c-12}{(c-2)(c+3)}$$

Exercise 13.9.10.47

$$\frac{3d}{d^2-9}+\frac{5}{d^2+6d+9}$$

Exercise 13.9.10.48

$$\frac{2x}{x^2+10x+24}+\frac{3x}{x^2+8x+16}$$

Answer

$$\frac{5x^2+26x}{(x+4)(x+4)(x+6)}$$

Exercise 13.9.10.49

$$\frac{5q}{p^2q-p^2}+\frac{4q}{q^2-1}$$

Subtract Rational Expressions with Different Denominators

In the following exercises, subtract and add.

Exercise 13.9.10.50

$$\frac{3v}{v+2}-\frac{v+2}{v+8}$$

Answer

$$\frac{2(v^2+10v-2)}{(v+2)(v+8)}$$

Exercise 13.9.10.51

$$\frac{-3w-15}{w^2+w-20}-\frac{w+2}{4-w}$$

Exercise 13.9.10.52

$$\frac{7m+3}{m+2}-5$$

Answer

$$\frac{2m-7}{m+2}$$

Exercise 13.9.10.53

$$\frac{n}{n+3}+\frac{2}{n-3}-\frac{n-9}{n^2-9}$$

Exercise 13.9.10.54

$$\frac{8d}{d^2-64}-\frac{4}{d+8}$$

Answer

$$4d-8$$

Exercise 13.9.10.55

$$\frac{5}{12x^2y}+\frac{7}{20xy^3}$$

Simplify Complex Rational Expressions

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify.

Exercise 13.9.10.56

$$\frac{\frac{5a}{a+2}}{\frac{10a^2}{a^2-4}}$$

Answer

$$\frac{a-2}{2a}$$

Exercise 13.9.10.57

$$\frac{\frac{2}{5}+\frac{5}{6}}{\frac{1}{3}+\frac{1}{4}}$$

Exercise 13.9.10.58

$$\frac{x-\frac{3x}{x+5}}{\frac{1}{x+5}+\frac{1}{x-5}}$$

Answer

$$\frac{(x-8)(x-5)}{2}$$

Exercise 13.9.10.59

$$\frac{\frac{2}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

Exercise 13.9.10.60

$$\frac{6+\frac{2}{q-4}}{\frac{5}{q}+4}$$

Answer

$$\frac{(q-2)(q+4)}{5(q-4)}$$

Exercise 13.9.10.61

$$\frac{\frac{3}{a^2}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b^2}}$$

Exercise 13.9.10.62

$$\frac{\frac{2}{z^2-49}+\frac{1}{z+7}}{\frac{9}{z+7}+\frac{12}{z-7}}$$

Answer

$$\frac{z-5}{21z+21}$$

Exercise 13.9.10.63

$$\frac{\frac{3}{y^2-4y-32}}{\frac{2}{y-8}+\frac{1}{y+4}}$$

Solve Rational Equations

Solve Rational Equations

In the following exercises, solve.

Exercise 13.9.10.64

$$\frac{1}{2}+\frac{2}{3}=\frac{1}{x}$$

Answer

$$\frac{6}{7}$$

Exercise 13.9.10.65

$$1-\frac{2}{m}=\frac{8}{m^2}$$

Exercise 13.9.10.66

$$\frac{1}{b-2}+\frac{1}{b+2}=\frac{3}{b^2-4}$$

Answer

$$\frac{3}{2}$$

Exercise 13.9.10.67

$$\frac{3}{q+8}-\frac{2}{q-2}=1$$

Exercise 13.9.10.68

$$\frac{v-15}{v^2-9v+18}=\frac{4}{v-3}+\frac{2}{v-6}$$

Answer

no solution

Exercise 13.9.10.69

$$\frac{z}{12}+\frac{z+3}{3z}=\frac{1}{z}$$

Solve a Rational Equation for a Specific Variable

In the following exercises, solve for the indicated variable.

Exercise 13.9.10.70

$\frac{V}{l} = hw$ for l

Answer

$$l = \frac{V}{hw}$$

Exercise 13.9.10.71

$\frac{1}{x} - \frac{2}{y} = 5$ for y

Exercise 13.9.10.72

$x = \frac{y+5}{z-7}$ for z

Answer

$$z = \frac{y+5+7x}{x}$$

Exercise 13.9.10.73

$P = \frac{k}{V}$ for V

Solve Proportion and Similar Figure Applications Similarity

Solve Proportions

In the following exercises, solve.

Exercise 13.9.10.74

$\frac{x}{4} = \frac{3}{5}$

Answer

$$\frac{12}{5}$$

Exercise 13.9.10.75

$\frac{3}{y} = \frac{9}{5}$

Exercise 13.9.10.76

$$\frac{s}{s+20}=\frac{3}{7}$$

Answer

15

Exercise 13.9.10.77

$$\frac{t-3}{5}=\frac{t+2}{9}$$

In the following exercises, solve using proportions.

Exercise 13.9.10.78

Rachael had a 21 ounce strawberry shake that has 739 calories. How many calories are there in a 32 ounce shake?

Answer

1161 calories

Exercise 13.9.10.79

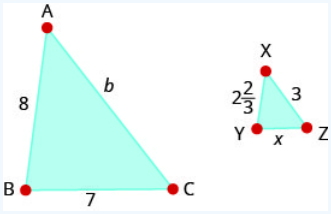
Leo went to Mexico over Christmas break and changed $525 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 16.25 Mexican pesos. How many Mexican pesos did he get for his trip?

Solve Similar Figure Applications

In the following exercises, solve.

Exercise 13.9.10.80

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure. Find the lengths of the third sides.



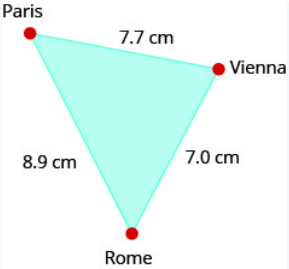
Answer

$b=9;\ x=2\frac{1}{3}$

Exercise 13.9.10.81

On a map of Europe, Paris, Rome, and Vienna form a triangle whose sides are shown in the figure below. If the actual distance from Rome to Vienna is 700 miles, find the distance from

1. a. Paris to Rome
2. b. Paris to Vienna



Exercise 13.9.10.82

Tony is 5.75 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a nearby tree was 32 feet long. Find the height of the tree.

Answer

23 feet

Exercise 13.9.10.83

The height of a lighthouse in Pensacola, Florida is 150 feet. Standing next to the statue, 5.5 foot tall Natalie cast a 1.1 foot shadow How long would the shadow of the lighthouse be?

Solve Uniform Motion and Work Applications Problems

Solve Uniform Motion Applications

In the following exercises, solve.

Exercise 13.9.10.84

When making the 5-hour drive home from visiting her parents, Lisa ran into bad weather. She was able to drive 176 miles while the weather was good, but then driving 10 mph slower, went 81 miles in the bad weather. How fast did she drive when the weather was bad?

Answer

45 mph

Exercise 13.9.10.85

Mark is riding on a plane that can fly 490 miles with a tailwind of 20 mph in the same time that it can fly 350 miles against a tailwind of 20 mph. What is the speed of the plane?

Exercise 13.9.10.86

John can ride his bicycle 8 mph faster than Luke can ride his bike. It takes Luke 3 hours longer than John to ride 48 miles. How fast can John ride his bike?

Answer

16 mph

Exercise 13.9.10.87

Mark was training for a triathlon. He ran 8 kilometers and biked 32 kilometers in a total of 3 hours. His running speed was 8 kilometers per hour less than his biking speed. What was his running speed?

Solve Work Applications

In the following exercises, solve.

Exercise 13.9.10.88

Jerry can frame a room in 1 hour, while Jake takes 4 hours. How long could they frame a room working together?

Answer

$\frac{4}{5}$ hour

Exercise 13.9.10.89

Lisa takes 3 hours to mow the lawn while her cousin, Barb, takes 2 hours. How long will it take them working together?

Exercise 13.9.10.90

Jeffrey can paint a house in 6 days, but if he gets a helper he can do it in 4 days. How long would it take the helper to paint the house alone?

Answer

12 days

Exercise 13.9.10.91

Sue and Deb work together writing a book that takes them 90 days. If Sue worked alone it would take her 120 days. How long would it take Deb to write the book alone?

Use Direct and Inverse Variation

Solve Direct Variation Problems

In the following exercises, solve.

Exercise 13.9.10.92

If y varies directly as x, when $y = 9$ and $x = 3$, find x when $y = 21$.

Answer

7

Exercise 13.9.10.93

If y varies directly as x, when $y = 20$ and $x = 2$, find y when $x = 4$.

Exercise 13.9.10.94

If m varies inversely with the square of n, when $m = 4$ and $n = 6$, find m when $n = 2$.

Answer

36

Exercise 13.9.10.95

Vanessa is traveling to see her fiancé. The distance, d, varies directly with the speed, v, she drives. If she travels 258 miles driving 60 mph, how far would she travel going 70 mph?

Exercise 13.9.10.96

If the cost of a pizza varies directly with its diameter, and if an 8” diameter pizza costs $\$12$, how much would a 6” diameter pizza cost?

Answer

$\$9$

Exercise 13.9.10.97

The distance to stop a car varies directly with the square of its speed. It takes 200 feet to stop a car going 50 mph. How many feet would it take to stop a car going 60 mph?

Solve Inverse Variation Problems

In the following exercises, solve.

Exercise 13.9.10.98

The number of tickets for a music fundraiser varies inversely with the price of the tickets. If Madelyn has just enough money to purchase 12 tickets for $\$6$, how many tickets can Madelyn afford to buy if the price increased to $\$8$?

Answer

97 tickets

Exercise 13.9.10.99

On a string instrument, the length of a string varies inversely with the frequency of its vibrations. If an 11-inch string on a violin has a frequency of 360 cycles per second, what frequency does a 12-inch string have?

Practice Test

In the following exercises, simplify.

Exercise 13.9.10.1

$\dfrac{3a^2b}{6ab^2}$

Answer

$\dfrac{a}{2b}$

Exercise 13.9.10.2

$$\frac{5b-25}{b^2-25}$$

In the following exercises, perform the indicated operation and simplify.

Exercise 13.9.10.3

$$\frac{4x}{x+2}\cdot\frac{x^2+5x+6}{12x^2}$$

Answer

$$\frac{x+3}{3x}$$

Exercise 13.9.10.4

$$\frac{5y}{4y-8}\cdot\frac{y^2-4}{10}$$

Exercise 13.9.10.5

$$\frac{4p}{q}+\frac{5}{p}$$

Answer

$$\frac{4+5q}{pq}$$

Exercise 13.9.10.6

$$\frac{1}{z-9}-\frac{3}{z+9}$$

Exercise 13.9.10.7

$$\frac{\frac{2}{3}+\frac{3}{5}}{\frac{2}{5}}$$

Answer

$$\frac{19}{16}$$

Exercise 13.9.10.8

$$\frac{\frac{1}{m}-\frac{1}{n}}{\frac{1}{n}+\frac{1}{m}}$$

In the following exercises, solve each equation.

Exercise 13.9.10.9

$$\frac{1}{2}+\frac{2}{7}=\frac{1}{x}$$

Answer

$$x=\frac{14}{11}$$

Exercise 13.9.10.10

$$\frac{5}{y-6}=\frac{3}{y+6}$$

Exercise 13.9.10.11

$$\frac{1}{z-5}+\frac{1}{z+5}=\frac{1}{z^2-25}$$

Answer

$$z=\frac{1}{2}$$

Exercise 13.9.10.12

$$\frac{t}{4}=\frac{3}{5}$$

Exercise 13.9.10.13

$$\frac{2}{r-2}=\frac{3}{r-1}$$

Answer

$$r=4$$

In the following exercises, solve.

Exercise 13.9.10.14

If y varies directly with x, and $x=5$ when $y=30$, find x when $y=42$.

Exercise 13.9.10.15

If y varies inversely with x and $x=6$ when $y=20$, find y when $x=2$.

Answer

$$y=60$$

Exercise 13.9.10.16

If y varies inversely with the square of x and $x=3$ when $y=9$, find y when $x=4$.

Exercise 13.9.10.17

The recommended erythromycin dosage for dogs, is 5 mg for every pound the dog weighs. If Daisy weighs 25 pounds, how many milligrams of erythromycin should her veterinarian prescribe?

Answer

125 mg

Exercise 13.9.10.18

Julia spent 4 hours Sunday afternoon exercising at the gym. She ran on the treadmill for 10 miles and then biked for 20 miles. Her biking speed was 5 mph faster than her running speed on the treadmill. What was her running speed?

Exercise 13.9.10.19

Kurt can ride his bike for 30 miles with the wind in the same amount of time that he can go 21 miles against the wind. If the wind's speed is 6 mph, what is Kurt's speed on his bike?

Answer

14 mph

Exercise 13.9.10.20

Amanda jogs to the park 8 miles using one route and then returns via a 14-mile route. The return trip takes her 1 hour longer than her jog to the park. Find her jogging rate.

Exercise 13.9.10.21

An experienced window washer can wash all the windows in Mike's house in 2 hours, while a new trainee can wash all the windows in 7 hours. How long would it take them working together?

Answer

$1\frac{5}{9}$ hour

Exercise 13.9.10.22

Josh can split a truckload of logs in 8 hours, but working with his dad they can get it done in 3 hours. How long would it take Josh's dad working alone to split the logs?

Exercise 13.9.10.23

The price that Tyler pays for gas varies directly with the number of gallons he buys. If 24 gallons cost him \$59.76, what would 30 gallons cost?

Answer

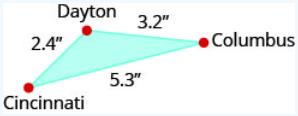
\$74.70

Exercise 13.9.10.24

The volume of a gas in a container varies inversely with the pressure on the gas. If a container of nitrogen has a volume of 29.5 liters with 2000 psi, what is the volume if the tank has a 14.7 psi rating? Round to the nearest whole number.

Exercise 13.9.10.25

The cities of Dayton, Columbus, and Cincinnati form a triangle in southern Ohio, as shown on the figure below, that gives the map distances between these cities in inches.



The actual distance from Dayton to Cincinnati is 48 miles. What is the actual distance between Dayton and Columbus?

Answer

64 miles

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13.E: Review Exercises 2

Exercise 13.E.1

Evaluate

1. Given $f(x) = 2x^2 - x + 6$, find $f(-3)$, $f(0)$, and $f(10)$.
2. Given $g(x) = -x^2 + 4x - 1$, find $g(-1)$, $g(0)$, and $g(3)$.
3. Given $h(t) = -t^3 - 2t^2 + 3$, find $h(-3)$, $h(0)$, and $h(2)$.
4. Given $p(x) = x^4 - 2x^2 + x$, find $p(-1)$, $p(0)$, and $p(2)$.
5. The following graph gives the height $h(t)$ in feet of a projectile over time t in seconds

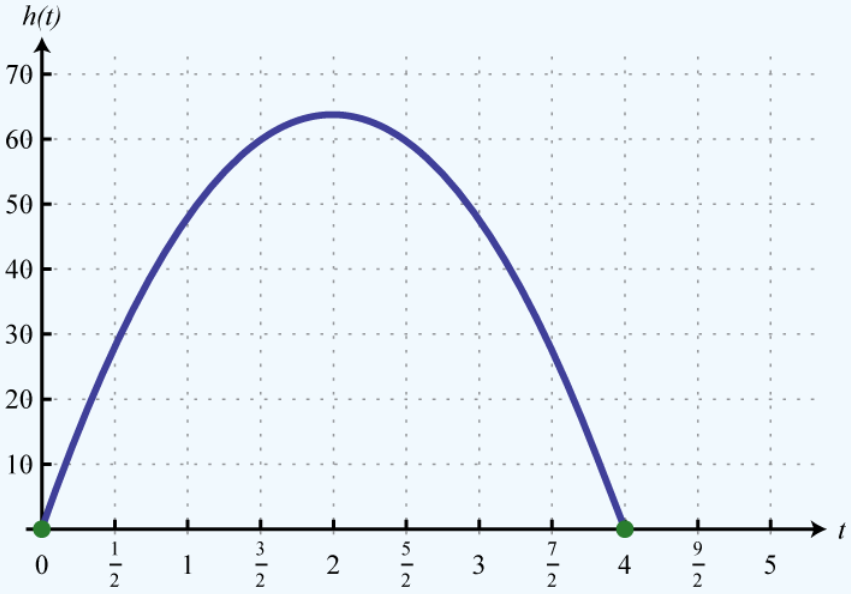

Figure 4.E.1

(a) Use the graph to determine the height of the projectile at 2.5 seconds.

(b) At what time does the projectile reach its maximum height?

(c) How long does it take the projectile to return to the ground?

6. Given the graph of the function f, find $f(-9)$, $f(-3)$, and $f(12)$.

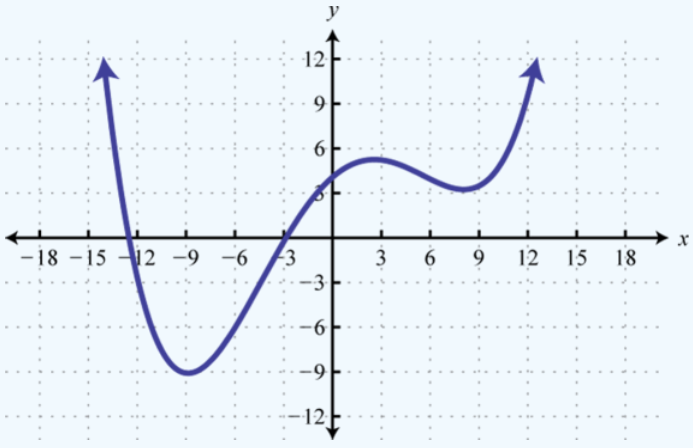

Figure 4.E.2

7. From the ground, a bullet is fired straight up into the air at 340 meters per second. Ignoring the effects of air friction, write a function that models the height of the bullet and use it to calculate the bullet's height after one-quarter of a second. (Round off to the nearest meter.)

8. An object is tossed into the air at an initial speed of 30 feet per second from a rooftop 10 feet high. Write a function that models the height of the object and use it to calculate the height of the object after 1 second.

Answer

1. $f(-3)=27; f(0)=6; f(10)=196$
3. $h(-3)=12; h(0)=3; h(2)=-13$
5. (a) 60 feet; (b) 2 seconds; (c) 4 seconds
7. $h(t)=-4.9t^2+340t$; at 0.25second, the bullet's height is about 85 meters.

Exercise $13.E.2$

Perform the operations.

1. Given $f(x)=5x^2-3x+1$ and $g(x)=2x^2-x-1$, find $(f+g)(x)$.
2. Given $f(x)=x^2+3x-8$ and $g(x)=x^2-5x-7$, find $(f-g)(x)$.
3. Given $f(x)=3x^2-x+2$ and $g(x)=2x-3$, find $(f\cdot g)(x)$.
4. Given $f(x)=27x^5-15x^3-3x^2$ and $g(x)=3x^2$, find $(f/g)(x)$.
5. Given $g(x)=x^2-x+1$, find $g(-3u)$.
6. GIven $g(x)=x^3-1$, find $g(x-1)$.

Answer

1. $(f+g)(x)=7x^2-4x$
3. $(f\cdot g)(x)=6x^3-11x^2+7x-6$
5. $g(-3u)=9u^2+3u+1$

Exercise $13.E.3$

Given $f(x)=16x^3-12x^2+4x, g(x)=x^2-x+1$, and $h(x)=4x$, find the following:

1. $(g\cdot h)(x)$
2. $(f-g)(x)$
3. $(g+f)(x)$
4. $(f/h)(x)$
5. $(f\cdot h)(-1)$
6. $(g+h)(-3)$
7. $(g-f)(2)$
8. $(f/h)\left(\frac{3}{2}\right)$

Answer

1. $(g\cdot h)(x)=4x^3-4x^2+4x$
3. $(g+f)(x)=16x^3-11x^2+3x+1$
5. $(f\cdot h)(-1)=128$
7. $(g-f)(2)=-85$

Exercise $13.E.4$

Factor our the greatest common factor (GCF).

1. $2x^4-12x^3-2x^2$
2. $18a^3b-3a^2b^2+3ab^3$

3. $x^4y^3 - 3x^3y + x^2y$
4. $x^{3n} - x^{2n} - x^n$

Answer

1. $2x^2\left(x^2 - 6x - 1\right)$
3. $x^2y\left(x^2y^2 - 3x + 1\right)$

Exercise 13.E.5

Factor by grouping.

1. $2x^3 - x^2 + 2x - 1$
2. $3x^3 - x^2 - 6x + 2$
3. $x^3 - 5x^2y + xy^2 - 5y^3$
4. $a^2b - a + ab^3 - b^2$
5. $2x^4 - 4xy^3 + 2x^2y^2 - 4x^3y$
6. $x^4y^2 - xy^5 + x^3y^4 - x^2y^3$

Answer

1. $\left(x^2 + 1\right)(2x - 1)$
3. $\left(x^2 + y^2\right)(x - 5y)$
5. $2x(x - 2y)\left(x^2 + y^2\right)$

Exercise 13.E.6

Factor the special binomials.

1. $64x^2 - 1$
2. $9 - 100y^2$
3. $x^2 - 36y^2$
4. $4 - (2x - 1)^2$
5. $a^3b^3 + 125$
6. $64x^3 - y^3$
7. $81x^4 - y^4$
8. $x^8 - 1$
9. $x^6 - 64y^6$
10. $1 - a^6b^6$

Answer

1. $(8x + 1)(8x - 1)$
3. $(x + 6y)(x - 6y)$
5. $(ab + 5)\left(a^2b^2 - 5ab + 25\right)$
7. $\left(9x^2 + y^2\right)(3x + y)(3x - y)$
9. $(x + 2y)\left(x^2 - 2xy + 4y^2\right)(x - 2y)\left(x^2 + 2xy + 4y^2\right)$

Exercise 13.E.7

Factor.

1. $x^2 - 8x - 48$
2. $x^2 - 15x + 54$

3. x^2-4x-6
4. $x^2-12xy+36y^2$
5. $x^2+20xy+75y^2$
6. $-x^2+5x+150$
7. $-2y^2+20y+48$
8. $28x^2+20x+3$
9. $150x^2-100x+6$
10. $24a^2-38ab+3b^2$
11. $27u^2-3uv-4v^2$
12. $16x^2y^2-78xy+27$
13. $16m^2+72mn+81n^2$
14. $4x^2-5x+20$
15. $25x^4-35x^2+6$
16. $2x^4+7x^2+3$
17. $x^6+3x^3y^3-10y^6$
18. $a^6-8a^3b^3+15b^6$
19. $x^{2n}-2x^n+1$
20. $6x^{2n}-x^n-2$

Answer

1. $(x-12)(x+4)$
3. Prime
5. $(x+5y)(x+15y)$
7. $-2(y-12)(y+2)$
9. $2(15x-1)(5x-3)$
11. $(3u+v)(9u-4v)$
13. $(4m+9n)^2$
15. $\left(5x^2-6\right)\left(5x^2-1\right)$
17. $\left(x^3+5y^3\right)\left(x^3-2y^3\right)$
19. $\left(x^n-1\right)^2$

Exercise 13.E.8

Factor completely.

1. $45x^3-20x$
2. $12x^4-70x^3+50x^2$
3. $-20x^2+32x-3$
4. $-x^3y+9xy^3$
5. $24a^4b^2+3ab^5$
6. $64a^6b^6-1$
7. $64x^2+1$
8. $x^3+x^2y-xy^2-y^3$

Answer

1. $5x(3x+2)(3x-2)$
3. $-(10x-1)(2x-3)$
5. $3ab^2(2a+b)\left(4a^2-2ab+b^2\right)$

7. Prime

Exercise 13.E.9

Solve by factoring.

1. $9x^2+8x=0$
2. $x^2-1=0$
3. $x^2-12x+20=0$
4. $x^2-2x-48=0$
5. $(2x+1)(x-2)=3$
6. $2-(x-4)^2=-7$
7. $(x-6)(x+3)=-18$
8. $(x+5)(2x-1)=3(2x-1)$
9. $\frac{1}{2}x^2+\frac{2}{3}x-\frac{1}{8}=0$
10. $\frac{1}{4}x^2-\frac{19}{12}x+\frac{1}{2}=0$
11. $x^3-2x^2-24x=0$
12. $x^4-5x^2+4=0$

Answer

1. $-\frac{8}{9}, 0$
3. $2, 10$
5. $-1, \frac{5}{2}$
7. $0, 3$
9. $-\frac{3}{2}, \frac{1}{6}$
11. $-4, 0, 6$

Exercise 13.E.10

Find the roots of the given functions.

1. $f(x)=12x^2-8x$
2. $g(x)=2x^3-18x$
3. $h(t)=-16t^2+64$
4. $p(x)=5x^2-21x+4$
5.

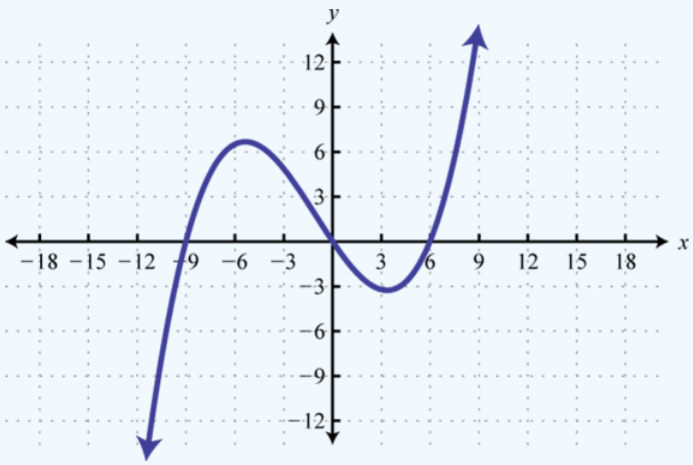

Figure 4.E.3

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6.

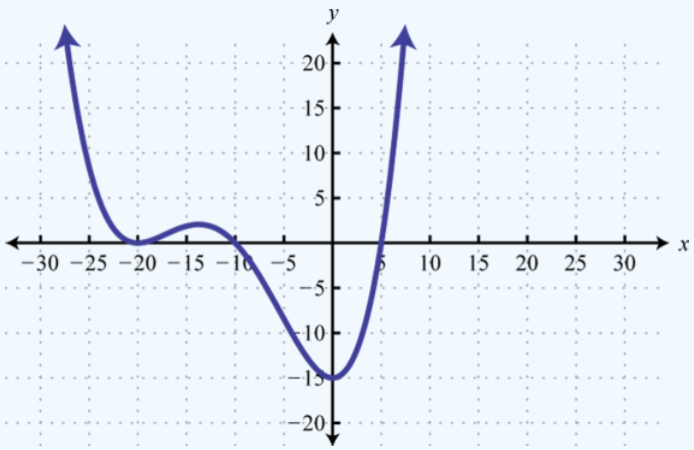

Figure 4.E.4

7. The height in feet of an object dropped from the top of a 16-foot ladder is given by $h(t) = -16t^2 + 16$, where t represents the time in seconds after the object has been dropped. How long will it take to hit the ground?

8. The length of a rectangle is 2 centimeters less than twice its width. If the area of the rectangle is 112 square centimeters, find its dimensions.

9. A triangle whose base is equal in measure to its height has an area of 72 square inches. Find the length of the base.

10. A box can be made by cutting out the corners and folding up the edges of a sheet of cardboard. A template for a rectangular cardboard box of height 2 inches is given.

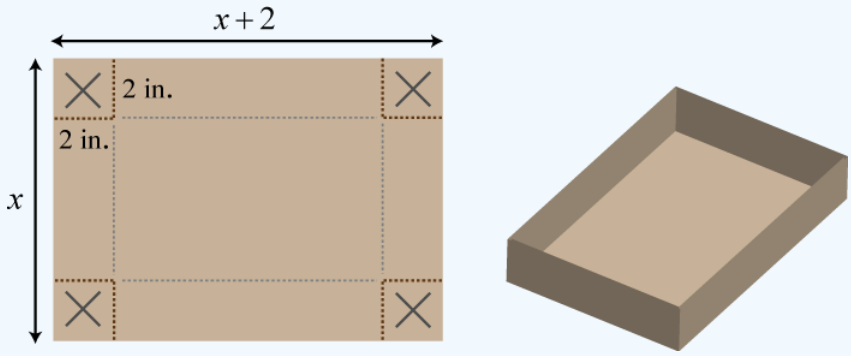

Figure 4.E.5

What are the dimensions of a cardboard sheet that will make a rectangular box with volume 240 cubic inches?

Answer

1. $0, \frac{2}{3}$
3. ± 2
5. $-9, 0, 6$
7. 1 second
9. 12 inches

Exercise 13.E.11

Solve or factor.

1. $x^2 - 25$
2. $x^2 - 121 = 0$
3. $16x^2 - 22x - 3 = 0$

4. $3x^2 - 14x - 5$
5. $x^3 - x^2 - 2x - 2$
6. $3x^2 = -15x$

Answer

1. Factor; $(x+5)(x-5)$
3. Solve; $-\frac{1}{8}, \frac{3}{2}$
5. Factor; $(x-1)\left(x^2-2\right)$

Exercise 13.E.12

Find a polynomial equation with integer coefficients, given the solutions.

1. $5, -2$
2. $\frac{2}{3}, -\frac{1}{2}$
3. $\pm\frac{4}{5}$
4. ± 10
5. $-4, 0, 3$
6. -8 double root

Answer

1. $x^2 - 3x - 10 = 0$
3. $25x^2 - 16 = 0$
5. $x^3 + x^2 - 12x = 0$

Exercise 13.E.13

State the restrictions and simplify.

1. $\frac{108x^3}{12x^2}$
2. $\frac{56x^2(x-2)^2}{8x(x-2)^3}$
3. $\frac{64-x^2}{2x^2-15x-8}$
4. $\frac{3x^2+28x+9}{81-x^2}$
5. $\frac{x^2-25}{5x^2} \cdot \frac{10x^2-15x}{2x^2+7x-15}$
6. $\frac{7x^2-41x-6}{(x-7)^2} \cdot \frac{49-x^2}{x^2+x-42}$
7. $\frac{28x^2(2x-3)}{4x^2-9} \div \frac{7x}{4x^2-12x+9}$
8. $\frac{x^2-10x+24}{x^2-8x+16} \div \frac{2x^2-13x+6}{x^2+2x-24}$

Answer

1. $9x; x \neq 0$
3. $-\frac{x+8}{2x+1}; x \neq -\frac{1}{2}, 8$
5. $\frac{x-5}{x}; x \neq -5, 0, \frac{3}{2}$
7. $\frac{4x(2x-3)^2}{2x+3}; x \neq \pm\frac{3}{2}, 0$

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Exercise 13.E.14

Perform the operations and simplify. Assume all variable expressions in the denominator are nonzero.

1. $\frac{a^2-b^2}{4a^2b^2+4ab^3} \cdot \frac{2ab}{a^2-2ab+b^2}$
2. $\frac{a^2-5ab+6b^2}{a^2-4ab+4b^2} \div \frac{9b^2-a^2}{3a^3b-6a^2b^2}$
3. $\frac{x^2+xy+y^2}{4x^2+3xy-y^2} \cdot \frac{x^2-y^2}{x^3-y^3} \div \frac{x+y}{12x^2y-3xy^2}$
4. $\frac{x^4-y^4}{x^2-2xy+y^2} \div \frac{x^2-4xy-5y^2}{10x^3} \cdot \frac{2x^2-11xy+5y^2}{2x^3y+2xy^3}$

Answer

1. $\frac{1}{2b(a-b)}$
3. $\frac{3xy}{x+y}$

Exercise 13.E.15

Perform the operations and state the restrictions.

1. Given $f(x)=\frac{4x^2+39x-10}{x^2+3x-10}$ and $g(x)=\frac{2x^2+7x-15}{x^2+13x+30}$, find $(f \cdot g)(x)$.
2. Given $f(x)=\frac{25-x^2}{3+x}$ and $g(x)=\frac{9-x^2}{5-x}$, find $(f \cdot g)(x)$.
3. Given $f(x)=\frac{42x^2}{2x^2+3x-2}$ and $g(x)=\frac{14x}{4x^2-4x+1}$, find $(f/g)(x)$.
4. Given $f(x)=\frac{x^2-20x+100}{x^2-1}$ and $g(x)=\frac{x^2-100}{x^2+2x+1}$, find $(f/g)(x)$.
5. The daily cost in dollars of running a small business is given by $C(x)=150+45x$ where x represents the number of hours the business is in operation. Determine the average cost per hour if the business is in operation for 8 hours in a day.
6. An electric bicycle manufacturer has determined that the cost of producing its product in dollars is given by the function $C(n)=2n^2+100n+2,500$ where n represents the number of electric bicycles produced in a day. Determine the average cost per bicycle if 10 and 20 are produced in a day.
7. Given $f(x)=3x-5$, simplify $\frac{f(x+h)-f(x)}{h}$.
8. Given $g(x)=2x^2-x+1$, simplify $\frac{g(x+h)-g(x)}{h}$.

Answer

1. $(f \cdot g)(x)=\frac{(4x-1)(2x-3)}{(x-2)(x+3)}; x \neq -10, -5, -3, 2$
3. $(f/g)(x)=\frac{3x(2x-1)}{x+2}; x \neq -2, 0, \frac{1}{2}$
5. $\$63.75$per hour
7. 3

Exercise 13.E.16

State the restrictions and simplify.

1. $\frac{5x-6}{x^2-36}-\frac{4x}{x^2-36}$
2. $\frac{2}{x}+5x$
3. $\frac{5}{x-5}+\frac{1}{2x}$
4. $\frac{x}{x-2}+\frac{3}{x+3}$
5. $\frac{7(x-1)}{4x^2-17x+15}-\frac{2}{x-3}$
6. $\frac{5}{x}-\frac{19x+25}{2x^2+5x}$
7. $\frac{x}{x-5}-\frac{2}{x-3}-\frac{5(x-3)}{x^2-8x+15}$

8. $\frac{3x}{2x-1}-\frac{x-4}{x+4}+\frac{12(2-x)}{2x^2+7x-4}$
9. $\frac{1}{t-1}+\frac{1}{(t-1)^2}-\frac{1}{t^2-1}$
10. $\frac{1}{t-1}-\frac{2t-5}{t^2-2t+1}-\frac{5t^2-3t-2}{(t-1)^3}$
11. $2x^{-1}+x^{-2}$
12. $(x-4)^{-1}-2x^{-2}$

Answer

1. $\frac{1}{x+6}; x \neq \pm 6$
3. $\frac{11x-5}{2x(x-5)}; x \neq 0, 5$
5. $-\frac{1}{4x-5}; x \neq \frac{5}{4}, 3$
7. $\frac{x-5}{x-3}; x \neq 3, 5$
9. $\frac{t^2+1}{(t+1)(t-1)^2}; t \neq \pm 1$
11. $\frac{2x+1}{x^2}; x \neq 0$

Exercise 13.E.17

Simplify. Assume that all variable expressions used as denominators are nonzero.

1. $\frac{\frac{1}{7}+\frac{1}{x}}{\frac{1}{49}-\frac{1}{x^2}}$
2. $\frac{\frac{1}{100}-\frac{1}{x^2}}{\frac{1}{x}-\frac{1}{10}}$
3. $\frac{\frac{3}{x}-\frac{1}{x-5}}{\frac{5}{x+2}-\frac{2}{x}}$
4. $\frac{1-\frac{12}{x}+\frac{35}{x^2}}{1-\frac{25}{x^2}}$
5. $\frac{x-4x^{-1}}{2-5x^{-1}+2x^{-2}}$
6. $\frac{8x^{-1}+y^{-1}}{y^{-2}-64x^{-2}}$

Answer

1. $\frac{7x}{x-7}$
3. $\frac{(x+2)(2x-15)}{(x-5)(3x-4)}$
5. $\frac{x(x+2)}{2x-1}$

Exercise 13.E.18

Perform the operations and state the restrictions.

1. Given $f(x)=\frac{3}{x-3}$ and $g(x)=\frac{x-2}{x+2}$, find $(f+g)(x)$.
2. Given $f(x)=\frac{1}{x^2+x}$ and $g(x)=\frac{2x}{x^2-1}$, find $(f+g)(x)$.
3. Given $f(x)=\frac{x-3}{x-5}$ and $g(x)=\frac{x^2-x}{x^2-25}$, find $(f-g)(x)$.
4. Given $f(x)=\frac{11x+4}{x^2-2x-8}$ and $g(x)=\frac{2x}{x-4}$, find $(f-g)(x)$.

Answer

1. $(f+g)(x)=\frac{x^2-2x+12}{(x-3)(x+2)}; x\neq -2,3$

3. $(f-g)(x)=\frac{3}{x+5}; x\neq \pm 5$

Exercise 13.E.19

Solve.

1. $\frac{3}{x}=\frac{1}{2x+15}$
2. $\frac{x}{x-4}=\frac{x+8}{x-8}$
3. $\frac{x+5}{2(x+2)}+\frac{x-2}{x+4}=1$
4. $\frac{2x}{x-5}+\frac{1}{x+1}=0$
5. $\frac{x+1}{x-4}+\frac{4}{x+6}=-\frac{10}{x^2+2x-24}$
6. $\frac{2}{x}-\frac{12}{2x+3}=\frac{2-3x^2}{2x^2+3x}$
7. $\frac{x+7}{x-2}-\frac{9}{x+7}=\frac{81}{x^2+5x-14}$
8. $\frac{x}{x+5}+\frac{1}{x-4}=\frac{4x-7}{x^2+x-20}$
9. $\frac{2}{3x-1}+\frac{x}{2x+1}=\frac{2(3-4x)}{6x^2+x-1}$
10. $\frac{x}{x-1}+\frac{1}{x+1}=\frac{2x}{x^2-1}$
11. $\frac{2x}{x+5}-\frac{1}{2x-3}=\frac{4-7x}{2x^2+7x-15}$
12. $\frac{x}{x+4}+\frac{1}{2x+7}=\frac{x+8}{2x^2+15x+28}$
13. $\frac{1}{t-1}-\frac{2}{2t+1}=\frac{1}{t-2}-\frac{2}{2t-1}$
14. $\frac{t-1}{t-2}-\frac{t-2}{t-3}=\frac{t-3}{t-4}-\frac{t-4}{t-5}$
15. Solve for a : $\frac{1}{a}=\frac{1}{b}-\frac{1}{c}$
16. Solve for y : $x=\frac{3y-1}{y-5}$
17. A positive integer is 4 less than another. If the reciprocal of the larger integer is subtracted from twice the reciprocal of the smaller, the result is $\frac{1}{6}$. Find the two integers.
18. If 3 times the reciprocal of the larger of two consecutive odd integers is added to 7 times the reciprocal of the smaller, the result is $\frac{4}{3}$. Find the integers.
19. If the reciprocal of the smaller of two consecutive integers is subtracted from three times the reciprocal of the larger, the result is $\frac{3}{10}$. Find the integers.
20. A positive integer is twice that of another. The sum of the reciprocals of the two positive integers is $\frac{1}{4}$. Find the two integers.

Answer

1. -9

3. $-1, 4$

5. $-11, 0$

7. Ø

9. -4

11. $-\frac{3}{2}$

13. $\frac{3}{4}$

15. $a=\frac{bc}{c-b}$

17. $8, 12$

19. $5, 6$

Exercise 13.E.20

Use algebra to solve the following applications.

1. Manuel traveled 8 miles on the bus and another 84 miles on a train. If the train was 16 miles per hour faster than the bus, and the total trip took 2 hours, what was the average speed of the train?
2. A boat can average 10 miles per hour in still water. On a trip downriver, the boat was able to travel 7.5 miles with the current. On the return trip, the boat was only able to travel 4.5 miles in the same amount of time against the current. What was the speed of the current?
3. Susan can jog, on average, $1\frac{1}{2}$ miles per hour faster than her husband Bill. Bill can jog 10 miles in the same amount of time it takes Susan to jog 13 miles. How fast, on average, can Susan jog?
4. In the morning, Raul drove 8 miles to visit his grandmother and then returned later that evening. Because of traffic, his average speed on the return trip was $\frac{1}{2}$ that of his average speed that morning. If the total driving time was $\frac{3}{4}$ of an hour, what was his average speed on the return trip?
5. One pipe can completely fill a water tank in 6 hours while another smaller pipe takes 8 hours to fill the same tank. How long will it take to fill the tank to $\frac{3}{4}$ capacity if both pipes are turned on?
6. It takes Bill 3 minutes longer than Jerry to fill an order. Working together they can fill 15 orders in 30 minutes. How long does it take Bill to fill an order by himself?
7. Manny takes twice as long as John to assemble a skateboard. If they work together, they can assemble a skateboard in 6 minutes. How long would it take Manny to assemble the skateboard without John's help?
8. Working alone, Joe can complete the yard work in 30 minutes. It takes Mike 45 minutes to complete work on the same yard. How long would it take them working together?

Answer

1. 48 miles per hour

3. 6.5 miles per hour

5. Approximately 2.6 hours

7. 18 minutes

Exercise 13.E.21

Construct a mathematical model given the following:

1. y varies directly as x, where $y = 30$ when $x = 5$.
2. y varies inversely as x, where $y = 3$ when $x = -2$.
3. y is jointly proportional to x and z, where $y = -50$ when $x = -2$ and $z = 5$.
4. y is directly proportional to the square of x and inversely proportional to z, where $y = -6$ when $x = 2$ and $z = -8$.
5. The distance an object in free fall varies directly with the square of the time that it has been falling. It is observed that an object falls 36 feet in $1\frac{1}{2}$ seconds. Find an equation that models the distance an object will fall, and use it to determine how far it will fall in $2\frac{1}{2}$ seconds.
6. After the brakes are applied, the stopping distance d of an automobile varies directly with the square of the speed s of the car. If a car traveling 55 miles per hour takes 181.5 feet to stop, how many feet will it take to stop if it is moving 65 miles per hour?
7. The weight of an object varies inversely as the square of its distance from the center of the Earth. If an object weighs 180 lbs on the surface of the Earth (approximately 4,000 miles from the center), then how much will it weigh at 2,000 miles above the Earth's surface?
8. The cost per person of renting a limousine varies inversely with the number of people renting it. If 5 people go in on the rental, the limousine will cost $112 per person. How much will the rental cost per person if 8 people go in on the rental?
9. To balance a seesaw, the distance from the fulcrum that a person must sit is inversely proportional to his weight. If a 52-pound boy is sitting 3 feet away from the fulcrum, then how far from the fulcrum must a 44-pound boy sit? Round to the nearest tenth of a foot.

Answer

1. $y = 6x$
3. $y = 5xz$
5. $d = 16t^2$; 100 feet
7. 80 lbs
9. Approximately 3.5 feet

Sample Exam

Exercise 13.E.22

Given $f(x) = x^2 - x + 4, g(x) = 5x - 1$, and $h(x) = 2x^2 + x - 3$, find the following:

1. $(g \cdot h)(x)$
2. $(h - f)(x)$
3. $(f + g)(-1)$

Answer

1. $(g \cdot h)(x) = 10x^3 + 3x^2 - 16x + 3$
3. $(f + g)(-1) = 0$

Exercise 13.E.23

Factor.

1. $x^3 + 16x - 2x^2 - 32$
2. $x^3 - 8y^3$
3. $x^4 - 81$
4. $25x^2y^2 - 40xy + 16$
5. $16x^3y + 12x^2y^2 - 18xy^3$

Answer

2. $(x - 2y)\left(x^2 + 2xy + 4y^2\right)$
4. $(5xy - 4)^2$

Exercise 13.E.24

Solve

1. $6x^2 + 24x = 0$
2. $(2x + 1)(3x + 2) = 12$
3. $(2x + 1)^2 = 23x + 6$
4. Find a quadratic equation with integer coefficients given the solutions $\left\{\frac{1}{2}, -3\right\}$.
5. Given $f(x) = 5x^2 - x + 4$, simplify $\frac{f(x+h) - f(x)}{h}$, where $h \neq 0$.

Answer

1. $-4, 0$
3. $-\frac{1}{4}, 5$
5. $10x + 5h - 1$

Exercise 13.E.25

Simplify and state the restrictions.

1. $\frac{4x^2-33x+8}{x^2-10x+16} \div \frac{16x^2-1}{x^2-4x+4}$
2. $\frac{x-1}{x-7} + \frac{1}{1-x} - \frac{2(x+11)}{x^2-8x+7}$

Answer

2. $\frac{x+2}{x-1}; x \neq 1, 7$

Exercise 13.E.26

Assume all variable expressions in the denominator are nonzero and simplify.

1. $\frac{\frac{3}{x}+\frac{1}{y}}{\frac{1}{y^2}-\frac{9}{x^2}}$

Exercise 13.E.27

Solve.

1. $\frac{6x-5}{3x+2} = \frac{2x}{x+1}$
2. $\frac{2x}{x+5} - \frac{1}{5-x} = \frac{2x}{x^2-25}$
3. Find the root of the function defined by $f(x) = \frac{1}{x+3} - 4$.
4. Solve for y: $x = \frac{4y}{3y-1}$

Answer

1. $-\frac{5}{3}$
3. $-\frac{11}{4}$

Exercise 13.E.28

Use algebra to solve.

1. The height of an object dropped from a 64-foot building is given by the function $h(t) = -16t^2 + 64$, where t represents time in seconds after it was dropped.
   1. Determine the height of the object at $\frac{3}{4}$ of a second.
   2. How long will it take the object to hit the ground?
2. One positive integer is 3 units more than another. When the reciprocal of the larger is subtracted from twice the reciprocal of the smaller, the result is $\frac{2}{9}$. Find the two positive integers.
3. A light airplane can average 126 miles per hour in still air. On a trip, the airplane traveled 222 miles with a tailwind. On the return trip, against a headwind of the same speed, the plane was only able to travel 156 miles in the same amount of time. What was the speed of the wind?
4. On the production line, it takes John 2 minutes less time than Mark to assemble a watch. Working together they can assemble 5 watches in 12 minutes. How long does it take John to assemble a watch working alone?
5. Write an equation that relates x and y, given that y varies inversely with the square of x, where $y = -\frac{1}{3}$ when $x = 3$. Use it to find y when $x = \frac{1}{2}$.

Answer

1. (1) 55 feet; (2) 2 seconds
3. 22 miles per hour

5. $y = -\frac{3}{x^2}; y = -12$

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CHAPTER OVERVIEW

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14.1: Prelude to Factoring

Figure 14.1.1: Scientists use factoring to calculate growth rates of infectious diseases such as viruses. (credit: “FotoshopTofs” / Pixabay)

An epidemic of a disease has broken out. Where did it start? How is it spreading? What can be done to control it? Answers to these and other questions can be found by scientists known as epidemiologists. They collect data and analyze it to study disease and consider possible control measures. Because diseases can spread at alarming rates, these scientists must use their knowledge of mathematics involving factoring. In this chapter, you will learn how to factor and apply factoring to real-life situations.

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14.2: Greatest Common Factor and Factor by Grouping

Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

Be Prepared

Before you get started, take this readiness quiz.

1. Factor 56 into primes.
   If you missed this problem, review [link].
2. Find the least common multiple (LCM) of 18 and 24.
   If you missed this problem, review [link].
3. Multiply: $-3a(7a+8b)$.
   If you missed this problem, review [link].

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a **product**. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.

We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

GREATEST COMMON FACTOR

The **greatest common factor** (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

We summarize the steps we use to find the greatest common factor.

FIND THE GREATEST COMMON FACTOR (GCF) OF TWO EXPRESSIONS.

1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. List all factors—matching common factors in a column. In each column, circle the common factors.
3. Bring down the common factors that all expressions share.
4. Multiply the factors.

The next example will show us the steps to find the greatest common factor of three expressions.

Example 14.2.1

Find the greatest common factor of $21x^3,\ 9x^2,\ 15x$.

Answer

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors.	$21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$ $\text{GCF} = 3 \cdot x$
Multiply the factors.	$\text{GCF} = 3x$
	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

Example 14.2.2

Find the greatest common factor: $25m^4,\ 35m^3,\ 20m^2$.

Answer

$5m^2$

Example 14.2.3

Find the greatest common factor: $14x^3,\ 70x^2,\ 105x$.

Answer

$7x$

Factor the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as $2 \cdot 6$ or $3 \cdot 4$. In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x$, and end with its factors, $3x(x + 5)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

DISTRIBUTIVE PROPERTY

If *a*, *b*, and *c* are real numbers, then

$$a(b+c) = ab + ac \quad \text{and} \quad ab + ac = a(b+c)$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a **polynomial**? You just find the GCF of all the terms and write the polynomial as a product!

Example 14.2.4: How to Use the Distributive Property to factor a polynomial

Factor: $8m^3 - 12m^2n + 20mn^2$.

Answer

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $8m^3$, $12m^2n$, $20mn^2$	$8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $GCF = 2 \cdot 2 \cdot m$ $GCF = 4m$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $8m^3$, $12m^2n$, $20mn^2$ as products of their GCF, $4m$. $8m^3 = 4m \cdot 2m^2$ $12m^2n = -4m \cdot 3m\,n$ $20mn^2 = 4m \cdot 5n^2$	$8m^3 - 12m^2n + 20mn^2$ $4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$
Step 3. Use the "reverse" Distributive Property to factor the expression.		$4m(2m^2 - 3m\,n + 5n^2)$
Step 4. Check by multiplying the factors.		$4m(2m^2 - 3m\,n + 5n^2)$ $4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$ $8m^3 - 12m^2n + 20mn^2$ ✓

Example 14.2.5

Factor: $9xy^2 + 6x^2y^2 + 21y^3$.

Answer

$3y^2(3x + 2x^2 + 7y)$

Example 14.2.6

Factor: $3p^3 - 6p^2q + 9pq^3$.

Answer

$3p(p^2 - 2pq + 3q^3)$

FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL.

1. Find the GCF of all the terms of the polynomial.
2. Rewrite each term as a product using the GCF.
3. Use the "reverse" Distributive Property to factor the expression.
4. Check by multiplying the factors.

FACTOR AS A NOUN AND A VERB

We use "factor" as both a noun and a verb:

Noun:	7 is a factor of 14
Verb:	factor 3 from $3a + 3$

Example 14.2.7

Factor: $5x^3 - 25x^2$.

Answer

Find the GCF of $5x^3$ and $25x^2$.		$5x^3 = \textcircled{5} \cdot \textcircled{x} \cdot \textcircled{x} \cdot x$ $25x^2 = \textcircled{5} \cdot 5 \cdot \textcircled{x} \cdot \textcircled{x}$ $\text{GCF} = 5 \cdot x \cdot x$
		$\text{GCF} = 5x^2$
		$5x^3 - 25x^2$
Rewrite each term.		$5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.		$5x^2(x - 5)$
Check: $5x^2(x-5)$ $5x^2 \cdot x - 5x^2 \cdot 5$ $5x^3 - 25x^2$ ✓		

Example 14.2.8

Factor: $2x^3 + 12x^2$.

Answer

$2x^2(x+6)$

Example 14.2.9

Factor: $6y^3 - 15y^2$.

Answer

$3y^2(2y-5)$

Example 14.2.10

Factor: $8x^3y - 10x^2y^2 + 12xy^3$.

Answer

The GCF of $8x^3y$, $-10x^2y^2$, and $12xy^3$ is $2xy$.	$8x^3y = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot y$ $10x^2y^2 = 2 \cdot 5 \cdot x \cdot x \cdot y \cdot y$ $12xy^3 = 2 \cdot 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$ $\text{GCF} = 2 \cdot x \cdot y$
	$\text{GCF} = 2xy$
	$8x^3y - 10x^2y^2 + 12xy^3$
Rewrite each term using the GCF, $2xy$.	$2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$
Factor the GCF.	$2xy(4x^2 - 5xy + 6y^2)$

Check: $$2xy(4x^2-5xy+6y^2)$$ $$2xy\cdot 4x^2-2xy\cdot 5xy+2xy\cdot 6y^2$$ $$8x^3y-10x^2y^2+12xy^3\checkmark$$	

? Example 14.2.11

Factor: $15x^3y-3x^2y^2+6xy^3$.

Answer

$3xy(5x^2-xy+2y^2)$

? Example 14.2.12

Factor: $8a^3b+2a^2b^2-6ab^3$.

Answer

$2ab(4a^2+ab-3b^2)$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

? Example 14.2.13

Factor: $-4a^3+36a^2-8a$.

Answer

The leading coefficient is negative, so the GCF will be negative.

	$-4a^3+36a^2-8a$
Rewrite each term using the GCF, $-4a$.	$-4a\cdot a^2-(-4a)\cdot 9a+(-4a)\cdot 2$
Factor the GCF.	$-4a(a^2-9a+2)$
Check: $$-4a(a^2-9a+2)$$ $$-4a\cdot a^2-(-4a)\cdot 9a+(-4a)\cdot 2$$ $$-4a^3+36a^2-8a\checkmark$$	

? Example 14.2.14

Factor: $-4b^3+16b^2-8b$.

Answer

$-4b(b^2-4b+2)$

Example 14.2.15

Factor: $-7a^3+21a^2-14a$.

Answer

$-7a(a^2-3a+2)$

So far our greatest common factors have been monomials. In the next example, the greatest common factor is a binomial.

Example 14.2.16

Factor: $3y(y+7)-4(y+7)$.

Answer

The GCF is the binomial $y+7$.

		$3y(y+7)-4(y+7)$
Factor the GCF, $(y+7)$.		$(y+7)(3y-4)$
Check on your own by multiplying.		

Example 14.2.17

Factor: $4m(m+3)-7(m+3)$.

Answer

$(m+3)(4m-7)$

Example 14.2.18

Factor: $8n(n-4)+5(n-4)$.

Answer

$(n-4)(8n+5)$

Factor by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the **GCF** in each part. If the polynomial can be factored, you will find a common factor emerges from both parts. Not all polynomials can be factored. Just like some numbers are **prime**, some polynomials are prime.

Example 14.2.19: How to Factor a Polynomial by Grouping

Factor by grouping: $xy+3y+2x+6$.

Answer

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms?	$xy + 3y + 2x + 6$
	No, so let's separate the first two terms from the second two.	$xy + 3y + 2x + 6$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms.	$y(x + 3) + 2x + 6$
	Factor the GCF from the second two terms.	$y(x + 3) + 2(x + 3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x + 3)$.	$y(x + 3) + 2(x + 3)$
	Factor out the common factor.	$(x + 3)(y + 2)$
Step 4. Check.	Multiply $(x + 3)(y + 2)$. Is the product the original expression?	$(x + 3)(y + 2)$ $xy + 2x + 3y + 6$ $xy + 3y + 2x + 6$ ✓

Example 14.2.20

Factor by grouping: $xy+8y+3x+24$.

Answer

$(x+8)(y+3)$

Example 14.2.21

Factor by grouping: $ab+7b+8a+56$.

Answer

$(a+7)(b+8)$

FACTOR BY GROUPING.

1. Group terms with common factors.
2. Factor out the common factor in each group.
3. Factor the common factor from the expression.
4. Check by multiplying the factors.

Example 14.2.22

Factor by grouping: ⓐ $x^2+3x-2x-6$ ⓑ $6x^2-3x-4x+2$.

Answer

ⓐ

There is no GCF in all four terms.	$x^2+3x-2x-6$
Separate into two parts.	$x^2+3x \quad -2x-6$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x(x+3)-2(x+3)$
Factor out the common factor.	$(x+3)(x-2)$
Check on your own by multiplying.	

ⓑ

There is no GCF in all four terms.	$6x^2-3x-4x+2$
Separate into two parts.	$6x^2-3x \quad -4x+2$
Factor the GCF from both parts.	$3x(2x-1)-2(2x-1)$
Factor out the common factor.	$(2x-1)(3x-2)$
Check on your own by multiplying.	

? Example 14.2.23

Factor by grouping: ⓐ $x^2+2x-5x-10$ ⓑ $20x^2-16x-15x+12$.

Answer

ⓐ $(x-5)(x+2)$
ⓑ $(5x-4)(4x-3)$

? Example 14.2.24

Factor by grouping: ⓐ $y^2+4y-7y-28$ ⓑ $42m^2-18m-35m+15$.

Answer

ⓐ $(y+4)(y-7)$
ⓑ $(7m-3)(6m-5)$

Key Concepts

- **How to find the greatest common factor (GCF) of two expressions.**
  1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
  2. List all factors—matching common factors in a column. In each column, circle the common factors.
  3. Bring down the common factors that all expressions share.
  4. Multiply the factors.
- **Distributive Property:** If a, b and c are real numbers, then

$$a(b+c)=ab+ac \quad \text{and} \quad ab+ac=a(b+c)$$

  The form on the left is used to multiply. The form on the right is used to factor.
- **How to factor the greatest common factor from a polynomial.**
  1. Find the GCF of all the terms of the polynomial.
  2. Rewrite each term as a product using the GCF.
  3. Use the "reverse" Distributive Property to factor the expression.
  4. Check by multiplying the factors.
- **Factor as a Noun and a Verb:** We use "factor" as both a noun and a verb.

Noun:	7 is a factor of 14
Verb:	factor 3 from $3a+3$

- **How to factor by grouping.**
  1. Group terms with common factors.
  2. Factor out the common factor in each group.
  3. Factor the common factor from the expression.
  4. Check by multiplying the factors.

Glossary

factoring

Splitting a product into factors is called factoring.

greatest common factor

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

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14.2E: Exercises

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. $10p^3q, 12pq^2$

Answer

$2pq$

2. $8a^2b^3, 10ab^2$

3. $12m^2n^3, 30m^5n^3$

Answer

$6m^2n^3$

4. $28x^2y^4, 42x^4y^4$

5. $10a^3, 12a^2, 14a$

Answer

$2a$

6. $20y^3, 28y^2, 40y$

7. $35x^3y^2, 10x^4y, 5x^5y^3$

Answer

$5x^3y$

8. $27p^2q^3, 45p^3q^4, 9p^4q^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

9. $6m+9$

Answer

$3(2m+3)$

10. $14p+35$

11. $9n-63$

Answer

$9(n-7)$

12. $45b-18$

13. $3x^2+6x-9$

Answer

$3(x^2+2x-3)$

14. $4y^2+8y-4$

15. $8p^2+4p+2$

Answer

$2(4p^2+2p+1)$

16. $10q^2+14q+20$

17. $8y^3+16y^2$

Answer

$8y^2(y+2)$

18. $12x^3-10x$

19. $5x^3-15x^2+20x$

Answer

$5x(x^2-3x+4)$

20. $8m^2-40m+16$

21. $24x^3-12x^2+15x$

Answer

$3x(8x^2-4x+5)$

22. $24y^3-18y^2-30y$

23. $12xy^2+18x^2y^2-30y^3$

Answer

$6y^2(2x+3x^2-5y)$

24. $21pq^2+35p^2q^2-28q^3$

25. $20x^3y-4x^2y^2+12xy^3$

Answer

$4xy(5x^2-xy+3y^2)$

26. $24a^3b+6a^2b^2-18ab^3$

27. $-2x-4$

Answer

$-2(x+4)$

28. $-3b+12$

29. $-2x^3+18x^2-8x$

Answer

$-2x(x^2-9x+4)$

30. $-5y^3+35y^2-15y$

31. $-4p^3q-12p^2q^2+16pq^2$

Answer

$-4pq(p^2+3pq-4q)$

32. $-6a^3b-12a^2b^2+18ab^2$

33. $5x(x+1)+3(x+1)$

Answer

$(x+1)(5x+3)$

34. $2x(x-1)+9(x-1)$

35. $3b(b-2)-13(b-2)$

Answer

$(b-2)(3b-13)$

36. $6m(m-5)-7(m-5)$

Factor by Grouping

In the following exercises, factor by grouping.

37. $ab+5a+3b+15$

Answer

$(b+5)(a+3)$

38. $cd+6c+4d+24$

39. $8y^2+y+40y+5$

Answer

$(y+5)(8y+1)$

40. $6y^2+7y+24y+28$

41. $uv-9u+2v-18$

Answer

$(u+2)(v-9)$

42. $pq-10p+8q-80$

43. $u^2-u+6u-6$

Answer

$(u-1)(u+6)$

44. $x^2-x+4x-4$

45. $9p^2-15p+12p-20$

Answer

$(3p-5)(3p+4)$

46. $16q^2+20q-28q-35$

47. $mn-6m-4n+24$

Answer

$(n-6)(m-4)$

48. $r^2-3r-r+3$

49. $2x^2-14x-5x+35$

Answer

$(x-7)(2x-5)$

50. $4x^2-36x-3x+27$

Mixed Practice

In the following exercises, factor.

51. $-18xy^2-27x^2y$

Answer

$-9xy(3x+2y)$

52. $-4x^3y^5-x^2y^3+12xy^4$

53. $3x^3 - 7x^2 + 6x - 14$

Answer

$(x^2 + 2)(3x - 7)$

54. $x^3 + x^2 - x - 1$

55. $x^2 + xy + 5x + 5y$

Answer

$(x + y)(x + 5)$

56. $5x^3 - 3x^2 + 5x - 3$

Writing Exercises

57. What does it mean to say a polynomial is in factored form?

Answer

Answers will vary.

58. How do you check result after factoring a polynomial?

59. The greatest common factor of 36 and 60 is 12. Explain what this means.

Answer

Answers will vary.

60. What is the GCF of y^4, y^5, and y^{10}? Write a general rule that tells you how to find the GCF of y^a, y^b, and y^c.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			
factor by grouping.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you

the help you need.

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14.3: Factor Trinomials

Learning Objectives

By the end of this section, you will be able to:

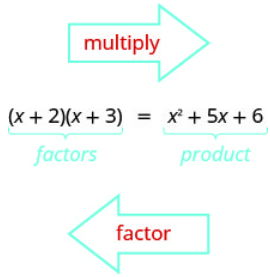
- Factor trinomials of the form x^2+bx+c
- Factor trinomials of the form ax^2+bx+c using trial and error
- Factor trinomials of the form ax^2+bx+c using the 'ac' method
- Factor using substitution

Before you get started, take this readiness quiz.

1. Find all the factors of 72.
   If you missed this problem, review [link].
2. Find the product: $(3y+4)(2y+5)$.
   If you missed this problem, review [link].
3. Simplify: $-9(6);\ -9(-6)$.
   If you missed this problem, review [link].

Factor Trinomials of the Form x^2+bx+c

You have already learned how to multiply binomials using **FOIL**. Now you'll need to "undo" this multiplication. To factor the trinomial means to start with the product, and end with the factors.



To figure out how we would factor a **trinomial** of the form x^2+bx+c, such as x^2+5x+6 and factor it to $(x+2)(x+3)$, let's start with two general binomials of the form $(x+m)$ and $(x+n)$.

	$(x+m)(x+n)$
Foil to find the product.	$x^2+mx+nx+mn$
Factor the GCF from the middle terms.	$x^2+(m+n)x+mn$
Our trinomial is of the form x^2+bx+c.	$\overbrace{x^2+(m+n)x+mn}^{x^2+bx+c}$

This tells us that to factor a trinomial of the form x^2+bx+c, we need two factors $(x+m)$ and $(x+n)$ where the two numbers m and n multiply to c and add to b.

Example 14.3.1: How to Factor a Trinomial of the form x^2+bx+c

Factor: $x^2+11x+24$.

Answer

Step 1. Write the factors as two binomials with first terms x.	Write two sets of parentheses and put x as the first term.	$x^2 + 11x + 24$ $(x \quad)(x \quad)$
Step 2. Find two numbers m and n that multiply to c, $m \cdot n = c$ add to b, $m + n = b$	Find two numbers that multiply to 24 and add to 11. **Factors of 24** / **Sum of factors** 1, 24 — $1 + 24 = 25$ 2, 12 — $2 + 12 = 14$ 3, 8 — $3 + 8 = 11$* 4, 6 — $4 + 6 = 10$	
Step 3. Use m and n as the last terms of the factors.	Use 3 and 8 as the last terms of the binomials.	$(x + 3)(x + 8)$
Step 4. Check by multiplying the factors.		$(x + 3)(x + 8)$ $x^2 + 8x + 3x + 24$ $x^2 + 11x + 24$ ✓

? Example 14.3.2

Factor: $q^2 + 10q + 24$.

Answer

$(q+4)(q+6)$

? Example 14.3.3

Factor: $t^2 + 14t + 24$.

Answer

$(t+2)(t+12)$

Let's summarize the steps we used to find the factors.

1. Write the factors as two binomials with first terms x. $x^2 + bx + c$ $(x \quad)(x \quad)$
2. Find two numbers m and n that
   - multiply to c, $m \cdot n = c$
   - add to b, $m + n = b$
3. Use m and n as the last terms of the factors. $(x+m)(x+n)$
4. Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

How do you get a *positive product* and a *negative sum*? We use two negative numbers.

? Example 14.3.4

Factor: $y^2 - 11y + 28$.

Answer

Again, with the positive last term, 28, and the negative middle term, $-11y$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

	$y^2-11y+28$
Write the factors as two binomials with first terms y.	$(y\quad)(y\quad)$
Find two numbers that: multiply to 28 and add to -11.	

Factors of 28	Sum of factors
$-1,\ -28$	$-1+(-28)=-29$
$-2,\ -14$	$-2+(-14)=-16$
$-4,\ -7$	$-4+(-7)=-11^*$

Use $-4,\ -7$ as the last terms of the binomials.	$(y-4)(y-7)$
Check:	
	$(y-4)(y-7)$
	$y^2-7y-4y+28$
	$y^2-11y+28\checkmark$

? Example 14.3.5

Factor: $u^2-9u+18$.

Answer

$(u-3)(u-6)$

? Example 14.3.6

Factor: $y^2-16y+63$.

Answer

$(y-7)(y-9)$

Now, what if the last term in the trinomial is negative? Think about **FOIL**. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

How do you get a *negative product* and a *positive sum*? We use one positive and one negative number.

When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

? Example 14.3.7

Factor: $2x+x^2-48$.

Answer

	$2x+x^2-48$
First we put the terms in decreasing degree order.	$x^2+2x-48$
Factors will be two binomials with first terms x.	$(x\quad)(x\quad)$

Factors of −48−48	Sum of factors

Factors of -48	Sum of factors
$-1,\ 48$	$-1+48=47$
$-2,\ 24$	$-2+24=22$
$-3,\ 16$	$-3+16=13$
$-4,\ 12$	$-4+12=8$
$-6,\ 8$	$-6+8=2^*$

Use $-6,\ 8$ as the last terms of the binomials. $(x-6)(x+8)$

Check:

$$(x-6)(x+8)$$
$$x^2-6q+8q-48$$
$$x^2+2x-48\checkmark$$

Example 14.3.8

Factor: $9m+m^2+18$.

Answer

$(m+3)(m+6)$

Example 14.3.9

Factor: $-7n+12+n^2$.

Answer

$(n-3)(n-4)$

Sometimes you'll need to factor trinomials of the form $x^2+bxy+cy^2$ with two variables, such as $x^2+12xy+36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x\cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain y. To get the coefficients b and c, you use the same process summarized in How To Factor trinomials.

Example 14.3.10

Factor: $r^2-8rs-9s^2$.

Answer

We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$r^2-8rs-9s^2$$
$$(r\quad s)(r\quad s)$$

Note that the first terms are r, last terms contain s.

Find the numbers that multiply to -9 and add to -8.

Factors of -9	Sum of factors
$1,\ -9$	$-1+9=8$
$-1,\ 9$	$1+(-9)=-8^*$
$3,\ -3$	$3+(-3)=0$

Use $1, -9$ as coefficients of the last terms. $(r+s)(r-9s)$

Check:

$$(r-9s)(r+s)$$
$$r^2+rs-9rs-9s^2$$
$$r^2-8rs-9s^2 \checkmark$$

Example 14.3.11

Factor: $a^2-11ab+10b^2$.

Answer

$(a-b)(a-10b)$

Example 14.3.12

Factor: $m^2-13mn+12n^2$.

Answer

$(m-n)(m-12n)$

Some trinomials are prime. The only way to be certain a trinomial is **prime** is to list all the possibilities and show that none of them work.

Example 14.3.13

Factor: $u^2-9uv-12v^2$.

Answer

We need u in the first term of each binomial and v in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$u^2-9uv-12v^2$$

Note that the first terms are u, last terms contain v. $(u \quad v)(u \quad v)$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	Sum of factors
$1, -12$	$1+(-12)=-11$
$-1, 12$	$-1+12=11$
$2, -6$	$2+(-6)=-4$
$-2, 6$	$-2+6=4$
$3, -4$	$3+(-4)=-1$
$-3, 4$	$-3+4=1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

Example 14.3.14

Factor: $x^2-7xy-10y^2$.

Answer

prime

Example 14.3.15

Factor: $p^2+15pq+20q^2$.

Answer

prime

Let's summarize the method we just developed to factor trinomials of the form x^2+bx+c .

STRATEGY FOR FACTORING TRINOMIALS OF THE FORM x^2+bx+c

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

	x^2+bx+c	
	$(x+m)(x+n)$	
When c is positive, m and n have the same sign.		
b positive		b negative
m, n positive		m, n negative
x^2+5x+6		x^2-6x+8
$(x+2)(x+3)$		$(x-4)(x-2)$
same signs		same signs
When c is negative, m and n have the opposite sign.		
x^2+x-12		$x^2-2x-15$
$(x+4)(x-3)$		$(x-5)(x+3)$
opposite signs		opposite signs

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

Factor Trinomials of the form $ax^2 + bx + c$ using Trial and Error

Our next step is to factor trinomials whose leading coefficient is not 1, trinomials of the form ax^2+bx+c .

Remember to always check for a **GCF** first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods we've used so far. Let's do an example to see how this works.

ExAMPLE 14.3.16

Factor completely: $4x^3+16x^2-20x$.

Answer

Is there a greatest common factor? Yes, $GCF = 4x$. Factor it.	$4x^3 + 16x^2 - 20x$ $4x(x^2 + 4x - 5)$
Binomial, trinomial, or more than three terms? It is a trinomial. So "undo FOIL."	$4x(x \quad)(x \quad)$
Use a table like the one shown to find two numbers that multiply to -5 and add to 4.	$4x(x-1)(x+5)$

Factors of -5	Sum of factors
$-1, 5$ $1, -5$	$-1+5=4^*$ $1+(-5)=-4$

Check:

$$4x(x-1)(x+5)$$
$$4x(x^2+5x-x-5)$$
$$4x(x^2+4x-5)$$
$$4x^3+16x2-20x\checkmark$$

? ExAMPLE 14.3.17

Factor completely: $5x^3 + 15x^2 - 20x$.

Answer

$5x(x-1)(x+4)$

? ExAMPLE 14.3.18

Factor completely: $6y^3 + 18y^2 - 60y$.

Answer

$6y(y-2)(y+5)$

What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2 + 5x + 2$.

From our earlier work, we expect this will factor into two binomials.

$$3x^2 + 5x + 2$$
$$(\quad)(\quad)$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x$, $3x$. We can place them in the binomials.

$$3x^2 + 5x + 2$$
$$1x,\ 3x$$
$$(x\quad)(3x\quad)$$

Check: Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1 and 2. But we now have two cases to consider as it will make a difference if we write 1, 2 or 2, 1.

$$3x^2 + 5x + 2 \quad (1x,\ 3x;\ 1,\ 2) \quad (x + 1)(3x + 2) \quad \text{or} \quad 3x^2 + 5x + 2 \quad (1x,\ 3x;\ 1,\ 2) \quad (x + 2)(3x + 1)$$

Which factors are correct? To decide that, we multiply the inner and outer terms.

$$(x + 1)(3x + 2): \quad 3x,\ 2x \rightarrow 5x \qquad \text{or} \qquad (x + 2)(3x + 1): \quad 6x,\ 1x \rightarrow 7x$$

Since the middle term of the trinomial is $5x$, the factors in the first case will work. Let's use FOIL to check.

$$(x+1)(3x+2)$$
$$3x^2+2x+3x+2$$
$$3x^2+5x+2\checkmark$$

Our result of the factoring is:

$$3x^2+5x+2$$
$$(x+1)(3x+2)$$

? ExAMPLE 14.3.19: How to Factor a Trinomial Using Trial and Error

Factor completely using trial and error: $3y^2+22y+7$.

Answer

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2 + 22y + 7$
Step 2. Factor any GCF.	There is no GCF.	
Step 3. Find all the factor pairs of the first term.	The only of $3y^2$ are $1y$, $3y$. Since there is only one pair, we can put them in the parentheses.	$3y^2 + 22y + 7$ ($1y,\ 3y$) $3y^2 + 22y + 7$ ($1y,\ 3y$) $(y\quad)(3y\quad)$
Step 4. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2 + 22y + 7$ ($1y,\ 3y$; $1,\ 7$) $(y\quad)(3y\quad)$

Step 5. Test all the possible combinations of the factors until the correct product is found.	$3y^2 + 22y + 7$ 1y, 3y 1, 7 $(y + 1)(3y + 7)$ 3y 7y 10y No! We need 22y $3y^2 + 22y + 7$ 1y, 3y 1, 7 $(y + 7)(3y + 1)$ 21y +y 22y	$3y^2 + 22y + 7$ **Possible factors** / **Product** $(y + 1)(3y + 7)$ / $3y^2 + 10y + 7$ $(y + 7)(3y + 1)$ / $3y^2 + 22y + 7$
Step 6. Check by multiplying.		$(y + 7)(3y + 1)$ $3y^2 + 22y + 7$ ✓

? ExAMPLE 14.3.20

Factor completely using trial and error: $2a^2+5a+3$.

Answer

$(a+1)(2a+3)$

? ExAMPLE 14.3.21

Factor completely using trial and error: $4b^2+5b+1$.

Answer

$(b+1)(4b+1)$

FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING TRIAL AND ERROR.

1. Write the trinomial in descending order of degrees as needed.
2. Factor any GCF.
3. Find all the factor pairs of the first term.
4. Find all the factor pairs of the third term.
5. Test all the possible combinations of the factors until the correct product is found.
6. Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

? ExAMPLE 14.3.22

Factor completely using trial and error: $6b^2-13b+5$.

Answer

The trinomial is already in descending order.	$6b^2 - 13b + 5$
Find the factors of the first term.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $2b \cdot 3b$

Find the factors of the last term. Consider the signs. Since the last term, 5, is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $2b \cdot 3b$ $-1, -5$

Consider all the combinations of factors.

$6b^2 - 13b + 5$

Possible factors	Product
$(b-1)(6b-5)$	$6b^2 - 11b + 5$
$(b-5)(6b-1)$	$6b^2 - 31b + 5$
$(2b-1)(3b-5)$	$6b^2 - 13b + 5^*$
$(2b-5)(3b-1)$	$6b^2 - 17b + 5$

The correct factors are those whose product is the original trinomial. $(2b-1)(3b-5)$

Check by multiplying:

$$(2b-1)(3b-5)$$
$$6b^2 - 10b - 3b + 5$$
$$6b^2 - 13b + 5 \checkmark$$

? ExAMPLE 14.3.23

Factor completely using trial and error: $8x^2 - 14x + 3$.

Answer

$(2x-3)(4x-1)$

? ExAMPLE 14.3.24

Factor completely using trial and error: $10y^2 - 37y + 7$.

Answer

$(2y-7)(5y-1)$

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

? ExAMPLE 14.3.25

Factor completely using trial and error: $18x^2 - 37xy + 15y^2$.

Answer

The trinomial is already in descending order.	$18x^2 - 37xy + 15y^2$
Find the factors of the first term.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $2x \cdot 9x$ $3x \cdot 6x$

Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative factors.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $\quad -1, -15$ $2x \cdot 9x$ $\quad -3, -5$ $3x \cdot 6x$

Consider all the combinations of factors.

$18x^2 - 37xy + 15y^2$	
Possible factors	**Product**
$(x - 1y)(18x - 15y)$	Not an option
$(x - 15y)(18x - 1y)$	$18x^2 - 271xy + 15y^2$
$(x - 3y)(18x - 5y)$	$18x^2 - 59xy + 15y^2$
$(x - 5y)(18x - 3y)$	Not an option
$(2x - 1y)(9x - 15y)$	Not an option
$(2x - 15y)(9x - 1y)$	$18x^2 - 137xy + 15y^2$
$(2x - 3y)(9x - 5y)$	$18x^2 - 37xy + 15y^2$*
$(2x - 5y)(9x - 3y)$	Not an option
$(3x - 1y)(6x - 15y)$	Not an option
$(3x - 15y)(6x - 1y)$	Not an option
$(3x - 3y)(6x - 5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial.
Check by multiplying:

$(2x - 3y)(9x - 5y)$

$$(2x - 3y)(9x - 5y)$$
$$18x^2 - 10xy - 27xy + 15y^2$$
$$18x^2 - 37xy + 15y^2 \checkmark$$

? ExAMPLE 14.3.26

Factor completely using trial and error $18x^2 - 3xy - 10y^2$.

Answer

$(3x + 2y)(6x - 5y)$

? ExAMPLE 14.3.27

Factor completely using trial and error: $30x^2 - 53xy - 21y^2$.

Answer

$(3x + y)(10x - 21y)$

Don't forget to look for a GCF first and remember if the leading coefficient is negative, so is the GCF.

? ExAMPLE 14.3.28

Factor completely using trial and error: $-10y^4 - 55y^3 - 60y^2$.

Answer

	$-10y^4 - 55y^3 - 60y^2$
Notice the greatest common factor, so factor it first.	$-5y^2(2y^2 + 11y + 12)$
Factor the trinomial.	$-5y^2\left(2y^2 + 11y + 12\right)$ $y \cdot 2y$ $1 \cdot 12$ $2 \cdot 6$ $3 \cdot 4$

Consider all the combinations.

$2y^2 + 11y + 12$	
Possible factors	**Product**
$(y + 1)(2y + 12)$	Not an option
$(y + 12)(2y + 1)$	$2y^2 + 25y + 12$
$(y + 2)(2y + 6)$	Not an option
$(y + 6)(2y + 2)$	Not an option
$(y + 3)(2y + 4)$	Not an option
$(y + 4)(2y + 3)$	$2y^2 + 11y + 12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor -5^y2.

$$-5y^2(y+4)(2y+3)$$

Check by multiplying:

$$-5y^2(y+4)(2y+3)$$
$$-5y^2(2y^2+8y+3y+12)$$
$$-10y^4 - 55y^3 - 60y^2 \checkmark$$

? ExAMPLE 14.3.29

Factor completely using trial and error: $15n^3 - 85n^2 + 100n$.

Answer

$5n(n-4)(3n-5)$

? ExAMPLE 14.3.30

Factor completely using trial and error: $56q^3 + 320q^2 - 96q$.

Answer

$8q(q+6)(7q-2)$

Factor Trinomials of the Form $ax^2 + bx + c$ using the "ac" Method

Another way to factor trinomials of the form $ax^2 + bx + c$ is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

Example 14.3.31: How to Factor Trinomials using the "ac" Method

Factor using the "ac" method: $6x^2+7x+2$.

Answer

Step 1. Factor any GCF.	Is there a greatest common factor? No!	$6x^2+7x+2$
Step 2. Find the product ac.	$a \cdot c$ $6 \cdot 2$ 12	ax^2+bx+c $6x^2+7x+2$
Step 3. Find two numbers m and n that: Multiply to ac. $m \cdot n = a \cdot c$ Add to b. $m+n=b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12 \quad 3+4=7$	
Step 4. Split the middle term using m and n. ax^2+bx+c $ax^2+\overbrace{mx+nx}^{bx}+c$	Rewrite $7x$ as $3x+4x$. It would also give the same result if we used $4x+3x$. Notice that $6x^2+3x+4x+2$ is equal to $6x^2+7x+2$. We just split the middle term to get a more useful form.	$6x^2+7x+2$ $6x^2+\underbrace{3x}+\underbrace{4x}+2$
Step 5. Factor by grouping.		$3x(2x+1)+2(2x+1)$ $(2x+1)(3x+2)$
Step 6. Check by multiplying the factors.		$(2x+1)(3x+2)$ $6x^2+4x+3x+2$ $6x^2+7x+2$ ✓

Example 14.3.32

Factor using the "ac" method: $6x^2+13x+2$.

Answer

$(x+2)(6x+1)$

Example 14.3.33

Factor using the "ac" method: $4y^2+8y+3$.

Answer

$(2y+1)(2y+3)$

The "ac" method is summarized here.

FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING THE "ac" METHOD.

1. Factor any GCF.
2. Find the product ac.
3. Find two numbers m and n that:
   Multiply to ac $m \cdot n = a \cdot c$
   Add to b $m + n = b$
   $ax^2 + bx + c$
4. Split the middle term using m and n. $ax^2 + mx + nx + c$
5. Factor by grouping.
6. Check by multiplying the factors.

Don't forget to look for a common factor!

Example 14.3.34

Factor using the "'ac" method: $10y^2 - 55y + 70$.

Answer

Is there a greatest common factor?		
Yes. The GCF is 5.		$10y^2 - 55y + 70$
Factor it.		$5(2y^2 - 11y + 14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.		$ax^2 + bx + c$ $5(2y^2 - 11y + 14)$
Find the product ac.	$ac = 28$	
Find two numbers that multiply to ac	$(-4)(-7) = 28$	
and add to b.	$-4(-7) = -11$	
Split the middle term.		$5(2y^2 - 11y + 14)$
		$5(2y^2 - 7y - 4y + 14)$
Factor the trinomial by grouping.		
		$5(y - 2)(2y - 7)$
Check by multiplying all three factors. $5(y-2)(2y-7)$ $5(2y^2 - 7y - 4y + 14)$ $5(2y^2 - 11y + 14)$ $10y^2 - 55y + 70$✓		

Example 14.3.35

Factor using the "ac" method: $16x^2 - 32x + 12$.

Answer

$4(2x - 3)(2x - 1)$

Example 14.3.36

Factor using the "ac" method: $18w^2 - 39w + 18$.

Answer

$3(3w - 2)(2w - 3)$

Factor Using Substitution

Sometimes a trinomial does not appear to be in the $ax^2 + bx + c$ form. However, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c$ form. This is called **factoring by substitution**. It is standard to use u for the substitution.

In the $ax^2 + bx + c$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

ExAMPLE 14.3.37

Factor by substitution: $x^4 - 4x^2 - 5$.

Answer

The variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $(x^2)^2 = x^4$). If we let $u = x^2$, we can put our trinomial in the $ax^2 + bx + c$ form we need to factor it.

	$x^4 - 4x^2 - 5$
Rewrite the trinomial to prepare for the substitution.	$(x^2)^2 - 4(x^2) - 5$
Let $u = x^2$ and substitute.	$(u)^2 - 4(u) - 5$
Factor the trinomial.	$(u + 1)(u - 5)$
Replace u with x^2.	$(x^2 + 1)(x^2 - 5)$
Check: $(x^2 + 1)(x^2 - 5)$ $x^4 - 5x^2 + x^2 - 5$ $x^4 - 4x^2 - 5$✓	

ExAMPLE 14.3.38

Factor by substitution: $h^4 + 4h^2 - 12$.

Answer

$(h^2 - 2)(h^2 + 6)$

ExAMPLE 14.3.39

Factor by substitution: $y^4 - y^2 - 20$.

Answer

$(y^2 + 4)(y^2 - 5)$

Sometimes the expression to be substituted is not a monomial.

ExAMPLE 14.3.40

Factor by substitution: $(x-2)^2+7(x-2)+12$

Answer

The binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u=x-2$ and substitute, our trinomial will be in ax^2+bx+c form.

	$(x-2)^2+7(x-2)+12$
Rewrite the trinomial to prepare for the substitution.	$(x-2)^2+7(x-2)+12$
Let $u=x-2$ and substitute.	$u^2+7u+12$
Factor the trinomial.	$(u+3)(u+4)$
Replace u with $x-2$.	$((x-2)+3)((x-2)+4)$
Simplify inside the parentheses.	$(x+1)(x+2)$

This could also be factored by first multiplying out the $(x-2)^2$ and the $7(x-2)$ and then combining like terms and then factoring. Most students prefer the substitution method.

ExAMPLE 14.3.41

Factor by substitution: $(x-5)^2+6(x-5)+8$.

Answer

$(x-3)(x-1)$

ExAMPLE 14.3.42

Factor by substitution: $(y-4)^2+8(y-4)+15$.

Answer

$(y-1)(y+1)$

Watch this video for additional instruction and practice with factoring.

Key Concepts

- **How to factor trinomials of the form x^2+bx+c.**
  1. Write the factors as two binomials with first terms x. $l)x^2+bx+c$ $(x\quad)(x\quad)$
  2. Find two numbers m and n that
     multiply to $c,\ m\cdot n=c$
     add to $b,\ m+n=b$
  3. Use m and n as the last terms of the factors. $(x+m)(x+n)$
  4. Check by multiplying the factors.
- **Strategy for Factoring Trinomials of the Form x^2+bx+c** : When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

  For trinomials of the form: $x^2+bx+c=(x+m)(x+n)$

  When c is positive, m and n must have the same sign (and this will be the sign of b).

  Examples: $x^2+5x+6=(x+2)(x+3)$, $x^2-6x+8=(x-4)(x-2)$

  When c is negative, m and n have opposite signs. The larger of m and n will have the sign of b.

  Examples: $x^2+x-12=(x+4)(x-3)$, $x^2-2x-15=(x-5)(x+3)$

  Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.
- **How to factor trinomials of the form ax^2+bx+c using trial and error.**
  1. Write the trinomial in descending order of degrees as needed.
  2. Factor any GCF.
  3. Find all the factor pairs of the first term.
  4. Find all the factor pairs of the third term.
  5. Test all the possible combinations of the factors until the correct product is found.
  6. Check by multiplying.
- **How to factor trinomials of the form ax^2+bx+c using the "ac" method.**

1. Factor any GCF.
2. Find the product ac.
3. Find two numbers m and n that:
   Multiply to ac. $m \cdot n = a \cdot c$
   Add to b. $m + n = b$
   $ax^2 + bx + c$
4. Split the middle term using m and n. $ax^2 + mx + nx + c$
5. Factor by grouping.
6. Check by multiplying the factors.

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14.3E: Exercises

Practice Makes Perfect

Factor Trinomials of the Form x^2+bx+c

In the following exercises, factor each trinomial of the form x^2+bx+c .

1. $p^2+11p+30$

Answer

$(p+5)(p+6)$

2. $w^2+10w+21$

3. $n^2+19n+48$

Answer

$(n+3)(n+16)$

4. $b^2+14b+48$

5. $a^2+25a+100$

Answer

$(a+5)(a+20)$

6. $u^2+101u+100$

7. $x^2-8x+12$

Answer

$(x-2)(x-6)$

8. $q^2-13q+36$

9. $y^2-18y+45$

Answer

$(y-3)(y-15)$

10. $m^2-13m+30$

11. x^2-8x+7

Answer

$(x-1)(x-7)$

12. y^2-5y+6

13. $5p-6+p^2$

Answer

$(p-1)(p+6)$

14. $6n-7+n^2$

15. $8-6x+x^2$

Answer

$(x-4)(x-2)$

16. $7x+x^2+6$

17. $x^2-12-11x$

Answer

$(x-12)(x+1)$

18. $-11-10x+x^2$

In the following exercises, factor each trinomial of the form $x^2+bxy+cy^2$.

19. $x^2-2xy-80y^2$

Answer

$(x+8y)(x-10y)$

20. $p^2-8pq-65q^2$

21. $m^2-64mn-65n^2$

Answer

$(m+n)(m-65n)$

22. $p^2-2pq-35q^2$

23. $a^2+5ab-24b^2$

Answer

$(a+8b)(a-3b)$

24. $r^2+3rs-28s^2$

25. $x^2-3xy-14y^2$

Answer

Prime

26. $u^2-8uv-24v^2$

27. $m^2 - 5mn + 30n^2$

Answer

Prime

28. $c^2 - 7cd + 18d^2$

Factor Trinomials of the Form $ax^2 + bx + c$ Using Trial and Error

In the following exercises, factor completely using trial and error.

29. $p^3 - 8p^2 - 20p$

Answer

$p(p-10)(p+2)$

30. $q^3 - 5q^2 - 24q$

31. $3m^3 - 21m^2 + 30m$

Answer

$3m(m-5)(m-2)$

32. $11n^3 - 55n^2 + 44n$

33. $5x^4 + 10x^3 - 75x^2$

Answer

$5x^2(x-3)(x+5)$

34. $6y^4 + 12y^3 - 48y^2$

35. $2t^2 + 7t + 5$

Answer

$(2t+5)(t+1)$

36. $5y^2 + 16y + 11$

37. $11x^2 + 34x + 3$

Answer

$(11x+1)(x+3)$

38. $7b^2 + 50b + 7$

39. $4w^2 - 5w + 1$

Answer

$(4w-1)(w-1)$

40. $5x^2 - 17x + 6$

41. $4q^2 - 7q - 2$

Answer

$(4q+1)(q-2)$

42. $10y^2 - 53y - 111$

43. $6p^2 - 19pq + 10q^2$

Answer

$(2p-5q)(3p-2q)$

44. $21m^2 - 29mn + 10n^2$

45. $4a^2 + 17ab - 15b^2$

Answer

$(4a-3b)(a+5b)$

46. $6u^2 + 5uv - 14v^2$

47. $-16x^2 - 32x - 16$

Answer

$-16(x+1)(x+1)$

48. $-81a^2 + 153a + 18$

49. $-30q^3 - 140q^2 - 80q$

Answer

$-10q(3q+2)(q+4)$

50. $-5y^3 - 30y^2 + 35y$

Factor Trinomials of the Form $ax^2 + bx + c$ using the 'ac' Method

In the following exercises, factor using the 'ac' method.

51. $5n^2 + 21n + 4$

Answer

$(5n+1)(n+4)$

52. $8w^2 + 25w + 3$

53. $4k^2 - 16k + 15$

Answer

$(2k-3)(2k-5)$

54. $5s^2-9s+4$

55. $6y^2+y-15$

Answer

$(3y+5)(2y-3)$

56. $6p^2+p-22$

57. $2n^2-27n-45$

Answer

$(2n+3)(n-15)$

58. $12z^2-41z-11$

59. $60y^2+290y-50$

Answer

$10(6y-1)(y+5)$

60. $6u^2-46u-16$

61. $48z^3-102z^2-45z$

Answer

$3z(8z+3)(2z-5)$

62. $90n^3+42n^2-216n$

63. $16s^2+40s+24$

Answer

$8(2s+3)(s+1)$

64. $24p^2+160p+96$

65. $48y^2+12y-36$

Answer

$12(4y-3)(y+1)$

66. $30x^2+105x-60$

Factor Using Substitution

In the following exercises, factor using substitution.

https://math.libretexts.org/@go/page/46374

67. $x^4 - x^2 - 12$

Answer

$(x^2 + 3)(x^2 - 4)$

68. $x^4 + 2x^2 - 8$

69. $x^4 - 3x^2 - 28$

Answer

$(x^2 - 7)(x^2 + 4)$

70. $x^4 - 13x^2 - 30$

71. $(x - 3)^2 - 5(x - 3) - 36$

Answer

$(x - 12)(x + 1)$

72. $(x - 2)^2 - 3(x - 2) - 54$

73. $(3y - 2)^2 - (3y - 2) - 2$

Answer

$(3y - 4)(3y - 1)$

74. $(5y - 1)^2 - 3(5y - 1) - 18$

Mixed Practice

In the following exercises, factor each expression using any method.

75. $u^2 - 12u + 36$

Answer

$(u - 6)(u - 6)$

76. $x^2 - 14x - 32$

77. $r^2 - 20rs + 64s^2$

Answer

$(r - 4s)(r - 16s)$

78. $q^2 - 29qr - 96r^2$

79. $12y^2 - 29y + 14$

Answer

$(4y - 7)(3y - 2)$

80. $12x^2+36y-24z$

81. $6n^2+5n-4$

Answer

$(2n-1)(3n+4)$

82. $3q^2+6q+2$

83. $13z^2+39z-26$

Answer

$13(z^2+3z-2)$

84. $5r^2+25r+30$

85. $3p^2+21p$

Answer

$3p(p+7)$

86. $7x^2-21x$

87. $6r^2+30r+36$

Answer

$6(r+2)(r+3)$

88. $18m^2+15m+3$

89. $24n^2+20n+4$

Answer

$4(2n+1)(3n+1)$

90. $4a^2+5a+2$

91. x^4-4x^2-12

Answer

$(x^2+2)(x^2-6)$

92. x^4-7x^2-8

93. $(x+3)^2-9(x+3)-36$

Answer

$(x-9)(x+6)$

94. $(x+2)^2-25(x+2)-54$

Writing Exercises

95. Many trinomials of the form x^2+bx+c factor into the product of two binomials $(x+m)(x+n)$. Explain how you find the values of m and n.

Answer

Answers will vary.

96. Tommy factored x^2-x-20 as $(x+5)(x-4)$. Sara factored it as $(x+4)(x-5)$. Ernesto factored it as $(x-5)(x-4)$. Who is correct? Explain why the other two are wrong.

97. List, in order, all the steps you take when using the "ac" method to factor a trinomial of the form ax^2+bx+c.

Answer

Answers will vary.

98. How is the "ac" method similar to the "undo FOIL" method? How is it different?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
factor trinomials of the form x^2+bx+c			
factor trinomials of the form ax^2+bx+c using trial and error.			
factor trinomials of the form ax^2+bx+c with using the "ac" method.			
factor using substitution.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

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14.4: Factor Special Products

Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

Before you get started, take this readiness quiz.

1. Simplify: $(3x^2)^3$.
2. Multiply: $(m+4)^2$.
3. Multiply: $(x-3)(x+3)$.

We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$\begin{matrix} (a & + & b)^2 \\ (3x & + & 4)^2 \end{matrix}$$

$$\begin{matrix} a^2 & + & 2 \cdot a & \cdot & b & + & b^2 \\ (3x)^2 & + & 2(3x & \cdot & 4) & + & 4^2 \end{matrix}$$

$$9x^2 + 24x + 16$$

The trinomial $9x^2+24x+16$ is called a *perfect square trinomial*. It is the square of the binomial $3x+4$.

In this chapter, you will start with a perfect square trinomial and factor it into its **prime** factors. You could factor this **trinomial** using the methods described in the last section, since it is of the form ax^2+bx+c. But if you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, you will save yourself a lot of work. Here is the pattern—the reverse of the binomial squares pattern.

PERFECT SQUARE TRINOMIALS PATTERN

If a and b are real numbers

$$a^2+2ab+b^2=(a+b)^2$$

$$a^2-2ab+b^2=(a-b)^2$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it the product, $2ab$? If everything checks, you can easily write the factors.

Example 14.4.1: How to Factor Perfect Square Trinomials

Factor: $9x^2+12x+4$.

Answer

Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$? • Is the first term a perfect square? Write it as a square, a^2. • Is the last term a perfect square? Write it as a square, b^2. • Check the middle term. Is it $2ab$?	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$. Is 4 a perfect square? Yes—write it as $(2)^2$. Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial!	$9x^2 + 12x + 4$ $(3x)^2$ $(3x)^2$ $(2)^2$ $2(3x)(2)$ $12x$
Step 2. Write the square of the binomial.		$9x^2 + 12x + 4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $(a + b)^2$ $(3x + 2)^2$
Step 3. Check.		$(3x + 2)^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $9x^2 + 12x + 4$ ✓

? Example 14.4.2

Factor: $4x^2 + 12x + 9$.

Answer

$(2x + 3)^2$

? Example 14.4.3

Factor: $9y^2 + 24y + 16$.

Answer

$(3y + 4)^2$

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a - b)^2$.

The steps are summarized here.

FACTOR PERFECT SQUARE TRINOMIALS

Step 1.	Does the trinomial fit the pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
	Are the first and last terms perfect squares?		
	Write them as squares.	$(a)^2$ $(b)^2$	$(a)^2$ $(b)^2$
	Check the middle term. Is it $2ab$?	$2 \cdot a \cdot b$	$2 \cdot a \cdot b$
Step 2.	Write the square of the binomial.	$(a + b)^2$	$(a - b)^2$
Step 3.	Check by multiplying.		

We'll work one now where the middle term is negative.

Example 14.4.4

Factor: $81y^2 - 72y + 16$.

Answer

The first and last terms are squares. See if the middle term fits the pattern of a **perfect square** trinomial. The middle term is negative, so the binomial square would be $(a-b)^2$.

	$81y^2 - 72y + 16$
Are the first and last terms perfect squares?	$(9y)^2 \qquad (4)^2$
Check the middle term.	$(9y)^2 \qquad (4)^2$ $2(9y)(4)$ $72y$
Does it match $(a-b)^2$? Yes.	$a^2 \quad -2 \quad a \quad b + b^2$ $(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$
Write as the square of a binomial.	$(9y-4)^2$
Check by multiplying:	$(9y-4)^2$ $(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$ $81y^2 - 72y + 16$✓

Example 14.4.5

Factor: $64y^2 - 80y + 25$.

Answer

$(8y-5)^2$

Example 14.4.6

Factor: $16z^2 - 72z + 81$.

Answer

$(4z-9)^2$

The next example will be a perfect square trinomial with two variables.

Example 14.4.7

Factor: $36x^2 + 84xy + 49y^2$.

Answer

	$36x^2 + 84xy + 49y^2$
Test each term to verify the pattern.	$a^2 \quad +2 \quad a \quad b + b^2$ $(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$
Factor.	$(6x+7y)^2$

Check by multiplying. $(6x+7y)^2$ $(6x)^2+2\cdot 6x\cdot 7y+(7y)^2$ $36x^2+84xy+49y^2$ ✓	

Example 14.4.8

Factor: $49x^2+84xy+36y^2$.

Answer

$(7x+6y)^2$

Example 14.4.9

Factor: $64m^2+112mn+49n^2$.

Answer

$(8m+7n)^2$

Remember the first step in factoring is to look for a greatest common factor. Perfect square trinomials may have a **GCF** in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

Example 14.4.10

Factor: $100x^2y-80xy+16y$.

Answer

	$100x^2y-80xy+16y$
Is there a GCF? Yes, $4y$, so factor it out.	$4y\left(25x^2-20x+4\right)$
Is this a perfect square trinomial?	
Verify the pattern.	$4y[(5x)^2-2\cdot 5x\cdot 2+2^2]$
Factor.	$4y(5x-2)^2$
Remember: Keep the factor $4y$ in the final product.	
Check: $4y(5x-2)^2$ $4y[(5x)2-2\cdot 5x\cdot 2+22]$ $4y(25x2-20x+4)$ 100x2y−80xy+16y\checkmark\]	

Example 14.4.11

Factor: $8x^2y - 24xy + 18y$.

Answer

$2y(2x-3)^2$

Example 14.4.12

Factor: $27p^2q + 90pq + 75q$.

Answer

$3q(3p+5)^2$

Factor Differences of Squares

The other special product you saw in the previous chapter was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here's an example:

$$\overset{(a\ -\ b)\ (a\ +\ b)}{(3x-4)(3x+4)}$$

$$\overset{(a)^2\ -\ (b)^2}{(3x)^2-(4)^2}$$

$$9x^2-16$$

A difference of squares factors to a product of conjugates.

DIFFERENCE OF SQUARES PATTERN

If a and b are real numbers,

$$a^2-b^2=(a-b)(a+b) \qquad \underset{\text{squares}}{a^2} \overset{\text{difference}}{-} \underset{\text{conjugates}}{b^2=\underbrace{(a-b)(a+b)}}$$

Remember, "difference" refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

Example 14.4.13: How to Factor a Trinomial Using the Difference of Squares

Factor: $64y^2 - 1$.

Answer

Step 1. Does the binomial fit the pattern?		$64y^2-1$
• Is this a difference?	Yes	$64y^2-1$
• Are the first and last terms perfect squares?	Yes	
Step 2. Write them as squares.	Write them as x^2 and 2^2.	$\overset{a^2\ -\ b^2}{(8y)^2-1^2}$

Step 3. Write the product of conjugates.		$(a-b)(a+b)$ $(8y-1)(8y+1)$
Step 4. Check.		$(8y-1)(8y+1)$ $64y^2-1$ ✓

Example 14.4.14

Factor: $121m^2-1$.

Answer

$(11m-1)(11m+1)$

Example 14.4.15

Factor: $81y^2-1$.

Answer

$(9y-1)(9y+1)$

FACTOR DIFFERENCES OF SQUARES.

Step 1.	Does the binomial fit the pattern?	a^2-b^2
	Is this a difference?	____−____
	Are the first and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2-(b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression a^2+b^2 is prime!

The next example shows variables in both terms.

Example 14.4.16

Factor: $144x^2-49y^2$.

Answer

	$144x^2-49y^2$
Is this a difference of squares? Yes.	$(12x)^2-(7y)^2$
Factor as the product of conjugates.	$(12x-7y)(12x+7y)$
Check by multiplying.	$(12x-7y)(12x+7y)$

Check by multiplying.

$(12x-7y)(12x+7y)$

$144x^2-49y^2$ ✓

Example 14.4.17

Factor: $196m^2 - 25n^2$.

Answer

$(14m - 5n)(14m + 5n)$

Example 14.4.18

Factor: $121p^2 - 9q^2$.

Answer

$(11p - 3q)(11p + 3q)$

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may "disguise" the difference of squares and you won't recognize the perfect squares until you factor the GCF.

Also, to completely factor the binomial in the next example, we'll factor a difference of squares twice!

Example 14.4.19

Factor: $48x^4y^2 - 243y^2$.

Answer

	$48x^4y^2 - 243y^2$
Is there a GCF? Yes, $3y^2$—factor it out!	$3y^2(16x^4 - 81)$
Is the binomial a difference of squares? Yes.	$3y^2\left((4x^2)^2 - (9)^2\right)$
Factor as a product of conjugates.	$3y^2(4x^2 - 9)(4x^2 + 9)$
Notice the first binomial is also a difference of squares!	$3y^2((2x)^2 - (3)^2)(4x^2 + 9)$
Factor it as the product of conjugates.	$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

The last factor, the sum of squares, cannot be factored.

Check by multiplying:

$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

$3y^2(4x^2 - 9)(4x^2 + 9)$

$3y^2(16x^4 - 81)$

$48x^4y^2 - 243y^2$ ✓

Example 14.4.20

Factor: $2x^4y^2 - 32y^2$.

Answer

$2y^2(x - 2)(x + 2)(x^2 + 4)$

Example 14.4.21

Factor: $7a^4c^2 - 7b^4c^2$.

Answer

$7c^2(a-b)(a+b)(a^2+b^2)$

The next example has a polynomial with 4 terms. So far, when this occurred we grouped the terms in twos and factored from there. Here we will notice that the first three terms form a perfect square trinomial.

Example 14.4.22

Factor: $x^2 - 6x + 9 - y^2$.

Answer

Notice that the first three terms form a perfect square trinomial.

	$x^2 - 6x + 9 - y^2$
Factor by grouping the first three terms.	$\underbrace{x^2 - 6x + 9}\ - y^2$
Use the perfect square trinomial pattern.	$(x-3)^2 - y^2$
Is this a difference of squares? Yes.	
Yes—write them as squares.	$\overset{a^2}{(x-3)^2} - \overset{b^2}{y^2}$
Factor as the product of conjugates.	$((\overset{a}{x-3}) - \overset{b}{y})((\overset{a}{x-3}) + \overset{b}{y})$
	$(x-3-y)(x-3+y)$

You may want to rewrite the solution as $(x-y-3)(x+y-3)$.

Example 14.4.23

Factor: $x^2 - 10x + 25 - y^2$.

Answer

$(x-5-y)(x-5+y)$

Example 14.4.24

Factor: $x^2 + 6x + 9 - 4y^2$.

Answer

$(x+3-2y)(x+3+2y)$

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

We'll check the first pattern and leave the second to you.

	$(a+b)\left(a^2-ab+b^2\right)$
Distribute.	$a\left(a^2-ab+b^2\right)+b\left(a^2-ab+b^2\right)$
Multiply.	$a^3-a^2b+ab^2+a^2b-ab^2+b^3$
Combine like terms.	a^3+b^3

SUM AND DIFFERENCE OF CUBES PATTERN

$$a^3+b^3=(a+b)(a^2-ab+b^2$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

The two patterns look very similar, don't they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

$$a^3 \underbrace{+}_{\text{same sign}} b^3 = (a \underbrace{+}_{\text{opposite signs}} b)(a^2 - ab + b^2)$$

$$a^3 \underbrace{-}_{\text{same sign}} b^3 = (a \underbrace{-}_{\text{opposite signs}} b)(a^2 + ab + b^2)$$

The trinomial factor in the sum and difference of cubes pattern cannot be factored.

It be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in Table.

n	1	2	3	4	5	6	7	8	9	10
n^3	1	8	27	64	125	216	343	512	729	1000

Example 14.4.25: How to Factor the Sum or Difference of Cubes

Factor: x^3+64.

Answer

Step 1. Does the binomial fit the sum or difference of cubes pattern? • Is it a sum or difference? • Are the first and last terms perfect cubes?	This is a sum. Yes.	x^3+64 x^3+64
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3.	a^3+b^3 x^3+4^3
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$(a+b)\left(a^2-ab+b^2\right)$ $(x+4)\left(x^2-4x+4^2\right)$

Step 4. Simplify inside the parentheses.	It is already simplified.	$(x+4)(x^2-4x+16)$
Step 5. Check by multiplying the factors.		$\begin{array}{r} x^2-4x+16 \\ x+4 \\ \hline 4x^2-16x+64 \checkmark \\ x^3-4x^2+16x \\ \hline x^3 \qquad +64 \end{array}$

? Example 14.4.26

Factor: x^3+27.

Answer

$(x+3)(x^2-3x+9)$

? Example 14.4.27

Factor: y^3+8.

Answer

$(y+2)(y^2-2y+4)$

FACTOR THE SUM OR DIFFERENCE OF CUBES.

1. Does the binomial fit the sum or difference of cubes pattern?
   Is it a sum or difference?
   Are the first and last terms perfect cubes?
2. Write them as cubes.
3. Use either the sum or difference of cubes pattern.
4. Simplify inside the parentheses.
5. Check by multiplying the factors.

? Example 14.4.28

Factor: $27u^3-125v^3$.

Answer

	$27u^3-125v^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(3u)^3} - \overset{b^3}{(5v)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{3u} - \overset{b}{5v}\right)\left(\overset{a^2}{(3u)^2} + \overset{ab}{3u \cdot 5v} + \overset{b^2}{(5v)^2}\right)$
Simplify.	$\left(\overset{a}{3u} - \overset{b}{5v}\right)\left(\overset{a^2}{9u^2} + \overset{ab}{15uv} + \overset{b^2}{25v^2}\right)$
Check by multiplying.	We'll leave the check to you.

Example 14.4.29

Factor: $8x^3 - 27y^3$.

Answer

$(2x - 3y)(4x^2 + 6xy + 9y^2)$

Example 14.4.30

Factor: $1000m^3 - 125n^3$.

Answer

$(10m - 5n)(100m^2 + 50mn + 25n^2)$

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

Example 14.4.31

Factor: $6x^3y + 48y^4$.

Answer

	$6x^3y + 48y^4$
Factor the common factor.	$6y\left(x^3 + 8y^3\right)$
This binomial is a sum The first and last terms are perfect cubes.	
Write the terms as cubes.	$6y\left(x^3 + (2y)^3\right)$
Use the sum of cubes pattern.	$6y\left(x + 2y\right)\left(x^2 - x \cdot 2y + (2y)^2\right)$
Simplify.	$6y(x + 2y)(x^2 - 2xy + 4y^2)$

Check:

To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by 6y.6y. We'll leave the multiplication for you.

Example 14.4.32

Factor: $500p^3 + 4q^3$.

Answer

$4(5p + q)(25p^2 - 5pq + q^2)$

Example 14.4.33

Factor: $432c^3 + 686d^3$.

Answer

$2(6c + 7d)(36c^2 - 42cd + 49d^2)$

The first term in the next example is a binomial cubed.

Example 14.4.34

Factor: $(x+5)^3-64x^3$.

Answer

	$(x+5)^3-64x^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(x+5)^3}-\overset{b^3}{(4x)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{(x+5)}-\overset{b}{4x}\right)\left(\overset{a^2}{(x+5)^2}+\overset{a}{(x+5)}\cdot\overset{b}{4x}+\overset{b^2}{(4x)^2}\right)$
Simplify.	$(x+5-4x)\left(x^2+10x+25+4x^2+20x+16x^2\right)$
	$(-3x+5)\left(21x^2+30x+25\right)$
Check by multiplying.	We'll leave the check to you.

Example 14.4.35

Factor: $(y+1)^3-27y^3$.

Answer

$(-2y+1)(13y^2+5y+1)$

Example 14.4.36

Factor: $(n+3)^3-125n^3$.

Answer

$(-4n+3)(31n^2+21n+9)$

Access this online resource for additional instruction and practice with factoring special products.

- Factoring Binomials-Cubes #2

Key Concepts

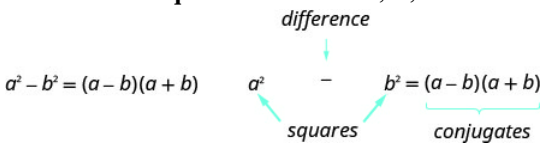
- **Perfect Square Trinomials Pattern:** If *a* and *b* are real numbers,

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

- **How to factor perfect square trinomials.**

Step 1.	Does the trinomial fit the pattern?	$a^2+2ab+b^2$	$a^2-2ab+b^2$
	Are the first and last terms perfect squares?		
	Write them as squares.	$(a)^2 \quad (b)^2$	$(a)^2 \quad (b)^2$
	Check the middle term. Is it $2ab$?	$\searrow 2\cdot a\cdot b \swarrow$	$\searrow 2\cdot a\cdot b \swarrow$
Step 2.	Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3.	Check by multiplying.		

- **Difference of Squares Pattern:** If a,ba,b are real numbers,



- **How to factor differences of squares.**

Step 1.	Does the binomial fit the pattern?	$a^2 - b^2$
	Is this a difference?	____−____
	Are the first and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2 - (b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

- **Sum and Difference of Cubes Pattern**
  $a^3 + b3 = (a+b)(a^2 - ab + b^2)$
  $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- **How to factor the sum or difference of cubes.**
  1. Does the binomial fit the sum or difference of cubes pattern?
     Is it a sum or difference?
     Are the first and last terms perfect cubes?
  2. Write them as cubes.
  3. Use either the sum or difference of cubes pattern.
  4. Simplify inside the parentheses
  5. Check by multiplying the factors.

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14.4E: Exercises

Practice Makes Perfect

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

1. $16y^2+24y+9$

Answer

$(4y+3)^2$

2. $25v^2+20v+4$

3. $36s^2+84s+49$

Answer

$(6s+7)^2$

4. $49s^2+154s+121$

5. $100x^2-20x+1$

Answer

$(10x-1)^2$

6. $64z^2-16z+1$

7. $25n^2-120n+144$

Answer

$(5n-12)^2$

8. $4p^2-52p+169$

9. $49x^2+28xy+4y^2$

Answer

$(7x+2y)^2$

10. $25r^2+60rs+36s^2$

11. $100y^2-52y+1$

Answer

$(50y-1)(2y-1)$

12. $64m^2-34m+1$

13. $10jk^2+80jk+160j$

https://math.libretexts.org/@go/page/46376

Answer

$10j(k+4)^2$

14. $64x^2y-96xy+36y$

15. $75u^4-30u^3v+3u^2v^2$

Answer

$3u^2(5u-v)^2$

16. $90p^4+300p^4q+250p^2q^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

17. $25v^2-1$

Answer

$(5v-1)(5v+1)$

18. $169q^2-1$

19. $4-49x^2$

Answer

$(7x-2)(7x+2)$

20. $121-25s^2$

21. $6p^2q^2-54p^2$

Answer

$6p^2(q-3)(q+3)$

22. $98r^3-72r$

23. $24p^2+54$

Answer

$6(4p^2+9)$

24. $20b^2+140$

25. $121x^2-144y^2$

Answer

$(11x-12y)(11x+12y)$

26. $49x^2-81y^2$

27. $169c^2 - 36d^2$

Answer

$(13c - 6d)(13c + 6d)$

28. $36p^2 - 49q^2$

29. $16z^4 - 1$

Answer

$(2z - 1)(2z + 1)(4z^2 + 1)$

30. $m^4 - n^4$

31. $162a^4b^2 - 32b^2$

Answer

$2b^2(3a - 2)(3a + 2)(9a^2 + 4)$

32. $48m^4n^2 - 243n^2$

33. $x^2 - 16x + 64 - y^2$

Answer

$(x - 8 - y)(x - 8 + y)$

34. $p^2 + 14p + 49 - q^2$

35. $a^2 + 6a + 9 - 9b^2$

Answer

$(a + 3 - 3b)(a + 3 + 3b)$

36. $m^2 - 6m + 9 - 16n^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

37. $x^3 + 125$

Answer

$(x + 5)(x^2 - 5x + 25)$

38. $n^6 + 512$

39. $z^6 - 27$

Answer

$(z^2 - 3)(z^4 + 3z^2 + 9)$

40. $v^3 - 216$

41. $8 - 343t^3$

Answer

$(2-7t)(4+14t+49t^2)$

42. $125 - 27w^3$

43. $8y^3 - 125z^3$

Answer

$(2y-5z)(4y^2+10yz+25z^2)$

44. $27x^3 - 64y^3$

45. $216a^3 + 125b^3$

Answer

$(6a+5b)(36a^2-30ab+25b^2)$

46. $27y^3 + 8z^3$

47. $7k^3 + 56$

Answer

$7(k+2)(k^2-2k+4)$

48. $6x^3 - 48y^3$

49. $2x^2 - 16x^2y^3$

Answer

$2x^2(1-2y)(1+2y+4y^2)$

50. $-2x^3y^2 - 16y^5$

51. $(x+3)^3 + 8x^3$

Answer

$9(x+1)(x^2+3)$

52. $(x+4)^3 - 27x^3$

53. $(y-5)^3 - 64y^3$

Answer

$-(3y+5)(21y^2-30y+25)$

54. $(y-5)^3+125y^3$

Mixed Practice

In the following exercises, factor completely.

55. $64a^2-25$

Answer

$(8a-5)(8a+5)$

56. $121x^2-144$

57. $27q^2-3$

Answer

$3(3q-1)(3q+1)$

58. $4p^2-100$

59. $16x^2-72x+81$

Answer

$(4x-9)^2$

60. $36y^2+12y+1$

61. $8p^2+2$

Answer

$2(4p^2+1)$

62. $81x^2+169$

63. $125-8y^3$

Answer

$(5-2y)(25+10y+4y^2)$

64. $27u^3+1000$

65. $45n^2+60n+20$

Answer

$5(3n+2)^2$

66. $48q^3-24q^2+3q$

67. $x^2-10x+25-y^2$

Answer

$(x+y-5)(x-y-5)$

68. $x^2+12x+36-y^2$

69. $(x+1)^3+8x^3$

Answer

$(3x+1)(3x^2+1)$

70. $(y-3)^3-64y^3$

Writing Exercises

71. Why was it important to practice using the binomial squares pattern in the chapter on multiplying polynomials?

Answer

Answers will vary.

72. How do you recognize the binomial squares pattern?

73. Explain why $n^2+25 \neq (n+5)^2$. Use algebra, words, or pictures.

Answer

Answers will vary.

74. Maribel factored $y^2-30y+81$ as $(y-9)^2$. Was she right or wrong? How do you know?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
factor perfect square trinomials.			
factor differences of squares.			
factor sums and differences of cubes.			

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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14.5: General Strategy for Factoring Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. The following chart summarizes all the factoring methods we have covered, and outlines a strategy you should use when factoring polynomials.

GENERAL STRATEGY FOR FACTORING POLYNOMIALS

GCF

Binomial	Trinomial	More than 3 terms
• **Difference of Squares** $a^2 - b^2 = (a - b)(a + b)$	• $x^2 + bx + c$ $(x \quad)(x \quad)$	• grouping
• **Sum of Squares** Sums of squares do not factor.	• $ax^2 + bx + c$ ◦ **'a' and 'c' squares** $(a + b)^2 = a^2 + 2ab + b^2$ $(a - b)^2 = a^2 - 2ab + b^2$	
• **Sum of Cubes** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	◦ **'ac' method**	
• **Difference of Cubes** $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$		

USE A GENERAL STRATEGY FOR FACTORING POLYNOMIALS.

1. Is there a greatest common factor?
   Factor it out.
2. Is the polynomial a binomial, trinomial, or are there more than three terms?
   If it is a binomial:
   - Is it a sum?
     Of squares? Sums of squares do not factor.
     Of cubes? Use the sum of cubes pattern.
   - Is it a difference?
     Of squares? Factor as the product of conjugates.
     Of cubes? Use the difference of cubes pattern.

   If it is a trinomial:
   - Is it of the form $x^2 + bx + c$? Undo FOIL.
   - Is it of the form $ax^2 + bx + c$?
     If a and c are squares, check if it fits the trinomial square pattern.
     Use the trial and error or "ac" method.

   If it has more than three terms:
   - Use the grouping method.
3. Check.
   Is it factored completely?
   Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are **prime**!

Example 14.5.1

Factor completely: $7x^3 - 21x^2 - 70x$.

Solution

$7x^3 - 21x^2 - 70x$

Is there a GCF? Yes, $7x$.

Factor out the GCF. $7x(x^2 - 3x - 10)$

In the parentheses, is it a binomial, trinomial, or are there more terms?

Trinomial with leading coefficient 1.

"Undo" FOIL. $7x(x \quad)(x \quad)$

$7x(x+2)(x-5)$

Is the expression factored completely? Yes.

Neither binomial can be factored.

Check your answer.

Multiply.

$$7x(x+2)(x-5)$$
$$7x(x^2 - 5x + 2x - 10)$$
$$7x(x^2 - 3x - 10)$$
$$7x^3 - 21x^2 - 70x \checkmark$$

Try It 14.5.1

Factor completely: $8y^3 + 16y^2 - 24y$.

Answer

$8y(y-1)(y+3)$

Try It 14.5.2

Factor completely: $5y^3 - 15y^2 - 270y$.

Answer

$5y(y-9)(y+6)$

Be careful when you are asked to factor a binomial as there are several options!

Example 14.5.2

Factor completely: $24y^2 - 150$

Solution

	$24y^2-150$
Is there a GCF? Yes, 6.	
Factor out the GCF.	$6(4y^2-25)$
In the parentheses, is it a binomial, trinomial or are there more than three terms? Binomial.	
Is it a sum? No.	
Is it a difference? Of squares or cubes? Yes, squares.	$6((2y)^2-(5)^2)$
Write as a product of conjugates.	$6(2y-5)(2y+5)$

Is the expression factored completely?
Neither binomial can be factored.

Check:

Multiply.

$6(2y-5)(2y+5)$

$6(4y^2-25)$
$24y^2-150$✓

? Try It 14.5.3

Factor completely: $16x^3-36x$.

Answer

$4x(2x-3)(2x+3)$

? Try It 14.5.4

Factor completely: $27y^2-48$.

Answer

$3(3y-4)(3y+4)$

The next example can be factored using several methods. Recognizing the trinomial squares pattern will make your work easier.

? Example 14.5.3

Factor completely: $4a^2-12ab+9b^2$.

Solution

$4a^2-12ab+9b^2$

Is there a GCF? No.

Is it a binomial, trinomial, or are there more terms?

Trinomial with $a \neq 1$. But the first term is a perfect square.

Is the last term a perfect square? Yes. $(2a)^2-12ab+(3b)^2$

Does it fit the pattern, $a^2-2ab+b^2$? Yes. $(2a)^2-12ab+(3b)^2$

$\searrow \quad -2(2a)(3b) \quad \swarrow$

Write it as a square. $(2a-3b)^2$

Is the expression factored completely? Yes.

The binomial cannot be factored.

Check your answer.

Multiply.

$$(2a-3b)^2$$
$$(2a)^2-2\cdot 2a\cdot 3b+(3b)^2$$
$$4a^2-12ab+9b^2 \checkmark$$

? Try It 14.5.5

Factor completely: $4x^2+20xy+25y^2$.

Answer

$(2x+5y)^2$

? Try It 14.5.6

Factor completely: $9x^2-24xy+16y^2$.

Answer

$(3x-4y)^2$

Remember, sums of squares do not factor, but sums of cubes do!

? Example 14.5.4

Factor completely $12x^3y^2+75xy^2$.

Solution

	$12x^3y^2+75xy^2$
Is there a GCF? Yes, $3xy^2$.	
Factor out the GCF.	$3xy^2(4x^2+25)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms? Binomial.	
Is it a sum? Of squares? Yes.	Sums of squares are prime.
Is the expression factored completely? Yes.	

Check:

Multiply.

$3xy^2(4x^2+25)$

$12x^3y^2+75xy^2$ ✓

? Try It 14.5.7

Factor completely: $50x^3y+72xy$.

Answer

$2xy(25x^2+36)$

? Try It 14.5.8

Factor completely: $27xy^3+48xy$.

Answer

$3xy(9y^2+16)$

When using the sum or difference of cubes pattern, being careful with the signs.

? Example 14.5.5

Factor completely: $24x^3+81y^3$.

Solution

Is there a GCF? Yes, 3.	$24x^3+81y^3$
Factor it out.	$3(8x^3+27y^3)$
In the parentheses, is it a binomial, trinomial, of are there more than three terms? Binomial.	
Is it a sum or difference? Sum.	
Of squares or cubes? Sum of cubes.	$3\left(\overset{a^3}{(2x)^3}+\overset{b^3}{(3y)^3}\right)$
Write it using the sum of cubes pattern.	$3\left(\overset{a}{2x}+\overset{b}{3y}\right)\left(\overset{a^2}{(2x)^2}-\overset{ab}{2x\cdot 3y}+\overset{b^2}{(3y)^3}\right)$
Is the expression factored completely? Yes.	$3(2x+3y)(4x^2-6xy+9y^2)$

Check by multiplying.

Try It 14.5.9

Factor completely: $250m^3+432n^3$.

Answer

$2(5m+6n)(25m^2-30mn+36n^2)$

Try It 14.5.10

Factor completely: $2p^3+54q^3$.

Answer

$2(p+3q)(p^2-3pq+9q^2)$

Example 14.5.6

Factor completely: $3x^5y-48xy$.

Solution

	$3x^5y-48xy$
Is there a GCF? Factor out $3xy$	$3xy(x^4-16)$
Is the binomial a sum or difference? Of squares or cubes? Write it as a difference of squares.	$3xy\left((x^2)^2-(4)2\right)$
Factor it as a product of conjugates	$3xy(x^2-4)(x^2+4)$
The first binomial is again a difference of squares.	$3xy\left((x)^2-(2)^2\right)(x^2+4)$
Factor it as a product of conjugates.	$3xy(x-2)(x+2)(x^2+4)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	
$3xy(x-2)(x+2)(x^2+4)$	
$3xy(x^2-4)(x^2+4)$	
$3xy(x^4-16)$	
$3x^5y-48xy$✓	

Try It 14.5.11

Factor completely: $4a^5b-64ab$.

Answer

$4ab(a^2+4)(a-2)(a+2)$

Try It 14.5.12

Factor completely: $7xy^5-7xy$.

Answer

$7xy(y^2+1)(y-1)(y+1)$

? Example 14.5.7

Factor completely: $4x^2+8bx-4ax-8ab$.

Solution

	$4x^2+8bx-4ax-8ab$
Is there a GCF? Factor out the GCF, 4.	$4(x^2+2bx-ax-2ab)$
There are four terms. Use grouping.	$4[x(x+2b)-a(x+2b)]4(x+2b)(x-a)$

Is the expression factored completely? Yes.

Check your answer.

Multiply.

$$4(x+2b)(x-a)$$
$$4(x^2-ax+2bx-2ab)$$
$$4x^2+8bx-4ax-8ab\checkmark$$

? Try It 14.5.13

Factor completely: $6x^2-12xc+6bx-12bc$.

Answer

$6(x+b)(x-2c)$

? Try It 14.5.14

Factor completely: $16x^2+24xy-4x-6y$.

Answer

$2(4x-1)(2x+3y)$

Taking out the complete GCF in the first step will always make your work easier.

? Example 14.5.8

Factor completely: $40x^2y+44xy-24y$.

Solution

	$40x^2y+44xy-24y$
Is there a GCF? Factor out the GCF, $4y$.	$4y(10x^2+11x-6)$
Factor the trinomial with $a\neq 1$.	$4y(10x^2+11x-6)$
	$4y(5x-2)(2x+3)$

Is the expression factored completely? Yes.

Check your answer.

Multiply.

$$4y(5x-2)(2x+3)$$
$$4y(10x^2+11x-6)$$
$$40x^2y+44xy-24y\checkmark$$

Try It 14.5.15

Factor completely: $4p^2q-16pq+12q$.

Answer

$4q(p-3)(p-1)$

Try It 14.5.16

Factor completely: $6pq^2-9pq-6p$.

Answer

$3p(2q+1)(q-2)$

When we have factored a polynomial with four terms, most often we separated it into two groups of two terms. Remember that we can also separate it into a trinomial and then one term.

Example 14.5.9

Factor completely: $9x^2-12xy+4y^2-49$.

Solution

	$9x^2-12xy+4y^2-49$
Is there a GCF? No.	
With more than 3 terms, use grouping. Last 2 terms have no GCF. Try grouping first 3 terms.	$9x^2-12xy+4y^2-49$
Factor the trinomial with $a\neq 1$. But the first term is a perfect square.	
Is the last term of the trinomial a perfect square? Yes.	$(3x)^2-12xy+(2y)^2-49$
Does the trinomial fit the pattern, $a^2-2ab+b^2$? Yes.	$(3x)^2-12xy+(2y)^2-49$ $\searrow \quad -2(3x)(2y) \quad \swarrow$
Write the trinomial as a square.	$(3x-2y)^2-49$
Is this binomial a sum or difference? Of squares or cubes? Write it as a difference of squares.	$(3x-2y)^2-72$
Write it as a product of conjugates.	$((3x-2y)-7)((3x-2y)+7)$ $(3x-2y-7)(3x-2y+7)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	
	$(3x-2y-7)(3x-2y+7)$
	$9x^2-6xy-21x-6xy+4y^2+14y+21x-14y-49$
	$9x^2-12xy+4y^2-49$✓

Try It 14.5.17

Factor completely: $4x^2-12xy+9y^2-25$.

Answer

$(2x-3y-5)(2x-3y+5)$

? Try It 14.5.18

Factor completely: $16x^2 - 24xy + 9y^2 - 64$.

Answer

$(4x - 3y - 8)(4x - 3y + 8)$

Key Concepts

General Strategy for Factoring Polynomials

GCF

Binomial	Trinomial	More than 3 terms
• **Difference of Squares** $a^2 - b^2 = (a - b)(a + b)$	• $x^2 + bx + c$ $(x \quad)(x \quad)$	• grouping
• **Sum of Squares** Sums of squares do not factor.	• $ax^2 + bx + c$ ∘ **'a' and 'c' squares** $(a + b)^2 = a^2 + 2ab + b^2$ $(a - b)^2 = a^2 - 2ab + b^2$	
• **Sum of Cubes** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	∘ **'ac' method**	
• **Difference of Cubes** $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$		

- **How to use a general strategy for factoring polynomials.**
  1. Is there a greatest common factor?
     Factor it out.
  2. Is the polynomial a binomial, trinomial, or are there more than three terms?
     If it is a binomial:
     Is it a sum?
     Of squares? Sums of squares do not factor.
     Of cubes? Use the sum of cubes pattern.
     Is it a difference?
     Of squares? Factor as the product of conjugates.
     Of cubes? Use the difference of cubes pattern.
     If it is a trinomial:
     Is it of the form $x^2 + bx + c$? Undo FOIL.
     Is it of the form $ax^2 + bx + c$?
     If *a* and *c* are squares, check if it fits the trinomial square pattern.
     Use the trial and error or "ac" method.
     If it has more than three terms:
     Use the grouping method.
  3. Check.
     Is it factored completely?
     Do the factors multiply back to the original polynomial?

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14.5E: Exercises

Practice Makes Perfect

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

1. $2n^2+13n-7$

Answer

$(2n-1)(n+7)$

2. $8x^2-9x-3$

3. a^5+9a^3

Answer

$a^3(a^2+9)$

4. $75m^3+12m$

5. $121r^2-s^2$

Answer

$(11r-s)(11r+s)$

6. $49b^2-36a^2$

7. $8m^2-32$

Answer

$8(m-2)(m+2)$

8. $36q^2-100$

9. $25w^2-60w+36$

Answer

$(5w-6)^2$

10. $49b^2-112b+64$

11. $m^2+14mn+49n^2$

Answer

$(m+7n)^2$

12. $64x^2+16xy+y^2$

13. $7b^2+7b-42$

Answer

$7(b+3)(b-2)$

14. $30n^2+30n+72$

15. $3x^4y-81xy$

Answer

$3xy(x-3)(x^2+3x+9)$

16. $4x^5y-32x^2y$

17. k^4-16

Answer

$(k-2)(k+2)(k^2+4)$

18. m^4-81

19. $5x5y^2-80xy^2$

Answer

$5xy^2(x^2+4)(x+2)(x-2)$

20. $48x^5y^2-243xy^2$

21. $15pq-15p+12q-12$

Answer

$3(5p+4)(q-1)$

22. $12ab-6a+10b-5$

23. $4x^2+40x+84$

Answer

$4(x+3)(x+7)$

24. $5q^2-15q-90$

25. $4u^5v+4u^2v^3$

Answer

$u^2(u+1)(u^2-u+1)$

26. $5m^4n+320mn^4$

27. $4c^2+20cd+81d^2$

Answer

prime

28. $25x^2+35xy+49y^2$

29. $10m^4-6250$

Answer

$10(m-5)(m+5)(m^2+25)$

30. $3v^4-768$

31. $36x^2y+15xy-6y$

Answer

$3y(3x+2)(4x-1)$

32. $60x^2y-75xy+30y$

33. $8x^3-27y^3$

Answer

$(2x-3y)(4x^2+6xy+9y^2)$

34. $64x^3+125y^3$

35. y^6-1

Answer

$(y+1)(y-1)(y^2-y+1)$

36. y^6+1

37. $9x^2-6xy+y^2-49$

Answer

$(3x-y+7)(3x-y-7)$

38. $16x^2-24xy+9y^2-64$

39. $(3x+1)^2-6(3x-1)+9$

Answer

$(3x-2)2$

40. $(4x-5)^2-7(4x-5)+12$

Writing Exercises

41. Explain what it mean to factor a polynomial completely.

Answer

Answers will vary.

42. The difference of squares $y^4 - 625$ can be factored as $(y^2 - 25)(y^2 + 25)$. But it is not completely factored. What more must be done to completely factor.

43. Of all the factoring methods covered in this chapter (GCF, grouping, undo FOIL, 'ac' method, special products) which is the easiest for you? Which is the hardest? Explain your answers.

Answer

Answers will vary.

44. Create three factoring problems that would be good test questions to measure your knowledge of factoring. Show the solutions.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize and use the appropriate method to factor a polynomial completely.			

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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14.6: Polynomial Equations

Learning Objectives

By the end of this section, you will be able to:

- Use the Zero Product Property
- Solve quadratic equations by factoring
- Solve equations with polynomial functions
- Solve applications modeled by polynomial equations

Before you get started, take this readiness quiz.

1. Solve: $5y-3=0$.
   If you missed this problem, review [link].
2. Factor completely: n^3-9n^2-22n.
   If you missed this problem, review [link].
3. If $f(x)=8x-16$, find $f(3)$ and solve $f(x)=0$.
   If you missed this problem, review [link].

We have spent considerable time learning how to factor polynomials. We will now look at polynomial equations and solve them using factoring, if possible.

A **polynomial equation** is an equation that contains a polynomial expression. The **degree of the polynomial equation** is the degree of the polynomial.

POLYNOMIAL EQUATION

A **polynomial equation** is an equation that contains a polynomial expression.

The **degree of the polynomial equation** is the degree of the polynomial.

We have already solved polynomial equations of **degree one**. Polynomial equations of degree one are linear equations are of the form $ax+b=c$.

We are now going to solve polynomial equations of **degree two**. A polynomial equation of degree two is called a **quadratic equation**. Listed below are some examples of quadratic equations:

$$x^2+5x+6=0 \qquad 3y^2+4y=10 \qquad 64u^2-81=0 \qquad n(n+1)=42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get n^2+n.

The general form of a quadratic equation is $ax^2+bx+c=0$, with $a\neq 0$. (If $a=0$, then $0\cdot x^2=0$ and we are left with no quadratic term.)

QUADRATIC EQUATION

An equation of the form $ax^2+bx+c=0$ is called a quadratic equation.

$$a, b, \text{ and } c \text{ are real numbers and } a\neq 0$$

To solve quadratic equations we need methods different from the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Use the Zero Product Property

We will first solve some quadratic equations by using the **Zero Product Property**. The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

ZERO PRODUCT PROPERTY

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.

We will now use the Zero Product Property, to solve a **quadratic equation**.

Example 14.6.1: How to Solve a Quadratic Equation Using the Zero Product Property

Solve: $(5n-2)(6n-1)=0$.

Answer

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(5n-2)(6n-1)=0$ $5n-2=0$ or $6n-1=0$
Step 2. Solve the linear equations.	Solve each equation.	$n=\frac{2}{5}$ $n=\frac{1}{6}$
Step 3. Check.	Substitute each solution separately into the original equation.	$n=\frac{2}{5}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0$ $(2-2)\left(\frac{12}{5}-1\right)\stackrel{?}{=}0$ $0\cdot\frac{7}{5}\stackrel{?}{=}0$ $0=0$ ✓ $n=\frac{1}{6}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{1}{6}-2\right)\left(6\cdot\frac{1}{6}-1\right)\stackrel{?}{=}0$ $\left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$ $\left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$ $0=0$ ✓

Example 14.6.2

Solve: $(3m-2)(2m+1)=0$.

Answer

$m=\frac{2}{3}$, $m=-\frac{1}{2}$

Example 14.6.3

Solve: $(4p+3)(4p-3)=0$.

Answer

$p=-\frac{3}{4}$, $p=\frac{3}{4}$

USE THE ZERO PRODUCT PROPERTY.

1. Set each factor equal to zero.
2. Solve the linear equations.
3. Check.

Solve Quadratic Equations by Factoring

The Zero Product Property works very nicely to solve quadratic equations. The quadratic equation must be factored, with zero isolated on one side. So we be sure to start with the quadratic equation in **standard form**, $ax^2+bx+c=0$. Then we factor the expression on the left.

Solve: $2y^2=13y+45$.

Answer

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.	Write the equation in standard form.	$2y^2=13y+45$ $2y^2-13y-45=0$
Step 2. Factor the quadratic expression.	Factor $2y^2-13y+45$ $(2y+5)(y-9)$	$(2y+5)(y-9)=0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero. We have two linear equations.	$2y+5=0$ $y-9=0$
Step 4. Solve the linear equations.		$y=-\frac{5}{2}$ $y=9$
Step 5. Check. Substitute each solution separately into the original equation.	Substitute each solution separately into the original equation.	$y=-\frac{5}{2}$ $2y^2=13y+45$ $2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right)+45$ $2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right)+\frac{90}{2}$ $\frac{25}{2}=\frac{25}{2}$ ✓ $y=9$ $2y^2=13y+45$ $2(9)^2 \stackrel{?}{=} 13(9)+45$ $2(81) \stackrel{?}{=} 117+45$ $162=162$ ✓

Example 14.6.5

Solve: $3c^2=10c-8$.

Answer

$c=2,\ c=\frac{4}{3}$

Example 14.6.6

Solve: $2d^2 - 5d = 3$.

Answer

$d = 3,\ d = -12$

SOLVE A QUADRATIC EQUATION BY FACTORING.

1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve the linear equations.
5. Check. Substitute each solution separately into the original equation.

Before we factor, we must make sure the **quadratic equation** is in **standard form**.

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

Example 14.6.7

Solve: $169q^2 = 49$.

Answer

	$169x^2 = 49$
Write the quadratic equation in standard form.	$169x^2 - 49 = 0$
Factor. It is a difference of squares.	$(13x - 7)(13x + 7) = 0$
Use the Zero Product Property to set each factor to 0.	
	$13x - 7 = 0 \quad 13x + 7 = 0$
Solve each equation.	$13x = 7 \quad 13x = -7$
	$x = \frac{7}{13} \quad x = -\frac{7}{13}$

Check:

We leave the check up to you.

Example 14.6.8

Solve: $25p^2 = 49$.

Answer

$p = \frac{7}{5}, p = -\frac{7}{5}$

Example 14.6.9

Solve: $36x^2 = 121$.

Answer

$x = \frac{11}{6}, x = -\frac{11}{6}$

In the next example, the left side of the equation is factored, but the right side is not zero. In order to use the **Zero Product Property**, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Example 14.6.10

Solve: $(3x-8)(x-1)=3x$.

Answer

	$(3x-8)(x-1)=3x$
Multiply the binomials.	$3x^2-11x+8=3x$
Write the quadratic equation in standard form.	$3x^2-14x+8=0$
Factor the trinomial.	$(3x-2)(x-4)=0$
Use the Zero Product Property to set each factor to 0.	$3x-2=0 \quad x-4=0$
Solve each equation.	$3x=2 \quad x=4$
	$x=\frac{2}{3}$
Check your answers.	The check is left to you.

Example 14.6.11

Solve: $(2m+1)(m+3)=12m$.

Answer

$m=1,\ m=\frac{3}{2}$

Example 14.6.12

Solve: $(k+1)(k-1)=8$.

Answer

$k=3,\ k=-3$

In the next example, when we factor the quadratic equation we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Example 14.6.13

Solve: $3x^2=12x+63$.

Answer

	$3x^2=12x+63$
Write the quadratic equation in standard form.	$3x^2-12x-63=0$
Factor the greatest common factor first.	$3(x^2-4x-21)=0$
Factor the trinomial.	$3(x-7)(x+3)=0$
Use the Zero Product Property to set each factor to 0.	$3\neq 0 \quad x-7=0 \quad x+3=0$
Solve each equation.	$3\neq 0 \quad x=7 \quad x=-3$
Check your answers.	The check is left to you.

Example 14.6.14

Solve: $18a^2-30=-33a$.

Answer

$a=-\frac{5}{2}, a=\frac{2}{3}$

Example 14.6.15

Solve: $123b = -6 - 60b^2$

Answer

$b = -2,\ b = -\frac{1}{20}$

The **Zero Product Property** also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree greater than two by using the Zero Product Property, just like we solved quadratic equations.

Example 14.6.16

Solve: $9m^3 + 100m = 60m^2$

Answer

	$9m^3 + 100m = 60m^2$
Bring all the terms to one side so that the other side is zero.	$9m^3 - 60m^2 + 100m = 0$
Factor the greatest common factor first.	$m(9m^2 - 60m + 100) = 0$
Factor the trinomial.	$m(3m - 10)^2 = 0$
Use the Zero Product Property to set each factor to 0.	
Solve each equation.	$m = 0 \quad 3m - 10 = 0$ $m = 0 \quad m = \frac{10}{3}$
Check your answers.	The check is left to you.

Example 14.6.17

Solve: $8x^3 = 24x^2 - 18x$.

Answer

$x = 0,\ x = \frac{3}{2}$

Example 14.6.18

Solve: $16y^2 = 32y^3 + 2y$.

Answer

$y = 0,\ y = 14$

Solve Equations with Polynomial Functions

As our study of polynomial functions continues, it will often be important to know when the function will have a certain value or what points lie on the graph of the function. Our work with the **Zero Product Property** will be help us find these answers.

Example 14.6.19

For the function $f(x) = x^2 + 2x - 2$,

ⓐ find x when $f(x) = 6$

ⓑ find two points that lie on the graph of the function.

Answer

ⓐ

	$f(x)=x^2+2x-2$
Substitute 6 for $f(x)$.	$6=x^2+2x-2$
Put the quadratic in standard form.	$x^2+2x-8=0$
Factor the trinomial.	$(x+4)(x-2)=0$
Use the zero product property.	$x+4=0 \quad \text{or} \quad x-2=0$
Solve.	$x=-4 \quad \text{or} \quad x=2$
Check:	

$f(x)=x^2+2x-2$	$f(x)=x^2+2x-2$
$f(-4)=(-4)^2+2(-4)-2$	$f(2)=2^2+2\cdot 2-2$
$f(-4)=16-8-2$	$f(2)=4+4-2$
$f(-4)=6$✓	$f(2)=6$✓

ⓑ Since $f(-4)=6$ and $f(2)=6$, the points $(-4,6)$ and $(2,6)$ lie on the graph of the function.

? Example 14.6.20

For the function $f(x)=x^2-2x-8$,

ⓐ find x when $f(x)=7$
ⓑ Find two points that lie on the graph of the function.

Answer

ⓐ $x=-3$ or $x=5$
ⓑ $(-3,7)$ $(5,7)$

? Example 14.6.21

For the function $f(x)=x^2-8x+3$,

ⓐ find x when $f(x)=-4$
ⓑ Find two points that lie on the graph of the function.

Answer

ⓐ $x=1$ or $x=7$
ⓑ $(1,-4)$ $(7,-4)$

The **Zero Product Property** also helps us determine where the function is zero. A value of x where the function is 0, is called a **zero of the function**.

ZERO OF A FUNCTION

For any function f, if $f(x)=0$, then x is a **zero of the function**.

When $f(x)=0$, the point $(x,0)$ is a point on the graph. This point is an x-**intercept** of the graph. It is often important to know where the graph of a function crosses the axes. We will see some examples later.

Example 14.6.22

For the function $f(x)=3x^2+10x-8$, find

ⓐ the zeros of the function,
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function

Answer

ⓐ To find the zeros of the function, we need to find when the function value is 0.

	$f(x)=3x^2+10x-8$
Substitute 0 for $f(x)$.	$0=3x^2+10x-8$
Factor the trinomial.	$(x+4)(3x-2)=0$
Use the zero product property.	$x+4=0 \quad \text{or} \quad 3x-2=0$
Solve.	$x=-4 \quad \text{or} \quad x=\frac{2}{3}$

ⓑ An x-intercept occurs when $y=0$. Since $f(-4)=0$ and $f(\frac{2}{3})=0$, the points $(-4,0)$ and $(\frac{2}{3},0)$ lie on the graph. These points are x-intercepts of the function.

ⓒ A y-intercept occurs when $x=0$. To find the y-intercepts we need to find $f(0)$.

	$f(x)=3x^2+10x-8$
Find $f(0)$ by substituting 0 for x.	$f(0)=3\cdot 0^2+10\cdot 0-8$
Simplify.	$f(0)=-8$

Since $f(0)=-8$, the point $(0,-8)$ lies on the graph. This point is the y-intercept of the function.

Example 14.6.23

For the function $f(x)=2x^2-7x+5$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

Answer

ⓐ $x=1$ or $x=\frac{5}{2}$
ⓑ $(1,0),\ (\frac{5}{2},0)$ ⓒ $(0,5)$

Example 14.6.24

For the function $f(x)=6x^2+13x-15$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

Answer

ⓐ $x=-3$ or $x=\frac{5}{6}$
ⓑ $(-3,0),\ (\frac{5}{6},0)$ ⓒ $(0,-15)$

Solve Applications Modeled by Polynomial Equations

The problem-solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to polynomial equations. We will copy the problem-solving strategy here so we can use it for reference.

USE A PROBLEM SOLVING STRATEGY TO SOLVE WORD PROBLEMS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using appropriate algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a polynomial equation.

Example 14.6.25

The product of two consecutive odd integers is 323. Find the integers.

Answer

Step 1. Read the problem.

Step 2. Identify what we are looking for. — We are looking for two consecutive integers.

Step 3. Name what we are looking for. — Let $n=$ the first integer.
$n+2=$ next consecutive odd integer

Step 4. Translate into an equation. Restate the problem in a sentence. — The product of the two consecutive odd integers is 323.
$n(n+2)=323$

Step 5. Solve the equation. $n^2+2n=323$

Bring all the terms to one side. $n^2+2n-323=0$

Factor the trinomial. $(n-17)(n+19)=0$

Use the Zero Product Property. $n-17=0$ $\quad$ $n+19=0$

Solve the equations. $n=17$ $\quad$ $n=-19$

There are two values for n that are solutions to this problem. So there are two sets of consecutive odd integers that will work.

If the first integer is $n=17$ then the next odd integer is

$$\begin{aligned} n+2 \\ 17+2 \\ 19 \\ 17,19 \end{aligned}$$

If the first integer is $n=-19$ then the next odd integer is

$$\begin{aligned} n+2 \\ -19+2 \\ -17 \\ -17,-19 \end{aligned}$$

Step 6. Check the answer.

The results are consecutive odd integers
$17,\ 19$ and $-19,\ -17$.

$17\cdot 19=323$✓ $\quad$ $-19(-17)=323$✓

Both pairs of consecutive integers are solutions.

Step 7. Answer the question — The consecutive integers are $17, 19$ and $-19, -17$.

? Example 14.6.26

The product of two consecutive odd integers is 255. Find the integers.

Answer

$-15, -17$ and $15, 17$

? Example 14.6.27

The product of two consecutive odd integers is 483 Find the integers.

Answer

$-23, -21$ and $21, 23$

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give positive results.

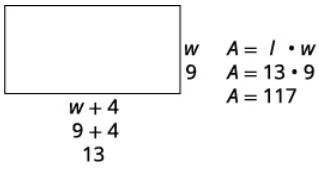
In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

? Example 14.6.28

A rectangular bedroom has an area 117 square feet. The length of the bedroom is four feet more than the width. Find the length and width of the bedroom.

Answer

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what you are looking for.	Let $w =$ the width of the bedroom.
The length is four feet more than the width.	$w + 4 =$ the length of the garden
Step 4. Translate into an equation.	
Restate the important information in a sentence.	The area of the bedroom is 117 square feet.
Use the formula for the area of a rectangle.	$A = l \cdot w$
Substitute in the variables.	$117 = (w + 4)w$
Step 5. Solve the equation Distribute first.	$117 = w^2 + 4w$
Get zero on one side.	$117 = w^2 + 4w$
Factor the trinomial.	$0 = w^2 + 4w - 117$
Use the Zero Product Property.	$0 = (w^2 + 13)(w - 9)$
Solve each equation.	$0 = w + 13 \quad 0 = w - 9$
Since w is the width of the bedroom, it does not make sense for it to be negative. We eliminate that value for w.	$\cancel{w = -13} \quad w = 9$
	$w = 9$ The width is 9 feet.
Find the value of the length.	$w + 4$ $9 + 4$ 13 The length is 13 feet.

Step 6. Check the answer. Does the answer make sense?  $A = l \cdot w$ $A = 13 \cdot 9$ $A = 117$ Yes, this makes sense.	
Step 7. Answer the question.	The width of the bedroom is 9 feet and the length is 13 feet.

Example 14.6.29

A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

Answer

The width is 5 feet and length is 6 feet.

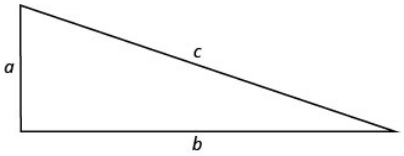
Example 14.6.30

A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

Answer

The length of the patio is 12 feet and the width 15 feet.

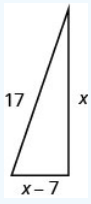
In the next example, we will use the Pythagorean Theorem $(a^2 + b^2 = c^2)$. This formula gives the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

Example 14.6.31

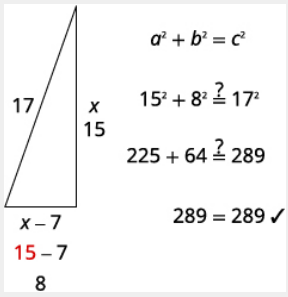
A boat's sail is in the shape of a right triangle as shown. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the sail.



Answer

Step 1. Read the problem

https://math.libretexts.org/@go/page/46379

Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the sail.
Step 3. Name what you are looking for. One side is 7 less than the other.	Let $x =$ length of a side of the sail. $x - 7 =$ length of other side
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Substitute in the variables.	$x^2 + (x-7)^2 = 17^2$
Step 5. Solve the equation Simplify.	$x^2 + x^2 - 14x + 49 = 289$
	$2x^2 - 14x + 49 = 289$
It is a quadratic equation, so get zero on one side.	$2x^2 - 14x - 240 = 0$
Factor the greatest common factor.	$2(x^2 - 7x - 120) = 0$
Factor the trinomial.	$2(x-15)(x+8) = 0$
Use the Zero Product Property.	$2 \neq 0 \quad x - 15 = 0 \quad x + 8 = 0$
Solve.	$2 \neq 0 \quad x = 15 \quad x = -8$
Since x is a side of the triangle, $x = -8$ does not make sense.	$2 \neq 0 \quad x = 15 \quad \cancel{x = -8}$
Find the length of the other side.	
If the length of one side is then the length of the other side is	$x = 15$ $x - 7$ $15 - 7$ 8 is the length of the other side.
Step 6. Check the answer in the problem Do these numbers make sense? 	
Step 7. Answer the question	The sides of the sail are 8, 15 and 17 feet.

? Example 14.6.32

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle. The length of one side of the deck is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the deck.

Answer

5 feet and 12 feet

Example 14.6.33

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of the other leg. Find the lengths of the hypotenuse and the other leg.

Answer

24 feet and 25 feet

The next example uses the function that gives the height of an object as a function of time when it is thrown from 80 feet above the ground.

Example 14.6.34

Dennis is going to throw his rubber band ball upward from the top of a campus building. When he throws the rubber band ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, h, of the ball above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tell us when the ball hits the ground
ⓑ when the ball will be 80 feet above the ground
ⓒ the height of the ball at $t = 2$ seconds.

Answer

ⓐ The zeros of this function are found by solving $h(t) = 0$. This will tell us when the ball will hit the ground.

	$h(t) = 0$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 0$
Factor the GCF, -16.	$-16(t^2 - 4t - 5) = 0$
Factor the trinomial.	$-16(t-5)(t+1) = 0$
Use the Zero Product Property.	$t-5=0 \quad t+1=0$
Solve.	$t=5 \quad t=-1$

The result $t = 5$ tells us the ball will hit the ground 5 seconds after it is thrown. Since time cannot be negative, the result $t = -1$ is discarded.

ⓑ The ball will be 80 feet above the ground when $h(t) = 80$.

	$h(t) = 80$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 80$
Subtract 80 from both sides.	$-16t^2 + 64t = 0$
Factor the GCF, $-16t$.	$-16t(t-4) = 0$
Use the Zero Product Property.	$-16t = 0 \quad t-4=0$
Solve.	$t=0 \quad t=4$
	The ball will be at 80 feet the moment Dennis tosses the ball and then 4 seconds later, when the ball is falling.

ⓒ To find the height ball at $t = 2$ seconds we find $h(2)$.

	$h(t) = -16t^2 + 64t + 80$
To find $h(2)$ substitute 2 for t.	$h(2) = -16(2)^2 + 64 \cdot 2 + 80$
Simplify.	$h(2) = 144$
	After 2 seconds, the ball will be at 144 feet.

? Example 14.6.35

Genevieve is going to throw a rock from the top a trail overlooking the ocean. When she throws the rock upward from 160 feet above the ocean, the function $h(t) = -16t^2 + 48t + 160$ models the height, h, of the rock above the ocean as a function of time, t. Find:

ⓐ the zeros of this function which tell us when the rock will hit the ocean
ⓑ when the rock will be 160 feet above the ocean.
ⓒ the height of the rock at $t = 1.5$ seconds.

Answer

ⓐ 5 ⓑ 0;3 ⓒ 196

? Example 14.6.36

Calib is going to throw his lucky penny from his balcony on a cruise ship. When he throws the penny upward from 128 feet above the ground, the function $h(t) = -16t^2 + 32t + 128$ models the height, h, of the penny above the ocean as a function of time, t. Find:

ⓐ the zeros of this function which is when the penny will hit the ocean
ⓑ when the penny will be 128 feet above the ocean.
ⓒ the height the penny will be at $t = 1$ seconds which is when the penny will be at its highest point.

Answer

ⓐ 4 ⓑ 0;2 ⓒ 144

Access this online resource for additional instruction and practice with quadratic equations.

- Beginning Algebra & Solving Quadratics with the Zero Property

Key Concepts

- **Polynomial Equation:** A polynomial equation is an equation that contains a polynomial expression. The degree of the polynomial equation is the degree of the polynomial.
- **Quadratic Equation:** An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation.

$$a, b, c \text{ are real numbers and } a \neq 0$$

- **Zero Product Property:** If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.
- **How to use the Zero Product Property**
  1. Set each factor equal to zero.
  2. Solve the linear equations.
  3. Check.
- **How to solve a quadratic equation by factoring.**
  1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
  2. Factor the quadratic expression.
  3. Use the Zero Product Property.
  4. Solve the linear equations.
  5. Check. Substitute each solution separately into the original equation.
- **Zero of a Function:** For any function f, if $f(x) = 0$, then x is a zero of the function.
- **How to use a problem solving strategy to solve word problems.**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.

4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using appropriate algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

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14.6E: Exercises

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.

1. $(3a-10)(2a-7)=0$

Answer

$a=\frac{10}{3},\ a=\frac{7}{2}$

2. $(5b+1)(6b+1)=0$

3. $6m(12m-5)=0$

Answer

$m=0,\ m=\frac{5}{12}$

4. $2x(6x-3)=0$

5. $(2x-1)^2=0$

Answer

$x=\frac{1}{2}$

6. $(3y+5)^2=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

7. $5a^2-26a=24$

Answer

$a=-\frac{4}{5},\ a=6$

8. $4b^2+7b=-3$

9. $4m^2=17m-15$

Answer

$m=\frac{5}{4},\ m=3$

10. $n^2=5-6n$

11. $7a^2+14a=7a$

Answer

$a=-1,\ a=0$

12. $12b^2 - 15b = -9b$

13. $49m^2 = 144$

Answer

$m = \frac{12}{7},\ m = -\frac{12}{7}$

14. $625 = x^2$

15. $16y^2 = 81$

Answer

$y = -\frac{9}{4},\ y = \frac{9}{4}$

16. $64p^2 = 225$

17. $121n^2 = 36$

Answer

$n = -\frac{6}{11},\ n = \frac{6}{11}$

18. $100y^2 = 9$

19. $(x+6)(x-3) = -8$

Answer

$x = 2,\ x = -5$

20. $(p-5)(p+3) = -7$

21. $(2x+1)(x-3) = -4x$

Answer

$x = \frac{3}{2},\ x = -1$

22. $(y-3)(y+2) = 4y$

23. $(3x-2)(x+4) = 12x$

Answer

$x = \frac{3}{2},\ x = -1$

24. $(2y-3)(3y-1) = 8y$

25. $20x^2 - 60x = -45$

Answer

$x = -\frac{2}{3}$

26. $3y^2 - 18y = -27$

27. $15x^2 - 10x = 40$

Answer

$x = 2,\ x = -\frac{4}{3}$

28. $14y^2 - 77y = -35$

29. $18x^2 - 9 = -21x$

Answer

$x = -\frac{3}{2},\ x = \frac{1}{3}$

30. $16y^2 + 12 = -32y$

31. $16p^3 = 24p^2 - 9p$

Answer

$p = 0,\ p = \frac{3}{4}$

32. $m^3 - 2m^2 = -m$

33. $2x^3 + 72x = 24x^2$

Answer

$x = 0,\ x = 6$

34. $3y^3 + 48y = 24y^2$

35. $36x^3 + 24x^2 = -4x$

Answer

$x = 0,\ x = \frac{1}{3}$

36. $2y^3 + 2y^2 = 12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

37. For the function, $f(x) = x^2 - 8x + 8$, ⓐ find when $f(x) = -4$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x = 2$ or $x = 6$ ⓑ $(2, -4)$ $(6, -4)$

38. For the function, $f(x) = x^2 + 11x + 20$, ⓐ find when $f(x) = -8$ ⓑ Use this information to find two points that lie on the graph of the function.

39. For the function, $f(x) = 8x^2 - 18x + 5$, ⓐ find when $f(x) = -4$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x = \frac{3}{2}$ or $x = \frac{3}{4}$
ⓑ $(\frac{3}{2}, -4)$ $(\frac{3}{4}, -4)$

40. For the function, $f(x) = 18x^2 + 15x - 10$, ⓐ find when $f(x) = 15$ ⓑ Use this information to find two points that lie on the graph of the function.

In the following exercises, for each function, find: ⓐ the zeros of the function ⓑ the x-intercepts of the graph of the function ⓒ the y-intercept of the graph of the function.

41. $f(x) = 9x^2 - 4$

Answer

ⓐ $x = \frac{2}{3}$ or $x = -\frac{2}{3}$
ⓑ $(\frac{2}{3}, 0)$, $(-\frac{2}{3}, 0)$
ⓒ $(0, -4)$

42. $f(x) = 25x^2 - 49$

43. $f(x) = 6x^2 - 7x - 5$

Answer

ⓐ $x = \frac{5}{3}$ or $x = -\frac{1}{2}$
ⓑ $(\frac{5}{3}, 0)$, $(-\frac{1}{2}, 0)$
ⓒ $(0, -5)$

44. $f(x) = 12x^2 - 11x + 2$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

45. The product of two consecutive odd integers is 143. Find the integers.

Answer

$-13,\ -11$ and $11,\ 13$

46. The product of two consecutive odd integers is 195. Find the integers.

47. The product of two consecutive even integers is 168. Find the integers.

Answer

$-14,\ -12$ and $12,\ 14$

48. The product of two consecutive even integers is 288. Find the integers.

49. The area of a rectangular carpet is 28 square feet. The length is three feet more than the width. Find the length and the width of the carpet.

Answer

-4 and 7

50. A rectangular retaining wall has area 15 square feet. The height of the wall is two feet less than its length. Find the height and the length of the wall.

51. The area of a bulletin board is 55 square feet. The length is four feet less than three times the width. Find the length and the width of the a bulletin board.

Answer

$5,\ 11$

52. A rectangular carport has area 150 square feet. The height of the carport is five feet less than twice its length. Find the height and the length of the carport.

53. A pennant is shaped like a right triangle, with hypotenuse 10 feet. The length of one side of the pennant is two feet longer than the length of the other side. Find the length of the two sides of the pennant.

Answer

$6,\ 8$

54. A stained glass window is shaped like a right triangle. The hypotenuse is 15 feet. One leg is three more than the other. Find the lengths of the legs.

55. A reflecting pool is shaped like a right triangle, with one leg along the wall of a building. The hypotenuse is 9 feet longer than the side along the building. The third side is 7 feet longer than the side along the building. Find the lengths of all three sides of the reflecting pool.

Answer

$8,\ 15,\ 17$

56. A goat enclosure is in the shape of a right triangle. One leg of the enclosure is built against the side of the barn. The other leg is 4 feet more than the leg against the barn. The hypotenuse is 8 feet more than the leg along the barn. Find the three sides of the goat enclosure.

57. Juli is going to launch a model rocket in her back yard. When she launches the rocket, the function $h(t) = -16t^2 + 32t$ models the height, h, of the rocket above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tells us when the rocket will hit the ground. ⓑ the time the rocket will be 16 feet above the ground.

Answer

ⓐ 0, 2 ⓑ 1

58. Gianna is going to throw a ball from the top floor of her middle school. When she throws the ball from 48 feet above the ground, the function $h(t) = -16t^2 + 32t + 48$ models the height, h, of the ball above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 48 feet above the ground. ⓒ the height the ball will be at $t = 1$ seconds which is when the ball will be at its highest point.

Writing Exercises

59. Explain how you solve a quadratic equation. How many answers do you expect to get for a quadratic equation?

Answer

Answers will vary.

60. Give an example of a quadratic equation that has a GCF and none of the solutions to the equation is zero.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations by using the Zero Product Property.			
solve quadratic equations by factoring.			
solve applications modeled by quadratic equations.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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14.E: Chapter 6 Review Exercises

Chapter Review Exercises

Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

$12a^2b^3,\ 15ab^2$

Answer

$3ab^2$

$12m^2n^3, 42m^5n^3$

$15y^3,\ 21y^2,\ 30y$

Answer

$3y$

$45x^3y^2,\ 15x^4y,\ 10x^5y^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

$35y+84$

Answer

$7(5y+12)$

$6y^2+12y-6$

$18x^3-15x$

Answer

$3x(6x^2-5)$

$15m^4+6m^2n$

$4x^3-12x^2+16x$

Answer

$4x(x^2-3x+4)$

$-3x+24$

$-3x^3+27x^2-12x$

Answer

$-3x(x^2-9x+4)$

$3x(x-1)+5(x-1)$

Factor by Grouping

In the following exercises, factor by grouping.

$ax-ay+bx-by$

Answer

$(a+b)(x-y)$

$x^2y-xy^2+2x-2y$

$x^2+7x-3x-21$

Answer

$(x-3)(x+7)$

$4x^2-16x+3x-12$

m^3+m^2+m+1

Answer

$(m^2+1)(m+1)$

$5x-5y-y+x$

Factor Trinomials

Factor Trinomials of the Form x^2+bx+c

In the following exercises, factor each trinomial of the form x^2+bx+c .

$a^2+14a+33$

Answer

$(a+3)(a+11)$

$k^2-16k+60$

$m^2+3m-54$

Answer

$(m+9)(m-6)$

$x^2-3x-10$

In the following examples, factor each trinomial of the form $x^2+bxy+cy^2$.

$x^2+12xy+35y^2$

Answer

$(x+5y)(x+7y)$

$r^2+3rs-28s^2$

$a^2+4ab-21b^2$

Answer

$(a+7b)(a-3b)$

$p^2-5pq-36q^2$

$m^2-5mn+30n^2$

Answer

Prime

Factor Trinomials of the Form ax2+bx+cax2+bx+c Using Trial and Error

In the following exercises, factor completely using trial and error.

x^3+5x^2-24x

$3y^3-21y^2+30y$

Answer

$3y(y-5)(y-2)$

$5x^4+10x^3-75x^2$

$5y^2+14y+9$

Answer

$(5y+9)(y+1)$

$8x^2+25x+3$

$10y^2-53y-11$

Answer

$(5y+1)(2y-11)$

$6p^2-19pq+10q^2$

$-81a^2+153a+18$

Answer

$-9(9a-1)(a+2)$

Factor Trinomials of the Form ax2+bx+cax2+bx+c using the 'ac' Method

In the following exercises, factor.

$2x^2+9x+4$

$18a^2-9a+1$

Answer

$(3a-1)(6a-1)$

$15p^2+2p-8$

$15x^2+6x-2$

Answer

$(3x-1)(5x+2)$

$8a^2+32a+24$

$3x^2+3x-36$

Answer

$3(x+4)(x-3)$

$48y^2+12y-36$

$18a^2-57a-21$

Answer

$3(2a-7)(3a+1)$

$3n^4-12n^3-96n^2$

Factor using substitution

In the following exercises, factor using substitution.

x^4-13x^2-30

Answer

$(x^2-15)(x^2+2)$

$(x-3)^2-5(x-3)-36$

Factor Special Products

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

$25x^2+30x+9$

Answer

$(5x+3)^2$

$36a^2 - 84ab + 49b^2$

$40x^2 + 360x + 810$

Answer

$10(2x + 9)^2$

$5k^3 - 70k^2 + 245k$

$75u^4 - 30u^3v + 3u^2v^2$

Answer

$3u^2(5u - v)^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

$81r^2 - 25$

$169m^2 - n^2$

Answer

$(13m + n)(13m - n)$

$25p^2 - 1$

$9 - 121y^2$

Answer

$(3 + 11y)(3 - 11y)$

$20x^2 - 125$

$169n^3 - n$

Answer

$n(13n + 1)(13n - 1)$

$6p^2q^2 - 54p^2$

$24p^2 + 54$

Answer

$6(4p^2 + 9)$

$49x^2 - 81y^2$

$16z^4 - 1$

Answer

$(2z-1)(2z+1)(4z^2+1)$

$48m^4n^2-243n^2$

$a^2+6a+9-9b^2$

Answer

$(a+3-3b)(a+3+3b)$

$x^2-16x+64-y^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

a^3-125

Answer

$(a-5)(a^2+5a+25)$

b^3-216

$2m^3+54$

Answer

$2(m+3)(m^2-3m+9)$

$81m^3+3$

General Strategy for Factoring Polynomials

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

$24x^3+44x^2$

Answer

$4x^2(6x+11)$

$24a^4-9a^3$

$16n^2-56mn+49m^2$

Answer

$(4n-7m)^2$

$6a^2-25a-9$

$5u^4-45u^2$

Answer

$5u^2(u+3)(u-3)$

n^4-81

$64j^2+225$

Answer

prime

$5x^2+5x-60$

b^3-64

Answer

$(b-4)(b^2+4b+16)$

m^3+125

$2b^2-2bc+5cb-5c^2$

Answer

$(2b+5c)(b-c)$

$48x^5y^2-243xy^2$

$5q^2-15q-90$

Answer

$5(q+3)(q-6)$

$4u^5v+4u^2v^3$

$10m^4-6250$

Answer

$10(m-5)(m+5)(m^2+25)$

$60x^2y-75xy+30y$

$16x^2-24xy+9y^2-64$

Answer

$(4x-3y+8)(4x-3y-8)$

Polynomial Equations

Use the Zero Product Property

In the following exercises, solve.

$(a-3)(a+7)=0$

https://math.libretexts.org/@go/page/46381

$(5b+1)(6b+1)=0$

Answer

$b=-\frac{1}{5},\ b=-\frac{1}{6}$

$6m(12m-5)=0$

$(2x-1)^2=0$

Answer

$x=\frac{1}{2}$

$3m(2m-5)(m+6)=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

$x^2+9x+20=0$

Answer

$x=-4,\ x=-5$

$y^2-y-72=0$

$2p^2-11p=40$

Answer

$p=-\frac{5}{2}, p=8$

$q^3+3q^2+2q=0$

$144m^2-25=0$

Answer

$m=\frac{5}{12},\ m=-\frac{5}{12}$

$4n^2=36$

$(x+6)(x-3)=-8$

Answer

$x=2,\ x=-5$

$(3x-2)(x+4)=12$

$16p^3=24p^2+9p$

Answer

$p = 0,\ p = \frac{3}{4}$

$2y^3 + 2y^2 = 12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

For the function, $f(x) = x^2 + 11x + 20$, ⓐ find when $f(x) = -8$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x = -7$ or $\backslash x = -4$
ⓑ $(-7, -8)$ $(-4, -8)$

For the function, $f(x) = 9x^2 - 18x + 5$, ⓐ find when $f(x) = -3$ ⓑ Use this information to find two points that lie on the graph of the function.

In each function, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒ the *y*-intercept of the graph of the function.

$f(x) = 64x^2 - 49$

Answer

ⓐ $x = \frac{7}{8}$ or $x = -\frac{7}{8}$
ⓑ $(\frac{7}{8}, 0),\ (-\frac{7}{8}, 0)$ ⓒ $(0, -49)$

$f(x) = 6x^2 - 13x - 5$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

The product of two consecutive numbers is 399. Find the numbers.

Answer

The numbers are -21 and -19 or 19 and 21.

The area of a rectangular shaped patio 432 square feet. The length of the patio is 6 feet more than its width. Find the length and width.

A ladder leans against the wall of a building. The length of the ladder is 9 feet longer than the distance of the bottom of the ladder from the building. The distance of the top of the ladder reaches up the side of the building is 7 feet longer than the distance of the bottom of the ladder from the building. Find the lengths of all three sides of the triangle formed by the ladder leaning against the building.

Answer

The lengths are 8, 15, and 17 ft.

Shruti is going to throw a ball from the top of a cliff. When she throws the ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐthe zeros of this

function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 80 feet above the ground. ⓒ the height the ball will be at $t = 2$ seconds which is when the ball will be at its highest point.

Chapter Practice Test

In the following exercises, factor completely.

$80a^2 + 120a^3$

Answer

$40a^2(2 + 3a)$

$5m(m - 1) + 3(m - 1)$

$x^2 + 13x + 36$

Answer

$(x + 7)(x + 6)$

$p^2 + pq - 12q^2$

$xy - 8y + 7x - 56$

Answer

$(x - 8)(y + 7)$

$40r^2 + 810$

$9s^2 - 12s + 4$

Answer

$(3s - 2)^2$

$6x^2 - 11x - 10$

$3x^2 - 75y^2$

Answer

$3(x + 5y)(x - 5y)$

$6u^2 + 3u - 18$

$x^3 + 125$

Answer

$(x + 5)(x^2 - 5x + 25)$

$32x^5y^2 - 162xy^2$

$6x^4 - 19x^2 + 15$

Answer

$(3x^2-5)(2x^2-3)$

$3x^3-36x^2+108x$

In the following exercises, solve

$5a^2+26a=24$

Answer

$a=\frac{4}{5},\ a=-6$

The product of two consecutive integers is 156. Find the integers.

The area of a rectangular place mat is 168 square inches. Its length is two inches longer than the width. Find the length and width of the placemat.

Answer

The width is 12 inches and the length is 14 inches.

Jing is going to throw a ball from the balcony of her condo. When she throws the ball from 80 feet above the ground, the function $h(t)=-16t^2+64t+80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 128 feet above the ground. ⓒ the height the ball will be at $t=4$ seconds.

For the function, $f(x)=x^2-7x+5$, ⓐ find when $f(x)=-7$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x=3$ or $x=4$ ⓑ $(3,-7)$ $(4,-7)$

For the function $f(x)=25x^2-81$, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒ the *y*-intercept of the graph of the function.

Glossary

degree of the polynomial equation

The degree of the polynomial equation is the degree of the polynomial.

polynomial equation

A polynomial equation is an equation that contains a polynomial expression.

quadratic equation

Polynomial equations of degree two are called quadratic equations.

zero of the function

A value of xx where the function is 0, is called a zero of the function.

Zero Product Property

The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero.

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CHAPTER OVERVIEW

15: Rational Expressions

- 15.1: Rational Expressions and Functions
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15.1: Rational Expressions and Functions

Twelve goals last season. Fifteen home runs. Nine touchdowns. Whatever the statistics, sports analysts know it. Their jobs depend on it. Compiling and analyzing sports data not only help fans appreciate their teams but also help owners and coaches decide which players to recruit, how to best use them in games, how much they should be paid, and which players to trade. Understanding this kind of data requires a knowledge of specific types of expressions and functions. In this chapter, you will work with rational expressions and perform operations on them. And you will use rational expressions and inequalities to solve real-world problems.

Figure 7.0.1: *American football is the most watched spectator sport in the United States. People around the country are constantly tracking statistics for football and other sports. (credit: "keijj44" / Pixabay)*

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15.2: Multiply and Divide Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Simplify rational expressions
- Multiply rational expressions
- Divide rational expressions
- Multiply and divide rational functions

Before you get started, take this readiness quiz.

1. Simplify: $\frac{90y}{15y^2}$.
   If you missed this problem, review [link].
2. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.
   If you missed this problem, review [link].
3. Divide: $\frac{12}{10} \div \frac{8}{25}$.
   If you missed this problem, review [link].

We previously reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

RATIONAL EXPRESSION

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Here are some examples of rational expressions:

$$-\frac{24}{56} \qquad \frac{5x}{12y} \qquad \frac{4x+1}{x^2-9} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{24}{56}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

When we work with a numerical fraction, it is easy to avoid dividing by zero because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

DETERMINE THE VALUES FOR WHICH A RATIONAL EXPRESSION IS UNDEFINED.

1. Set the denominator equal to zero.
2. Solve the equation.

EXAMPLE 15.2.1

Determine the value for which each rational expression is undefined:

a. $\frac{8a^2b}{3c}$ b. $\frac{4b-3}{2b+5}$ c. $\frac{x+4}{x^2+5x+6}$.

Solution

The expression will be undefined when the denominator is zero.

a.

$$\frac{8a^2b}{3c}$$

Set the denominator equal to zero and solve for the variable.

$$3c=0$$

$$c=0$$

$\frac{8a^2b}{3c}$ is undefined for $c=0$

b.

$$\frac{4b-3}{2b+5}$$

Set the denominator equal to zero and solve for the variable.

$$2b+5=0$$

$$2b=-5$$

$$b=-\frac{5}{2}$$

$\frac{4b-3}{2b+5}$ is undefined for $b=-\frac{5}{2}$

c.

$$\frac{x+4}{x^2+5x+6}$$

Set the denominator equal to zero and solve for the variable.

$$x^2+5x+6=0$$

$$(x+2)(x+3)=0$$

$$x+2=0 \text{ or } x+3=0$$

$$x=-2 \text{ or } x=-3$$

$\frac{x+4}{x^2+5x+6}$ is undefined for $x=-2$ or $x=-3$

Try It 15.2.2

Determine the value for which each rational expression is undefined.

a. $\frac{3y^2}{8x}$ b. $\frac{8n-5}{3n+1}$ c. $\frac{a+10}{a^2+4a+3}$

Answer

a. $x = 0$

b. $n = -\frac{1}{3}$

c. $a = -1, a = -3$

Try It 15.2.3

Determine the value for which each rational expression is undefined.

a. $\frac{4p}{5q}$ b. $\frac{y-1}{3y+2}$ c. $\frac{m-5}{m^2+m-6}$

Answer

a. $q = 0$

b. $y = -\frac{2}{3}$

c. $m = 2, m = -3$

Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1, in its numerator and denominator.

SIMPLIFIED RATIONAL EXPRESSION

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,

$\frac{x+2}{x+3}$ is simplified because there are no common factors of $x+2$ and $x+3$.

$\frac{2x}{3x}$ is not simplified because x is a common factor of $2x$ and $3x$.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

EQUIVALENT FRACTIONS PROPERTY

If a, b, and c are numbers where $b \neq 0, c \neq 0$,

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0, c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$ $\frac{2}{5}$

We removed the common factors 3 and 7. They are factors of the product.

$\frac{3x\cancel{(x-9)}}{5\cancel{(x-9)}}$ where $x \neq 9$ $\frac{3x}{5}$

We removed the common factor $(x-9)$. It is a factor of the product.

$\frac{x+5}{x}$ **NO COMMON FACTORS**

While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

How to Simplify a Rational Expression

EXAMPLE 15.2.4

Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$

Solution

Step 1. Factor the numerator and denominator completely.	Factor x^2+5x+6 and $x^2+8x+12$	$\frac{x^2+5x+6}{x^2+8x+12}$ $\frac{(x+2)(x+3)}{(x+2)(x+6)}$
Step 2. Simplify by dividing out common factors.	Remove the common factor $x+2$ from the numerator and the denominator.	$\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$ $\frac{(x+3)}{(x+6)}$ $x \neq -2 \quad x \neq -6$

Try It 15.2.5

Simplify: $\frac{x^2-x-2}{x^2-3x+2}$.

Answer

$\frac{x+1}{x-1}, x \neq 2, x \neq 1$

Try It 15.2.6

Simplify: $\frac{x^2-3x-10}{x^2+x-2}$.

Answer

$\frac{x-5}{x-1}, x \neq -2, x \neq 1$

We now summarize the steps you should follow to simplify rational expressions.

SIMPLIFY A RATIONAL EXPRESSION.

1. Factor the numerator and denominator completely.
2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all* the common factors.

We'll use the methods we have learned to factor the polynomials in the numerators and denominators in the following examples.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

EXAMPLE 15.2.7

Simplify: $\frac{3a^2-12ab+12b^2}{6a^2-24b^2}$.

Solution

$$\frac{3a^2-12ab+12b^2}{6a^2-24b^2}$$

Factor the numerator and denominator, first factoring out the GCF.

$$\frac{3(a^2-4ab+4b^2)}{6(a^2-4b^2)}$$

$$\frac{3(a-2b)(a-2b)}{6(a+2b)(a-2b)}$$

Remove the common factors of $a-2b$ and 3.

$$\frac{\cancel{3}(a-2b)\ \cancel{(a-2b)}}{\cancel{3}\cdot 2(a+2b)\ \cancel{(a-2b)}}$$

$$\frac{a-2b}{2(a+2b)}$$

Try It 15.2.8

Simplify: $\frac{2x^2-12xy+18y^2}{3x^2-27y^2}$.

Answer

$$\frac{2(x-3y)}{3(x+3y)}$$

Try It 15.2.9

Simplify: $\frac{5x^2-30xy+25y^2}{2x^2-50y^2}$.

Answer

$$\frac{5(x-y)}{2(x+5y)}$$

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of a is $-a$ and $-a = -1 \cdot a$.

The numerical fraction, say $\frac{7}{-7}$ simplifies to -1. We also recognize that the numerator and denominator are opposites.

The fraction $\frac{a}{-a}$, whose numerator and denominator are opposites also simplifies to -1.

Let's look at the expression $b-a$.	$b-a$
Rewrite.	$-a+b$
Factor out -1.	$-1(a-b)$

This tells us that $b-a$ is the opposite of $a-b$.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $\frac{a-b}{b-a}$ simplifies to -1.

OPPOSITES IN A RATIONAL EXPRESSION

The opposite of $a-b$ is $b-a$.

$$\frac{a-b}{b-a} = -1 \quad a \neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat $a+b$ and $b+a$ as opposites. Recall that in addition, order doesn't matter so $a+b=b+a$. So if $a \neq -b$, then $\frac{a+b}{b+a} = 1$.

EXAMPLE 15.2.10

Simplify: $\frac{x^2-4x-32}{64-x^2}$

Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and the denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Simplify.	$-\frac{x+4}{x+8}$

Try It 15.2.11

Simplify: $\frac{x^2-4x-5}{25-x^2}$

Answer

$-\frac{x+1}{x+5}$

Try It 15.2.12

Simplify: $\dfrac{x^2+x-2}{1-x^2}$.

Answer

$-\dfrac{x+2}{x+1}$

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

MULTIPLICATION OF RATIONAL EXPRESSIONS

If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$, $x \neq 3$, and $x \neq 4$.

EXAMPLE 15.2.13: How to Multiply Rational Expressions

Simplify: $\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$.

Solution

Step 1. Factor each numerator and denominator completely.	Factor x^2-9 and $x^2-7x+12$.	$\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$ $\dfrac{2x}{(x-3)(x-4)} \cdot \dfrac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\dfrac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\dfrac{2 \cdot x \cdot (x-3)(x+3)}{2 \cdot 3 \cdot x \cdot x \cdot (x-3)(x-4)}$ $\dfrac{(x+3)}{3x(x-4)}$

Try It 15.2.14

Simplify: $\dfrac{5x}{x^2+5x+6} \cdot \dfrac{x^2-4}{10x}$.

Answer

$\dfrac{x-2}{2(x+3)}$

Try It 15.2.15

Simplify: $\frac{9x^2}{x^2+11x+30} \cdot \frac{x^2-36}{3x^2}$.

Answer

$\frac{3(x-6)}{x+5}$

MULTIPLY RATIONAL EXPRESSIONS.

1. Factor each numerator and denominator completely.
2. Multiply the numerators and denominators.
3. Simplify by dividing out common factors.

EXAMPLE 15.2.16

Multiply: $\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$.

Solution

$$\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$$

Factor the numerators and denominators and then multiply.

$$\frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$$

Simplify by dividing out common factors.

$$\frac{\cancel{(3a+1)}(a-3)\ \cancel{(a+5)}(a+5)}{(a-5)\ \cancel{(a+5)}\ \cancel{(3a+1)}(a-5)}$$

Simplify.

$$\frac{(a-3)(a+5)}{(a-5)(a-5)}$$

Rewrite $(a-5)(a-5)$ using an exponent.

$$\frac{(a-3)(a+5)}{(a-5)^2}$$

Try It 15.2.17

Simplify: $\frac{2x^2+5x-12}{x^2-16} \cdot \frac{x^2-8x+16}{2x^2-13x+15}$.

Answer

$\frac{x-4}{x-5}$

Try It 15.2.18

Simplify: $\frac{4b^2+7b-2}{1-b^2} \cdot \frac{b^2-2b+1}{4b^2+15b-4}$.

Answer

$$-\frac{(b+2)(b-1)}{(1+b)(b+4)}$$

Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

DIVISION OF RATIONAL EXPRESSIONS

If p, q, r, and s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

EXAMPLE 15.2.19: How to Divide Rational Expressions

Divide: $\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$.

Solution

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$ $\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$
Step 2. Factor the numerators and denominators completely.	Factor the numerators and denominators.	$\frac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \frac{2 \cdot 3}{(p-q)(p+q)}$
Step 3. Multiply the numerators and denominators.	Multiply the numerators and multiply the denominators.	$\frac{(p+q)(p^2-pq+q^2)2 \cdot 3}{2(p^2+pq+q^2)(p-q)(p+q)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2} \cdot 3}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$ $\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Try It 15.2.20

Simplify: $\frac{x^3-8}{3x^2-6x+12} \div \frac{x^2-4}{6}$.

Answer

$$\frac{2(x^2+2x+4)}{(x+2)(x^2-2x+4)}$$

Try It 15.2.21

Simplify: $\frac{2z^2}{z^2-1} \div \frac{z^3-z^2+z}{z^3+1}$.

Answer

$\frac{2z}{z-1}$

DIVISION OF RATIONAL EXPRESSIONS

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

Recall from Use the Language of Algebra that a complex fraction is a fraction that contains a fraction in the numerator, the denominator or both. Also, remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

EXAMPLE 15.2.22

Divide: $\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$.

Solution

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$$

Rewrite with a division sign. $\frac{6x^2-7x+2}{4x-8} \div \frac{2x^2-7x+3}{x^2-5x+6}$

Rewrite as product of first times reciprocal of second. $\frac{6x^2-7x+2}{4x-8} \cdot \frac{x^2-5x+6}{2x^2-7x+3}$

Factor the numerators and the denominators, and then multiply. $\frac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$

Simplify by dividing out common factors. $\frac{\cancel{(2x-1)}(3x-2)\ \cancel{(x-2)}\ \cancel{(x-3)}}{4\ \cancel{(x-2)}\ \cancel{(2x-1)}\ \cancel{(x-3)}}$

Simplify. $\frac{3x-2}{4}$

Try It 15.2.23

Simplify: $\dfrac{\dfrac{3x^2+7x+2}{4x+24}}{\dfrac{3x^2-14x-5}{x^2+x-30}}$.

Answer

$\dfrac{x+2}{4}$

Try It 15.2.24

Simplify: $\dfrac{\dfrac{y^2-36}{2y^2+11y-6}}{\dfrac{2y^2-2y-60}{8y-4}}$.

Answer

$\dfrac{2}{y+5}$

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then, we factor and multiply.

EXAMPLE 15.2.25

Perform the indicated operations: $\dfrac{3x-6}{4x-4}\cdot\dfrac{x^2+2x-3}{x^2-3x-10}\div\dfrac{2x+12}{8x+16}$.

Solution

	$\frac{3x-6}{4x-4}\cdot\frac{x^2+2x-3}{x^2-3x-10}\div\frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4}\cdot\frac{x^2+2x-3}{x^2-3x-10}\cdot\frac{8x+16}{2x+12}$
Factor the numerators and the denominators.	$\frac{3(x-2)}{4(x-1)}\cdot\frac{(x+3)(x-1)}{(x+2)(x-5)}\cdot\frac{8(x+2)}{2(x+6)}$
Multiply the fractions. Bringing the constants to the front will help when removing common factors.	
Simplify by dividing out common factors.	$\frac{3\cdot 8(x-2)(x+3)(x-1)(x+2)}{4\cdot 2(x-1)(x+2)(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

Try It 15.2.26

Perform the indicated operations: $\dfrac{4m+4}{3m-15}\cdot\dfrac{m^2-3m-10}{m^2-4m-32}\div\dfrac{12m-36}{6m-48}$.

Answer

$\dfrac{2(m+1)(m+2)}{3(m+4)(m-3)}$

Try It 15.2.27

Perform the indicated operations: $\frac{2n^2+10n}{n-1} \div \frac{n^2+10n+24}{n^2+8n-9} \cdot \frac{n+4}{8n^2+12n}$.

Answer

$$\frac{(n+5)(n+9)}{2(n+6)(2n+3)}$$

Multiply and Divide Rational Functions

We started this section stating that a **rational expression** is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$. Similarly, we define a **rational function** as a function of the form $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

RATIONAL FUNCTION

A rational function is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

The domain of a rational function is all real numbers except for those values that would cause division by zero. We must eliminate any values that make $q(x) = 0$.

DETERMINE THE DOMAIN OF A RATIONAL FUNCTION.

1. Set the denominator equal to zero.
2. Solve the equation.
3. The domain is all real numbers excluding the values found in Step 2.

EXAMPLE 15.2.28

Find the domain of $R(x) = \frac{2x^2-14x}{4x^2-16x-48}$.

Solution

The domain will be all real numbers except those values that make the denominator zero. We will set the denominator equal to zero , solve that equation, and then exclude those values from the domain.

Set the denominator to zero.	$4x^2-16x-48=0$
Factor, first factor out the GCF.	$4(x^2-4x-12)=0$
	$4(x-6)(x+2)=0$
Use the Zero Product Property.	$4 \neq 0 \quad x-6=0 \quad x+2=0$
Solve.	$x=6 \quad x=-2$
	The domain of $R(x)$ is all real numbers where $x \neq 6$ and $x \neq -2$

Try It 15.2.29

Find the domain of $R(x) = \dfrac{2x^2 - 10x}{4x^2 - 16x - 20}$.

Answer

The domain of $R(x)$ is all real numbers where $x \neq 5$ and $x \neq -1$.

Try It 15.2.30

Find the domain of $R(x) = \dfrac{4x^2 - 16x}{8x^2 - 16x - 64}$.

Answer

The domain of $R(x)$ is all real numbers where $x \neq 4$ and $x \neq -2$.

To multiply rational functions, we multiply the resulting rational expressions on the right side of the equation using the same techniques we used to multiply rational expressions.

EXAMPLE 15.2.31

Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{2x-6}{x^2-8x+15}$ and $g(x) = \dfrac{x^2-25}{2x+10}$.

Solution

$$R(x) = f(x) \cdot g(x)$$

$$R(x) = \frac{2x-6}{x^2-8x+15} \cdot \frac{x^2-25}{2x+10}$$

Factor each numerator and denominator. $$R(x) = \frac{2(x-3)}{(x-3)(x-5)} \cdot \frac{(x-5)(x+5)}{2(x+5)}$$

Multiply the numerators and denominators. $$R(x) = \frac{2(x-3)(x-5)(x+5)}{2(x-3)(x-5)(x+5)}$$

Remove common factors. $$R(x) = \frac{\cancel{2}\,\cancel{(x-3)}\,\cancel{(x-5)}\,\cancel{(x+5)}}{\cancel{2}\,\cancel{(x-3)}\,\cancel{(x-5)}\,\cancel{(x+5)}}$$

Simplify. $$R(x) = 1$$

Try It 15.2.32

Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{3x-21}{x^2-9x+14}$ and $g(x) = \dfrac{2x^2-8}{3x+6}$.

Answer

$R(x) = 2$

Try It 15.2.33

Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{x^2 - x}{3x^2 + 27x - 30}$ and $g(x) = \dfrac{x^2 - 100}{x^2 - 10x}$.

Answer

$$R(x) = \frac{1}{3}$$

To divide rational functions, we divide the resulting rational expressions on the right side of the equation using the same techniques we used to divide rational expressions.

EXAMPLE 15.2.34

Find $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x) = \dfrac{3x^2}{x^2 - 4x}$ and $g(x) = \dfrac{9x^2 - 45x}{x^2 - 7x + 10}$.

Solution

$$R(x) = \frac{f(x)}{g(x)}$$

Substitute in the functions $f(x),\ g(x)$.

$$R(x) = \frac{\dfrac{3x^2}{x^2 - 4x}}{\dfrac{9x^2 - 45x}{x^2 - 7x + 10}}$$

Rewrite the division as the product of $f(x)$ and the reciprocal of $g(x)$.

$$R(x) = \frac{3x^2}{x^2 - 4x} \cdot \frac{x^2 - 7x + 10}{9x^2 - 45x}$$

Factor the numerators and denominators and then multiply.

$$R(x) = \frac{3 \cdot x \cdot x \cdot (x-5)(x-2)}{x(x-4) \cdot 3 \cdot 3 \cdot x \cdot (x-5)}$$

Simplify by dividing out common factors.

$$R(x) = \frac{\cancel{3} \cdot \cancel{x} \cdot \cancel{x}\,\cancel{(x-5)}(x-2)}{\cancel{x}(x-4) \cdot \cancel{3} \cdot 3 \cdot \cancel{x}\,\cancel{(x-5)}}$$

$$R(x) = \frac{x-2}{3(x-4)}$$

Try It 15.2.35

Find $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x) = \dfrac{2x^2}{x^2 - 8x}$ and $g(x) = \dfrac{8x^2 + 24x}{x^2 + x - 6}$.

Answer

$$R(x) = \frac{x-2}{4(x-8)}$$

Try It 15.2.36

Find $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x) = \dfrac{15x^2}{3x^2 + 33x}$ and $g(x) = \dfrac{5x - 5}{x^2 + 9x - 22}$.

Answer

$$R(x) = \frac{x(x-2)}{x-1}$$

Key Concepts

- **Determine the values for which a rational expression is undefined.**
  1. Set the denominator equal to zero.
  2. Solve the equation.
- **Equivalent Fractions Property**
  If a, b, and c are numbers where $b \neq 0$, $c \neq 0$, then

  $$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$
- **How to simplify a rational expression.**
  1. Factor the numerator and denominator completely.
  2. Simplify by dividing out common factors.
- **Opposites in a Rational Expression**
  The opposite of $a - b$ is $b - a$.

  $$\frac{a-b}{b-a} = -1 \qquad a \neq b$$

  An expression and its opposite divide to -1.
- **Multiplication of Rational Expressions**
  If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then

  $$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$
- **How to multiply rational expressions.**
  1. Factor each numerator and denominator completely.
  2. Multiply the numerators and denominators.
  3. Simplify by dividing out common factors.
- **Division of Rational Expressions**
  If p, q, r, and s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then

  $$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$
- **How to divide rational expressions.**
  1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
  2. Factor the numerators and denominators completely.
  3. Multiply the numerators and denominators together.
  4. Simplify by dividing out common factors.
- **How to determine the domain of a rational function.**
  1. Set the denominator equal to zero.
  2. Solve the equation.
  3. The domain is all real numbers excluding the values found in Step 2.

Glossary

rational expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

simplified rational expression

A simplified rational expression has no common factors, other than 1, in its numerator and denominator.

rational function

A rational function is a function of the form $R(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

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15.2E: Exercises

Practice Makes Perfect

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

1. a. $\frac{2x^2}{z}$ b. $\frac{4p-1}{6p-5}$ c. $\frac{n-3}{n^2+2n-8}$

Answer

a. $z=0$
b. $p=\frac{5}{6}$
c. $n=-4, n=2$

2. a. $\frac{10m}{11n}$ b. $\frac{6y+13}{4y-9}$ c. $\frac{b-8}{b^2-36}$

3. a. $\frac{4x^2y}{3y}$ b. $\frac{3x-2}{2x+1}$ c. $\frac{u-1}{u^2-3u-28}$

Answer

a. $y=0$
b. $x=-\frac{1}{2}$
c. $u=-4, u=7$

4. a. $\frac{5pq^2}{9q}$ b. $\frac{7a-4}{3a+5}$ c. $\frac{1}{x^2-4}$

Simplify Rational Expressions

In the following exercises, simplify each rational expression.

5. $-\frac{44}{55}$

Answer

$-\frac{4}{5}$

6. $\frac{56}{63}$

7. $\frac{8m^3n}{12mn^2}$

Answer

$\frac{2m^2}{3n}$

8. $\frac{36v^3w^2}{27vw^3}$

9. $\dfrac{8n-96}{3n-36}$

Answer

$\dfrac{8}{3}$

10. $\dfrac{12p-240}{5p-100}$

11. $\dfrac{x^2+4x-5}{x^2-2x+1}$

Answer

$\dfrac{x+5}{x-1}$

12. $\dfrac{y^2+3y-4}{y^2-6y+5}$

13. $\dfrac{a^2-4}{a^2+6a-16}$

Answer

$\dfrac{a+2}{a+8}$

14. $\dfrac{y^2-2y-3}{y^2-9}$

15. $\dfrac{p^3+3p^2+4p+12}{p^2+p-6}$

Answer

$\dfrac{p^2+4}{p-2}$

16. $\dfrac{x^3-2x^2-25x+50}{x^2-25}$

17. $\dfrac{8b^2-32b}{2b^2-6b-80}$

Answer

$\dfrac{4b(b-4)}{(b+5)(b-8)}$

18. $\dfrac{-5c^2-10c}{-10c^2+30c+100}$

19. $\frac{3m^2+30mn+75n^2}{4m^2-100n^2}$

Answer

$\frac{3(m+5n)}{4(m-5n)}$

20. $\frac{5r^2+30rs-35s^2}{r^2-49s^2}$

21. $\frac{a-5}{5-a}$

Answer

-1

22. $\frac{5-d}{d-5}$

23. $\frac{20-5y}{y^2-16}$

Answer

$\frac{-5}{y+4}$

24. $\frac{4v-32}{64-v^2}$

25. $\frac{w^3+21}{6w^2-36}$

Answer

$\frac{w^2-6w+3}{6w-6}$

26. $\frac{v^3+125}{v^2-25}$

27. $\frac{z^2-9z+20}{16-z^2}$

Answer

$\frac{-z-5}{4+z}$

28. $\frac{a^2-5z-36}{81-a^2}$

Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

29. $\frac{12}{16} \cdot \frac{4}{10}$

Answer

$\frac{3}{10}$

30. $\frac{3}{25} \cdot \frac{16}{24}$

31. $\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2}$

Answer

$\frac{x^3}{8y}$

32. $\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3}$

33. $\frac{5p^2}{p^2-5p-36} \cdot \frac{p^2-16}{10p}$

Answer

$\frac{p(p-4)}{2(p-9)}$

34. $\frac{3q^2}{q^2+q-6} \cdot \frac{q^2-9}{9q}$

35. $\frac{2y^2-10y}{y^2+10y+25} \cdot \frac{y+5}{6y}$

Answer

$\frac{y-5}{3(y+5)}$

36. $\frac{z^2+3z}{z^2-3z-4} \cdot \frac{z-4}{z^2}$

37. $\frac{28-4b}{3b-3} \cdot \frac{b^2+8b-9}{b^2-49}$

Answer

$\frac{-4(b+9)}{3(b+7)}$

38. $\frac{72m-12m^2}{8m+32} \cdot \frac{m^2+10m+24}{m^2-36}$

39. $\frac{c^2-10c+25}{c^2-25} \cdot \frac{c^2+10c+25}{3c^2-14c-5}$

Answer

$\frac{c+5}{3c+1}$

40. $\frac{2d^2+d-3}{d^2-16} \cdot \frac{d^2-8d+16}{2d^2-9d-18}$

41. $\frac{2m^2-3m-2}{2m2+7m+3} \cdot \frac{3m^2-14m+15}{3m^2+17m-20}$

Answer

$\frac{(m-3)(m-2)}{(m+4)(m+3)}$

42. $\frac{2n^2-3n-14}{25-n^2} \cdot \frac{n^2-10n+25}{2n^2-13n+21}$

Divide Rational Expressions

In the following exercises, divide the rational expressions.

43. $\frac{v-5}{11-v} \div \frac{v^2-25}{v-11}$

Answer

$-\frac{1}{v+5}$

44. $\frac{10+w}{w-8} \div \frac{100-w^2}{8-w}$

45. $\frac{3s^2}{s^2-16} \div \frac{s^3-4s^2+16s}{s^3-64}$

Answer

$\frac{3s}{s+4}$

46. $\frac{r^2-9}{15} \div \frac{r^3-27}{5r^2+15r+45}$

47. $\frac{p^3+q^3}{3p^2+3pq+3q^2} \div \frac{p^2-q^2}{12}$

Answer

$\frac{4(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

48. $\dfrac{v^3-8w^3}{2v^2+4vw+8w^2} \div \dfrac{v^2-4w^2}{4}$

49. $\dfrac{x^2+3x-10}{4x} \div (2x^2+20x+50)$

Answer

$$\frac{x-2}{8x(x+5)}$$

50. $\dfrac{2y^2-10yz-48z^2}{2y-1} \div (4y^2-32yz)$

51. $\dfrac{\dfrac{2a^2-a-21}{5a+20}}{\dfrac{a^2+7a+12}{a^2+8a+16}}$

Answer

$$\frac{2a-7}{5}$$

52. $\dfrac{\dfrac{3b^2+2b-8}{12b+18}}{\dfrac{3b^2+2b-8}{2b^2-7b-15}}$

53. $\dfrac{\dfrac{12c^2-12}{2c^2-3c+1}}{\dfrac{4c+4}{6c^2-13c+5}}$

Answer

$$3(3c-5)$$

54. $\dfrac{\dfrac{4d^2+7d-2}{35d+10}}{\dfrac{d^2-4}{7d^2-12d-4}}$

For the following exercises, perform the indicated operations.

55. $\dfrac{10m^2+80m}{3m-9} \cdot \dfrac{m^2+4m-21}{m^2-9m+20} \div \dfrac{5m^2+10m}{2m-10}$

Answer

$$\frac{4(m+8)(m+7)}{3(m-4)(m+2)}$$

56. $\dfrac{4n^2+32n}{3n+2} \cdot \dfrac{3n^2-n-2}{n^2+n-30} \div \dfrac{108n^2-24n}{n+6}$

57. $\dfrac{12p^2+3p}{p+3} \div \dfrac{p^2+2p-63}{p^2-p-12} \cdot \dfrac{p-7}{9p^3-9p^2}$

Answer

$$\frac{(4p+1)(p-4)}{3p(p+9)(p-1)}$$

58. $\dfrac{6q+3}{9q^2-9q} \div \dfrac{q^2+14q+33}{q^2+4q-5} \cdot \dfrac{4q^2+12q}{12q+6}$

Multiply and Divide Rational Functions

In the following exercises, find the domain of each function.

59. $R(x) = \dfrac{x^3-2x^2-25x+50}{x^2-25}$

Answer

$x \neq 5$ and $x \neq -5$

60. $R(x) = \dfrac{x^3+3x^2-4x-12}{x^2-4}$

61. $R(x) = \dfrac{3x^2+15x}{6x^2+6x-36}$

Answer

$x \neq 2$ and $x \neq -3$

62. $R(x) = \dfrac{8x^2-32x}{2x^2-6x-80}$

For the following exercises, find $R(x) = f(x) \cdot g(x)$ where $f(x)$ and $g(x)$ are given.

63. $f(x) = \dfrac{6x^2-12x}{x^2+7x-18} \quad g(x) = \dfrac{x^2-81}{3x^2-27x}$

Answer

$R(x) = 2$

64. $f(x) = \dfrac{x^2-2x}{x^2+6x-16} \quad g(x) = \dfrac{x^2-64}{x^2-8x}$

65. $f(x) = \dfrac{4x}{x^2-3x-10} \quad g(x) = \dfrac{x^2-25}{8x^2}$

Answer

$$R(x) = \frac{x+5}{2x(x+2)}$$

66. $f(x) = \dfrac{2x^2+8x}{x^2-9x+20} \quad g(x) = \dfrac{x-5}{x^2}$

For the following exercises, find $R(x) = f(x)g(x)$ where $f(x)$ and $g(x)$ are given.

67. $f(x) = \dfrac{27x^2}{3x-21} \quad g(x) = \dfrac{3x^2+18x}{x^2+13x+42}$

Answer

$R(x) = \dfrac{3x(x+7)}{x-7}$

68. $f(x) = \dfrac{24x^2}{2x-8} \quad g(x) = \dfrac{4x^3+28x^2}{x^2+11x+28}$

69. $f(x) = \dfrac{16x^2}{4x+36} \quad g(x) = \dfrac{4x^2-24x}{x^2+4x-45}$

Answer

$R(x) = \dfrac{x(x-5)}{x-6}$

70. $f(x) = \dfrac{24x^2}{2x-4} \quad g(x) = \dfrac{12x^2+36x}{x^2-11x+18}$

Writing Exercises

71. Explain how you find the values of *x* for which the rational expression $\dfrac{x^2-x-20}{x^2-4}$ is undefined.

Answer

Answers will vary.

72. Explain all the steps you take to simplify the rational expression $\dfrac{p^2+4p-21}{9-p^2}$.

73. a. Multiply $\dfrac{7}{4} \cdot \dfrac{9}{10}$ and explain all your steps.
b. Multiply $\dfrac{n}{n-3} \cdot \dfrac{9}{n+3}$ and explain all your steps.
c. Evaluate your answer to part b. when $n = 7$. Did you get the same answer you got in part a.? Why or why not?

Answer

Answers will vary.

74. a. Divide $\dfrac{24}{5} \div 6$ and explain all your steps.
b. Divide $\dfrac{x^2-1}{x} \div (x+1)$ and explain all your steps.
c. Evaluate your answer to part b. when $x = 5$. Did you get the same answer you got in part a.? Why or why not?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
simplify rational expressions.			
multiply rational expressions.			
divide rational expressions.			
multiply and divide rational functions.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

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15.3: Add and Subtract Rational Expressions

Summary

By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{10}+\frac{8}{15}$.
   If you missed this problem, review [link].
2. Subtract: $\frac{3x}{4}-\frac{8}{9}$.
   If you missed this problem, review [link].
3. Subtract: $6(2x+1)-4(x-5)$.
   If you missed this problem, review [link].

Add and Subtract Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

RATIONAL EXPRESSION ADDITION AND SUBTRACTION

If p, q, and r are polynomials where $r\neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Remember, too, we do not allow values that would make the denominator zero. What value of x should be excluded in the next example?

Example 15.3.1

Add: $\frac{11x+28}{x+4}+\frac{x^2}{x+4}$.

Solution

Since the denominator is $x+4$, we must exclude the value $x=-4$.

	$\frac{11x+28}{x+4}+\frac{x^2}{x+4},\ x\neq -4$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{11x+28+x^2}{x+4}$
Write the degrees in descending order.	$\frac{x^2+11x+28}{x+4}$
Factor the numerator.	$\frac{(x+4)(x+7)}{x+4}$
Simplify by removing common factors.	$\frac{\cancel{(x+4)}(x+7)}{\cancel{x+4}}$
Simplify.	$x+7$

The expression simplifies to $x+7$ but the original expression had a denominator of $x+4$ so $x\neq -4$.

? Try It 15.3.2

Simplify: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

Answer

$x+2$

? Try It 15.3.3

Simplify: $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Answer

$x+3$

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator. Be careful of the signs when you subtract a binomial or trinomial.

✓ Example 15.3.4

Subtract: $\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$.

Solution

	$\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$
Subtract the numerators and place the difference over the common denominator.	$\frac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\frac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\frac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\frac{(x-2)\ \cancel{(x-6)}}{(x+3)\ \cancel{(x-6)}}$
	$(x-2)(x+3)$

? Try It 15.3.5

Subtract: $\frac{4x^2-11x+8}{x^2-3x+2}-\frac{3x^2+x-3}{x^2-3x+2}$.

Answer

$\frac{x-11}{x-2}$

? Try It 15.3.6

Subtract: $\frac{6x^2-x+20}{x^2-81}-\frac{5x^2+11x-7}{x^2-81}$.

Answer

$\frac{x-3}{x+9}$

Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works.

	$\frac{7}{d}+\frac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{7}{d}+\frac{(-1)5}{(-1)(-d)}$
The denominators are the same.	$\frac{7}{d}+\frac{-5}{d}$

Simplify.	$\frac{2}{d}$

Be careful with the signs as you work with the opposites when the fractions are being subtracted.

Example 15.3.7

Subtract: $\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$.

Solution

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	$\frac{m^2-6m+3m+2}{m^2-1}$
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify by removing common factors.	$\frac{\cancel{(m-1)}(m-2)}{\cancel{(m-1)}(m+1)}$
Simplify.	$\frac{m-2}{m+1}$

Try It 15.3.8

Subtract: $\frac{y^2-5y}{y^2-4}-\frac{6y-6}{4-y^2}$.

Answer

$\frac{y+3}{y+2}$

Try It 15.3.9

Subtract: $\frac{2n^2+8n-1}{n^2-1}-\frac{n^2-7n-1}{1-n^2}$.

Answer

$\frac{3n-2}{n-1}$

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at this example: $\frac{7}{12}+\frac{5}{18}$. Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied the factors to find the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number. We are now ready to add.

$$\frac{7}{12}+\frac{5}{18}$$
$$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$$
$$\frac{21}{36}+\frac{10}{36}$$

$$\begin{aligned}12&=2\cdot 2\cdot 3\\18&=2\cdot \quad 3\cdot 3\\\text{LCD}&=2\cdot 2\cdot 3\cdot 3\\\text{LCD}&=36\end{aligned}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

FIND THE LEAST COMMON DENOMINATOR OF RATIONAL EXPRESSIONS.

1. Factor each denominator completely.
2. List the factors of each denominator. Match factors vertically when possible.
3. Bring down the columns by including all factors, but do not include common factors twice.
4. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of xx should we exclude in this next example?

Example 15.3.10

a. Find the LCD for the expressions $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$ and b. rewrite them as equivalent rational expressions with the lowest common denominator.

Solution

a.

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.	
Factor each denominator completely, lining up common factors. Bring down the columns.	$x^2-2x-3=(x+1)(x-3)$ $x^2+4x+3=(x+1)(x+3)$ $\text{LCD}\quad =(x+1)(x-3)(x+3)$
Write the LCD as the product of the factors.	The LCD is $(x+1)(x-3)(x+3)$.

b.

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Multiply each denominator by the 'missing' LCD factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

Try It 15.3.11

a. Find the LCD for the expressions $\frac{2}{x^2-x-12}$, $\frac{1}{x^2-16}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x-4)(x+3)(x+4)$

b. $\frac{2x+8}{(x-4)(x+3)(x+4)}$, $\frac{x+3}{(x-4)(x+3)(x+4)}$

Try It 15.3.12

a. Find the LCD for the expressions $\frac{3x}{x^2-3x+10}$, $\frac{5}{x^2+3x+2}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x+2)(x-5)(x+1)$

b. $\frac{3x^2+3x}{(x+2)(x-5)(x+1)}$, $\frac{5x-25}{(x+2)(x-5)(x+1)}$

Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

Example 15.3.13: How to Add Rational Expressions with Unlike Denominators

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Solution

Step 1. Determine if the expressions have a common denominator. • Yes–go to step 2. • No–Rewrite each rational expression with the LCD. ◦ Find the LCD. ◦ Rewrite each rational expression as an equivalent rational expression with the LCD.	No. Find the LCD of $(x-3)$ and $(x-2)$. Change into equivalent rational expressions with the LCD, $(x-3)$ and $(x-2)$. Keep the denominators factored!	$x-3:(x-3)$ $x-2$: $(x-2)$ LCD : $(x-3)(x-2)$ $\frac{3}{x-3}+\frac{2}{x-2}$ $\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$ $\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$
Step 2. Add or subtract the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	$\frac{5x-12}{(x-3)(x-2)}$

Try It 15.3.14

Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

Answer

$$\frac{7x-4}{(x-2)(x+3)}$$

Try It 15.3.15

Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

Answer

$$\frac{7m+25}{(m+3)(m+4)}$$

The steps used to add rational expressions are summarized here.

ADD OR SUBTRACT RATIONAL EXPRESSIONS.

1. Determine if the expressions have a common denominator.
   - **Yes** – go to step 2.
   - **No** – Rewrite each rational expression with the LCD.
     - Find the LCD.
     - Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add or subtract the rational expressions.
3. Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational expression as $\frac{3x-6}{(x-3)(x-2)}$ to be able to add it to $\frac{2x-6}{(x-2)(x-3)}$. Simplify *only* after you have combined the numerators.

Example 15.3.16

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Solution

	$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $x^2-2x-3=(x+1)(x-3)$, $x^2+4x+3=(x+1)(x+3)$, $LCD=(x+1)(x-3)(x+3)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x+24}{(x+1)(x-3)(x+3)}$
	The numerator is prime, so there are no common factors.

Try It 15.3.17

Add: $\frac{1}{m^2-m-2}+\frac{5m}{m^2+3m+2}$.

Answer

$$\frac{5m^2-9m+2}{(m+1)(m-2)(m+2)}$$

Try It 15.3.18

Add: $\frac{2n}{n^2-3n-10}+\frac{6}{n^2+5n+6}$.

Answer

$$\frac{2n^2+12n-30}{(n+2)(n-5)(n+3)}$$

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

Example 15.3.19

Subtract: $\frac{8y}{y^2-16}-\frac{4}{y-4}$.

Solution

	$\frac{8y}{y^2-16}-\frac{4}{y-4}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $y^2-16=(y-4)(y+4)$, $y-4=y-4$, $LCD=(y-4)(y+4)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)}-\frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)}-\frac{4y+16}{(y-4)(y+4)}$

Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerator.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors	$\frac{4(\cancel{y-4})}{(\cancel{y-4})(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

Try It 15.3.20

Subtract: $\frac{2x}{x^2-4}-\frac{1}{x+2}$.

Answer

$\frac{1}{x-2}$

Try It 15.3.21

Subtract: $\frac{3}{z+3}-\frac{6z}{z^2-9}$.

Answer

$\frac{-3}{z-3}$

There are lots of negative signs in the next example. Be extra careful.

Example 15.3.22

Subtract: $\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$.

Solution

	$\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$
Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will multiply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(-1)(2-n)}$
Write $(-1)(2-n)$ as $n-2$.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(n-2)}$
Simplify. Remember, $a-(-b)=a+b$.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{(n+3)}{(n-2)}$
Do the rational expressions have a common denominator? No.	

Find the LCD. $n^2+n-6=(n-2)(n+3)$ $n-2=(n-2)$ $LCD=\quad(n-2)(n+3)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{(n+3)(n+3)}{(n-2)(n+3)}$
Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{n^2+6n+9}{(n-2)(n+3)}$
Add the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Simplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n(n+3)}{(n-2)(n+3)}$
Simplify.	$\frac{n}{(n-2)}$

Try It 15.3.23

Subtract: $\frac{3x-1}{x^2-5x-6}-\frac{2}{6-x}$.

Answer

$\frac{5x+1}{(x-6)(x+1)}$

Try It 15.3.24

Subtract: $\frac{-2y-2}{y^2+2y-8}-\frac{y-1}{2-y}$.

Answer

$\frac{y+3}{y+4}$

Things can get very messy when both fractions must be multiplied by a binomial to get the common denominator.

Example 15.3.25

Subtract: $\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$.

Solution

	$\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$
Factor the denominators.	$\frac{4}{(a+1)(a+5)}-\frac{3}{(a+2)(a+5)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $a^2+6a+5=(a+1)(a+5)$ $a^2+7a+10=(a+5)(a+2)$ $LCD=(a+1)(a+5)(a+2)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{4(a+2)}{(a+1)(a+5)(a+2)} - \frac{3(a+1)}{(a+2)(a+5)(a+1)}$
Simplify the numerators.	$\frac{4a+8}{(a+1)(a+5)(a+2)} - \frac{3a+3}{(a+2)(a+5)(a+1)}$
Subtract the rational expressions.	$\frac{4a+8-(3a+3)}{(a+1)(a+5)(a+2)}$
Simplify the numerator.	$\frac{4a+8-3a+3}{(a+1)(a+5)(a+2)}$
	$\frac{a+5}{(a+1)(a+5)(a+2)}$
Look for common factors.	$\frac{\cancel{(a+5)}}{(a+1)\cancel{(a+5)}(a+2)}$
Simplify.	$\frac{1}{(a+1)(a+2)}$

Try It 15.3.26

Subtract: $\frac{3}{b^2-4b-5} - \frac{2}{b^2-6b+5}$.

Answer

$$\frac{1}{(b+1)(b-1)}$$

Try It 15.3.27

Subtract: $\frac{4}{x^2-4} - \frac{3}{x^2-x-2}$.

Answer

$$\frac{1}{(x+2)(x+1)}$$

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example, we will start by factoring all three denominators to find their LCD.

Example 15.3.28

Simplify: $\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$.

Solution

	$\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $u-1=(u-1)$ $u=u$ $u^2-u=u(u-1)$ $LCD=u(u-1)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2u \cdot u}{(u-1)u} + \frac{1 \cdot (u-1)}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$
	$\frac{2u^2}{(u-1)u} + \frac{u-1}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$
Write as one rational expression.	$\frac{2u^2+u-1-2u+1}{u(u-1)}$
Simplify.	$\frac{2u^2-u}{u(u-1)}$
Factor the numerator, and remove common factors.	$\frac{\cancel{u}(2u-1)}{\cancel{u}(u-1)}$
Simplify.	$\frac{2u-1}{u-1}$

? Try It 15.3.29

Simplify: $\frac{v}{v+1} + \frac{3}{v-1} - \frac{6}{v^2-1}$.

Answer

$\frac{v+3}{v+1}$

? Try It 15.3.30

Simplify: $\frac{3w}{w+2} + \frac{2}{w+7} - \frac{17w+4}{w^2+9w+14}$.

Answer

$\frac{3w}{w+7}$

Add and subtract rational functions

To add or subtract rational functions, we use the same techniques we used to add or subtract polynomial functions.

? Example 15.3.31

Find $R(x) = f(x) - g(x)$ where $f(x) = \frac{x+5}{x-2}$ and $g(x) = \frac{5x+18}{x^2-4}$.

xolution

	$R(x) = f(x) - g(x)$
Substitute in the functions $f(x)$, $g(x)$.	$R(x) = \frac{x+5}{x-2} - \frac{5x+18}{x^2-4}$
Factor the denominators.	$R(x) = \frac{x+5}{x-2} - \frac{5x+18}{(x-2)(x+2)}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $x-2 = (x-2)$, $x^2-4 = (x-2)(x+2)$, $LCD = (x-2)(x+2)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$R(x)=\frac{(x+5)(x+2)}{(x-2)(x+2)}-\frac{5x+18}{(x-2)(x+2)}$
Write as one rational expression.	$R(x)=\frac{(x+5)(x+2)-(5x+18)}{(x-2)(x+2)}$
Simplify.	$R(x)=\frac{x^2+7x+10-5x-18}{(x-2)(x+2)}$
	$R(x)=\frac{x^2+2x-8}{(x-2)(x+2)}$
Factor the numerator, and remove common factors.	$R(x)=\frac{(x+4)(x-2)}{(x-2)(x+2)}$
Simplify.	$R(x)=\frac{(x+4)}{(x+2)}$

Try It 15.3.32

Find $R(x)=f(x)-g(x)$ where $f(x)=\frac{x+1}{x+3}$ and $g(x)=\frac{x+17}{x^2-x-12}$.

Answer

$\frac{x-7}{x-4}$

Try It 15.3.33

Find $R(x)=f(x)+g(x)$ where $f(x)=\frac{x-4}{x+3}$ and $g(x)=\frac{4x+6}{x^2-9}$.

Answer

$\frac{x^2-3x+18}{(x+3)(x-3)}$

Access this online resource for additional instruction and practice with adding and subtracting rational expressions.

- Add and Subtract Rational Expressions- Unlike Denominators

Key Concepts

- **Rational Expression Addition and Subtraction**
  If p, q, and r are polynomials where $r\neq 0$, then
  $$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$
- **How to find the least common denominator of rational expressions.**
  1. Factor each expression completely.
  2. List the factors of each expression. Match factors vertically when possible.
  3. Bring down the columns.
  4. Write the LCD as the product of the factors.
- **How to add or subtract rational expressions.**
  1. Determine if the expressions have a common denominator.
     - Yes – go to step 2.
     - No – Rewrite each rational expression with the LCD.
       - Find the LCD.

- Rewrite each rational expression as an equivalent rational expression with the LCD.

2. Add or subtract the rational expressions.
3. Simplify, if possible.

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15.3E: Exercises

Practice Makes Perfect

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, add.

1. $\frac{2}{15}+\frac{7}{15}$

Answer

$\frac{3}{5}$

2. $\frac{7}{24}+\frac{11}{24}$

3. $\frac{3c}{4c-5}+\frac{5}{4c-5}$

Answer

$\frac{3c+5}{4c-5}$

4. $\frac{7m}{2m+n}+\frac{4}{2m+n}$

5. $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

Answer

$r+8$

6. $\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

7. $\frac{2w^2}{w^2-16}+\frac{8w}{w^2-16}$

Answer

$\frac{2w}{w-4}$

8. $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

In the following exercises, subtract.

9. $\frac{9a^2}{3a-7}-\frac{49}{3a-7}$

Answer

$3a+7$

10. $\frac{25b^2}{5b-6} - \frac{36}{5b-6}$

11. $\frac{3m^2}{6m-30} - \frac{21m-30}{6m-30}$

Answer

$\frac{m-2}{2}$

12. $\frac{2n^2}{4n-32} - \frac{18n-16}{4n-32}$

13. $\frac{6p^2+3p+4}{p^2+4p-5} - \frac{5p^2+p+7}{p^2+4p-5}$

Answer

$\frac{p+3}{p+5}$

14. $\frac{5q^2+3q-9}{q^2+6q+8} - \frac{4q^2+9q+7}{q^2+6q+8}$

15. $\frac{5r^2+7r-33}{r^2-49} - \frac{4r^2+5r+30}{r^2-49}$

Answer

$\frac{r+9}{r+7}$

16. $\frac{7t^2-t-4}{t^2-25} - \frac{6t^2+12t-44}{t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add or subtract.

17. $\frac{10v}{2v-1} + \frac{2v+4}{1-2v}$

Answer

4

18. $\frac{20w}{5w-2} + \frac{5w+6}{2-5w}$

19. $\frac{10x^2+16x-7}{8x-3} + \frac{2x^2+3x-1}{3-8x}$

Answer

$x+2$

20. $\frac{6y^2+2y-11}{3y-7}+\frac{3y^2-3y+17}{7-3y}$

21. $\frac{z^2+6z}{z^2-25}-\frac{3z+20}{25-z^2}$

Answer

$\frac{z+4}{z-5}$

22. $\frac{a^2+3a}{a^2-9}-\frac{3a-27}{9-a^2}$

23. $\frac{2b^2+30b-13}{b^2-49}-\frac{2b^2-5b-8}{49-b^2}$

Answer

$\frac{4b-3}{b-7}$

24. $\frac{c^2+5c-10}{c^2-16}-\frac{c^2-8c-10}{16-c^2}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, a. find the LCD for the given rational expressions b. rewrite them as equivalent rational expressions with the lowest common denominator.

25. $\frac{5}{x^2-2x-8}, \frac{2x}{x^2-x-12}$

Answer

a. $(x+2)(x-4)(x+3)$

b. $\frac{5x+15}{(x+2)(x-4)(x+3)}, \frac{2x^2+4x}{(x+2)(x-4)(x+3)}$

26. $\frac{8}{y^2+12y+35}, \frac{3y}{y^2+y-42}$

27. $\frac{9}{z^2+2z-8}, \frac{4z}{z^2-4}$

Answer

a. $(z-2)(z+4)(z-4)$

b. $\frac{9z-36}{(z-2)(z+4)(z-4)}, \frac{4z^2-8z}{(z-2)(z+4)(z-4)}$

28. $\frac{6}{a^2+14a+45}, \frac{5a}{a^2-81}$

29. $\frac{4}{b^2+6b+9}, \frac{2b}{b^2-2b-15}$

Answer

a. $(b+3)(b+3)(b-5)$

b. $\frac{4b-20}{(b+3)(b+3)(b-5)}, \frac{2b^2+6b}{(b+3)(b+3)(b-5)}$

30. $\frac{5}{c^2-4c+4}, \frac{3c}{c^2-7c+10}$

31. $\frac{2}{3d^2+14d-5}, \frac{5d}{3d^2-19d+6}$

Answer

a. $(d+5)(3d-1)(d-6)$

b. $\frac{2d-12}{(d+5)(3d-1)(d-6)}, \frac{5d^2+25d}{(d+5)(3d-1)(d-6)}$

32. $\frac{3}{5m^2-3m-2}, \frac{6m}{5m^2+17m+6}$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

33. $\frac{7}{10x^2y}+\frac{4}{15xy^2}$

Answer

$\frac{21y+8x}{30x^2y^2}$

34. $\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3}$

35. $\frac{3}{r+4}+\frac{2}{r-5}$

Answer

$\frac{5r-7}{(r+4)(r-5)}$

36. $\frac{4}{s-7}+\frac{5}{s+3}$

37. $\frac{5}{3w-2}+\frac{2}{w+1}$

Answer

$$\frac{11w+1}{(3w-2)(w+1)}$$

38. $\frac{4}{2x+5}+\frac{2}{x-1}$

39. $\frac{2y}{y+3}+\frac{3}{y-1}$

Answer

$$\frac{2y^2+y+9}{(y+3)(y-1)}$$

40. $\frac{3z}{z-2}+\frac{1}{z+5}$

41. $\frac{5b}{a^2b-2a^2}+\frac{2b}{b^2-4}$

Answer

$$\frac{b(5b+10+2a^2)}{a^2(b-2)(b+2)}$$

42. $\frac{4}{cd+3c}+\frac{1}{d^2-9}$

43. $\frac{-3m}{3m-3}+\frac{5m}{m^2+3m-4}$

Answer

$$-\frac{m}{m+4}$$

44. $\frac{8}{4n+4}+\frac{6}{n^2-n-2}$

45. $\frac{3r}{r^2+7r+6}+\frac{9}{r^2+4r+3}$

Answer

$$\frac{3(r^2+6r+18)}{(r+1)(r+6)(r+3)}$$

46. $\frac{2s}{s^2+2s-8}+\frac{4}{s^2+3s-10}$

47. $\frac{t}{t-6}-\frac{t-2}{t+6}$

Answer

$$\frac{2(7t-6)}{(t-6)(t+6)}$$

48. $\frac{x-3}{x+6}-\frac{x}{x+3}$

49. $\frac{5a}{a+3}-\frac{a+2}{a+6}$

Answer

$\frac{4a^2+25a-6}{(a+3)(a+6)}$

50. $\frac{3b}{b-2}-\frac{b-6}{b-8}$

51. $\frac{6}{m+6}-\frac{12m}{m^2-36}$

Answer

$\frac{-6}{m-6}$

52. $\frac{4}{n+4}-\frac{8n}{n^2-16}$

53. $\frac{-9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$

Answer

$\frac{p+2}{p+3}$

54. $\frac{-13q-8}{q^2+2q-24}-\frac{q+2}{4-q}$

55. $\frac{-2r-16}{r^2+6r-16}-\frac{5}{2-r}$

Answer

$\frac{3}{r-2}$

56. $\frac{2t-30}{t^2+6t-27}-\frac{2}{3-t}$

57. $\frac{2x+7}{10x-1}+3$

Answer

$\frac{4(8x+1)}{10x-1}$

58. $\frac{8y-4}{5y+2}-6$

59. $\dfrac{3}{x^2-3x-4}-\dfrac{2}{x^2-5x+4}$

Answer

$$\frac{x-5}{(x-4)(x+1)(x-1)}$$

60. $\dfrac{4}{x^2-6x+5}-\dfrac{3}{x^2-7x+10}$

61. $\dfrac{5}{x^2+8x-9}-\dfrac{4}{x^2+10x+9}$

Answer

$$\frac{1}{(x-1)(x+1)}$$

62. $\dfrac{3}{2x^2+5x+2}-\dfrac{1}{2x^2+3x+1}$

63. $\dfrac{5a}{a-2}+\dfrac{9}{a}-\dfrac{2a+18}{a^2-2a}$

Answer

$$\frac{5a^2+7a-36}{a(a-2)}$$

64. $\dfrac{2b}{b-5}+\dfrac{3}{2b}-\dfrac{2b-15}{2b^2-10b}$

65. $\dfrac{c}{c+2}+\dfrac{5}{c-2}-\dfrac{10c}{c^2-4}$

Answer

$$\frac{c-5}{c+2}$$

66. $\dfrac{6d}{d-5}+\dfrac{1}{d+4}+\dfrac{7d-5}{d^2-d-20}$

67. $\dfrac{3d}{d+2}+\dfrac{4}{d}-\dfrac{d+8}{d^2+2d}$

Answer

$$\frac{3(d+1)}{d+2}$$

68. $\dfrac{2q}{q+5}+\dfrac{3}{q-3}-\dfrac{13q+15}{q^2+2q-15}$

Add and Subtract Rational Functions

In the following exercises, find a. $R(x)=f(x)+g(x)$ b. $R(x)=f(x)-g(x)$.

69. $f(x)=\dfrac{-5x-5}{x^2+x-6}$ and $g(x)=\dfrac{x+1}{2-x}$

Answer

a. $R(x)=-\dfrac{(x+8)(x+1)}{(x-2)(x+3)}$

b. $R(x)=\dfrac{x+1}{x+3}$

70. $f(x)=\dfrac{-4x-24}{x^2+x-30}$ and $g(x)=\dfrac{x+7}{5-x}$

71. $f(x)=\dfrac{6x}{x^2-64}$ and $g(x)=\dfrac{3}{x-8}$

Answer

a. $R(x)=\dfrac{3(3x+8)}{(x-8)(x+8)}$

b. $R(x)=\dfrac{3}{x+8}$

72. $f(x)=\dfrac{5}{x+7}$ and $g(x)=\dfrac{10x}{x^2-49}$

Writing Exercises

73. Donald thinks that $\dfrac{3}{x}+\dfrac{4}{x}$ is $\dfrac{7}{2x}$. Is Donald correct? Explain.

Answer

Answers will vary.

74. Explain how you find the Least Common Denominator of x^2+5x+4 and x^2-16.

75. Felipe thinks $\dfrac{1}{x}+\dfrac{1}{y}$ is $\dfrac{2}{x+y}$.

a. Choose numerical values for *x* and *y* and evaluate $\dfrac{1}{x}+\dfrac{1}{y}$.

b. Evaluate $\dfrac{2}{x+y}$ for the same values of *x* and *y* you used in part a..

c. Explain why Felipe is wrong.

d. Find the correct expression for $1x+1y$.

Answer

a. Answers will vary.

b. Answers will vary.

c. Answers will vary.

d. $\dfrac{x+y}{x}$

76. Simplify the expression $\dfrac{4}{n^2+6n+9}-\dfrac{1}{n^2-9}$ and explain all your steps.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites.			
find the least common denominator of rational expressions.			
add and subtract rational expressions with unlike denominators.			
add or subtract rational functions.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

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15.4: Simplify Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Before you get started, take this readiness quiz.

1. Simplify: $\dfrac{\frac{3}{5}}{\frac{9}{10}}$.
   If you missed this problem, review [link].
2. Simplify: $\dfrac{1-\frac{1}{3}}{4^2+4\cdot 5}$.
   If you missed this problem, review [link].
3. Solve: $\dfrac{1}{2x}+\dfrac{1}{4}=\dfrac{1}{8}$.
   If you missed this problem, review [link].

Complex Fraction

A **complex fraction** is a fraction in which the numerator and/or the denominator contains a fraction.

We previously simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

Complex Rational Expression

A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

We have already seen this complex rational expression earlier in this chapter.

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem:

$$\left(\frac{6x^2-7x+2}{4x-8}\right)\div\left(\frac{2x^2-8x+3}{x^2-5x+6}\right)$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify complex rational expressions. We make sure the complex rational expression is of the form where one fraction is over one fraction. We then write it as if we were dividing two fractions.

Example 15.4.1

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$$

Solution

Rewrite the complex fraction as division.

$$\frac{6}{x-4}\div\frac{3}{x^2-16}$$

Rewrite as the product of first times the reciprocal of the second.Rewrite as the product of first times the reciprocal of the second.Rewrite as the product of first times the reciprocal of the second.

$$\frac{6}{x-4}\cdot\frac{x^2-16}{3}$$

Factor.

$$\frac{3\cdot 2}{x-4}\cdot\frac{(x-4)(x+4)}{3}$$

Multiply.

$$\frac{3\cdot 2(x-4)(x+4)}{3(x-4)}$$

Remove common factors.

$$\frac{\cancel{3}\cdot 2\ \cancel{(x-4)}(x+4)}{\cancel{3}\ \cancel{(x-4)}}$$

Simplify.

$$2(x+4)$$

Are there any value(s) of x that should not be allowed? The original complex rational expression had denominators of $x-4$ and x^2-16. This expression would be undefined if $x=4$ or $x=-4$.

Try It 15.4.1

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$$

Answer

$$\frac{2}{3(x-1)}$$

Try It 15.4.2

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$$

Answer

$$\frac{1}{2(x-3)}$$

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

Example 15.4.2

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$$

Solution

Simplify the numerator and denominator. Find the LCD and add the fractions in the numerator. Find the LCD and subtract the fractions in the denominator.

$$\frac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$$

Simplify the numerator and denominator.

$$\frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$$

Rewrite the complex rational expression as a division problem.

$$\frac{3}{6}\div\frac{1}{6}$$

Multiply the first by the reciprocal of the second.

$$\frac{3}{6} \cdot \frac{6}{1}$$

Simplify.

$$3$$

? Try It 15.4.3

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$$

Answer

$$\frac{14}{11}$$

? Try It 15.4.4

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$$

Answer

$$\frac{10}{23}$$

We follow the same procedure when the complex rational expression contains variables.

✓ Example 15.4.3: How to Simplify a Complex Rational Expression using Division

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Solution

Step 1. Simplify the numerator.

We will simplify the sum in and denominator. the numerator and difference in the denominator.

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Find a common denominator and add the fractions in the numerator.

$$\frac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$$

$$\frac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$$

Find a common denominator and subtract the fractions in the denominator.

$$\frac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$$

We now have just one rational expression in the numerator and one in the denominator.

Step 2. Rewrite the complex rational expression as a division problem.

We write the numerator divided by the denominator.

$$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$$

Step 3. Divide the expressions.

Multiply the first by the reciprocal of the second.

$$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$$

Factor any expressions if possible.

$$\frac{xy(y+x)}{xy(x-y)(x+y)}$$

Remove common factors.

$$\frac{\cancel{xy}\ \cancel{(y+x)}}{\cancel{xy}(x-y)\ \cancel{(x+y)}}$$

Simplify.

$$\frac{1}{x-y}$$

? Try It 15.4.5

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$$

Answer

$$\frac{y+x}{y-x}$$

Try It 15.4.6

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$$

Answer

$$\frac{ab}{b-a}$$

We summarize the steps here.

How to simplify a complex rational expression by writing it as division.

1. Rewrite the complex rational expression as a division problem.
2. Divide the expressions.

Example 15.4.4

Simplify the complex rational expression by writing it as division:

$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$$

Solution

Simplify the numerator and denominator. Find common denominators for the numerator and denominator.

$$\frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$$

Simplify the numerators.

$$\frac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$$

Subtract the rational expressions in the numerator and add in the denominator.

$$\frac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$$

Simplify. (We now have one rational expression over one rational expression.)

$$\frac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$$

Rewrite as fraction division.

$$\frac{n^2+n}{n+5} \div \frac{2n}{(n+5)(n-5)}$$

Multiply the first times the reciprocal of the second.

$$\frac{n^2+n}{n+5} \cdot \frac{(n+5)(n-5)}{2n}$$

Factor any expressions if possible.

$$\frac{n(n+1)(n+5)(n-5)}{(n+5)2n}$$

Remove common factors.

$$\frac{\cancel{n}(n+1)\ \cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\ \cancel{n}}$$

Simplify.

$$\frac{(n+1)(n-5)}{2}$$

? Try It 15.4.7

Simplify the complex rational expression by writing it as division:

$$\frac{b-\frac{3b}{b+5}}{\frac{2}{b+5}+\frac{1}{b-5}}$$

Answer

$$\frac{b(b+2)(b-5)}{3b-5}$$

? Try It 15.4.8

Simplify the complex rational expression by writing it as division:

$$\frac{1-\frac{3}{c+4}}{\frac{1}{c+4}+\frac{c}{3}}$$

Answer

$$\frac{3}{c+3}$$

Simplify a Complex Rational Expression by Using the LCD

We "cleared" the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Let's look at the complex rational expression we simplified one way in Example 7.4.2. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{LCD}{LCD}$ we are multiplying by 1, so the value stays the same.

Example 15.4.5

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$$

Solution

The LCD of all the fractions in the whole expression is 6.

Clear the fractions by multiplying the numerator and denominator by that LCD.

$$\frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$$

Distribute.

$$\frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$$

Simplify.

$$\frac{2+1}{3-2}$$

$$\frac{3}{1}$$

$$3$$

Try It 15.4.9

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}$$

Answer

$\frac{7}{3}$

Try It 15.4.10

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}$$

Answer

$$\frac{10}{3}$$

We will use the same example as in Example 7.4.3. Decide which method works better for you.

Example 15.4.6: How to Simplify a Complex Rational Expressing using the LCD

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Solution

Step 1. Find the LCD of all fractions in the is complex rational expression.

The LCD of all the fractions xy.

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Step 2. Multiply the numerator and denominator by the LCD.

Multiply both the numerator and denominator by xy.

$$\frac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$$

Step 3. Simplify the expression.

Distribute.

$$\frac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$$

$$\frac{y+x}{x^2-y^2}$$

Simplify.

$$\frac{(y+x)}{(x-y)\ (x+y)}$$

Remove common factors.

$$\frac{1}{x-y}$$

Try It 15.4.11

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}$$

Answer

$$\frac{b+a}{a^2+b^2}$$

Try It 15.4.12

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$$

Answer

$$\frac{y-x}{xy}$$

How to simplify a complex rational expression by using the LCD.

1. Multiply the numerator and denominator by the LCD.
2. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

Example 15.4.7

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Solution

Find the LCD of all fractions in the complex rational expression. The LCD is:

$$x^2-36=(x+6)(x-6)$$

Multiply the numerator and denominator by the LCD.

$$\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$$

Simplify the expression.

Distribute in the denominator.

$$\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right)-(x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$$

Simplify.

$$\frac{\cancel{(x+6)}(x-6)\frac{2}{\cancel{x+6}}}{\cancel{(x+6)}\ \cancel{(x-6)}\left(\frac{4}{x-6}\right)-\ \cancel{(x+6)(x-6)}\left(\frac{4}{\cancel{(x+6)(x-6)}}\right)}$$

Simplify.

$$\frac{2(x-6)}{4(x+6)-4}$$

To simplify the denominator, distribute and combine like terms.

$$\frac{2(x-6)}{4x+20}$$

Factor the denominator.

$$\frac{2(x-6)}{4(x+5)}$$

Remove common factors.

$$\frac{\cancel{2}(x-6)}{\cancel{2}\cdot 2(x+5)}$$

Simplify.

$$\frac{x-6}{2(x+5)}$$

Notice that there are no more factors common to the numerator and denominator.

Try It 15.4.13

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}$$

Answer

$$\frac{3(x-2)}{5x+7}$$

Try It 15.4.14

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}$$

Answer

$$\frac{x+21}{6x-43}$$

Be sure to factor the denominators first. Proceed carefully as the math can get messy!

Example 15.4.8

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$$

Solution

Find the LCD of all fractions in the complex rational expression.

The LCD is $(m-3)(m-4)$.

Multiply the numerator and denominator by the LCD.

$$\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$$

Simplify.

$$\frac{\cancel{(m-3)(m-4)}\frac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\frac{3}{\cancel{m-3}}\right)-(m-3)\ \cancel{(m-4)}\left(\frac{2}{\cancel{m-4}}\right)}$$

Simplify.

$$\frac{4}{3(m-4)-2(m-3)}$$

Distribute.

$$\frac{4}{3m-12-2m+6}$$

Combine like terms.

$$\frac{4}{m-6}$$

Try It 15.4.15

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}$$

Answer

$$\frac{3}{5x+22}$$

Try It 15.4.16

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$$

Answer

$$\frac{2\left(2y^2+13y+5\right)}{3y}$$

Example 15.4.9

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$$

Solution

Find the LCD of all fractions in the complex rational expression.

The LCD is $(y+1)(y-1)$.

Multiply the numerator and denominator by the LCD.

$$\frac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$$

Distribute in the denominator and simplify.

$$\frac{\cancel{(y+1)}(y-1)\frac{y}{\cancel{y+1}}}{(y+1)(y-1)(1)+(y+1)\ \cancel{(y-1)}\left(\frac{1}{\cancel{(y-1)}}\right)}$$

Simplify.

$$\frac{(y-1)y}{(y+1)(y-1)+(y+1)}$$

Simplify the denominator and leave the numerator factored.

$$\frac{y(y-1)}{y^2-1+y+1}$$

$$\frac{y(y-1)}{y^2+y}$$

Factor the denominator and remove factors common with the numerator.

$$\frac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$$

Simplify.

$$\frac{y-1}{y+1}$$

? Try It 15.4.17

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{x}{x+3}}{1+\frac{1}{x+3}}$$

Answer

$$\frac{x}{x+4}$$

? Try It 15.4.18

Simplify the complex rational expression by using the LCD:

$$\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$$

Answer

$$\frac{x(x+1)}{3(x-1)}$$

Access this online resource for additional instruction and practice with complex fractions.

- Complex Fractions

Key Concepts

- **How to simplify a complex rational expression by writing it as division.**
  1. Simplify the numerator and denominator.
  2. Rewrite the complex rational expression as a division problem.
  3. Divide the expressions.
- **How to simplify a complex rational expression by using the LCD.**
  1. Find the LCD of all fractions in the complex rational expression.
  2. Multiply the numerator and denominator by the LCD.
  3. Simplify the expression.

Glossary

complex rational expression

A complex rational expression is a rational expression in which the numerator and/or denominator contains a rational expression.

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15.4E: Exercises

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify each complex rational expression by writing it as division.

1. $\dfrac{\dfrac{2a}{a+4}}{\dfrac{4a^2}{a^2-16}}$

Answer

$\dfrac{a-4}{2a}$

2. $\dfrac{\dfrac{3b}{b-5}}{\dfrac{b^2}{b^2-25}}$

3. $\dfrac{\dfrac{5}{c^2+5c-14}}{\dfrac{10}{c+7}}$

Answer

$\dfrac{1}{2(c-2)}$

4. $\dfrac{\dfrac{8}{d^2+9d+18}}{\dfrac{12}{d+6}}$

5. $\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

Answer

$\dfrac{12}{13}$

6. $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{3}{5}+\frac{7}{10}}$

7. $\dfrac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

Answer

$$\frac{20}{57}$$

8. $\dfrac{\frac{1}{2}-\frac{1}{6}}{\frac{2}{3}+\frac{3}{4}}$

9. $\dfrac{\frac{n}{m}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

Answer

$$\frac{n^2+m}{m-n^2}$$

10. $\dfrac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

11. $\dfrac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

Answer

$$\frac{rt}{t-r}$$

12. $\dfrac{\frac{2}{v}+\frac{2}{w}}{\frac{1}{v^2}-\frac{1}{w^2}}$

13. $\dfrac{x-\frac{2x}{x+3}}{\frac{1}{x+3}+\frac{1}{x-3}}$

Answer

$$\frac{(x+1)(x-3)}{2}$$

14. $\dfrac{y-\frac{2y}{y-4}}{\frac{2}{y-4}+\frac{2}{y+4}}$

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15. $\dfrac{2-\dfrac{2}{a+3}}{\dfrac{1}{a+3}+\dfrac{a}{2}}$

Answer

$\dfrac{4}{a+1}$

16. $\dfrac{4+\dfrac{4}{b-5}}{\dfrac{1}{b-5}+\dfrac{b}{4}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify each complex rational expression by using the LCD.

17. $\dfrac{\dfrac{1}{3}+\dfrac{1}{8}}{\dfrac{1}{4}+\dfrac{1}{12}}$

Answer

$\dfrac{11}{8}$

18. $\dfrac{\dfrac{1}{4}+\dfrac{1}{9}}{\dfrac{1}{6}+\dfrac{1}{12}}$

19. $\dfrac{\dfrac{5}{6}+\dfrac{2}{9}}{\dfrac{7}{18}-\dfrac{1}{3}}$

Answer

19

20. $\dfrac{\dfrac{1}{6}+\dfrac{4}{15}}{\dfrac{3}{5}-\dfrac{1}{2}}$

21. $\dfrac{\dfrac{c}{d}+\dfrac{1}{d}}{\dfrac{1}{d}-\dfrac{d}{c}}$

Answer

$\dfrac{c^2+c}{c-d^2}$

22. $\dfrac{\frac{1}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

23. $\dfrac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$

Answer

$\dfrac{pq}{q-p}$

24. $\dfrac{\frac{2}{r}+\frac{2}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

25. $\dfrac{\frac{2}{x+5}}{\frac{3}{x-5}+\frac{1}{x^2-25}}$

Answer

$\dfrac{2x-10}{3x+16}$

26. $\dfrac{\frac{5}{y-4}}{\frac{3}{y+4}+\frac{2}{y^2-16}}$

27. $\dfrac{\frac{5}{z^2-64}+\frac{3}{z+8}}{\frac{1}{z+8}+\frac{2}{z-8}}$

Answer

$\dfrac{3z-19}{3z+8}$

28. $\dfrac{\frac{3}{s+6}+\frac{5}{s-6}}{\frac{1}{s^2-36}+\frac{4}{s+6}}$

29. $\dfrac{\frac{4}{a^2-2a-15}}{\frac{1}{a-5}+\frac{2}{a+3}}$

Answer

$$\frac{4}{3a-7}$$

30. $$\frac{\frac{5}{b^2-6b-27}}{\frac{3}{b-9}+\frac{1}{b+3}}$$

31. $$\frac{\frac{5}{c+2}-\frac{3}{c+7}}{\frac{5c}{c^2+9c+14}}$$

Answer

$$\frac{2c+29}{5c}$$

32. $$\frac{\frac{6}{d-4}-\frac{2}{d+7}}{\frac{2d}{d^2+3d-28}}$$

33. $$\frac{2+\frac{1}{p-3}}{\frac{5}{p-3}}$$

Answer

$$\frac{2p-5}{5}$$

34. $$\frac{\frac{n}{n-2}}{3+\frac{5}{n-2}}$$

35. $$\frac{\frac{m}{m+5}}{4+\frac{1}{m-5}}$$

Answer

$$\frac{m(m-5)}{(4m-19)(m+5)}$$

36. $$\frac{7+\frac{2}{q-2}}{\frac{1}{q+2}}$$

In the following exercises, simplify each complex rational expression using either method.

37. $\dfrac{\frac{3}{4}-\frac{2}{7}}{\frac{1}{2}+\frac{5}{14}}$

Answer

$\frac{13}{24}$

38. $\dfrac{\frac{v}{w}+\frac{1}{v}}{\frac{1}{v}-\frac{v}{w}}$

39. $\dfrac{\frac{2}{a+4}}{\frac{1}{a^2-16}}$

Answer

$2(a-4)$

40. $\dfrac{\frac{3}{b^2-3b-40}}{\frac{5}{b+5}-\frac{2}{b-8}}$

41. $\dfrac{\frac{3}{m}+\frac{3}{n}}{\frac{1}{m^2}-\frac{1}{n^2}}$

Answer

$\frac{3mn}{n-m}$

42. $\dfrac{\frac{2}{r-9}}{\frac{1}{r+9}+\frac{3}{r^2-81}}$

43. $\dfrac{x-\frac{3x}{x+2}}{\frac{3}{x+2}+\frac{3}{x-2}}$

Answer

$\frac{(x-1)(x-2)}{6}$

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44. $\dfrac{\dfrac{y}{y+3}}{2+\dfrac{1}{y-3}}$

Writing Exercises

45. In this section, you learned to simplify the complex fraction $\dfrac{\dfrac{3}{x+2}}{\dfrac{x}{x^2-4}}$ two ways: rewriting it as a division problem or multiplying the numerator and denominator by the LCD. Which method do you prefer? Why?

Answer

Answers will vary.

44. Efraim wants to start simplifying the complex fraction $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$ by cancelling the variables from the numerator and denominator, $\dfrac{\frac{1}{\cancel{a}}+\frac{1}{\cancel{b}}}{\frac{1}{\cancel{a}}-\frac{1}{\cancel{b}}}$. Explain what is wrong with Efraim's plan.

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15.5: Solve Rational Equations

Learning Objectives

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Before you get started, take this readiness quiz.

1. Solve: $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$
   If you missed this problem, review Example 2.2.9.
2. Solve: $n^2-5n-36=0$
   If you missed this problem, review Example 6.6.4.
3. Solve the formula $5x+2y=10$ for y.
   If you missed this problem, review Example 2.4.10.

After defining the terms ‘expression’ and ‘equation’ earlier, we have used them throughout this book. We have simplified many kinds of expressions and solved many kinds of equations. We have simplified many rational expressions so far in this chapter. Now we will solve a **rational equation**.

Rational Equation

A **rational equation** is an equation that contains a rational expression.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don’t end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

Extraneous Solution to a Rational Equation

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.

Example 15.5.1: How to Solve a Rational Equation

Solve:

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

Solution

Step 1. Note any value of the variable that would make any denominator zero.

If $x=0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}, x \neq 0$$

Step 2. Find the least common denominator of all denominators in the equation.

Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$

The LCD is $6x$.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.

Multiply both sides of the equation by the LCD, $6x$.

$$6x \cdot \left(\frac{1}{x}+\frac{1}{3}\right)=6x \cdot \left(\frac{5}{6}\right)$$

Use the Distributive Property.

$$6x \cdot \frac{1}{x}+6x \cdot \frac{1}{3}=6x \cdot \left(\frac{5}{6}\right)$$

Simplify - and notice, no more fractions!

$$6+2x=5x$$

Step 4. Solve the resulting equation.

Simplify.

$$6=3x$$
$$2=x$$

Step 5. Check.

If any values found in Step 1 are algebraic solutions, discard them. Check any remaining solutions in the original equation.

We did not get 0 as an algebraic solution.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

We substitute $x=2$ into the original equation.

$$\frac{1}{2}+\frac{1}{3} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{3}{6}+\frac{2}{6} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{5}{6}=\frac{5}{6}\checkmark$$

The solution is $x=2$

Exercise 15.5.1

Solve:

$$\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$$

Answer

$$y=-\frac{15}{7}$$

Exercise 15.5.2

Solve:

$$\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$$

Answer

$$x=\frac{15}{3}$$

The steps of this method are shown.

how to Solve equations with rational expressions.

- Step 1. Note any value of the variable that would make any denominator zero.
- Step 2. Find the least common denominator of all denominators in the equation.
- Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
- Step 4. Solve the resulting equation.
- Step 5. Check:
  - If any values found in Step 1 are algebraic solutions, discard them.
  - Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

Example 15.5.2: How to Solve a Rational Equation using the Zero Product Property

Solve:

$$1-\frac{5}{y}=-\frac{6}{y^2}$$

Solution

Note any value of the variable that would make any denominator zero.

$$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$$

Find the least common denominator of all denominators in the equation. The LCD is y^2.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Distribute.

$$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Multiply.

$$y^2-5y=-6$$

Solve the resulting equation. First write the quadratic equation in standard form.

$$y^2-5y+6=0$$

Factor.

$$(y-2)(y-3)=0$$

Use the Zero Product Property.

$$y-2=0 \text{ or } y-3=0$$

Solve.

$$y=2 \text{ or } y=3$$

Check. We did not get 0 as an algebraic solution.

Check $y=2$ and $y=3$ in the original equation.

$$1-\frac{5}{y}=-\frac{6}{y^2} \qquad 1-\frac{5}{y}=-\frac{6}{y^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$\frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad \frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}\stackrel{?}{=}-\frac{6}{4} \qquad -\frac{2}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}=-\frac{3}{2}\surd \qquad -\frac{2}{3}=-\frac{2}{3}\surd$$

The solution is $y=2, y=3$

? Exercise 15.5.3

Solve:

$$1-\frac{2}{x}=\frac{15}{x^2}$$

Answer

$x=-3, x=5$

Exercise 15.5.4

Solve:

$$1-\frac{4}{y}=\frac{12}{y^2}$$

Answer

$y=-2, y=6$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

Example 15.5.3

Solve:

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$

Solution

Note any value of the variable that would make any denominator zero.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{(x+2)(x-2)}, x\neq -2, x\neq 2$$

Find the least common denominator of all denominators in the equation. The LCD is $(x+2)(x-2)$.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(x+2)(x-2)\left(\frac{2}{x+2}+\frac{4}{x-2}\right)=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Distribute.

$$(x+2)(x-2)\frac{2}{x+2}+(x+2)(x-2)\frac{4}{x-2}=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Remove common factors.

$$\cancel{(x+2)}(x-2)\frac{2}{\cancel{x+2}}+(x+2)\ \cancel{(x-2)}\frac{4}{\cancel{x-2}}=\ \cancel{(x+2)(x-2)}\left(\frac{x-1}{\cancel{x^2-4}}\right)$$

Simplify.

$$2(x-2)+4(x+2)=x-1$$

Distribute.

$$2x-4+4x+8=x-1$$

Solve.

$$\begin{aligned}6x+4&=x-1\\5x&=-5\\x&=-1\end{aligned}$$

Check: We did not get 2 or −2 as algebraic solutions.

Check $x=-1$ in the original equation.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$
$$\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\stackrel{?}{=}\frac{(-1)-1}{(-1)^2-4}$$
$$\frac{2}{1}+\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$
$$\frac{6}{3}-\frac{4}{3}\stackrel{?}{=}\frac{2}{3}$$
$$\frac{2}{3}=\frac{2}{3}\surd$$

The solution is $x=-1$.

? Exercise 15.5.5

Solve:

$$\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$$

Answer

$x=\frac{2}{3}$

? Exercise 15.5.6

Solve:

$$\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$$

Answer

$y=2$

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

✓ Example 15.5.4

Solve:

$$\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$$

Solution

Note any value of the variable that would make any denominator zero. Use the factored form of the quadratic denominator.

$$\frac{m+11}{(m-4)(m-1)}=\frac{5}{m-4}-\frac{3}{m-1}, m\neq 4, m\neq 1$$

Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\left(\frac{5}{m-4}-\frac{3}{m-1}\right)$$

Distribute.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-1}$$

Remove common factors.

$$\cancel{(m-4)(m-1)}\left(\frac{m+11}{\cancel{(m-4)(m-1)}}\right)=\cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}}-(m-4)\ \cancel{(m-1)}\frac{3}{\cancel{m-1}}$$

Simplify.

$$m+11=5(m-1)-3(m-4)$$

Solve the resulting equation.

$$\begin{aligned} m+11 &= 5m-5-3m+12 \\ 4 &= m \end{aligned}$$

Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

? Exercise 15.5.7

Solve:

$$\frac{x+13}{x^2-7x+10}=\frac{6}{x-5}-\frac{4}{x-2}$$

Answer

There is no solution.

? Exercise 15.5.8

Solve:

$$\frac{y-6}{y^2+3y-4}=\frac{2}{y+4}+\frac{7}{y-1}$$

Answer

There is no solution.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

✓ Example 15.5.5

Solve:

$$\frac{y}{y+6}=\frac{72}{y^2-36}+4$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$$

Find the least common denominator. The LCD is $(y-6)(y+6)$

Clear the fractions.

$$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$$

Simplify.

$$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$$

Simplify.

$$y(y-6)=72+4\left(y^2-36\right)$$

Solve the resulting equation.

$$\begin{aligned} y^2-6y &= 72+4y^2-144 \\ 0 &= 3y^2+6y-72 \\ 0 &= 3\left(y^2+2y-24\right) \\ 0 &= 3(y+6)(y-4) \\ y &= -6, y=4 \end{aligned}$$

Check.

$y=-6$ is an extraneous solution. Check $y=4$ in the original equation.

$$\begin{aligned} \frac{y}{y+6} &= \frac{72}{y^2-36}+4 \\ \frac{4}{4+6} &\stackrel{?}{=} \frac{72}{4^2-36}+4 \\ \frac{4}{10} &\stackrel{?}{=} \frac{72}{-20}+4 \\ \frac{4}{10} &\stackrel{?}{=} -\frac{36}{10}+\frac{40}{10} \\ \frac{4}{10} &= \frac{4}{10}\surd \end{aligned}$$

The solution is $y=4$.

Exercise 15.5.9

Solve:

$$\frac{x}{x+4}=\frac{32}{x^2-16}+5$$

Answer

$x=3$

Exercise 15.5.10

Solve:

$$\frac{y}{y+8}=\frac{128}{y^2-64}+9$$

Answer

$y=7$

In some cases, all the algebraic solutions are extraneous.

Example 15.5.6

Solve:

$$\frac{x}{2x-2}-\frac{2}{3x+3}=\frac{5x^2-2x+9}{12x^2-12}$$

Solution

We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}$$

Note any value of the variable that would make any denominator zero.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}, x\neq 1, x\neq -1$$

Find the least common denominator. The LCD is $12(x-1)(x+1)$.

Clear the fractions.

$$12(x-1)(x+1)\left(\frac{x}{2(x-1)}-\frac{2}{3(x+1)}\right)=12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$$

Simplify.

$$6(x+1)\cdot x-4(x-1)\cdot 2=5x^2-2x+9$$

Simplify.

$$6x(x+1)-4\cdot 2(x-1)=5x^2-2x+9$$

Solve the resulting equation.

$$\begin{aligned} 6x^2+6x-8x+8 &= 5x^2-2x+9 \\ x^2-1 &= 0 \\ (x-1)(x+1) &= 0 \\ x &= 1 \text{ or } x=-1 \end{aligned}$$

Check.

$x=1$ and $x=-1$ are extraneous solutions.

The equation has no solution.

Exercise 15.5.11

Solve:

$$\frac{y}{5y-10}-\frac{5}{3y+6}=\frac{2y^2-19y+54}{15y^2-60}$$

Answer

There is no solution.

Exercise 15.5.12

Solve:

$$\frac{z}{2z+8}-\frac{3}{4z-8}=\frac{3z^2-16z-16}{8z^2+2z-64}$$

Answer

There is no solution.

Example 15.5.7

Solve:

$$\frac{4}{3x^2-10x+3}+\frac{3}{3x^2+2x-1}=\frac{2}{x^2-2x-3}$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}=\frac{2}{(x-3)(x+1)}, x\neq -1, x\neq \frac{1}{3}, x\neq 3$$

Find the least common denominator. The LCD is $(3x-1)(x+1)(x-3)$.

Clear the fractions.

$$(3x-1)(x+1)(x-3)\left(\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}\right)=(3x-1)(x+1)(x-3)\left(\frac{2}{(x-3)(x+1)}\right)$$

Simplify.

$$4(x+1)+3(x-3)=2(3x-1)$$

Distribute.

$$4x+4+3x-9=6x-2$$

Simplify.

$$7x-5=6x-2$$

$$x=3$$

The only algebraic solution was $x=3$, but we said that $x=3$ would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

Exercise 15.5.13

Solve:

$$\frac{15}{x^2+x-6}-\frac{3}{x-2}=\frac{2}{x+3}$$

Answer

There is no solution.

Exercise 15.5.14

Solve:

$$\frac{5}{x^2+2x-3}-\frac{3}{x^2+x-2}=\frac{1}{x^2+5x+6}$$

Answer

There is no solution.

Use Rational Functions

Working with functions that are defined by rational expressions often lead to rational equations. Again, we use the same techniques to solve them.

Example 15.5.8

For rational function, $f(x)=\frac{2x-6}{x^2-8x+15}$:

a. Find the domain of the function
b. Solve $f(x)=1$
c. Find the points on the graph at this function value.

Solution

a. The domain of a rational function is all real numbers except those that make the rational expression undefined. So to find them, we will set the denominator equal to zero and solve.

$$\begin{aligned} x^2-8x+15 &= 0 \\ (x-3)(x-5) &= 0 \quad \text{Factor the trinomial.} \\ x-3 &= 0 \quad \text{Use the Zero Product Property.} \\ x-5 &= 0 \quad \text{Use the Zero Product Property.} \\ x=3 \quad x &= 5 \quad \text{Solve.} \end{aligned}$$

The domain is all real numbers except $x\neq 3, x\neq 5$

b. $$f(x)=1$$

Substitute in the rational expression.

$$\frac{2x-6}{x^2-8x+15}=1$$

Factor the denominator.

$$\frac{2x-6}{(x-3)(x-5)}=1$$

Multiply both sides by the LCD, $(x-3)(x-5)$

$$(x-3)(x-5)\left(\frac{2x-6}{(x-3)(x-5)}\right)=(x-3)(x-5)(1)$$

Simplify.

$$2x-6=x^2-8x+15$$

Solve.

$$0=x^2-10x+21$$

Factor.

$$0 = (x-7)(x-3)$$

Use the Zero Product Property.

$$x-7=0 \quad x-3=0$$

Solve.

$$x=7 \quad x=3$$

c. The value of the function is 1 when $x=7, x=3$. So the points on the graph of this function when $f(x)=1$, will be $(7,1),(3,1)$. However, now we need to check for extraneous solutions. Since $x=3$ is not in the domain of the original function, we must remove $(3,1)$, since it would be considered extraneous.

? Exercise 15.5.15

For rational function, $f(x)=\dfrac{8-x}{x^2-7x+12}$

a. Find the domain of the function.
b. Solve $f(x)=3$.
c. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x \neq 3$ and $x \neq 4$
b. $x=2, x=\dfrac{14}{3}$
c. $(2,3), \left(\dfrac{14}{3},3\right)$

? Exercise 15.5.16

For rational function, $f(x)=\dfrac{x-1}{x^2-6x+5}$

a. Solve $f(x)=4$.
b. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x \neq 1$ and $x \neq 5$
b. $x=\dfrac{21}{4}$
c. $\left(\dfrac{21}{4},4\right)$

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

$$
\begin{aligned}
m &= \frac{y-y_1}{x-x_1} \\
m\,(x-x_1) &= \left(\frac{y-y_1}{x-x_1}\right)(x-x_1) \quad \text{Multiply both sides of the equation by } x-x_1. \\
m\,(x-x_1) &= y-y_1 \quad \text{Simplify.} \\
y-y_1 &= m\,(x-x_1) \quad \text{Rewrite the equation with the y terms on the left.}
\end{aligned}
$$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point $(2,3)$. We will add one more step to solve for y.

Example 15.5.9

Solve: $m=\dfrac{y-2}{x-3}$ for y.

Solution

$$m=\frac{y-2}{x-3}$$

Note any value of the variable that would make any denominator zero.

$$m=\frac{y-2}{x-3}, x\neq 3$$

Clear the fractions by multiplying both sides of the equation by the LCD, $x-3$.

$$(x-3)m=(x-3)\left(\frac{y-2}{x-3}\right)$$

Simplify.

$$xm-3m=y-2$$

Isolate the term with y.

$$xm-3m+2=y$$

Exercise 15.5.17

Solve: $m=\dfrac{y-5}{x-4}$ for y.

Answer

$y=mx-4m+5$

Exercise 15.5.18

Solve: $m=\dfrac{y-1}{x+5}$ for y.

Answer

$y=mx+5m+1$

Remember to multiply both sides by the LCD in the next example.

Example 15.5.10

Solve: $\frac{1}{c}+\frac{1}{m}=1$ for c

Solution

$$\frac{1}{c}+\frac{1}{m}=1 \text{ for } c$$

Note any value of the variable that would make any denominator zero.

$$\frac{1}{c}+\frac{1}{m}=1, c\neq 0, m\neq 0$$

Clear the fractions by multiplying both sides of the equations by the LCD, cm.

$$cm\left(\frac{1}{c}+\frac{1}{m}\right)=cm(1)$$

Distribute.

$$cm\left(\frac{1}{c}\right)+cm\frac{1}{m}=cm(1)$$

Simplify.

$$m+c=cm$$

Collect the terms with c to the right.

$$m=cm-c$$

Factor the expression on the right.

$$m=c(m-1)$$

To isolate c, divide both sides by $m-1$.

$$\frac{m}{m-1}=\frac{c(m-1)}{m-1}$$

Simplify by removing common factors.

$$\frac{m}{m-1}=c$$

Notice that even though we excluded $c=0$ and $m=0$ from the original equation, we must also now state that $m\neq 1$.

Exercise 15.5.19

Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

Answer

$$a=\frac{b}{cb-1}$$

Exercise 15.5.20

Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y

Answer

$$y = \frac{3x}{x+6}$$

Media Access Additional Online Resources

Access this online resource for additional instruction and practice with equations with rational expressions.

- Equations with Rational Expressions

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15.5E: Exercises

Solve Rational Equations

In the following exercises, solve each rational equation.

1. $\frac{1}{a}+\frac{2}{5}=\frac{1}{2}$

Answer

$a=10$

2. $\frac{6}{3}-\frac{2}{d}=\frac{4}{9}$

3. $\frac{4}{5}+\frac{1}{4}=\frac{2}{v}$

Answer

$v=\frac{40}{21}$

4. $\frac{3}{8}+\frac{2}{y}=\frac{1}{4}$

5. $1-\frac{2}{m}=\frac{8}{m^2}$

Answer

$m=-2,\ m=4$

6. $1+\frac{4}{n}=\frac{21}{n^2}$

7. $1+\frac{9}{p}=\frac{-20}{p^2}$

Answer

$p=-5,\ p=-4$

8. $1-\frac{7}{q}=\frac{-6}{q^2}$

9. $\frac{5}{3v-2}=\frac{7}{4v}$

Answer

$v=14$

10. $\frac{8}{2w+1}=\frac{3}{w}$

11. $\frac{3}{x+4}+\frac{7}{x-4}=\frac{8}{x^2-16}$

Answer

$x=-\frac{4}{5}$

12. $\frac{5}{y-9}+\frac{1}{y+9}=\frac{18}{y^2-81}$

13. $\frac{8}{z-10}-\frac{7}{z+10}=\frac{5}{z^2-100}$

Answer

$z=-145$

14. $\frac{9}{a+11}-\frac{6}{a-11}=\frac{6}{a^2-121}$

15. $\frac{-10}{q-2}-\frac{7}{q+4}=1$

Answer

$q=-18,\ q=-1$

16. $\frac{2}{s+7}-\frac{3}{s-3}=1$

17. $\frac{v-10}{v^2-5v+4}=\frac{3}{v-1}-\frac{6}{v-4}$

Answer

no solution

18. $\frac{w+8}{w^2-11w+28}=\frac{5}{w-7}+\frac{2}{w-4}$

19. $\frac{x-10}{x^2+8x+12}=\frac{3}{x+2}+\frac{4}{x+6}$

Answer

no solution

20. $\frac{y-5}{y^2-4y-5}=\frac{1}{y+1}+\frac{1}{y-5}$

21. $\frac{b+3}{3b}+\frac{b}{24}=\frac{1}{b}$

Answer

$b=-8$

https://math.libretexts.org/@go/page/46391

22. $\frac{c+3}{12c}+\frac{c}{36}=\frac{1}{4c}$

23. $\frac{d}{d+3}=\frac{18}{d^2-9}+4$

Answer

$d=2$

24. $\frac{m}{m+5}=\frac{50}{m^2-25}+6$

25. $\frac{n}{n+2}-3=\frac{8}{n^2-4}$

Answer

$m=1$

26. $\frac{p}{p+7}-8=\frac{98}{p^2-49}$

27. $\frac{q}{3q-9}-\frac{3}{4q+12}=\frac{7q^2+6q+63}{24q^2-216}$

Answer

no solution

28. $\frac{r}{3r-15}-\frac{1}{4r+20}=\frac{3r^2+17r+40}{12r^2-300}$

29. $\frac{s}{2s+6}-\frac{2}{5s+5}=\frac{5s^2-3s-7}{10s^2+40s+30}$

Answer

$s=\frac{5}{4}$

30. $\frac{t}{6t-12}-\frac{5}{2t+10}=\frac{t^2-23t+70}{12t^2+36t-120}$

31. $\frac{2}{x^2+2x-8}-\frac{1}{x^2+9x+20}=\frac{4}{x^2+3x-10}$

Answer

$x=-\frac{4}{3}$

32. $\frac{5}{x^2+4x+3}+\frac{2}{x^2+x-6}=\frac{3}{x^2-x-2}$

33. $\frac{3}{x^2-5x-6}+\frac{3}{x^2-7x+6}=\frac{6}{x^2-1}$

Answer

no solution

34. $\dfrac{2}{x^2+2x-3}+\dfrac{3}{x^2+4x+3}=\dfrac{6}{x^2-1}$

Solve Rational Equations that Involve Functions

35. For rational function, $f(x)=\dfrac{x-2}{x^2+6x+8}$:

a. Find the domain of the function
b. Solve $f(x)=5$
c. Find the points on the graph at this function value

Answer

a. The domain is all real numbers except $x\neq -2$ and $x\neq -4$
b. $x=-3, x=-\dfrac{14}{5}$
c. $(-3,5), \left(-\dfrac{14}{5},5\right)$

36. For rational function, $f(x)=\dfrac{x+1}{x^2-2x-3}$:

a. Find the domain of the function
b. Solve $f(x)=1$
c. Find the points on the graph at this function value

37. For rational function, $f(x)=\dfrac{2-x}{x^2-7x+10}$:

a. Find the domain of the function
b. Solve $f(x)=2$
c. Find the points on the graph at this function value

Answer

a. The domain is all real numbers except $x\neq 2$ and $x\neq 5$
b. $x=\dfrac{9}{2}$
c. $\left(\dfrac{9}{2},2\right)$

38. For rational function, $f(x)=\dfrac{5-x}{x^2+5x+6}$:

a. Find the domain of the function
b. Solve $f(x)=3$
c. Find the points on the graph at this function value

Solve a Rational Equation for a Specific Variable

In the following exercises, solve:

39. $\dfrac{c}{r}=2\pi$ for r

Answer

$$r = \frac{C}{2\pi}$$

40. $\frac{I}{r} = P$ for r

41. $\frac{v+3}{w-1} = \frac{1}{2}$ for w

Answer

$$w = 2v+7$$

42. $\frac{x+5}{2-y} = \frac{4}{3}$ for y

43. $a = \frac{b+3}{c-2}$ for c

Answer

$$c = \frac{b+3+2a}{a}$$

44. $m = \frac{n}{2-n}$ for n

45. $\frac{1}{p} + \frac{2}{q} = 4$ for p

Answer

$$p = \frac{q}{4q-2}$$

46. $\frac{3}{s} + \frac{1}{t} = 2$ for s

47. $\frac{2}{v} + \frac{1}{5} = \frac{3}{w}$ for w

Answer

$$w = \frac{15v}{10+v}$$

48. $\frac{6}{x} + \frac{2}{3} = \frac{1}{y}$ for y

49. $\frac{m+3}{n-2} = \frac{4}{5}$ for n

Answer

$$n = \frac{5m+23}{4}$$

50. $r = \dfrac{s}{3-t}$ for t

51. $\dfrac{E}{c} = m^2$ for c

Answer

$$c = \frac{E}{m^2}$$

52. $\dfrac{R}{T} = W$ for T

53. $\dfrac{3}{x} - \dfrac{5}{y} = \dfrac{1}{4}$ for y

Answer

$$y = \frac{20x}{12-x}$$

54. $c = \dfrac{2}{a} + \dfrac{b}{5}$ for a

Writing Exercises

55. Your class mate is having trouble in this section. Write down the steps you would use to explain how to solve a rational equation.

Answer

Answers will vary.

56. Alek thinks the equation $\dfrac{y}{y+6} = \dfrac{72}{y^2-36} + 4$ has two solutions, $y = -6$ and $y = 4$. Explain why Alek is wrong.

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15.6: Solve Applications with Rational Equations

Learning Objectives

- Solve proportions
- Solve similar figure applications
- Solve uniform motion applications
- Solve work applications
- Solve direct variation problems
- Solve inverse variation problems

Note

Before you get started, take this readiness quiz.

1. Example 2.2.13. Example 2.5.13. Example 2.2.9.

Solve Proportions

When two rational expressions are equal, the equation relating them is called a **proportion**.

Proportion

A proportion is an equation of the form $\frac{a}{b}=\frac{c}{d}$, where $b\neq 0, d\neq 0$.

The proportion is read "a is to b as c is to d."

The equation $\frac{1}{2}=\frac{4}{8}$ is a proportion because the two fractions are equal. The proportion $\frac{1}{2}=\frac{4}{8}$ is read "1 is to 2 as 4 is to 8."

Since a proportion is an equation with rational expressions, we will solve proportions the same way we solved rational equations. We'll multiply both sides of the equation by the LCD to clear the fractions and then solve the resulting equation.

Example 15.6.1

Solve: $\frac{n}{n+14}=\frac{5}{7}$.

Solution

$$\frac{n}{n+14}=\frac{5}{7},\quad n\neq -14$$

Multiply both sides by LCD.

$$7(n+14)\left(\frac{n}{n+14}\right)=7(n+14)\left(\frac{5}{7}\right)$$

Remove common factors on each side.

$$7n=5(n+14)$$

Simplify.

$$7n=5n+70$$

Solve for n.

$$\begin{aligned}2n&=70\\ n&=35\end{aligned}$$

Check.

$$\frac{n}{n+14}=\frac{5}{7}$$

Substitute $n=35$

$$\frac{35}{35+14}\stackrel{?}{=}\frac{5}{7}$$

Simplify.

$$\frac{35}{49}\stackrel{?}{=}\frac{5}{7}$$

Show common factors.

$$\frac{5\cdot 7}{7\cdot 7}\stackrel{?}{=}\frac{5}{7}$$

Simplify.

$$\frac{5}{7}=\frac{5}{7}\ \surd$$

Exercise 15.6.1

Solve the proportion: $\frac{y}{y+55}=\frac{3}{8}$.

Answer

$y=33$

Exercise 15.6.2

Solve the proportion: $\frac{z}{z-84}=-\frac{1}{5}$.

Answer

$z=14$

Notice in the last example that when we cleared the fractions by multiplying by the LCD, the result is the same as if we had cross-multiplied.

$$\begin{array}{ll} \frac{n}{n+14}=\frac{5}{7} & \frac{n}{n+14}=\frac{5}{7} \\ 7(n+14)\left(\frac{n}{n+14}\right)=7(n+14)\left(\frac{5}{7}\right) & \frac{n}{n+14}=\frac{5}{7} \\ 7n=5(n+14) & 7n=5(n+14) \end{array}$$

For any proportion, $\frac{a}{b}=\frac{c}{d}$, we get the same result when we clear the fractions by multiplying by the LCD as when we cross-multiply.

$$\begin{array}{ll} \frac{a}{b}=\frac{c}{d} & \frac{a}{b}=\frac{c}{d} \\ bd\left(\frac{a}{b}=\frac{c}{d}\right)bd & \frac{a}{b}=\frac{c}{d} \\ ad=bc & ad=bc \end{array}$$

To solve applications with proportions, we will follow our usual strategy for solving applications But when we set up the proportion, we must make sure to have the units correct—the units in the numerators must match each other and the units in the denominators must also match each other.

When pediatricians prescribe acetaminophen to children, they prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how many milliliters of acetaminophen will her doctor prescribe?

Solution

Identify what we are asked to find, and choose a variable to represent it.

How many ml of acetaminophen will the doctor prescribe?

Let $a=ml$ of acetaminophen.

Write a sentence that gives the information to find it.

If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?

Translate into a proportion—be careful of the units.

Step 1. Write the inequality as one quotient on the left and zero on the right. Our inequality is in this form.

$$\frac{x-1}{x+3}\geq 0$$

Step 2. Determine the critical points-the points where the rational expression will be zero or undefined.

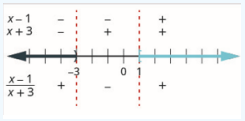
The rational expression will be zero when the numerator is zero. Since $x-1=0$ when $x=1$, then 1 is a critical point. The rational expression will be undefined when the denominator is zero. Since $x+3=0$ when $x=-3$, then -3 is a critical point.

Step 3. Use the critical points to divide the number line into intervals.

Step 4. Above the number line show the sign of each factor of the rational expression in each interval. Below the number line show the sign of the quotient.

Use values in each interval to determine the value of each factor in the interval. In the interval (-3,1), zero is a good value to test. For example, when $x=0$ then $x-1=-1$ and $x+3=3$ The factor $x-1$ is marked negative and $x+3$ marked positive. Since a negative divided by a positive is negative, the quotient is marked negative in that interval.

Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.



We want the quotient to be greater than or equal to zero, so the numbers in the intervals $(-\infty,-3)$ and $(1,\infty)$ are solutions. Since 3 must be excluded since it makes the rational expression 0, we cannot include it in the solution. We can include 1 in our solution.

$$(-\infty,-3)\cup[1,\infty)$$

Multiply both sides by the LCD, 400. Remove common factors on each side. Simplify, but don't multiply on the left. Notice what the next step will be.

$$16\cdot 5=5a$$

Solve for a.

$$\begin{array}{c} \frac{16\cdot 5}{5}=\frac{5a}{5} \\ 16=a \end{array}$$

Check. Is the answer reasonable? Write a complete sentence.

The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

Exercise 15.6.3

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

Answer

The pediatrician will prescribe 12 ml of acetaminophen to Emilia.

Exercise 15.6.4

For every 1 kilogram (kg) of a child's weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

Answer

The pediatrician will prescribe 180 mg of fever reducer to Isabella.

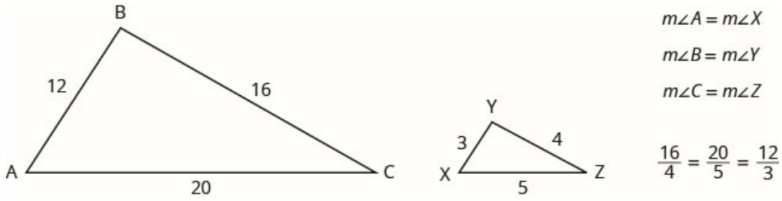
Solve similar figure applications

When you shrink or enlarge a photo on a phone or tablet, figure out a distance on a map, or use a pattern to build a bookcase or sew a dress, you are working with similar figures. If two figures have exactly the same shape, but different sizes, they are said to be similar. One is a scale model of the other. All their corresponding angles have the same measures and their corresponding sides have the same ratio.

Similar Figures

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

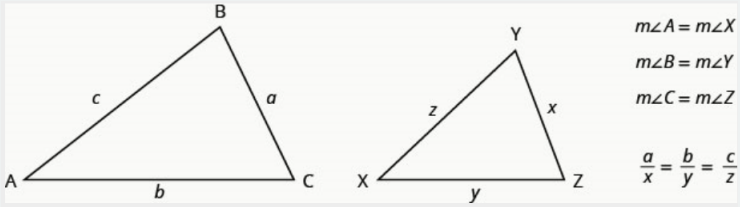
For example, the two triangles in Figure below are similar. Each side of ΔABC is four times the length of the corresponding side of ΔXYZ.



This is summed up in the Property of Similar Triangles.

Property of Similar Triangles

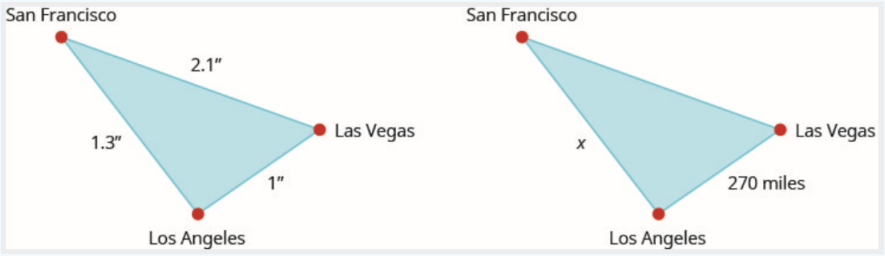
If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides have the same ratio.



To solve applications with similar figures we will follow the Problem-Solving Strategy for Geometry Applications we used earlier.

Example 15.6.3

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle. The distance between the cities is measured in inches. The figure on the left below represents the triangle formed by the cities on the map. If the actual distance from Los Angeles to Las Vegas is 270 miles, find the distance from Los Angeles to San Francisco.



Solution

Since the triangles are similar, the corresponding sides are proportional.

Read the problem. Draw the figures and label it with the given information. The figures are shown above.

Identify what we are looking for: the actual distance from Los Angeles to San Francisco

Name the variables: Let x = distance from Los Angeles to San Francisco.

Translate into an equation. Since the triangles are similar, the corresponding sides are proportional. We'll make the numerators "miles" and the denominators "inches".

$$\$\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}\$$$

Solve the equation.

$$1.3\left(\frac{x}{1.3}\right) = 1.3\left(\frac{270}{1}\right)$$
$$x = 351$$

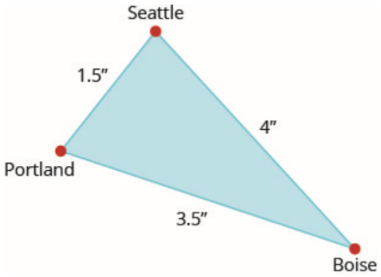
Check. On the map, the distance from Los Angeles to San Francisco is more than the distance from Los Angeles to Las Vegas. Since 351 is more than 270 the answer makes sense.

Check $x = 351$ in the original proportion. Use a calculator.

$$\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$$
$$\frac{351 \text{ miles}}{1.3 \text{ inches}} \stackrel{?}{=} \frac{270 \text{ miles}}{1 \text{ inch}}$$
$$\frac{270 \text{ miles}}{1 \text{ inch}} = \frac{270 \text{ miles}}{1 \text{ inch}} \surd$$

Answer the question: The distance from Los Angeles to San Francisco is 351 miles.

On the map, Seattle, Portland, and Boise form a triangle. The distance between the cities is measured in inches. The figure on the left below represents the triangle formed by the cities on the map. The actual distance from Seattle to Boise is 400 miles.



? Exercise 15.6.5

Find the actual distance from Seattle to Portland.

Answer

The distance is 150 miles.

? Exercise 15.6.6

Find the actual distance from Portland to Boise.

Answer

The distance is 350 miles.

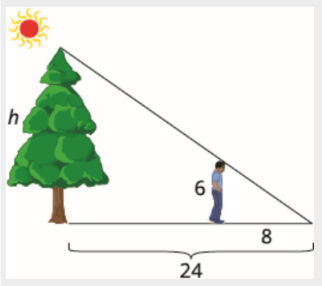
We can use similar figures to find heights that we cannot directly measure.

✓ Example 15.6.4

Tyler is 6 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a tree was 24 feet long. Find the height of the tree.

Solution

Read the problem and draw a figure. We are looking for h, the height of the tree.



We will use similar triangles to write an equation. The small triangle is similar to the large triangle.

$$\frac{h}{24} = \frac{6}{8}$$

Solve the proportion.

$$24\left(\frac{6}{8}\right) = 24\left(\frac{h}{24}\right)$$
$$18 = h$$

Simplify. Check.

Tyler's height is less than his shadow's length so it makes sense that the tree's height is less than the length of its shadow. Check $h = 18$ in the original proportion.

$$\frac{6}{8} = \frac{h}{24}$$
$$\frac{6}{8} \stackrel{?}{=} \frac{18}{24}$$
$$\frac{3}{4} = \frac{3}{4}\surd$$

Exercise 15.6.7

A telephone pole casts a shadow that is 50 feet long. Nearby, an 8 foot tall traffic sign casts a shadow that is 10 feet long. How tall is the telephone pole?

Answer

The telephone pole is 40 feet tall.

Exercise 15.6.8

A pine tree casts a shadow of 80 feet next to a 30 foot tall building which casts a 40 feet shadow. How tall is the pine tree?

Answer

The pine tree is 60 feet tall.

Solve Uniform Motion Applications

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate · Time = Distance		

The formula $D = rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula $t = \frac{D}{r}$.

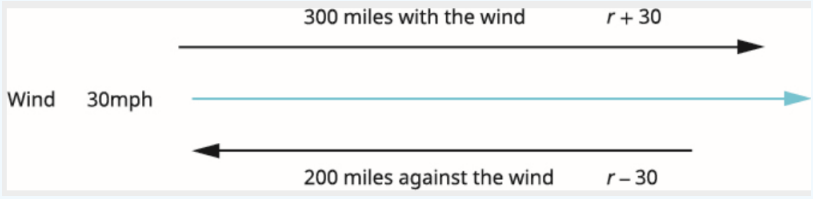
We have also explained how flying with or against the wind affects the speed of a plane. We will revisit that idea in the next example.

Example 15.6.5

An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the airplane. Let r = the speed of the airplane.

When the plane flies with the wind, the wind increases its speed and so the rate is $r + 30$.

When the plane flies against the wind, the wind decreases its speed and the rate is $r - 30$.

Write in the rates. Write in the distances. Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.

	Rate · Time = Distance		
Headwind	$r - 30$	$\frac{200}{r-30}$	200
Tailwind	$r + 30$	$\frac{300}{r+30}$	300

We know the times are equal and so we write our equation.

$$\frac{200}{r-30} = \frac{300}{r+30}$$

We multiply both sides by the LCD.

$$(r+30)(r-30)\left(\frac{200}{r-30}\right) = (r+30)(r-30)\left(\frac{300}{r+30}\right)$$

Simplify and solve.

$$\begin{aligned}(r+30)(200) &= (r-30)300\\ 200r+6000 &= 300r-9000\\ 15000 &= 100r\end{aligned}$$

Check.

Is 150mph a reasonable speed for an airplane? Yes. If the plane is traveling 150mph and the wind is 30mph,

$$\text{Tailwind} \quad 150+30=180\text{mph} \quad \frac{300}{180}=\frac{5}{3} \text{ hours}$$

$$\text{Headwind } 150-30=120\text{mph} \frac{200}{120}=\frac{5}{3} \text{ hours}$$

The times are equal, so it checks. The plane was traveling 150mph.

Exercise 15.6.9

Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link's biking speed?

Answer

Link's biking speed is 15 mph.

Exercise 15.6.10

Danica can sail her boat 5 miles into a 7 mph headwind in the same amount of time she can sail 12 miles with a 7 mph tailwind. What is the speed of Danica's boat without a wind?

Answer

The speed of Danica's boat is 17 mph.

In the next example, we will know the total time resulting from traveling different distances at different speeds.

Example 15.6.6

Jazmine trained for 3 hours on Saturday. She ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. What is her running speed?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.

We fill in the chart to organize the information. We are looking for Jazmine's running speed. Let r = Jazmine's running speed.

Her biking speed is 4 miles faster than her running speed. $r+4$ = her biking speed

The distances are given, enter them into the chart. Since $D=r\cdot t$, we solve for t and get $t=\frac{D}{r}$.We divide the distance by the rate in each row, and place the expression in the time column.

	Rate · Time = Distance		
Run	r	$\frac{8}{r}$	8
Bike	$r+4$	$\frac{24}{r+4}$	24
		3	

Write a word sentence: Her time plus the time biking is 3 hours.

Translate the sentence to get the equation.

$$\frac{8}{r}+\frac{24}{r+4}=3$$

Solve.

$$\begin{aligned} r(r+4)\left(\frac{8}{r}+\frac{24}{r+4}\right) &= 3\cdot r(r+4) \\ 8(r+4)+24r &= 3r(r+4) \\ 8r+32+24r &= 3r^2+12r \\ 32+32r &= 3r^2+12r \\ 0 &= 3r^2-20r-32 \\ 0 &= (3r+4)(r-8) \end{aligned}$$

$$(3r+4)=0 \quad (r-8)=0$$

$$r=\cancel{-\frac{4}{3}} \qquad r=8$$

Check.

A negative speed does not make sense in this problem, so $r=8$ is the solution.

Is 8 mph a reasonable running speed? Yes.

If Jazmine's running rate is 4, then her biking rate, $r+4$, which is $8+4=12$.

$$\text{Run } 8\text{mph} \quad \frac{8\text{miles}}{8\text{mph}}=1 \text{ hour}$$

$$\text{Bike } 12 \text{ mph} \quad \frac{24 \text{ miles}}{12\text{mph}}=2 \text{ hours}$$

Total 3 hours.

Jazmine's running speed is 8 mph.

Exercise 15.6.11

Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhil

Answer

Dennis's uphill speed was 10 mph and his downhill speed was 5 mph.

Exercise 15.6.12

Joon drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the countr

Answer

Joon's rate on the country roads is 50 mph.

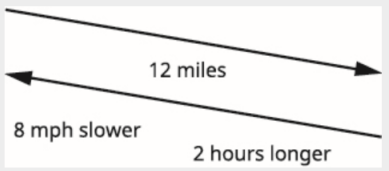
Once again, we will use the uniform motion formula solved for the variable t.

Example 15.6.7

Hamilton rode his bike downhill 12 miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was 8 miles per hour slower than his downhill speed. It too

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Hamilton's downhill speed. Let h= Hamilton's downhill speed.

His uphill speed is 8 miles per hour slower. $h-8$ = Hamilton's uphill speed.

Enter the rates into the chart.

The distance is the same in both directions: 12 miles.

Since $D=r\cdot t$, we solve for t and get $t=\frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.

	Rate · Time = Distance		
Downhill	h	$\frac{12}{h}$	12
Uphill	$h-8$	$\frac{12}{h-8}$	12

Write a word sentence about the line: He took 2 hours longer uphill than downhill. The uphill time is 2 more than the downhill time.

Translate the sentence to get the equation.

$$\frac{12}{h-8}=\frac{12}{h}+2$$

Solve.

$$\begin{aligned} h(h-8)\left(\frac{12}{h-8}\right) &= h(h-8)\left(\frac{12}{h}+2\right) \\ 12h &= 12(h-8)+2h(h-8) \\ 12h &= 12h-96+2h^2-16h \\ 0 &= 2h^2-16h-96 \\ 0 &= 2\left(h^2-8h-48\right) \\ 0 &= 2(h-12)(h+4) \end{aligned}$$

$$\begin{aligned} h-12=0 \quad & \quad h+4=0 \\ h=12 \quad & \quad h=\cancel{-4} \end{aligned}$$

Check. Is 12mph a reasonable speed for biking downhill? Yes.

$$\text{Downhill } 12\text{mph} \quad \frac{12 \text{ miles}}{12\text{mph}}=1 \text{ hour}$$

$$\text{Uphill } 12-8=4\text{mph} \quad \frac{12 \text{ miles}}{4\text{mph}}=3 \text{ hours}$$

The uphill time is 2 hours more that the downhill time.

Hamilton's downhill speed is 12mph.

? Exercise 15.6.13

Kayla rode her bike 75 miles home from college one weekend and then rode the bus back to college. It took her 2 hours less to ride back to college on the bus than it took her to ride home on her bike,

Answer

Kayla's biking speed was 15 mph.

? Exercise 15.6.14

Victoria jogs 12 miles to the park along a flat trail and then returns by jogging on an 20 mile hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes h

Answer

Victoria jogged 6 mph on the flat trail.

Solve Work Applications

The weekly gossip magazine has a big story about the Princess' baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the

This is a typical 'work' application. There are three quantities involved here—the time it would take each of the two presses to do the job alone and the time it would take for them to do the job together.

If Press #1 can complete the job in 6 hours, in one hour it would complete $\frac{1}{6}$ of the job.

If Press #2 can complete the job in 12 hours, in one hour it would complete $\frac{1}{12}$ of the job.

We will let t be the number of hours it would take the presses to print the magazines with both presses running together. So in 1 hour working together they have completed $\frac{1}{t}$ of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take the presses to complete the job if they worked together, we solve for t.

Follow the steps to organize the information. We are looking for how many hours it would take to complete the job with both presses running together.

Step 1: Let t = the number of hours needed to complete the job together.

Step 2: Enter the hours per job for Press #1, Press #2, and when they work together.

If a job on Press #1 takes 6 hours, then in 1 hour $\frac{1}{6}$ of the job is completed.

Similarly find the part of the job completed/hours for Press #2 and when they both together.

	Number of hours to complete the job.	Part of job completed/hour
Press #1	6	$\frac{1}{6}$
Press #2	12	$\frac{1}{12}$
Together	t	$\frac{1}{t}$

Write a word sentence. The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.

Step 3: Translate into an equation.

$$\underbrace{\text{Press \#1} + \text{Press \#2} = \text{Together}}_{\text{Work completed by}}$$

$$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$$

Step 4: Solve. Simplify.

$$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$$

Multiply by the LCD, $12t$ and simplify.

$$\begin{aligned} 12t\left(\frac{1}{6} + \frac{1}{12}\right) &= 12t\left(\frac{1}{t}\right) \\ 2t + t &= 12 \\ 3t &= 12 \\ t &= 4 \end{aligned}$$

When both presses are running it takes 4 hours to do the job.

Keep in mind, it should take less time for two presses to complete a job working together than for either press to do it alone.

✓ Example 15.6.8

Suppose Pete can paint a room in 10 hours. If he works at a steady pace, in 1 hour he would paint $\frac{1}{10}$ of the room. If Alicia would take 8 hours to paint the same room, then in 1 hour she would paint

Solution

This is a 'work' application. The steps below will help us organize the information. We are looking for the numbers of hours it will take them to paint the room together.

In one hour Pete did $\frac{1}{10}$ of the job. Alicia did $\frac{1}{8}$ of the job. And together they did $\frac{1}{t}$ of the job.

Step 1: Let t be the number of hours needed to paint the room together.

Step 2: Enter the hours per job for Pete, Alicia, and when they work together. In 1 hour working together, they have completed $\frac{1}{t}$ of the job. Similarly, find the part of the job completed/hour by Pete

	Number of hours to complete the job.	Part of job completed/hour

	Number of hours to complete the job.	Part of job completed/hour
Pete	10	$\frac{1}{10}$
Alicia	8	$\frac{1}{8}$
Together	t	$\frac{1}{t}$

Write a word sentence. The work completed by Pete plus the work completed by Alicia equals the total work completed.

Step 3: Translate into an equation.

$$\underbrace{\text{Work completed by Pete}}+\underbrace{\text{Alicia}}=\underbrace{\text{Together}}$$

$$\frac{1}{10}+\frac{1}{8}=\frac{1}{t}$$

Step 4: Simplify. Solve.

Multiply by the LCD, $40t$.

$$40t\left(\frac{1}{10}+\frac{1}{8}\right)=40t\left(\frac{1}{t}\right)$$

Distribute.

$$40t\cdot\frac{1}{10}+40t\cdot\frac{1}{8}=40t\left(\frac{1}{t}\right)$$

Simplify and solve.

$$\begin{aligned}4t+5t&=40\\9t&=40\\t&=\frac{40}{9}\end{aligned}$$

We'll write as a mixed number so that we can convert it to hours and minutes.

$$t=4\frac{4}{9}\text{ hours}$$

Remember, 1 hour = 60 minutes.

$$t=4\text{ hours}+\frac{4}{9}(60\text{ minutes})$$

Multiply, and then round to the nearest minute.

$$t=4\text{ hours}+27\text{ minutes}$$

It would take Pete and Alica about 4 hours and 27 minutes to paint the room.

? Exercise 15.6.15

One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course

Answer

When the two gardeners work together it takes 2 hours and 24 minutes.

? Exercise 15.6.16

Daria can weed the garden in 7 hours, while her mother can do it in 3. How long will it take the two of them working together?

Answer

When Daria and her mother work together it takes 2 hours and 6 minutes.

✓ Example 15.6.9

Ra'shon can clean the house in 7 hours. When his sister helps him it takes 3 hours. How long does it take his sister when she cleans the house alone?

Solution

This is a work problem. The steps below will help us organize the information. We are looking for how many hours it would take Ra'shon's sister to complete the job by herself.

Step 1: Let s be the number of hours Ra'shon's sister takes to clean the house alone.

Step 2: Enter the hours per job for Ra'shon, his sister, and when they work together. If Ra'shon takes 7 hours, then in 1 hour $\frac{1}{s}$ of the job is completed. If Ra'shon's sister takes s hours, then in 1 hour

	Number of hours to complete the job.	Part of job completed/hour
Ra'shon	7	$\frac{1}{7}$
His sister	s	$\frac{1}{s}$
Together	3	$\frac{1}{3}$

Write a word sentence. The part completed by Ra'shon plus the part by his sister equals the amount completed together.

Step 3: Translate to an equation.

$$\underbrace{\text{Ra'shon} + \text{His sister} = \text{Together}}_{\text{Work completed by}}$$
$$\frac{1}{7} + \frac{1}{s} = \frac{1}{3}$$

Step 4: Simplify. Solve.

$$\frac{1}{7}+\frac{1}{5}=\frac{1}{3}$$

Multiply by the LCD, 21s.

$$21s\left(\frac{1}{7}+\frac{1}{s}\right)=\left(\frac{1}{3}\right)21s$$
$$3s+21=7s$$

Simplify.

$$-4s=-21$$
$$s=\frac{-21}{-4}=\frac{21}{4}$$

Write as a mixed number to convert it to hours and minutes.

$$s=5\frac{1}{4}\text{ hours}$$

There are 60 minutes in 1 hour.

$$s=5\text{ hours}+\frac{1}{4}(60\text{ minutes})$$
$$s=5\text{ hours}+15\text{ minutes}$$

It would take Ra'shon's sister 5 hours and 15 minutes to clean the house alone.

Exercise 15.6.17

Alice can paint a room in 6 hours. If Kristina helps her it takes them 4 hours to paint the room. How long would it take Kristina to paint the room by herself?

Answer

Kristina can paint the room in 12 hours.

Exercise 15.6.18

Tracy can lay a slab of concrete in 3 hours, with Jordan's help they can do it in 2 hours. If Jordan works alone, how long will it take?

Answer

It will take Jordan 6 hours.

Solve Direct Variation Problems

When two quantities are related by a proportion, we say they are proportional to each other. Another way to express this relation is to talk about the variation of the two quantities. We will discuss direct

Lindsay gets paid $15 per hour at her job. If we let s be her salary and h be the number of hours she has worked, we could model this situation with the equation

$$s=15h$$

Lindsay's salary is the product of a constant, 15, and the number of hours she works. We say that Lindsay's salary varies directly with the number of hours she works. Two variables vary directly if one is

Direct Variation

For any two variables x and y, y varies directly with x if

$$y=kx\text{, where } k\neq 0$$

The constant k is called the constant of variation.

In applications using direct variation, generally we will know values of one pair of the variables and will be asked to find the equation that relates x and y. Then we can use that equation to find values o

We'll list the steps here.

How to Solve direct variation problems

Step 1. Write the formula for direct variation.

Step 2. Substitute the given values for the variables.

Step 3. Solve for the constant of variation.

Step 4. Write the equation that relates x and y using the constant of variation.

Now we'll solve an application of direct variation.

Example 15.6.10

When Raoul runs on the treadmill at the gym, the number of calories, c, he burns varies directly with the number of minutes, m, he uses the treadmill. He burned 315 calories when he used the treadm

a. Write the equation that relates c and m.
b. How many calories would he burn if he ran on the treadmill for 25 minutes?

Solution

a.

The number of calories, c, varies directly with the number of minutes, m, on the treadmill, and $c = 315$ when $m = 18$.

Write the formula for direct variation.

$$y = kx$$

We will use c in place of y and m in place of x.

$$c = km$$

Substitute the given values for the variables.

$$315 = k \cdot 18$$

Solve for the constant of variation.

$$\frac{315}{18} = \frac{k \cdot 18}{18}$$
$$17.5 = k$$

Write the equation that relates c and m.

$$c = km$$

Substitute in the constant of variation.

$$c = 17.5m$$

b.

Find c when $m = 25$.

Write the equation that relates c and m.

$$c = 17.5m$$

Substitute the given value for m.

$$c = 17.5(25)$$

Simplify.

$$c = 437.5$$

Raoul would burn 437.5 calories if he used the treadmill for 25 minutes.

? Exercise **15.6.19**

The number of calories, c, burned varies directly with the amount of time, t, spent exercising. Arnold burned 312 calories in 65 minutes exercising.

a. Write the equation that relates c and t.
b. How many calories would he burn if he exercises for 90 minutes?

Answer

a. $c = 4.8t$
b. He would burn 432 calories.

? Exercise **15.6.20**

The distance a moving body travels, d, varies directly with time, t, it moves. A train travels 100 miles in 2 hours.

a. Write the equation that relates d and t.
b. How many miles would it travel in 5 hours?

Answer

a. $d = 50t$
b. It would travel 250 miles.

Solve Inverse Variation Problems

Many applications involve two variable that vary inversely. As one variable increases, the other decreases. The equation that relates them is $y = \frac{k}{x}$.

Inverse Variation

For any two variables x and y, y varies inversely with x if

$$y = \frac{k}{x}, \text{ where } k \neq 0$$

The constant k is called the constant of variation.

The word ‘inverse’ in inverse variation refers to the multiplicative inverse. The multiplicative inverse of x is $\frac{1}{x}$.

We solve inverse variation problems in the same way we solved direct variation problems. Only the general form of the equation has changed. We will copy the procedure box here and just change ‘direc

how to Solve inverse variation problems

Step 1. Write the formula for inverse variation.

Step 2. Substitute the given values for the variables.

Step 3. Solve for the constant of variation.

Step 4. Write the equation that relates x and y using the constant of variation.

Example 15.6.11

The frequency of a guitar string varies inversely with its length. A 26 in.-long string has a frequency of 440 vibrations per second.

a. Write the equation of variation.
b. How many vibrations per second will there be if the string's length is reduced to 20 inches by putting a finger on a fret?

Solution

a.

The frequency varies inversely with the length.

Name the variables. Let f = frequency. L = length

Write the formula for inverse variation.

$$y = \frac{k}{x}$$

We will use f in place of y and L in place of x.

$$f = \frac{k}{L}$$

$$f = 440 \text{ when } L = 26$$

Substitute the given values for the variables.

$$440 = \frac{k}{26}$$

Solve for the constant of variation.

$$26(440) = 26\left(\frac{k}{26}\right)$$

$$11,440 = k$$

Write the equation that relates f and L.

$$f = \frac{k}{L}$$

Substitute the constant of variation.

$$f = \frac{11,440}{L}$$

b.

Find f when $L = 20$.

Write the equation that relates f and L.

$$f = \frac{11,440}{L}$$

Substitute the given value forL.

$$f = \frac{11,440}{20}$$

Simplify.

$$f = 572$$

A 20"-guitar string has frequency 572 vibrations per second.

Exercise 15.6.21

The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees Celsius.

a. Write the equation of variation.
b. How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

Answer

a. $h = \frac{130}{t}$
b. $1\frac{2}{3}$ hours

Exercise 15.6.22

Xander's new business found that the daily demand for its product was inversely proportional to the price, p. When the price is \$5, the demand is 700 units.

a. Write the equation of variation.
b. What is the demand if the price is raised to \$7?

Answer

a. $x = \frac{3500}{p}$
b. 500 units

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15.6.1: More Examples of Applications

Learning Objectives

- Solve applications involving uniform motion (distance problems).
- Solve work-rate applications.
- Set up and solve applications involving direct, inverse, and joint variation.

Solving Uniform Motion Problems

Uniform motion (or distance)[37] problems involve the formula $D = rt$, where the distance D is given as the product of the average rate r and the time t traveled at that rate. If we divide both sides by the average rate r, then we obtain the formula

$$t = \frac{D}{r}$$

For this reason, when the unknown quantity is time, the algebraic setup for distance problems often results in a rational equation. We begin any uniform motion problem by first organizing our data with a chart. Use this information to set up an algebraic equation that models the application.

Example 15.6.1.1:

Sally traveled 15 miles on the bus and then another 72 miles on a train. The train was 18 miles per hour faster than the bus, and the total trip took 2 hours. What was the average speed of the train?

Solution

First, identify the unknown quantity and organize the data.

Let x represent the average speed (in miles per hour) of the bus.

Let $x + 18$ represent the average speed of the train.

	Distance =	***Rate*** ×	***Time***
Bus trip	15 mi	x	
Train trip	72 mi	$x + 18$	
Total			2 hours

Figure 15.6.1.1

To avoid introducing two more variables for the time column, use the formula $t = \frac{D}{r}$. The time for each leg of the trip is calculated as follows:

$$\textit{Time spent on the bus}: t = \frac{D}{r} = \frac{15}{x}$$

$$\textit{Time spent on the train}: t = \frac{D}{r} = \frac{72}{x+18}$$

Use these expressions to complete the chart.

	Distance =	Rate ×	Time
Bus trip	15 mi	x	$\frac{15}{x}$
Train trip	72 mi	$x+18$	$\frac{72}{x+18}$
Total			2 hours

$$\frac{15}{x}+\frac{72}{x+18}=2$$

Figure 15.6.1.2

The algebraic setup is defined by the time column. Add the time spent on each leg of the trip to obtain a total of 2 hours:

$$\overbrace{\frac{15}{x}}^{\text{time spent on the bus}} + \overbrace{\frac{72}{x+18}}^{\text{time spent on the train}} = \overbrace{2}^{\text{total time of trip}}$$

Figure 15.6.1.3

We begin solving this equation by first multiplying both sides by the LCD, $x(x+18)$.

$$\begin{aligned}
\frac{15}{x}+\frac{72}{x+18} &= 2 \\
x(x+18)\cdot\left(\frac{15}{x}+\frac{72}{x+18}\right) &= x(x+18)\cdot 2 \\
x(x+18)\cdot\frac{15}{x}+x(x+18)\cdot\frac{72}{x+18} &= x(x+18)\cdot 2 \\
15(x+18)\cdot 72x &= 2x(x+18) \\
15x+270+72x &= 2x^2+36x \\
87x+270 &= 2x^2+36x \\
0 &= 2x^2-51x-270
\end{aligned}$$

Solve the resulting quadratic equation by factoring.

$$\begin{aligned}
0 &= 2x^2-51x-270 \\
0 &= (2x+9)(x-30) \\
2x+9=0 \quad &\text{or} \quad x-30=0 \\
x=-\frac{9}{2} \quad &\qquad\quad x=30
\end{aligned}$$

Since we are looking for an average speed we will disregard the negative answer and conclude the bus averaged 30 mph. Substitute $x=30$ in the expression identified as the speed of the train.

$$x+18=30+18=48$$

Answer:

The speed of the train was 48 mph.

Example 15.6.1.2:

A boat can average 12 miles per hour in still water. On a trip downriver the boat was able to travel 29 miles with the current. On the return trip the boat was only able to travel 19 miles in the same amount of time against the current. What was the speed of the current?

Solution

First, identify the unknown quantities and organize the data.

Let c represent the speed of the river current.

Next, organize the given data in a chart. Traveling downstream, the current will increase the speed of the boat, so it adds to the average speed of the boat. Traveling upstream, the current slows the boat, so it will subtract from the average speed of the boat.

	Distance	= **Rate**	× **Time**
trip downriver	29 mi	$12 + c$	
trip upriver	19 mi	$12 - c$	
Total			

Figure 15.6.1.4

Use the formula $t = \frac{D}{r}$ to fill in the time column.

$$\textit{trip downriver}: t = \frac{D}{r} = \frac{29}{12+c}$$
$$\textit{trip upriver}: t = \frac{D}{r} = \frac{19}{12-c}$$

	Distance	= **Rate**	× **Time**
trip downriver	29 mi	$12 + c$	$\frac{29}{12 + c}$
trip upriver	19 mi	$12 - c$	$\frac{19}{12 - c}$
Total			

Figure 15.6.1.5

Because the boat traveled the same amount of time downriver as it did upriver, finish the algebraic setup by setting the expressions that represent the times equal to each other.

$$\frac{29}{12+c} = \frac{19}{12-c}$$

Since there is a single algebraic fraction on each side, we can solve this equation using cross multiplication.

$$\begin{aligned} \frac{29}{12+c} &= \frac{19}{12-c} \\ 29(12-c) &= 19(12+c) \\ 348 - 29c &= 228 + 19c \\ 120 &= 48c \\ \frac{120}{48} &= c \\ \frac{5}{2} &= c \end{aligned}$$

Answer:

The speed of the current was $2\frac{1}{2}$ miles per hour.

Exercise 15.6.1.1

A jet aircraft can average 160 miles per hour in calm air. On a trip, the aircraft traveled 600 miles with a tailwind and returned the 600 miles against a headwind of the same speed. If the total round trip took 8 hours, then what was the speed of the wind?

Answer

40 miles per hour

www.youtube.com/v/0NglBthTwss

Solving Work-Rate Problems

The rate at which a task can be performed is called a **work rate**[38]. For example, if a painter can paint a room in 6 hours, then the task is to paint the room, and we can write

$$\frac{1 \text{ task}}{6 \text{ hours}} \quad work\ rate$$

In other words, the painter can complete $\frac{1}{6}$ of the task per hour. If he works for less than 6 hours, then he will perform a fraction of the task. If he works for more than 6 hours, then he can complete more than one task. For example,

$$work-rate \times time = amount\ of\ task\ completed$$
$$\frac{1}{6} \times 3hrs = \frac{1}{2} \quad one-half\ of\ the\ room\ painted$$
$$\frac{1}{6} \times 6hrs = 1 \quad one\ whole\ room\ painted$$
$$\frac{1}{6} \times 12 \text{ hrs} = 2 \quad two\ whole\ rooms\ painted$$

Obtain the amount of the task completed by multiplying the work rate by the amount of time the painter works. Typically, work-rate problems involve people or machines working together to complete tasks. In general, if t represents the time two people work together, then we have the following **work-rate formula**[39]:

$$\frac{1}{t_1}t + \frac{1}{t_2}t = amount\ of\ task\ completed\ together$$

Here $\frac{1}{t_1}$ and $\frac{1}{t_2}$ are the individual work rates.

Example 15.6.1.3:

Joe can paint a typical room in 2 hours less time than Mark. If Joe and Mark can paint 5 rooms working together in a 12 hour shift, how long does it take each to paint a single room?

Solution

Let x represent the time it takes Mark to paint a typical room.

Let $x-2$ represent the time it takes Joe to paint a typical room.

Therefore, Mark's individual work-rate is $\frac{1}{x}$ rooms per hour and Joe's is $\frac{1}{x-2}$ rooms per hour. Both men worked for 12 hours. We can organize the data in a chart, just as we did with distance problems.

	Amount of Task Completed =	***Work - Rate*** ×	***Time***
Joe	$\frac{1}{x-2}\cdot 12$	$\frac{1}{x-2}$	12
Mark	$\frac{1}{x}\cdot 12$	$\frac{1}{x}$	12
Total	5		

$$\frac{1}{x-2}\cdot 12 + \frac{1}{x}\cdot 12 = 5$$

Figure 15.6.1.6

Working together, they can paint 5 total rooms in 12 hours. This leads us to the following algebraic setup:

$$\frac{12}{x-2}+\frac{12}{x}=5$$

Multiply both sides by the LCD, $x(x-2)$.

$$\begin{aligned} x(x-2)\cdot\left(\frac{12}{x-2}+\frac{12}{x}\right) &= x(x-2)\cdot 5 \\ x(x-2)\cdot\frac{12}{x-2}+x(x-2)\cdot\frac{12}{x} &= x(x-2)\cdot 5 \\ 12x+12(x-2) &= 5x(x-2) \\ 12x+12x-24 &= 5x^2-10x \\ 0 &= 5x^2-34x+24 \end{aligned}$$

Solve the resulting quadratic equation by factoring.

$$\begin{aligned} 0 &= 5x^2-34x+24 \\ 0 &= (5x-4)(x-6) \end{aligned}$$

$$\begin{aligned} 5x-4=0 \quad &\text{or} \quad x-6=0 \\ 5x=4 \quad &\quad\quad x=6 \\ x=\frac{4}{5} \quad &\end{aligned}$$

We can disregard $\frac{4}{5}$ because back substituting into $x-2$ would yield a negative time to paint a room. Take $x=6$ to be the only solution and use it to find the time it takes Joe to paint a typical room.

$$x-2=6-2=4$$

Answer:

Joe can paint a typical room in 4 hours and Mark can paint a typical room in 6 hours. As a check we can multiply both work rates by 12 hours to see that together they can paint 5 rooms.

$$\left.\begin{array}{l} Joe\ \frac{1\text{ room}}{4\text{hrs}}\cdot 12\text{ hrs}=3\text{ rooms} \\ Mark\ \frac{1\text{ room}}{6\text{hrs}}\cdot 12\text{hrs}=2\text{ rooms} \end{array}\right\} Total\ 5\ rooms\ \checkmark$$

Example 15.6.1.4:

It takes Bill twice as long to lay a tile floor by himself as it does Manny. After working together with Bill for 4 hours, Manny was able to complete the job in 2 additional hours. How long would it have taken Manny working alone?

Solution

Let x represent the time it takes Manny to lay the floor alone.

Let $2x$ represent the time it takes Bill to lay the floor alone.

Manny's work rate is $\frac{1}{x}$ of the floor per hour and Bill's work rate is $\frac{1}{2x}$. Bill worked on the job for 4 hours and Manny worked on the job for 6 hours.

	Amount of Task Completed =	***Work - Rate*** ×	***Time***
Manny	$\frac{1}{x}\cdot 6$	$\frac{1}{x}$	6 hours
Bill	$\frac{1}{2x}\cdot 4$	$\frac{1}{2x}$	4 hours
Total	1		

Figure 15.6.1.7

This leads us to the following algebraic setup:

$$\frac{1}{x}\cdot 6+\frac{1}{2x}\cdot 4=1$$

Solve.

$$\begin{aligned}\frac{6}{x}+\frac{4}{2x}&=1\\ x\cdot\left(\frac{6}{x}+\frac{2}{x}\right)&=x\cdot 1\\ 6+2&=x\\ 8&=x\end{aligned}$$

Answer:

It would have taken Manny 8 hours to complete the floor by himself.

Consider the work-rate formula where one task is to be completed.

$$\frac{1}{t_1}t+\frac{1}{t_2}t=1$$

Factor out the time t and then divide both sides by t. This will result in equivalent specialized work-rate formulas:

$$\begin{aligned}t\left(\frac{1}{t_1}+\frac{1}{t_2}\right)&=1\\ \frac{1}{t_1}+\frac{1}{t_2}&=\frac{1}{t}\end{aligned}$$

In summary, we have the following equivalent work-rate formulas:

Work rate formulas

$$\frac{1}{t_1}t+\frac{1}{t_2}t=1 \quad \text{or} \quad \frac{t}{t_1}+\frac{t}{t_2}=1 \quad \text{or} \quad \frac{1}{t_1}+\frac{1}{t_2}=\frac{1}{t}$$

Exercise 15.6.1.2

Matt can tile a countertop in 2 hours, and his assistant can do the same job in 3 hours. If Matt starts the job and his assistant joins him 1 hour later, then how long will it take to tile the countertop?

Answer

$1\frac{3}{5}$ hours

www.youtube.com/v/5g6sSFWGb7M

Solving Problems involving Direct, Inverse, and Joint variation

Many real-world problems encountered in the sciences involve two types of functional relationships. The first type can be explored using the fact that the distance s in feet an object falls from rest, without regard to air resistance, can be approximated using the following formula:

$$s=16t^2$$

Here t represents the time in seconds the object has been falling. For example, after 2 seconds the object will have fallen $s=16(2)^2=16\cdot 4=64$ feet.

Table 15.6.1.1

Time t in seconds	Distance $s=16t^2$ in feet
0	0
1	16
2	64
3	144

Time t in seconds	Distance $s = 16t^2$ in feet
4	256

In this example, we can see that the distance varies over time as the product of a constant 16 and the square of the time t. This relationship is described as **direct variation**[40] and 16 is called the **constant of variation**[41]. Furthermore, if we divide both sides of $s = 16t^2$ by t^2 we have

$$\frac{s}{t^2} = 16$$

In this form, it is reasonable to say that s is proportional to t^2, and 16 is called the **constant of proportionality**[42]. In general, we have

Table 15.6.1.2

Key words	Translation
"y varies x directly as x"	$y = kx$
"y is **directly proportional**[43] to x"	
"y is proportional to x"	

Here k is nonzero and is called the constant of variation or the constant of proportionality. Typically, we will be given information from which we can determine this constant.

Example 15.6.1.5:

An object's weight on Earth varies directly to its weight on the Moon. If a man weighs 180 pounds on Earth, then he will weigh 30 pounds on the Moon. Set up an algebraic equation that expresses the weight on Earth in terms of the weight on the Moon and use it to determine the weight of a woman on the Moon if she weighs 120 pounds on Earth.

Solution

Let y represent weight on Earth.

Let x represent weight on the Moon.

We are given that the "weight on Earth varies directly to the weight on the Moon."

$$y = kx$$

To find the constant of variation k, use the given information. A 180-lb man on Earth weighs 30 pounds on the Moon, or $y = 180$ when $x = 30$.

$$180 = k \cdot 30$$

Solve for k.

$$\frac{180}{30} = k$$
$$6 = k$$

Next, set up a formula that models the given information.

$$y = 6x$$

This implies that a person's weight on Earth is 6 times his weight on the Moon. To answer the question, use the woman's weight on Earth, $y = 120$ lbs, and solve for x.

$$120 = 6x$$
$$\frac{120}{6} = x$$
$$20 = x$$

Answer:

The woman weighs 20 pounds on the Moon.

The second functional relationship can be explored using the formula that relates the intensity of light I to the distance from its source d.

$$I = \frac{k}{d^2}$$

Here k represents some constant. A foot-candle is a measurement of the intensity of light. One foot-candle is defined to be equal to the amount of illumination produced by a standard candle measured one foot away. For example, a 125-Watt fluorescent growing light is advertised to produce 525 foot-candles of illumination. This means that at a distance $d = 1$ foot, $I = 525$ foot-candles and we have:

$$525 = \frac{k}{(1)^2}$$
$$525 = k$$

Using $k = 525$ we can construct a formula which gives the light intensity produced by the bulb:

$$I = \frac{525}{d^2}$$

Here d represents the distance the growing light is from the plants. In the following chart, we can see that the amount of illumination fades quickly as the distance from the plants increases.

Table 15.6.1.3

Distance t in feet	Light Intensity $I = \frac{525}{d^2}$
1	525
2	131.25
3	58.33
4	32.81
5	21

This type of relationship is described as an **inverse variation**[44]. We say that I is **inversely proportional**[45] to the square of the distance d, where 525 is the constant of proportionality. In general, we have

Table 15.6.1.4

Key words	Translation
"y varies inversely as x"	$y = \frac{k}{x}$
"y is inversely proportional to x"	

Again, k is nonzero and is called the constant of variation or the constant of proportionality.

Example 15.6.1.6:

The weight of an object varies inversely as the square of its distance from the center of Earth. If an object weighs 100 pounds on the surface of Earth (approximately $4,000$ miles from the center), how much will it weigh at $1,000$ miles above Earth's surface?

Solution

Let w represent the weight of the object.

Let d represent the object's distance from the center of Earth.

Since "w varies inversely as the square of d," we can write

$$w = \frac{k}{d^2}$$

Use the given information to find k. An object weighs 100 pounds on the surface of Earth, approximately $4,000$ miles from the center. In other words, $w = 100$ when $d = 4,000$:

$$100 = \frac{k}{(4{,}000)^2}$$

Solve for k.

$$(4,000)^2 \cdot 100 = (4,000)^2 \cdot \frac{k}{(4,000)^2}$$
$$1,600,000,000 = k$$
$$1.6 \times 10^9 = k$$

Therefore, we can model the problem with the following formula:

$$w = \frac{1.6 \times 10^9}{d^2}$$

To use the formula to find the weight, we need the distance from the center of Earth. Since the object is $1,000$ miles above the surface, find the distance from the center of Earth by adding $4,000$ miles:

$$d = 4,000 + 1,000 = 5,000 \text{ miles}$$

To answer the question, use the formula with $d = 5,000$.

$$\begin{aligned} y &= \frac{1.6 \times 10^9}{(5,000)^2} \\ &= \frac{1.6 \times 10^9}{25,000,000} \\ &= \frac{1.6 \times 10^9}{2.5 \times 10^9} \\ &= 0.64 \times 10^2 \\ &= 64 \end{aligned}$$

Answer:

The object will weigh 64 pounds at a distance $1,000$ miles above the surface of Earth.

Lastly, we define relationships between multiple variables, described as **joint variation**[46]. In general, we have

Table 15.6.1.5

Key Words	Translation
"y varies jointly as x and z"	$y = kxz$
"y is **jointly proportional**[47] to x and z"	

Here k is nonzero and is called the constant of variation or the constant of proportionality.

Example 15.6.1.7:

The area of an ellipse varies jointly as a, half of the ellipse's major axis, and b, half of the ellipse's minor axis as pictured. If the area of an ellipse is 300π cm^2, where $a = 10$ cm and $b = 30$ cm, what is the constant of proportionality? Give a formula for the area of an ellipse.

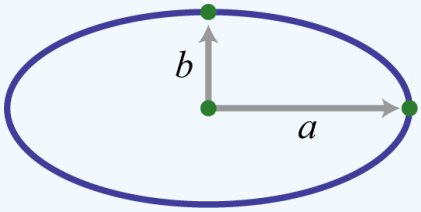

Figure 15.6.1.8

Solution

If we let A represent the area of an ellipse, then we can use the statement "area varies jointly as a and b" to write

$$A = kab$$

To find the constant of variation k, use the fact that the area is 300π when $a = 10$ and $b = 30$.

$$300\pi = k(10)(30)$$
$$300\pi = 300k$$
$$\pi = k$$

Therefore, the formula for the area of an ellipse is

$$A = \pi ab$$

Answer:

The constant of proportionality is π and the formula for the area of an ellipse is $A = ab\pi$.

Exercise 15.6.1.3

Given that y varies directly as the square of x and inversely with z, where $y = 2$ when $x = 3$ and $z = 27$, find y when $x = 2$ and $z = 16$.

Answer

$\frac{3}{2}$

www.youtube.com/v/ee3AFf7b6Kg

Key Takeaways

- When solving distance problems where the time element is unknown, use the equivalent form of the uniform motion formula, $t = \frac{D}{r}$, to avoid introducing more variables.
- When solving work-rate problems, multiply the individual work rate by the time to obtain the portion of the task completed. The sum of the portions of the task results in the total amount of work completed.
- The setup of variation problems usually requires multiple steps. First, identify the key words to set up an equation and then use the given information to find the constant of variation k. After determining the constant of variation, write a formula that models the problem. Once a formula is found, use it to answer the question.

Exercise 15.6.1.4

Use algebra to solve the following applications.

1. Every morning Jim spends 1 hour exercising. He runs 2 miles and then he bikes 16 miles. If Jim can bike twice as fast as he can run, at what speed does he average on his bike?
2. Sally runs 3 times as fast as she walks. She ran for $\frac{3}{4}$ of a mile and then walked another $3\frac{1}{2}$ miles. The total workout took $1\frac{1}{2}$ hours. What was Sally's average walking speed?
3. On a business trip, an executive traveled 720 miles by jet and then another 80 miles by helicopter. If the jet averaged 3 times the speed of the helicopter, and the total trip took 4 hours, what was the average speed of the jet?
4. A triathlete can run 3 times as fast as she can swim and bike 6 times as fast as she can swim. The race consists of a $\frac{1}{4}$ mile swim, 3 mile run, and a 12 mile bike race. If she can complete all of these events in $1\frac{5}{8}$ hour, then how fast can she swim, run and bike?
5. On a road trip, Marty was able to drive an average 4 miles per hour faster than George. If Marty was able to drive 39 miles in the same amount of time George drove 36 miles, what was Marty's average speed?
6. The bus is 8 miles per hour faster than the trolley. If the bus travels 9 miles in the same amount of time the trolley can travel 7 miles, what is the average speed of each?
7. Terry decided to jog the 5 miles to town. On the return trip, she walked the 5 miles home at half of the speed that she was able to jog. If the total trip took 3 hours, what was her average jogging speed?

8. James drove the 24 miles to town and back in 1 hour. On the return trip, he was able to average 20 miles per hour faster than he averaged on the trip to town. What was his average speed on the trip to town?
9. A light aircraft was able to travel 189 miles with a 14 mile per hour tailwind in the same time it was able to travel 147 miles against it. What was the speed of the aircraft in calm air?
10. A jet flew 875 miles with a 30 mile per hour tailwind. On the return trip, against a 30 mile per hour headwind, it was able to cover only 725 miles in the same amount of time. How fast was the jet in calm air?
11. A helicopter averaged 90 miles per hour in calm air. Flying with the wind it was able to travel 250 miles in the same amount of time it took to travel 200 miles against it. What is the speed of the wind?
12. Mary and Joe took a road-trip on separate motorcycles. Mary's average speed was 12 miles per hour less than Joe's average speed. If Mary drove 115 miles in the same time it took Joe to drive 145 miles, what was Mary's average speed?
13. A boat averaged 12 miles per hour in still water. On a trip downstream, with the current, the boat was able to travel 26 miles. The boat then turned around and returned upstream 33 miles. How fast was the current if the total trip took 5 hours?
14. If the river current flows at an average 3 miles per hour, a tour boat can make an 18-mile tour downstream with the current and back the 18 miles against the current in $4\frac{1}{2}$ hours. What is the average speed of the boat in still water?
15. Jose drove 10 miles to his grandmother's house for dinner and back that same evening. Because of traffic, he averaged 20 miles per hour less on the return trip. If it took $\frac{1}{4}$ hour longer to get home, what was his average speed driving to his grandmother's house?
16. Jerry paddled his kayak, upstream against a 1 mph current, for 12 miles. The return trip, downstream with the 1 mph current, took one hour less time. How fast did Jerry paddle the kayak in still water?
17. James and Mildred left the same location in separate cars and met in Los Angeles 300 miles away. James was able to average 10 miles an hour faster than Mildred on the trip. If James arrived 1 hour earlier than Mildred, what was Mildred's average speed?
18. A bus is 20 miles per hour faster than a bicycle. If Bill boards a bus at the same time and place that Mary departs on her bicycle, Bill will arrive downtown 5 miles away $\frac{1}{3}$ hour earlier than Mary. What is the average speed of the bus?

Answer

1. 20 miles per hour

3. 240 miles per hour

5. 52 miles per hour

7. 5 miles per hour

9. 112 miles per hour

11. 10 miles per hour

13. 1 mile per hour

15. 40 miles per hour

17. 50 miles per hour

Exercise 15.6.1.5

Use algebra to solve the following applications.

1. Mike can paint the office by himself in $4\frac{1}{2}$ hours. Jordan can paint the office in 6 hours. How long will it take them to paint the office working together?
2. Barry can lay a brick driveway by himself in $3\frac{1}{2}$ days. Robert does the same job in 5 days. How long will it take them to lay the brick driveway working together?
3. A larger pipe fills a water tank twice as fast as a smaller pipe. When both pipes are used, they fill the tank in 10 hours. If the larger pipe is left off, how long would it take the smaller pipe to fill the tank?
4. A newer printer can print twice as fast as an older printer. If both printers working together can print a batch of flyers in 45 minutes, then how long would it take the older printer to print the batch working alone?

5. Mary can assemble a bicycle for display in 2 hours. It takes Jane 3 hours to assemble a bicycle. How long will it take Mary and Jane, working together, to assemble 5 bicycles?
6. Working alone, James takes twice as long to assemble a computer as it takes Bill. In one 8-hour shift, working together, James and Bill can assemble 6 computers. How long would it take James to assemble a computer if he were working alone?
7. Working alone, it takes Harry one hour longer than Mike to install a fountain. Together they can install 10 fountains in 12 hours. How long would it take Mike to install 10 fountains by himself?
8. Working alone, it takes Henry 2 hours longer than Bill to paint a room. Working together they painted $2\frac{1}{2}$ rooms in 6 hours. How long would it have taken Henry to paint the same amount if he were working alone?
9. Manny, working alone, can install a custom cabinet in 3 hours less time than his assistant. Working together they can install the cabinet in 2 hours. How long would it take Manny to install the cabinet working alone?
10. Working alone, Garret can assemble a garden shed in 5 hours less time than his brother. Working together, they need 6 hours to build the garden shed. How long would it take Garret to build the shed working alone?
11. Working alone, the assistant-manager takes 2 more hours than the manager to record the inventory of the entire shop. After working together for 2 hours, it took the assistant-manager 1 additional hour to complete the inventory. How long would it have taken the manager to complete the inventory working alone?
12. An older printer can print a batch of sales brochures in 16 minutes. A newer printer can print the same batch in 10 minutes. After working together for some time, the newer printer was shut down and it took the older printer 3 more minutes to complete the job. How long was the newer printer operating?

Answer

1. $2\frac{4}{7}$ hours
3. 30 hours
5. 6 hours
7. 20 hours
9. 3 hours
11. 4 hours

Exercise 15.6.1.6

Translate each of the following sentences into a mathematical formula.

1. The distance D an automobile can travel is directly proportional to the time t that it travels at a constant speed.
2. The extension of a hanging spring d is directly proportional to the weight w attached to it.
3. An automobile's braking distance d is directly proportional to the square of the automobile's speed v.
4. The volume V of a sphere varies directly as the cube of its radius r.
5. The volume V of a given mass of gas is inversely proportional to the pressure p exerted on it.
6. Every particle of matter in the universe attracts every other particle with a force F that is directly proportional to the product of the masses m_1 and m_2 of the particles, and it is inversely proportional to the square of the distance d between them.
7. Simple interest I is jointly proportional to the annual interest rate r and the time t in years a fixed amount of money is invested.
8. The time t it takes an object to fall is directly proportional to the square root of the distance d it falls.

Answer

1. $D = kt$
3. $d = kv^2$
5. $V = \frac{k}{p}$
7. $I = krt$

Exercise 15.6.1.7

Construct a mathematical model given the following:

1. y varies directly as x, and $y = 30$ when $x = 6$.
2. y varies directly as x, and $y = 52$ when $x = 4$.
3. y is directly proportional to x, and $y = 12$ when $x = 3$.
4. y is directly proportional to x, and $y = 120$ when $x = 20$.
5. y is directly proportional to x, and $y = 3$ when $x = 9$.
6. y is directly proportional to x, and $y = 21$ when $x = 3$.
7. y varies inversely as x, and $y = 2$ when $x = \frac{1}{8}$.
8. y varies inversely as x, and $y = \frac{3}{2}$ when $x = \frac{1}{9}$.
9. y is jointly proportional to x and z, where $y = 2$ when $x = 1$ and $z = 3$.
10. y is jointly proportional to x and z, where $y = 15$ when $x = 3$ and $z = 7$.
11. y varies jointly as x and z, where $y = \frac{2}{3}$ when $x = \frac{1}{2}$ and $z = 12$.
12. y varies jointly as x and z, where $y = 5$ when $x = \frac{3}{2}$ and $z = \frac{2}{9}$.
13. y varies directly as the square of x, where $y = 45$ when $x = 3$.
14. y varies directly as the square of x, where $y = 3$ when $x = \frac{1}{2}$.
15. y is inversely proportional to the square of x, where $y = 27$ when $x = \frac{1}{3}$.
16. y is inversely proportional to the square of x, where $y = 9$ when $x = \frac{2}{3}$.
17. y varies jointly as x and the square of z, where $y = 6$ when $x = \frac{1}{4}$ and $z = \frac{2}{3}$.
18. y varies jointly as x and z and inversely as the square of w, where $y = 5$ when $z = 1, z = 3$, and $w = \frac{1}{2}$.
19. y varies directly as the square root of x and inversely as the square of z, where $y = 15$ when $x = 25$ and $z = 2$.
20. y varies directly as the square of x and inversely as z and the square of w, where $y = 14$ when $x = 4, w = 2$ and $z = 2$.

Answer

1. $y = 5x$
3. $y = 4x$
5. $y = \frac{27}{x}$
7. $y = \frac{1}{4x}$
9. $y = \frac{2}{3}xz$
11. $y = \frac{1}{9}xz$
13. $y = 5x^2$
15. $y = \frac{3}{x^2}$
17. $y = 54xz^2$
19. $y = \frac{12\sqrt{x}}{z^2}$

Exercise 15.6.1.8

Solve applications involving variation.

1. Revenue in dollars is directly proportional to the number of branded sweatshirts sold. The revenue earned from selling 25 sweatshirts is $\$318.75$. Determine the revenue if 30 sweatshirts are sold.
2. The sales tax on the purchase of a new car varies directly as the price of the car. If an $\$18,000$ new car is purchased, then the sales tax is $\$1,350$. How much sales tax is charged if the new car is priced at $\$22,000$?
3. The price of a share of common stock in a company is directly proportional to the earnings per share (EPS) of the previous 12 months. If the price of a share of common stock in a company is $22.55, and the EPS is published to be $\$1.10$, determine the value of the stock if the EPS increases by $\$0.20$.

4. The distance traveled on a road trip varies directly with the time spent on the road. If a 126-mile trip can be made in 3 hours, then what distance can be traveled in 4 hours?
5. The circumference of a circle is directly proportional to its radius. The circumference of a circle with radius 7 centimeters is measured as 14π centimeters. What is the constant of proportionality?
6. The area of circle varies directly as the square of its radius. The area of a circle with radius 7 centimeters is determined to be 49π square centimeters. What is the constant of proportionality?
7. The surface area of a sphere varies directly as the square of its radius. When the radius of a sphere measures 2 meters, the surface area measures 16π square meters. Find the surface area of a sphere with radius 3 meters.
8. The volume of a sphere varies directly as the cube of its radius. When the radius of a sphere measures 3 meters, the volume is 36π cubic meters. Find the volume of a sphere with radius 1 meter.
9. With a fixed height, the volume of a cone is directly proportional to the square of the radius at the base. When the radius at the base measures 10 centimeters, the volume is 200 cubic centimeters. Determine the volume of the cone if the radius of the base is halved.
10. The distance d an object in free fall drops varies directly with the square of the time t that it has been falling. If an object in free fall drops 36 feet in 1.5 seconds, then how far will it have fallen in 3 seconds?

Answer

1. $\$382.50$
3. $\$26.65$
5. 2π
7. 36π square meters
9. 50 cubic centimeters

Exercise 15.6.1.9

Hooke's law suggests that the extension of a hanging spring is directly proportional to the weight attached to it. The constant of variation is called the spring constant.

Figure 15.6.1.9: Robert Hooke (1635-1703)

1. A hanging spring is stretched 5 inches when a 20-pound weight is attached to it. Determine its spring constant.
2. A hanging spring is stretched 3 centimeters when a 2-kilogram weight is attached to it. Determine the spring constant.
3. If a hanging spring is stretched 3 inches when a 2-pound weight is attached, how far will it stretch with a 5-pound weight attached?
4. If a hanging spring is stretched 6 centimeters when a 4-kilogram weight is attached to it, how far will it stretch with a 2-kilogram weight attached?

Answer

1. $\frac{1}{4}$
3. 7.5 inches

Exercise 15.6.1.10

The braking distance of an automobile is directly proportional to the square of its speed.

1. It takes 36 feet to stop a particular automobile moving at a speed of 30 miles per hour. How much breaking distance is required if the speed is 35 miles per hour?

2. After an accident, it was determined that it took a driver 80 feet to stop his car. In an experiment under similar conditions, it takes 45 feet to stop the car moving at a speed of 30 miles per hour. Estimate how fast the driver was moving before the accident.

Figure 15.6.1.10: Robert Boyle (1627-1691)

Answer

1. 49 feet

Exercise 15.6.1.11

Boyle's law states that if the temperature remains constant, the volume V of a given mass of gas is inversely proportional to the pressure p exerted on it.

1. A balloon is filled to a volume of 216 cubic inches on a diving boat under 1 atmosphere of pressure. If the balloon is taken underwater approximately 33 feet, where the pressure measures 2 atmospheres, then what is the volume of the balloon?
2. A balloon is filled to 216 cubic inches under a pressure of 3 atmospheres at a depth of 66 feet. What would the volume be at the surface, where the pressure is 1 atmosphere?
3. To balance a seesaw, the distance from the fulcrum that a person must sit is inversely proportional to his weight. If a 72-pound boy is sitting 3 feet from the fulcrum, how far from the fulcrum must a 54-pound boy sit to balance the seesaw?
4. The current I in an electrical conductor is inversely proportional to its resistance R. If the current is $\frac{1}{4}$ ampere when the resistance is 100 ohms, what is the current when the resistance is 150 ohms?
5. The amount of illumination I is inversely proportional to the square of the distance d from a light source. If 70 foot-candles of illumination is measured 2 feet away from a lamp, what level of illumination might we expect $\frac{1}{2}$ foot away from the lamp?
6. The amount of illumination I is inversely proportional to the square of the distance d from a light source. If 40 foot-candles of illumination is measured 3 feet away from a lamp, at what distance can we expect 10 foot-candles of illumination?
7. The number of men, represented by y, needed to lay a cobblestone driveway is directly proportional to the area A of the driveway and inversely proportional to the amount of time t allowed to complete the job. Typically, 3 men can lay $1,200$ square feet of cobblestone in 4 hours. How many men will be required to lay $2,400$ square feet of cobblestone in 6 hours?
8. The volume of a right circular cylinder varies jointly as the square of its radius and its height. A right circular cylinder with a 3-centimeter radius and a height of 4 centimeters has a volume of 36π cubic centimeters. Find a formula for the volume of a right circular cylinder in terms of its radius and height.
9. The period T of a pendulum is directly proportional to the square root of its length L. If the length of a pendulum is 1 meter, then the period is approximately 2 seconds. Approximate the period of a pendulum that is 0.5 meter in length.
10. The time t it takes an object to fall is directly proportional to the square root of the distance d it falls. An object dropped from 4 feet will take $\frac{1}{2}$ second to hit the ground. How long will it take an object dropped from 16 feet to hit the ground?

Answer

1. 108 cubic inches

3. 4 feet

5. $1,120$ foot-candles

7. 4 men

9. 1.4 seconds

Exercise 15.6.1.12

Newton's universal law of gravitation states that every particle of matter in the universe attracts every other particle with a force F that is directly proportional to the product of the masses m_1 and m_2 of the particles and inversely proportional to the square of the distance d between them. The constant of proportionality is called the gravitational constant.

Figure 15.6.1.11: Sir Isaac Newton (1643-1727)

1. If two objects with masses 50 kilograms and 100 kilograms are $\frac{1}{2}$ meter apart, then they produce approximately 1.34×10^{-6} newtons (N) of force. Calculate the gravitational constant.
2. Use the gravitational constant from the previous exercise to write a formula that approximates the force F in newtons between two masses m_1 and m_2, expressed in kilograms, given the distance d between them in meters.
3. Calculate the force in newtons between Earth and the Moon, given that the mass of the Moon is approximately 7.3×10^{22} kilograms, the mass of Earth is approximately 6.0×10^{24} kilograms, and the distance between them is on average 1.5×10^{11} meters.
4. Calculate the force in newtons between Earth and the Sun, given that the mass of the Sun is approximately 2.0×10^{30} kilograms, the mass of Earth is approximately 6.0×10^{24} kilograms, and the distance between them is on average 3.85×10^{8} meters.
5. If y varies directly as the square of x, then how does y change if x is doubled?
6. If y varies inversely as square of t, then how does y change if t is doubled?
7. If y varies directly as the square of x and inversely as the square of t, then how does y change if both x and t are doubled?

Answer

1. $6.7 \times 10^{-11} \mathrm{Nm}^2/\mathrm{kg}^2$
3. $1.98 \times 10^{20} \mathrm{N}$
5. y changes by a factor of 4
7. y remains unchanged

Footnotes

[37] Described by the formula $D = rt$, where the distance D is given as the product of the average rate r and the time t traveled at that rate.

[38] The rate at which a task can be performed.

[39] $\frac{1}{t_1} \cdot t + \frac{1}{t_2} \cdot t = 1$, where $\frac{1}{t_1}$ and $\frac{1}{t_2}$ are the individual work rates and t is the time it takes to complete the task working together.

[40] Describes two quantities x and y that are constant multiples of each other: $y = kx$.

[41] The nonzero multiple k, when quantities vary directly or inversely.

[42] Used when referring to the constant of variation.

[43] Used when referring to direct variation.

[44] Describes two quantities x and y, where one variable is directly proportional to the reciprocal of the other: $y = \frac{k}{x}$.

[45] Used when referring to inverse variation.

[46] Describes a quantity y that varies directly as the product of two other quantities x and z : $y = kxz$.

[47] Used when referring to joint variation.

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15.6E: Exercises

Solve Proportions

In the following exercises, solve each proportion.

1. $\frac{x}{56}=\frac{7}{8}$

Answer

$x=49$

2. $\frac{56}{72}=\frac{y}{9}$

3. $\frac{98}{154}=\frac{-7}{p}$

Answer

$p=-11$

4. $\frac{72}{156}=\frac{-6}{q}$

5. $\frac{a}{a+12}=\frac{4}{7}$

Answer

$a=16$

6. $\frac{b}{b-16}=\frac{11}{9}$

7. $\frac{m+90}{25}=\frac{m+30}{15}$

Answer

$m=60$

8. $\frac{n+10}{4}=\frac{40-n}{6}$

9. $\frac{2p+4}{8}=\frac{p+18}{6}$

Answer

$p=30$

10. $\frac{q-2}{2}=\frac{2q-7}{18}$

In the following exercises, solve.

11. Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds.

a. How many beats per minute is this?
b. Has Kevin met his target heart rate?

Answer

a. 162 beats per minute
b. yes

12. Jesse's car gets 30 miles per gallon of gas.
    a. If Las Vegas is 285 miles away, how many gallons of gas are needed to get there and then home?
    b. If gas is $3.09 per gallon, what is the total cost of the gas for the trip?
13. Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?

Answer

9 ml

14. A veterinarian prescribed Sunny, a 65-pound dog, an antibacterial medicine in case an infection emerges after her teeth were cleaned. If the dosage is 5 mg for every pound, how much medicine was Sunny given?
15. A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?

Answer

159 calories

16. One 12-ounce can of soda has 150 calories. If Josiah drinks the big 32-ounce size from the local mini-mart, how many calories does he get?
17. Kyra is traveling to Canada and will change $250 US dollars into Canadian dollars. At the current exchange rate, $1 US is equal to $1.3 Canadian. How many Canadian dollars will she get for her trip?

Answer

325 Canadian Dollars

18. Maurice is traveling to Mexico and needs to exchange $450 into Mexican pesos. If each dollar is worth 12.29 pesos, how many pesos will he get for his trip?
19. Ronald needs a morning breakfast drink that will give him at least 390 calories. Orange juice has 130 calories in one cup. How many cups does he need to drink to reach his calorie goal?

Answer

3 cups

20. Sonya drinks a 32-ounce energy drink containing 80 calories per 12 ounce. How many calories did she drink?
21. Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil's lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?

Answer

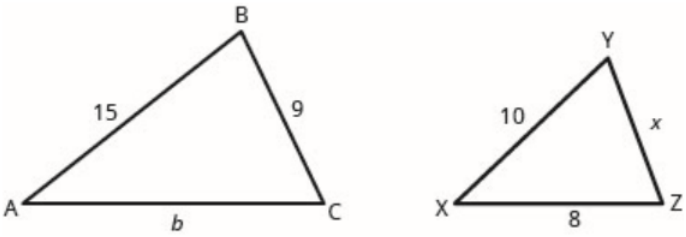
4 bags

22. An oatmeal cookie recipe calls for $\frac{1}{2}$ cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?

Solve Similar Figure Applications

In the following exercises, the triangles are similar. Find the length of the indicated side.

23.

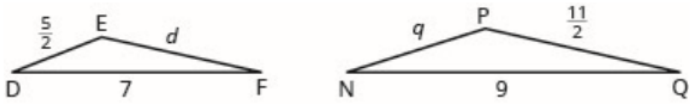


a. side x
b. side b

Answer

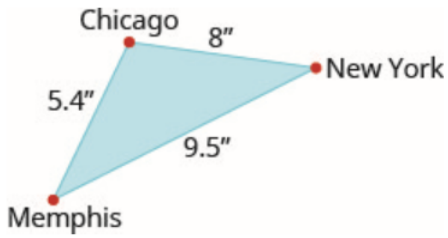
a. 6
b. 12

24.



a. side d
b. side q

In the following exercises, use the map shown. On the map, New York City, Chicago, and Memphis form a triangle. The actual distance from New York to Chicago is 800 miles.



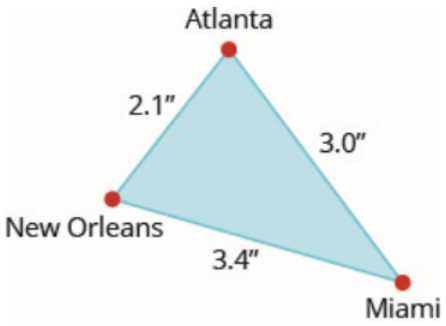
25. Find the actual distance from New York to Memphis.

Answer

950 miles

26. Find the actual distance from Chicago to Memphis.

In the following exercises, use the map shown. On the map, Atlanta, Miami, and New Orleans form a triangle. The actual distance from Atlanta to New Orleans is 420 miles.



27. Find the actual distance from New Orleans to Miami.

Answer

680 miles

28. Find the actual distance from Atlanta to Miami.

In the following exercises, answer each question.

29. A 2-foot-tall dog casts a 3-foot shadow at the same time a cat casts a one foot shadow. How tall is the cat?

Answer

$\frac{2}{3}$ foot (8 in.)

30. Larry and Tom were standing next to each other in the backyard when Tom challenged Larry to guess how tall he was. Larry knew his own height is 6.5 feet and when they measured their shadows, Larry's shadow was 8 feet and Tom's was 7.75 feet long. What is Tom's height?
31. The tower portion of a windmill is 212 feet tall. A six foot tall person standing next to the tower casts a seven-foot shadow. How long is the windmill's shadow?

Answer

247.3 feet

32. The height of the Statue of Liberty is 305 feet. Nikia, who is standing next to the statue, casts a 6-foot shadow and she is 5 feet tall. How long should the shadow of the statue be?

Solve Uniform Motion Applications

In the following exercises, solve the application problem provided.

33. Mary takes a sightseeing tour on a helicopter that can fly 450 miles against a 35-mph headwind in the same amount of time it can travel 702 miles with a 35-mph tailwind. Find the speed of the helicopter.

Answer

160 mph

34. A private jet can fly 1,210 miles against a 25-mph headwind in the same amount of time it can fly 1694 miles with a 25-mph tailwind. Find the speed of the jet.
35. A boat travels 140 miles downstream in the same time as it travels 92 miles upstream. The speed of the current is 6mph. What is the speed of the boat?

Answer

29 mph

36. Darrin can skateboard 2 miles against a 4-mph wind in the same amount of time he skateboards 6 miles with a 4-mph wind. Find the speed Darrin skateboards with no wind.
37. Jane spent 2 hours exploring a mountain with a dirt bike. First, she rode 40 miles uphill. After she reached the peak she rode for 12 miles along the summit. While going uphill, she went 5 mph slower than when she was on the summit. What was her rate along the summit?

Answer

30 mph

38. Laney wanted to lose some weight so she planned a day of exercising. She spent a total of 2 hours riding her bike and jogging. She biked for 12 miles and jogged for 6 miles. Her rate for jogging was 10 mph less than biking rate. What was her rate when jogging?

39. Byron wanted to try out different water craft. He went 62 miles downstream in a motor boat and 27 miles downstream on a jet ski. His speed on the jet ski was 10 mph faster than in the motor boat. Bill spent a total of 4 hours on the water. What was his rate of speed in the motor boat?

Answer

20 mph

40. Nancy took a 3-hour drive. She went 50 miles before she got caught in a storm. Then she drove 68 miles at 9 mph less than she had driven when the weather was good. What was her speed driving in the storm?
41. Chester rode his bike uphill 24 miles and then back downhill at 2 mph faster than his uphill. If it took him 2 hours longer to ride uphill than downhill, what was his uphill rate?

Answer

4 mph

42. Matthew jogged to his friend's house 12 miles away and then got a ride back home. It took him 2 hours longer to jog there than ride back. His jogging rate was 25 mph slower than the rate when he was riding. What was his jogging rate?
43. Hudson travels 1080 miles in a jet and then 240 miles by car to get to a business meeting. The jet goes 300 mph faster than the rate of the car, and the car ride takes 1 hour longer than the jet. What is the speed of the car?

Answer

60 mph

44. Nathan walked on an asphalt pathway for 12 miles. He walked the 12 miles back to his car on a gravel road through the forest. On the asphalt he walked 2 miles per hour faster than on the gravel. The walk on the gravel took one hour longer than the walk on the asphalt. How fast did he walk on the gravel.
45. John can fly his airplane 2800 miles with a wind speed of 50 mph in the same time he can travel 2400 miles against the wind. If the speed of the wind is 50 mph, find the speed of his airplane.

Answer

650 mph

46. Jim's speedboat can travel 20 miles upstream against a 3-mph current in the same amount of time it travels 22 miles downstream with a 3-mph current speed . Find the speed of the Jim's boat.
47. Hazel needs to get to her granddaughter's house by taking an airplane and a rental car. She travels 900 miles by plane and 250 miles by car. The plane travels 250 mph faster than the car. If she drives the rental car for 2 hours more than she rode the plane, find the speed of the car.

Answer

50 mph

48. Stu trained for 3 hours yesterday. He ran 14 miles and then biked 40 miles. His biking speed is 6 mph faster than his running speed. What is his running speed?
49. When driving the 9-hour trip home, Sharon drove 390 miles on the interstate and 150 miles on country roads. Her speed on the interstate was 15 more than on country roads. What was her speed on country roads?

Answer

50 mph

50. Two sisters like to compete on their bike rides. Tamara can go 4 mph faster than her sister, Samantha. If it takes Samantha 1 hours longer than Tamara to go 80 miles, how fast can Samantha ride her bike?
51. Dana enjoys taking her dog for a walk, but sometimes her dog gets away, and she has to run after him. Dana walked her dog for 7 miles but then had to run for 1 mile, spending a total time of 2.5 hours with her dog. Her running speed was 3 mph faster than her walking speed. Find her walking speed.

Answer

4.2 mph

52. Ken and Joe leave their apartment to go to a football game 45 miles away. Ken drives his car 30 mph faster Joe can ride his bike. If it takes Joe 2 hours longer than Ken to get to the game, what is Joe's speed?

Solve Work Applications

53. Mike, an experienced bricklayer, can build a wall in 3 hours, while his son, who is learning, can do the job in 6 hours. How long does it take for them to build a wall together?

Answer

2 hours

54. It takes Sam 4 hours to rake the front lawn while his brother, Dave, can rake the lawn in 2 hours. How long will it take them to rake the lawn working together?
55. Mia can clean her apartment in 6 hours while her roommate can clean the apartment in 5 hours. If they work together, how long would it take them to clean the apartment?

Answer

2 hours and 44 minutes

56. Brian can lay a slab of concrete in 6 hours, while Greg can do it in 4 hours. If Brian and Greg work together, how long will it take?
57. Josephine can correct her students test papers in 5 hours, but if her teacher's assistant helps, it would take them 3 hours. How long would it take the assistant to do it alone?

Answer

7 hours and 30 minutes

58. Washing his dad's car alone, eight year old Levi takes 2.5 hours. If his dad helps him, then it takes 1 hour. How long does it take Levi's dad to wash the car by himself?
59. At the end of the day Dodie can clean her hair salon in 15 minutes. Ann, who works with her, can clean the salon in 30 minutes. How long would it take them to clean the shop if they work together?

Answer

10 min

60. Ronald can shovel the driveway in 4 hours, but if his brother Donald helps it would take 2 hours. How long would it take Donald to shovel the driveway alone?

Solve Direct Variation Problems

In the following exercises, solve.

61. If y varies directly as x and $y = 14$, when $x = 3$. Find the equation that relates x and y.

Answer

$y = \frac{14}{3}x$

62. If a varies directly as b and $a = 16$, when $b = 4$. Find the equation that relates a and b.
63. If p varies directly as q and $p = 9$, when $q = 3$. Find the equation that relates p and q.

Answer

$p = 3.2q$

64. If v varies directly as w and $v = 8$, when $w = 12$. Find the equation that relates v and w.
65. The price, P, that Eric pays for gas varies directly with the number of gallons, g, he buys. It costs him \$50 to buy 20 gallons of gas.
    a. Write the equation that relates P and g.
    b. How much would 33 gallons cost Eric?

Answer

a. $P = 2.5g$
b. $\$82.50$

66. Joseph is traveling on a road trip. The distance, d, he travels before stopping for lunch varies directly with the speed, v, he travels. He can travel 120 miles at a speed of 60 mph.
    a. Write the equation that relates d and v.
    b. How far would he travel before stopping for lunch at a rate of 65 mph?
67. The mass of a liquid varies directly with its volume. A liquid with mass 16 kilograms has a volume of 2 liters.
    a. Write the equation that relates the mass to the volume.
    b. What is the volume of this liquid if its mass is 128 kilograms?

Answer

a. $m = 8v$
b. 16 liters

68. The length that a spring stretches varies directly with a weight placed at the end of the spring. When Sarah placed a 10-pound watermelon on a hanging scale, the spring stretched 5 inches.
    a. Write the equation that relates the length of the spring to the weight.
    b. What weight of watermelon would stretch the spring 6 inches?
69. The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 6 inch will support a maximum load of 108 pounds.
    a. Write the equation that relates the load to the diagonal of the cross-section.
    b. What load will a beam with a 10-inch diagonal support?

Answer

a. $L = 3d^2$
b. 300 pounds

70. The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.
    a. Write the equation that relates the area to the radius.
    b. What is the area of a personal pizza with a radius 4 inches?

Solve Inverse Variation Problems

In the following exercises, solve.

71. If y varies inversely with x and $y = 5$, when $x = 4$. Find the equation that relates x and y.

Answer

$$y = \frac{20}{x}$$

72. If p varies inversely with q and $p = 2$, when $q = 1$. Find the equation that relates p and q.
73. If v varies inversely with w and $v = 6$, when $w = 12$. Find the equation that relates v and w.

Answer

$v = \frac{3}{w}$

74. If a varies inversely with b and $a = 12$, when $b = 13$. Find the equation that relates a and b.

In the following exercises, write an inverse variation equation to solve the following problems.

75. The fuel consumption (mpg) of a car varies inversely with its weight. A Toyota Corolla weighs 2800 pounds getting 33 mpg on the highway.
    a. Write the equation that relates the mpg to the car's weight.
    b. What would the fuel consumption be for a Toyota Sequoia that weighs 5500 pounds?

Answer

a. $g = \frac{92,400}{w}$
b. 16.8 mpg

76. A car's value varies inversely with its age. Jackie bought a 10-year-old car for $2,400.
    a. Write the equation that relates the car's value to its age.
    b. What will be the value of Jackie's car when it is 15 years old?

77. The time required to empty a tank varies inversely as the rate of pumping. It took Ada 5 hours to pump her flooded basement using a pump that was rated at 200 gpm (gallons per minute).
    a. Write the equation that relates the number of hours to the pump rate.
    b. How long would it take Ada to pump her basement if she used a pump rated at 400 gpm?

Answer

1. $t = \frac{1000}{r}$
2. 2.5 hours

78. On a string instrument, the length of a string varies inversely as the frequency of its vibrations. An 11-inch string on a violin has a frequency of 400 cycles per second.
    a. Write the equation that relates the string length to its frequency.
    b. What is the frequency of a 10 inch string?

79. Paul, a dentist, determined that the number of cavities that develops in his patient's mouth each year varies inversely to the number of minutes spent brushing each night. His patient, Lori, had four cavities when brushing her teeth 30 seconds (0.5 minutes) each night.
    a. Write the equation that relates the number of cavities to the time spent brushing.
    b. How many cavities would Paul expect Lori to have if she had brushed her teeth for 2 minutes each night?

Answer

a. $c = \frac{2}{t}$
b. 1 cavity

80. Boyle's law states that if the temperature of a gas stays constant, then the pressure varies inversely to the volume of the gas. Braydon, a scuba diver, has a tank that holds 6 liters of air under a pressure of 220 psi.
    a. Write the equation that relates pressure to volume.
    b. If the pressure increases to 330 psi, how much air can Braydon's tank hold?

81. The cost of a ride service varies directly with the distance traveled. It costs $35 for a ride from the city center to the airport, 14 miles away.
    a. Write the equation that relates the cost, c, with the number of miles, m.
    b. What would it cost to travel 22 miles with this service?

Answer

a. $c = 2.5m$
b. $\$55$

82. The number of hours it takes Jack to drive from Boston to Bangor is inversely proportional to his average driving speed. When he drives at an average speed of 40 miles per hour, it takes him 6 hours for the trip.
    a. Write the equation that relates the number of hours, h, with the speed, s.
    b. How long would the trip take if his average speed was 75 miles per hour?

Writing Exercises

83. Marisol solves the proportion $\frac{144}{a} = \frac{9}{4}$ by 'cross multiplying,' so her first step looks like $4 \cdot 144 = 9 \cdot a$. Explain how this differs from the method of solution shown in Example 7.6.2.

Answer

Answers will vary.

84. Paula and Yuki are roommates. It takes Paula 3 hours to clean their apartment. It takes Yuki 4 hours to clean the apartment. The equation $\frac{1}{3} + \frac{1}{4} = \frac{1}{t}$ can be used to find t, the number of hours it would take both of them, working together, to clean their apartment. Explain how this equation models the situation.
85. In your own words, explain the difference between direct variation and inverse variation.

Answer

Answers will vary.

86. Make up an example from your life experience of inverse variation.

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15.E: Review Exercises

Simplify, Multiply, and Divide Rational Expressions

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

1. $\frac{5a+3}{3a-2}$

Answer

$a \neq \frac{2}{3}$

2. $\frac{b-7}{b^2-25}$

3. $\frac{5x^2y^2}{8y}$

Answer

$y \neq 0$

4. $\frac{x-3}{x^2-x-30}$

Simplify Rational Expressions

In the following exercises, simplify.

5. $\frac{18}{24}$

Answer

$\frac{3}{4}$

6. $\frac{9m^4}{18mn^3}$

7. $\frac{x^2+7x+12}{x^2+8x+16}$

Answer

$\frac{x+3}{x+4}$

8. $\frac{7v-35}{25-v^2}$

Multiply Rational Expressions

In the following exercises, multiply.

9. $\frac{5}{8} \cdot \frac{4}{15}$

Answer

$\frac{1}{6}$

10. $\frac{3xy^2}{8y^3} \cdot \frac{16y^2}{24x}$

11. $\frac{72x-12x^2}{8x+32} \cdot \frac{x^2+10x+24}{x^2-36}$

Answer

$\frac{-3x}{2}$

12. $\frac{6y^2-2y-10}{9-y^2} \cdot \frac{y^2-6y+9}{6y^2+29y-20}$

Divide Rational Expressions

In the following exercises, divide.

13. $\frac{x^2-4x-12}{x^2+8x+12} \div \frac{x^2-36}{3x}$

Answer

$\frac{3x}{(x+6)(x+6)}$

14. $\frac{y^2-16}{4} \div \frac{y^3-64}{2y^2+8y+32}$

15. $\frac{11+w}{w-9} \div \frac{121-w^2}{9-w}$

Answer

$\frac{1}{11+w}$

16. $\frac{3y^2-12y-63}{4y+3} \div \left(6y^2-42y\right)$

17. $\frac{\frac{c^2-64}{3c^2+26c+16}}{\frac{c^2-4c-32}{15c+10}}$

Answer

$\frac{5}{c+4}$

18. $\frac{8a^2+16a}{a-4} \cdot \frac{a^2+2a-24}{a^2+7a+10} \div \frac{2a^2-6a}{a+5}$

Multiply and Divide Rational Functions

19. Find $R(x)=f(x)\cdot g(x)$ where $f(x)=\frac{9x^2+9x}{x^2-3x-4}$ and $g(x)=\frac{x^2-16}{3x^2+12x}$.

Answer

$R(x)=3$

20. Find $R(x)=\frac{f(x)}{g(x)}$ where $f(x)=\frac{27x^2}{3x-21}$ and $g(x)=\frac{9x^2+54x}{x^2-x-42}$.

Add and Subtract Rational Expressions

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, perform the indicated operations.

21. $\frac{7}{15}+\frac{8}{15}$

Answer

1

22. $\frac{4a^2}{2a-1}-\frac{1}{2a-1}$

23. $\frac{y^2+10y}{y+5}+\frac{25}{y+5}$

Answer

$y+5$

24. $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

25. $\frac{x^2}{x-7}-\frac{3x+28}{x-7}$

Answer

$x+4$

26. $\frac{y^2}{y+11}-\frac{121}{y+11}$

27. $\frac{4q^2-q+3}{q^2+6q+5}-\frac{3q^2-q-6}{q^2+6q+5}$

Answer

$\frac{q-3}{q+5}$

28. $\frac{5t+4t+3}{t^2-25}-\frac{4t^2-8t-32}{t^2-25}$

Add and Subtract Rational Expressions Whose Denominators Are Opposites

In the following exercises, add and subtract.

29. $\frac{18w}{6w-1}+\frac{3w-2}{1-6w}$

Answer

$\frac{15w+2}{6w-1}$

30. $\frac{a^2+3a}{a^2-4}-\frac{3a-8}{4-a^2}$

31. $\frac{2b^2+3b-15}{b^2-49}-\frac{b^2+16b-1}{49-b^2}$

Answer

$\frac{3b-2}{b+7}$

32. $\frac{8y^2-10y+7}{2y-5}+\frac{2y^2+7y+2}{5-2y}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, find the LCD.

33. $\frac{7}{a^2-3a-10}, \frac{3a}{a^2-a-20}$

Answer

$(a+2)(a-5)(a+4)$

34. $\frac{6}{n^2-4}, \frac{2n}{n^2-4n+4}$

35. $\frac{5}{3p^2+17p-6}, \frac{2m}{3p^2-23p-8}$

Answer

$(3p+1)(p+6)(p+8)$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

36. $\frac{7}{5a}+\frac{3}{2b}$

37. $\frac{2}{c-2}+\frac{9}{c+3}$

Answer

$$\frac{11c-12}{(c-2)(c+3)}$$

38. $\frac{3x}{x^2-9}+\frac{5}{x^2+6x+9}$

39. $\frac{2x}{x^2+10x+24}+\frac{3x}{x^2+8x+16}$

Answer

$$\frac{5x^2+26x}{(x+4)(x+4)(x+6)}$$

40. $\frac{5q}{p^2q-p^2}+\frac{4q}{q^2-1}$

41. $\frac{3y}{y+2}-\frac{y+2}{y+8}$

Answer

$$\frac{2\left(y^2+10y-2\right)}{(y+2)(y+8)}$$

42. $\frac{-3w-15}{w^2+w-20}-\frac{w+2}{4-w}$

43. $\frac{7m+3}{m+2}-5$

Answer

$$\frac{2m-7}{m+2}$$

44. $\frac{n}{n+3}+\frac{2}{n-3}-\frac{n-9}{n^2-9}$

45. $\frac{8a}{a^2-64}-\frac{4}{a+8}$

Answer

$$\frac{4}{a-8}$$

46. $\frac{5}{12x^2y}+\frac{7}{20xy^3}$

Add and Subtract Rational Functions

In the following exercises, find $R(x)=f(x)+g(x)$ where $f(x)$ and $g(x)$ are given.

47. $f(x)=\frac{2x^2+12x-11}{x^2+3x-10}, g(x)=\frac{x+1}{2-x}$

Answer

$$R(x) = \frac{x+8}{x+5}$$

48. $f(x) = \frac{-4x+31}{x^2+x-30}, g(x) = \frac{5}{x+6}$

In the following exercises, find $R(x) = f(x) - g(x)$ where $f(x)$ and $g(x)$ are given.

49. $f(x) = \frac{4x}{x^2-121}, g(x) = \frac{2}{x-11}$

Answer

$$R(x) = \frac{2}{x+11}$$

50. $f(x) = \frac{7}{x+6}, g(x) = \frac{14x}{x^2-36}$

Simplify Complex Rational Expressions

Simplify a Complex Rational Expression by Writing It as Division

In the following exercises, simplify.

51. $\frac{\frac{7x}{x+2}}{\frac{14x^2}{x^2-4}}$

Answer

$$\frac{x-2}{2x}$$

52. $\frac{\frac{2}{5}+\frac{5}{6}}{\frac{1}{3}+\frac{1}{4}}$

53. $\frac{x-\frac{3x}{x+5}}{\frac{1}{x+5}+\frac{1}{x-5}}$

Answer

$$\frac{(x-8)(x-5)}{2}$$

54. $\frac{\frac{2}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

55. $\dfrac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$

Answer

$\frac{11}{8}$

56. $\dfrac{\frac{3}{a^2}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b^2}}$

57. $\dfrac{\frac{2}{z^2-49}+\frac{1}{z+7}}{\frac{9}{z+7}+\frac{12}{z-7}}$

Answer

$\frac{z-5}{21z+21}$

58. $\dfrac{\frac{3}{y^2-4y-32}}{\frac{2}{y-8}+\frac{1}{y+4}}$

Solve Rational Equations

Solve Rational Equations

In the following exercises, solve.

59. $\frac{1}{2}+\frac{2}{3}=\frac{1}{x}$

Answer

$x=\frac{6}{7}$

60. $1-\frac{2}{m}=\frac{8}{m^2}$

61. $\frac{1}{b-2}+\frac{1}{b+2}=\frac{3}{b^2-4}$

Answer

$b=\frac{3}{2}$

62. $\frac{3}{q+8}-\frac{2}{q-2}=1$

63. $\frac{v-15}{v^2-9v+18}=\frac{4}{v-3}+\frac{2}{v-6}$

Answer

no solution

64. $\frac{z}{12}+\frac{z+3}{3z}=\frac{1}{z}$

Solve Rational Equations that Involve Functions

65. For rational function, $f(x)=\frac{x+2}{x^2-6x+8}$,

a. Find the domain of the function
b. Solve $f(x)=1$
c. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x\neq 2$ and $x\neq 4$
b. $x=1, x=6$
c. $(1,1),(6,1)$

66. For rational function, $f(x)=\frac{2-x}{x^2+7x+10}$,

a. Solve $f(x)=2$
b. Find the points on the graph at this function value.

Solve a Rational Equation for a Specific Variable

In the following exercises, solve for the indicated variable.

67. $\frac{V}{l}=hw$ for l

Answer

$l=\frac{V}{hw}$

68. $\frac{1}{x}-\frac{2}{y}=5$ for y

69. $x=\frac{y+5}{z-7}$ for z

Answer

$z=\frac{y+5+7x}{x}$

70. $P=\frac{k}{V}$ for V

Solve Applications with Rational Equations

Solve Proportions

In the following exercises, solve.

71. $\frac{x}{4} = \frac{3}{5}$

Answer

$x = \frac{12}{5}$

72. $\frac{3}{y} = \frac{9}{5}$

73. $\frac{s}{s+20} = \frac{3}{7}$

Answer

$s = 15$

74. $\frac{t-3}{5} = \frac{t+2}{9}$

Solve Applications Using Proportions

In the following exercises, solve.

75. Rachael had a 21-ounce strawberry shake that has 739 calories. How many calories are there in a 32-ounce shake?

Answer

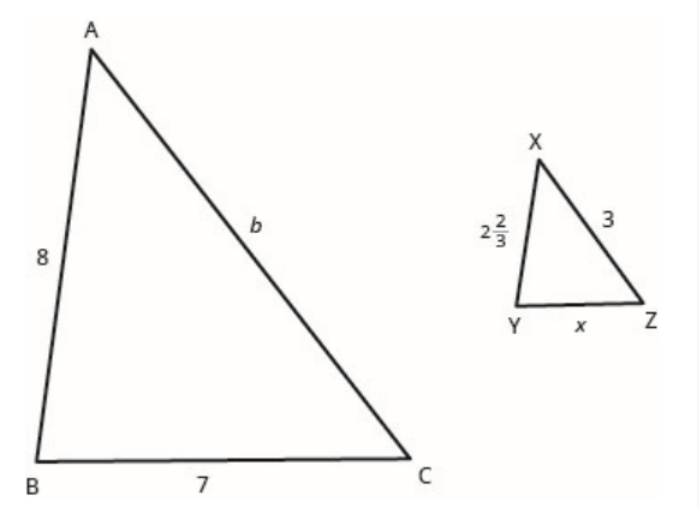
1161 calories

76. Leo went to Mexico over Christmas break and changed $525 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 16.25 Mexican pesos. How many Mexican pesos did he get for his trip?

Solve Similar Figure Applications

In the following exercises, solve.

77. ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure. Find the lengths of the third sides.

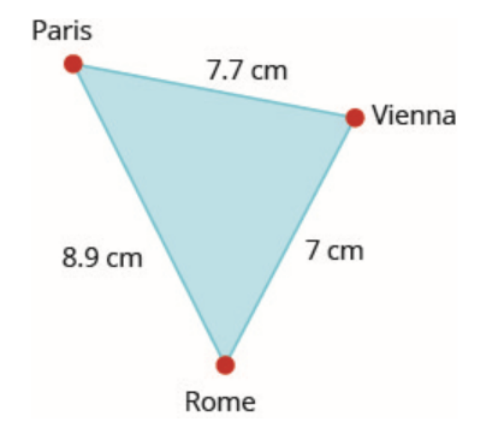


Answer

$b = 9;\ x = 2\frac{1}{3}$

78. On a map of Europe, Paris, Rome, and Vienna form a triangle whose sides are shown in the figure below. If the actual distance from Rome to Vienna is 700 miles, find the distance from

a. Paris to Rome
b. Paris to Vienna



79. Francesca is 5.75 feet tall. Late one afternoon, her shadow was 8 feet long. At the same time, the shadow of a nearby tree was 32 feet long. Find the height of the tree.

Answer

23 feet

80. The height of a lighthouse in Pensacola, Florida is 150 feet. Standing next to the statue, 5.5-foot-tall Natasha cast a 1.1-foot shadow. How long would the shadow of the lighthouse be?

Solve Uniform Motion Applications

In the following exercises, solve.

81. When making the 5-hour drive home from visiting her parents, Lolo ran into bad weather. She was able to drive 176 miles while the weather was good, but then driving 10 mph slower, went 81 miles when it turned bad. How fast did she drive when the weather was bad?

Answer

45 mph

82. Mark is riding on a plane that can fly 490 miles with a tailwind of 20 mph in the same time that it can fly 350 miles against a tailwind of 20 mph. What is the speed of the plane?

83. Josue can ride his bicycle 8 mph faster than Arjun can ride his bike. It takes Luke 3 hours longer than Josue to ride 48 miles. How fast can John ride his bike?

Answer

16 mph

84. Curtis was training for a triathlon. He ran 8 kilometers and biked 32 kilometers in a total of 3 hours. His running speed was 8 kilometers per hour less than his biking speed. What was his running speed?

Solve Work Applications

In the following exercises, solve.

85. Brandy can frame a room in 1 hour, while Jake takes 4 hours. How long could they frame a room working together?

Answer

$\frac{4}{5}$ hour

86. Prem takes 3 hours to mow the lawn while her cousin, Barb, takes 2 hours. How long will it take them working together?

87. Jeffrey can paint a house in 6 days, but if he gets a helper he can do it in 4 days. How long would it take the helper to paint the house alone?

Answer

12 days

88. Marta and Deb work together writing a book that takes them 90 days. If Sue worked alone it would take her 120 days. How long would it take Deb to write the book alone?

Solve Direct Variation Problems

In the following exercises, solve.

89. If y varies directly as x when $y = 9$ and $x = 3$, find x when $y = 21$.

Answer

7

90. If y varies inversely as x when $y = 20$ and $x = 2$ find y when $x = 4$.

91. Vanessa is traveling to see her fiancé. The distance, d, varies directly with the speed, v, she drives. If she travels 258 miles driving 60 mph, how far would she travel going 70 mph?

Answer

301 mph

92. If the cost of a pizza varies directly with its diameter, and if an 8” diameter pizza costs $12, how much would a 6” diameter pizza cost?

93. The distance to stop a car varies directly with the square of its speed. It takes 200 feet to stop a car going 50 mph. How many feet would it take to stop a car going 60 mph?

Answer

288 feet

Solve Inverse Variation Problems

In the following exercises, solve.

94. If m varies inversely with the square of n, when $m = 4$ and $n = 6$ find m when $n = 2$.

95. The number of tickets for a music fundraiser varies inversely with the price of the tickets. If Madelyn has just enough money to purchase 12 tickets for $6, how many tickets can Madelyn afford to buy if the price increased to $8?

Answer

97 tickets

96. On a string instrument, the length of a string varies inversely with the frequency of its vibrations. If an 11-inch string on a violin has a frequency of 360 cycles per second, what frequency does a 12-inch string have?

Solve Rational Inequalities

Solve Rational Inequalities

In the following exercises, solve each rational inequality and write the solution in interval notation.

97. $\frac{x-3}{x+4} \leq 0$

Answer

$(-4, 3]$

98. $\frac{5x}{x-2} > 1$

99. $\frac{3x-2}{x-4} \leq 2$

Answer

$[-6, 4)$

100. $\frac{1}{x^2-4x-12} < 0$

101. $\frac{1}{2} - \frac{4}{x^2} \geq \frac{1}{x}$

Answer

$(-\infty, -2] \cup [4, \infty)$

102. $\dfrac{4}{x-2} < \dfrac{3}{x+1}$

Solve an Inequality with Rational Functions

In the following exercises, solve each rational function inequality and write the solution in interval notation

103. Given the function, $R(x) = \dfrac{x-5}{x-2}$, find the values of x that make the function greater than or equal to 0.

Answer

$(-\infty, 2) \cup [5, \infty)$

104. Given the function, $R(x) = \dfrac{x+1}{x+3}$, find the values of x that make the function greater than or equal to 0.

105. The function $C(x) = 150x + 100,000$ represents the cost to produce x, number of items. Find

a. The average cost function, $c(x)$
b. How many items should be produced so that the average cost is less than $160.

Answer

a. $c(x) = \dfrac{150x + 100000}{x}$
b. More than 10,000 items must be produced to keep the average cost below $160 per item.

106. Tillman is starting his own business by selling tacos at the beach. Accounting for the cost of his food truck and ingredients for the tacos, the function $C(x) = 2x + 6,000$ represents the cost for Tillman to produce x, tacos. Find

a. The average cost function, $c(x)$ for Tillman's Tacos
b. How many tacos should Tillman produce so that the average cost is less than $4.

Practice Test

In the following exercises, simplify.

1. $\dfrac{4a^2b}{12ab^2}$

Answer

$\dfrac{a}{3b}$

2. $\dfrac{6x-18}{x^2-9}$

In the following exercises, perform the indicated operation and simplify.

3. $\dfrac{4x}{x+2} \cdot \dfrac{x^2+5x+6}{12x^2}$

Answer

$\frac{x+3}{3x}$

4. $\frac{2y^2}{y^2-1} \div \frac{y^3-y^2+y}{y^3-1}$

5. $\frac{6x^2-x+20}{x^2-81} - \frac{5x^2+11x-7}{x^2-81}$

Answer

$\frac{x-3}{x+9}$

6. $\frac{-3a}{3a-3} + \frac{5a}{a^2+3a-4}$

7. $\frac{2n^2+8n-1}{n^2-1} - \frac{n^2-7n-1}{1-n^2}$

Answer

$\frac{3n-2}{n-1}$

8. $\frac{10x^2+16x-7}{8x-3} + \frac{2x^2+3x-1}{3-8x}$

9. $\frac{\frac{1}{m}-\frac{1}{n}}{\frac{1}{n}+\frac{1}{m}}$

Answer

$\frac{n-m}{m+n}$

In the following exercises, solve each equation.

10. $\frac{1}{x} + \frac{3}{4} = \frac{5}{8}$

11. $\frac{1}{z-5} + \frac{1}{z+5} = \frac{1}{z^2-25}$

Answer

$z = \frac{1}{2}$

12. $\frac{z}{2z+8} - \frac{3}{4z-8} = \frac{3z^2-16z-16}{8z^2+2z-64}$

In the following exercises, solve each rational inequality and write the solution in interval notation.

13. $\frac{6x}{x-6} \leq 2$

Answer

$[-3, 6)$

14. $\dfrac{2x+3}{x-6} > 1$

15. $\dfrac{1}{2} + \dfrac{12}{x^2} \geq \dfrac{5}{x}$

Answer

$(-\infty, 0) \cup (0, 4] \cup [6, \infty)$

In the following exercises, find $R(x)$ given $f(x) = \dfrac{x-4}{x^2-3x-10}$ and $g(x) = \dfrac{x-5}{x^2-2x-8}$.

16. $R(x) = f(x) - g(x)$

17. $R(x) = f(x) \cdot g(x)$

Answer

$R(x) = \dfrac{1}{(x+2)(x+2)}$

18. $R(x) = f(x) \div g(x)$

19. Given the function, $R(x) = \dfrac{2}{2x^2+x-15}$, find the values of x that make the function less than or equal to 0.

Answer

$(2, 5]$

In the following exercises, solve.

20. If y varies directly with x, and $x = 5$ when $y = 30$, find x when $y = 42$.

21. If y varies inversely with the square of x and $x = 3$ when $y = 9$, find y when $x = 4$.

Answer

$y = \dfrac{81}{16}$

22. Matheus can ride his bike for 30 miles with the wind in the same amount of time that he can go 21 miles against the wind. If the wind's speed is 6 mph, what is Matheus' speed on his bike?

23. Oliver can split a truckload of logs in 8 hours, but working with his dad they can get it done in 3 hours. How long would it take Oliver's dad working alone to split the logs?

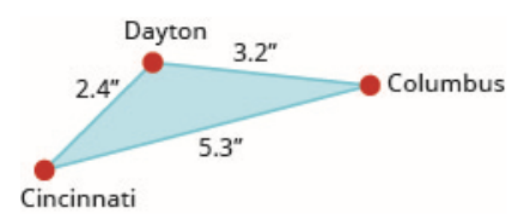
Answer

Oliver's dad would take $4\dfrac{4}{5}$ hours to split the logs himself.

24. The volume of a gas in a container varies inversely with the pressure on the gas. If a container of nitrogen has a volume of 29.5 liters with 2000 psi, what is the volume if the tank has a 14.7 psi rating? Round to the nearest whole number.

25.

The cities of Dayton, Columbus, and Cincinnati form a triangle in southern Ohio. The diagram gives the map distances between these cities in inches.



The actual distance from Dayton to Cincinnati is 48 miles. What is the actual distance between Dayton and Columbus?

Answer

The distance between Dayton and Columbus is 64 miles.

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CHAPTER OVERVIEW

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16.1: Relations and Functions

Learning Objectives

By the end of this section, you will be able to:

- Find the domain and range of a relation
- Determine if a relation is a function
- Find the value of a function

Before you get started, take this readiness quiz.

1. Evaluate $3x-5$ when $x=-2$.
   If you missed this problem, review [link].
2. Evaluate $2x^2-x-3$ when $x=a$.
   If you missed this problem, review [link].
3. Simplify: $7x-1-4x+5$.
   If you missed this problem, review [link].

Find the Domain and Range of a Relation

As we go about our daily lives, we have many data items or quantities that are paired to our names. Our social security number, student ID number, email address, phone number and our birthday are matched to our name. There is a relationship between our name and each of those items.

When your professor gets her class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a **relation**.

$$(\text{Student name},\ \text{Student ID }\#)$$

The set of all the names of the students in the class is called the **domain** of the relation and the set of all student ID numbers paired with these students is the range of the relation.

There are many similar situations where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation.

Definition: Relation

A **relation** is any set of ordered pairs, (x,y). All the x-values in the ordered pairs together make up the **domain**. All the y-values in the ordered pairs together make up the **range**.

Example 16.1.1

For the relation $(1,1),(2,4),(3,9),(4,16),(5,25)$:

a. Find the domain of the relation.
b. Find the range of the relation.

Answer

$\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$

ⓐ The domain is the set of all x-values of the relation. $\{1,2,3,4,5\}$

ⓑ The range is the set of all y-values of the relation. $\{1,4,9,16,25\}$

Example 16.1.2

For the relation $\{(1,1),(2,8),(3,27),(4,64),(5,125)\}$

a. Find the domain of the relation.
b. Find the range of the relation.

Answer a

$\{1,2,3,4,5\}$

Answer b

$\{1,8,27,64,125\}$

Example 16.1.3

For the relation $\{(1,3),(2,6),(3,9),(4,12),(5,15)\}$

a. Find the domain of the relation.
b. Find the range of the relation.

Answer a

$\{1,2,3,4,5\}$

Answer b

$\{3,6,9,12,15\}$

MAPPING

A **mapping** is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

Example 16.1.4

Use the **mapping** of the relation shown to

a. list the ordered pairs of the relation,
b. find the domain of the relation, and
c. find the range of the relation.

Answer

ⓐ The arrow shows the matching of the person to their birthday. We create ordered pairs with the person's name as the *x*-value and their birthday as the *y*-value.

{(Alison, April 25), (Penelope, May 23), (June, August 2), (Gregory, September 15), (Geoffrey, January 12), (Lauren, May 10), (Stephen, July 24), (Alice, February 3), (Liz, August 2), (Danny, July 24)}

ⓑ The domain is the set of all x-values of the relation.

{Alison, Penelope, June, Gregory, Geoffrey, Lauren, Stephen, Alice, Liz, Danny}

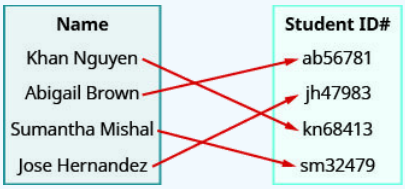
ⓒ The range is the set of all y-values of the relation.

{January 12, February 3, April 25, May 10, May 23, July 24, August 2, September 15}

Example 16.1.5

Use the mapping of the relation shown to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ (Khanh Nguyen, kn68413), (Abigail Brown, ab56781), (Sumantha Mishal, sm32479), (Jose Hern and ez, jh47983)

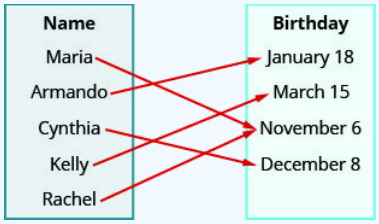
ⓑ {Khanh Nguyen, Abigail Brown, Sumantha Mishal, Jose Hern and ez}

ⓒ {kn68413, ab56781, sm32479, jh47983}

Example 16.1.6

Use the mapping of the relation shown to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ (Maria, November 6), (Arm and o, January 18), (Cynthia, December 8), (Kelly, March 15), (Rachel, November 6)

ⓑ {Maria, Arm and o, Cynthia, Kelly, Rachel}

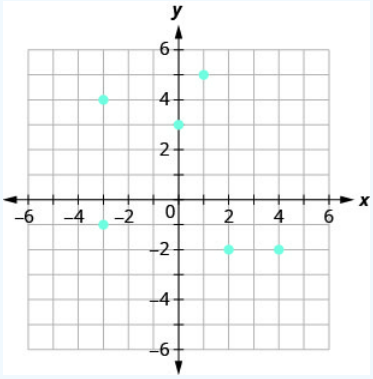
ⓒ{November 6, January 18, December 8, March 15}

A graph is yet another way that a relation can be represented. The set of ordered pairs of all the points plotted is the relation. The set of all x-coordinates is the domain of the relation and the set of all y-coordinates is the range. Generally we write the numbers in ascending order for both the domain and range.

Example 16.1.7

Use the graph of the relation to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ The ordered pairs of the relation are:

$$\{(1,5),(-3,-1),(4,-2),(0,3),(2,-2),(-3,4)\}.$$

ⓑ The domain is the set of all x-values of the relation: $\{-3,0,1,2,4\}$

Notice that while -3 repeats, it is only listed once.

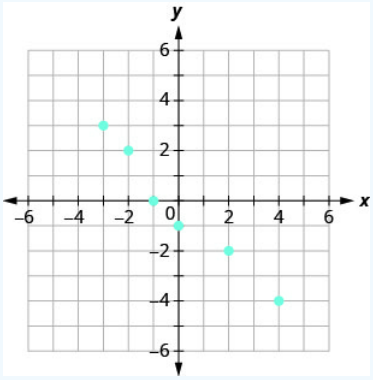
ⓒ The range is the set of all y-values of the relation: $\{-2,-1,3,4,5\}$

Notice that while -2 repeats, it is only listed once.

Example 16.1.8

Use the graph of the relation to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

a. $(-3,3),(-2,2),(-1,0),(0,-1),(2,-2),(4,-4)$

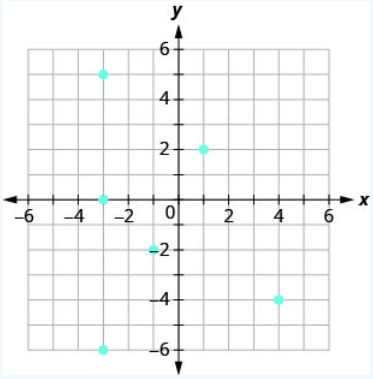
b. $\{-3,-2,-1,0,2,4\}$

c. $\{3, 2, 0, -1, -2, -4\}$

Example 16.1.9

Use the graph of the relation to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ $(-3, 0), (-3, 5), (-3, -6), (-1, -2), (1, 2), (4, -4)$
ⓑ $\{-3, -1, 1, 4\}$
ⓒ $\{-6, 0, 5, -2, 2, -4\}$

Determine if a Relation is a Function

A special type of relation, called a **function**, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

Definition: Function

A **function** is a relation that assigns to each element in its domain exactly one element in the range.

The birthday example from Example helps us understand this definition. Every person has a birthday but no one has two birthdays. It is okay for two people to share a birthday. It is okay that Danny and Stephen share July 24th as their birthday and that June and Liz share August 2nd. Since each person has exactly one birthday, the relation in Example is a function.

The relation shown by the graph in Example includes the ordered pairs $(-3, -1)$ and $(-3, 4)$. Is that okay in a function? No, as this is like one person having two different birthdays.

Example 16.1.10

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

a. $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$
b. $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

Answer

ⓐ $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

(i) Each x-value is matched with only one y-value. So this relation is a function.

(ii) The domain is the set of all x-values in the relation.
The domain is: $\{-3, -2, -1, 0, 1, 2, 3\}$

(iii) The range is the set of all y-values in the relation. Notice we do not list range values twice.
The range is: $\{27, 8, 1, 0\}$

ⓑ $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

(i) The x-value 9 is matched with two y-values, both 3 and -3. So this relation is not a function.

(ii) The domain is the set of all x-values in the relation. Notice we do not list domain values twice.
The domain is: $\{0, 1, 2, 4, 9\}$

(iii) The range is the set of all y-values in the relation.
The range is: $\{-3, -2, -1, 0, 1, 2, 3\}$

Example 16.1.11

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the function.

a. $\{(-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6)\}$
b. $\{(8, -4), (4, -2), (2, -1), (0, 0), (2, 1), (4, 2), (8, 4)\}$

Answer

ⓐ Yes; $\{-3, -2, -1, 0, 1, 2, 3\}$
$\{-6, -4, -2, 0, 2, 4, 6\}$
ⓑ No; $\{0, 2, 4, 8\}$
$\{-4, -2, -1, 0, 1, 2, 4\}$

Example 16.1.12

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

a. $\{(27, -3), (8, -2), (1, -1), (0, 0), (1, 1), (8, 2), (27, 3)\}$
b. $\{(7, -3), (-5, -4), (8, -0), (0, 0), (-6, 4), (-2, 2), (-1, 3)\}$

Answer

ⓐ No; $\{0, 1, 8, 27\}$
$\{-3, -2, -1, 0, 2, 2, 3\}$
ⓑ Yes; $\{7, -5, 8, 0, -6, -2, -1\}$
$\{-3, -4, 0, 4, 2, 3\}$

Example 16.1.13

Use the mapping to

a. determine whether the relation is a function
b. find the domain of the relation
c. find the range of the relation.

NAME	PHONE

Lydia	321-549-3327 427-658-2314
Eugene	321-964-7324
Janet	684-358-7961
Rick	
Marty	684-369-7231 798-367-8541

Answer

ⓐ Both Lydia and Marty have two phone numbers. So each x-value is not matched with only one y-value. So this relation is not a function.

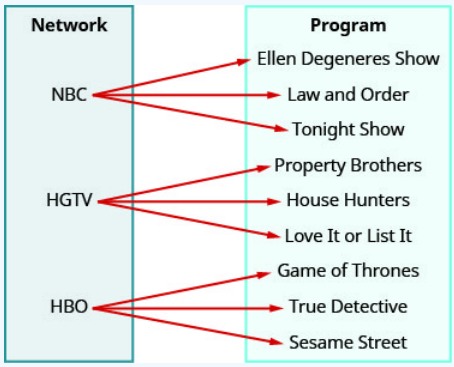
ⓑ The domain is the set of all x-values in the relation. The domain is: {Lydia, Eugene, Janet, Rick, Marty}

ⓒ The range is the set of all y-values in the relation. The range is:

$\{321-549-3327, 427-658-2314, 321-964-7324, 684-358-7961, 684-369-7231, 798-367-8541\}$

Example 16.1.14

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



Answer

ⓐ no ⓑ {NBC, HGTV, HBO} ⓒ {Ellen Degeneres Show, Law and Order, Tonight Show, Property Brothers, House Hunters, Love it or List it, Game of Thrones, True Detective, Sesame Street}

Example 16.1.15

Use the mapping to

a. determine whether the relation is a function
b. find the domain of the relation
c. find the range of the relation.

Neal	123-567-4839 work 231-378-5941 cell
Krystal	743-469-9731 cell
Kelvin	567-534-2970 work
George	684-369-7231 cell
Christa	684-369-7231 cell
Mike	639-847-6971 cell

Answer

ⓐ No ⓑ {Neal, Krystal, Kelvin, George, Christa, Mike} ⓒ {123-567-4839 work, 231-378-5941 cell, 743-469-9731 cell, 567-534-2970 work, 684-369-7231 cell, 798-367-8541 cell, 639-847-6971 cell}

In algebra, more often than not, functions will be represented by an equation. It is easiest to see if the equation is a function when it is solved for *y*. If each value of *x* results in only one value of *y*, then the equation defines a function.

? Example 16.1.16

Determine whether each equation is a function.

a. $2x+y=7$
b. $y=x^2+1$
c. $x+y^2=3$

Answer

ⓐ $2x+y=7$

For each value of *x*, we multiply it by -2 and then add 7 to get the *y*-value

	$y=-2x+7$
For example, if $x=3$:	$y=-2\cdot 3+7$
	$y=1$

We have that when $x=3$, then $y=1$. It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓑ $y=x^2+1$

For each value of *x*, we square it and then add 1 to get the *y*-value.

	$y=x^2+1$
For example, if $x=2$:	$y=2^2+1$
	$y=5$

We have that when $x=2$, then $y=5$. It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓒ

	$x+y^2=3$
Isolate the y term.	$y^2=-x+3$
Let's substitute $x=2$.	$y^2=-2+3$
	$y^2=1$
This give us two values for y.	$y=1\ y=-1$

We have shown that when $x=2$, then $y=1$ and $y=-1$. It would work similarly for any value of x. Since each value of x does not corresponds to only one value of y the equation does not define a function.

Example 16.1.17

Determine whether each equation is a function.

a. $4x+y=-3$
b. $x+y^2=1$
c. $y-x^2=2$

Answer

ⓐ yes ⓑ no ⓒ yes

Example 16.1.18

Determine whether each equation is a function.

a. $x+y^2=4$
b. $y=x^2-7$
c. $y=5x-4$

Answer

ⓐ no ⓑ yes ⓒ yes

Find the Value of a Function

It is very convenient to name a function and most often we name it f, g, h, F, G, or H. In any function, for each x-value from the domain we get a corresponding y-value in the range. For the function f, we write this range value y as $f(x)$. This is called **function notation** and is read f of x or the value of f at x. In this case the parentheses does not indicate multiplication.

Definition: Function Notation

For the function $y=f(x)$

f is the name of the function
x is the domain value
$f(x)$ is the range value y corresponding to the value x

We read $f(x)$ as f of x or the value of f at x.

We call x the independent variable as it can be any value in the domain. We call y the dependent variable as its value depends on x.

INDEPENDENT AND DEPENDENT VARIABLES

For the function $y = f(x)$,

x is the independent variable as it can be any value in the domain
y the dependent variable as its value depends on x

Much as when you first encountered the variable *x*, function notation may be rather unsettling. It seems strange because it is new. You will feel more comfortable with the notation as you use it.

Let's look at the equation $y = 4x - 5$. To find the value of *y* when $x = 2$, we know to substitute $x = 2$ into the equation and then simplify.

	$y = 4x - 5$
Let x=2.	$y = 4 \cdot 2 - 5$
	$y = 3$

The value of the function at $x = 2$ is 3.

We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f(x) = 4x - 5$. To find the value when $x = 2$, we write:

	$f(x) = 4x - 5$
Let x=2.	$f(2) = 4 \cdot 2 - 5$
	$f(2) = 3$

The value of the function at $x = 2$ is 3.

This process of finding the value of $f(x)$ for a given value of *x* is called *evaluating the function.*

Example 16.1.19

For the function $f(x) = 2x^2 + 3x - 1$, evaluate the function.

a. $f(3)$
b. $f(-2)$
c. $f(a)$

Answer

ⓐ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(3)$, substitute 3 for *x*.	$f(3) = 2(3)^2 + 3 \cdot 3 - 1$
Simplify.	$f(3) = 2 \cdot 9 + 3 \cdot 3 - 1$
	$f(3) = 18 + 9 - 1$
	$f(3) = 26$

ⓑ

		$f(x) = 2x^2 + 3x - 1$
To evaluate $f(-2)$, substitute -2 for *x*.		$f(-2) = 2(-2)^2 + 3(-2) - 1$

Simplify.		$f(-2) = 2 \cdot 4 + (-6) - 1$
		$f(-2) = 8 + (-6) - 1$
		$f(-2) = 1$

ⓒ

	$f(x) = 2x^2 + 3x - 1$
To evaluate f(a),f(a), substitute a for x.	$f(a) = 2(a)^2 + 3 \cdot a - 1$
Simplify.	$f(a) = 2a^2 + 3a - 1$

Example 16.1.20

For the function $f(x) = 3x^2 - 2x + 1$, evaluate the function.

a. $f(3)$
b. $f(-1)$
c. $f(t)$

Answer

ⓐ $f(3) = 22$ ⓑ $f(-1) = 6$ ⓒ $f(t) = 3t^2 - 2t - 1$

Example 16.1.21

For the function $f(x) = 2x^2 + 4x - 3$, evaluate the function.

a. $f(2)$
b. $f(-3)$
c. $f(h)$

Answer

ⓐ $(2) = 13$ ⓑ $f(-3) = 3$
ⓒ $f(h) = 2h2 + 4h - 3$

In the last example, we found $f(x)$ for a constant value of x. In the next example, we are asked to find $g(x)$ with values of x that are variables. We still follow the same procedure and substitute the variables in for the x.

Example 16.1.22

For the function $g(x) = 3x - 5$, evaluate the function.

a. $g(h^2)$
b. $g(x + 2)$
c. $g(x) + g(2)$

Answer

ⓐ

			$g(x) = 3x - 5$
To evaluate $g(h^2)$, substitute h^2 for x.			$g(h^2) = 3h^2 - 5$

			$g(h^2) = 3h^2 - 5$

ⓑ

	$g(x) = 3x - 5$
To evaluate $g(x+2)$, substitute $x+2$ for x.	$g(x+2) = 3(x+2) - 5$
Simplify.	$g(x+2) = 3x + 6 - 5$
	$g(x+2) = 3x + 1$

ⓒ

		$g(x) = 3x - 5$
To evaluate $g(x) + g(2)$, first find $g(2)$.		$g(2) = 3 \cdot 2 - 5$
		$g(2) = 1$
Now find $g(x) + g(2)$		$g(x) + g(2) = \underbrace{3x - 5}_{g(x)} + \underbrace{1}_{g(2)}$
Simplify.		$g(x) + g(2) = 3x - 5 + 1$
		$g(x) + g(2) = 3x - 4$

Notice the difference between part ⓑ and ⓒ. We get $g(x+2) = 3x+1$ and $g(x) + g(2) = 3x - 4$. So we see that $g(x+2) \neq g(x) + g(2)$.

Example 16.1.23

For the function $g(x) = 4x - 7$, evaluate the function.

a. $g(m^2)$
b. $g(x-3)$
c. $g(x) - g(3)$

Answer

ⓐ $4m^2 - 7$ ⓑ $4x - 19$
ⓒ $x - 12$

Example 16.1.24

For the function $h(x) = 2x + 1$, evaluate the function.

a. $h(k^2)$
b. $h(x+1)$
c. $h(x) + h(1)$

Answer

ⓐ $2k^2 + 1$ ⓑ $2x + 3$
ⓒ $2x + 4$

Many everyday situations can be modeled using functions.

Example 16.1.25

The number of unread emails in Sylvia's account is 75. This number grows by 10 unread emails a day. The function $N(t) = 75 + 10t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(5)$. Explain what this result means.

Answer

ⓐ The number of unread emails is a function of the number of days. The number of unread emails, *N*, depends on the number of days, *t*. Therefore, the variable *N*, is the dependent variable and the variable tt is the independent variable.

ⓑ Find $N(5)$. Explain what this result means.

	$N(t) = 75 + 10t$
Substitute in t=5.t=5.	$N(5) = 75 + 10 \cdot 5$
Simplify.	$N(5) = 75 + 50$
	$N(5) = 125$

Since 5 is the number of days, $N(5)$, is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in the account.

Example 16.1.26

The number of unread emails in Bryan's account is 100. This number grows by 15 unread emails a day. The function $N(t) = 100 + 15t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(7)]$. Explain what this result means.

Answer

ⓐ *t* IND; *N* DEP ⓑ 205; the number of unread emails in Bryan's account on the seventh day.

Example 16.1.27

The number of unread emails in Anthony's account is 110. This number grows by 25 unread emails a day. The function $N(t) = 110 + 25t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(14)$. Explain what this result means.

Answer

ⓐ *t* IND; *N* DEP ⓑ 460; the number of unread emails in Anthony's account on the fourteenth day

Access this online resource for additional instruction and practice with relations and functions.

- Introduction to Functions

Key Concepts

- **Function Notation:** For the function $y = f(x)$
  - f is the name of the function
  - x is the domain value

- $f(x)$ is the range value y corresponding to the value x
  We read $f(x)$ as f of x or the value of f at x.

- **Independent and Dependent Variables:** For the function $y = f(x)$,
  - x is the independent variable as it can be any value in the domain
  - y is the dependent variable as its value depends on x

Glossary

domain of a relation

The domain of a relation is all the x-values in the ordered pairs of the relation.

function

A function is a relation that assigns to each element in its domain exactly one element in the range.

mapping

A mapping is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

range of a relation

The range of a relation is all the y-values in the ordered pairs of the relation.

relation

A relation is any set of ordered pairs,(x,y).(x,y). All the x-values in the ordered pairs together make up the domain. All the y-values in the ordered pairs together make up the range.

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16.1E: Exercises

Practice Makes Perfect

Find the Domain and Range of a Relation

In the following exercises, for each relation a. find the domain of the relation b. find the range of the relation.

1. $\{(1,4),(2,8),(3,12),(4,16),(5,20)\}$

Answer

a. $\{1,2,3,4,5\}$b. $\{4,8,12,16,20\}$

2. $\{(1,-2),(2,-4),(3,-6),(4,-8),(5,-10)\}$

3. $\{(1,7),(5,3),(7,9),(-2,-3),(-2,8)\}$

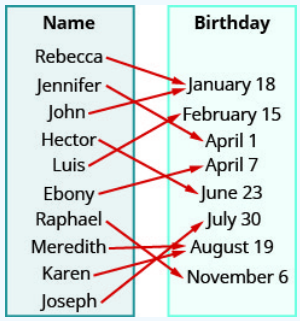
Answer

a. $\{1,5,7,-2\}$b. $\{7,3,9,-3,8\}$

4. $\{(11,3),(-2,-7),(4,-8),(4,17),(-6,9)\}$

In the following exercises, use the mapping of the relation to a. list the ordered pairs of the relation, b. find the domain of the relation, and c. find the range of the relation.

5.

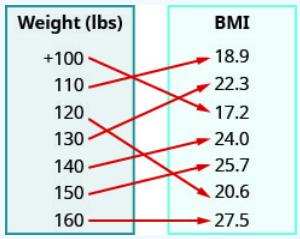


Answer

a. (Rebecca, January 18), (Jennifer, April 1), (John, January 18), (Hector, June 23), (Luis, February 15), (Ebony, April 7), (Raphael, November 6), (Meredith, August 19), (Karen, August 19), (Joseph, July 30)
b. {Rebecca, Jennifer, John, Hector, Luis, Ebony, Raphael, Meredith, Karen, Joseph}
c. {January 18, April 1, June 23, February 15, April 7, November 6, August 19, July 30}

6.

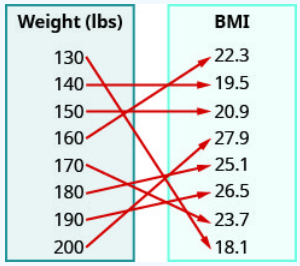
7. For a woman of height $5'4''$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5–24.9$is considered healthy.



Answer

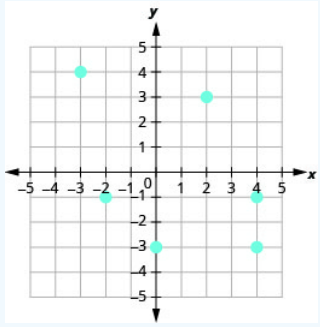
a. $(+100, 17.2), (110, 18.9), (120, 20.6), (130, 22.3), (140, 24.0), (150, 25.7), (160, 27.5)$ b. $\{+100, 110, 120, 130, 140, 150, 160\}$c. $\{17.2, 18.9, 20.6, 22.3, 24.0, 25.7, 27.5\}$

8. For a man of height $5'11''$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5–24.9$is considered healthy.



In the following exercises, use the graph of the relation to a. list the ordered pairs of the relation b. find the domain of the relation c. find the range of the relation.

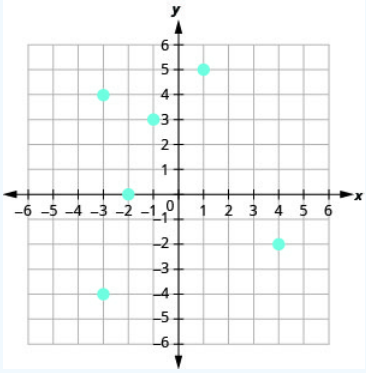
9.



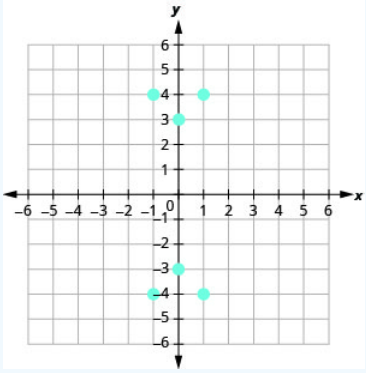
Answer

a. $(2, 3), (4, -3), (-2, -1), (-3, 4), (4, -1), (0, -3)$b. $\{-3, -2, 0, 2, 4\}$
c. $\{-3, -1, 3, 4\}$

10.



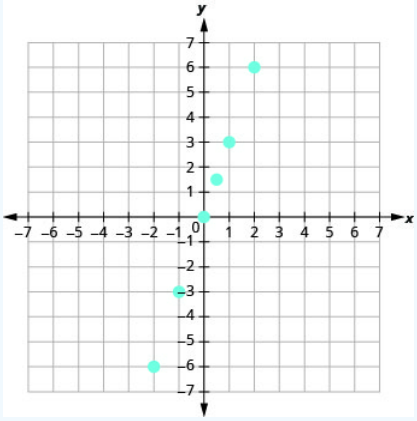
11.



Answer

a. $(1,4),(1,-4),(-1,4),(-1,-4),(0,3),(0,-3)$b. $\{-1,0,1\}$c. $\{-4,-3,3,4\}$

12.



Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to a. determine whether the relation is a function, b. find the domain of the relation, and c. find the range of the relation.

13. $\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

Answer

a. yes b. $\{-3,-2,-1,0,1,2,3\}$c. $\{9,4,1,0\}$

14. $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

15. $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

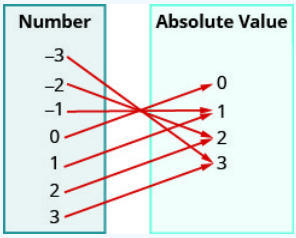
Answer

a. yes b. $\{-3,-2,-1,0,1,2,3\}$c. $\{0,1,8,27\}$

16. $\{(-3,-27),(-2,-8),(-1,-1),(0,0),(1,1),(2,8),(3,27)\}$

In the following exercises, use the mapping to a. determine whether the relation is a function, b. find the domain of the function, and c. find the range of the function.

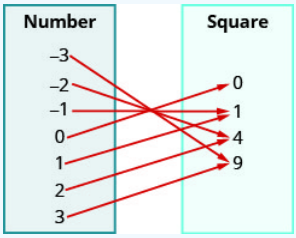
17.



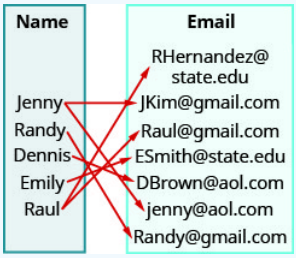
Answer

a. yes b. $\{-3,-2,-1,0,1,2,3\}$c. $\{0,1,2,3\}$

18.



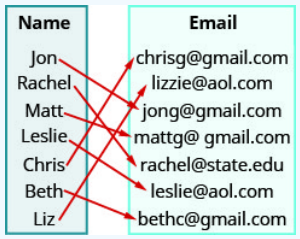
19.



Answer

a. no b. {Jenny, R and y, Dennis, Emily, Raul} c. {RHern and ez@state.edu, JKim@gmail.com, Raul@gmail.com, ESmith@state.edu, DBroen@aol.com, jenny@aol.cvom, R and y@gmail.com}

20.



In the following exercises, determine whether each equation is a function.

21. a. $2x+y=-3$
b. $y=x^2$
c. $x+y^2=-5$

Answer

a. yes b. yes c. no

22. a. $y=3x-5$
b. $y=x^3$
c. $2x+y^2=4$

23. a. $y-3x^3=2$
b. $x+y^2=3$
c. $3x-2y=6$

Answer

a. yes b. no c. yes

24. a. $2x-4y=8$
b. $-4=x^2-y$
c. $y^2=-x+5$

Find the Value of a Function

In the following exercises, evaluate the function: a. $f(2)$ b. $f(-1)$ c. $f(a)$.

25. $f(x)=5x-3$

Answer

a. $f(2)=7$ b. $f(-1)=-8$ c. $f(a)=5a-3$

26. $f(x)=3x+4$

27. $f(x)=-4x+2$

Answer

a. $f(2)=-6$ b. $f(-1)=6$ c. $f(a)=-4a+2$

28. $f(x)=-6x-3$

29. $f(x)=x^2-x+3$

Answer

a. $f(2)=5$ b. $f(-1)=5$
c. $f(a)=a^2-a+3$

30. $f(x)=x^2+x-2$

31. $f(x)=2x^2-x+3$

Answer

a. $f(2)=9$ b. $f(-1)=6$
c. $f(a)=2a^2-a+3$

32. $f(x)=3x^2+x-2$

In the following exercises, evaluate the function: a. $g(h^2)$ b. $g(x+2)$ c. $g(x)+g(2)$.

33. $g(x)=2x+1$

Answer

a. $g(h^2)=2h^2+1$
b. $g(x+2)=4x+5$
c. $g(x)+g(2)=2x+6$

34. $g(x)=5x-8$

35. $g(x)=-3x-2$

Answer

a. $g(h^2)=-3h^2-2$
b. $g(x+2)=-3x-8$
c. $g(x)+g(2)=-3x-10$

36. $g(x)=-8x+2$

37. $g(x)=3-x$

Answer

a. $g(h^2)=3-h^2$
b. $g(x+2)=1-x$
c. $g(x)+g(2)=4-x$

38. $g(x)=7-5x$

In the following exercises, evaluate the function.

39. $f(x)=3x^2-5x;\ f(2)$

Answer

2

40. $g(x) = 4x^2 - 3x$; $g(3)$

41. $F(x) = 2x^2 - 3x + 1$; $F(-1)$

Answer

6

42. $G(x) = 3x^2 - 5x + 2$; $G(-2)$

43. $h(t) = 2|t - 5| + 4$; $f(-4)$

Answer

22

44. $h(y) = 3|y - 1| - 3$; $h(-4)$

45. $f(x) = x + 2x - 1$; $f(2)$

Answer

4

46. $g(x) = x - 2x + 2$; $g(4)$

In the following exercises, solve.

47. The number of unwatched shows in Sylvia's DVR is 85. This number grows by 20 unwatched shows per week. The function $N(t) = 85 + 20t$ represents the relation between the number of unwatched shows, *N*, and the time, *t*, measured in weeks.

a. Determine the independent and dependent variable.

b. Find $N(4)$. Explain what this result means

Answer

a. *t* IND; *N* DEP
b. $N(4) = 165$ the number of unwatched shows in Sylvia's DVR at the fourth week.

48. Every day a new puzzle is downloaded into Ken's account. Right now he has 43 puzzles in his account. The function $N(t) = 43 + t$ represents the relation between the number of puzzles, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.

b. Find $N(30)$. Explain what this result means.

49. The daily cost to the printing company to print a book is modeled by the function $C(x) = 3.25x + 1500$ where *C* is the total daily cost and *x* is the number of books printed.

a. Determine the independent and dependent variable.

b. Find $N(0)$. Explain what this result means.

c. Find $N(1000)$. Explain what this result means.

Answer

a. x IND; C DEP
b. $N(0) = 1500$ the daily cost if no books are printed
c. $N(1000) = 4750$the daily cost of printing 1000 books

50. The daily cost to the manufacturing company is modeled by the function $C(x) = 7.25x + 2500$ where $C(x)$ is the total daily cost and x is the number of items manufactured.

a. Determine the independent and dependent variable.

b. Find $C(0)$. Explain what this result means.

c. Find $C(1000)$. Explain what this result means.

Writing Exercises

51. In your own words, explain the difference between a relation and a function.

52. In your own words, explain what is meant by domain and range.

53. Is every relation a function? Is every function a relation?

54. How do you find the value of a function?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the domain and range of a relation.			
determine if a relation is a function.			
find the value of a function.			

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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16.2: Basic Classes of Functions

We have studied the general characteristics of functions, so now let's examine some specific classes of functions. We begin by reviewing the basic properties of linear and quadratic functions, and then generalize to include higher-degree polynomials. By combining root functions with polynomials, we can define general algebraic functions and distinguish them from the transcendental functions we examine later in this chapter. We finish the section with examples of piecewise-defined functions and take a look at how to sketch the graph of a function that has been shifted, stretched, or reflected from its initial form.

Linear Functions and Slope

The easiest type of function to consider is a **linear function**. Linear functions have the form $f(x) = ax + b$, where a and b are constants. In Figure 16.2.1, we see examples of linear functions when a is positive, negative, and zero. Note that if $a > 0$, the graph of the line rises as x increases. In other words, $f(x) = ax + b$ is increasing on $(-\infty, \infty)$. If $a < 0$, the graph of the line falls as x increases. In this case, $f(x) = ax + b$ is decreasing on $(-\infty, \infty)$. If $a = 0$, the line is horizontal.

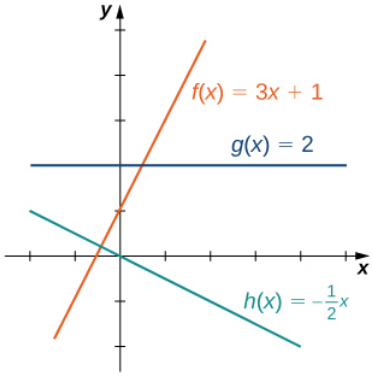


***Figure* 16.2.1:** *These linear functions are increasing or decreasing on (∞, ∞) and one function is a horizontal line.*

As suggested by Figure, the graph of any linear function is a line. One of the distinguishing features of a line is its slope. The **slope** is the change in y for each unit change in x. The slope measures both the steepness and the direction of a line. If the slope is positive, the line points upward when moving from left to right. If the slope is negative, the line points downward when moving from left to right. If the slope is zero, the line is horizontal. To calculate the slope of a line, we need to determine the ratio of the change in y versus the change in x. To do so, we choose any two points (x_1, y_1) and (x_2, y_2) on the line and calculate $\dfrac{y_2 - y_1}{x_2 - x_1}$. In Figure 16.2.2, we see this ratio is independent of the points chosen.

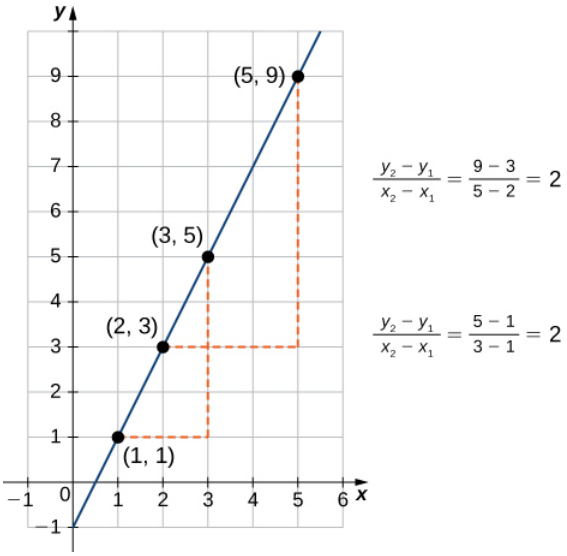


***Figure* 16.2.2**: *For any linear function, the slope $(y_2 - y_1)/(x_2 - x_1)$ is independent of the choice of points (x_1, y_1) and (x_2, y_2) on the line.*

Definition: Linear Functions

Consider line L passing through points (x_1, y_1) and (x_2, y_2). Let $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$ denote the changes in y and x,respectively. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \tag{16.2.1}$$

We now examine the relationship between slope and the formula for a linear function. Consider the linear function given by the formula $f(x) = ax + b$. As discussed earlier, we know the graph of a linear function is given by a line. We can use our definition of slope to calculate the slope of this line. As shown, we can determine the slope by calculating $(y_2 - y_1)/(x_2 - x_1)$ for any points (x_1, y_1) and (x_2, y_2) on the line. Evaluating the function f at $x = 0$, we see that $(0, b)$ is a point on this line. Evaluating this function at $x = 1$, we see that $(1, a + b)$ is also a point on this line. Therefore, the slope of this line is

$$\frac{(a+b) - b}{1 - 0} = a. \tag{16.2.2}$$

We have shown that the coefficient a is the slope of the line. We can conclude that the formula $f(x) = ax + b$ describes a line with slope a. Furthermore, because this line intersects the y-axis at the point $(0, b)$, we see that the y-intercept for this linear function is $(0, b)$. We conclude that the formula $f(x) = ax + b$ tells us the slope, a, and the y-intercept, $(0, b)$, for this line. Since we often use the symbol m to denote the slope of a line, we can write

$$f(x) = mx + b \tag{16.2.3}$$

to denote the **slope-intercept form** of a linear function.

Sometimes it is convenient to express a linear function in different ways. For example, suppose the graph of a linear function passes through the point (x_1, y_1) and the slope of the line is m. Since any other point $(x, f(x))$ on the graph of f must satisfy the equation

$$m = \frac{f(x) - y_1}{x - x_1}, \tag{16.2.4}$$

this linear function can be expressed by writing

$$f(x) - y_1 = m(x - x_1). \tag{16.2.5}$$

We call this equation the **point-slope equation** for that linear function.

Since every nonvertical line is the graph of a linear function, the points on a nonvertical line can be described using the slope-intercept or point-slope equations. However, a vertical line does not represent the graph of a function and cannot be expressed in either of these forms. Instead, a vertical line is described by the equation $x = k$ for some constant k. Since neither the slope-intercept form nor the point-slope form allows for vertical lines, we use the notation

$$ax + by = c, \tag{16.2.6}$$

where a, b are both not zero, to denote the **standard form of a line.**

Definition: point-slope equation, point-slope equation and the standard form of a line

Consider a line passing through the point (x_1, y_1) with slope m. The equation

$$y - y_1 = m(x - x_1) \tag{16.2.7}$$

is the *point-slope equation* for that line.

Consider a line with slope m and y-intercept $(0, b)$. The equation

$$y = mx + b \tag{16.2.8}$$

is an equation for that line in *point-slope equation.*

The *standard form of a line* is given by the equation

$$ax + by = c, \tag{16.2.9}$$

where a and b are both not zero. This form is more general because it allows for a vertical line, $x = k$.

Example 16.2.1: Finding the Slope and Equations of Lines

Consider the line passing through the points $(11, -4)$ and $(-4, 5)$, as shown in Figure.

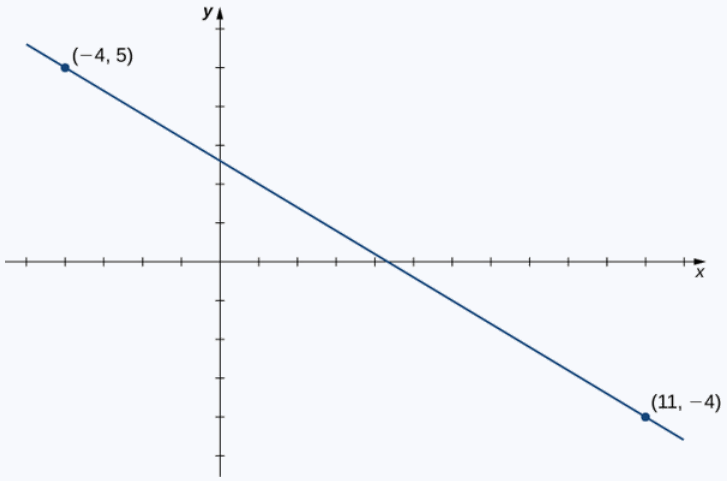


***Figure* 16.2.3:** *Finding the equation of a linear function with a graph that is a line between two given points.*

1. Find the slope of the line.
2. Find an equation for this linear function in point-slope form.
3. Find an equation for this linear function in slope-intercept form.

Solution

1. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-4)}{-4 - 11} = -\frac{9}{15} = -\frac{3}{5}. \tag{16.2.10}$$

2. To find an equation for the linear function in point-slope form, use the slope $m = -3/5$ and choose any point on the line. If we choose the point $(11, -4)$, we get the equation

$$f(x) + 4 = -\frac{3}{5}(x - 11). \tag{16.2.11}$$

3. To find an equation for the linear function in slope-intercept form, solve the equation in part b. for $f(x)$. When we do this, we get the equation

$$f(x) = -\frac{3}{5}x + \frac{13}{5}. \tag{16.2.12}$$

Exercise 16.2.1

Consider the line passing through points $(-3, 2)$ and $(1, 4)$.

a. Find the slope of the line.
b. Find an equation of that line in point-slope form.
c. Find an equation of that line in slope-intercept form.

Hint

The slope $m = \Delta y / \Delta x$.

Answer a

$m = 1/2$.

Answer b

The point-slope form is $y-4=\frac{1}{2}(x-1)$.

Answer c

The slope-intercept form is $y=\frac{1}{2}x+\frac{7}{2}$.

Example 16.2.2:

Jessica leaves her house at 5:50 a.m. and goes for a 9-mile run. She returns to her house at 7:08 a.m. Answer the following questions, assuming Jessica runs at a constant pace.

a. Describe the distance D (in miles) Jessica runs as a linear function of her run time t (in minutes).
b. Sketch a graph of D.
c. Interpret the meaning of the slope.

Solution:

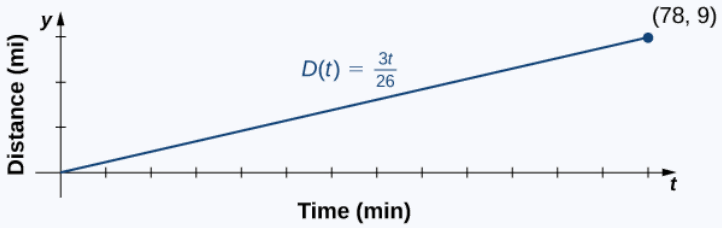
a. At time $t=0$, Jessica is at her house, so $D(0)=0$. At time $t=78$ minutes, Jessica has finished running 9 mi, so $D(78)=9$. The slope of the linear function is

$$m=\frac{9-0}{78-0}=\frac{3}{26}.$$

The y-intercept is $(0,0)$, so the equation for this linear function is

$$D(t)=\frac{3}{26}t.$$

b. To graph D, use the fact that the graph passes through the origin and has slope $m=3/26$.



c. The slope $m=3/26\approx 0.115$ describes the distance (in miles) Jessica runs per minute, or her average velocity.

Polynomials

A linear function is a special type of a more general class of functions: polynomials. A polynomial function is any function that can be written in the form

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0 \tag{16.2.13}$$

for some integer $n\geq 0$ and constants $a_n, a+n-1, \ldots, a_0$, where $a_n\neq 0$. In the case when $n=0$, we allow for $a_0=0$; if $a_0=0$, the function $f(x)=0$ is called the **zero function**. The value n is called the **degree** of the polynomial; the constant an is called the **leading coefficient**. A linear function of the form $f(x)=mx+b$ is a polynomial of degree 1 if $m\neq 0$ and degree 0 if $m=0$. A polynomial of degree 0 is also called a **constant function**. A polynomial function of degree 2 is called a **quadratic function**. In particular, a quadratic function has the form $f(x)=ax^2+bx+c$, where $a\neq 0$. A polynomial function of degree 3 is called a **cubic function**.

Power Functions

Some polynomial functions are power functions. A **power function** is any function of the form $f(x)=ax^b$, where a and b are any real numbers. The exponent in a power function can be any real number, but here we consider the case when the exponent is a positive integer. (We consider other cases later.) If the exponent is a positive integer, then $f(x)=ax^n$ is a polynomial. If n is even, then $f(x)=ax^n$ is an even function because $f(-x)=a(-x)^n=ax^n$ if n is even. If n is odd, then $f(x)=ax^n$ is an odd function because $f(-x)=a(-x)^n=-ax^n$ if n is odd (Figure 16.2.3).

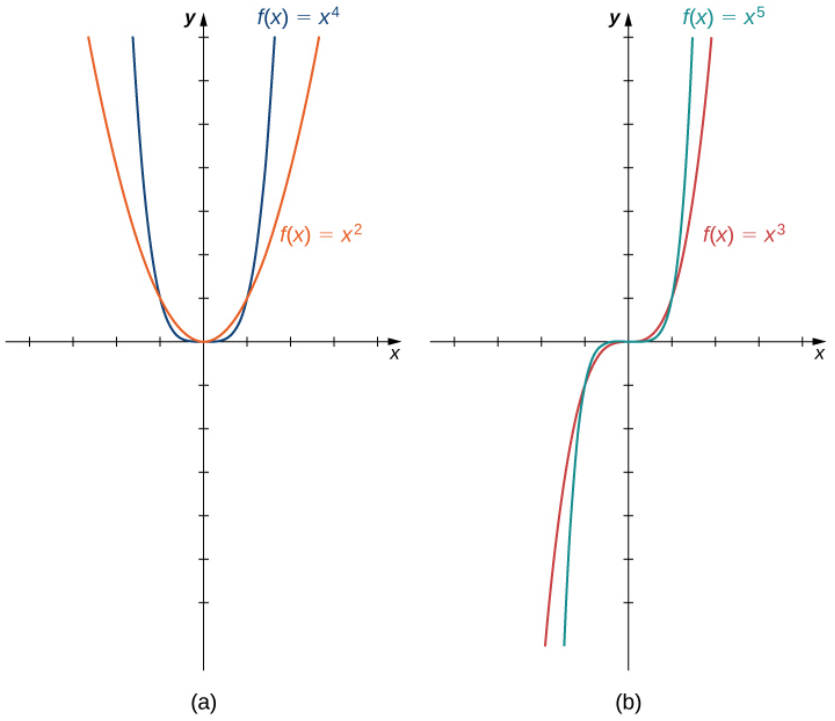


***Figure* 16.2.4:** *(a) For any even integer $n, f(x) = ax^n$ is an even function. (b) For any odd integer $n, f(x) = ax^n$ is an odd function.*

Mathematical Models

A large variety of real-world situations can be described using **mathematical models**. A mathematical model is a method of simulating real-life situations with mathematical equations. Physicists, engineers, economists, and other researchers develop models by combining observation with quantitative data to develop equations, functions, graphs, and other mathematical tools to describe the behavior of various systems accurately. Models are useful because they help predict future outcomes. Examples of mathematical models include the study of population dynamics, investigations of weather patterns, and predictions of product sales.

As an example, let's consider a mathematical model that a company could use to describe its revenue for the sale of a particular item. The amount of revenue R a company receives for the sale of n items sold at a price of p dollars per item is described by the equation $R = p \cdot n$. The company is interested in how the sales change as the price of the item changes. Suppose the data in Table show the number of units a company sells as a function of the price per item.

Number of Units Sold n (in Thousands) as a Function of Price per Unit p (in Dollars)

p	6	8	10	12	14
n	19.4	18.5	16.2	13.8	12.2

In Figure, we see the graph the number of units sold (in thousands) as a function of price (in dollars). We note from the shape of the graph that the number of units sold is likely a linear function of price per item, and the data can be closely approximated by the linear function $n = -1.04p + 26$ for $0 \le p \le 25$, where n predicts the number of units sold in thousands. Using this linear function, the revenue (in thousands of dollars) can be estimated by the quadratic function

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p \tag{16.2.14}$$

for $0 \le p \le 25$ In Example 16.2.1, we use this quadratic function to predict the amount of revenue the company receives depending on the price the company charges per item. Note that we cannot conclude definitively the actual number of units sold for

values of p, for which no data are collected. However, given the other data values and the graph shown, it seems reasonable that the number of units sold (in thousands) if the price charged is p dollars may be close to the values predicted by the linear function $n = -1.04p + 26$.

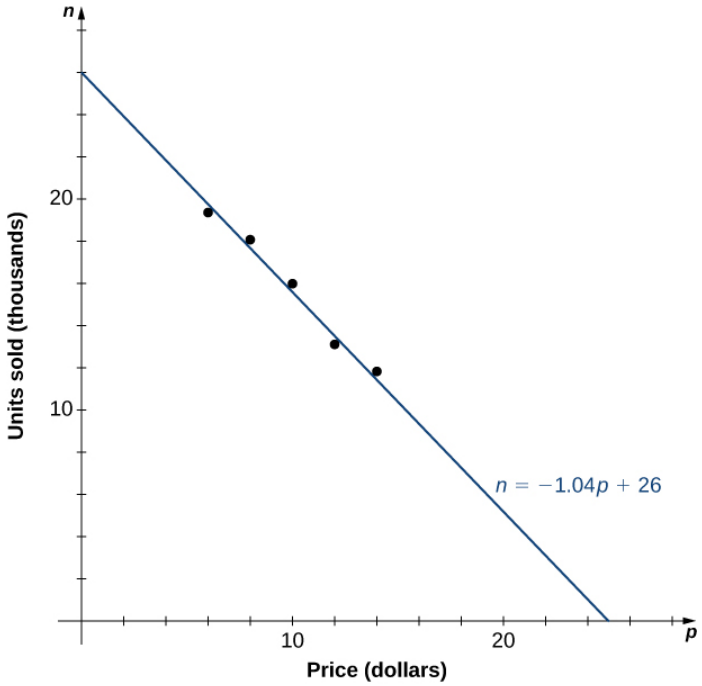


***Figure* 16.2.6:** *The data collected for the number of items sold as a function of price is roughly linear. We use the linear function $n = -1.04p + 26$ to estimate this function.*

Algebraic Functions

By allowing for quotients and fractional powers in polynomial functions, we create a larger class of functions. An **algebraic function** is one that involves addition, subtraction, multiplication, division, rational powers, and roots. Two types of algebraic functions are rational functions and root functions.

Just as rational numbers are quotients of integers, rational functions are quotients of polynomials. In particular, a **rational function** is any function of the form $f(x) = p(x)/q(x)$,where $p(x)$ and $q(x)$ are polynomials. For example,

$$f(x) = \frac{3x-1}{5x+2} \text{ and } g(x) = \frac{4}{x^2+1}$$

are rational functions. A **root function** is a power function of the form $f(x) = x^{1/n}$, where n is a positive integer greater than one. For example, f(x)=x1/2=x√ is the square-root function and $g(x) = x^{1/3} = \sqrt[3]{x})$ is the cube-root function. By allowing for compositions of root functions and rational functions, we can create other algebraic functions. For example, $f(x) = \sqrt{4-x^2}$ is an algebraic function.

Example 16.2.5: Finding Domain and Range for Algebraic Functions

For each of the following functions, find the domain and range.

a. $f(x) = \dfrac{3x-1}{5x+2}$
b. $f(x) = \sqrt{4-x^2}$

Solution

1.It is not possible to divide by zero, so the domain is the set of real numbers x such that $x \neq -2/5$. To find the range, we need to find the values y for which there exists a real number x such that

$$y = \frac{3x-1}{5x+2}$$

When we multiply both sides of this equation by $5x+2$, we see that x must satisfy the equation

$$5xy+2y=3x-1.$$

From this equation, we can see that x must satisfy

$$2y+1=x(3-5y).$$

If y=$3/5$, this equation has no solution. On the other hand, as long as $y \neq 3/5$,

$$x=\frac{2y+1}{3-5y}$$

satisfies this equation. We can conclude that the range of f is $y|y \neq 3/5$.

2. To find the domain of f, we need $4-x^2 \geq 0$. When we factor, we write $4-x^2=(2-x)(2+x) \geq 0$. This inequality holds if and only if both terms are positive or both terms are negative. For both terms to be positive, we need to find x such that

$$2-x \geq 0 \text{ and } 2+x \geq 0.$$

These two inequalities reduce to $2 \geq x$ and $x \geq -2$. Therefore, the set $x|-2 \leq x \leq 2$ must be part of the domain. For both terms to be negative, we need

$$2-x \leq 0 \text{ and } 2+x \geq 0.$$

These two inequalities also reduce to $2 \leq x$ and $x \geq -2$. There are no values of x that satisfy both of these inequalities. Thus, we can conclude the domain of this function is $x|-2 \leq x \leq 2$.

If $-2 \leq x \leq 2$, then $0 \leq 4-x^2 \leq 4$. Therefore, $0 \leq \sqrt{4-x2} \leq 2$, and the range of f is $y|0 \leq y \leq 2$.

Exercise 16.2.3

Find the domain and range for the function $f(x)=(5x+2)/(2x-1)$.

Hint

The denominator cannot be zero. Solve the equation $y=(5x+2)/(2x-1)$ for x to find the range.

Answer

The domain is the set of real numbers x such that $x \neq 1/2$. The range is the set $\{y|y \neq 5/2\}$

The root functions $f(x)=x^{1/n}$ have defining characteristics depending on whether n is odd or even. For all even integers $n \geq 2$, the domain of $f(x)=x^{1/n}$ is the interval $[0,\infty)$. For all odd integers $n \geq 1$, the domain of $f(x)=x^{1/n}$ is the set of all real numbers. Since $x^{1/n}=(-x)^{1/n}$ for odd integers n ,$f(x)=x^{1/n}$ is an odd function if$n$ is odd. See the graphs of root functions for different values of n in Figure.

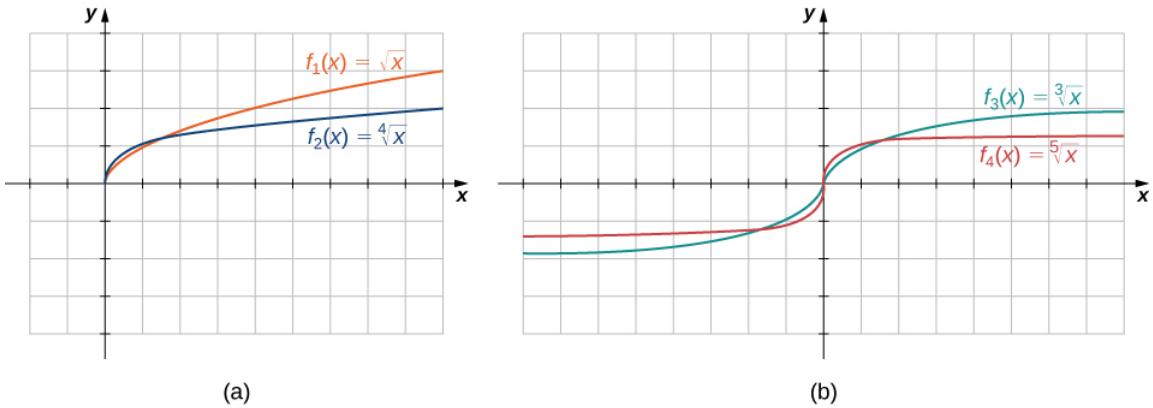


Figure* 16.2.7*: *(a) If n is even, the domain of $f(x)=\sqrt[n]{x}$ is $[0,\infty)$. (b) If n is odd, the domain of $f(x)=\frac{[}{n}]x$ is $(-\infty,\infty)$ and the function $f(x)=\frac{[}{n}]x$ is an odd function.*

Example 16.2.6: Finding Domains for Algebraic Functions

For each of the following functions, determine the domain of the function.

a. $f(x) = \dfrac{3}{x^2 - 1}$
b. $f(x) = \dfrac{2x + 5}{3x^2 + 4}$
c. $f(x) = \sqrt{4 - 3x}$
d. $f(x) = \sqrt[3]{2x - 1}$

Solution

a. You cannot divide by zero, so the domain is the set of values x such that $x^2 - 1 \neq 0$. Therefore, the domain is $x|x \neq \pm 1$.
b. You need to determine the values of x for which the denominator is zero. Since $3x^2 + 4 \geq 4$ for all real numbers x, the denominator is never zero. Therefore, the domain is $(-\infty, \infty)$.
c. Since the square root of a negative number is not a real number, the domain is the set of values x for which $4 - 3x \geq 0$. Therefore, the domain is $x|x \leq 4/3$.
d. The cube root is defined for all real numbers, so the domain is the interval $(-\infty, \infty)$.

Exercise 16.2.4

Find the domain for each of the following functions: $f(x) = (5 - 2x)/(x^2 + 2)$ and $g(x) = \sqrt{5x - 1}$.

Hint

Determine the values of x when the expression in the denominator of f is nonzero, and find the values of x when the expression inside the radical of g is nonnegative.

Answer

The domain of f is $(-\infty, \infty)$. The domain of g is $x|x \geq 1/5$.

Transformations of Functions

We have seen several cases in which we have added, subtracted, or multiplied constants to form variations of simple functions. In the previous example, for instance, we subtracted 2 from the argument of the function $y = x^2$ to get the function $f(x) = (x - 2)^2$). $This subtraction represents a shift of the function \(y = x^2$ two units to the right. A shift, horizontally or vertically, is a type of **transformation of a function**. Other transformations include horizontal and vertical scalings, and reflections about the axes.

A vertical shift of a function occurs if we add or subtract the same constant to each output y. For $c > 0$, the graph of $f(x) + c$ is a shift of the graph of $f(x)$ up c units, whereas the graph of $f(x) - c$ is a shift of the graph of $f(x)$ down c units. For example, the graph of the function $f(x) = x^3 + 4$ is the graph of $y = x^3$ shifted up 4 units; the graph of the function $f(x) = x^3 - 4$ is the graph of $y = x^3$ shifted down 4 units (Figure 16.2.6).

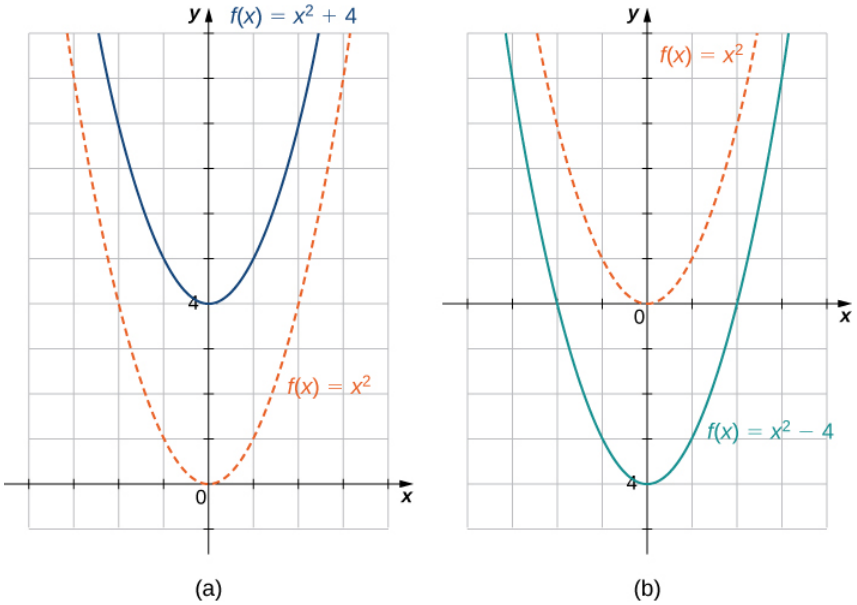


***Figure* 16.2.9:** *(a) For $c > 0$, the graph of $y = f(x) + c$ is a vertical shift up c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x) - c$ is a vertical shift down c units of the graph of $y = f(x)$.*

A horizontal shift of a function occurs if we add or subtract the same constant to each input x. For $c > 0$, the graph of $f(x + c)$ is a shift of the graph of $f(x)$ to the left c units; the graph of $f(x - c)$ is a shift of the graph of $f(x)$ to the right c units. Why does the graph shift left when adding a constant and shift right when subtracting a constant? To answer this question, let's look at an example.

Consider the function $f(x) = |x + 3|$ and evaluate this function at $x - 3$. Since $f(x - 3) = |x|$ and $x - 3 < x$, the graph of $f(x) = |x + 3|$ is the graph of $y = |x|$ shifted left 3 units. Similarly, the graph of $f(x) = |x - 3|$ is the graph of $y = |x|$ shifted right 3 units (Figure 16.2.7).

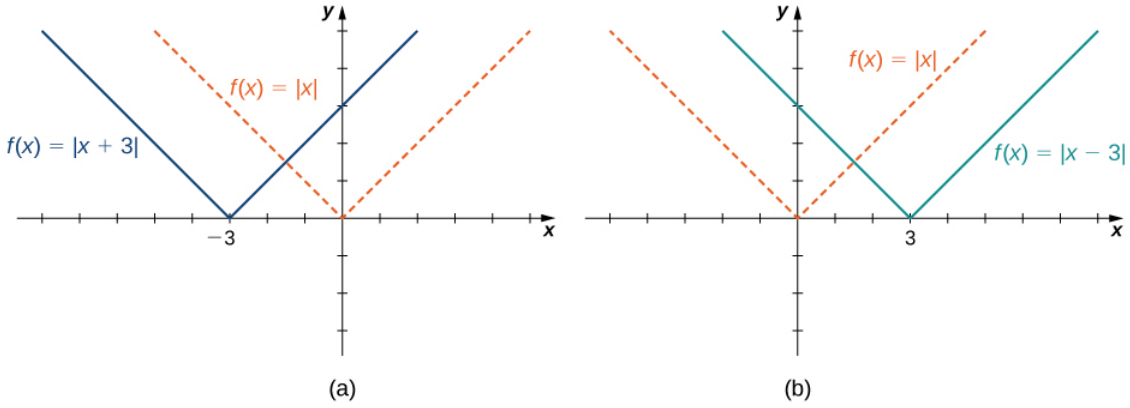


***Figure* 16.2.10:** *(a) For $c > 0$, the graph of $y = f(x + c)$ is a horizontal shift left c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x - c)$ is a horizontal shift right c units of the graph of $y = f(x)$.*

A vertical scaling of a graph occurs if we multiply all outputs y of a function by the same positive constant. For $c > 0$, the graph of the function $cf(x)$ is the graph of $f(x)$ scaled vertically by a factor of c. If $c > 1$, the values of the outputs for the function $cf(x)$ are larger than the values of the outputs for the function $f(x)$; therefore, the graph has been stretched vertically. If $0 < c < 1$, then the outputs of the function $cf(x)$ are smaller, so the graph has been compressed. For example, the graph of the function $f(x) = 3x^2$ is the graph of $y = x^2$ stretched vertically by a factor of 3, whereas the graph of $f(x) = x^2/3$ is the graph of $y = x^2$ compressed vertically by a factor of 3 (Figure 16.2.8).

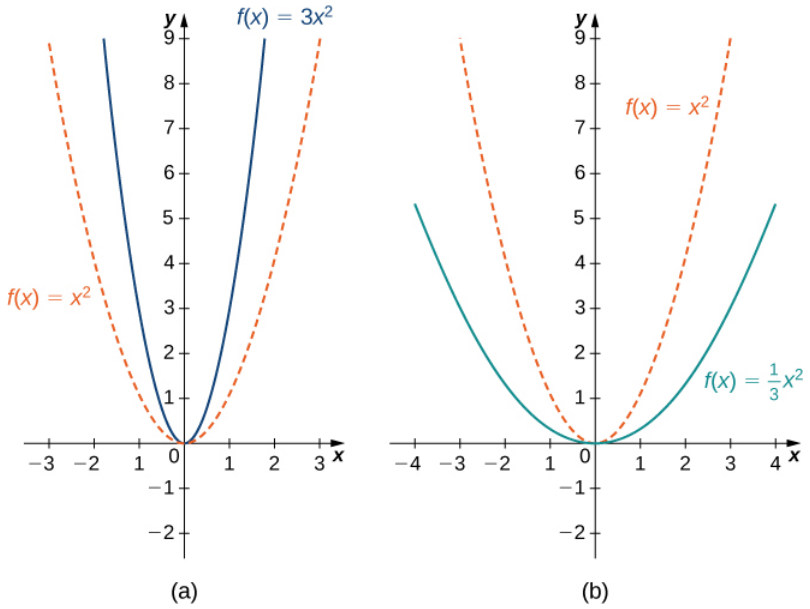


***Figure* 16.2.11:** *(a) If $c > 1$, the graph of $y = cf(x)$ is a vertical stretch of the graph of $y = f(x)$. (b) If $0 < c < 1$, the graph of $y = cf(x)$ is a vertical compression of the graph of $y = f(x)$.*

The horizontal scaling of a function occurs if we multiply the inputs x by the same positive constant. For $c > 0$, the graph of the function $f(cx)$ is the graph of $f(x)$ scaled horizontally by a factor of c. If $c > 1$, the graph of $f(cx)$ is the graph of $f(x)$ compressed horizontally. If $0 < c < 1$, the graph of $f(cx)$ is the graph of $f(x)$ stretched horizontally. For example, consider the function $f(x) = \sqrt{2x}$ and evaluate f at $x/2$. Since $f(x/2) = \sqrt{x}$, the graph of $f(x) = \sqrt{2x}$ is the graph of $y = \sqrt{x}$ compressed horizontally. The graph of $y = \sqrt{x/2}$ is a horizontal stretch of the graph of $y = \sqrt{x}$ (Figure 16.2.9).

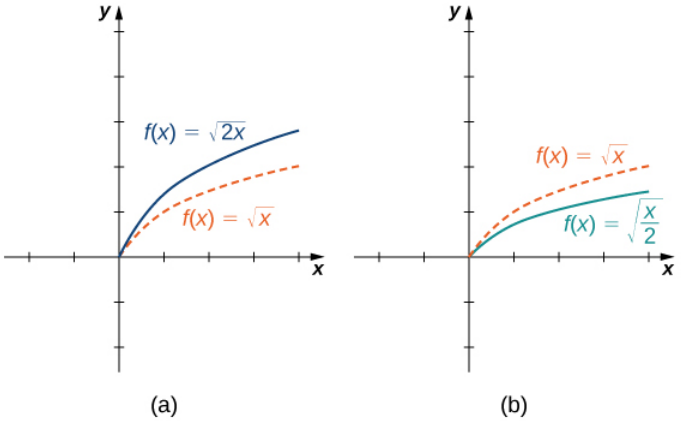


***Figure* 16.2.12:** *(a) If $c > 1$, the graph of $y = f(cx)$ is a horizontal compression of the graph of $y = f(x)$. (b) If $0 < c < 1$, the graph of $y = f(cx)$ is a horizontal stretch of the graph of $y = f(x)$.*

We have explored what happens to the graph of a function f when we multiply f by a constant $c > 0$ to get a new function $cf(x)$. We have also discussed what happens to the graph of a function $f$when we multiply the independent variable x by $c > 0$ to get a new function $f(cx)$. However, we have not addressed what happens to the graph of the function if the constant c is negative. If we have a constant $c < 0), we can write \(c$ as a positive number multiplied by -1; but, what kind of transformation do we get when we multiply the function or its argument by -1? When we multiply all the outputs by -1, we get a reflection about the x-axis. When we multiply all inputs by -1, we get a reflection about the y-axis. For example, the graph of $f(x) = -(x^3 + 1)$ is the graph of $y = (x^3 + 1)$ reflected about the x-axis. The graph of $f(x) = (-x)^3 + 1$ is the graph of $y = x^3 + 1$ reflected about the y-axis (Figure 16.2.10).

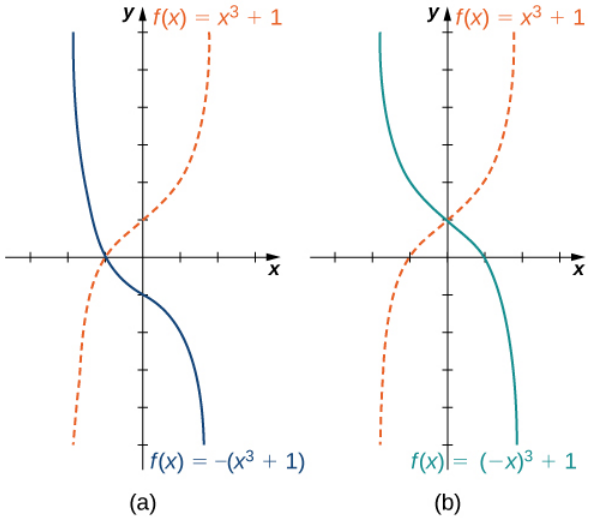


***Figure* 16.2.13:** (a) The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis. (b) The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y-axis.

If the graph of a function consists of more than one transformation of another graph, it is important to transform the graph in the correct order. Given a function $f(x)$, the graph of the related function $y = cf(a(x+b))+d$ can be obtained from the graph of $y = f(x)$by performing the transformations in the following order.

- Horizontal shift of the graph of $y = f(x)$. If $b > 0$, shift left. If $b < 0$ shift right.
- Horizontal scaling of the graph of $y = f(x+b)$ by a factor of $|a|$. If $a < 0$, reflect the graph about the y-axis.
- Vertical scaling of the graph of $y = f(a(x+b))$ by a factor of $|c|$. If $c < 0$, reflect the graph about the x -axis.
- Vertical shift of the graph of $y = cf(a(x+b))$. If $d > 0$, shift up. If $d < 0$, shift down.

We can summarize the different transformations and their related effects on the graph of a function in the following table.

Transformation of $f(c > 0)$	**Effect of the graph of f**
$f(x)+c$	Vertical shift up c units
$f(x)-c$	Vertical shift down c units
$f(x+c)$	Shift left by c units
$f(x-c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if $0 < c < 1$
$f(cx)$	Horizontal stretch if $0 < c < 1$; horizontal compression if $c > 1$
$-f(x)$	Reflection about the x-axis
$-f(x)$	Reflection about the y-axis

Example 16.2.10: Transforming a Function

For each of the following functions, a. and b., sketch a graph by using a sequence of transformations of a well-known function.

a. $f(x) = -|x+2|-3$
b. $f(x) = \sqrt[3]{x}+1$

Solution:

1.Starting with the graph of $y = |x|$, shift 2 units to the left, reflect about the x-axis, and then shift down 3 units.

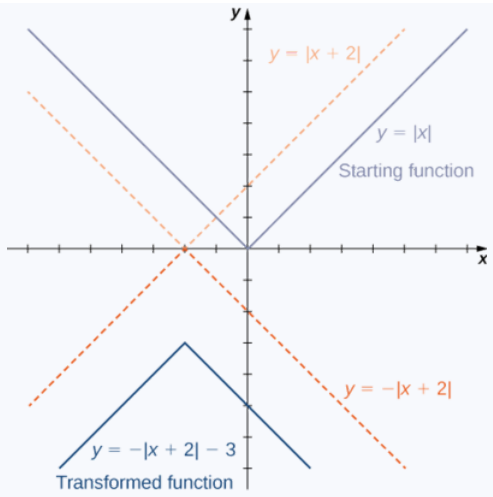


***Figure* 16.2.14:** *The function* $f(x) = -|x+2| - 3$ *can be viewed as a sequence of three transformations of the function* $y = |x|$.

2. Starting with the graph of y=x√, reflect about the y-axis, stretch the graph vertically by a factor of 3, and move up 1 unit.

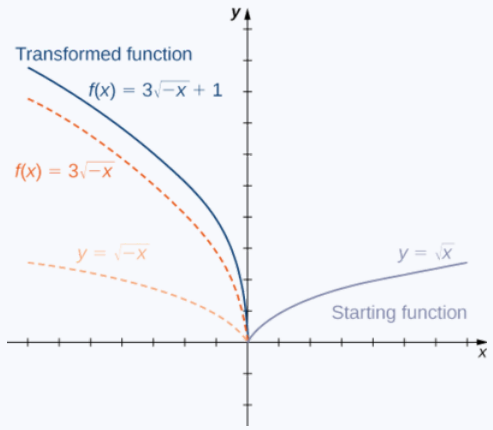


***Figure* 16.2.15:** *The function* $f(x) = \sqrt[3]{x} + 1$ *can be viewed as a sequence of three transformations of the function* $y = \sqrt{x}$.

Exercise 16.2.7

Describe how the function $f(x) = -(x+1)^2 - 4$ can be graphed using the graph of $y = x^2$ and a sequence of transformations

Answer

Shift the graph $y = x^2$ to the left 1 unit, reflect about the x-axis, then shift down 4 units.

Key Concepts

- The power function $f(x) = x^n$ is an even function if n is even and $n \neq 0$, and it is an odd function if n is odd.
- The root function $f(x) = x^{1/n}$ has the domain $[0, \infty)$ if n is even and the domain $(-\infty, \infty)$ if n is odd. If n is odd, then $f(x) = x^{1/n}$ is an odd function.
- The domain of the rational function $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomial functions, is the set of x such that $q(x) \neq 0$.
- Functions that involve the basic operations of addition, subtraction, multiplication, division, and powers are algebraic functions.
- Vertical and horizontal shifts, vertical and horizontal scalings, and reflections about the x- and y-axes are examples of transformations of functions.

Key Equations

- **Point-slope equation of a line**

  $y - y1 = m(x - x_1)$

- **Slope-intercept form of a line**

  $y = mx + b$

- **Standard form of a line**

  $ax + by = c$

- **Polynomial function**

  $f(x) = a_n^{x^n} + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$

Glossary

algebraic function

a function involving any combination of only the basic operations of addition, subtraction, multiplication, division, powers, and roots applied to an input variable x

cubic function

a polynomial of degree 3; that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

degree

for a polynomial function, the value of the largest exponent of any term

linear function

a function that can be written in the form $f(x) = mx + b$

mathematical model

A method of simulating real-life situations with mathematical equations

point-slope equation

equation of a linear function indicating its slope and a point on the graph of the function

polynomial function

a function of the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$

power function

a function of the form $f(x) = x^n$ for any positive integer $n \geq 1$

quadratic function

a polynomial of degree 2; that is, a function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$

rational function

a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials

root function

a function of the form $f(x) = x^{1/n}$ for any integer $n \geq 2$

slope

the change in *y* for each unit change in *x*

slope-intercept form

equation of a linear function indicating its slope and *y*-intercept

transformation of a function

a shift, scaling, or reflection of a function

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16.2.1: Linear Functions and Slope

In the previous section on Linear Models, we saw that if the dependent variable was changing at a constant rate with respect to the independent variable, then the graph was a line. If the rate was positive, then as we swept our eyes from left to right, the line rose upward, the dependent variable increasing with increasing changes in the independent variable. If the rate was negative, then the graph fell downward, the dependent variable decreasing with increasing changes in the independent variable. You may have also learned that higher rates led to steeper lines (lines that rose more quickly) and lower rates led to lines that were less steep.

In this section, we will connect the intuitive concept of rate developed in the previous section with a formal definition of the slope of a line. To start, let's state up front what is meant by the slope of a line.

Definition

Slope is a number that tells us how quickly a line rises or falls.

If slope is a number that is directly connected to the "steepness" of a line, then we should have certain expectations.

Expectations

1. Lines with positive slope should slant uphill (as our eyes sweep from left to right).
2. Lines with negative slope should slant downhill (as our eyes sweep from left to right).
3. Because any horizontal line neither slants uphill nor downhill, we expect that it should have slope equal to zero.
4. Lines with a larger positive slope should rise more quickly than lines with a smaller positive slope.
5. If two lines have negative slope, then the line having the slope with larger absolute value should fall more quickly than the other line.

It remains to define how to compute the slope of a particular line. Whatever definition we choose, it should conform with the expectations outlined above. We also would like the definition of slope to conform with the concept of rate developed in the previous section. Thus, we make the following definition.

Definition: Slope

The **slope** of a line is the rate at which the dependent variable is changing with respect to the independent variable.

Note how the word "change" is used Definition. It is important to understand that the change in some quantity can be positive, negative, or zero. For example, if the temperature outside is $40^{\circ}\mathrm{F}$ when I leave my home at 6 AM, and at noon the temperature is $65\circ$ F, then the change in temperature is a positive $25^{\circ}\mathrm{F}$. On the other hand, if the temperature outside is $65^{\circ}\mathrm{F}$ at noon, and the temperature is $50^{\circ}\mathrm{F}$ when I return home in the evening, then the change in temperature is a negative 15 degrees Fahrenheit.

In calculating the change in a quantity, follow this rule.

Definition: Change in Quantity

Change in Quantity = Latter Measurement − Former Measurement.

Thus, if T represents the temperature and ΔT represents the change in the temperature, then in our first case (taking the temperature in the morning then later at noon), the change in temperature is

$$\Delta T = \text{ Latter } - \text{ Former } = 65^{\circ}\mathrm{F} - 40^{\circ}\mathrm{F} = 25^{\circ}\mathrm{F}$$

This positive result represents an increase in the temperature of $25^{\circ}\mathrm{F}$.

In the second case (taking the temperature at noon then later in the evening), the change in temperature is

$$\begin{aligned} \Delta T &= \text{Latter} - \text{Former} \\ &= 50^{\circ}\mathrm{F} - 65^{\circ}\mathrm{F} \\ &= -15^{\circ}\mathrm{F} \end{aligned}$$

This negative result represents a decrease in the temperature of $15^\circ \mathrm{F}$

Tip 3

Readers should note that the direction of subtraction is extremely important. To detect the change in a quantity, always subtract the former (earlier) measurement from the latter (later) measurement.

Example 16.2.1.1

A ball is perched at rest at the top of a long ramp. It's given a little tap and it begins to roll down the ramp. The speed v of the ball (in meters per second) is plotted versus the time t (in seconds) in Figure 16.2.1.1.

Determine the slope of the line.

Solution

We've defined the slope as the rate at which the dependent variable is changing with respect to the independent variable. In this case, the speed v of the ball "depends" upon the amount of time t that has elapsed. Consequently, v is the dependent variable and has been placed on the vertical axis.3 On the other hand, t is the independent variable and has been assigned the horizontal axis.

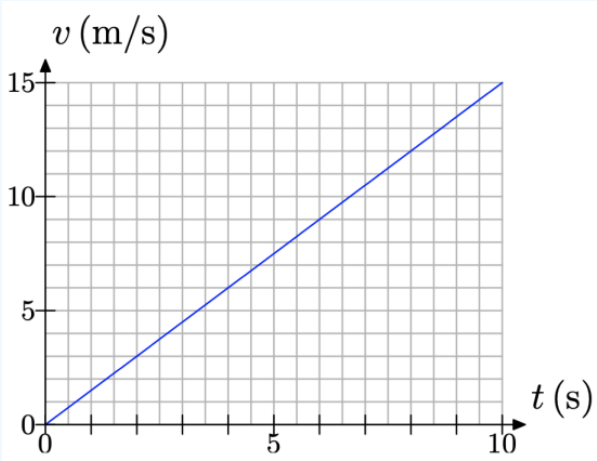


Figure 16.2.1.1. Speed versus time.

To determine the rate at which v is changing with respect to t (the slope of the line), we first select two points P(2, 3) and Q(8, 12) on the line, as shown in Figure 16.2.1.2 As we sweep our eyes from left to right (a convention we will always follow when dealing with slope), the point P occurs before the point Q. Hence, we consider P the "former" measurement and point Q the "latter" measurement.

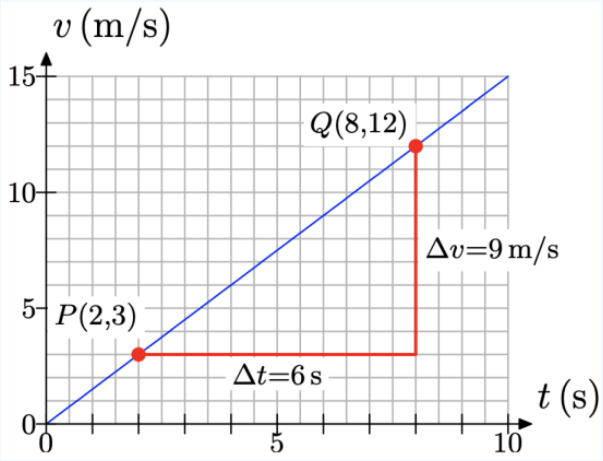


Figure 16.2.1.2. Determining the slope of the line.

At point P, the time is t = 2 seconds, then at point Q the time is t = 8 seconds. The change in t is found by subtracting the former measurement from the latter measurement.

$$\Delta t = 8s - 2s = 6s$$

At point P, the speed is v = 3 meters per second, then at point Q the speed is v = 12 meters per second. Hence, the change in v is

$$\Delta v = 12\text{m/s} - 3\text{m/s} = 9\text{m/s}$$

Finally, the slope of the line is defined as the rate at which the dependent variable v is changing with respect to the independent variable t. That is,

$$\text{Slope} = \frac{\Delta v}{\Delta t} = \frac{9\text{m/s}}{6\text{s}} = \frac{3\text{m/s}}{\text{s}}$$

Scientists prefer to write this as 1.5 m/s^2, but this might not be as intuitive as writing 1.5 (m/s)/s, which indicates that the speed is increasing at a rate of 1.5 m/s every second. This makes good sense as a ball rolling down a ramp will pick up speed with the passage of time. The slope provides an exact numerical description of how the speed increases with respect to time.

Note that our definition of the slope of the line satisfies one of our goals: the slope is precisely the same as the notion of rate described in the previous section. Indeed, note the right triangle we've drawn in Figure 16.2.1.2 The bottom edge of the triangle is 12 boxes long, but every 2 boxes represents one second, so this displacement in the time t direction is 6 seconds. The vertical side of the right triangle is 9 boxes in height where each box represents 1 meter per second. Consequently, this vertical edge of the right triangle represents a positive displacement of 9 meters per second. Thus, every 6 seconds, there is an increase in speed of 9 meters per second. Hence, the ball is picking up speed at the rate of 9 meters per second every 6 seconds, or equivalently, 1.5 meters per second every second.

Note

In Figure 16.2.1.2 the rate at which the speed is increasing with respect to time is equivalent to the slope of the line.

Suppose that we had labeled our points $P\left(t_{\text{initial}}, v_{\text{initial}}\right)$ and $Q\left(t_{\text{final}}, v_{\text{final}}\right)$ as shown in Figure 16.2.1.3

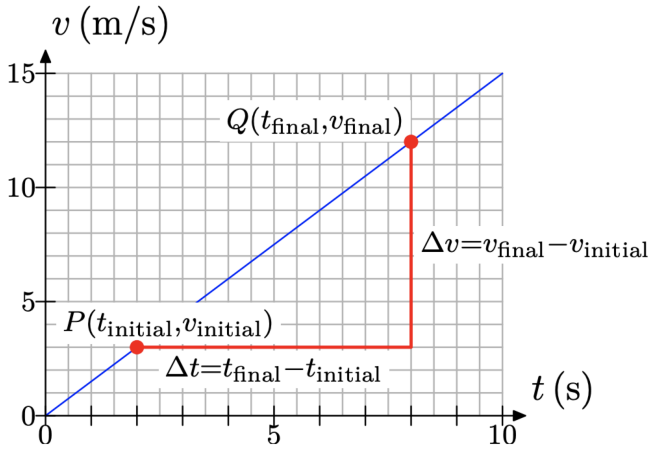


Figure 16.2.1.3. Initial and final measurements.

Now the change in speed v would be

$$\Delta v = v_{\text{final}} - v_{\text{initial}}$$

and the change in time t would be

$$\Delta t = t_{\text{final}} - t_{\text{initial}}$$

Therefore, the slope of the line would be computed with the following formula.

$$\text{Slope} = \frac{\Delta v}{\Delta t} = \frac{v_{\text{final}} - v_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}}$$

With $P\left(t_{\text{initial}}, v_{\text{initial}}\right) = (2\text{s}, 3\text{m/s})$ and $Q\left(t_{\text{final}}, v_{\text{final}}\right) = (8\text{s}, 12\text{m/s})$, this becomes

$$\text{Slope} = \frac{12\text{m/s} - 3\text{m/s}}{8\text{s} - 2\text{s}} = \frac{9\text{m/s}}{6\text{s}} = 1.5\text{m/s}^2$$

The Slope Formula

The last calculation in Example 16.2.1.1 allows us to discuss the slope of a line as a purely mathematical concept, one that is not rooted in a supporting application as in Example 16.2.1.1. Take, for example, the line shown in Figure 16.2.1.4 that passes through the points $P(-3, -3)$ and $Q(2, 1)$.

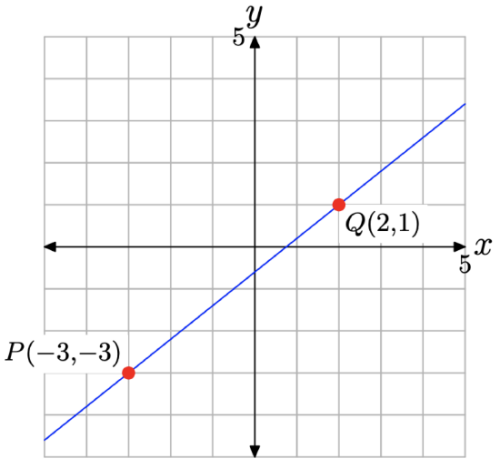


Figure 16.2.1.4. Computing the slope of a line in an xy-coordinate system.

In this example, the dependent variable is y and the independent variable is x, so the slope of the line is Δy (the change in y) divided by Δx (the change in x).

$$\text{Slope} = \frac{\Delta y}{\Delta x}$$

Sweeping our eyes from left to right, the point P comes first, followed by the point Q. Keeping "latter minus former" in mind, the change in y is computed by subtracting the y-value of point P from the y-value of point Q. That is,

$$\Delta y = 1 - (-3) = 4$$

Similarly, the change in x is computed by subtracting the x-value of point P from the x-value of point Q. That is,

$$\Delta x = 2 - (-3) = 5$$

Thus, the slope of the line is

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{4}{5}$$

Alternatively, we can use the points P and Q as vertices of a right triangle with sides parallel to the axes (shown in Figure 16.2.1.5(a)). The horizontal edge of the right triangle is 5 boxes (each representing 1 unit), so the displacement in x is 5 units. The vertical edge is 4 boxes (each representing 1 unit), so the displacement in y is 4 units. Hence, each time x is increased by 5 units, y experiences an increase of 4 units. Therefore, the slope of the line is again 4/5.

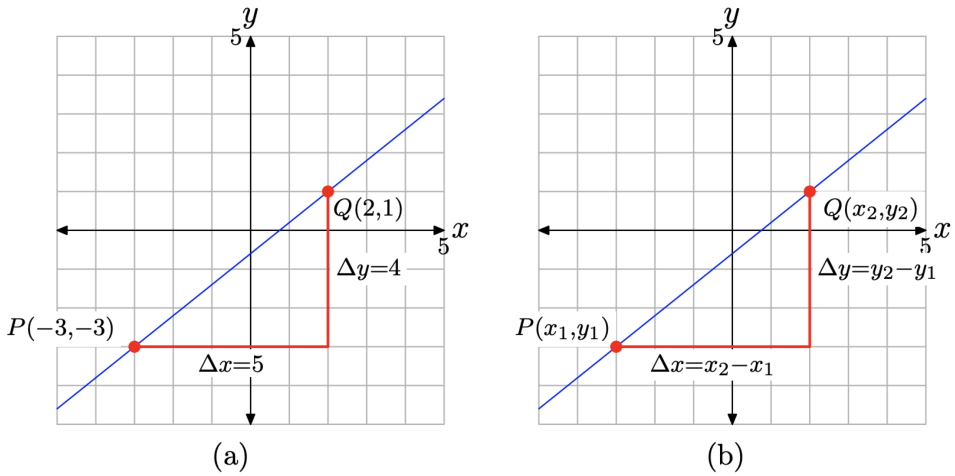


Figure 16.2.1.5. Using a right triangle to determine the slope.

Suppose that we had labeled our points $P\left(x_1, y_1\right)$ and $Q\left(x_2, y_2\right)$ as shown in Figure 16.2.1.5(b). Now the change in y would be

$$\Delta y = y_2 - y_1$$

and the change in x would be

$$\Delta x = x_2 - x_1$$

Therefore, the slope of the line would be computed with the following formula.

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

With $P\left(x_1, y_1\right) = (-3, -3)$ and $Q\left(x_2, y_2\right) = (2, 1)$, this becomes

$$\text{Slope } = \frac{1 - (-3)}{2 - (-3)} = \frac{4}{5}$$

The slope formula is worth summarizing in a definition.

Definition 5

The slope of the line that passes through the points $P\left(x_1, y_1\right)$ and $Q\left(x_2, y_2\right)$ is given by the formula

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Let's look at some more examples.

Example 16.2.1.2

Find the slope of the line passing through the points P(−3, −2) and Q(3, 1).

Solution

We can use the slope formula in Definition 5 to determine the slope. With $(x_1, y_1) = P(-3, -2)$ and $(x_2, y_2) = Q(3, 1)$,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-2)}{3 - (-3)} = \frac{3}{6} = \frac{1}{2}$$

Readers will sometimes ask, "Which point should be (x_1, y_1) and which should be (x_2, y_2)?" The short answer is, "It doesn't matter!" Suppose instead, that we let $(x_1, y_1) = Q(3, 1)$ and $(x_2, y_2) = P(-3, -2)$. Then,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 1}{-3 - 3} = \frac{-3}{-6} = \frac{1}{2}$$

Because the change in any quantity is found by subtracting the earlier measurement from the later measurement, we will continue to stress the first order. However, if we reverse the points as we did in our second calculation, both numerator and denominator reverse sign with this interchange, so we get the same answer.

Of course, we can also determine the slope by plotting P(−3, −2) and Q(3, 1) and the line that passes through P and Q, as we've done in Figure 16.2.1.6

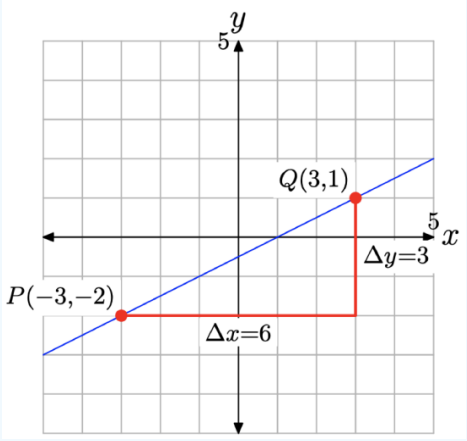


Figure 16.2.1.6. Determining the slope from the graph.

Starting at the point P, to get to the point Q, we move 6 boxes to the right, then 3 boxes up, as shown in Figure 16.2.1.6 Hence, the slope of the line is

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{3}{6} = \frac{1}{2}$$

Note that two of our expectations regarding the slope of a line are met with this example.

1. The line through P(−3, −2) and Q(3, 1) in Figure 16.2.1.6has slope 1/2. This is a positive number and the line slants uphill (as expected) as we sweep our eyes from left to right.
2. The slope in this example is 1/2, which is less than the slope of the line in Figure 16.2.1.5(a), which was 4/5. Note that the line in Figure 16.2.1.6is less steep than the line in Figure 16.2.1.5(a), which was another of our earlier expectations regarding the slope of a line.

Example 16.2.1.3

Find the slope of the line passing through the points P(−4, 4) and Q(4, −2).

Solution

We can use the slope formula in Definition 5 to determine the slope. With $(x_1, y_1) = P(-4, 4)$ and $(x_2, y_2) = Q(4, -2)$,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 4}{4 - (-4)} = \frac{-6}{8} = -\frac{3}{4}$$

We can also get the slope of the line from the graph in Figure 16.2.1.7. Starting at the point P(−4, 4), move 8 units to the right, then 6 units downward, as shown in Figure 16.2.1.7.

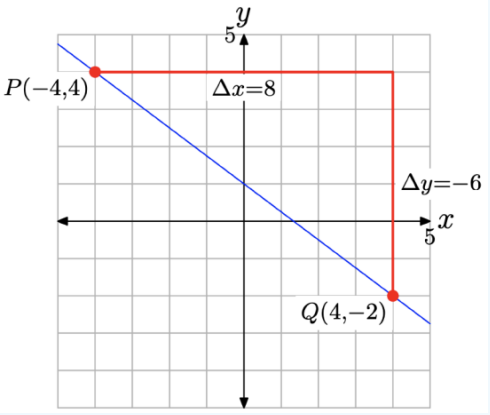


Figure 16.2.1.7. Determining the slope from the graph.

Thus, the slope of the line is

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{-6}{8} = -\frac{3}{4}$$

Again, one of our earlier expectations regarding the slope of a line is met in this example. The slope is −3/4, which is a negative number, and the line in Figure 16.2.1.7slants downhill (as we sweep our eyes from left to right).

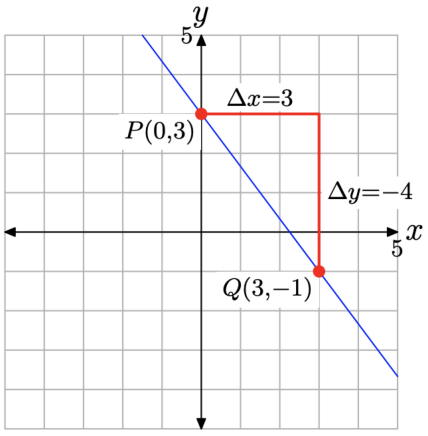
Example 16.2.1.4

Draw a line that intercepts the y-axis at (0, 3) so that the line has slope −4/3. Draw a second line that passes through the point P(−1, −1) with slope 3/5.

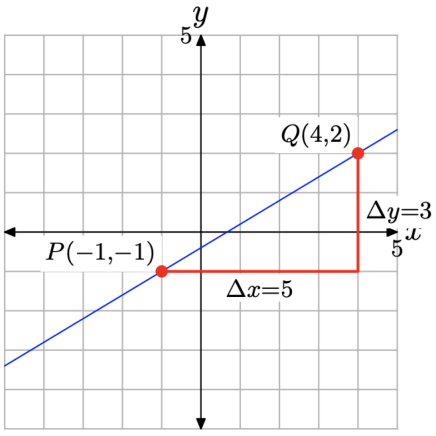
Solution

The slope of the first line is −4/3. This means that our line must slant downhill (as we sweep our eyes from left to right). The slope is the change in y over the change in x. Therefore, every time x increases by 3 units, y must decrease by 4 units. Plot the point P(0, 3), as shown in Figure 16.2.1.8(a). Then, starting at P, move 3 units to the right, followed by 4 units downward to the point Q(3, −1), as shown in Figure 16.2.1.8(a). Draw the required line, which must pass through the points P and Q.

To draw the second line, first plot the point P(−1, −1), as shown in Figure 16.2.1.8(b). Starting at the point P, move 5 units to the right, then upward 3 units to the point Q(4, 2), as shown in Figure 16.2.1.8(b). Draw the required line passing through the points P and Q.



(a) A line having y-intercept at $(0, 3)$ with slope $-4/3$.



(b) A line having slope $3/5$ that passes through the point $P(-1, -1)$.

Figure 16.2.1.8

Parallel Lines

Because slope controls the “steepness” of a line, it is a simple matter to see that parallel lines must have the same slope.

property

Let L_1 be a line having slope m_1. Let L_2 be a line having slope m_2. If L_1 and L_2 are parallel, then

$$m_1 = m_2$$

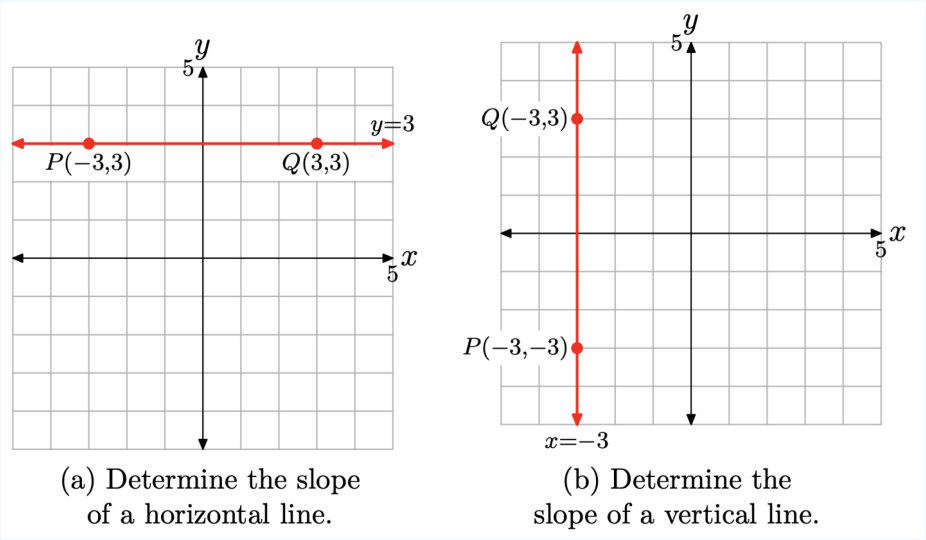
That is, any two parallel lines have the same slope.

Example 16.2.1.5

What is the slope of any horizontal line? What is the slope of any vertical line?

Solution

One would expect that our definition would verify that the slope of any horizontal line is zero. Select, for example, the horizontal line shown in Figure 9(a). Select the points (−3, 3) and (3, 3) on this line.



(a) Determine the slope of a horizontal line.

(b) Determine the slope of a vertical line.

Figure 16.2.1.9

With $(x_1, y_1) = (-3, 3)$ and $(x_2, y_2) = (3, 3)$

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{3-(-3)} = \frac{0}{6} = 0$$

Thus, the horizontal line in Figure 16.2.1.9(a) has slope equal to zero, exactly as expected. Further, all horizontal lines are parallel to this horizontal line and have the same slope. Therefore, all horizontal lines have slope zero.

We would surmise that the vertical line in Figure 16.2.1.9(b) has undefined slope (we’ll explore this more fully in the exercises). In Figure 16.2.1.9(b), we’ve selected the points P(−3, −3) and Q(−3, 3) on the vertical line. With $(x_1, y_1) = P(-3, -3)$ and $(x_2, y_2) = Q(-3, -3)$,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-3)}{-3-(-3)} = \frac{6}{0}, \text{ which is undefined.}$$

The slope of the vertical line in Figure 16.2.1.9(b) is undefined because division by zero is meaningless. Further, all vertical lines are parallel to this vertical line and have undefined slope.

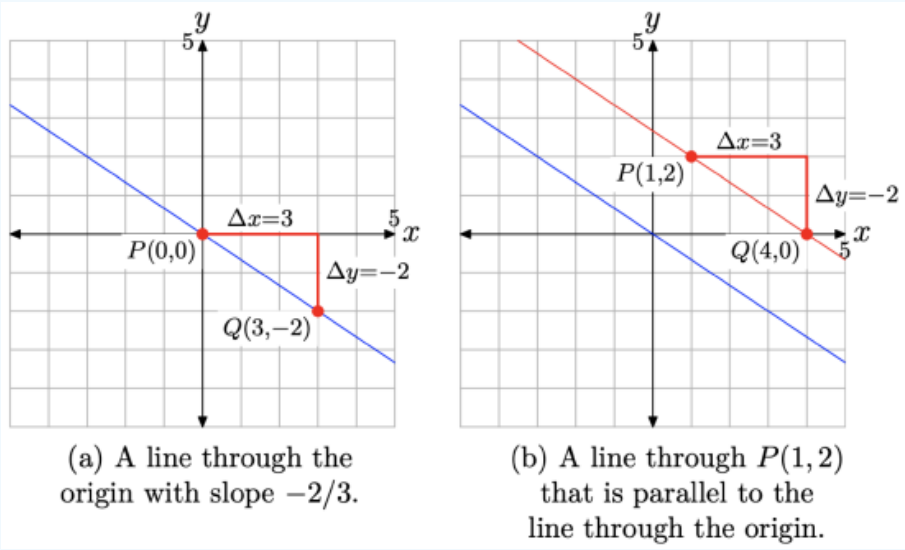
Example 16.2.1.6

Draw a line through the point P(1, 2) that is parallel to the line passing through the origin with slope −2/3.

Solution

We will first draw a line through the origin with slope −2/3. Plot the point P(0, 0), then move 3 units to the right and 2 units downward to the point Q(3, −2), as shown in Figure 16.2.1.10(a). Draw a line through the points P and Q as shown in Figure 16.2.1.10(a).

Next, plot the point P(1, 2) as shown in Figure 16.2.1.10(b). To draw a line through this point that is parallel to the line through the origin, this second line must have the same slope as the first line. Therefore, start at the point P(1, 2), as shown in Figure 16.2.1.10(b), then move 3 units to the right and 2 units downward to the point Q(4, 0). Draw a line through the points P and Q as shown in Figure 16.2.1.10(b). Note that this second line is parallel to the first.



(a) A line through the origin with slope −2/3.

(b) A line through $P(1,2)$ that is parallel to the line through the origin.

Figure 16.2.1.10

Perpendicular Lines

The relationship between the slopes of two perpendicular lines is not as straightforward as the relation between the slopes of two parallel lines. Let's begin by stating the pertinent property.

Property 13

Let L_1 be a line having slope m_1. Let L_2 be a line having slope m_2. If L_1 and L_2 are perpendicular, then

$$m_1 m_2 = -1$$

That is, the product of the slopes of two perpendicular lines is −1.

We can solve equation (14) for m_1 in terms of m_2.

$$m_1 = -\frac{1}{m_2}$$

Equation (15) tells us that the slope of the first line is the negative reciprocal of the slope of the second line.

For example, suppose that L_1 and L_2 are perpendicular lines with slopes m_1 and m_2, respectively

- If $m_2 = 2$, then $m_1 = -\frac{1}{2}$
- If $m_2 = \frac{3}{5}$, then $m_1 = -\frac{5}{3}$
- If $m_2 = -\frac{2}{3}$, then $m_1 = \frac{3}{2}$

Note that in each bulleted item, the product of the slopes is −1.

We won't provide a proof of equation (15), but we will provide some motivating evidence in the form of a graph.

Example 16.2.1.7

Sketch the graphs of the lines passing through the origin having slopes 2 and −1/2.

Solution

In Figure 16.2.1.11(a), we've plotted the point P(0, 0) at the origin, then moved 1 unit to the right and 2 units upward to the point Q(1, 2). The resulting line passes through the origin and has slope $m_1 = 2$ (alternatively, $m_1 = 2/1$).

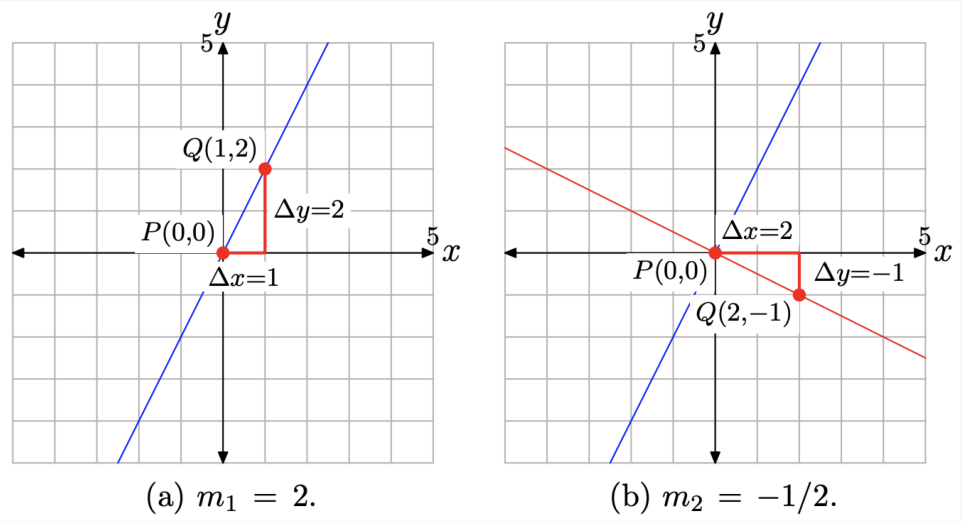
In Figure 16.2.1.11(b), we've again plotted the point P(0, 0) at the origin, then moved 2 units to the right and 1 unit downward to the point Q(2, −1). The resulting line passes through the origin and has slope $m_2 = -1/2$.

There are two important points that need to be made about the lines in Figure 16.2.1.11(b).

The two lines in Figure 16.2.1.11(b) are perpendicular. They meet and form a right angle of 90°. If you have a protractor available, you might want to measure the angle between the two lines and note that the measure of the angle is 90°.

The product of the two slopes is

$$m_1 m_2 = 2 \cdot \left(-\frac{1}{2}\right) = -1$$



(a) $m_1 = 2$. (b) $m_2 = -1/2$.

Figure 16.2.1.11. Sketching perpendicular lines.

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16.2.2: Equations of Lines

In this section we will develop the slope-intercept form of a line. When you have completed the work in this section, you should be able to look at the graph of a line and determine its equation in slope-intercept form.

The Slope-Intercept Form

In the previous section, we developed the formula for the slope of a line. Let's assume that the dependent variable is y and the independent variable is x and we have a line passing through the points $P\left(x_1, y_1\right)$ and $Q\left(x_2, y_2\right)$, as shown in Figure 16.2.2.1.

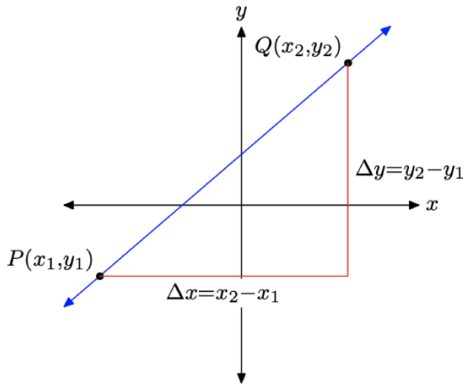


Figure 16.2.2.1. Determining the slope of a line through two points.

As we sweep our eyes from left to right, note that the change in x is $\Delta x = x_2 - x_1$ and the change in y is $\Delta y = y_2 - y_1$. Thus, the slope of the line is determined by the formula

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Now consider the line in Figure 16.2.2.2 Suppose that we are given two facts about this line:

1. The point where the line crosses the y-axis (the y-intercept) is (0, b).
2. The "slope" of the line is some number m.

To find the equation of the line pictured in Figure 16.2.2.2, select an arbitrary point Q(x, y) on the line, then compute the slope of the line using $(x_1, y_1) = P(0, b)$ and $(x_2, y_2) = Q(x, y)$ in the slope formula (1).

$$\text{Slope } = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - b}{x - 0}$$

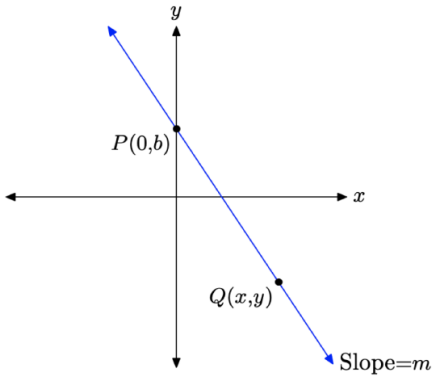


Figure 16.2.2.2. Find the equation of the line in slope-intercept form.

Simplify.

$$\text{Slope } = \frac{y - b}{x}$$

We're given that the slope is the number m, so substitute this number for the word "Slope" in the last result.

$$m = \frac{y-b}{x}$$

Multiply both sides of the last equation by x.

$$mx = y - b$$

Add b to both sides of the last equation to obtain

$$mx + b = y$$

or upon exchanging sides of the equation,

$$y = mx + b$$

The above discussion leads to the following result.

Definition: The Slope-Intercept Form of a Line

If the line L intercepts the y-axis at the point (0, b) and has slope m, then the equation of the line is

$$y = mx + b. \tag{16.2.2.1}$$

This form of the equation of a line is called the slope-intercept form. The function defined by the equation

$$f(x) = mx + b$$

is called a **linear function**.

It is important to note two key facts about the slope-intercept form y = mx + b.

- The coefficient of x (the m in y = mx + b) is the slope of the line.
- The constant term (the b in y = mx+b) is the y-coordinate of the y-intercept (0, b).

Procedure for Using the Slope-Intercept Form of a Line

When given the slope of a line and the y-intercept of the line, use the slope-intercept form as follows:

1. Substitute the given slope for m in the formula $y = mx + b$.
2. Substitute the y-coordinate of the y-intercept for b in the formula $y = mx + b$.

For example, if the line has slope −2 and the y-intercept (the point where the line crosses the y-axis) is (0, 3), then substitute m = −2 and b = 3 into Equation 16.2.2.1to obtain

$$y = -2x + 3$$

.

Let's look at some examples of use of this all-important formula.

Example 16.2.2.1

What is the equation of the line having slope −2/3 and y-intercept at (0, 3)? Sketch the line on graph paper.

Solution

The equation of the line is

$$y = mx + b$$

We're given that the slope is −2/3. Hence, m = −2/3. Secondly, we're given that the line intercepts the y-axis at the point (0, 3). In the slope-intercept form y = mx + b, recall that b represents the y-coordinate of the y-intercept. Hence, b = 3. Substitute m = −2/3 and b = 3 in equation (4), obtaining

$$y = -\frac{2}{3}x + 3$$

To sketch the graph of the line, first locate the y-intercept at P(0, 3), as shown in Figure 16.2.2.3 Starting from the y-intercept at P(0, 3), move 3 units to the right and 2 units downward to the point Q(3, 1). The required line passes through the points P and Q.

Note that the line “intercepts” the y-axis at 3 and slants downhill, in accordance with the fact that the slope is negative in this example.

Example 16.2.2.2

Given the graph of the line in Figure 16.2.2.4(a), determine the equation of the line.

Solution

First, locate the y-intercept of the line, which we’ve labeled P(0, −1) in Figure 16.2.2.4(b). In the formula y = mx + b, recall that b represents the y-coordinate of the y-intercept. Thus, b = −1.

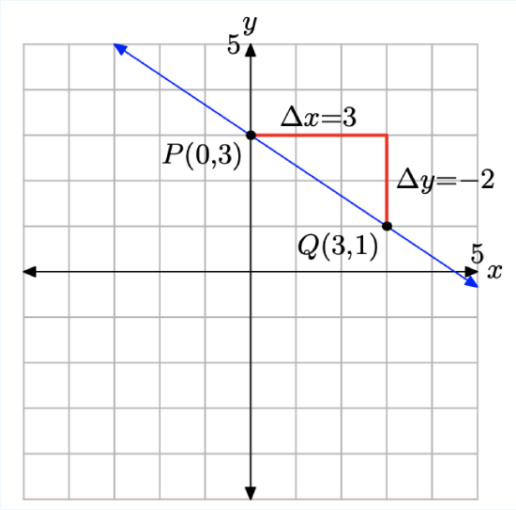


Figure 16.2.2.3. The line has y-intercept at (0, 3) and slope −2/3.

Secondly, we need to determine the slope of the line. In Figure 16.2.2.4(b), start at the point P, move 2 units to the right, then 3 units upward to the point Q(2, 2). This makes the slope

$$m = \frac{\Delta y}{\Delta x} = \frac{3}{2}$$

Substitute m = 3/2 and b = −1 into the slope-intercept form y = mx + b to obtain

$$y = \frac{3}{2}x - 1$$

which is the desired equation of the line.

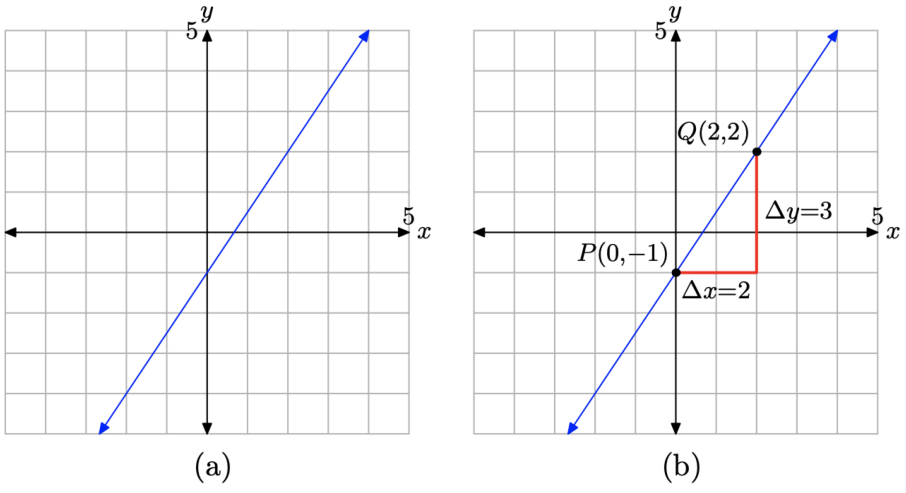


Figure 16.2.2.4. Determining the equation of a line from its graph.

Making Connections

If the connection between rate and slope is still not clear, let's recall an example we did earlier in the chapter.

Sebastian waves good-bye to his brother, who is talking to a group of his friends approximately 20 feet away. Sebastian then begins to walk away from his brother at a constant rate of 4 feet per second.

The distance between the brothers depends upon the amount of time that has passed, so we set distance d on the vertical axis and time t on the horizontal axis, as shown in Figure 16.2.2.5 Note that d and t are taking the "usual" place of y and x, respectively. The distance separating the brothers at time t = 0 is d = 20 feet. This is indicated with the "d-intercept" at P(0, 20) in Figure 16.2.2.5

Next, the distance between the brothers is increasing at a rate of 4 feet per second. Starting at the point P(0, 20), move 1 second to the right (2 boxes) and 4 units up (1 box) to the point Q(1, 24), as shown in Figure 16.2.2.5 The line through the points P and Q then models the distance between the brothers as a function of time.

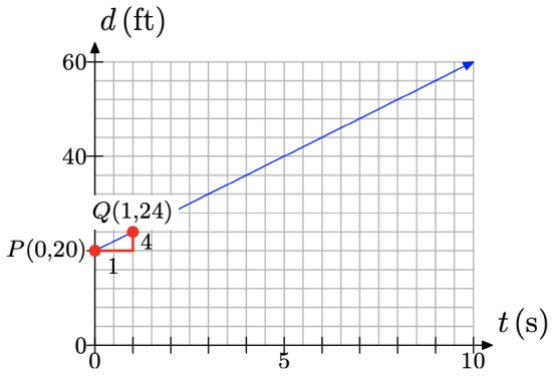


Figure 16.2.2.5. Distance between brothers as a function of time.

If you recall, we then determined a relation between distance d and time t by examining the distance between the brothers at times t = 0, 1, 2, and 3, and summarizing the results in Table 16.2.2.1

Table 16.2.2.1 Determining a model equation.

t	d
0	20
1	20+4(1)
2	20+4(2)

t	d
3	20+4(3)

Intuition then led to the following model, which provides the distance d between the brothers as a function of time t.

$$d = 20 + 4t$$

Again, readers should check that equation (6) produces the results in Table $16.2.2.1$at t = 0, 1, 2, and 3.

Alternatively, with the theory developed in this section, we would develop the equation of the line by using the slope-intercept form of the line; that is,

$$y = mx + b$$

However, in this case, the dependent variable is d, not y, and the independent variable is t, not x. So, replace y and x in equation (7) with d and t, respectively, obtaining

$$d = mt + b$$

Next, the line intercepts the d-axis at P(0, 20), so b = 20. Furthermore, the slope of the line is 4 feet per second, so m = 4. Substitute m = 4 and b = 20 in equation (8) to obtain

$$d = 4t + 20$$

or using function notation, $d(t) = 4t + 20$. Note that equation (9) is identical to the intuitively generated model of equation (6).

Hopefully, this development should cement for all time the idea that slope of the line is the rate at which the dependent variable is changing with respect to the independent variable.

The Standard Form of a Line

We now know that if our equation has the form $y = mx + b$ (or can be manipulated into this form), the graph will be a line. Let's take a moment to demonstrate that the graph of the equation $Ax + By = C$, where A, B, and C are constants, is a line.

If we can place the form $Ax + By = C$ into slope-intercept form $y = mx + b$, then that will demonstrate that the graph of $Ax + By = C$ is a line. So, start with $Ax + By = C$ and subtract Ax from both sides of the equation.

$$By = -Ax + C$$

Divide both sides of this last equation by B. Note that there is an assumption here that $B \neq 0$. We will handle the case when B = 0 separately, at the end of this section.

$$\frac{By}{B} = \frac{-Ax + C}{B}$$
$$y = -\frac{A}{B}x + \frac{C}{B}$$

When we compare $y = -(A/B)x + (C/B)$ with y = mx + b, we note that the slope is m = −A/B and the y-coordinate of the y-intercept is b = C/B. Because we were successful in placing the equation $Ax + By = C$ into slope-intercept form, we now know that the graph of $Ax + By = C$ is a line (we'll need this result in later work).

The Standard Form of a Line

The graph of the equation Ax + By = C is a line. The form

$$Ax + By = C$$

is called the standard form of a line.

Let's look at an example.

Example 16.2.2.3

The equation 3x+4y = 12 is in standard form. Place this equation in slope-intercept form, determine the slope and y-intercept, then use these results to draw the graph of the line.

Solution

First, solve the equation 3x + 4y = 12 for y.

$$\begin{aligned} 3x+4y &= 12 \\ 4y &= -3x+12 \\ y &= -\frac{3}{4}x+3 \end{aligned}$$

Note in the last step how the distributive property came into play. When we divided −3x + 12 by 4, we divided each term by 4, getting (−3/4)x + 3.

When we compare y = (−3/4)x+3 with the general slope-intercept form y = mx+b, we determine that the slope is m = −3/4 and the y-coordinate of the y-intercept is b = 3. To sketch the graph of the line, as we've done in Figure 16.2.2.6, plot the y-intercept at P(0, 3), then move 4 units to the right and 3 units down to the point Q(4, 0). The line passing through the points P and Q is the required line.

Note again that the slope is m = −3/4 and the line slants "downhill." Also, b = 3 and the line "intercepts" the y-axis at P(0, 3).

Let's look at another example.

Example 16.2.2.4

In Example 16.2.2.1, we determined that the given line had the equation

$$y = \frac{3}{2}x - 1$$

Place this equation in standard from $Ax + By = C$, where A, B, and C are integers and $A > 0$.

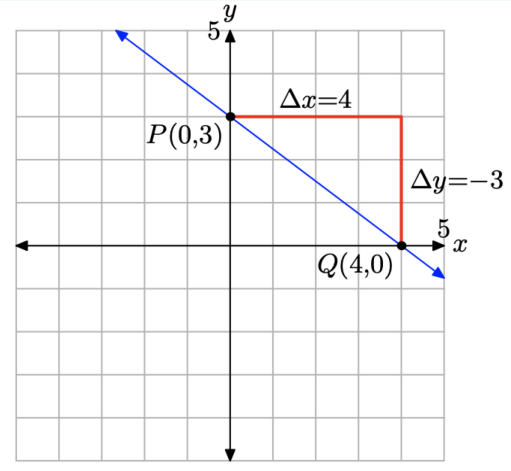


Figure 16.2.2.6. The graph of $3x + 4y = 12$.

Solution

We're requested to place the equation $y = (3/2)x - 1$ in the form $Ax + By = C$, where A, B, and C are integers, so let's begin by clearing fractions from the equation. Multiply both sides of the equation by the common denominator 2.

$$\begin{aligned} y &= \frac{3}{2}x - 1 \\ 2y &= 2\left(\frac{3}{2}x - 1\right) \\ 2y &= 3x - 2 \end{aligned}$$

Now, subtract 2y from both sides of the equation, then add 2 to both sides of the equation to obtain

$$2 = 3x - 2y$$

or equivalently,

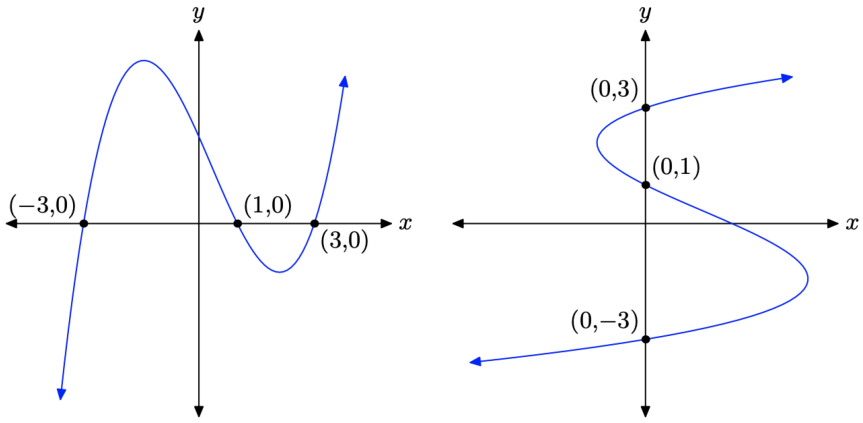
$$3x - 2y = 2$$

Note that this last result is in standard form $Ax + By = C$, where A, B, and C are integers and A > 0.

Intercepts

We now know that the graph of the equation $Ax + By = C$, where A, B, and C are constants, is a line. Because the graph of $Ax + By = C$ is a line, to draw the graph of the line, we need only find two points that satisfy the equation, plot them, then draw a line through them. Our two favorite points to work with are the x- and y-intercepts, because each involves the number zero, an easy number to work with.

Consider the graph in Figure 16.2.2.7(a). Note that the graph passes through the x-axis three times. The points where the graph intercepts the x-axis are called x-intercepts. Note that each of these points has a defining value in common: the y-value of each of these x-intercepts is equal to zero.



(a) x-intercepts. (b) y-intercepts.

Figure 16.2.2.7.

How to Find an x-intercept

To find an x intercept, let y = 0 in the equation and solve for x.

On the other hand, consider the graph in Figure 16.2.2.7(b). Note that this is not a function (fails the vertical line test) but the graph intercepts the y-axis three separate times. The points where the graph intercepts the y-axis are called y-intercepts. Note that each of the y-intercepts in Figure 16.2.2.7(b) has a defining value in common: the x-value of each of the y-intercepts is equal to zero.

How to Find a y-intercept

To find a y-intercept, let x = 0 in the equation and solve for y.

Let's put what we've learned into practice.

Example 16.2.2.5

Sketch the graph of 3x + 4y = 12.

Solution

We drew the graph of the equation 3x + 4y = 12 in Figure 16.2.2.6 There we solved the equation for y to determine the slope and the y-intercept. These, in turn, were used to draw the graph of 3x + 4y = 12 in Figure 16.2.2.6

Here, our approach will be different. We will first determine the x- and y-intercepts, plot them, then draw a line through these intercepts. Hopefully, we will get a result that matches that in Figure 16.2.2.6

To find the x-intercept, let $y = 0$ in $3x + 4y = 12$ and solve for x.

$$\begin{aligned} 3x + 4y &= 12 \\ 3x + 4(0) &= 12 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

Hence, the x-intercept is the point Q(4, 0). To find the y-intercept, let x = 0 in $3x + 4y = 12$ and solve for y.

$$\begin{aligned} 3x + 4y &= 12 \\ 3(0) + 4y &= 12 \\ 4y &= 12 \\ y &= 3 \end{aligned}$$

Hence, the y-intercept is the point P(0, 3). In Figure 16.2.2.8, we've plotted both x- and y-intercepts and drawn a line through them. Note that the resulting line in Figure 16.2.2.8matches the same line drawn in Figure 16.2.2.6(where we used a different method).

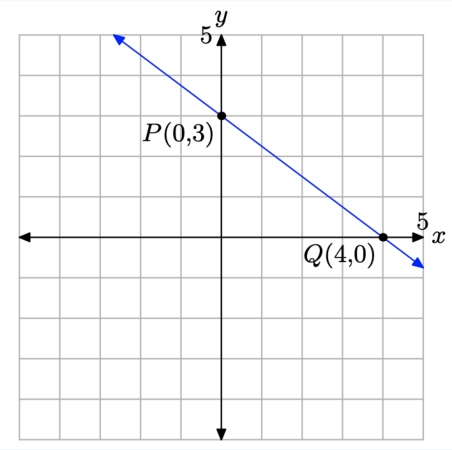


Figure 16.2.2.8 Plotting the x- and y-intercepts.

We recommend that whenever the line is given in Standard Form $Ax + By = C$, find the x- and y-intercepts, plot them, then draw a line through them. This technique is quite efficient because working with the number zero greatly simplifies the calculations.

Horizontal and Vertical Lines

We've introduced the standard form of the line $Ax + By = C$. The case where A and B are simultaneously equal to zero is not very interesting.4 However, the following two cases are of interest.

1. If we let A = 0 and B 6= 0 in the standard form $Ax + By = C$, then By = C, or equivalently y = C/B. Note that this has the form y = b, where b is some constant.
2. Similarly, if we let B = 0 and A 6= 0 in the standard form $Ax + By = C$, then Ax = C, or equivalently, x = C/A. Note that this has the form x = a, where a is some constant.

The lines having the form x = a and y = b are two of the easiest lines to plot. Let's look at an example of each.

Example 16.2.2.6

Sketch the graph of the equation x = 3.

Solution

The direction “sketch the graph of the equation x = 3” can be quite vexing unless one remembers that a graph of an equation is the set of all points that satisfy the equation. Thus, the direction is better posed if we say “sketch the set of all points (x, y) that satisfy x = 3,” or equivalently, “sketch the set of all points (x, y) that have an x-value of 3.” Then it is an easy matter to sketch the vertical line shown in Figure 16.2.2.9

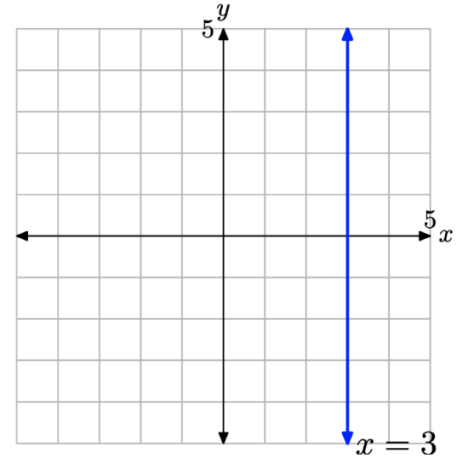


Figure 16.2.2.9. The graph of the equation x = 3.

Note that each point on the line has an x-value equal to 3. Also, note that the slope of this vertical line is undefined.

Example 16.2.2.7

Sketch the graph of the equation y = 3.

Solution

This direction is better posed if we say “sketch the set of all points (x, y) that satisfy y = 3,” or equivalently, “sketch the set of all points (x, y) that have a y-value of 3.” Then it is an easy matter to sketch the horizontal line shown in Figure 16.2.2.10

Note that each point on the line has a y-value equal to 3. Also, note that this horizontal line has slope zero.

Two final comments are in order. Because the line in Figure 10 has slope zero and y-intercept (0, 3), we can insert m = 0 and b = 3 into the slope intercept form y = mx + b and obtain

$$y = 0x + 3$$

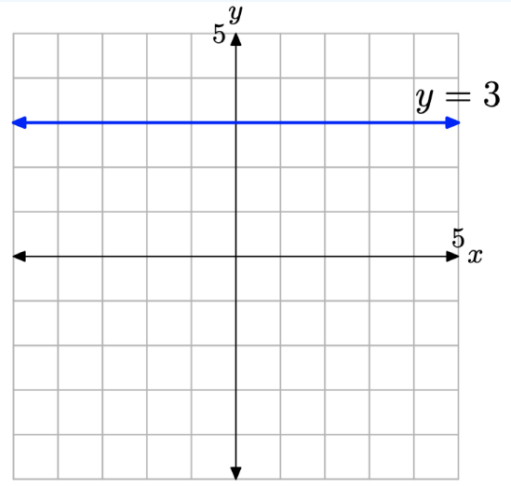


Figure 16.2.2.10. The graph of the equation y = 3.

which of course, is equivalent to y = 3. However, the vertical line shown in Figure 16.2.2.9 has “undefined” slope, so this approach is unavailable. We must simply recognize that the vertical line in Figure 16.2.2.9 consists of all points having an x-value equal to 3, and then intuit that its equation is x = 3.

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16.2.3: Rational Functions

Learning Objectives

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, x, produced. This is given by the equation $C(x) = 15,000x - 0.1x^2 + 1000$. If we want to know the average cost for producing x items, we would divide the cost function by the number of items, x. The average cost function, which yields the average cost per item for x items produced, is

$$f(x) = \frac{15,000x - 0.1x^2 + 1000}{x}$$

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

Using Arrow Notation

We have seen the graphs of the basic **reciprocal function** and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure 16.2.3.1, and notice some of their features.

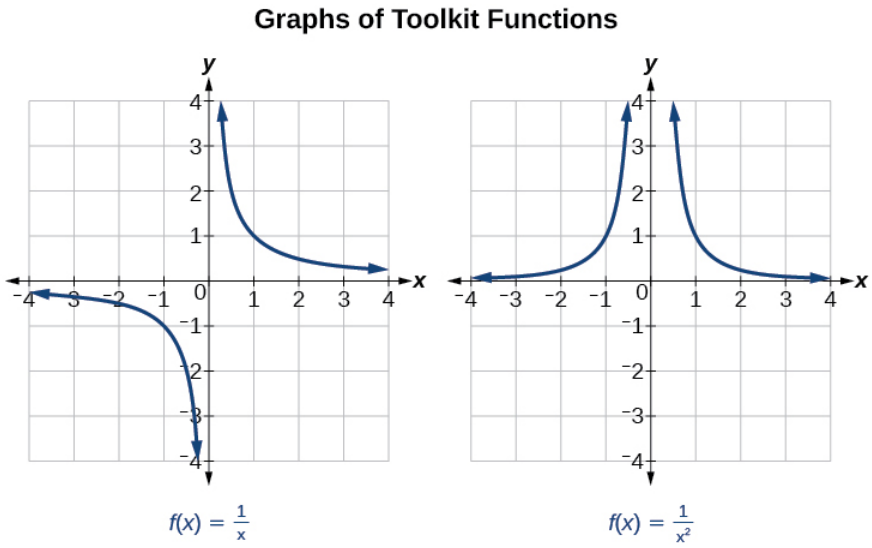

Figure 16.2.3.1

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

- On the left branch of the graph, the curve approaches the x-axis $(y = 0)$ as $x \to -\infty$.
- As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
- Finally, on the right branch of the graph, the curves approaches the x-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that x or $f(x)$ is approaching a particular value (Table 16.2.3.1).

Table 16.2.3.1

Symbol	Meaning

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table 16.2.3.2

Table 16.2.3.2

x	–0.1	–0.01	–0.001	–0.0001
$f(x) = \frac{1}{x}$	–10	–100	–1000	–10,000

We write in arrow notation

$$\text{as } x \to 0^-, f(x) \to -\infty$$

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table 16.2.3.3

Table 16.2.3.3

x	0.1	0.01	0.001	0.0001
$f(x) = \frac{1}{x}$	10	100	1000	10,000

We write in arrow notation

$$\text{As } x \to 0^+, f(x) \to \infty \text{ .}$$

See Figure 16.2.3.2

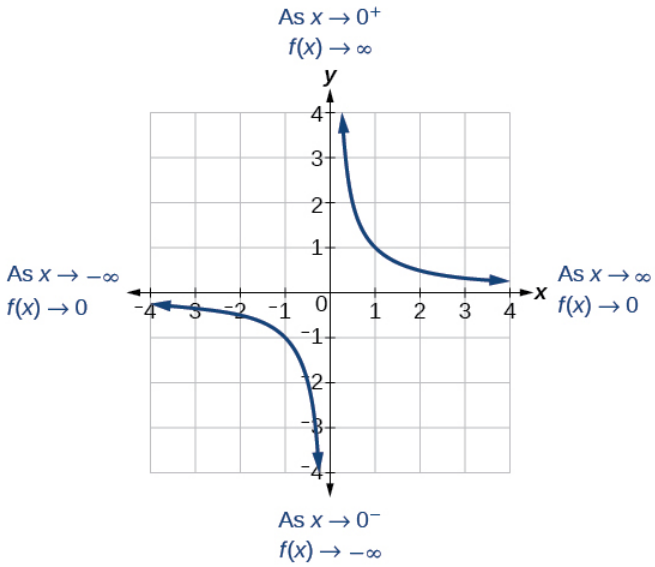

Figure 16.2.3.2.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero (Figure 16.2.3.3).

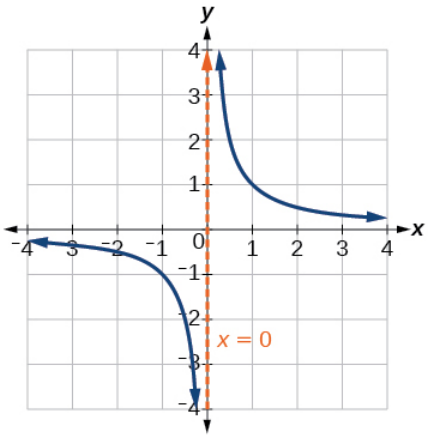

Figure 16.2.3.3.

Definition: VERTICAL ASYMPTOTE

A vertical asymptote of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a. We write

As $x \to a$, $f(x) \to \infty$, or as $x \to a$, $f(x) \to -\infty$.

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. As the values of x approach negative infinity, the function values approach 0 (Figure 16.2.3.4). Symbolically, using arrow notation

As $x \to \infty$, $f(x) \to 0$,and as $x \to -\infty$, $f(x) \to 0$.

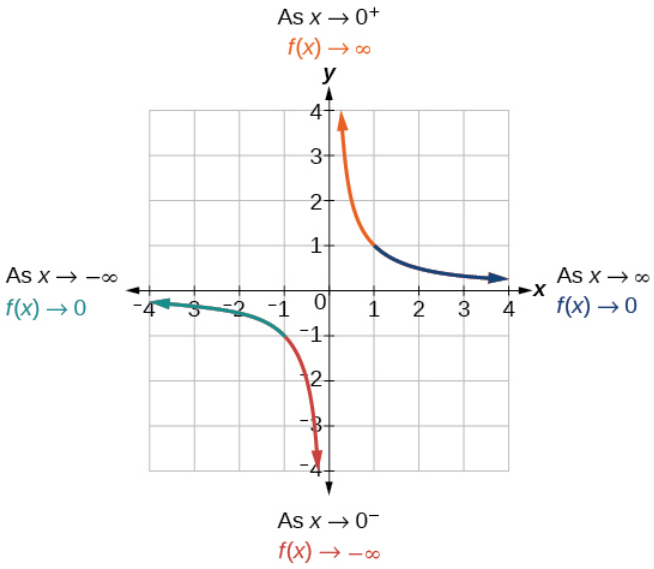

Figure 16.2.3.4.

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure 16.2.3.5

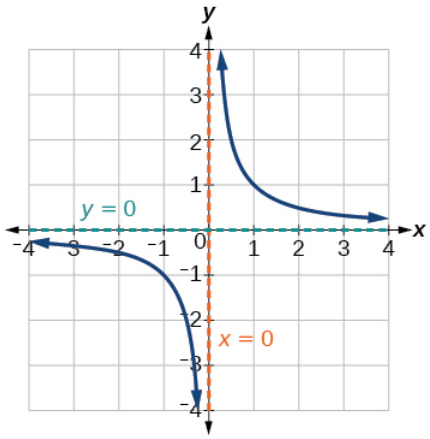

Figure 16.2.3.5.

Definition: HORIZONTAL ASYMPTOTE

A *horizontal asymptote* of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \to \infty \text{ or } x \to -\infty \text{ , } f(x) \to b.$$

Example 16.2.3.1: Using Arrow Notation.

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure 16.2.3.6

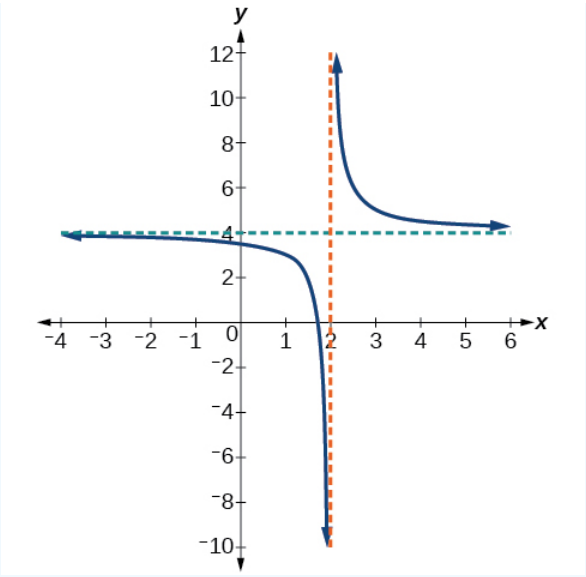

Figure 16.2.3.6.

Solution

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$.

$$\text{As } x \to 2^-,\ f(x) \to -\infty, \text{ and as } x \to 2^+,\ f(x) \to \infty.$$

And as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4$. As the inputs increase without bound, the graph levels off at 4.

$$\text{As } x \to \infty,\ f(x) \to 4 \text{ and as } x \to -\infty,\ f(x) \to 4.$$

Exercises 16.2.3.1

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

Answer

End behavior: as $x \to \pm\infty$, $f(x) \to 0$;

Local behavior: as $x \to 0$, $f(x) \to \infty$ (there are no x- or y-intercepts)

Example 16.2.3.2: Using Transformations to Graph a Rational Function.

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3 \tag{16.2.3.1}$$

or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x+7}{x+2} \tag{16.2.3.2}$$

The graph of the shifted function is displayed in Figure 16.2.3.7.

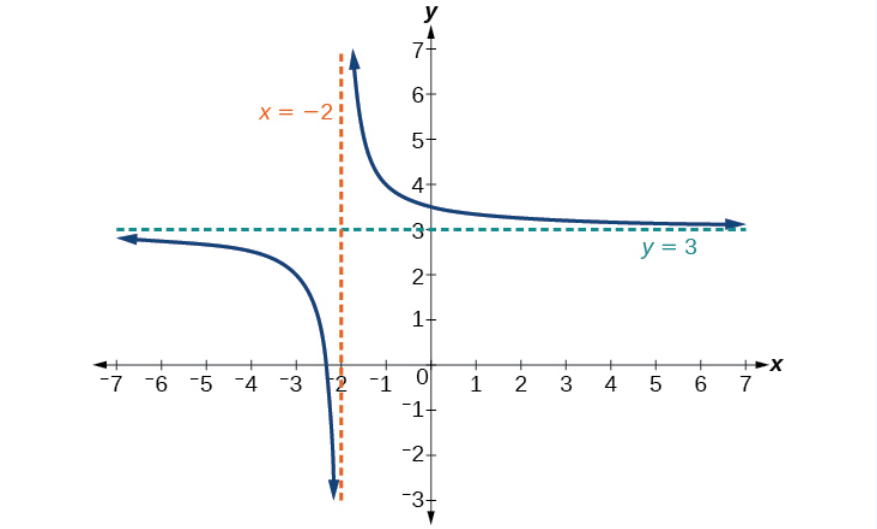

Figure 16.2.3.7.

Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

$$\text{As } x \to -2^{-}\text{, } f(x) \to -\infty\text{, and as } x \to -2^{+}\text{, } f(x) \to \infty\text{.}$$

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

$$\text{As } x \to \pm\infty\text{, } f(x) \to 3\text{.}$$

Analysis

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

Exercise 16.2.3.2

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Solution

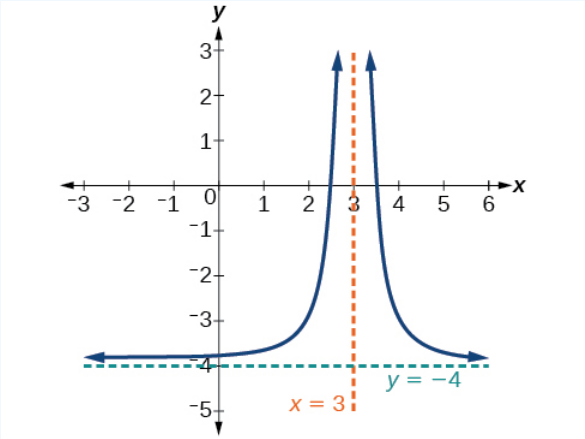

Figure 16.2.3.8.

The function and the asymptotes are shifted 3 units right and 4 units down. As $x \to 3$, $f(x) \to \infty$, and as $x \to \pm\infty$, $f(x) \to -4$.

The function is $f(x) = \frac{1}{(x-3)^2} - 4$.

Solving Applied Problems Involving Rational Functions

In Example 16.2.3.2 we shifted a toolkit function in a way that resulted in the function $f(x)=\frac{3x+7}{x+2}$. This is an example of a rational function. A *rational function* is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

Definition: RATIONAL FUNCTION

A ***rational function*** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x)=\frac{P(x)}{Q(x)}=\frac{a_p x^p+a_{p-1}x^{p-1}+\ldots+a_1 x+a_0}{b_q x^q+b_{q-1}x^{q-1}+\ldots+b_1 x+b_0},\ Q(x)\neq 0 \tag{16.2.3.3}$$

Example 16.2.3.3: Solving an Applied Problem Involving a Rational Function

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

Solution

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\begin{array}{c}\text{water: } W(t)=100+10t \text{ in gallons}\\ \text{sugar: } S(t)=5+1t \text{ in pounds}\end{array}$$

The concentration, C, will be the ratio of pounds of sugar to gallons of water

$$C(t)=\frac{5+t}{100+10t} \tag{16.2.3.4}$$

The concentration after 12 minutes is given by evaluating $C(t)$ at $t=12$.

$$C(12)=\frac{5+12}{100+10(12)} \tag{16.2.3.5}$$

$$=\frac{17}{220} \tag{16.2.3.6}$$

This means the concentration is 17 pounds of sugar to 220 gallons of water.

At the beginning, the concentration is

$$C(0)=\frac{5+0}{100+10(0)} \tag{16.2.3.7}$$

$$=\frac{1}{20} \tag{16.2.3.8}$$

Since $\frac{17}{220}\approx 0.08>\frac{1}{20}=0.05$, the concentration is greater after 12 minutes than at the beginning.

Analysis

To find the horizontal asymptote, divide the leading coefficient in the numerator by the leading coefficient in the denominator:

$$\frac{1}{10}=0.1 \tag{16.2.3.9}$$

Notice the horizontal asymptote is $y=0.1$. This means the concentration, C, the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

Exercise 16.2.3.3

There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

Answer

$\frac{12}{11}$

Finding the Domains of Rational Functions

A *vertical asymptote* represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Definition: DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

How To: Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Example 16.2.3.4: Finding the Domain of a Rational Function

Find the domain of $f(x) = \dfrac{x+3}{x^2-9}$.

Solution

Begin by setting the denominator equal to zero and solving.

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

Analysis

A graph of this function, as shown in Figure 16.2.3.9, confirms that the function is not defined when $x = \pm 3$.

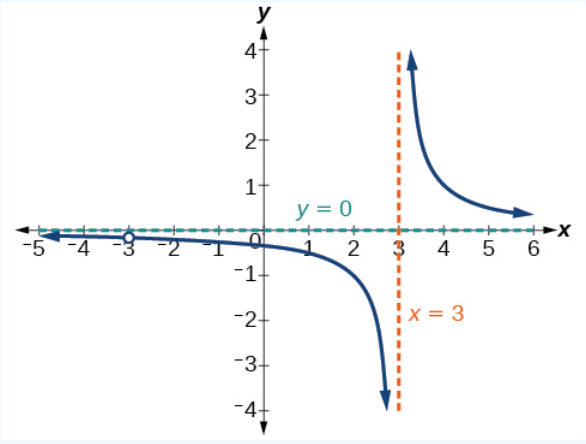

Figure 16.2.3.9.

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

Exercise 16.2.3.4

Find the domain of $f(x) = \dfrac{4x}{5(x-1)(x-5)}$.

Answer

The domain is all real numbers except $x = 1$ and $x = 5$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

How To: Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or "holes."

Example 16.2.3.5: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \dfrac{5+2x^2}{2-x-x^2}$.

Solution

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2}$$
$$= \frac{5+2x^2}{(2+x)(1-x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0$$
$$x = -2,\ x = 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 16.2.3.10confirms the location of the two vertical asymptotes.

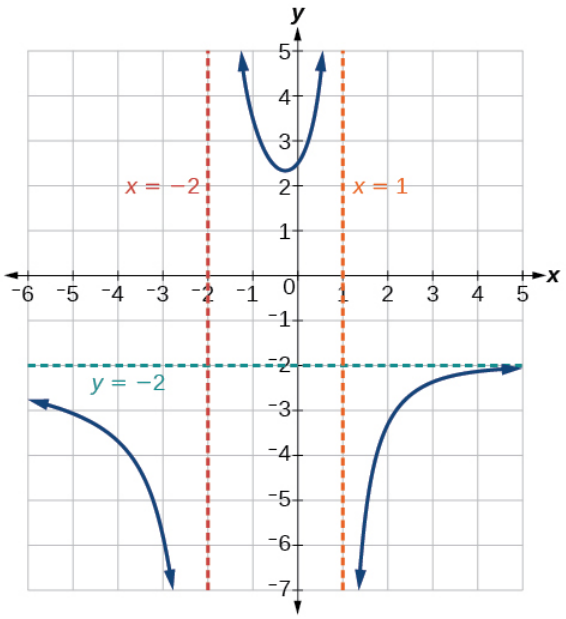

***Figure* 16.2.3.10.**

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x) = \dfrac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x=-1$, is the location of the removable discontinuity. Notice also that $(x-3)$ is not a factor in both the numerator and denominator. The zero of this factor, $x=3$, is the vertical asymptote. See Figure 16.2.3.11 [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

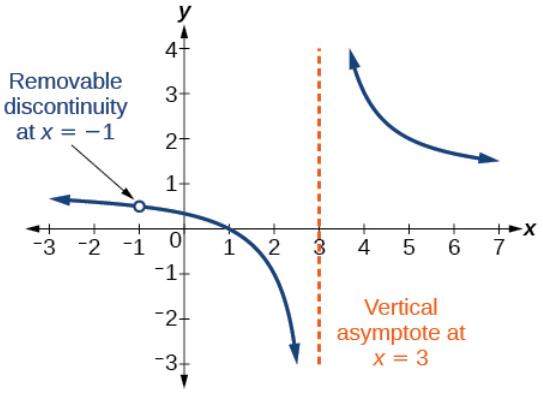

Figure 16.2.3.11.

REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x=a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the

multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Example 16.2.3.6: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \dfrac{x-2}{x^2-4}$.

Solution

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x-2$. The zero for this factor is $x = 2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2$. The zero for this factor is $x = -2$. The vertical asymptote is $x = -2$. See Figure 16.2.3.12

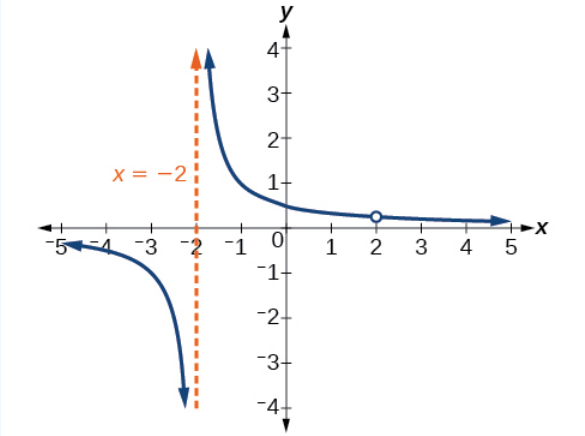

Figure 16.2.3.12.

The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole.

Exercise 16.2.3.5

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \dfrac{x^2-25}{x^3-6x^2+5x}$.

Answer

Removable discontinuity at $x = 5$.

Vertical asymptotes: $x = 0$, $x = 1$.

Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator > degree of the numerator, there is a **horizontal asymptote** at $y = 0$.

Example: $f(x)=\dfrac{4x+2}{x^2+4x-5}$

In this case, the end behavior is $f(x)\approx\dfrac{4x}{x^2}=\dfrac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x)=\dfrac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y=0$. See **Figure** 16.2.3.13 Note that this graph crosses the horizontal asymptote.

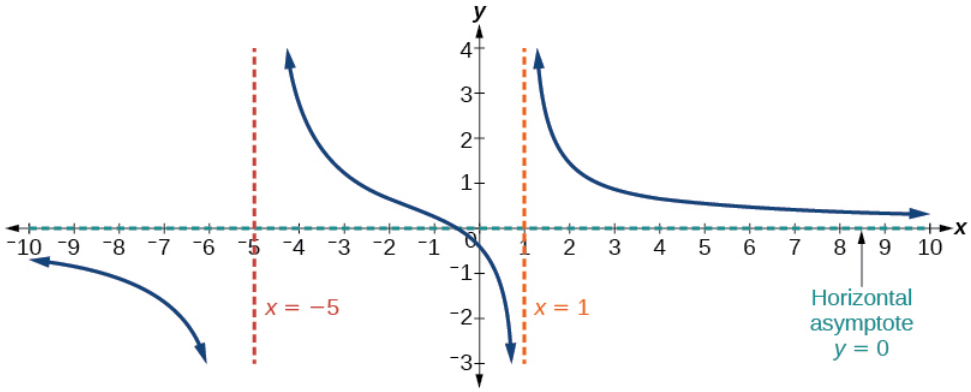

Figure 16.2.3.13: Horizontal asymptote $y=0$ when $f(x)=\dfrac{p(x)}{q(x)}$, $q(x)\neq 0$ where degree of p < degree of q.

Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

Example: $f(x)=\dfrac{3x^2-2x+1}{x-1}$

In this case, the end behavior is $f(x)\approx\dfrac{3x^2}{x}=3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x)=3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x)=3x$ looks like a diagonal line, and since f will behave similarly to g, it will approach a line close to $y=3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\dfrac{3x^2-2x+1}{x-1}$. The quotient is $3x+1$, and the remainder is 2. The slant asymptote is the graph of the line $g(x)=3x+1$. See Figure 16.2.3.14

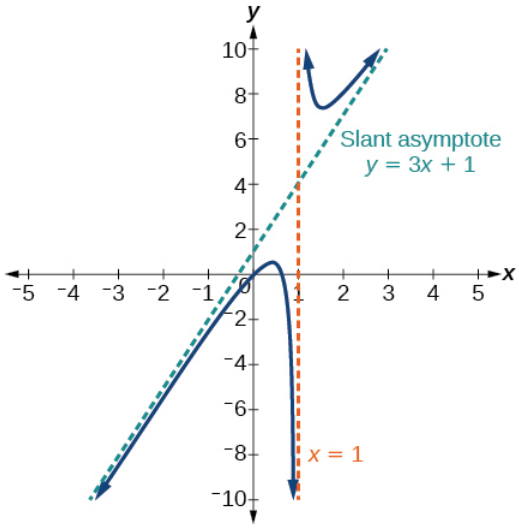

Figure 16.2.3.14: Slant asymptote when $f(x)=\dfrac{p(x)}{q(x)}$, $q(x)\neq 0$ where degree of p >degree of q by 1.

Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \frac{a_n}{b_n}$, where a_n and b_n are respectively the leading coefficients of $p(x)$ and $q(x)$ for $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$.

Example: $f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$. See Figure 16.2.3.15 Note that this graph crosses the horizontal asymptote.

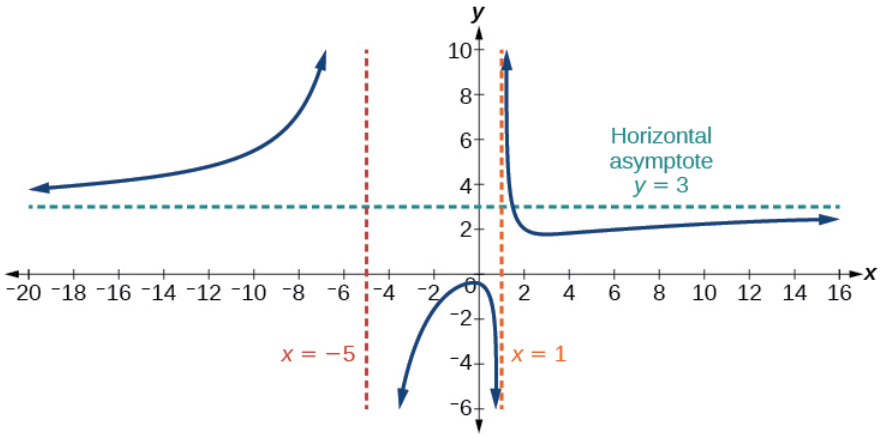

Figure 16.2.3.15: Horizontal asymptote when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of p = degree of q.

Notice that, while the graph of a rational function will never cross a **vertical asymptote**, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the **end behavior** of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty, f(x) \to \infty$$

HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

Example 16.2.3.7: Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$
3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$

Solution

For these solutions, we will use $f(x) = \dfrac{p(x)}{q(x)},\ q(x) \neq 0$.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of $p =$ degree of $q = 3$, so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$: The degree of $p = 2$ and degree of $q = 1$. Since $p > q$ by 1, there is a slant asymptote found at $\dfrac{x^2 - 4x + 1}{x + 2}$.

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$: The degree of $p = 2 <$ degree of $q = 3$, so there is a horizontal asymptote $y = 0$.

Example 16.2.3.8 Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \dfrac{5 + t}{100 + 10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is 10t, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty,\ C(t) \to \tfrac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example 16.2.3.9: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

Solution

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2$,and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure $16.2.3.16$

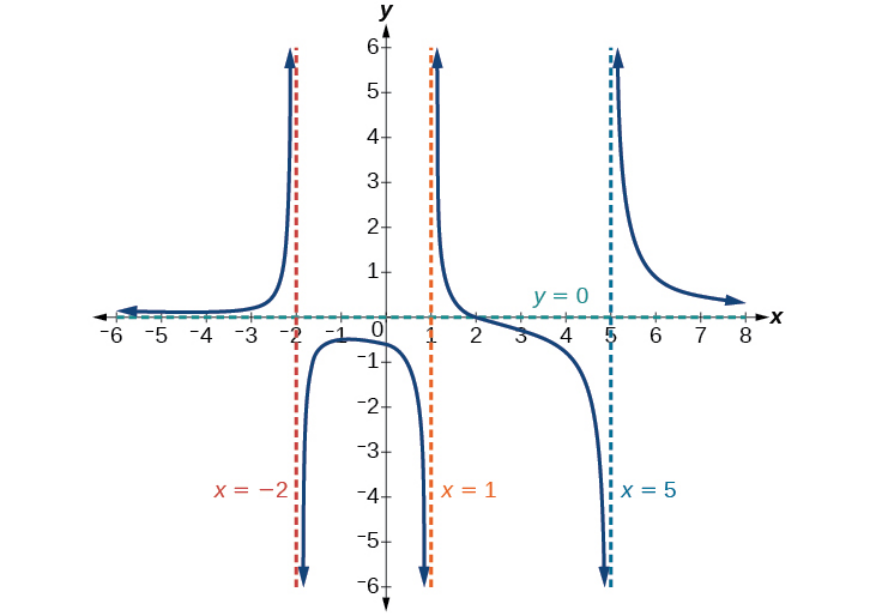

Figure 16.2.3.16.

Exercise 16.2.3.6

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Answer

Vertical asymptotes at $x = 2$ and $x = -3$

horizontal asymptote at $y = 4$.

INTERCEPTS OF RATIONAL FUNCTIONS

A *rational function* will have a y-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a y-intercept if the function is not defined at zero.

Likewise, a rational function will have x-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

Example 16.2.3.10: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution

We can find the y-intercept by evaluating the function at zero

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)}$$
$$= -\frac{6}{10}$$
$$= -\frac{3}{5}$$
$$= -0.6$$

The x-intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \text{ This is zero when the numerator is zero.}$$

$$0 = (x-2)(x+3)$$

$$x = 2, x = -3$$

The y-intercept is $(0, -0.6)$, the x-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 16.2.3.17.

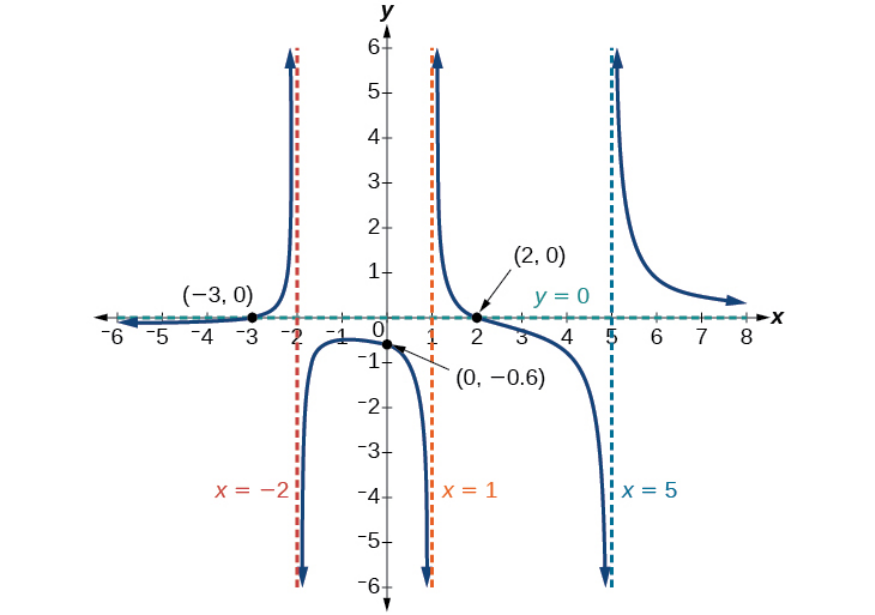

Figure 16.2.3.17.

Exercise 16.2.3.7

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

Answer

For the transformed reciprocal squared function, we find the rational form.

$$f(x) = \frac{1}{(x-3)^2} - 4 = \frac{1-4(x-3)^2}{(x-3)^2} = \frac{1-4(x^2-6x+9)}{(x-3)(x-3)} = \frac{-4x^2+24x-35}{x^2-6x+9}$$

Because the numerator is the same degree as the denominator we know that as $x \to \pm\infty$, $f(x) \to -4$; so $y = -4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x = 3$, because as $x \to 3$, $f(x) \to \infty$. We then set the numerator equal to 0 and find the x-intercepts are at $(2.5, 0)$ and $(3.5, 0)$. Finally, we evaluate the function at 0 and find the y-intercept to be at $(0, -\frac{35}{9})$.

Graphing Rational Functions

In Example 16.2.3.10 we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 16.2.3.18

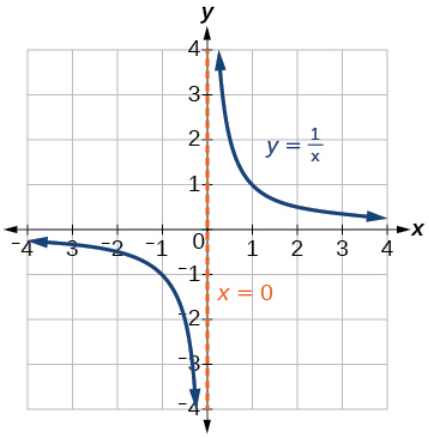

Figure 16.2.3.18.

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 16.2.3.19

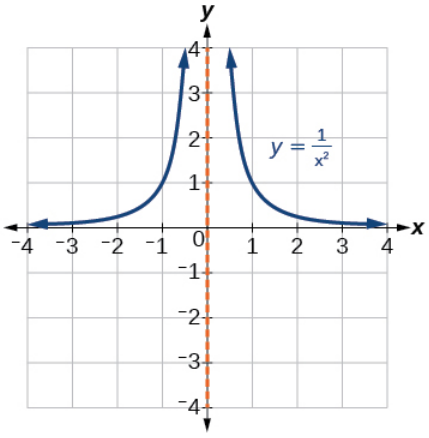

Figure 16.2.3.19.

For example, the graph of $f(x) = \dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure 16.2.3.20

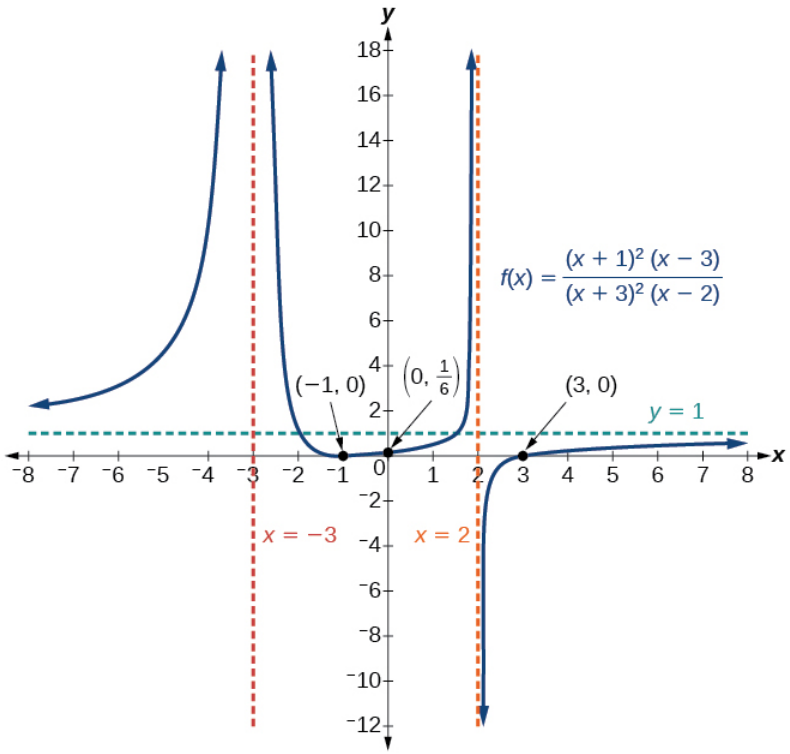

Figure 16.2.3.20.

- At the x-intercept $x=-1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the x-intercept $x=3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x=-3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x)=\dfrac{1}{x^2}$.
- At the vertical asymptote $x=2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x)=\dfrac{1}{x}$.

Howto: Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Example 16.2.3.11: Graphing a Rational Function

Sketch a graph of $f(x)=\frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0)=\frac{(0+2)(0-3)}{(0+1)^2(0-2)}=3$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x=-2$ and $x=3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a y-intercept at $(0,3)$ and x-intercepts at $(-2,0)$ and $(3,0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x+1=0$ and when $x-2=0$, giving us vertical asymptotes at $x=-1$ and $x=2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y=0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 16.2.3.21

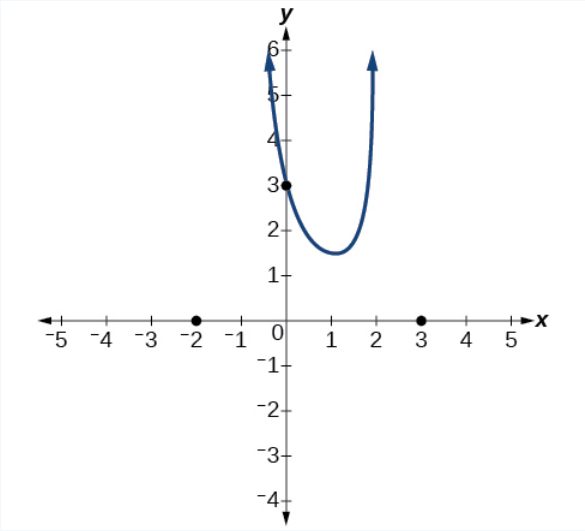

Figure 16.2.3.21.

The factor associated with the vertical asymptote at $x=-1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x=2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 16.2.3.22 After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

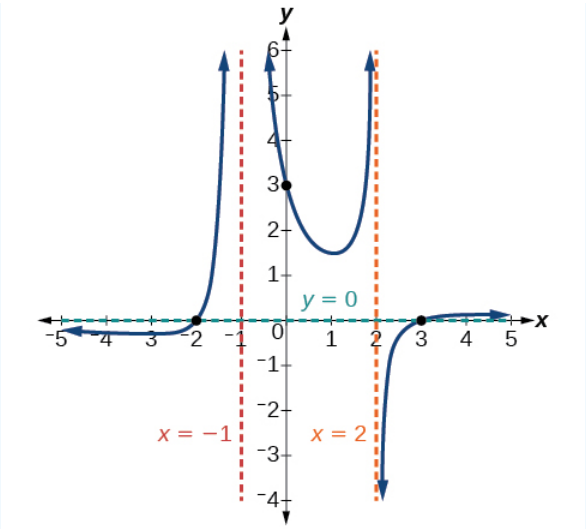

Figure 16.2.3.22.

Exercise 16.2.3.8

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Answer

Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ and $x = 3$. y-intercept at $(0, \frac{4}{3})$.

x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity 2, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the axis at this point.

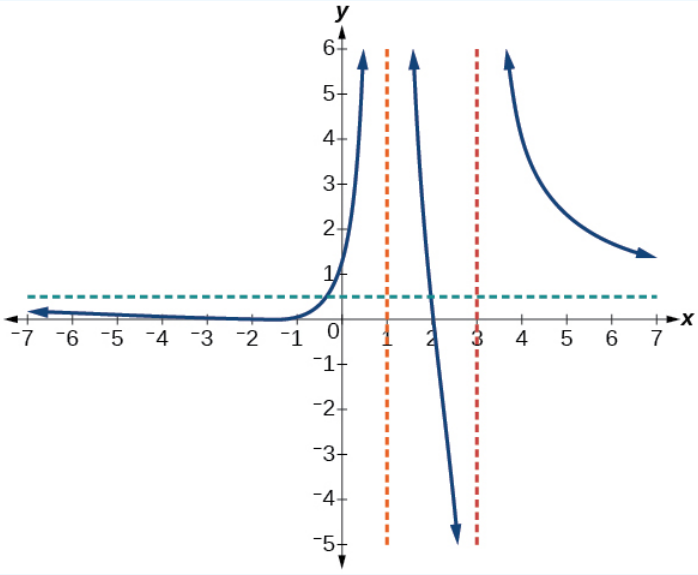

Figure 16.2.3.23

Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an *x*-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a

numerator of a function whose graph will pass through a set of *x*-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

WRITING RATIONAL FUNCTIONS FROM INTERCEPTS AND ASYMPTOTES

If a **rational function** has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any v_j, then the function can be written in the form:

$$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the x-intercept or by the horizontal asymptote if it is nonzero.

Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the *x*-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the "simplest" function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

Example 16.2.3.12: Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure 16.2.3.24

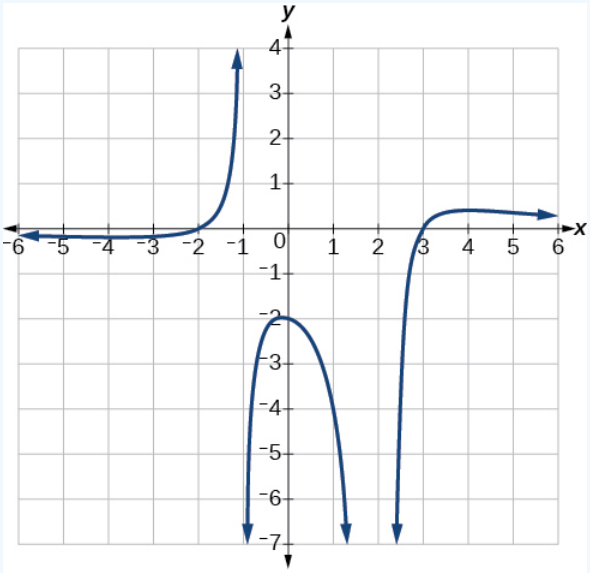

Figure 16.2.3.24.

Solution

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure 16.2.3.25

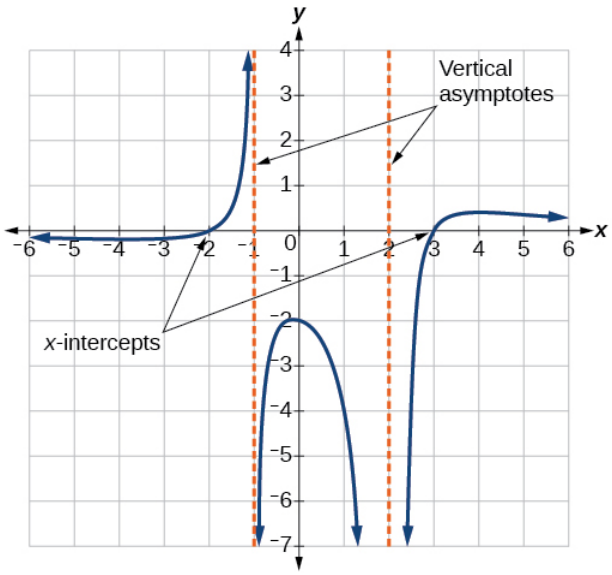

Figure 16.2.3.25.

We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $(0,-2)$.

$$-2 = a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$
$$-2 = a\frac{-6}{4}$$
$$a = \frac{-8}{-6} = \frac{4}{3}$$

This gives us a final function of $f(x) = \dfrac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

Media

Access these online resources for additional instruction and practice with rational functions.

- Graphing Rational Functions
- Find the Equation of a Rational Function
- Determining Vertical and Horizontal Asymptotes
- Find the Intercepts, Asymptotes, and Hole of a Rational Function

Key Equations

Rational Function $\quad f(x) = \dfrac{P(x)}{Q(x)} = \dfrac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0},\ Q(x) \neq 0$

Key Concepts

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. See Example 16.2.3.1.
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See Example.

- Application problems involving rates and concentrations often involve rational functions. See Example.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See Example.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See Example.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See Example.
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See Example, Example, Example, and Example.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See Example.
- If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any v_j, then the function can be written in the form

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16.2.4: Power Functions and Polynomial Functions

Learning Objectives

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table 16.2.4.1

Table 16.2.4.1

Year	2009	2010	2011	2012	2013
Bird Population	800	897	992	1,083	1,169

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800$, where $P(t)$ represents the bird population on the island t years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.

Figure 16.2.4.1: (credit: Jason Bay, Flickr)

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius r is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius r is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Definition: Power Function

A *power function* is a function that can be represented in the form

$$f(x) = kx^p \tag{16.2.4.1}$$

where k and p are real numbers, and k is known as the *coefficient*.

Q&A: Is $f(x) = 2^x$ a power function?

No. A power function contains a variable base raised to a fixed power (Equation 16.2.4.1). This function has a constant base raised to a variable power. This is called an exponential function, not a power function. This function will be discussed later.

Example 16.2.4.1: Identifying Power Functions

Which of the following functions are power functions?

$f(x) = 1$	Constant function
$f(x) = x$	Identify function
$f(x) = x^2$	Quadratic function
$f(x) = x^3$	Cubic function
$f(x) = \frac{1}{x}$	Reciprocal function
$f(x) = \frac{1}{x^2}$	Reciprocal squared function
$f(x) = \sqrt{x}$	Square root function
$f(x) = \sqrt[3]{x}$	Cube root function

Solution

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The **reciprocal** and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and **cube root** functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Exercise 16.2.4.1

Which functions are power functions?

- $f(x) = 2x^2 \cdot 4x^3$
- $g(x) = -x^5 + 5x^3 - 4x$
- $h(x) = \frac{2x^5 - 1}{3x^2 + 4}$

Answer

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.

Identifying End Behavior of Power Functions

Figure 16.2.4.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

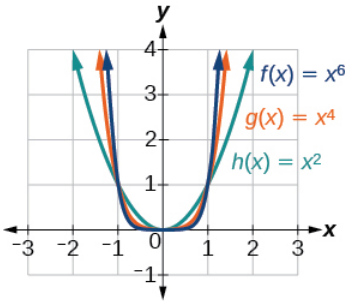


Figure 16.2.4.2: Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty,\ f(x) \to \infty$$

Figure 16.2.4.3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

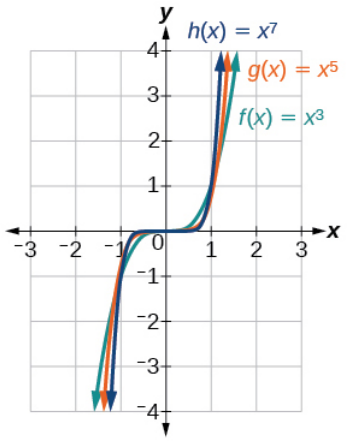


Figure 16.2.4.3: Odd-power function

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure 16.2.4.2 we see that even functions of the form $f(x) = x^n$, n even, are symmetric about the y-axis. In Figure 16.2.4.3 we see that odd functions of the form $f(x) = x^n$, n odd, are symmetric about the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$
$$\text{as } x \to \infty,\ f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small $(x \to -\infty)$ and get very large $x \to \infty$ is referred to as the *end behavior* of the function. We can use words or symbols to describe end behavior.

Figure 16.2.4.4 shows the end behavior of power functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power and the constant.

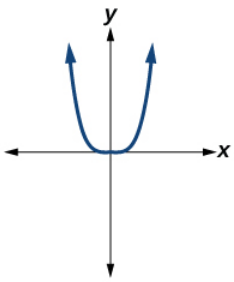
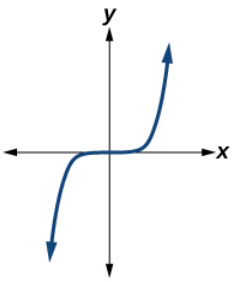
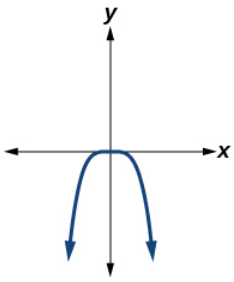
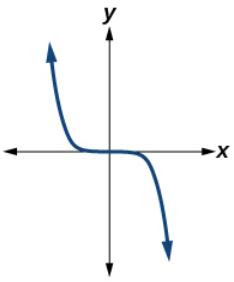
	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to \infty$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to -\infty$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to -\infty$

Figure 16.2.4.4

How To: Given a power function $f(x) = kx^n$ where n is a non-negative integer, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use Figure 16.2.4.4to identify the end behavior.

Example 16.2.4.2: Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. We write as $x \to \infty$, $f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty$, $f(x) \to \infty$. We can graphically represent the function as shown in Figure 16.2.4.5

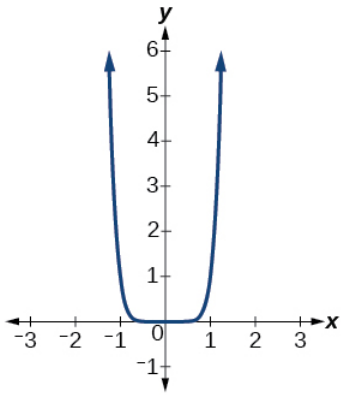


Figure 16.2.4.5: Graph of $f(x) = x^8$.

Example 16.2.4.3: Identifying the End Behavior of a Power Function.

Describe the end behavior of the graph of $f(x) = -x^9$.

Solution

The exponent of the power function is 9 (an odd number). Because the coefficient is –1 (negative), the graph is the reflection about the x-axis of the graph of $f(x) = x^9$. Figure 16.2.4.6 shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty$$
$$\text{as } x \to \infty,\ f(x) \to -\infty$$

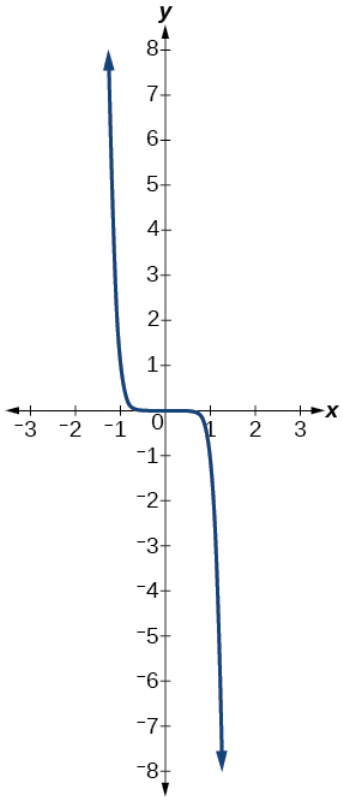


Figure 16.2.4.6: Graph of $f(x) = -x^9$.

Analysis

We can check our work by using the table feature on a graphing utility.

Table 16.2.4.2

x	$f(x)$
-10	1,000,000,000
-5	1,953,125
0	0
5	-1,953,125
10	-1,000,000,000

We can see from Table $16.2.4.2$ that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is very small (meaning that it is a very large negative value).

Exercise 16.2.4.2

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Answer

As x approaches positive or negative infinity, $f(x)$ decreases without bound: as $x \to \pm\infty$, $f(x) \to -\infty$ because of the negative coefficient.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius r of the spill depends on the number of weeks w that have passed. This relationship is linear.

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24 + 8w) \\ &= \pi(24 + 8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Definition: Polynomial Functions

Let n be a non-negative integer. A *polynomial function* is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \qquad (16.2.4.2)$$

This is called the general form of a polynomial function. Each a_i is a coefficient and can be any real number. Each product $a_i x^i$ is a *term of a polynomial function.*

Example 16.2.4.4: Identifying Polynomial Functions

Which of the following are polynomial functions?

- $f(x) = 2x^3 \cdot 3x + 4$
- $g(x) = -x(x^2 - 4)$
- $h(x) = 5\sqrt{x} + 2$

Solution

The first two functions are examples of polynomial functions because they can be written in the form of Equation 16.2.4.2, where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the highest power of the variable, or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.

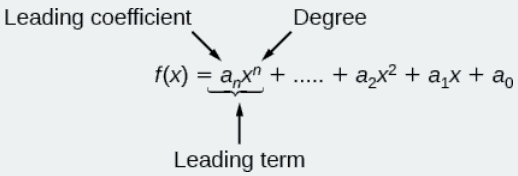


Figure 16.2.4.7

When a polynomial is written in this way, we say that it is in *general form*.

How To: Given a polynomial function, identify the degree and leading coefficient

1. Find the highest power of x to determine the degree function.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Example 16.2.4.5: Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$f(x) = 3 + 2x^2 - 4x^3$$

$$g(t) = 5t^5 - 2t^3 + 7t$$

$$h(p) = 6p - p^3 - 2$$

Solution

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, −4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, -1.

Exercise 16.2.4.3

Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$.

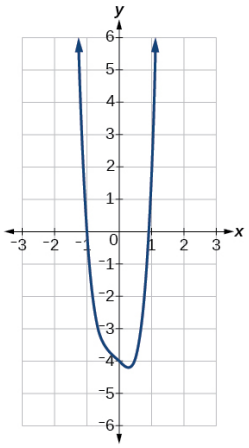
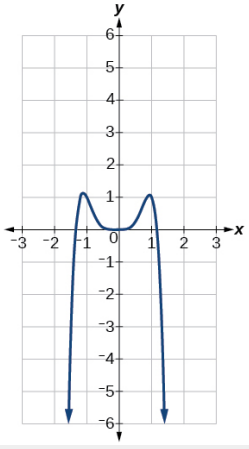
Answer

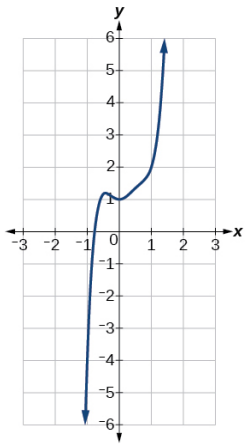
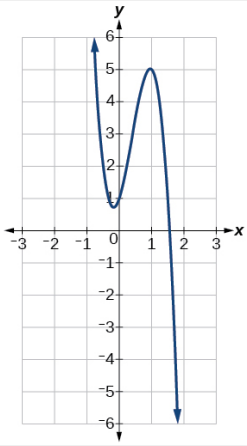
The degree is 6. The leading term is $-x^6$. The leading coefficient is -1.

Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree (Table $16.2.4.3$).

Table 16.2.4.3

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$	
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$	

Example 16.2.4.6: Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 16.2.4.8

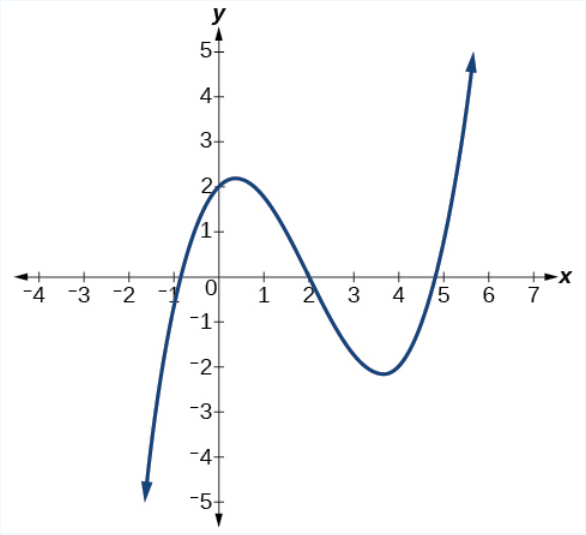


Figure 16.2.4.8.

Solution

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x\to\infty,\ f(x)\to\infty$$

$$\text{as } x\to-\infty,\ f(x)\to-\infty$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

Exercise 16.2.4.1

Describe the end behavior, and determine a possible degree of the polynomial function in Figure 16.2.4.9

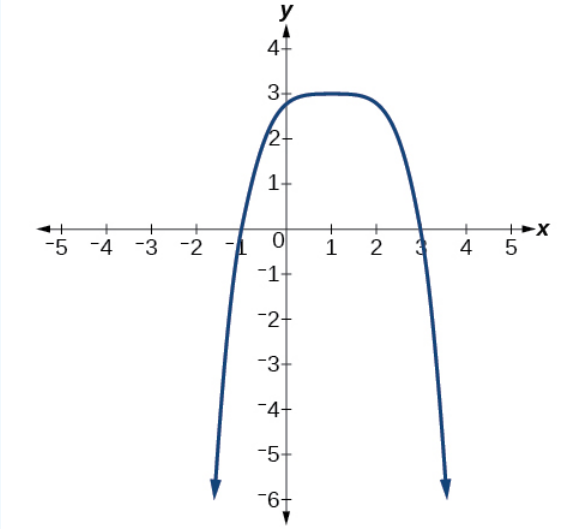


Figure 16.2.4.9

Answer

As $x\to\infty$, $f(x)\to-\infty$; as $x\to-\infty$, $f(x)\to-\infty$. It has the shape of an even degree power function with a negative coefficient.

Example 16.2.4.7: Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x)=-3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4-9x^3+12x^2 \end{aligned}$$

The general form is $f(x)=-3x^4-9x^3+12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x\to-\infty,\ f(x)\to-\infty$$

$$\text{as } x\to\infty,\ f(x)\to-\infty$$

Exercise 16.2.4.7

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

Answer

The leading term is $0.2x^3$, so it is a degree 3 polynomial. As x approaches positive infinity, $f(x)$ increases without bound; as x approaches negative infinity, $f(x)$ decreases without bound.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the y-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one y-intercept $(0, a_0)$. The x-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See Figure 16.2.4.10

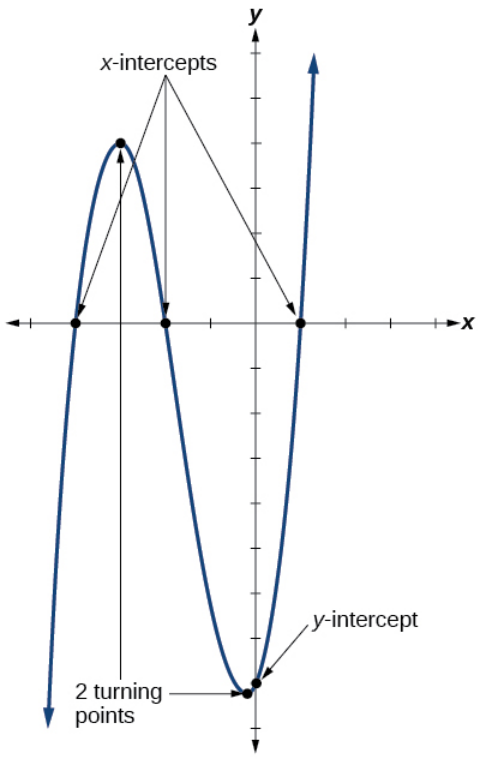


Figure 16.2.4.10

Defintion: Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The y-intercept is the point at which the function has an input value of zero. The x-intercepts are the points at which the output value is zero.

How To: Given a polynomial function, determine the intercepts.

1. Determine the y-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the x-intercepts by solving for the input values that yield an output value of zero.

Example 16.2.4.8: Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the y- and x-intercepts.

Solution

The y-intercept occurs when the input is zero, so substitute 0 for x.

$$\begin{aligned} f(0) &= (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The y-intercept is $(0, 8)$.

The x-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{aligned} x-2 &= 0 \quad \text{or} \quad x+1 = 0 \quad \text{or} \quad x-4 = 0 \\ x &= 2 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 4 \end{aligned}$$

The x-intercepts are $(2, 0)$,$(-1, 0)$, and $(4, 0)$.

We can see these intercepts on the graph of the function shown in Figure 16.2.4.11

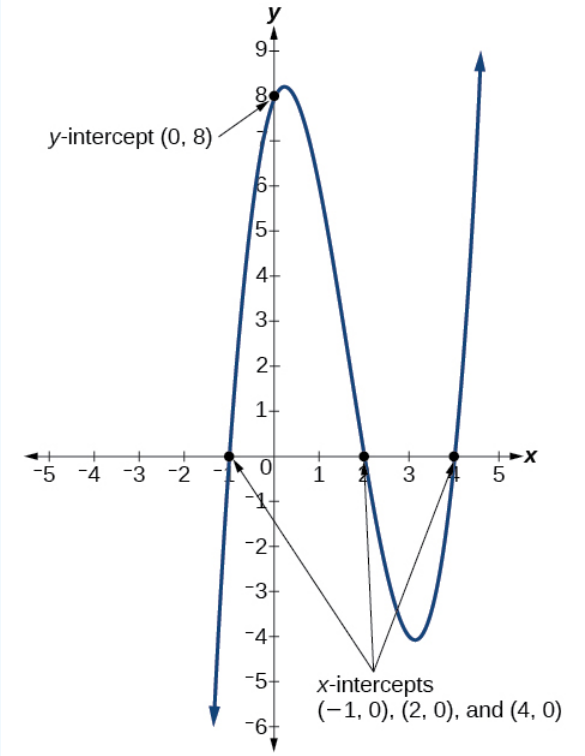


Figure 16.2.4.11: Graph of $f(x) = (x-2)(x+1)(x-4)$.

Example 16.2.4.9: Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the y- and x-intercepts.

Solution

The y-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^2 - 45 \\ &= -45 \end{aligned}$$

The y-intercept is $(0, -45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x^4 - 4x^2 - 45 \\ &= (x^2 - 9)(x^2 + 5) \\ &= (x-3)(x+3)(x^2+5) \end{aligned}$$

$$0 = (x-3)(x+3)(x^2+5)$$

$$\begin{array}{lllll} x-3=0 & \text{or} & x+3=0 & \text{or} & x^2+5=0 \\ x=3 & \text{or} & x=-3 & \text{or} & \text{(no real solution)} \end{array}$$

The x-intercepts are $(3, 0)$ and $(-3, 0)$.

We can see these intercepts on the graph of the function shown in Figure 16.2.4.12 We can see that the function is even because $f(x) = f(-x)$.

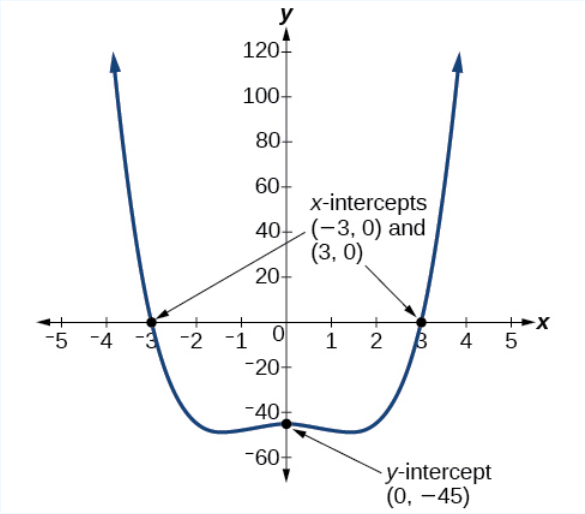


Figure 16.2.4.12: Graph of $f(x) = x^4 - 4x^2 - 45$.

Exercise 16.2.4.5

16.2.4.5: Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the y- and x-intercepts.

Solution

y-intercept $(0, 0)$; x-intercepts $(0, 0)$,$(-2, 0)$, and $(5, 0)$

Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of x-intercepts and the number of turning points. A polynomial function of nth degree is the product of n factors, so it will have at most n roots or zeros, or x-intercepts. The graph of the polynomial function of degree n must have at most $n-1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A smooth curve is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Intercepts and Turning Points of Polynomials

A polynomial of degree n will have, at most, n x-intercepts and $n-1$ turning points.

Example 16.2.4.10: Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of x-intercepts and turning points for $f(x)=-3x^{10}+4x^7-x^4+2x^3$.

Solution

The polynomial has a degree of **10**, so there are at most n x-intercepts and at most $n-1$ turning points.

Exercise 16.2.4.6

Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x)=108-13x^9-8x^4+14x^{12}+2x^3$

Answer

There are at most 12 x-intercepts and at most 11 turning points.

Example 16.2.4.11: Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure 16.2.4.12based on its intercepts and turning points?

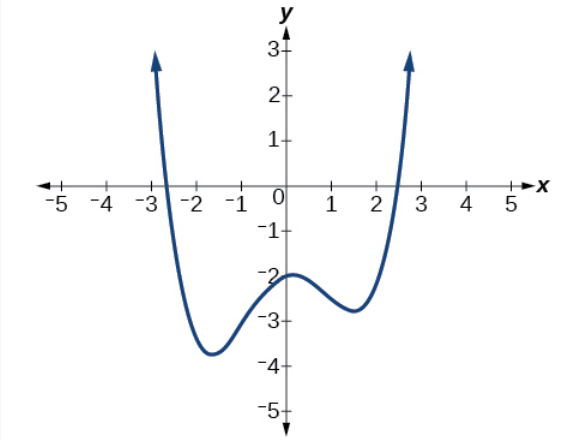


Figure 16.2.4.13.

Solution

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure 16.2.4.14

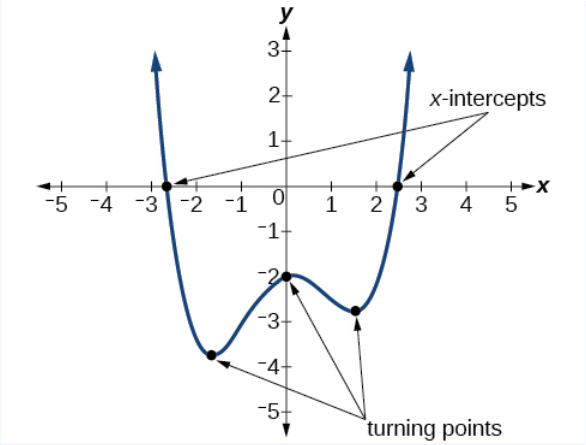


Figure 16.2.4.14: Graph of an even-degree polynomial.

The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

Exercise 16.2.4.7

What can we conclude about the polynomial represented by the graph shown in Figure 16.2.4.15based on its intercepts and turning points?

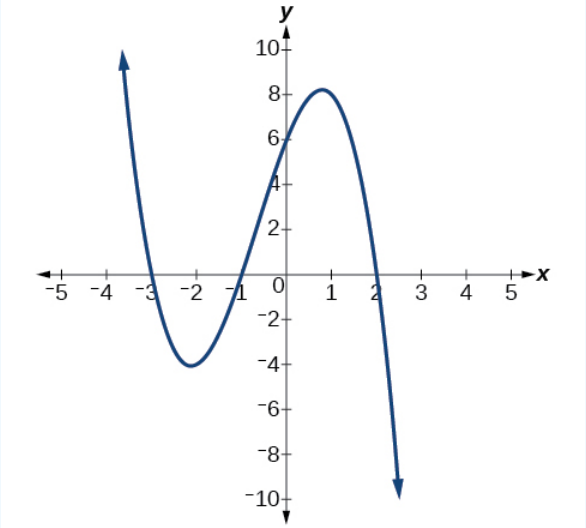


Figure 16.2.4.15.

Answer

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Solution

The end behavior indicates an odd-degree polynomial function; there are 3 x-intercepts and 2 turning points, so the degree is odd and at least 3. Because of the end behavior, we know that the lead coefficient must be negative.

Example 16.2.4.12: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

Solution

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The y-intercept is $(0, 0)$.

The x-intercepts are found by determining the zeros of the function.

$$\begin{aligned} 0 &= -4x(x+3)(x-4) \\ x &= 0 \quad \text{or} \quad x+3 = 0 \quad \text{or} \quad x-4 = 0 \\ x &= 0 \quad \text{or} \quad x = -3 \quad \text{or} \quad x = 4 \end{aligned}$$

The x-intercepts are $(0, 0)$,$(-3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

Exercise 16.2.4.8

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

Answer

The x-intercepts are $(2, 0)$, $(-1, 0)$, and $(5, 0)$, the y-intercept is $(0, 2)$, and the graph has at most 2 turning points.

Key Equations

- general form of a polynomial function: $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

Key Concepts

- A power function is a variable base raised to a number power.
- The behavior of a graph as the input decreases without bound and increases without bound is called the end behavior.
- The end behavior depends on whether the power is even or odd.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function.
- A polynomial of degree n will have at most n x-intercepts and at most $n-1$ turning points.

Glossary

coefficient

a nonzero real number that is multiplied by a variable raised to an exponent (only the number factor is the coefficient)

continuous function

a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

degree

the highest power of the variable that occurs in a polynomial

end behavior

the behavior of the graph of a function as the input decreases without bound and increases without bound

leading coefficient

the coefficient of the leading term

leading term

the term containing the highest power of the variable

polynomial function

a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

power function

a function that can be represented in the form $f(x) = kx^p$ where k is a constant, the base is a variable, and the exponent, p, is a constant

smooth curve

a graph with no sharp corners

term of a polynomial function

any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

turning point

the location at which the graph of a function changes direction

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16.2E: Exercises

1.2E: Exercises

Exercises

For the following exercises, for each pair of points, a. find the slope of the line passing through the points and b. indicate whether the line is increasing, decreasing, horizontal, or vertical.

59) $(-2, 4)$ and $(1, 1)$

Answer:

a. -1 b. Decreasing

Answer:

$a. -1$
$b. Decreasing$

60) $(-1, 4)$ and $(3, -1)$

61) $(3, 5)$ and $(-1, 2)$

Answer:

a. 3/4 b. Increasing

62) $(6, 4)$ and $(4, -3)$

63) $(2, 3)$ and $(5, 7)$

Answer:

a. 4/3 b. Decreasing

64) $(1, 9)$ and $(-8, 5)$

65) $(2, 4)$ and $(1, 4)$

Answer:

a. 0 b. Horizontal

66) $(1, 4)$ and $(1, 0)$

For the following exercises, write the equation of the line satisfying the given conditions in slope-intercept form.

67) Slope $=-6$, passes through $(1, 3)$

Answer:

$y = -6x + 9$

68) Slope $=3$, passes through $(-3, 2)$

69) Slope $=\frac{1}{3}$, passes through $(0, 4)$

Answer:

$y = \frac{1}{3}x + 4$

70) Slope $=\frac{2}{5}$, x-intercept $=8$

71) Passing through $(2, 1$ and $(-2, -1)$

Answer:

$y = \frac{1}{2}x$

Solution: $y = \frac{1}{2}x$

72) Passing through $(-3, 7)$ and $(1, 2)$

73) x-intercept =5 and y-intercept =-3

Answer:

$y = \frac{3}{5}x - 3$

74) x-Intercept =-6 and y-intercept =9

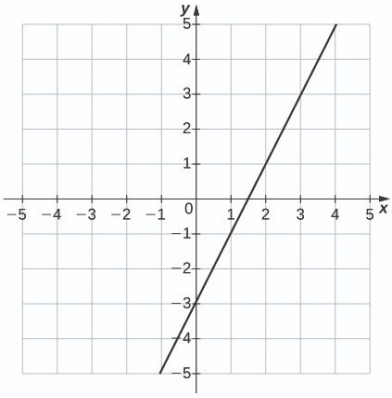
For the following exercises, for each linear equation, a. give the slope m and y-intercept b, if any, and b. graph the line.

75) $y = 2x - 3$

Answer:

a. $(m = 2, b = -3)$

b.



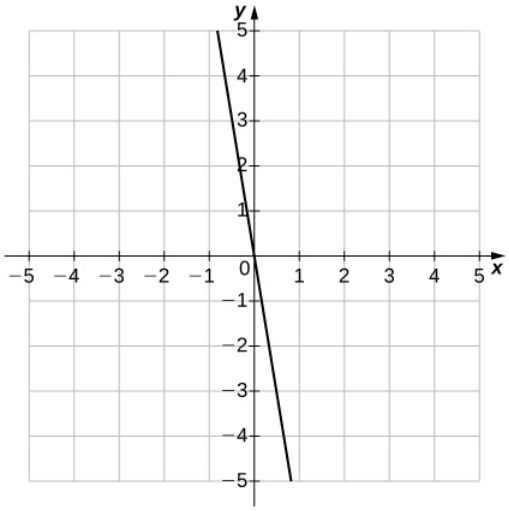
76) $y = -\frac{1}{7}x + 1$

77) $f(x) = -6x$

Answer:

a. $(m = -6, b = 0)$

b.



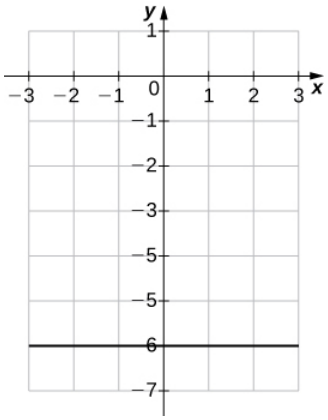
a. $(m=-6, b=0)$

b.

78) $f(x)=-5x+4$

79) $4y+24=0$

Answer:



a. $m=0$ b.

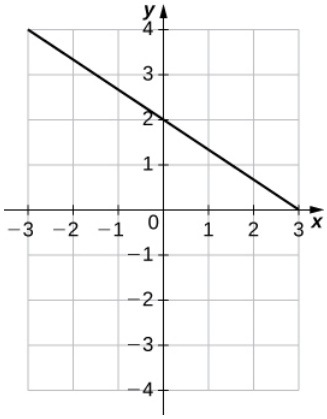
80) $8x-4=0$

81) $2x+3y=6$

Answer:

a. $(m=-\frac{2}{3}, b=2)$

b.



82) $6x-5y+15=0$

For the following exercises, for each polynomial, a. find the degree; b. find the zeros, if any; c. find the y-intercept(s), if any; d. use the leading coefficient to determine the graph's end behavior; and e. determine algebraically whether the polynomial is even, odd, or neither.

83) $f(x)=2x^2-3x-5$

Answer:

$a.\ 2$
$b.\ \frac{5}{2}, -1;$
$c.\ -5$
$d.\ Both\ ends\ rise$
$e.\ Neither$

84) $f(x) = -3x^2 + 6x$

85) $f(x) = \frac{1}{2}x^2 - 1$

Answer:

$a.\ 2$
$b.\ \pm\sqrt{2}$
$c.\ -1$
$d.\ Both\ ends\ rise$
$e.\ Even$

86) $f(x) = x^3 + 3x^2 - x - 3$

87) $f(x) = 3x - x^3$

Answer:

$a.\ 3$
$b.\ 0, \pm\sqrt{3}$
$c.\ 0$
$d.$ Left end rises, right end falls
$e.\ Odd$

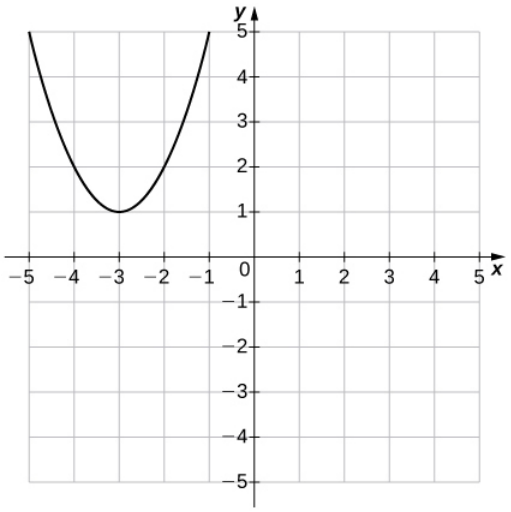
For the following exercises, use the graph of $f(x) = x^2$ to graph each transformed function g.

Exercise:

88) $g(x) = x^2 - 1$

89) $g(x) = (x+3)^2 + 1$

Answer:



For the following exercises, use the graph of $f(x) = \sqrt{x}$ to graph each transformed function g.

90) $g(x) = \sqrt{x+2}$

91) $g(x) = -\sqrt{x} - 1$

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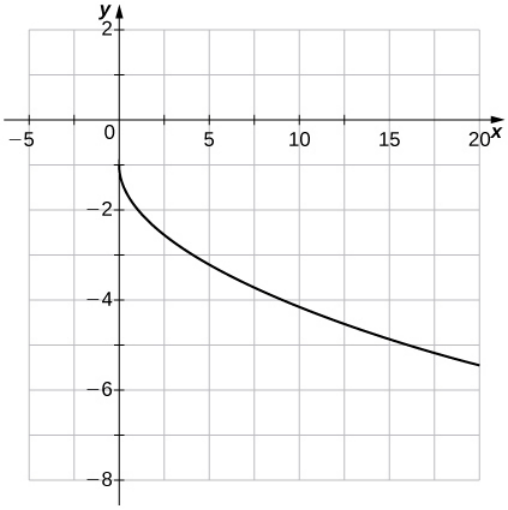
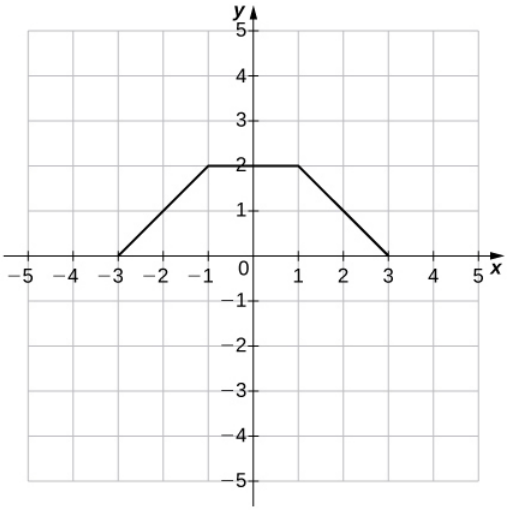
Answer:



For the following exercises, use the graph of $y = f(x)$ to graph each transformed function g.



92) $g(x) = f(x) + 1$

93) $g(x) = f(x - 1) + 2$

Answer:

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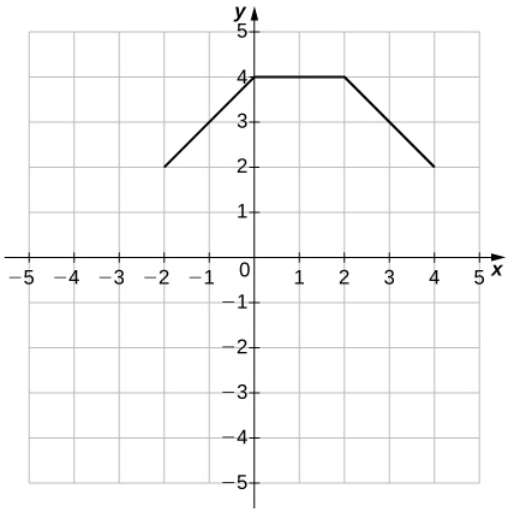


For the following exercises, for each of the piecewise-defined functions, a. evaluate at the given values of the independent variable and b. sketch the graph.

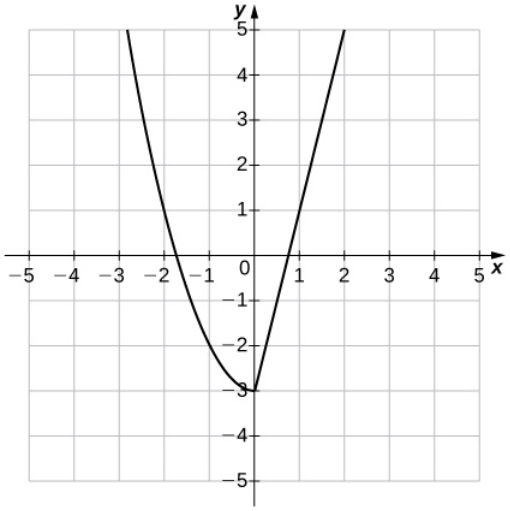
94) $f(x)=\begin{cases}4x+3, & x\le 0\\ -x+1, & x>0\end{cases}$; $f(-3); f(0); f(2)$

95) $f(x)=\begin{cases}x^2-3, & x\le 0\\ 4x+3, & x>0\end{cases}$; $f(-4); f(0); f(2)$

Answer:

a. $13, -3, 5$

b.



96) $h(x)=\begin{cases}x+1, & x\le 5\\ 4, & x>5\end{cases}$; $h(0); h(\pi); h(5)$

97) $g(x)=\begin{cases}\frac{3}{x-2}, & x\ne 2\\ 4, & x=2\end{cases}$; $g(0); g(-4); g(2)$

Answer:

a. $\frac{-3}{2}, \frac{-1}{2}, 4$

b.

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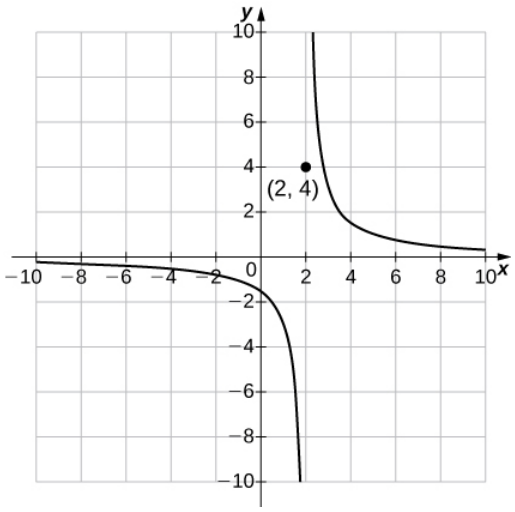

For the following exercises, determine whether the statement is *true or false*. Explain why.

98) $f(x) = (4x+1)/(7x-2)$ is a transcendental function.

99) $g(x) = \sqrt[3]{x}$ is an odd root function

Answer:

True; $n = 3$

100) A logarithmic function is an algebraic function.

101) A function of the form $f(x) = x^b$, where b is a real valued constant, is an exponential function.

Answer:

False; $f(x) = x^b$, where b is a real-valued constant, is a power function

102) The domain of an even root function is all real numbers.

103) [T] A company purchases some computer equipment for $20,500. At the end of a 3-year period, the value of the equipment has decreased linearly to $12,300.

a. Find a function $y = V(t)$ that determines the value V of the equipment at the end of t years.

b. Find and interpret the meaning of the x- and y-intercepts for this situation.

c. What is the value of the equipment at the end of 5 years?

d. When will the value of the equipment be $3000?

Answer:

$a.\ V(t) = -2733t + 20500$

$b.\ (0, 20,500)$ means that the initial purchase price of the equipment is $\$20,500$;
$(7.5, 0)$ means that in 7.5 years the computer equipment has no value.

$c.\ \$6835$

$d.$ In approximately 6.4 years.

104) [T] Total online shopping during the Christmas holidays has increased dramatically during the past 5 years. In 2012 $(t = 0)$,total online holiday sales were $42.3 billion, whereas in 2013 they were $48.1 billion.

a. Find a linear function S that estimates the total online holiday sales in the year t.

b. Interpret the slope of the graph of S.

c. Use part a. to predict the year when online shopping during Christmas will reach $60 billion.

105) [T] A family bakery makes cupcakes and sells them at local outdoor festivals. For a music festival, there is a fixed cost of $125 to set up a cupcake stand. The owner estimates that it costs $0.75 to make each cupcake. The owner is interested in determining the total cost C as a function of number of cupcakes made.

a. Find a linear function that relates cost C to x, the number of cupcakes made.

b. Find the cost to bake 160 cupcakes.

c. If the owner sells the cupcakes for $1.50 apiece, how many cupcakes does she need to sell to start making profit? (Hint: Use the INTERSECTION function on a calculator to find this number.)

Answer:

$a. C = 0.75x + 125$
$b. \$245$
$c. 167 cupcakes$

106) [T] A house purchased for $250,000 is expected to be worth twice its purchase price in 18 years.

a. Find a linear function that models the price P of the house versus the number of years t since the original purchase.

b. Interpret the slope of the graph of P.

c. Find the price of the house 15 years from when it was originally purchased.

107) [T] A car was purchased for $26,000. The value of the car depreciates by $1500 per year.

a. Find a linear function that models the value V of the car after t years.

b. Find and interpret $V(4)$.

Answer:

a. $V(t) = -1500t + 26,000$b. In 4 years, the value of the car is $20,000.

108) [T] A condominium in an upscale part of the city was purchased for $432,000. In 35 years it is worth $60,500. Find the rate of depreciation.

109) [T] The total cost C (in thousands of dollars) to produce a certain item is modeled by the function $C(x) = 10.50x + 28,500$, where x is the number of items produced. Determine the cost to produce 175 items.

Answer:

$30,337.50

110) [T] A professor asks her class to report the amount of time t they spent writing two assignments. Most students report that it takes them about 45 minutes to type a four-page assignment and about 1.5 hours to type a nine-page assignment.

a. Find the linear function $y = N(t)$ that models this situation, where N is the number of pages typed and t is the time in minutes.

b. Use part a. to determine how many pages can be typed in 2 hours.

c. Use part a. to determine how long it takes to type a 20-page assignment.

111) [T] The output (as a percent of total capacity) of nuclear power plants in the United States can be modeled by the function $P(t) = 1.8576t + 68.052$, where t is time in years and $t = 0$ corresponds to the beginning of 2000. Use the model to predict the percentage output in 2015.

Answer:

96% of the total capacity

112) [T] The admissions office at a public university estimates that 65% of the students offered admission to the class of 2019 will actually enroll.

a. Find the linear function $y = N(x)$, where N is the number of students that actually enroll and x is the number of all students offered admission to the class of 2019.

b. If the university wants the 2019 freshman class size to be 1350, determine how many students should be admitted.

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16.3: Graphs of Functions

Learning Objectives

By the end of this section, you will be able to:

- Use the vertical line test
- Identify graphs of basic functions
- Read information from a graph of a function

Before you get started, take this readiness quiz.

1. Evaluate: ⓐ 2^3 ⓑ 3^2.
   If you missed this problem, review [link].
2. Evaluate: ⓐ $|7|$ ⓑ $|-3|$.
   If you missed this problem, review [link].
3. Evaluate: ⓐ $\sqrt{4}$ ⓑ $\sqrt{16}$.
   If you missed this problem, review [link].

Use the Vertical Line Test

In the last section we learned how to determine if a relation is a function. The relations we looked at were expressed as a set of ordered pairs, a mapping or an equation. We will now look at how to tell if a graph is that of a function.

An ordered pair (x, y) is a solution of a linear equation, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

The graph of a linear equation is a straight line where every point on the line is a solution of the equation and every solution of this equation is a point on this line.

In Figure, we can see that, in graph of the equation $y = 2x - 3$, for every *x*-value there is only one *y*-value, as shown in the accompanying table.

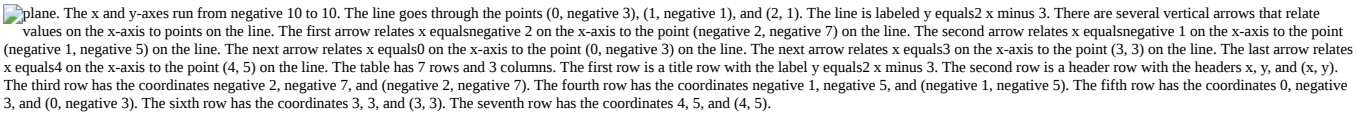

plane. The x and y-axes run from negative 10 to 10. The line goes through the points (0, negative 3), (1, negative 1), and (2, 1). The line is labeled y equals2 x minus 3. There are several vertical arrows that relate values on the x-axis to points on the line. The first arrow relates x equalsnegative 2 on the x-axis to the point (negative 2, negative 7) on the line. The second arrow relates x equalsnegative 1 on the x-axis to the point (negative 1, negative 5) on the line. The next arrow relates x equals0 on the x-axis to the point (0, negative 3) on the line. The next arrow relates x equals3 on the x-axis to the point (3, 3) on the line. The last arrow relates x equals4 on the x-axis to the point (4, 5) on the line. The table has 7 rows and 3 columns. The first row is a title row with the label y equals2 x minus 3. The second row is a header row with the headers x, y, and (x, y). The third row has the coordinates negative 2, negative 7, and (negative 2, negative 7). The fourth row has the coordinates negative 1, negative 5, and (negative 1, negative 5). The fifth row has the coordinates 0, negative 3, and (0, negative 3). The sixth row has the coordinates 3, 3, and (3, 3). The seventh row has the coordinates 4, 5, and (4, 5).

Figure 16.3.1

A relation is a function if every element of the domain has exactly one value in the range. So the relation defined by the equation $y = 2x - 3$ is a function.

If we look at the graph, each vertical dashed line only intersects the line at one point. This makes sense as in a function, for every *x*-value there is only one *y*-value.

If the vertical line hit the graph twice, the *x*-value would be mapped to two *y*-values, and so the graph would not represent a function.

This leads us to the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

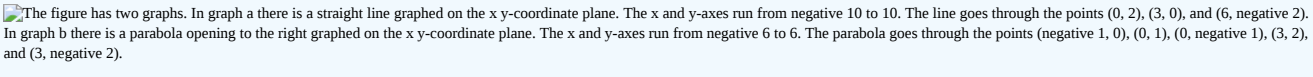
VERTICAL LINE TEST

A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.

If any vertical line intersects the graph in more than one point, the graph does not represent a function.

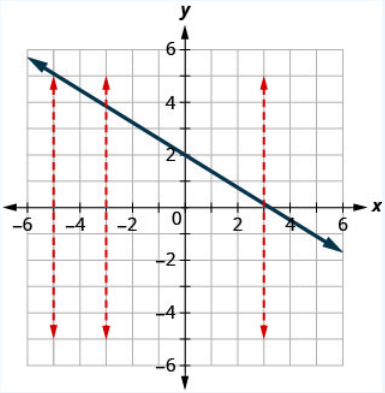
Example 16.3.1

Determine whether each graph is the graph of a function.

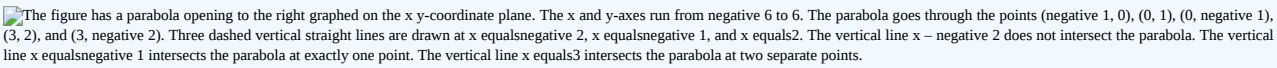

The figure has two graphs. In graph a there is a straight line graphed on the x y-coordinate plane. The x and y-axes run from negative 10 to 10. The line goes through the points (0, 2), (3, 0), and (6, negative 2). In graph b there is a parabola opening to the right graphed on the x y-coordinate plane. The x and y-axes run from negative 6 to 6. The parabola goes through the points (negative 1, 0), (0, 1), (0, negative 1), (3, 2), and (3, negative 2).

Answer

ⓐ Since any vertical line intersects the graph in at most one point, the graph is the graph of a function.

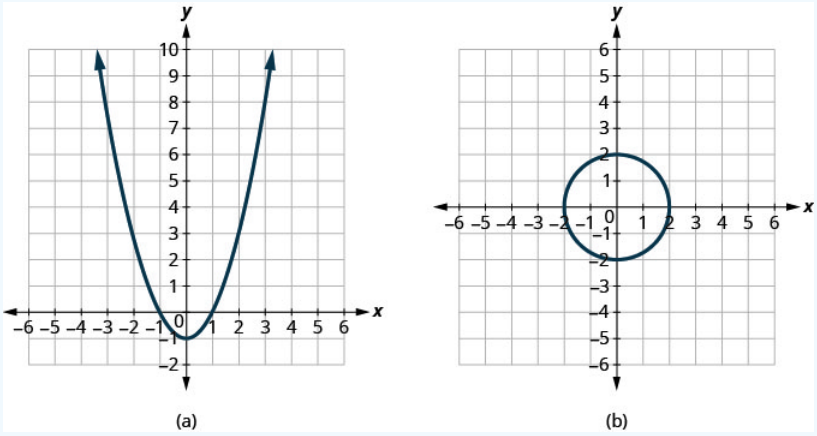


ⓑ One of the vertical lines shown on the graph, intersects it in two points. This graph does not represent a function.


The figure has a parabola opening to the right graphed on the x y-coordinate plane. The x and y-axes run from negative 6 to 6. The parabola goes through the points (negative 1, 0), (0, 1), (0, negative 1), (3, 2), and (3, negative 2). Three dashed vertical straight lines are drawn at x equalsnegative 2, x equalsnegative 1, and x equals2. The vertical line x – negative 2 does not intersect the parabola. The vertical line x equalsnegative 1 intersects the parabola at exactly one point. The vertical line x equals3 intersects the parabola at two separate points.

Example 16.3.2

Determine whether each graph is the graph of a function.

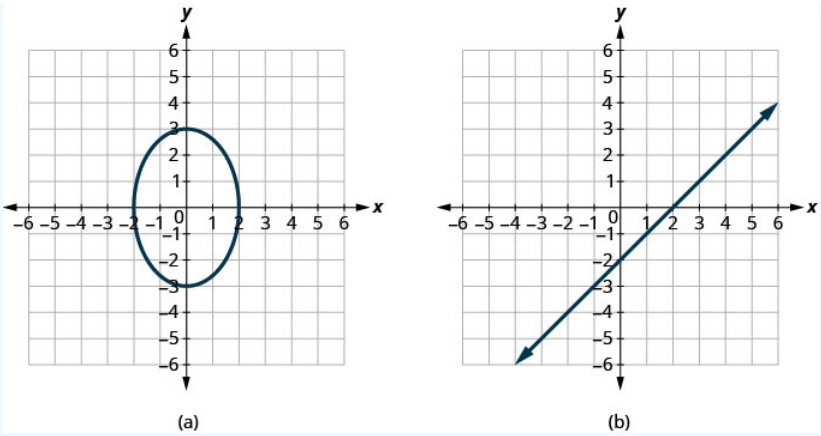


Answer

ⓐ yes ⓑ no

Example 16.3.3

Determine whether each graph is the graph of a function.



Answer

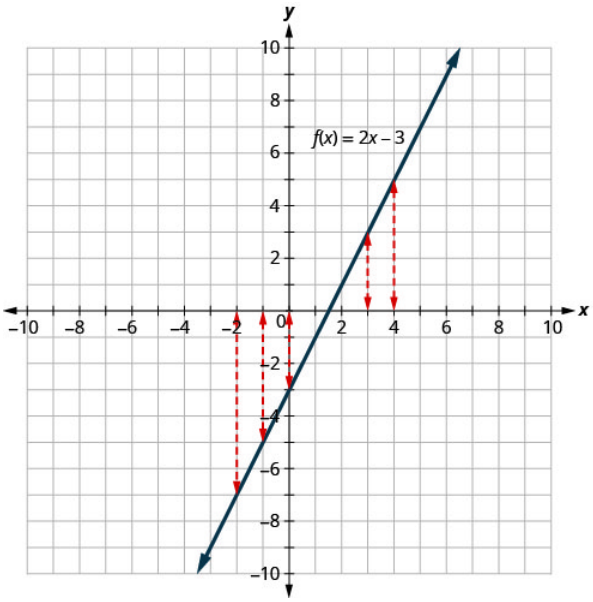
ⓐ no ⓑ yes

Identify Graphs of Basic Functions

We used the equation $y = 2x - 3$ and its graph as we developed the vertical line test. We said that the relation defined by the equation $y = 2x - 3$ is a function.

We can write this as in function notation as $f(x) = 2x - 3$. It still means the same thing. The graph of the function is the graph of all ordered pairs (x, y) where $y = f(x)$. So we can write the ordered pairs as $(x, f(x))$. It looks different but the graph will be the same.

Compare the graph of $y = 2x - 3$ previously shown in Figure with the graph of $f(x) = 2x - 3$ shown in Figure. Nothing has changed but the notation.



$f(x) = 2x - 3$		
x	$f(x)$	$(x, f(x))$
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

Figure 16.3.2

GRAPH OF A FUNCTION

The graph of a function is the graph of all its ordered pairs, (x,y)(x,y) or using function notation, (x,f(x))(x,f(x)) where y=f(x).y=f(x).

f	name of function
x	x-coordinate of the ordered pair
$f(x)$	y-coordinate of the ordered pair

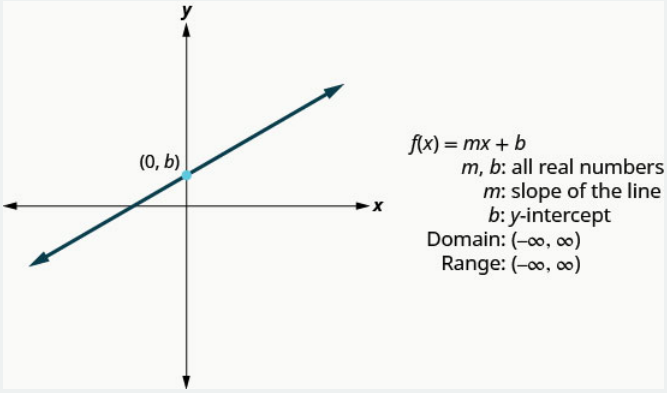
As we move forward in our study, it is helpful to be familiar with the graphs of several basic functions and be able to identify them.

Through our earlier work, we are familiar with the graphs of linear equations. The process we used to decide if $y = 2x - 3$ is a function would apply to all linear equations. All non-vertical linear equations are functions. Vertical lines are not functions as the *x*-value has infinitely many *y*-values.

We wrote linear equations in several forms, but it will be most helpful for us here to use the slope-intercept form of the linear equation. The slope-intercept form of a linear equation is $y = mx + b$. In function notation, this linear function becomes $f(x) = mx + b$ where *m* is the slope of the line and *b* is the *y*-intercept.

The domain is the set of all real numbers, and the range is also the set of all real numbers.

LINEAR FUNCTION



We will use the graphing techniques we used earlier, to graph the basic functions.

Example 16.3.4

Graph: $f(x) = -2x - 4$.

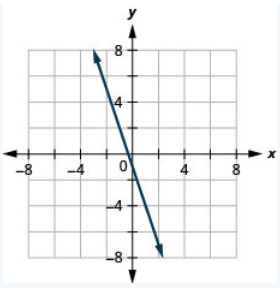
Answer

	$f(x) = -2x - 4$
We recognize this as a linear function.	
Find the slope and *y*-intercept.	$m = -2$ $b = -4$
Graph using the slope intercept.	

Example 16.3.5

Graph: $f(x) = -3x - 1$

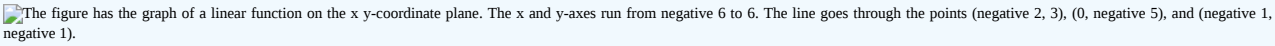
Answer



Example 16.3.6

Graph: $f(x) = -4x - 5$

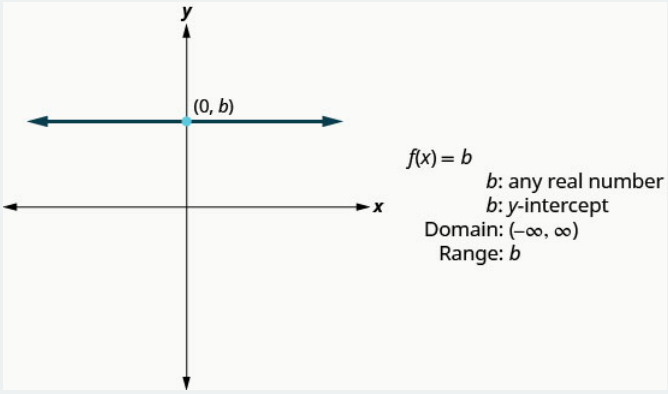
Answer


The figure has the graph of a linear function on the x y-coordinate plane. The x and y-axes run from negative 6 to 6. The line goes through the points (negative 2, 3), (0, negative 5), and (negative 1, negative 1).

The next function whose graph we will look at is called the constant function and its equation is of the form $f(x) = b$, where b is any real number. If we replace the $f(x)$ with y, we get $y = b$. We recognize this as the horizontal line whose *y*-intercept is b. The graph of the function $f(x) = b$, is also the horizontal line whose *y*-intercept is b.

Notice that for any real number we put in the function, the function value will be b. This tells us the range has only one value, b.

CONSTANT FUNCTION



Example 16.3.7

Graph: $f(x) = 4$.

Answer

	$f(x) = 4$
We recognize this as a constant function.	

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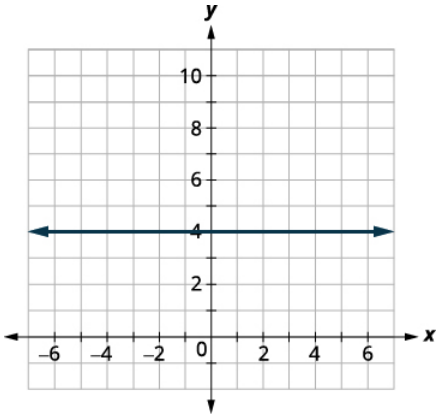
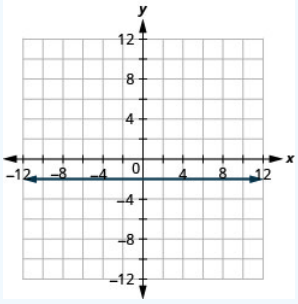
The graph will be a horizontal line through $(0, 4)$.



Example 16.3.8

Graph: $f(x) = -2$.

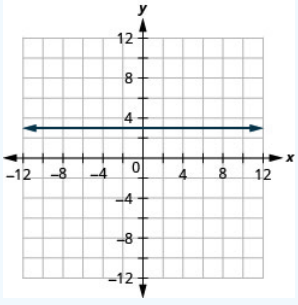
Answer



Example 16.3.9

Graph: $f(x) = 3$.

Answer



The identity function, $f(x) = x$ is a special case of the linear function. If we write it in linear function form, $f(x) = 1x + 0$, we see the slope is 1 and the *y*-intercept is 0.

IDENTITY FUNCTION

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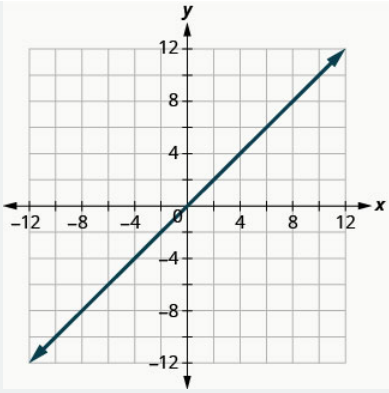


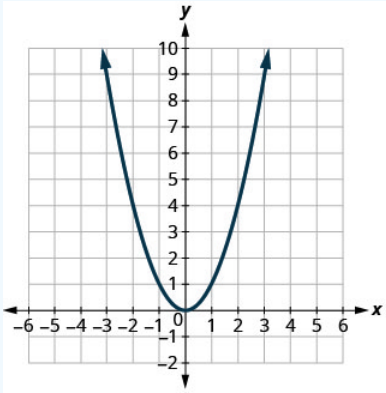
$f(x) = x$
m: 1
b: 0
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

The next function we will look at is not a linear function. So the graph will not be a line. The only method we have to graph this function is point plotting. Because this is an unfamiliar function, we make sure to choose several positive and negative values as well as 0 for our x-values.

Graph: $f(x) = x^2$.

Answer

We choose x-values. We substitute them in and then create a chart as shown.

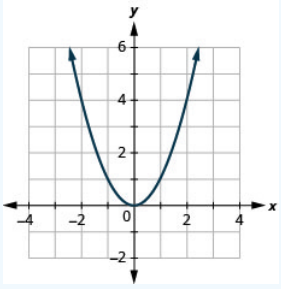


x	$f(x) = x^2$	$(x, f(x))$
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Example 16.3.11

Graph: $f(x) = x^2$.

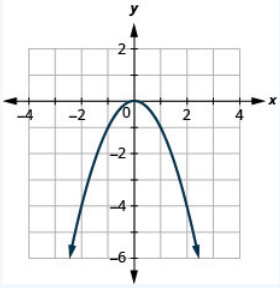
Answer



Example 16.3.12

$f(x) = -x^2$

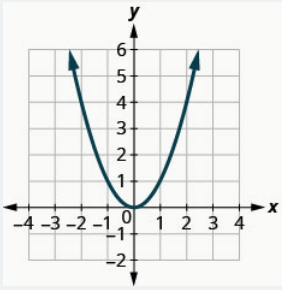
Answer



Looking at the result in Example, we can summarize the features of the square function. We call this graph a parabola. As we consider the domain, notice any real number can be used as an *x*-value. The domain is all real numbers.

The range is not all real numbers. Notice the graph consists of values of *y* never go below zero. This makes sense as the square of any number cannot be negative. So, the range of the square function is all non-negative real numbers.

SQUARE FUNCTION



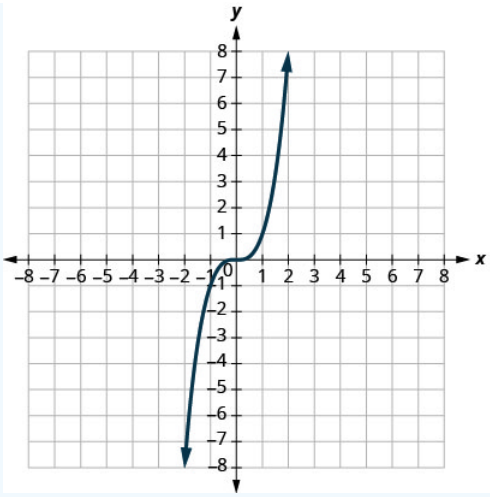
$f(x) = x^2$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

The next function we will look at is also not a linear function so the graph will not be a line. Again we will use point plotting, and make sure to choose several positive and negative values as well as 0 for our *x*-values.

Graph: $f(x) = x^3$.

Answer

We choose *x*-values. We substitute them in and then create a chart.

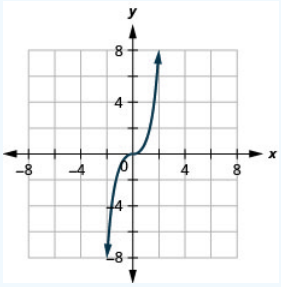


x	$f(x)=x^3$	$(x, f(x))$
–2	–8	(–2, –8)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
2	8	(2, 8)

Example 16.3.14

Graph: $f(x)=x^3$.

Answer



Example 16.3.15

Graph: $f(x)=-x^3$.

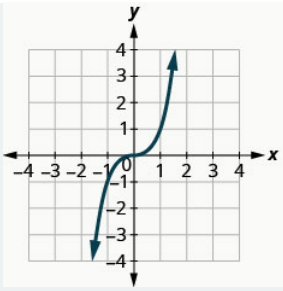
Answer

This figure has a curved line graphed on the x y-coordinate plane. The x-axis runs from negative 6 to 6. The y-axis runs from negative 6 to 6. The curved line goes through the points (negative 2, 8), (negative 1, 1), (0, 0), (1, negative 1), and (2, negative 8).

Looking at the result in Example, we can summarize the features of the cube function. As we consider the domain, notice any real number can be used as an x-value. The domain is all real numbers.

The range is all real numbers. This makes sense as the cube of any non-zero number can be positive or negative. So, the range of the cube function is all real numbers.

CUBE FUNCTION



$f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

The next function we will look at does not square or cube the input values, but rather takes the square root of those values.

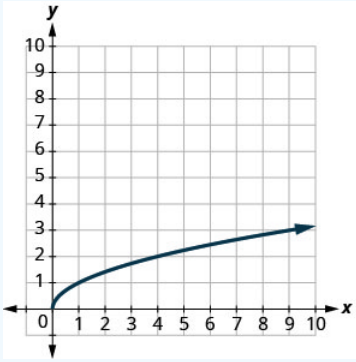
Let's graph the function $f(x) = \sqrt{x}$ and then summarize the features of the function. Remember, we can only take the square root of non-negative real numbers, so our domain will be the non-negative real numbers.

Example 16.3.16

$f(x) = \sqrt{x}$

Answer

We choose *x*-values. Since we will be taking the square root, we choose numbers that are perfect squares, to make our work easier. We substitute them in and then create a chart.

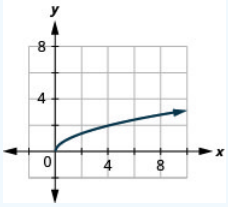


x	$f(x) = \sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Example 16.3.17

Graph: $f(x) = x$.

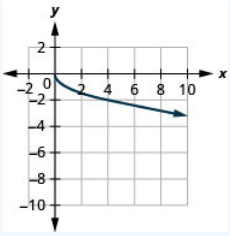
Answer



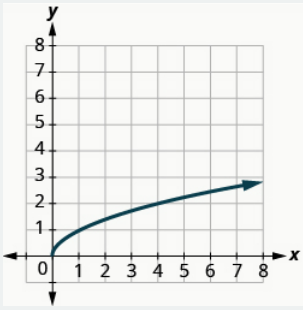
Example 16.3.18

Graph: $f(x) = -\sqrt{x}$.

Answer



SQUARE ROOT FUNCTION



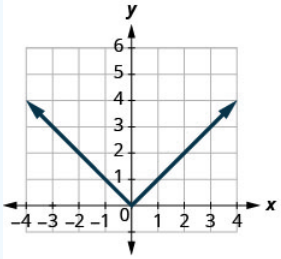
$f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$

Our last basic function is the absolute value function, $f(x) = |x|$. Keep in mind that the absolute value of a number is its distance from zero. Since we never measure distance as a negative number, we will never get a negative number in the range.

Graph: $f(x) = |x|$.

Answer

We choose *x*-values. We substitute them in and then create a chart.

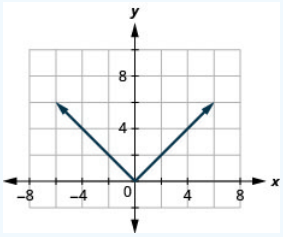


x	$f(x) = \lvert x \rvert$	$(x, f(x))$
−3	3	(−3, 3)
−2	2	(−2, 2)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	2	(2, 2)
3	3	(3, 3)

Example 16.3.20

Graph: $f(x) = |x|$.

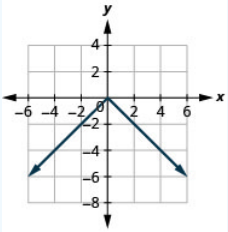
Answer



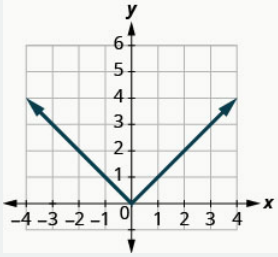
Example 16.3.21

Graph: $f(x) = -|x|$.

Answer



ABSOLUTE VALUE FUNCTION



$f(x) = |x|$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Read Information from a Graph of a Function

In the sciences and business, data is often collected and then graphed. The graph is analyzed, information is obtained from the graph and then often predictions are made from the data.

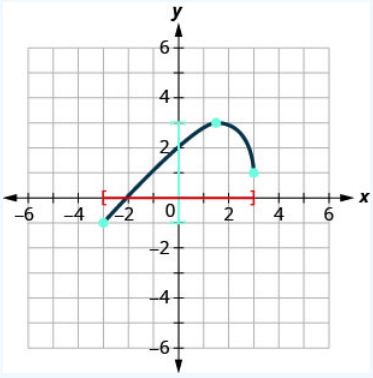
We will start by reading the domain and range of a function from its graph.

Remember the domain is the set of all the *x*-values in the ordered pairs in the function. To find the domain we look at the graph and find all the values of *x* that have a corresponding value on the graph. Follow the value *x* up or down vertically. If you hit the graph of the function then *x* is in the domain.

Remember the range is the set of all the *y*-values in the ordered pairs in the function. To find the range we look at the graph and find all the values of *y* that have a corresponding value on the graph. Follow the value *y* left or right horizontally. If you hit the graph of the function then *y* is in the range.

Example 16.3.22

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



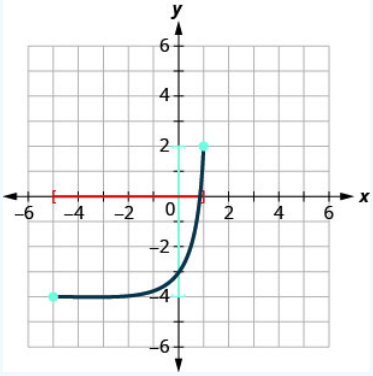
Answer

To find the domain we look at the graph and find all the values of x that correspond to a point on the graph. The domain is highlighted in red on the graph. The domain is $[-3, 3]$.

To find the range we look at the graph and find all the values of y that correspond to a point on the graph. The range is highlighted in blue on the graph. The range is $[-1, 3]$.

Example 16.3.23

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.

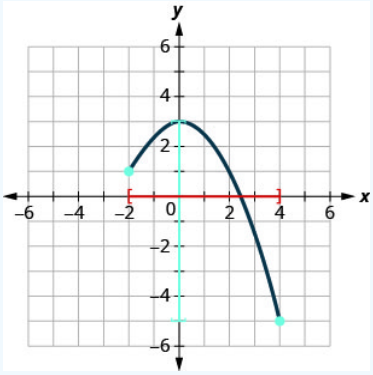


Answer

The domain is $[-5, 1]$. The range is $[-4, 2]$.

Example 16.3.24

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



Answer

The domain is $[-2, 4]$. The range is $[-5, 3]$.

Access this online resource for additional instruction and practice with graphs of functions.

- Find Domain and Range

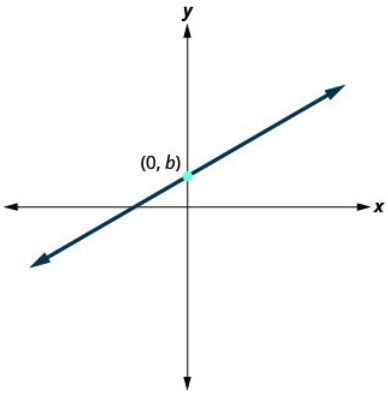
Key Concepts

- **Vertical Line Test**
  - A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.
  - If any vertical line intersects the graph in more than one point, the graph does not represent a function.

- **Graph of a Function**
  - The graph of a function is the graph of all its ordered pairs, (x,y)(x,y) or using function notation, (x,f(x))(x,f(x)) where y=f(x).y=f(x).

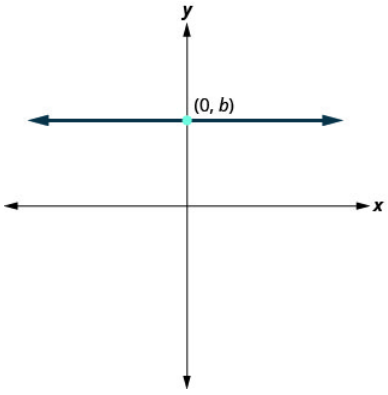
    fxf(x)name of functionx-coordinate of the ordered pairy-coordinate of the ordered pairfname of functionxx-coordinate of the ordered pairf(x)y-coordinate of the ordered pair

- **Linear Function**



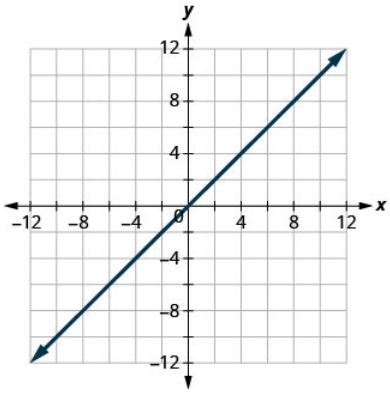
$f(x) = mx + b$
m, b: all real numbers
m: slope of the line
b: y-intercept
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

- **Constant Function**



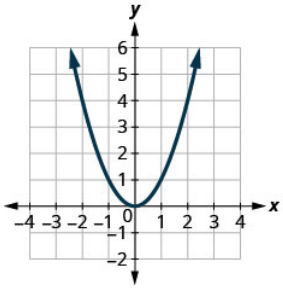
$f(x) = b$
b: any real number
b: y-intercept
Domain: $(-\infty, \infty)$
Range: b

- **Identity Function**



$f(x) = x$
m: 1
b: 0
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

- **Square Function**



$f(x) = x^2$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

- **Cube Function**

This figure has a curved line graphed on the x y-coordinate plane. The x-axis runs from negative 4 to 4. The y-axis runs from negative 4 to 4. The curved line goes through the points (negative 2, negative 8), (negative 1, negative 1), (0, 0), (1, 1), and (2, 8).). Next to the graph are the following: “f of x equalsx cubed”, “Domain: (negative infinity, infinity)”, and “Range: (negative infinity, infinity)”.

- **Square Root Function**

This figure has a curved half-line graphed on the x y-coordinate plane. The x-axis runs from 0 to 8. The y-axis runs from 0 to 8. The curved half-line starts at the point (0, 0) and then goes up and to the right. The curved half line goes through the points (1, 1) and (4, 2). Next to the graph are the following: “f of x equalssquare root of x”, “Domain: [0, infinity)”, and “Range: [0, infinity)”.

- **Absolute Value Function**

This figure has a v-shaped line graphed on the x y-coordinate plane. The x-axis runs from negative 4 to 4. The y-axis runs from negative 1 to 6. The v-shaped line goes through the points (negative 3, 3), (negative 2, 2), (negative 1, 1), (0, 0), (1, 1), (2, 2), and (3, 3). The point (0, 0) where the line changes slope is called the vertex. Next to the graph are the following: “f of x equalsabsolute value of x”, “Domain: (negative infinity, infinity)”, and “Range: [0, infinity)”.

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16.3.1: Domain and Range

Learning Objectives

- Find the domain of a function defined by an equation.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—I am Legend, Hannibal, The Ring, The Grudge, and The Conjuring. Figure 16.3.1.1shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

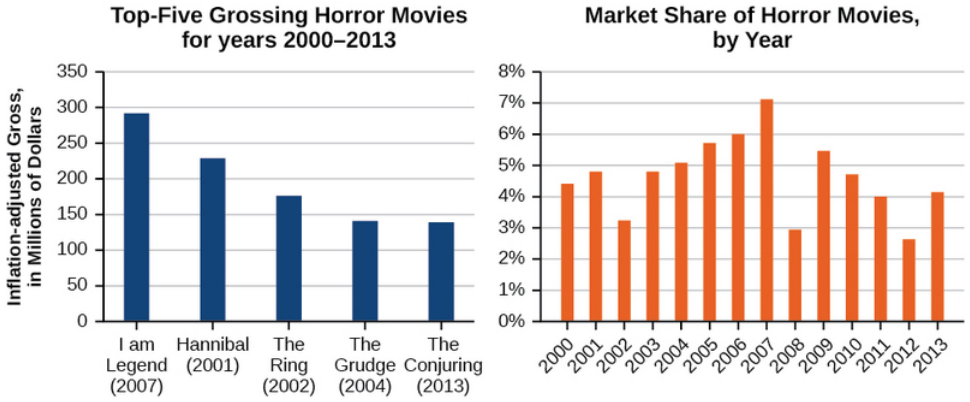


Figure 16.3.1.1: Graph of the Top-Five Grossing Horror Movies for years 2000-2003, and a Graph of the Market Share of Horror Movies by Year. Based on data compiled by www.the-numbers.com.

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of **domain and range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (Figure 16.3.1.2).

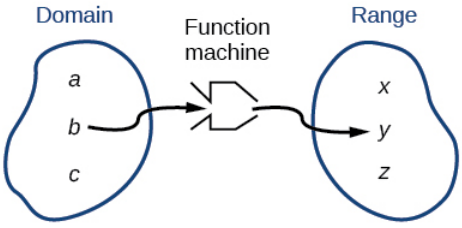


Figure 16.3.1.2: Diagram of how a function relates two relations

We can write the **domain and range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether

the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint is included, called inclusive.

See Figure 16.3.1.3for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Figure 16.3.1.3: Summary of interval notation.

Example 16.3.1.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)\}$

Solution

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2, 3, 4, 5, 6\}$$

Exercse 16.3.1.1

Find the domain of the function:

$$\{(-5,4),(0,0),(5,-4),(10,-8),(15,-12)\}$$

Answer

$\{-5,0,5,10,15\}$

How To: Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example 16.3.1.2: Finding the Domain of a Function

Find the domain of the function $f(x)=x^2-1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty,\infty)$.

Exercse 16.3.1.2

Find the domain of the function:

$$f(x)=5-x+x^3$$

Answer

$(-\infty,\infty)$

Howto: Given a function written in an equation form that includes a fraction, find the domain

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example 16.3.1.3: Finding the Domain of a Function Involving a Denominator

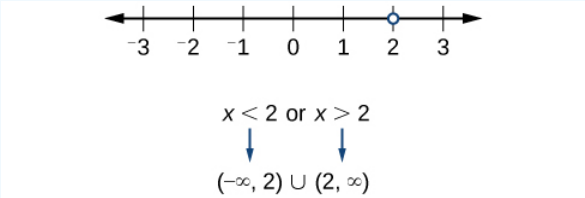
Find the domain of the function $f(x)=\dfrac{x+1}{2-x}$.

Solution

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2-x&=0\\ -x&=-2\\ x&=2 \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty, 2) \cup (2, \infty)$.



In interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Exercse 16.3.1.3

Find the domain of the function:

$$f(x) = \frac{1+4x}{2x-1}$$

Answer

$$(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$$

How To: Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example 16.3.1.4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x) = \sqrt{7-x}.$$

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7 - x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

Exercse 16.3.1.4

Find the domain of the function

$$f(x) = \sqrt{5+2x}.$$

Answer

$$[-2.5, \infty)$$

Q&A: Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function $f(x) = -\frac{1}{\sqrt{x}}$ *has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in set-builder notation. For example, $\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure $16.3.1.4$compares inequality notation, set-builder notation, and interval notation.

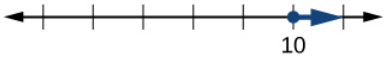
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 16.3.1.4: Summary of notations for inequalities, set-builder, and interval.

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$ It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \geq 3\} = (-\infty, -3] \cup [3, \infty) \tag{16.3.1.1}$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|$ statement about x$\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x|4 < x \leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

Example 16.3.1.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure 16.3.1.5using inequality notation, set-builder notation, and interval notation.

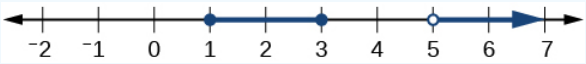


Figure 16.3.1.5: Line graph of $1 \leq x \leq 3$ or $x > 5$.

Solution

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality

$$1 \leq x \leq 3 \text{ or } x > 5$$

Set-builder Notation

$$\{x|1 \leq x \leq 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Exercse 16.3.1.5

Given Figure 16.3.1.6, specify the graphed set in

a. words
b. set-builder notation
c. interval notation

Figure 16.3.1.6: Line graph of $-2 \leq x$, $-1 \leq x < 3$ .leq}x\), $-1 \leq x < 3$.

Answer a

Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;

Answer b

$\{x|x \leq -2 or -1 \leq x < 3\}$

Answer c

$(-\infty, -2] \cup [-1, 3)$

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure 16.3.1.7.

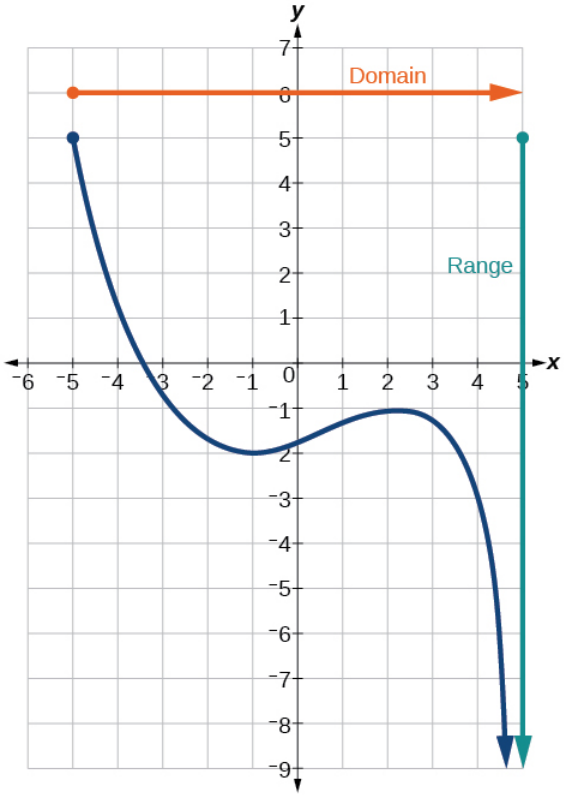


Figure 16.3.1.7: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from −5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example 16.3.1.6A: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 1.2.8.

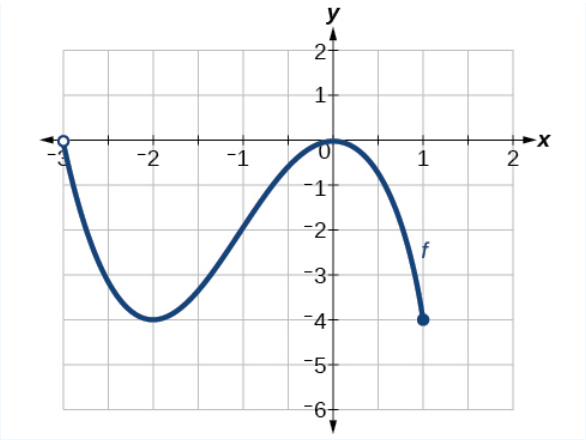


Figure 16.3.1.8: Graph of a function from (-3, 1].

Solution

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4, 0)$. See Figure 16.3.1.9

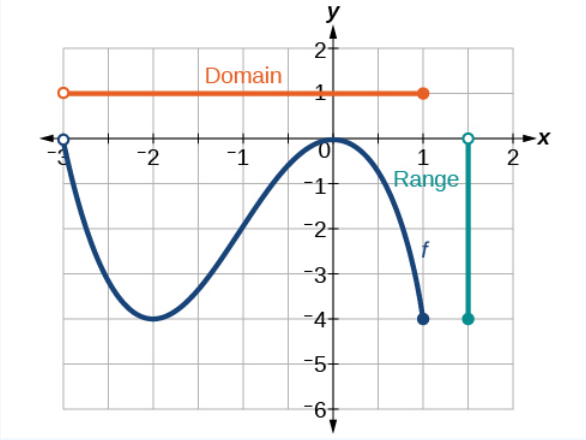


Figure 16.3.1.9: Graph of the previous function shows the domain and range

Example 16.3.1.6*B*: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure 16.3.1.10

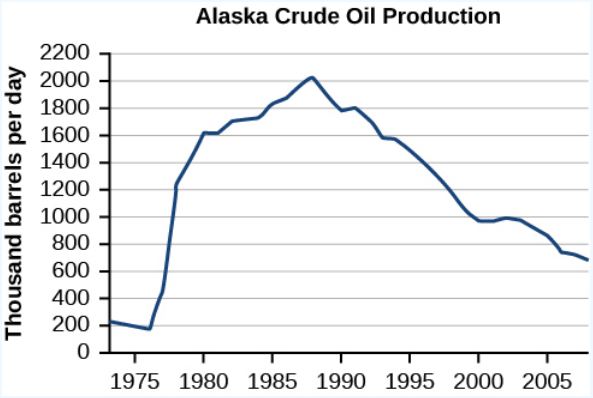


Figure 16.3.1.10: Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

Solution

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is $[1973, 2008]$ and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Exercse 16.3.1.6

Given Figure 16.3.1.11, identify the domain and range using interval notation.

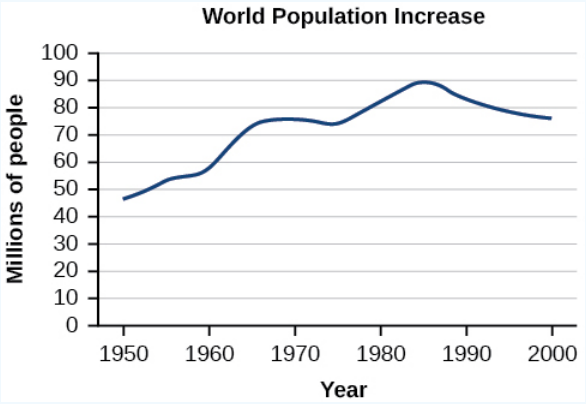


Figure 16.3.1.11: Graph of World Population Increase where the y-axis represents millions of people and the x-axis represents the year.

Answer

domain =$[1950, 2002]$

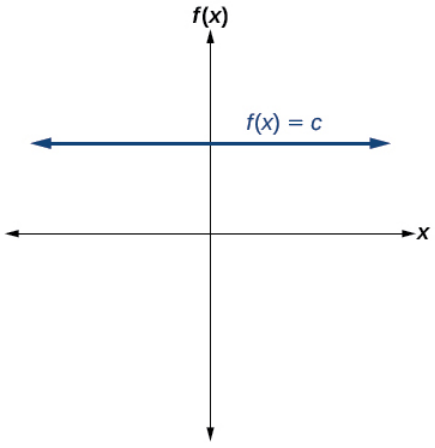
range = $[47,000,000, 89,000,000]$

Can a function’s domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

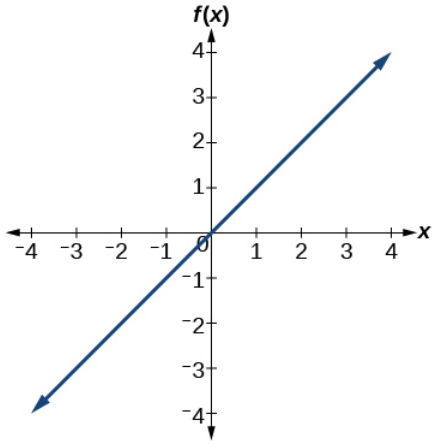
We will now return to our set of toolkit functions to determine the domain and range of each.



Domain: $(-\infty, \infty)$
Range: $[c,c]$

Figure 16.3.1.12: Constant function $f(x) = c$.

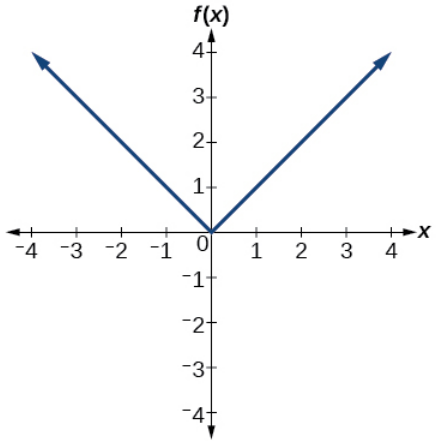
For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure* 16.3.1.13*: *Identity function f(x)=x.*

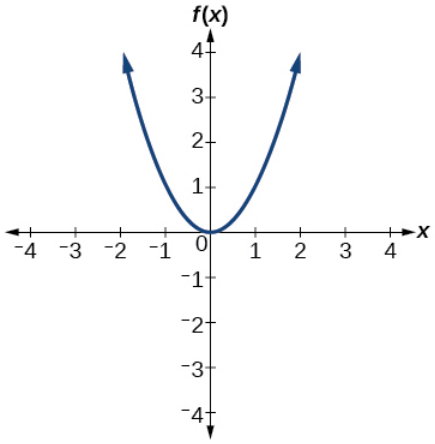
For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 16.3.1.14: Absolute function $f(x) = |x|$.

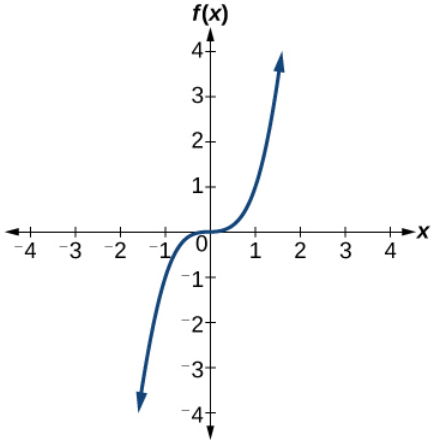
For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 16.3.1.15: Quadratic function $f(x) = x^2$.

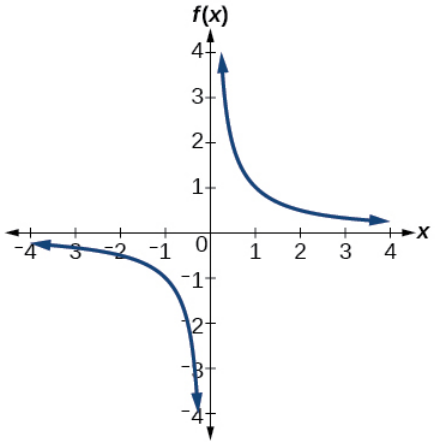
For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 16.3.1.16: Cubic function $f(x) - x^3$.

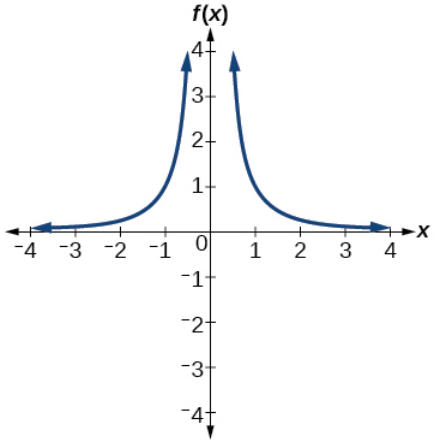
For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 16.3.1.17: Reciprocal function $f(x) = \frac{1}{x}$.

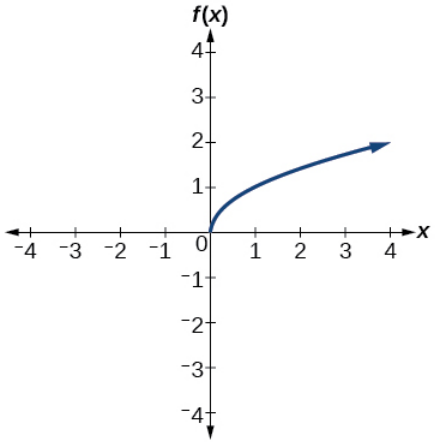
For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x | x \neq 0\}$,the set of all real numbers that are not zero.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

Figure 16.3.1.18: Reciprocal squared function $f(x) = \frac{1}{x^2}$

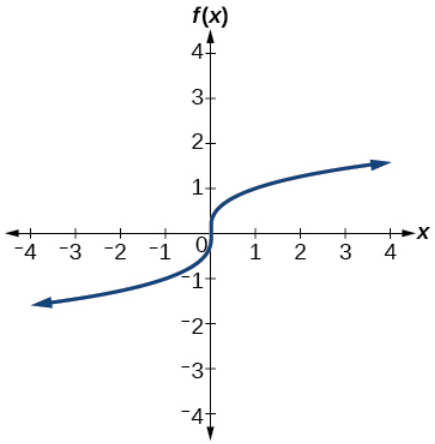
For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.



Domain: [0, ∞)
Range: [0, ∞)

***Figure* 16.3.1.19:** *Square root function* $f(x) = \sqrt{(x)}$.

For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 16.3.1.20: Cube root function $f(x) = \sqrt[3]{x}$.

For the **cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

How To: Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example 16.3.1.7B: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example 16.3.1.7C: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$x + 4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Analysis

Figure 16.3.1.19represents the function f.

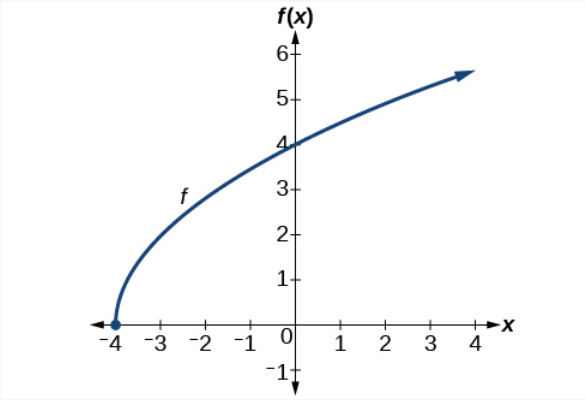


Figure 16.3.1.19: Graph of a square root function at $(-4, 0)$.

Exercise 16.3.1.7

Find the domain and range of

$f(x) = \sqrt{-2 - x}$.

Answer

domain: $(-\infty, -2]$

range: $[0, \infty)$

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.

- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.
- An understanding of toolkit functions can be used to find the domain and range of related functions.

Footnotes

1 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014
2 www.eia.gov/dnav/pet/hist/Lea...s=MCRFPAK2&f=A.

Glossary

interval notation

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

set-builder notation

a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

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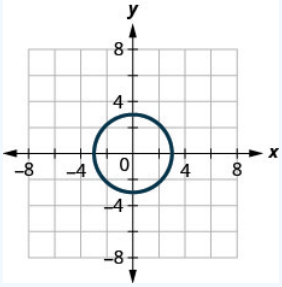
16.3E: Exercises

Practice Makes Perfect

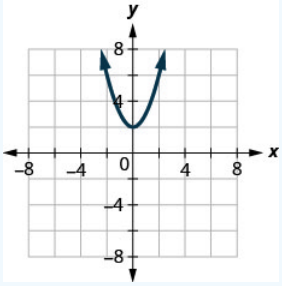
Use the Vertical Line Test

In the following exercises, determine whether each graph is the graph of a function.

1. ⓐ



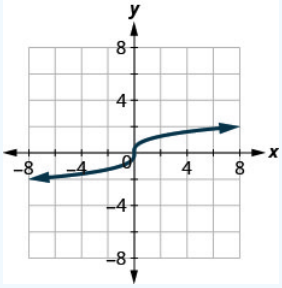
ⓑ



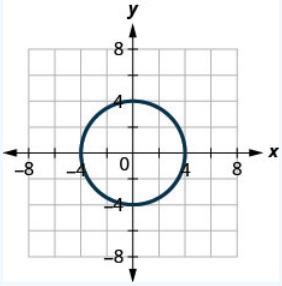
Answer

ⓐ no ⓑ yes

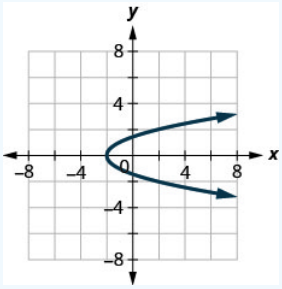
2. ⓐ



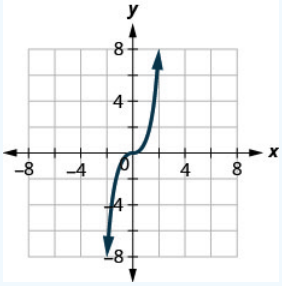
ⓑ



3. ⓐ



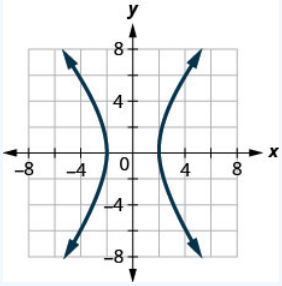
ⓑ



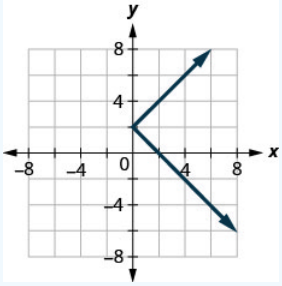
Answer

ⓐ no ⓑ yes

4. ⓐ



ⓑ



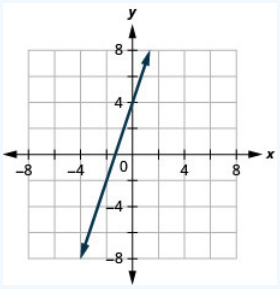
Identify Graphs of Basic Functions

In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

5. $f(x) = 3x + 4$

Answer

ⓐ



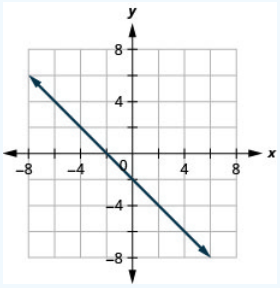
ⓑ $D: (-\text{inf}, \text{inf}),\ R: (-\text{inf}, \text{inf})$

6. $f(x) = 2x + 5$

7. $f(x) = -x - 2$

Answer

ⓐ



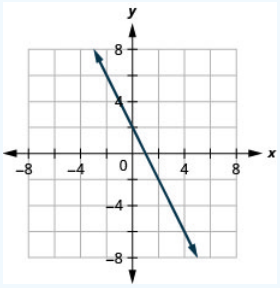
ⓑ $D: (-\text{inf}, \text{inf}),\ R: (-\text{inf}, \text{inf})$

8. $f(x) = -4x - 3$

9. $f(x) = -2x + 2$

Answer

ⓐ



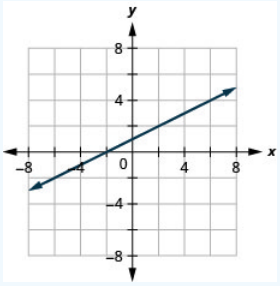
ⓑ $D: (-\text{inf}, \text{inf}),\ R: (-\text{inf}, \text{inf})$

10. $f(x) = -3x + 3$

11. $f(x) = \frac{1}{2}x + 1$

Answer

ⓐ



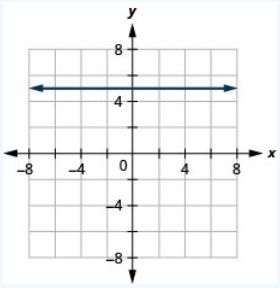
ⓑ $D : (-\text{inf}, \text{inf}),\ R : (-\text{inf}, \text{inf})$

12. $f(x) = \frac{2}{3}x - 2$

13. $f(x) = 5$

Answer

ⓐ



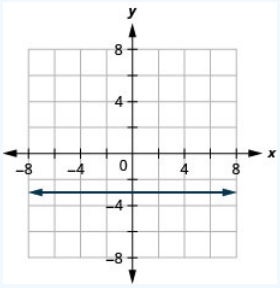
ⓑ $D : (-\text{inf}, \text{inf}), R : 5$

14. $f(x) = 2$

15. $f(x) = -3$

Answer

ⓐ



ⓑ $D : (-\text{inf}, \text{inf}),\ R : -3$

16. $f(x) = -1$

17. $f(x) = 2x$

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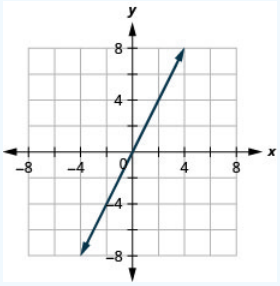
Answer

ⓐ



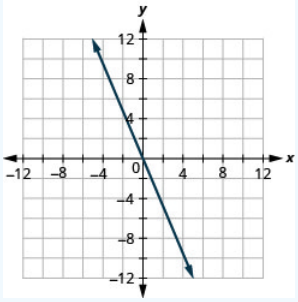
ⓑ $D: (-\text{inf}, \text{inf}),\ R: (-\text{inf}, \text{inf})$

18. $f(x) = 3x$

19. $f(x) = -2x$

Answer

ⓐ



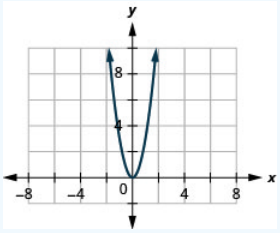
ⓑ $D: (-\text{inf}, \text{inf}), R: (-\text{inf}, \text{inf})$

20. $f(x) = -3x$

21. $f(x) = 3x^2$

Answer

ⓐ



ⓑ $D: (-\text{inf}, \text{inf}),\ R: [0, \text{inf})$

22. $f(x) = 2x^2$

23. $f(x) = -3x^2$

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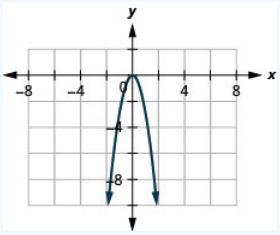
Answer

ⓐ



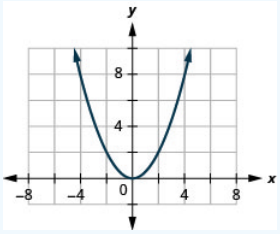
ⓑ $D : (-\text{inf}, \text{inf}),\ R : (-\text{inf}, 0]$

24. $f(x) = -2x^2$

25. $f(x) = 12x^2$

Answer

ⓐ



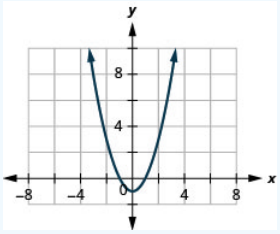
ⓑ $D : (-\text{inf}, \text{inf}),\ R : [-\text{inf}, 0)$

26. $f(x) = \frac{1}{3}x^2$

27. $f(x) = x^2 - 1$

Answer

ⓐ



ⓑ $D : (-\text{inf}, \text{inf}),\ R : [-1, \text{inf})$

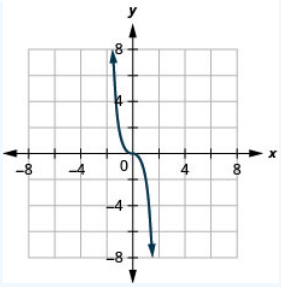
28. $f(x) = x^2 + 1$

29. $f(x) = -2x^3$

Answer

ⓐ

https://math.libretexts.org/@go/page/46405



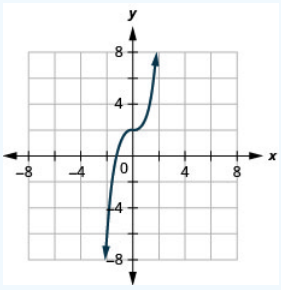
ⓑ $D: (-\text{inf}, \text{inf}),\ R: (-\text{inf}, \text{inf})$

30. $f(x) = 2x^3$

31. $f(x) = x^3 + 2$

Answer

ⓐ



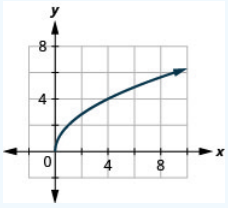
ⓑ $D: (-\text{inf}, \text{inf}), R: (-\text{inf}, \text{inf})$

32. $f(x) = x^3 - 2$

33. $f(x) = 2\sqrt{x}$

Answer

ⓐ



ⓑ $D: [0, \text{inf}), R: [0, \text{inf})$

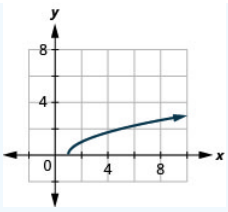
34. $f(x) = -2\sqrt{x}$

35. $f(x) = \sqrt{x-1}$

Answer

ⓐ

https://math.libretexts.org/@go/page/46405



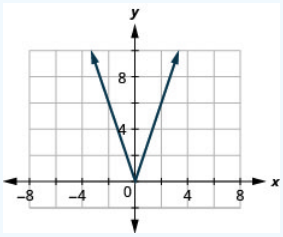
ⓑ $D: [1, \text{inf}), R: [0, \text{inf})$

36. $f(x) = \sqrt{x+1}$

37. $f(x) = 3|x|$

Answer

ⓐ



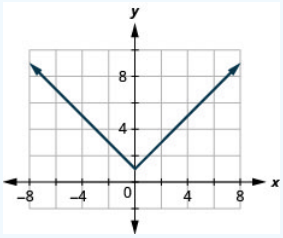
ⓑ $D: [-1, -1, \text{inf}), R: [-\text{inf}, \text{inf})$

38. $f(x) = -2|x|$

39. $f(x) = |x| + 1$

Answer

ⓐ



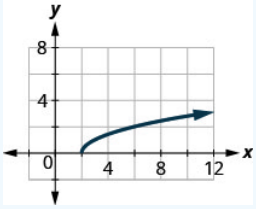
ⓑ $D: (-\text{inf}, \text{inf}), R: [1, \text{inf})$

40. $f(x) = |x| - 1$

Read Information from a Graph of a Function

In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation.

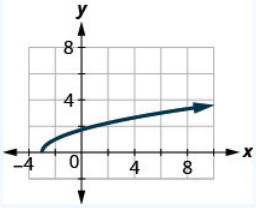
41.



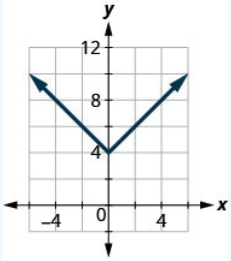
Answer

$D : [2, \text{inf}),\ R : [0, \text{inf})$

42.



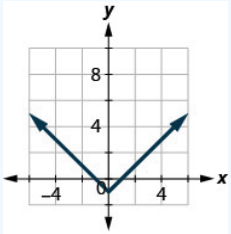
43.



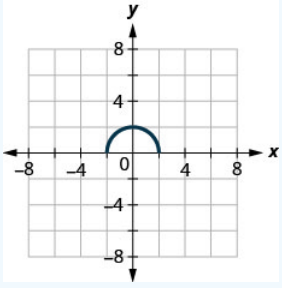
Answer

$D : (-\text{inf}, \text{inf}),\ R : [4, \text{inf})$

44.



45.



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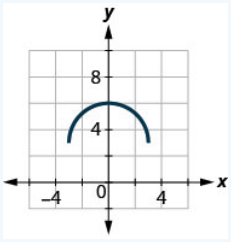
Answer

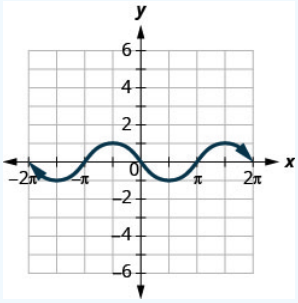
$D: [-2, 2],\ R: [0, 2]$

46.



In the following exercises, use the graph of the function to find the indicated values.

47.

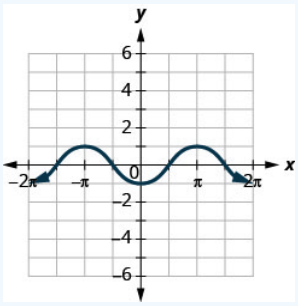


ⓐ Find: $f(0)$.
ⓑ Find: $f(12\pi)$.
ⓒ Find: $f(-32\pi)$.
ⓓ Find the values for x when $f(x) = 0$.
ⓔ Find the x-intercepts.
ⓕ Find the y-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Answer

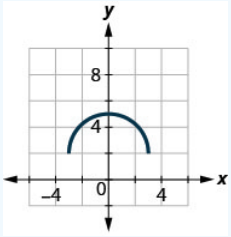
ⓐ $f(0) = 0$ ⓑ $(\pi/2) = -1$
ⓒ $f(-3\pi/2) = -1$ ⓓ $f(x) = 0$ for $x = -2\pi, -\pi, 0, \pi, 2\pi$
ⓔ $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)(f)(0, 0)$
ⓖ $[-2\pi, 2\pi]$ ⓗ $[-1, 1]$

48.



ⓐ Find: $f(0)$.
ⓑ Find: $f(\pi)$.
ⓒ Find: $f(-\pi)$.
ⓓ Find the values for x when $f(x)=0$.
ⓔ Find the x-intercepts.
ⓕ Find the y-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation

49.

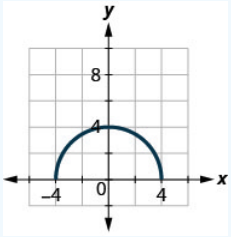


ⓐ Find: $f(0)$.
ⓑ Find: $f(-3)$.
ⓒ Find: $f(3)$.
ⓓ Find the values for x when $f(x)=0$.
ⓔ Find the x-intercepts.
ⓕ Find the y-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Answer

ⓐ $f(0)=-6$ ⓑ $f(-3)=3$ ⓒ $f(3)=3$ ⓓ $f(x)=0$ for no x ⓔ none ⓕ $y=6$ ⓖ $[-3,3]$ ⓗ $[-3,6]$

50.



ⓐ Find: $f(0)$.
ⓑ Find the values for x when $f(x)=0$.
ⓒ Find the x-intercepts.
ⓓ Find the y-intercepts.
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation

Writing Exercises

51. Explain in your own words how to find the domain from a graph.

52. Explain in your own words how to find the range from a graph.

53. Explain in your own words how to use the vertical line test.

54. Draw a sketch of the square and cube functions. What are the similarities and differences in the graphs?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the vertical line test.			
identify graphs of basic functions.			
read information from a graph.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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16.4: Graph Quadratic Functions Using Properties

Learning Objectives

By the end of this section, you will be able to:

- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic functions using properties
- Solve maximum and minimum applications

Before you get started, take this readiness quiz.

1. Graph the function $f(x) = x^2$ by plotting points.
   If you missed this problem, review Example 3.54.
2. Solve: $2x^2 + 3x - 2 = 0$.
   If you missed this problem, review Example 6.45.
3. Evaluate $-\frac{b}{2a}$ when $a = 3$ and $b = -6$.
   If you missed this problem, review Example 1.21.

Recognize the Graph of a Quadratic Function

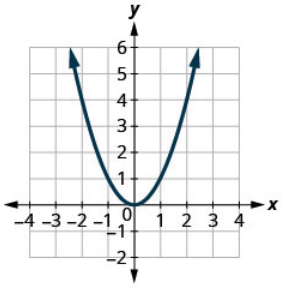
Previously we very briefly looked at the function $f(x) = x^2$, which we called the square function. It was one of the first non-linear functions we looked at. Now we will graph functions of the form $f(x) = ax^2 + bx + c$ if $a \neq 0$. We call this kind of function a quadratic function.

Definition 16.4.1

A **quadratic function**, where a, b, and c are real numbers and $a \neq 0$, is a function of the form

$$f(x) = ax^2 + bx + c$$

We graphed the quadratic function $f(x) = x^2$ by plotting points.



x	$f(x) = x^2$	$(x, f(x))$
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Figure 9.6.1

Every quadratic function has a graph that looks like this. We call this figure a **parabola**. Let's practice graphing a parabola by plotting a few points.

Example 16.4.1

Graph: $f(x) = x^2 - 1$.

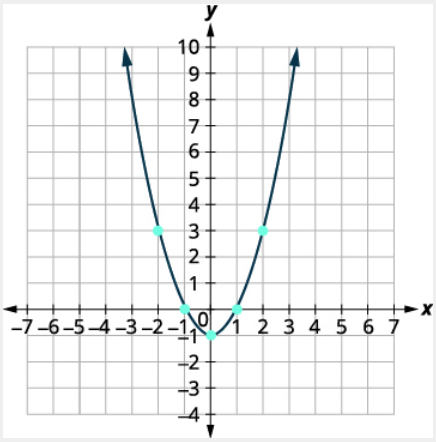
Solution:

We will graph the function by plotting points.

Choose integer values for x, substitute them into the equation and simplify to find $f(x)$. Record the values of the ordered pairs in the chart.

$f(x) = x^2 - 1$	
x	$f(x)$
0	–1
1	0
–1	0
2	3
–2	3

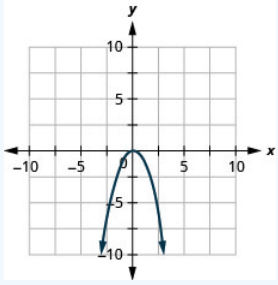
Plot the points, and then connect them with a smooth curve. The result will be the graph of the function $f(x) = x^2 - 1$.



Exercise 16.4.1

Graph $f(x) = -x^2$.

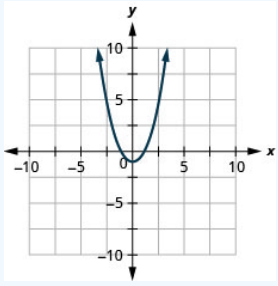
Answer



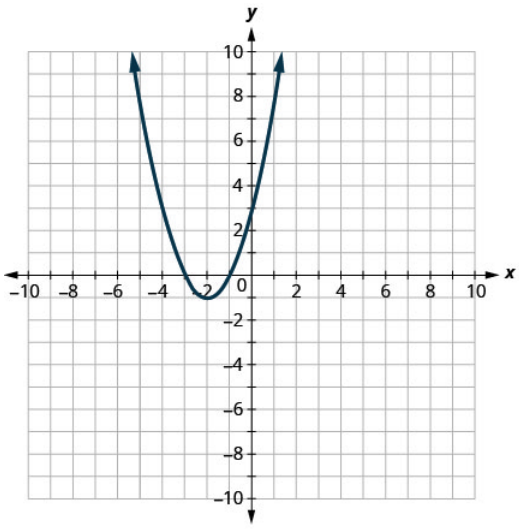
Exercise 16.4.2

Graph $f(x) = x^2 - 1$.

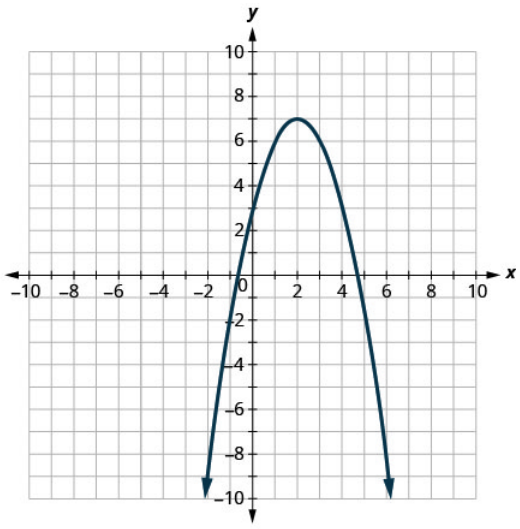
Answer



All graphs of quadratic functions of the form $f(x) = ax^2 + bx + c$ are parabolas that open upward or downward. See *Figure 9.6.6*



$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 3$
$a > 0$
opens upward



$f(x) = ax^2 + bx + c$
$f(x) = -x^2 + 4x + 3$
$a < 0$
opens downward

Notice that the only difference in the two functions is the negative sign before the quadratic term (x^2 in the equation of the graph in *Figure 9.6.6*). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Definition 16.4.2

Parabola Orientation

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if

- $a > 0$, the parabola opens upward
- $a < 0$, the parabola opens downward

Example 16.4.2

Determine whether each parabola opens upward or downward:

a. $f(x) = -3x^2 + 2x - 4$
b. $f(x) = 6x^2 + 7x - 9$

Solution:

a. Find the value of a.

$f(x) = ax^2 + bx + c$
$f(x) = -3x^2 + 2x - 4$
$a = -3$

Since the a is negative, the parabola will open downward.

b. Find the value of a.

$f(x) = ax^2 + bx + c$
$f(x) = 6x^2 + 7x - 9$
$a = 6$

Since the a is positive, the parabola will open upward.

Exercise 16.4.3

Determine whether the graph of each function is a parabola that opens upward or downward:

a. $f(x) = 2x^2 + 5x - 2$
b. $f(x) = -3x^2 - 4x + 7$

Answer

a. up
b. down

Exercise 16.4.4

Determine whether the graph of each function is a parabola that opens upward or downward:

a. $f(x) = -2x^2 - 2x - 3$
b. $f(x) = 5x^2 - 2x - 1$

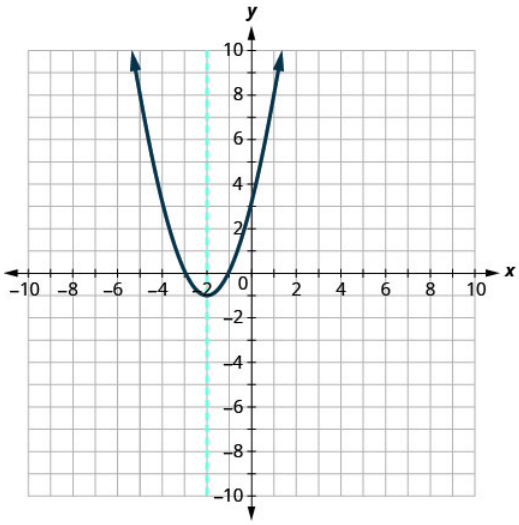
Answer

a. down
b. up

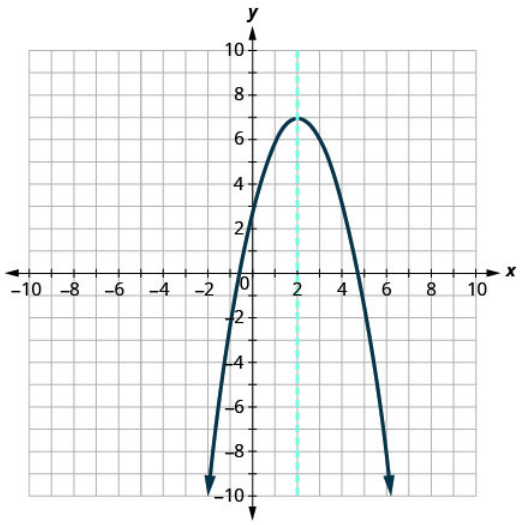
Find the Axis of Symmetry and Vertex of a Parabola

Look again at *Figure 9.6.10*. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry.



$f(x) = x^2 + 4x + 3$



$f(x) = -x^2 + 4x + 3$

The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

$f(x) = ax^2 + bx + c$	$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry	axis of symmetry
$x = -\frac{b}{2a}$	$x = -\frac{b}{2a}$
$x = -\frac{4}{2 \cdot 1}$	$x = -\frac{4}{2(-1)}$
$x = -2$	$x = 2$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its x-coordinate is $-\frac{b}{2a}$. To find the y-coordinate of the vertex we substitute the value of the x-coordinate into the quadratic function.

$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry is $x = -2$	axis of symmetry is $x = 2$
vertex is $(-2, \_\_\_)$	vertex is $(2, \_\_\_)$
$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
$f(x) = (-2)^2 + 4(-2) + 3$	$f(x) = -(2)^2 + 4(2) + 3$
$f(x) = -1$	$f(x) = 7$
vertex is $(-2, -1)$	vertex is $(2, 7)$

Axis of Symmetry and Vertex of a Parabola

The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:

- the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$
- the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.

Example 16.4.3

For the graph of $f(x) = 3x^2 - 6x + 2$ find:

a. the axis of symmetry
b. the vertex

Solution:

a.

	$f(x) = ax^2 + bx + c$ $f(x) = 3x^2 - 6x + 2$
The axis of symmetry is the vertical line $x = -\frac{b}{2a}$.	
Substitute the values a, b into the equation.	$x = -\frac{-6}{2 \cdot 3}$
Simplify.	$x = 1$
	The axis of symmetry is the line $x = 1$.

b.

	$f(x) = 3x^2 - 6x + 2$
The vertex is a point on the line of symmetry, so its x-coordinate will be $x = 1$. Find $f(1)$.	$f(1) = 3(1)^2 - 6(1) + 2$
Simplify.	$f(1) = 3 \cdot 1 - 6 + 2$

The result is the y-coordinate.	$f(1) = -1$
	The vertex is $(1, -1)$.

Exercise 16.4.5

For the graph of $f(x) = 2x^2 - 8x + 1$ find:

a. the axis of symmetry
b. the vertex

Answer

a. $x = 2$
b. $(2, -7)$

Exercise 16.4.6

For the graph of $f(x) = 2x^2 - 4x - 3$ find:

a. the axis of symmetry
b. the vertex

Answer

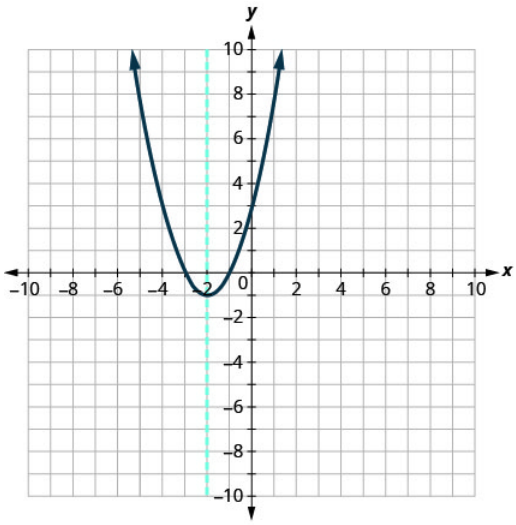
a. $x = 1$
b. $(1, -5)$

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the x- and y-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the y-intercept the value of x is zero. So to find the y-intercept, we substitute $x = 0$ into the function.

Let's find the y-intercepts of the two parabolas shown in *Figure 9.6.20*.

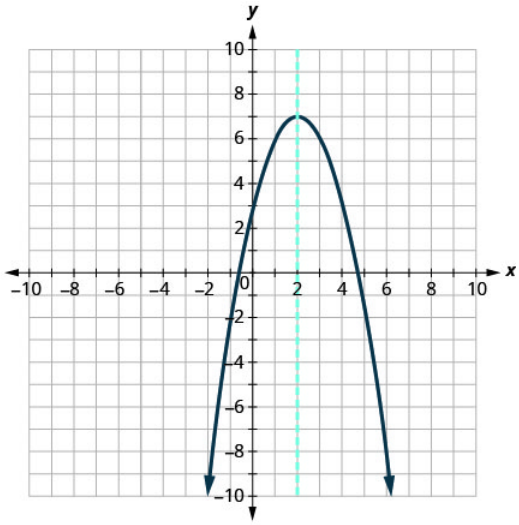


$f(x) = x^2 + 4x + 3$

$x = 0 \quad f(0) = 0^2 + 4 \cdot 0 + 3$

$f(0) = 3$

y-intercept $(0, 3)$



$f(x) = -x^2 + 4x + 3$

$x = 0 \quad f(0) = -0^2 + 4 \cdot 0 + 3$

$f(0) = 3$

y-intercept $(0, 3)$

An x-intercept results when the value of $f(x)$ is zero. To find an x-intercept, we let $f(x)=0$. In other words, we will need to solve the equation $0=ax^2+bx+c$ for x.

$$f(x)=ax^2+bx+c$$
$$0=ax^2+bx+c$$

Solving quadratic equations like this is exactly what we have done earlier in this chapter!

We can now find the x-intercepts of the two parabolas we looked at. First we will find the x-intercepts of the parabola whose function is $f(x)=x^2+4x+3$.

	$f(x)=x^2+4x+3$
Let $f(x)=0$.	$0=x^2+4x+3$
Factor.	$0=(x+1)(x+3)$
Use the Zero Product Property.	$x+1=0 \quad x+3=0$
Solve.	$x=-1 \quad x=-3$
	The x-intercepts are $(-1,0)$ and $(-3,0)$.

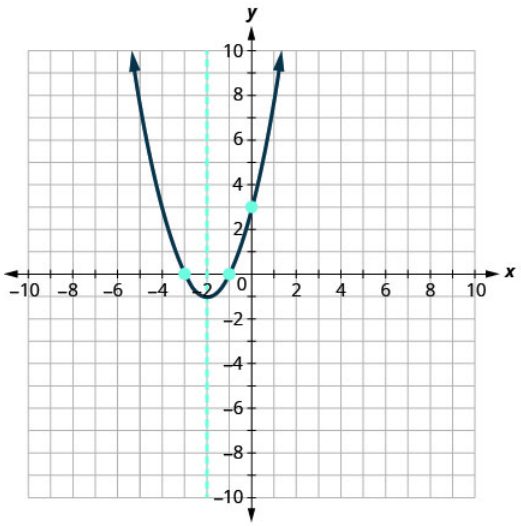
Now we will find the x-intercepts of the parabola whose function is $f(x)=-x^2+4x+3$.

	$f(x)=-x^2+4x+3$
Let $f(x)=0$.	$0=-x^2+4x+3$
This quadratic does not factor, so we use the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$a=-1, b=4, c=3$	$x=\frac{-4\pm\sqrt{4^2-4(-1)(3)}}{2(-1)}$
Simplify.	$x=\frac{-4\pm\sqrt{28}}{-2}$
	$x=\frac{-4\pm2\sqrt{7}}{-2}$
	$x=\frac{-2(2\pm\sqrt{7})}{-2}$
	$x=2\pm\sqrt{7}$
	The x-intercepts are $(2+\sqrt{7},0)$ and $(2-\sqrt{7},0)$.

We will use the decimal approximations of the x-intercepts, so that we can locate these points on the graph,

$$(2+\sqrt{7},0)\approx(4.6,0) \quad (2-\sqrt{7},0)\approx(-0.6,0)$$

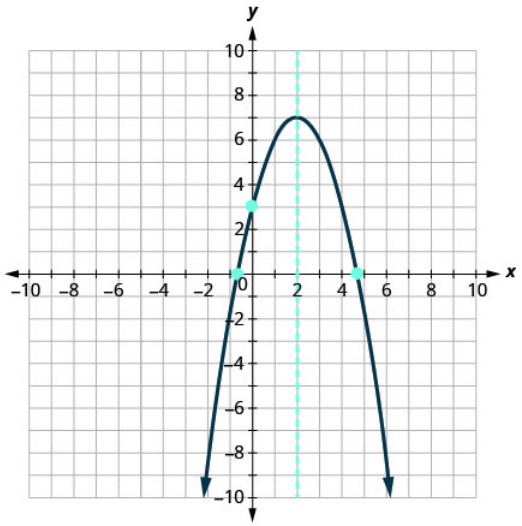
Do these results agree with our graphs? See *Figure 9.6.34*



$f(x) = x^2 + 4x + 3$

y-intercept $(0, 3)$

x-intercepts $(-1, 0)$ and $(-3, 0)$



$f(x) = -x^2 + 4x + 3$

y-intercept $(0, 3)$

x-intercepts $(2 + \sqrt{7}, 0) \approx (4.6, 0)$

$(2 - \sqrt{7}, 0) \approx (-0.6, 0)$

Find the Intercepts of a Parabola

To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

y-intercept

Let $x = 0$ and solve for $f(x)$.

x-intercepts

Let $f(x) = 0$ and solve for x

Example 16.4.4

Find the intercepts of the parabola whose function is $f(x) = x^2 - 2x - 8$.

Solution:

To find the y-intercept, let $x = 0$ and solve for $f(x)$.	$f(x) = x^2 - 2x - 8$
	$f(0) = 0^2 - 2 \cdot 0 - 8$
	$f(0) = -8$
	When $x = 0$, then $f(0) = -8$. The y-intercept is the point $(0, -8)$.
To find the x-intercept, let $f(x) = 0$ and solve for x.	$f(x) = x^2 - 2x - 8$
	$0 = x^2 - 2x - 8$
Solve by factoring.	$0 = (x - 4)(x + 2)$
	$0 = x - 4 \quad 0 = x + 2$
	$4 = x \quad -2 = x$
	When $f(x) = 0$, then $x = 4$ or $x = -2$. The x-intercepts are the points $(4, 0)$ and $(-2, 0)$.

Exercise 16.4.7

Find the intercepts of the parabola whose function is $f(x) = x^2 + 2x - 8$.

Answer

y -intercept: $(0, -8)$ x -intercepts $(-4, 0), (2, 0)$

Exercise 16.4.8

Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 12$.

Answer

y -intercept: $(0, -12)$ x -intercepts $(-2, 0), (6, 0)$

In this chapter, we have been solving quadratic equations of the form $ax^2 + bx + c = 0$. We solved for x and the results were the solutions to the equation.

We are now looking at quadratic functions of the form $f(x) = ax^2 + bx + c$. The graphs of these functions are parabolas. The x-intercepts of the parabolas occur where $f(x) = 0$.

For example:

Quadratic equation

$$\begin{aligned} x^2 - 2x - 15 &= 0 \quad \text{Let } f(x) = 0 \\ (x-5)(x+3) &= 0 \\ x - 5 = 0 \quad x + 3 &= 0 \\ x = 5 \quad x &= -3 \end{aligned}$$

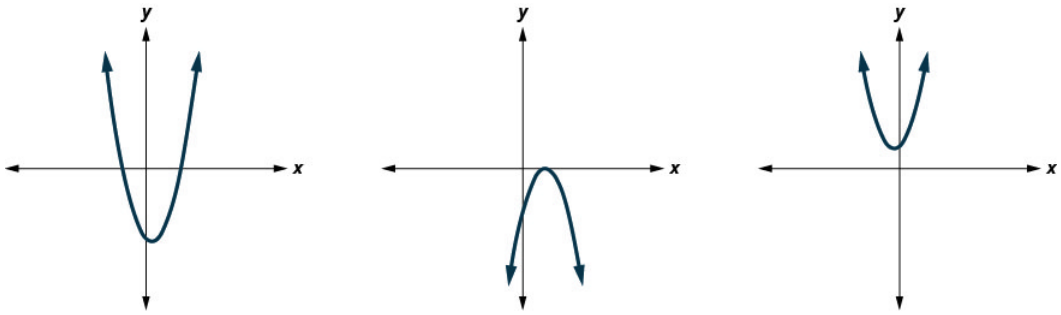
Quadratic function

$$\begin{aligned} f(x) &= x^2 - 2x - 15 \\ 0 &= x^2 - 2x - 15 \\ 0 &= (x-5)(x+3) \\ x - 5 = 0 \quad & x + 3 = 0 \\ x = 5 \quad & x = -3 \\ (5, 0) \text{ and } &(-3, 0) \\ x \text{ -intercepts} & \end{aligned}$$

The solutions of the quadratic function are the x values of the x-intercepts.

Earlier, we saw that quadratic equations have $2, 1$, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the x-intercepts of the graphs, the number of x-intercepts is the same as the number of solutions.

Previously, we used the **discriminant** to determine the number of solutions of a quadratic function of the form $ax^2 + bx + c = 0$. Now we can use the discriminant to tell us how many x-intercepts there are on the graph.



$b^2 - 4ac > 0$
Two solutions
Two x-intercepts

$b^2 - 4ac = 0$
One solution
One x-intercept

$b^2 - 4ac < 0$
No real solution
No x-intercept

Before you to find the values of the x-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

Example 16.4.5

Find the intercepts of the parabola for the function $f(x) = 5x^2 + x + 4$.

Solution:

	$f(x) = 5x^2 + x + 4$
To find the y-intercept, let $x = 0$ and solve for $f(x)$.	$f(0) = 5 \cdot 0^2 + 0 + 4$
	$f(0) = 4$
	When $x = 0$, then $f(0) = 4$. The y-intercept is the point $(0, 4)$.
To find the x-intercept, let $f(x) = 0$ and solve for x.	$f(x) = 5x^2 + x + 4$
	$0 = 5x^2 + x + 4$
Find the value of the discriminant to predict the number of solutions which is also the number of x-intercepts.	
$b^2 - 4ac$ $1^2 - 4 \cdot 5 \cdot 4$ $1 - 80$ -79	
	Since the value of the discriminant is negative, there is no real solution to the equation. There are no x-intercepts.

Exercise 16.4.9

Find the intercepts of the parabola whose function is $f(x) = 3x^2 + 4x + 4$.

Answer

y-intercept: $(0, 4)$ no x-intercept

Exercise 16.4.10

Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 5$

Answer

y-intercept: $(0,-5)$ x-intercepts $(-1,0),(5,0)$

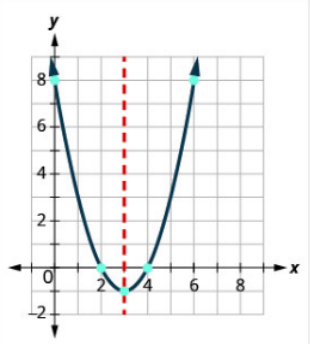
Graph Quadratic Functions Using Properties

Now we have all the pieces we need in order to graph a quadratic function. We just need to put them together. In the next example we will see how to do this.

Example 16.4.6 How to Graph a Quadratic Function Using Properties

Graph $f(x)=x^2-6x+8$ by using its properties.

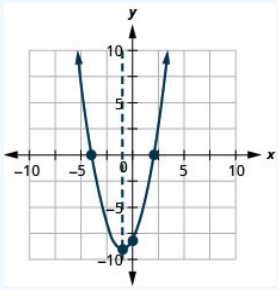
Solution:

Step 1: Determine whether the parabola opens upward or downward.	Look a in the equation $f(x)=x^2-6x+8$ Since a is positive, the parabola opens upward.	$f(x)=x^2-6x+8$ $a=1, b=-6, c=8$ **The parabola opens upward.**
Step 2: Find the axis of symmetry.	$f(x)=x^2-6x+8$ The axis of symmetry is the line $x=-\frac{b}{2a}$.	Axis of Symmetry $x=-\frac{b}{2a}$ $x=-\frac{(-6)}{2\cdot 1}$ $x=3$ **The axis of symmetry is the line $x=3$.**
Step 3: Find the vertex.	The vertex is on the axis of symmetry. Substitute $x=3$ into the function.	Vertex $f(x)=x^2-6x+8$ $f(3)=(3)^2-6(3)+8$ $f(3)=-1$ **The vertex is $(3,-1)$.**
Step 4: Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We find $f(0)$. We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x=3$. A point 3 units to the right of the axis of symmetry has $x=6$.	y-intercept $f(x)=x^2-6x+8$ $f(0)=(0)^2-6(0)+8$ $f(0)=8$ **The y-intercept is $(0,8)$.** Point symmetric to y-intercept: **The point is $(6,8)$.**
Step 5: Find the x-intercepts. Find additional points if needed.	We solve $f(x)=0$. We can solve this quadratic equation by factoring.	x-intercepts $f(x)=x^2-6x+8$ $0=x^2-6x+8$ $0=(x-2)(x-4)$ $x=2 or x=4$ **The x-intercepts are $(2,0)$ and $(4,0)$.**
Step 6: Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

Exercise 16.4.11

Graph $f(x) = x^2 + 2x - 8$ by using its properties.

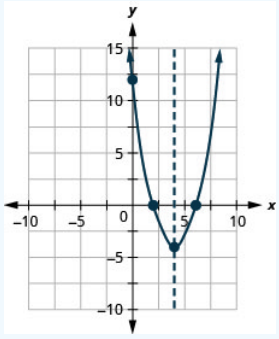
Answer



Exercise 16.4.12

Graph $f(x) = x^2 - 8x + 12$ by using its properties.

Answer



We list the steps to take in order to graph a quadratic function here.

To Graph a Quadratic Function Using Properties

1. Determine whether the parabola opens upward or downward.
2. Find the equation of the axis of symmetry.
3. Find the vertex.
4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
5. Find the x-intercepts. Find additional points if needed.
6. Graph the parabola.

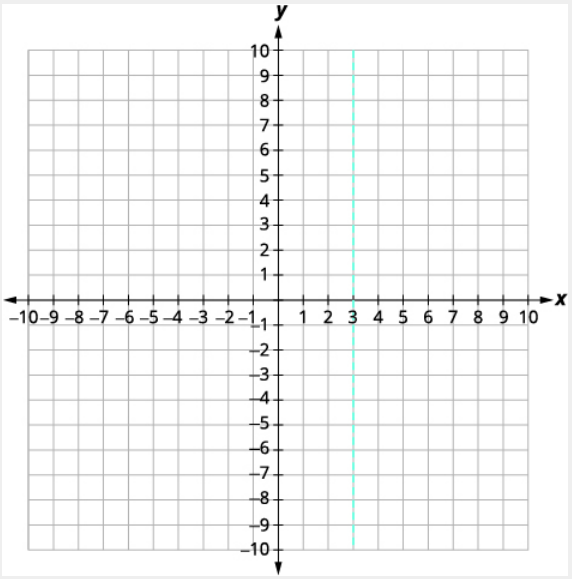
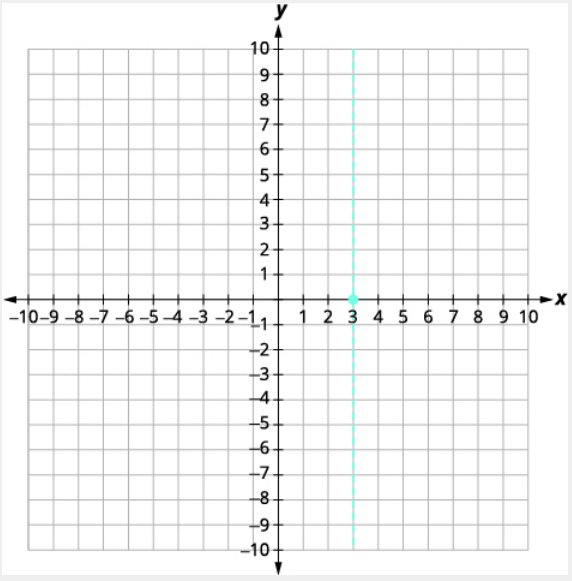
We were able to find the x-intercepts in the last example by factoring. We find the x-intercepts in the next example by factoring, too.

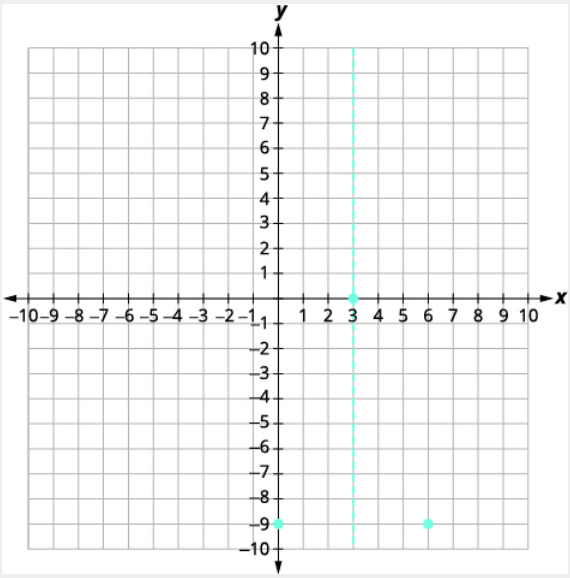
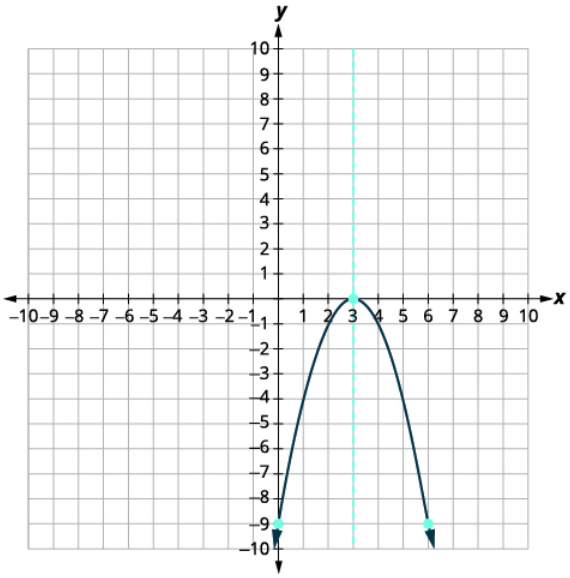
Example 16.4.7

Graph $f(x) = -x^2 + 6x - 9$ by using its properties.

Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	

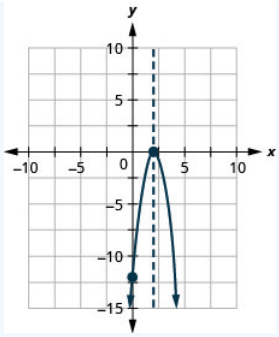
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{6}{2(-1)}$
	$x = 3$
	The axis of symmetry is $x = 3$. The vertex is on the line $x = 3$.
	
Find $f(3)$.	$f(x) = -x^2 + 6x - 9$
	$f(3) = -3^2 + 6 \cdot 3 - 9$
	$f(3) = -9 + 18 - 9$
	$f(3) = 0$
	The vertex is $(3, 0)$.
	
The y-intercept occurs when $x = 0$. Find $f(0)$.	$f(x) = -x^2 + 6x - 9$
Substitute $x = 0$.	$f(0) = -0^2 + 6 \cdot 0 - 9$

Simplify.	$f(0) = -9$
The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$.	
	Point symmetric to the y-intercept is $(6, -9)$
The x-intercept occurs when $f(x) = 0$.	$f(x) = -x^2 + 6x - 9$
Find $f(x) = 0$.	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for x.	$0 = 3$
Connect the points to graph the parabola.	

? Exercise 16.4.13

Graph $f(x) = 3x^2 + 12x - 12$ by using its properties.

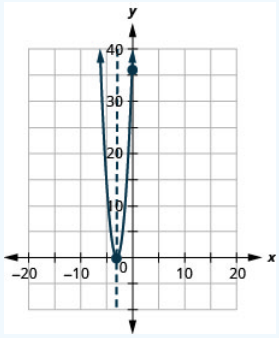
Answer



Exercise 16.4.14

Graph $f(x) = 4x^2 + 24x + 36$ by using its properties.

Answer



For the graph of $f(x) = -x^2 + 6x - 9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is only one solution. That means there is only one x-intercept, and it is the vertex of the parabola.

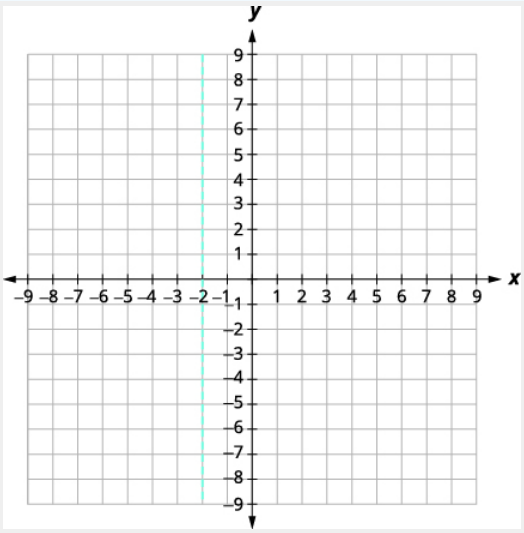
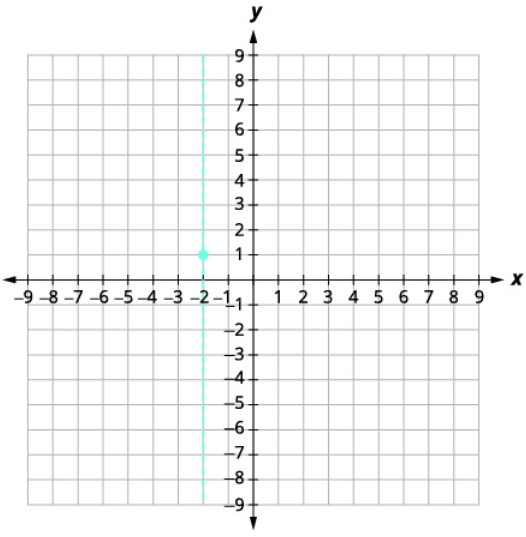
How many x-intercepts would you expect to see on the graph of $f(x) = x^2 + 4x + 5$?

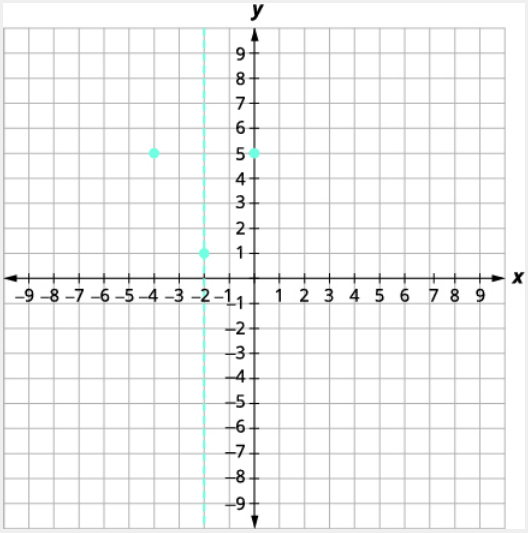
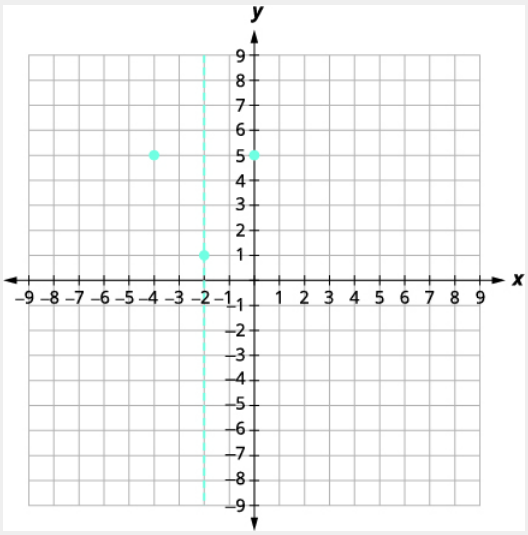
Example 16.4.8

Graph $f(x) = x^2 + 4x + 5$ by using its properties.

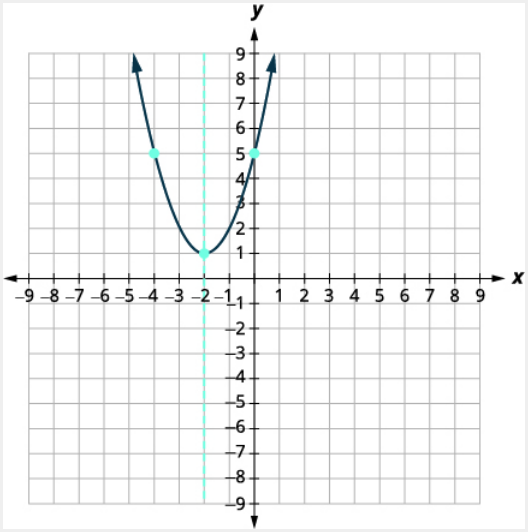
Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{4}{(2)1}$
	$x = -2$
	The equation of the axis of symmetry is \(x=-2).

	
The vertex is on the line $x = -2$.	
Find $f(x)$ when $x = -2$.	$f(x) = x^2 + 4x + 5$
	$f(-2) = (-2)^2 + 4(-2) + 5$
	$f(-2) = 4 - 8 + 5$
	$f(-2) = 1$
	The vertex is $(-2, 1)$.
	
The y-intercept occurs when $x = 0$.	$f(x) = x^2 + 4x - 5$
Find $f(0)$.	$f(0) = 5$
Simplify.	$f(0) = 5$
	The y-intercept is $(0, 5)$.

The point $(-4, 5)$ is two units to the left of the line of symmetry. The point to units to the right of the line of symmetry is \((0,5)\.	
	Point symmetric to the y-intercept is $(-4, 5)$.
The x-intercept occurs when $f(x) = 0$.	
Find $f(x) = 0$.	$0 = x^2 + 4x + 5$
Test the discriminant.	
	$b^2 - 4ac$
	$4^2 - 4 \cdot 1 \cdot 5$
	$16 - 20$
	-4
Since the value of the discriminant is negative, there is no real solution and so no x-intercept.	

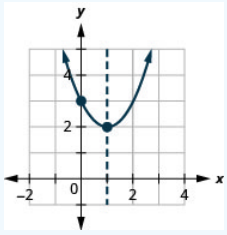
Connect the points to graph the parabola. You may want to choose two more points for greater accuracy.



Exercise 16.4.15

Graph $f(x) = x^2 - 2x + 3$ by using its properties.

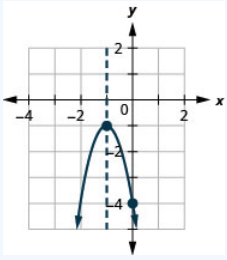
Answer



Exercise 16.4.16

Graph $f(x) = -3x^2 - 6x - 4$ by using its properties.

Answer



Finding the y-intercept by finding $f(0)$ is easy, isn't it? Sometimes we need to use the **Quadratic Formula** to find the x-intercepts.

Example 16.4.9

Graph $f(x) = 2x^2 - 4x - 3$ by using its properties.

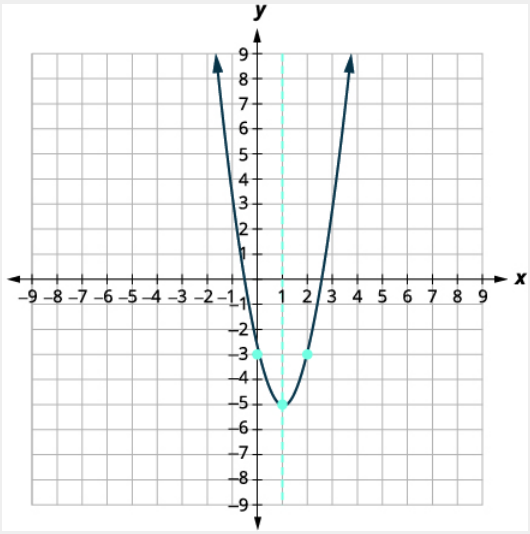
Solution:

$f(x) = ax^2 + bx + c$

$f(x) = 2x^2 - 4x - 3$

Since a is 2, the parabola opens upward.	
To find the equation of the axis of symmetry, use $x=-\frac{b}{2a}$.	$x=-\frac{b}{2a}$
	$x=-\frac{-4}{2\cdot 2}$
	$x=1$
	The equation of the axis of symmetry is $x=1$.
The vertex is on the line $x=1$.	$f(x)=2x^2-4x-3$
Find $f(1)$.	$f(x)=2(1)^2-4(1)-3$
	$f(1)=2-4-3$
	\(\f(1)=-5)
	The vertex is $(1,-5)$.
The y-intercept occurs when $x=0$.	$f(x)=2x^2-4x-3$
Find $f(0)$.	$f(0)=2(0)^2-4(0)-3$
Simplify.	$f(0)=-3$
	The y-intercept is $(0,-3)$.
The point $(0,-3)$ is one unit to the left of the line of symmetry.	Point symmetric to the y-intercept is $(2,-3)$
The point one unit to the right of the line of symmetry is $(2,3)$.	
The x-intercept occurs when $y=0$.	$f(x)=2x^2-4x-3$
Find $f(x)=0$.	$0=2x^2-4x-3$
Use the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Substitute in the values of a, b and c.	$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(3)}}{2(2)}$
Simplify.	$x=\frac{-4\pm\sqrt{16+24}}{4}$
Simplify inside the radical.	$x=\frac{4\pm\sqrt{40}}{4}$
Simplify the radical.	$x=\frac{4\pm2\sqrt{10}}{4}$
Factor the GCF.	$x=\frac{2(2\pm\sqrt{10})}{4}$
Remove common factors.	$x=\frac{2\pm\sqrt{10}}{2}$
Write as two equations.	$x=\frac{2+\sqrt{10}}{2}, \quad x=\frac{2-\sqrt{10}}{2}$
Approximate the values.	$x\approx 2.5, \quad x\approx -0.6$
	The approximate values of the x-intercepts are $(2.5,0)$ and $(-0.6,0)$.

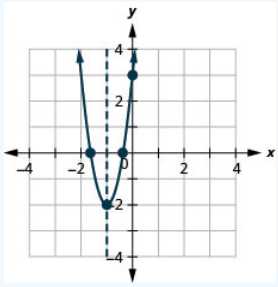
Graph the parabola using the points found.



Exercise 16.4.17

Graph $f(x) = 5x^2 + 10x + 3$ by using its properties.

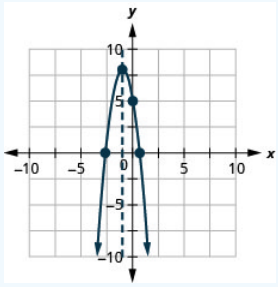
Answer



Exercise 16.4.18

Graph $f(x) = -3x^2 - 6x + 5$ by using its properties.

Answer



Solve Maximum and Minimum Applications

Knowing that the **vertex** of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function. The *y*-coordinate of the vertex is the **minimum** value of a parabola that opens upward. It is the **maximum** value of a parabola that opens downward. See *Figure 9.6.124*.

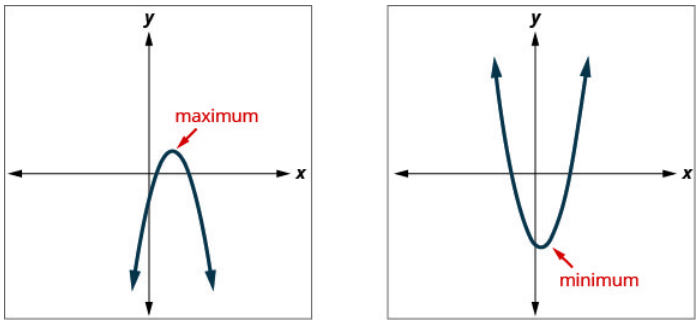


Figure 9.6.124

Minimum or Maximum Values of a Quadratic Function

The ***y*-coordinate of the vertex** of the graph of a quadratic function is the

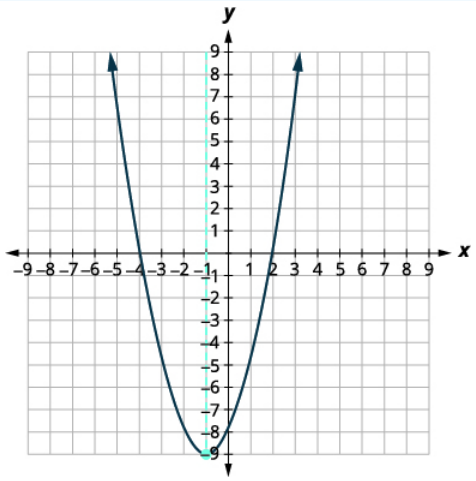
- *minimum* value of the quadratic equation if the parabola opens *upward*.
- *maximum* value of the quadratic equation if the parabola opens *downward*.

Example 16.4.10

Find the minimum or maximum value of the quadratic function $f(x) = x^2 + 2x - 8$.

Solution:

	$f(x) = x^2 + 2x - 8$
Since a is positive, the parabola opens upward. The quadratic equation has a minimum.	
Find the equation of the axis of symmetry.	$x = -\frac{b}{2a}$
	$x = -\frac{2}{2\times 1}$
	$x = -1$
	The equation of the axis of symmetry is $x = -1$.
The vertex is on the line $x = -1$.	$f(x) = x^2 + 2x - 8$
Find $f(-1)$.	$f(-1) = (-1)^2 + 2(-1) - 8$
	$f(-1) = 1 - 2 - 8$
	$f(-1) = -9$
	The vertex is $(-1, -9)$.
Since the parabola has a minimum, the y-coordinate of the vertex is the minimum y-value of the quadratic equation. The minimum value of the quadratic is -9 and it occurs when $x = -1$.	



Show the graph to verify the result.

? Exercise 16.4.19

Find the maximum or minimum value of the quadratic function $f(x) = x^2 - 8x + 12$.

Answer

The minimum value of the quadratic function is -4 and it occurs when $x = 4$.

? Exercise 16.4.20

Find the maximum or minimum value of the quadratic function $f(x) = -4x^2 + 16x - 11$.

Answer

The maximum value of the quadratic function is 5 and it occurs when $x = 2$.

We have used the formula

$$h(t) = -16t^2 + v_0 t + h_0$$

to calculate the height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds .

This formula is a quadratic function, so its graph is a parabola. By solving for the coordinates of the vertex (t, h), we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

✓ Example 16.4.11

The quadratic equation $h(t) = -16t^2 + 176t + 4$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

a. How many seconds will it take the volleyball to reach its maximum height?
b. Find the maximum height of the volleyball.

Solution:

$$h(t) = -16t^2 + 176t + 4$$

Since a is negative, the parabola opens downward. The quadratic function has a maximum.

a. Find the equation of the axis of symmetry.

$$t = -\frac{b}{2a}$$
$$t = -\frac{176}{2(-16)}$$
$$t = 5.5$$

The equation of the axis of symmetry is $t = 5.5$.

The vertex is on the line $t = 5.5$.

The maximum occurs when $t = 5.5$ seconds.

b. Find $h(5.5)$.

$$h(t) = -16t^2 + 176t + 4$$
$$h(t) = -16(5.5)^2 + 176(5.5) + 4$$

Use a calculator to simplify.

$$h(t) = 488$$

The vertex is $(5.5, 488)$.

Since the parabola has a maximum, the h-coordinate of the vertex is the maximum value of the quadratic function.

The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.

After 5.5 seconds, the volleyball will reach its maximum height of 488 feet.

? Exercise 16.4.21

Solve, rounding answers to the nearest tenth.

The quadratic function $h(t) = -16t^2 + 128t + 32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height?

Answer

It will take 4 seconds for the stone to reach its maximum height of 288 feet.

? Exercise 16.4.22

A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function of $h(t) = -16t^2 + 208t$. When will the rocket reach its maximum height? What will be the maximum height?

Answer

It will take 6.5 seconds for the rocket to reach its maximum height of 676 feet.

Key Concepts

- Parabola Orientation
  - For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if
    - $a > 0$, the parabola opens upward.
    - $a < 0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:
  - the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
  - the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$.
  - the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.
- Find the Intercepts of a Parabola
  - To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

- **y-intercept**
  - Let $x = 0$ and solve for $f(x)$.
- **x-intercepts**
  - Let \(f(x)=0\) and solve for x.

- How to graph a quadratic function using properties.
  1. Determine whether the parabola opens upward or downward.
  2. Find the equation of the axis of symmetry.
  3. Find the vertex.
  4. Find the y-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  5. Find the x-intercepts. Find additional points if needed.
  6. Graph the parabola.
- Minimum or Maximum Values of a Quadratic Equation
  - The y-coordinate of the vertex of the graph of a quadratic equation is the
  - *minimum* value of the quadratic equation if the parabola opens *upward*.
  - *maximum* value of the quadratic equation if the parabola opens *downward*.

Glossary

quadratic function

A quadratic function, where a, b, and c are real numbers and $a \neq 0$, is a function of the form $f(x) = ax^2 + bx + c$.

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16.4E: Exercises

Practice Makes Perfect

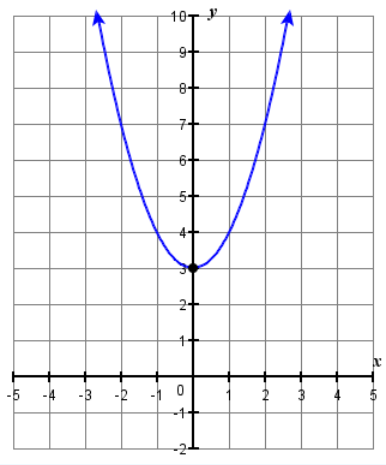
Exercises 1 - 4: Recognize the Graph of a Quadratic Function

In the following exercises, graph the functions by plotting points.

1. $f(x) = x^2 + 3$
2. $f(x) = x^2 - 3$
3. $y = -x^2 + 1$
4. $f(x) = -x^2 - 1$

Answer

1.



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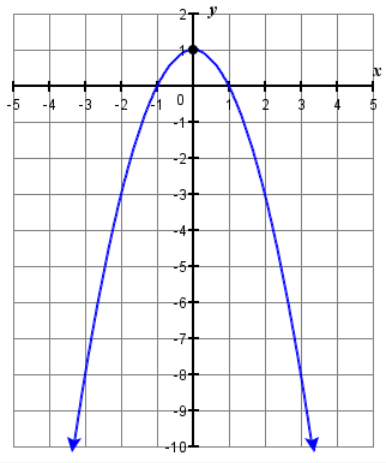

Exercises 5 - 8: Recognize the Graph of a Quadratic Function

For each of the following exercises, determine if the parabola opens up or down.

5. a. $f(x) = -2x^2 - 6x - 7$ b. $f(x) = 6x^2 + 2x + 3$
6. a. $f(x) = 4x^2 + x - 4$ b. $f(x) = -9x^2 - 24x - 16$

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7. a. $f(x) = -3x^2 + 5x - 1$ b. $f(x) = 2x^2 - 4x + 5$

8. a. $f(x) = x^2 + 3x - 4$ b. $f(x) = -4x^2 - 12x - 9$

Answer

5. a. down b. up

7. a. down b. up

Exercises 9 - 12: Find the Axis of Symmetry and Vertex of a Parabola

In the following functions, find

a. The equation of the axis of symmetry
b. The vertex of its graph

9. $f(x) = x^2 + 8x - 1$

10. $f(x) = x^2 + 10x + 25$

11. $f(x) = -x^2 + 2x + 5$

12. $f(x) = -2x^2 - 8x - 3$

Answer

9. a. Axis of symmetry: $x = -4$ b. Vertex: $(-4, -17)$

11. a. Axis of symmetry: $x = 1$ b. Vertex: $(1, 2)$

Exercises 13 - 24: Find the Intercepts of a Parabola

In the following exercises, find the intercepts of the parabola whose function is given.

13. $f(x) = x^2 + 7x + 6$

14. $f(x) = x^2 + 10x - 11$

15. $f(x) = x^2 + 8x + 12$

16. $f(x) = x^2 + 5x + 6$

17. $f(x) = -x^2 + 8x - 19$

18. $f(x) = -3x^2 + x - 1$

19. $f(x) = x^2 + 6x + 13$

20. $f(x) = x^2 + 8x + 12$

21. $f(x) = 4x^2 - 20x + 25$

22. $f(x) = -x^2 - 14x - 49$

23. $f(x) = -x^2 - 6x - 9$

24. $f(x) = 4x^2 + 4x + 1$

Answer

13. y-intercept: $(0, 6)$; x-intercept(s): $(-1, 0), (-6, 0)$

15. y-intercept: $(0, 12)$; x-intercept(s): $(-2, 0), (-6, 0)$

17. y-intercept: $(0, -19)$; x-intercept(s): none

19. y-intercept: $(0, 13)$; x-intercept(s): none

21. y-intercept: $(0, -16)$; x-intercept(s): $(\frac{5}{2}, 0)$

23. y-intercept: $(0, 9)$; x-intercept(s): $(-3, 0)$

Exercises 25 - 42: Graph Quadratic Functions Using Properties

In the following exercises, graph the function by using its properties.

25. $f(x) = x^2 + 6x + 5$
26. $f(x) = x^2 + 4x - 12$
27. $f(x) = x^2 + 4x + 3$
28. $f(x) = x^2 - 6x + 8$
29. $f(x) = 9x^2 + 12x + 4$
30. $f(x) = -x^2 + 8x - 16$
31. $f(x) = -x^2 + 2x - 7$
32. $f(x) = 5x^2 + 2$
33. $f(x) = 2x^2 - 4x + 1$
34. $f(x) = 3x^2 - 6x - 1$
35. $f(x) = 2x^2 - 4x + 2$
36. $f(x) = -4x^2 - 6x - 2$
37. $f(x) = -x^2 - 4x + 2$
38. $f(x) = x^2 + 6x + 8$
39. $f(x) = 5x^2 - 10x + 8$
40. $f(x) = -16x^2 + 24x - 9$
41. $f(x) = 3x^2 + 18x + 20$
42. $f(x) = -2x^2 + 8x - 10$

Answer

25.

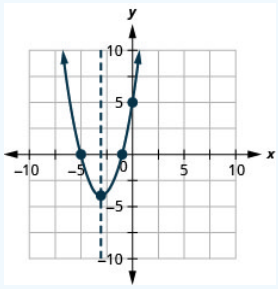

Figure 9.6.136

27.

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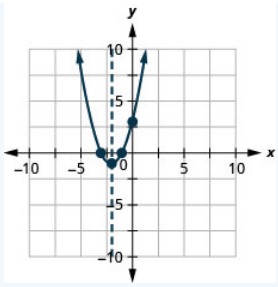

Figure 9.6.137

29.

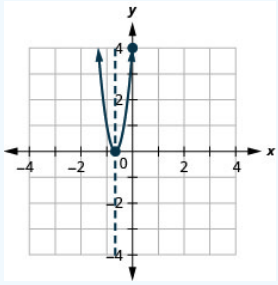

Figure 9.6.138

31.

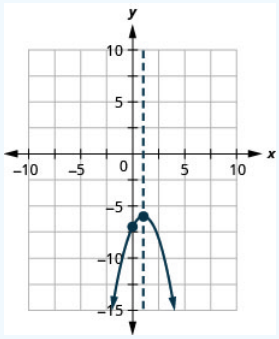

Figure 9.6.139

33.

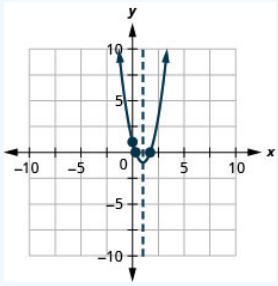

Figure 9.6.140

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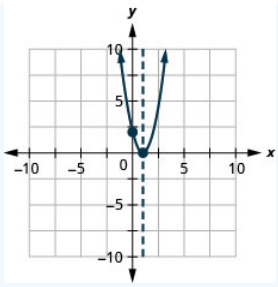

Figure 9.6.141

37.

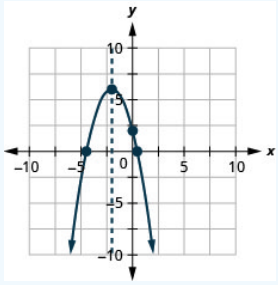

Figure 9.6.142

39.

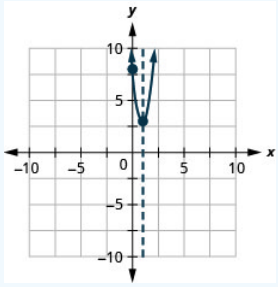

Figure 9.6.143

41.

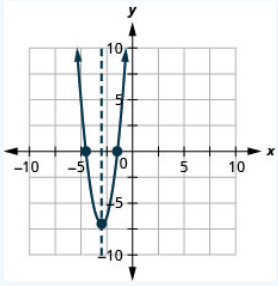

Figure 9.6.144

Exercises 43 - 48: Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value of each function.

43. $f(x) = 2x^2 + x - 1$

44. $y = -4x^2 + 12x - 5$

45. $y = x^2 - 6x + 15$

46. $y = -x^2 + 4x - 5$

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47. $y = -9x^2 + 16$

48. $y = 4x^2 - 49$

Answer

43. The minimum value is $-\frac{9}{8}$ when $x = -\frac{1}{4}$.

45. The maximum value is 6 when $x = 3$.

47. The maximum value is 16 when $x = 0$.

Exercises 49 - 60: Solve Maximum and Minimum Applications

In the following exercises, solve. Round answers to the nearest tenth.

49. An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic function $h(t) = -16t^2 + 168t + 45$ find how long it will take the arrow to reach its maximum height, and then find the maximum height.

50. A stone is thrown vertically upward from a platform that is 20 feet height at a rate of 160 ft/sec. Use the quadratic function $h(t) = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

51. A ball is thrown vertically upward from the ground with an initial velocity of 109 ft/sec. Use the quadratic function $h(t) = -16t^2 + 109t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

52. A ball is thrown vertically upward from the ground with an initial velocity of 122 ft/sec. Use the quadratic function $h(t) = -16t^2 + 122t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

53. A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic function $R(x) = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

54. A retailer who sells backpacks estimates that by selling them for x dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic function $R(x) = -x^2 + 100x$ is used to find the R, received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

55. A retailer who sells fashion boots estimates that by selling them for x dollars each, he will be able to sell $70 - x$ boots a week. Use the quadratic function $R(x) = -x^2 + 70x$ to find the revenue received when the average selling price of a pair of fashion boots is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue per day.

56. A cell phone company estimates that by charging x dollars each for a certain cell phone, they can sell $8 - x$ cell phones per day. Use the quadratic function $R(x) = -x^2 + 8x$ to find the revenue received per day when the selling price of a cell phone is x. Find the selling price that will give them the maximum revenue per day, and then find the amount of the maximum revenue.

57. A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The quadratic equation $A(x) = x(240 - 2x)$ gives the area of the corral, A, for the length, x, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

58. A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic function $A(x) = x(100 - 2x)$ gives the area, A, of the dog run for the length, x, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

59. A land owner is planning to build a fenced in rectangular patio behind his garage, using his garage as one of the "walls." He wants to maximize the area using 80 feet of fencing. The quadratic function $A(x) = x(80 - 2x)$ gives the area of the patio, where x is the width of one side. Find the maximum area of the patio.

60. A family of three young children just moved into a house with a yard that is not fenced in. The previous owner gave them 300 feet of fencing to use to enclose part of their backyard. Use the quadratic function $A(x) = x(300 - 2x)$ to determine the maximum area of the fenced in yard.

Answer

49. In 5.3 sec the arrow will reach maximum height of 486 ft.

51. In 3.4 seconds the ball will reach its maximum height of 185.6 feet.

53. 20 computers will give the maximum of $\$400$ in receipts.

55. He will be able to sell 35 pairs of boots at the maximum revenue of $\$1,225$

57. The length of the side along the river of the corral is 120 feet and the maximum area is $7,200$ square feet.

59. The maximum area of the patio is 800 feet.

? Exercises 61 - 64: Writing Exercises

61. How do the graphs of the functions $f(x) = x^2$ and $f(x) = x^2 - 1$ differ? We graphed them at the start of this section. What is the difference between their graphs? How are their graphs the same?

62. Explain the process of finding the vertex of a parabola.

63. Explain how to find the intercepts of a parabola.

64. How can you use the discriminant when you are graphing a quadratic function?

Answer

1. Answers will vary.

3. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize the graph of a quadratic equation.			
find the axis of symmetry and vertex of a parabola.			
find the intercepts of a parabola.			
graph quadratic equations in two variables.			
solve maximum and minimum applications.			

Figure 9.6.145

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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16.5: Transformation of Functions

Learning Objectives

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

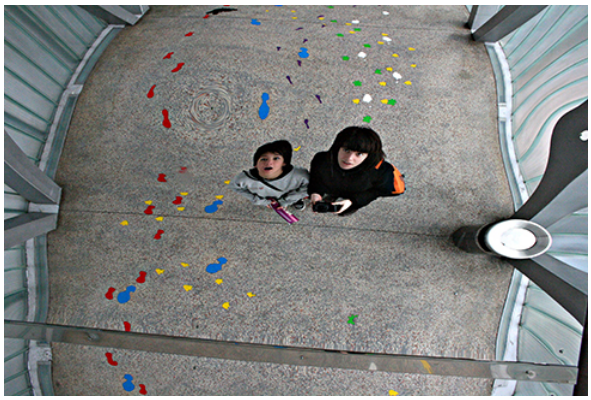

Figure 16.5.1: (credit: "Misko"/Flickr)

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See Figure $16.5.2$ for an example.

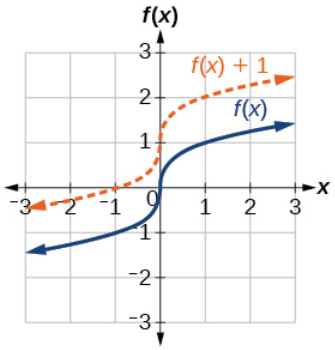

Figure 16.5.2: Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Definition: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example 16.5.1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 16.5.3 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

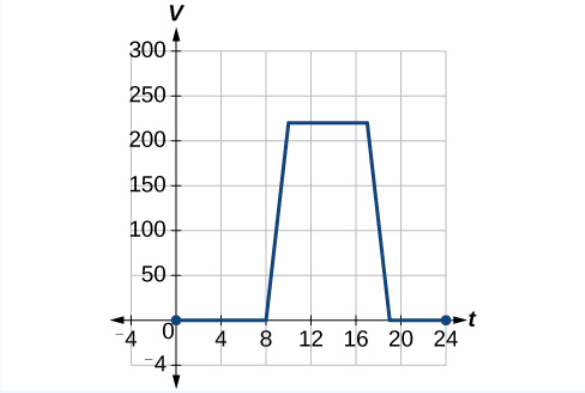

Figure 16.5.3

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 16.5.4

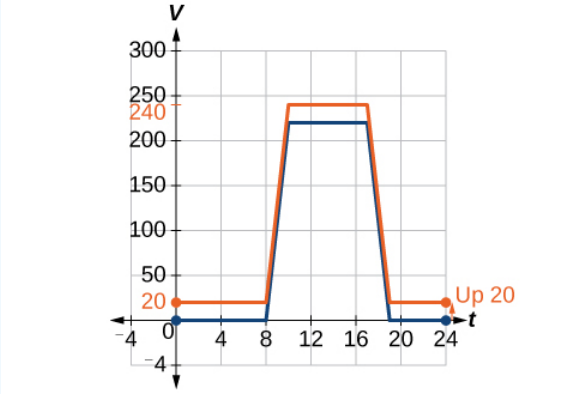

Figure 16.5.4

Notice that in Figure 16.5.4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$,we could write

$$S(t) = V(t) + 20 \tag{16.5.1}$$

This notation tells us that, for any value of t,$S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 16.5.1.

Table 16.5.1

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0

$S(t)$	20	20	240	240	20	20

How To...

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example 16.5.2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 16.5.2 Create a table for the function $g(x) = f(x) - 3$.

Table 16.5.2

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$$\begin{aligned} f(x) &= 1 && \text{Given} \\ g(x) &= f(x) - 3 && \text{Given Transformation} \\ g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 16.5.3

Table 16.5.3

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	-2	0	4	8

Analysis

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Exercise 16.5.1

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Answer

$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function

left or right in what is known as a **horizontal shift**, shown in Figure 16.5.4

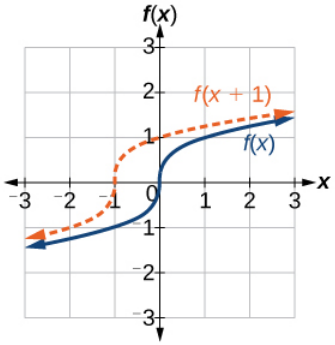

Figure 16.5.4: Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = -1$ shifts the graph to the left, that is, towards negative values of x.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Definition: Horizontal Shift

Given a function f, a new function $g(x) = f(x-h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 16.5.4: Adding a Constant to an Input

Returning to our building airflow example from Figure 16.5.2, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$

$$F(t) = \text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure 16.5.5. Notice also that the vents first opened to 220ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$.

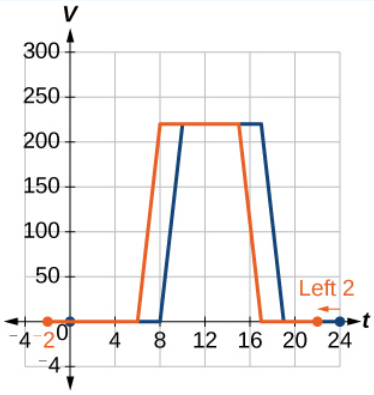

Figure 16.5.5

Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for F : $F(t) = V(t+2)$.

How To...

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Example 16.5.5: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 16.5.4 Create a table for the function $g(x) = f(x-3)$.

Table 16.5.4

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 larger. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 16.5.5

Table 16.5.5

x	5	7	9	11
$x-3$	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis

Figure 16.5.6represents both of the functions. We can see the horizontal shift in each point.

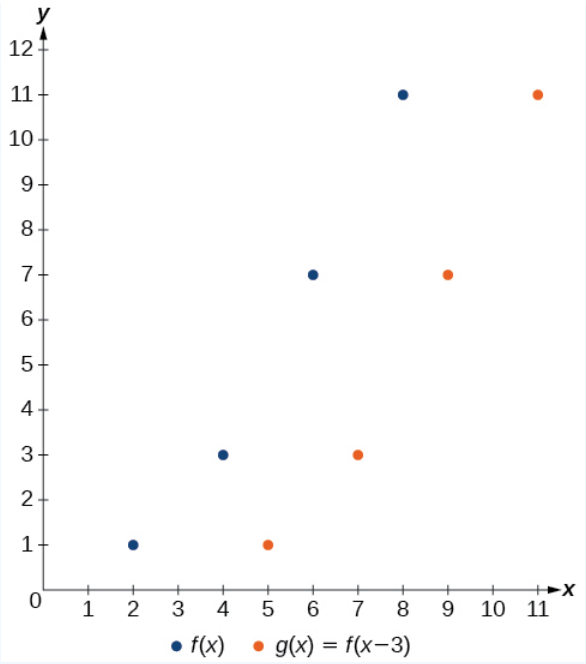

Figure 16.5.6: Graph of the points from the previous table for $f(x)$ and $g(x) = f(x-3)$

Example 16.5.6: Identifying a Horizontal Shift of a Toolkit Function

Figure 16.5.7 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

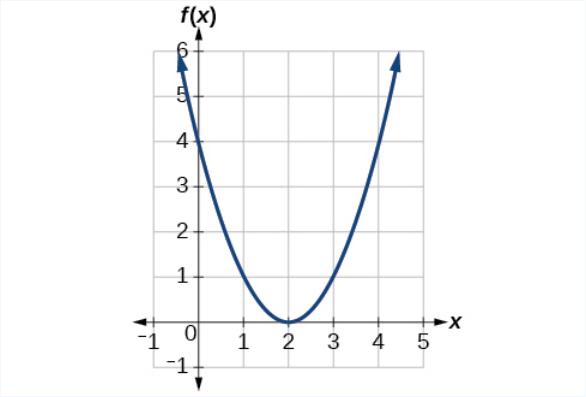

Figure 16.5.7: Graph of a parabola

Solution

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0, 0)$, but now the vertex is at $(2, 0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x-2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x-2) \\ g(x) &= f(x-2) = (x-2)^2 \end{aligned}$$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0)=0$. In our shifted function, $g(2)=0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2)=f(x-2)=f(0)=0$. For this to work, we will need to subtract 2 units from our input values.

Example 16.5.7: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)+10$ and $G(m+10)$

Solution

$G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Exercise 16.5.7

Given the function $f(x)=\sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x)=f(x+2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Answer

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

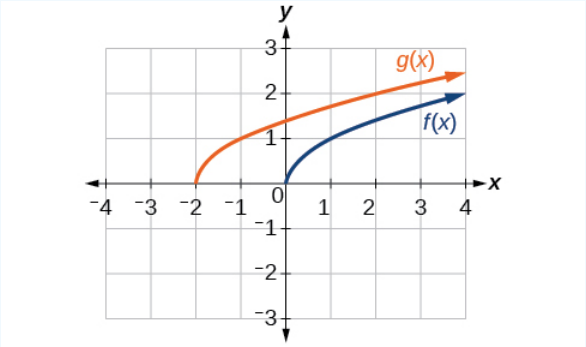

Figure 16.5.8

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output $(y-)$ axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input $(x-)$ axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

How To...

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Example 16.5.8: Graphing Combined Vertical and Horizontal Shifts

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x+1)-3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 16.5.9.

Let us follow one point of the graph of $f(x)=|x|$.

- The point $(0,0)$ is transformed first by shifting left 1 unit:$(0,0)\rightarrow(-1,0)$
- The point $(-1,0)$ is transformed next by shifting down 3 units:$(-1,0)\rightarrow(-1,-3)$

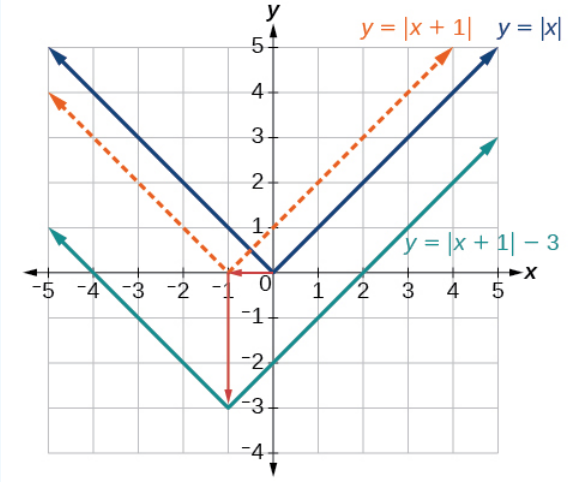

Figure 16.5.9: Graph of an absolute function, $y=|x|$, and how it was transformed to $y=|x+1|-3$

Figure 16.5.10shows the graph of h.

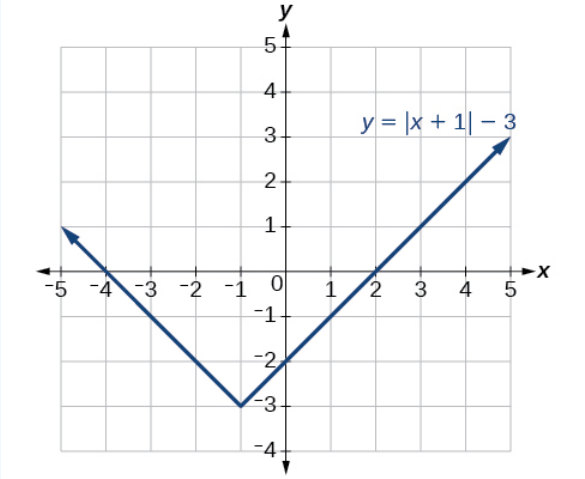

Figure 16.5.10: The final function $y=|x+1|-3$.

Exercise 16.5.8

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x-2)+4$.

Answer

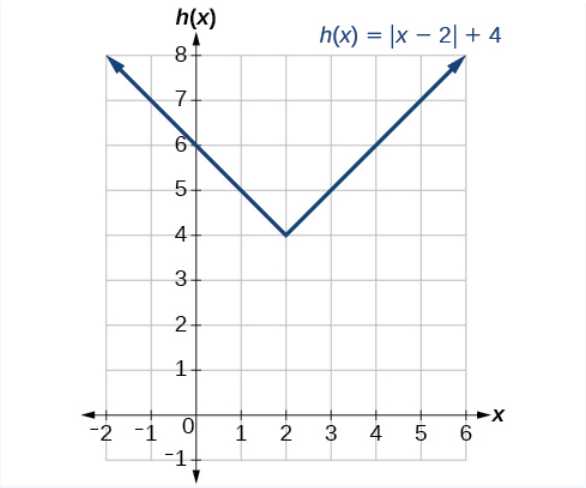

Figure 16.5.11

Example 16.5.9: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 16.5.12 which is a transformation of the toolkit square root function.

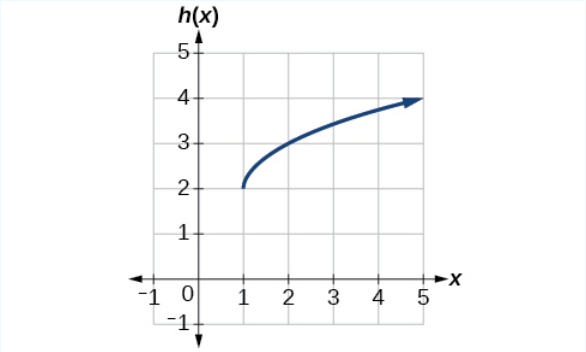

Figure 16.5.12: Graph of a square root function transposed right one unit and up 2.

Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1)+2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1}+2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Exercise 16.5.9

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Answer

$$g(x) = \frac{1}{x-1}+1$$

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 16.5.13

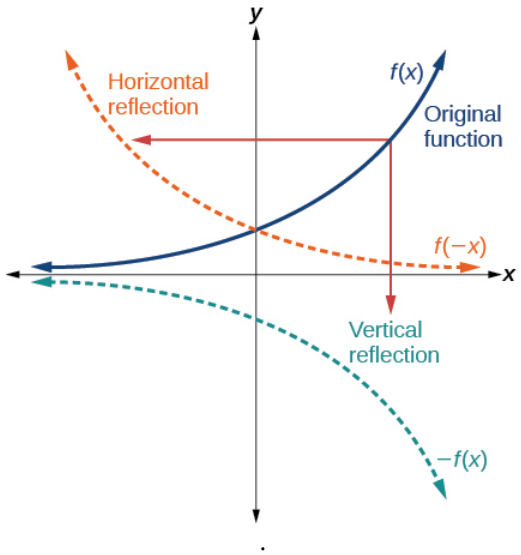

Figure 16.5.13: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

Definitions: Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

How To...

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

Example 16.5.10: Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 16.5.14

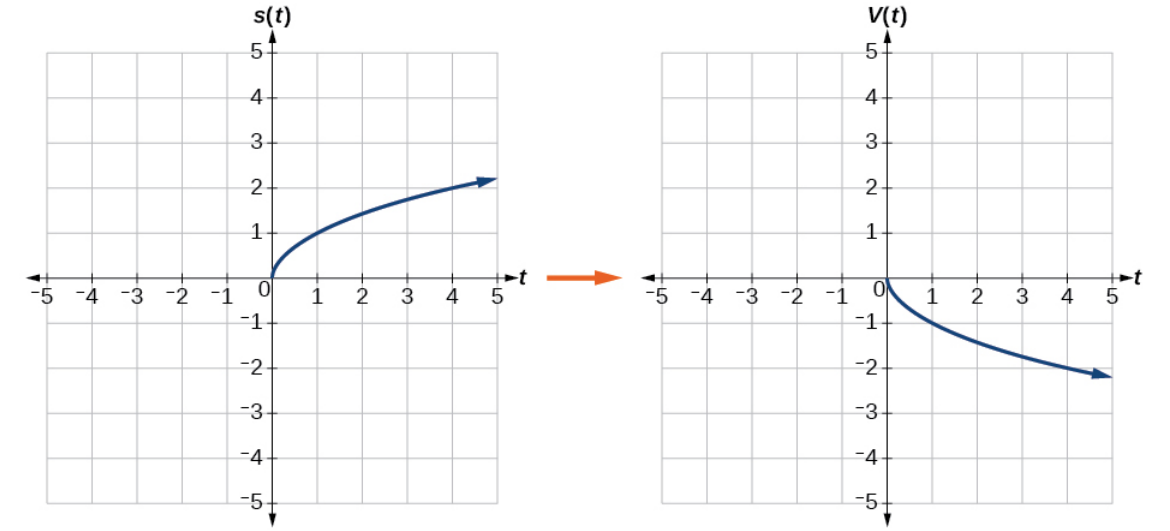

Figure 16.5.14: Graph of the vertical reflection of the square root function.

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 16.5.15

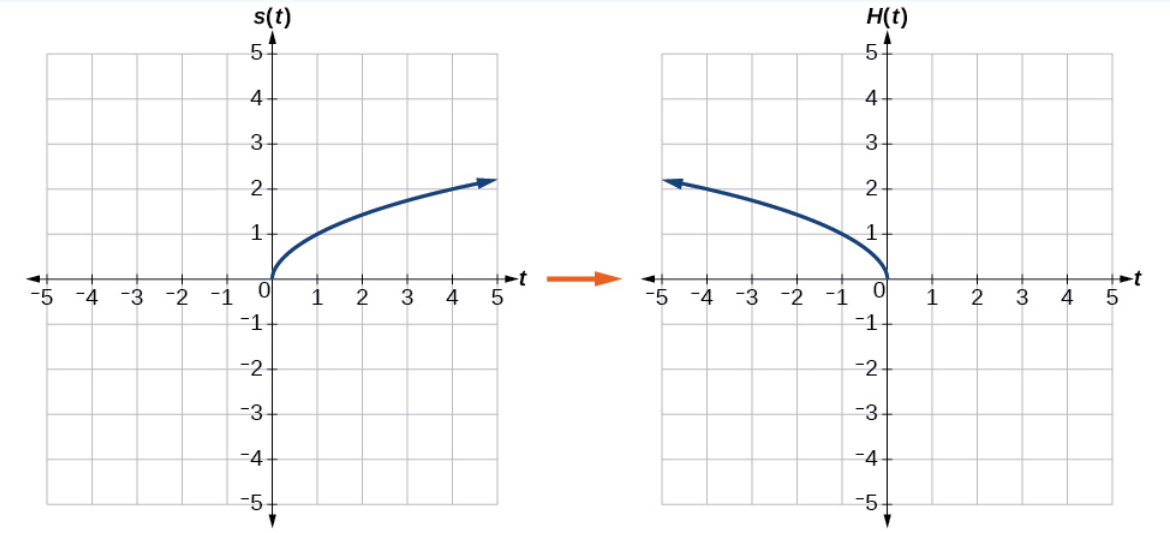

Figure 16.5.15: Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

Exercise 16.5.5

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

Answer

a.

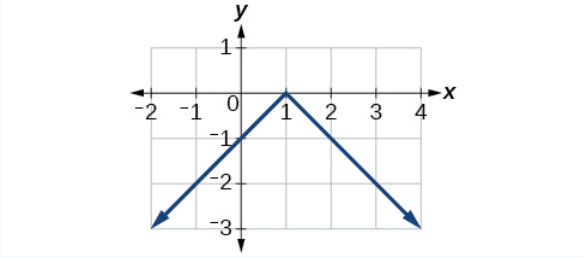

Figure 16.5.16: Graph of a vertically reflected absolute function.

b.

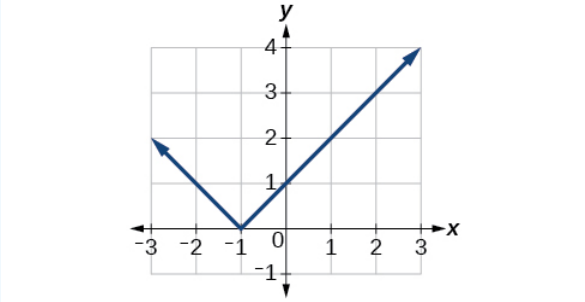

Figure 16.5.17: Graph of an absolute function translated one unit left.

Example 16.5.11: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 16.5.6. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 16.5.6

x	2	4	6	8
$f(x)$	1	3	7	11

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 16.5.7.

Table 16.5.7

x	2	4	6	8
$g(x)$	-1	-3	-7	-11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 16.5.8

Table 16.5.8

x	-2	-4	-6	-8
$h(x)$	1	3	7	11

Exercise 16.5.6

A function $f(x)$ is given as Table 16.5.9 Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 16.5.9

x	-2	0	2	4
$f(x)$	5	10	15	20

Answer

a. $g(x) = -f(x)$

Table 16.5.10

x	-2	0	2	4
$g(x)$	-5	-10	-15	-20

b. $h(x) = f(-x)$

Table 16.5.11

x	-2	0	2	-4
$h(x)$	15	10	5	20

Example 16.5.12: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure 16.5.18 Sketch a graph of $k(t)$.

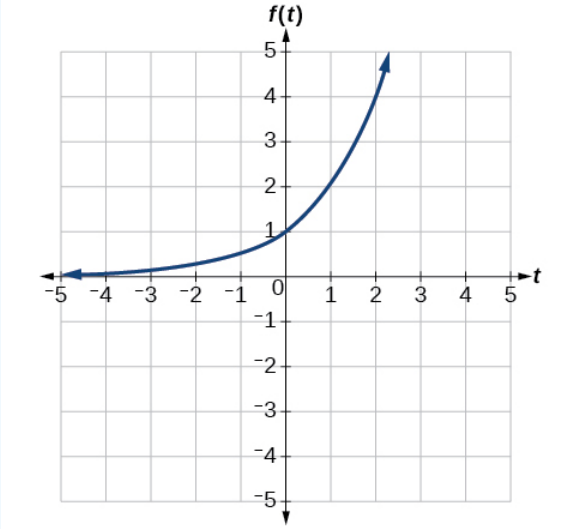

Figure 16.5.18: Graph of $k(t)$

Solution

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$

- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t)+1 = -2^{-t}+1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0, 1)$ and $(1, 2)$.

- First, we apply a horizontal reflection: $(0, 1)$ $(-1, 2)$.
- Then, we apply a vertical reflection: $(0, -1)$ $(-1, -2)$.
- Finally, we apply a vertical shift: $(0, 0)$ $(-1, -1)$.

This means that the original points, $(0, 1)$ and $(1, 2)$ become $(0, 0)$ and $(-1, -1)$ after we apply the transformations.

In Figure 16.5.19, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

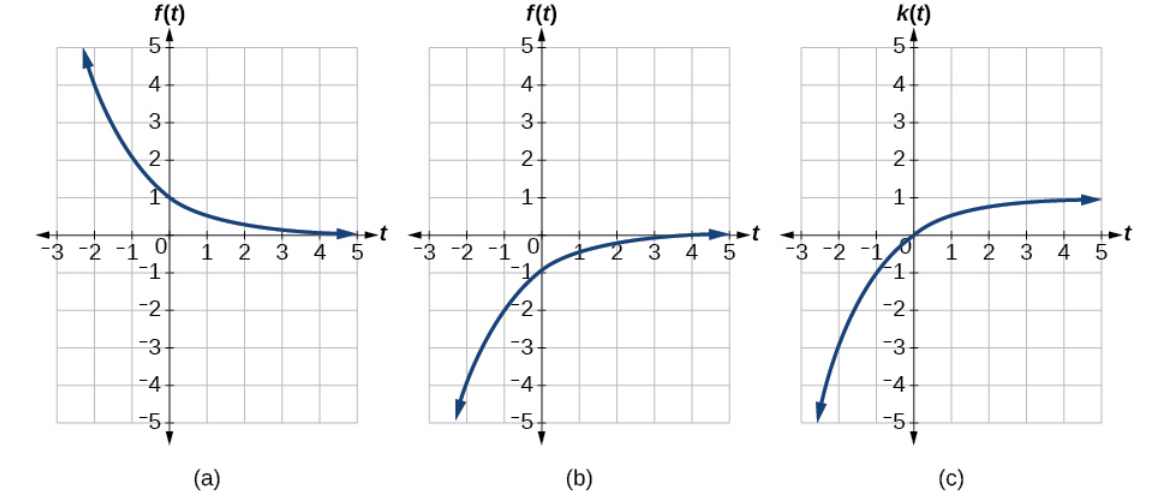

Figure 16.5.19: Graphs of all the transformations.

Analysis

As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

Exercise 16.5.7

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Answer

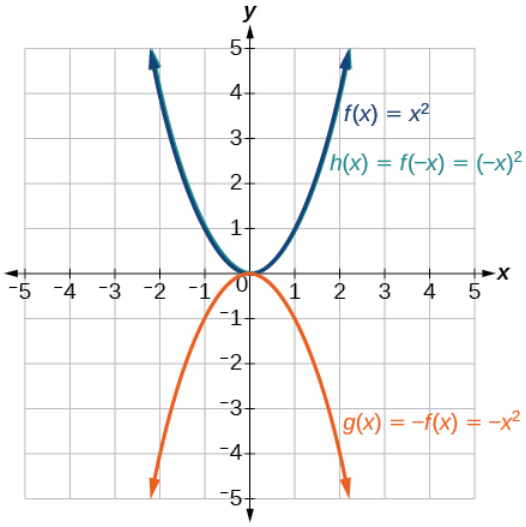

Figure 16.5.20: Graph of x^2 and its reflections.

Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph, as shown in Figure 16.5.21

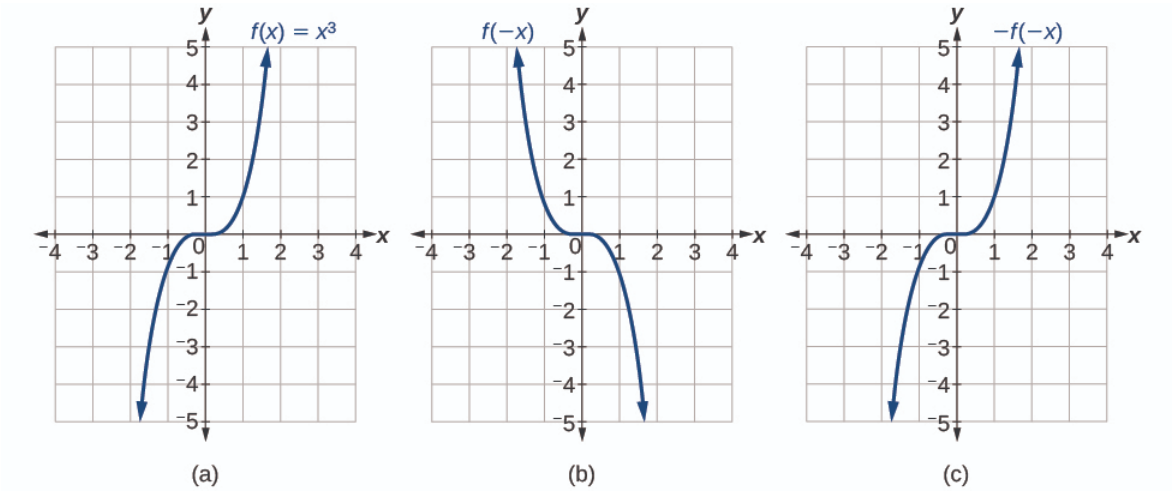

Figure 16.5.21: (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Definitions: Even and Odd Functions

A function is called an **even function** if for every input x

$f(x) = f(-x)$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input x

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin.

How To...

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Example 16.5.13: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis

Consider the graph of f in Figure 16.5.22 Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

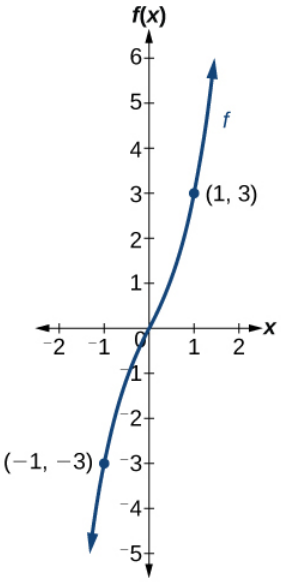

Figure 16.5.22: Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.

Exercise 16.5.8

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Answer

even

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 16.5.23shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

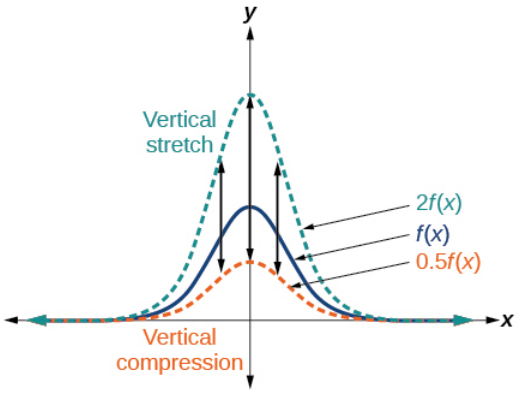

Figure 16.5.23: Vertical stretch and compression

Definitions: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To...

Given a function, graph its vertical stretch.

a. Identify the value of a.
b. Multiply all range values by a
c. If $a > 1$, the graph is stretched by a factor of a.
d. If $0 < a < 1$, the graph is compressed by a factor of a.
e. If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

Example 1.5.14: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure $16.5.24$

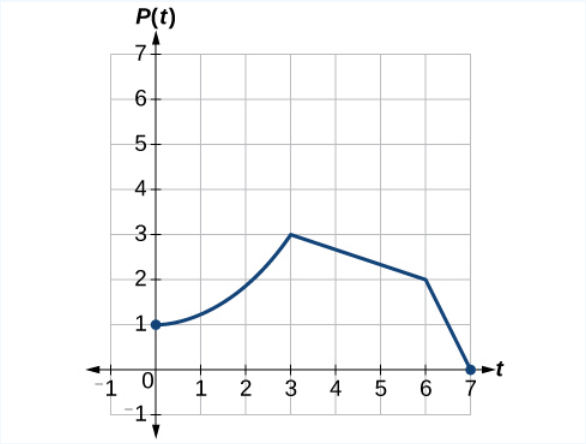

Figure 16.5.24: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure $16.5.25$

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2) \tag{16.5.2}$$
$$(3, 3) \to (3, 6) \tag{16.5.3}$$
$$(6, 2) \to (6, 4) \tag{16.5.4}$$
$$(7, 0) \to (7, 0) \tag{16.5.5}$$

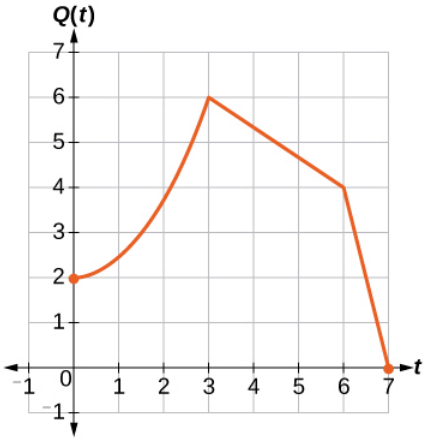

Figure 16.5.25: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How To...

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 16.5.15: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 16.5.12 Create a table for the function $g(x) = \frac{1}{2}f(x)$.

Table 16.5.12

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 16.5.13

Table 16.5.13

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Exercise 16.5.9

A function f is given as Table 16.5.14 Create a table for the function $g(x) = \frac{3}{4}f(x)$.

Table 16.5.14

x	2	4	6	8
$f(x)$	12	16	20	0

Answer

Table 16.5.15

x	2	4	6	8
$g(x)$	9	12	15	0

Example 16.5.16: Recognizing a Vertical Stretch

The graph in Figure 16.5.26is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

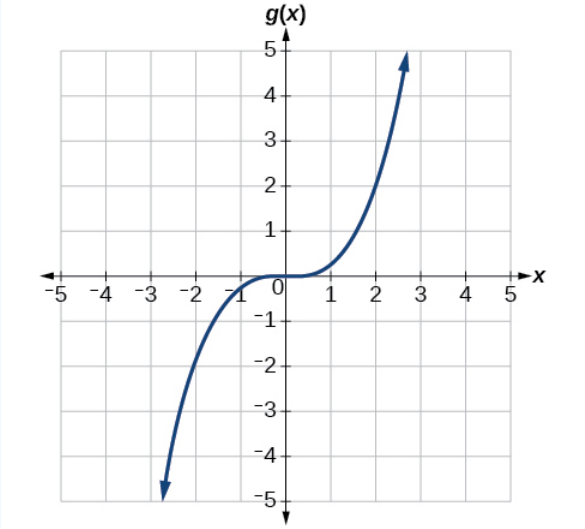

Figure 16.5.26: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3. \tag{16.5.6}$$

Exercise 16.5.1

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Answer

$g(x) = 3x - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

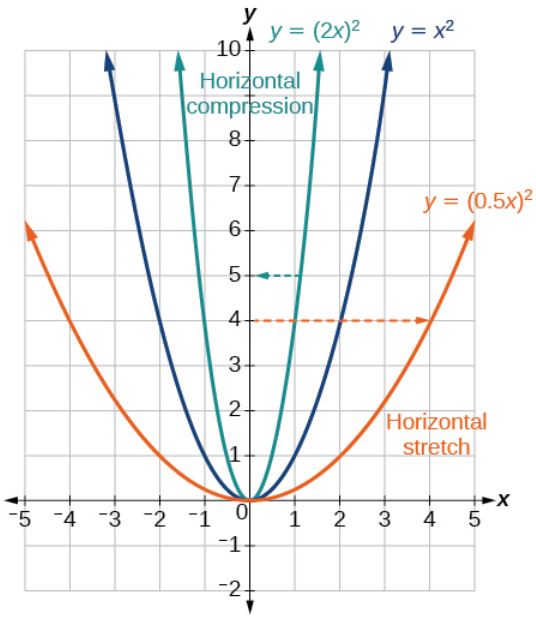

Figure 16.5.27: Graph of the vertical stretch and compression of x^2.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 16.5.27. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

Definitions: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How To...

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example 16.5.17: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$R(1) = P(2), \tag{16.5.7}$$
$$R(2) = P(4), \quad \text{and in general,} \tag{16.5.8}$$
$$R(t) = P(2t). \tag{16.5.9}$$

See Figure 16.5.28for a graphical comparison of the original population and the compressed population.

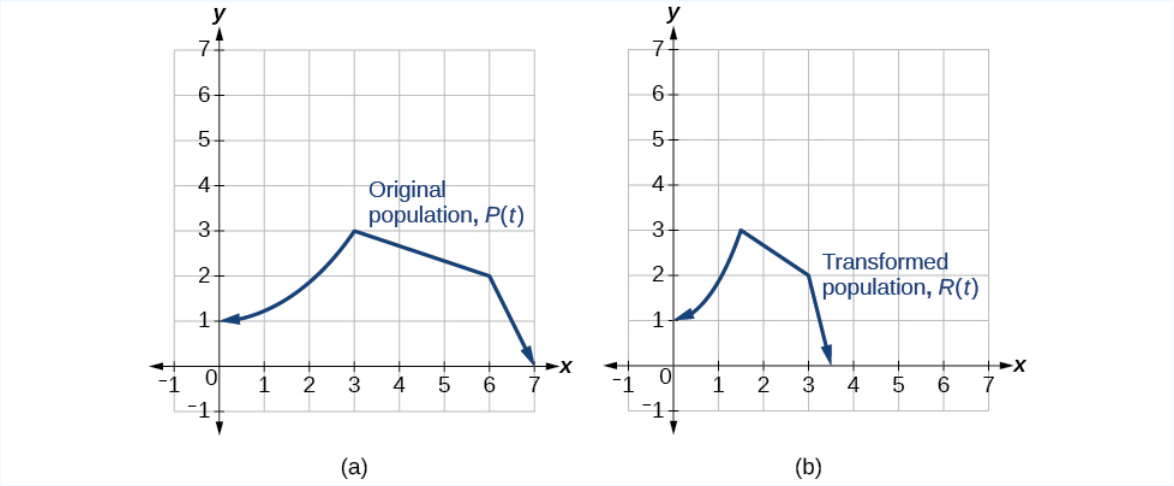

Figure 16.5.28: (a) Original population graph (b) Compressed population graph

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 16.5.18: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 16.5.16 Create a table for the function $g(x) = f(\frac{1}{2}x)$.

Table 16.5.16

x	2	4	6	8
$f(x)$	1	3	7	11

The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1 \tag{16.5.10}$$

We do the same for the other values to produce Table 16.5.17.

Table 16.5.17

x	4	8	12	16

$g(x)$	1	3	7	11

Figure 16.5.29shows the graphs of both of these sets of points.

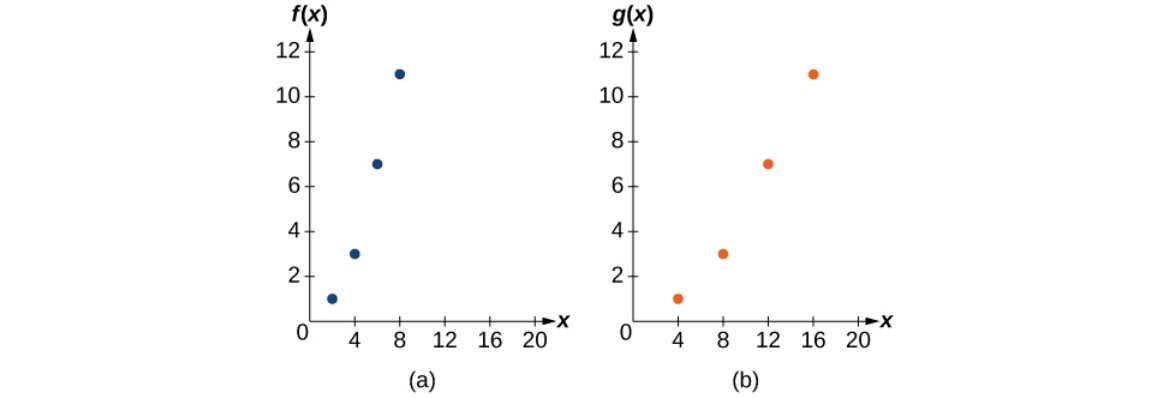

Figure 16.5.29: Graph of the previous table.

Analysis

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example 16.5.19: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 16.5.30

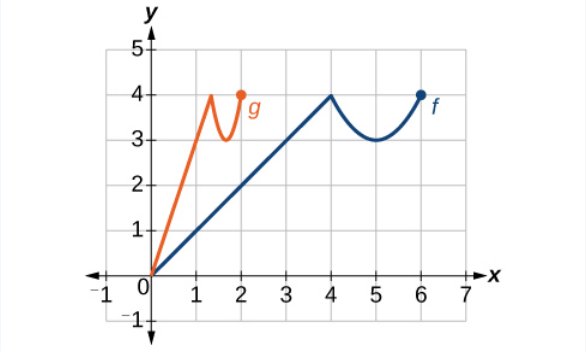

Figure 16.5.30: Graph of $f(x)$ being vertically compressed to $g(x)$.

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Exercise 16.5.11

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Answer

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x+3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7) = 12$. What input to g would produce that output? In other words, what value of x will allow $g(x) = f(2x+3) = 12$? We would need $2x+3 = 7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p) = f(b(x+\frac{p}{b}))$$

Let's work through an example.

$$f(x) = (2x+4)^2$$

We can factor out a 2.

$$f(x) = (2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations

- When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.
- When combining horizontal transformations written in the form $f(bx+h)$, first horizontally shift by h and then horizontally stretch by $\frac{1}{b}$.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 16.5.20: Finding a Triple Transformation of a Tabular Function

Given Table 16.5.18for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x)+1$.

Table 16.5.18

x	6	12	18	24
$f(x)$	10	14	15	17

Solution

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$.See Table 16.5.19

Table 16.5.19

x	2	4	6	8
$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 16.5.20

Table 16.5.20

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 16.5.21

Table 16.5.21

x	2	4	6	8
$g(x)=2f(3x)+1+1$	21	29	31	35

Example 16.5.21: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 16.5.31to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$.

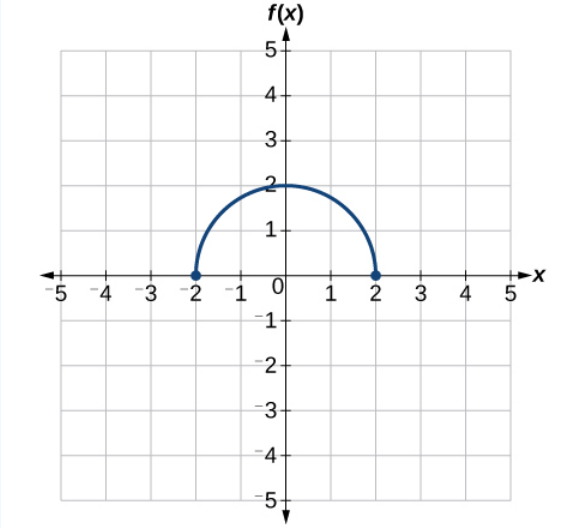

Figure 16.5.31: Graph of a half-circle.

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3 \tag{16.5.11}$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure 16.5.32

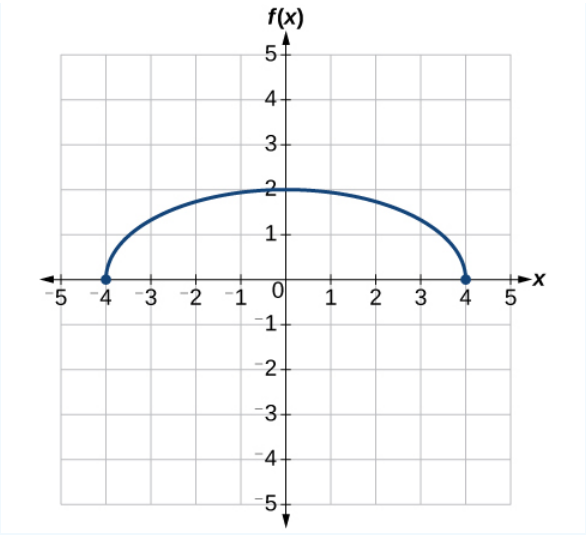

Figure 16.5.32: Graph of a vertically stretch half-circle.

Next, we horizontally shift left by 2 units, as indicated by $x+2$. See Figure 16.5.33

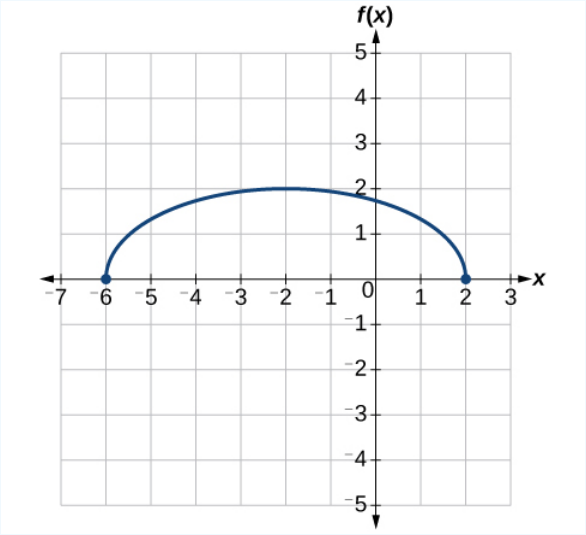

Figure 16.5.33: Graph of a vertically stretch and translated half-circle.

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See Figure 16.5.34

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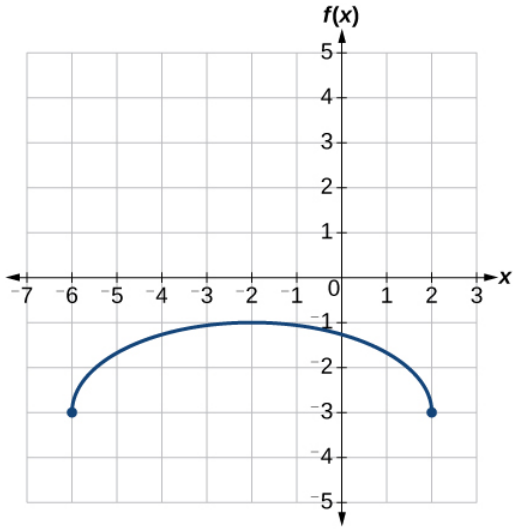

Figure 16.5.34: Graph of a vertically stretch and translated half-circle.

Key Equations

- Vertical shift $g(x) = f(x) + k$ (up for $k > 0$)
- Horizontal shift $g(x) = f(x - h)$ (right) for $h > 0$
- Vertical reflection $g(x) = -f(x)$
- Horizontal reflection $g(x) = f(-x)$
- Vertical stretch $g(x) = af(x)$ (a>0)
- Vertical compression $g(x) = af(x)$ (0<a<1)
- Horizontal stretch $g(x) = f(bx)(0 < b < 1)$
- Horizontal compression $g(x) = f(bx)$ (b>1)

Key Concepts

- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts.
- Vertical and horizontal shifts are often combined.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither.
- A function can be compressed or stretched vertically by multiplying the output by a constant.
- A function can be compressed or stretched horizontally by multiplying the input by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

Glossary

even function

a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis

horizontal compression
a transformation that compresses a function's graph horizontally, by multiplying the input by a constant b>1

horizontal reflection
a transformation that reflects a function's graph across the y-axis by multiplying the input by −1

horizontal shift
a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input

horizontal stretch
a transformation that stretches a function's graph horizontally by multiplying the input by a constant 0<b<1

odd function
a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$, and is symmetric about the origin

vertical compression
a function transformation that compresses the function's graph vertically by multiplying the output by a constant 0<a<1

vertical reflection
a transformation that reflects a function's graph across the x-axis by multiplying the output by −1

vertical shift
a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output

vertical stretch
a transformation that stretches a function's graph vertically by multiplying the output by a constant a>1

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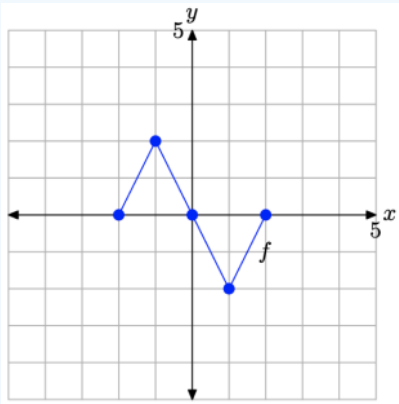
16.5.1: Vertical Transformations

In this section we study the art of transformations: scalings, reflections, and translations. We will restrict our attention to transformations in the vertical or y-direction. Our goal is to apply certain transformations to the equation of a function, then ask what effect it has on the graph of the function.

We begin our task with an example that requires that we read the graph of a function to capture several key points that lie on the graph of the function.

Example 16.5.1.1

Consider the graph of f presented in Figure 16.5.1.1(a). Use the graph of f to complete the table in Figure 16.5.1.1(b).



(a) The graph of f.

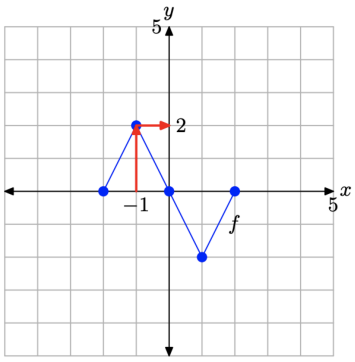
x	$f(x)$	$(x, f(x))$
-2		
-1		
0		
1		
2		

(b) The table.

Figure 16.5.1.1 Reading key values from the graph of f.

Solution

To compute f(−1), we would locate −1 on the x-axis, draw a vertical arrow to the graph of f, then a horizontal arrow to the y-axis, as shown in Figure 16.5.1.2(a). The y-value of this final destination is the value of f(−1). That is, f(−1) = 2. This allows us to complete one entry in the table, as shown in Figure 16.5.1.2(b). Continue in this manner to complete all of the entries in the table. The result is shown in Figure 16.5.1.2(c).



(a) The graph of f.

x	$f(x)$	$(x, f(x))$
-2		
-1	2	$(-1, 2)$
0		
1		
2		

(b) Recording $f(-1) = 2$.

x	$f(x)$	$(x, f(x))$
-2	0	$(-2, 0)$
-1	2	$(-1, 2)$
0	0	$(0, 0)$
1	-2	$(1, -2)$
2	0	$(2, 0)$

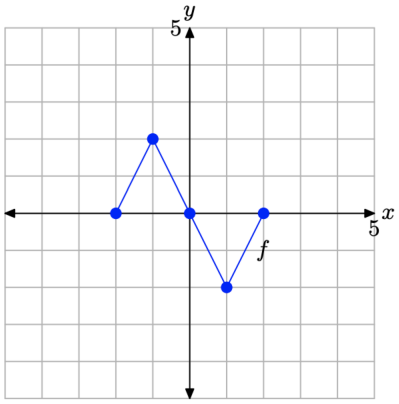
(c) Completed table.

Figure 16.5.1.2 Recording coordinates of points on the graph of f in the tables.

Vertical Scaling

In the narrative that follows, we will have repeated need of the graph in Figure 16.5.1.2(a) and the table in Figure 16.5.1.2(c). They characterize the basic function that will be the starting point for the concepts of scaling, reflection, and translation that we

develop in this section. Consequently, let's place them side-by-side for emphasis in Figure 16.5.1.3



(a)

x	$f(x)$	$(x, f(x))$
-2	0	$(-2, 0)$
-1	2	$(-1, 2)$
0	0	$(0, 0)$
1	-2	$(1, -2)$
2	0	$(2, 0)$

(b)

Figure 16.5.1.3 The original graph of f and a table of key points on the graph of f.

We are now going to scale the graph of f in the vertical direction.

Example 16.5.1.2

If y = f(x) has the graph shown in Figure 16.5.1.3(a), sketch the graph of y = 2f(x).

Solution

What do we do when we meet a graph whose shape we are unsure of? The answer to this question is we plot some points that satisfy the equation in order to get an idea of the shape of the graph. With that thought in mind, let's evaluate the function y = 2f(x) at x = −2.

The letter f refers to the original function shown in Figure 16.5.1.3(a) and the table in Figure 16.5.1.3(b) contains the values of that function at the given values of x. Thus, in computing y = 2f(−2), the first step is to look up the value of f(−2) in the table in Figure 16.5.1.3(b). There we find that f(−2) = 0. Thus, we can write

$$y = 2f(-2) = 2(0) = 0$$

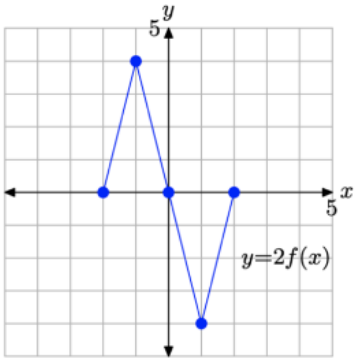
In similar fashion, let's evaluate the function y = 2f(x) at x = −1. First, look up the value of f(−1) in the table in Figure 16.5.1.3(b). There we find that f(−1) = 2. Thus, we can write

$$y = 2f(-1) = 2(2) = 4$$

We finish by evaluating the function y = 2f(x) at x = 0, 1, and 2. Each time you need to evaluate the function f at a number, take the result from the table or graph in Figure 3. What follows are the evaluations of y = 2f(x) at x = −2, −1, 0, 1, and 2.

$$y = 2f(-2) = 2(0) = 0$$
$$y = 2f(-1) = 2(2) = 4$$
$$y = 2f(0) = 2(0) = 0$$
$$y = 2f(1) = 2(-2) = -4$$
$$y = 2f(2) = 2(0) = 0$$

We can arrange these results in a table shown in Figure 16.5.1.4(b), then plot them in the figure shown in Figure 16.5.1.4(a).



(a)

x	$y = 2f(x)$	$(x, 2f(x))$
-2	0	$(-2, 0)$
-1	4	$(-1, 4)$
0	0	$(0, 0)$
1	-4	$(1, -4)$
2	0	$(2, 0)$

(b)

Figure 16.5.1.4 The points in the table are points on the graph of y = 2f(x).

At this point, there are a number of comparisons you can make.

1. Compare the data in the tables in Figure 16.5.1.3(b) and Figure 16.5.1.4(b). Note that the x-values are identical. In both tables, x = −2, −1, 0, 1, and 2. However, note that each y-value in the table in Figure 16.5.1.4(b) is precisely double the corresponding y-value in the table in Figure 16.5.1.3(b).
2. Compare the graphs in Figure 16.5.1.3(a) and Figure 16.5.1.4(a). Note that the y-value of each point in the graph of y = 2f(x) in Figure 16.5.1.4(a) is precisely double the y-value of the corresponding point in Figure 16.5.1.3(a).

Note the result. The graph of y = 2f(x) has been stretched vertically (away from the x-axis) , both positively and negatively, by a factor of 2.

Let's look at another example.

Example 16.5.1.3

If y = f(x) has the graph shown in Figure 16.5.1.3(a), sketch the graph of y = (1/2)f(x).

Solution

Let's begin by evaluating the function y = (1/2)f(x) at x = −2. First, look up the value of f(−2) in the table in Figure 16.5.1.3(b). There we find that f(−2) = 0. Thus, we can write

$$y = (1/2)f(-2) = (1/2)(0) = 0$$

In similar fashion, let's evaluate the function y = (1/2)f(x) at x = −1. First, look up the value of f(−1) in the table in Figure 16.5.1.3(b). There we find that f(−1) = 2. Thus, we can write

$$y = (1/2)f(-1) = (1/2)(2) = 1$$

Continuing in this manner, we can evaluate the function y = (1/2)f(x) at x = 0, 1, and 2.

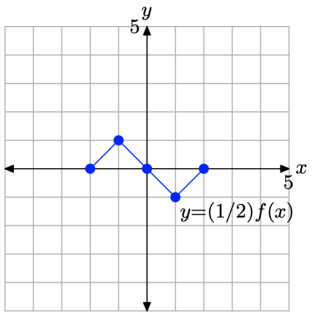
$$y = (1/2)f(0) = (1/2)(0) = 0$$
$$y = (1/2)f(1) = (1/2)(-2) = -1$$
$$y = (1/2)f(2) = (1/2)(0) = 0$$

The results are recorded in the table in Figure 16.5.1.5(b). Rather than double each value of y as did the function y = 2f(x) in Example 16.5.1.2, this function y = (1/2)f(x) halves each value of y. The graph of y = (1/2)f(x) and a table of key points on the graph are presented in Figures 16.5.1.5(a) and (b), respectively.

Again, there are a number of comparisons.

1. Compare the data in the tables in Figure 16.5.1.5(b) and Figure 16.5.1.3(b). Note that the x-values are identical. In both tables x = −2, −1, 0, 1, and 2. However, note that each y-value in the table in Figure 16.5.1.5(b) is precisely half the corresponding y-value in the table in Figure 16.5.1.3(b).

2. When you compare the graph of y = (1/2)f(x) in Figure 16.5.1.5(a) with the original graph of y = f(x) in Figure 16.5.1.3(a), note that each point on the graph of y = (1/2)f(x) has a y-value that is precisely half of the corresponding y-value on the original graph of y = f(x) in Figure 16.5.1.3(a).



(a)

x	$y=(1/2)f(x)$	$(x,(1/2)f(x))$
-2	0	$(-2,0)$
-1	1	$(-1,1)$
0	0	$(0,0)$
1	-1	$(1,-1)$
2	0	$(2,0)$

(b)

Figure 16.5.1.5 The points in the table are points on the graph of y = (1/2)f(x).

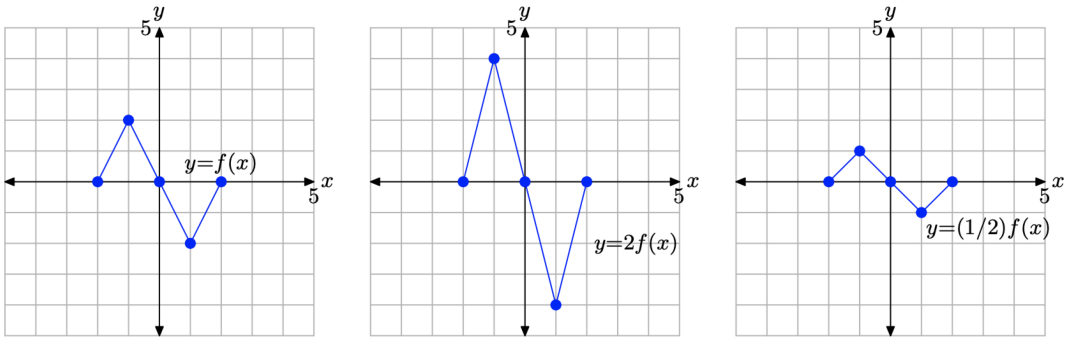
Note the result. The graph of f has been compressed vertically (toward the x-axis), both positively and negatively, by a factor of 2.

Let's summarize our findings.

A Visual Summary — Vertical Scaling.

Consider the images in Figure 16.5.1.6

- In Figure 16.5.1.6(a), we see pictured the graph of the original function y = f(x).
- In Figure 16.5.1.6(b), note that each key point on the graph of y = 2f(x) has a y-value that is precisely double the y-value of the corresponding point on the graph of y = f(x) in Figure 16.5.1.6(a).
- In Figure 16.5.1.6(c), note that each key point on the graph of y = (1/2)f(x) has a y-value that is precisely half the y-value of the corresponding point on the graph of y = f(x) in Figure 16.5.1.6(a).
- Note that the x-value of each transformed point remains the same.



(a) $y = f(x)$ (b) $y = 2f(x)$ (c) $y = (1/2)f(x)$

Figure 16.5.1.6 The graph of y = 2f(x) stretches vertically (away from the x-axis) by a factor of 2. The graph of y = (1/2)f(x) compresses vertically (toward the x-axis) by a factor of 2.

The visual summary in Figure 16.5.1.6makes sketching the graphs of y = 2f(x) and y = (1/2)f(x) an easy task.

- Given the graph of y = f(x), to sketch the graph of y = 2f(x), simply take each point on the graph of y = f(x) and double its y-value, keeping the same x-value.
- Given the graph of y = f(x), to sketch the graph of y = (1/2)f(x), simply take each point on the graph of y = f(x) and halve its y-value, keeping the same x-value.

Follow the same procedures for other scaling factors. For example, in the case of y = 3f(x), take each point on the graph of y = f(x) and multiply its y-value by 3, keeping the same x-value. On the other hand, to draw the graph of y = (1/3)f(x), take each point on the graph of f and multiply its y-value by 1/3, keeping the same y-value.

In general, we can state the following.

summary

Suppose we are given the graph of y = f(x).

- If a > 1, then the graph of y = af(x) is stretched vertically (away from the x-axis), both positively and negatively, by a factor of a.
- If 0 < a < 1, then the graph of y = af(x) is compressed vertically (toward the x-axis), both positively and negatively, by a factor of 1/a.

The second item in Summary warrants a word of explanation. Compare the general form y = af(x) with the function of Example 16.5.1.3, y = (1/2)f(x). In this case, a = 1/2, so

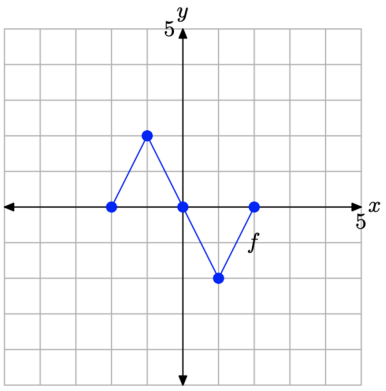
$$\frac{1}{a} = \frac{1}{1/2} = 1 \times 2 = 2$$

The second item says that when 0 < a < 1, the graph of y = af(x) is compressed vertically by a factor of 1/a. Indeed, this is exactly what happens in the case of y = (1/2)f(x), which is compressed by a factor of 1/(1/2), or 2.

Vertical Reflections

For convenience, we begin by repeating the original graph of y = f(x) and its accompanying data.

We are now going to reflect the graph in the vertical direction (across the x-axis).



(a)

x	$f(x)$	$(x, f(x))$
-2	0	$(-2, 0)$
-1	2	$(-1, 2)$
0	0	$(0, 0)$
1	-2	$(1, -2)$
2	0	$(2, 0)$

(b)

Figure 16.5.1.7. The original graph of f and a table of key points on the graph of f.

Example 16.5.1.4

If y = f(x) has the graph shown in Figure 16.5.1.7(a), sketch the graph of y = −f(x).

Solution

To set up a table of points in preparation for the plot of y = −f(x), we'll use exactly the same values of x that you see in the table in Figure 16.5.1.7(b), namely x = −2, −1, 0, 1, and 2. To evaluate y = −f(x) at the first value of x, namely x = −2, we make the following calculation,

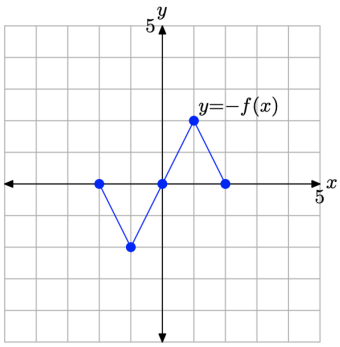
$$y = -f(-2) = -(0) = 0$$

where we've used the fact that f(−2) = 0 from the table in Figure 16.5.1.7(b). In similar fashion, we evaluate y = −f(x) at each of the remaining values of x, namely x = −1, 0, 1, and 2.

$$y = -f(-1) = -(2) = -2$$
$$y = -f(0) = -(0) = 0$$
$$y = -f(1) = -(-2) = 2$$
$$y = -f(2) = -(0) = 0$$

We assemble these points in the table in Figure 16.5.1.8(b) and plot them in Figure 16.5.1.8(a).

Note that the graph of y = −f(x) in Figure 16.5.1.8(a) is a reflection of the graph of y = f(x) in Figure 16.5.1.7(a) across the x-axis.



(a)

x	$y = -f(x)$	$(x, -f(x))$
-2	0	$(-2, 0)$
-1	-2	$(-1, -2)$
0	0	$(0, 0)$
1	2	$(1, 2)$
2	0	$(2, 0)$

(b)

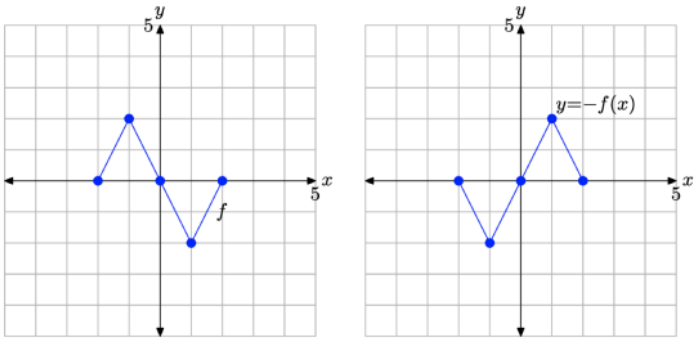
Figure 16.5.1.8 The graph of y = −f(x) and a table of key points on the graph.

Let's summarize what we've learned about vertical reflections.

A Visual Summary — Vertical Reflections

Consider the images in Figure 16.5.1.9

- In Figure 16.5.1.9(a), we see pictured the original graph of y = f(x).
- In Figure 16.5.1.9(b), the graph of y = −f(x) is a reflection of the graph of y = f(x) across the x-axis.



(a) $y = f(x)$ (b) $y = -f(x)$

Figure 16.5.1.9 The graph of y = −f(x) is a reflection of the graph of y = f(x) across the x-axis.

Thus, given the graph of y = f(x), it is a simple task to draw the graph of y = −f(x).

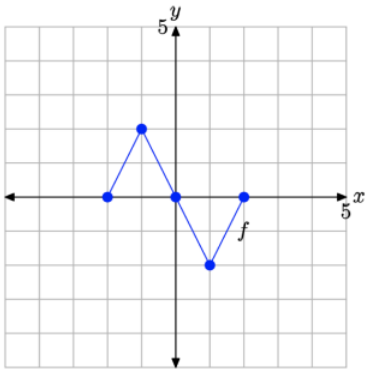
- To draw the graph of y = −f(x), take each point on the graph of y = f(x) and reflect it across the x-axis, keeping the x-value the same, but negating the y-value.

Vertical Translations

Translations are perhaps the easiest transformation of all. A translation is a "shift" or a "slide." Pretend, for a moment, that you've placed a transparent sheet of thin plastic over a sheet of graph paper. You've drawn a Cartesian coordinate system on your graph paper, but you've plotted your graph on the transparent sheet of plastic. Now, "shift" or "slide" the transparency over your graph

paper in a constant direction without rotating the transparency. This is what we mean by a “translation.” In this section, we will focus strictly on vertical translations.

For convenience, we begin by repeating the original graph of y = f(x) and its accompanying data in Figure 16.5.1.10(a) and(b), respectively. We will now translate this graph in the vertical direction.



(a)

x	$f(x)$	$(x, f(x))$
-2	0	$(-2,0)$
-1	2	$(-1,2)$
0	0	$(0,0)$
1	-2	$(1,-2)$
2	0	$(2,0)$

(b)

Figure 16.5.1.10 The original graph of f and a table of key points on the graph of f.

Example 16.5.1.5

If y = f(x) has the graph shown in Figure 16.5.1.10(a), sketch the graph of y = f(x) + 1.

Solution

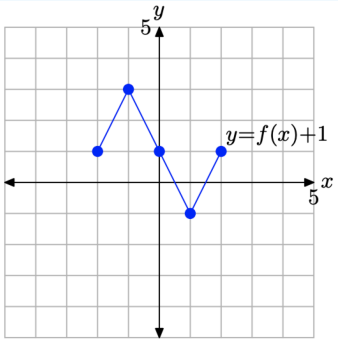
We will evaluate y = f(x)+1 at the same values shown in the table in Figure 16.5.1.10(b), namely x = −2, −1, 0, 1, and 2. To evaluate y = f(x)+1 at the first value of x, namely x = −2, we make the following calculation

$$y = f(-2) + 1 = 0 + 1 = 1$$

where we’ve used that fact that f(−2) = 0 from the table in Figure 16.5.1.10(b). In similar fashion, we can evaluate y = f(x) + 1 at each of the remaining values of x, namely x = −1, 0, 1, and 2.

$$y = f(-1) + 1 = 2 + 1 = 3$$
$$y = f(0) + 1 = 0 + 1 = 1$$
$$y = f(1) + 1 = -2 + 1 = -1$$
$$y = f(2) + 1 = 0 + 1 = 1$$

We assemble these points in the table in Figure 16.5.1.11(b) and plot them in Figure 16.5.1.11(a).



(a)

x	$y = f(x) + 1$	$(x, f(x) + 1)$
-2	1	$(-2,1)$
-1	3	$(-1,3)$
0	1	$(0,1)$
1	-1	$(1,-1)$
2	1	$(2,1)$

(b)

Figure 16.5.1.11 The graph of y = f(x) + 1 and a table of key points on the graph.

When you compare the entries in the table in Figure 16.5.1.11(b) with the original values in the table in Figure 16.5.1.10(b), you’ll note that the x-values in each table are identical, but the y-values in the table in Figure 16.5.1.11(b) are all increased by

1. This makes sense, because these are the y-values of the points associated with the function y = f(x) + 1. Of course, all the y-values should be 1 larger than the y-values associated with the original equation y = f(x).

Note the result. The graph of y = f(x) + 1 in Figure 16.5.1.11(a), when compared with the graph of y = f(x) in Figure 16.5.1.10(a), is shifted 1 unit upwards.

Let's look at another example.

Example 16.5.1.6

If y = f(x) has the graph shown in Figure 16.5.1.10(a), sketch the graph of y = f(x) − 2.

Solution

Evaluate the function y = f(x) − 2 at each value of x in the table in Figure 16.5.1.10(b). At x = −2,

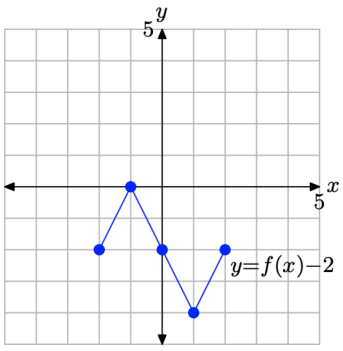
$$y = f(-2) - 2 = 0 - 2 = -2$$

In similar fashion, evaluate y = f(x) − 2 at each remaining x-value in the table in Figure 16.5.1.10(b).

$$\begin{aligned} y &= f(-1) - 2 = 2 - 2 = 0 \\ y &= f(0) - 2 = 0 - 2 = -2 \\ y &= f(1) - 2 = -2 - 2 = -4 \\ y &= f(2) - 2 = 0 - 2 = -2 \end{aligned}$$

We assemble these points in the table in Figure 16.5.1.12(b) and plot them in Figure 16.5.1.12(a).

When you compare the entries in the table in Figure 16.5.1.12(b) with the original values in the table in Figure 16.5.1.10(b), you'll note that the x-values in each table are identical, but the y-values in the table in Figure 16.5.1.12(b) are all decremented by 2. This makes



(a)

x	$y = f(x) - 2$	$(x, f(x) - 2)$
-2	-2	$(-2, -2)$
-1	0	$(-1, 0)$
0	-2	$(0, -2)$
1	-4	$(1, -4)$
2	-2	$(2, -2)$

(b)

Figure 16.5.1.12 The graph of y = f(x) − 2 and a table of key points on the graph.

sense, because these are the y-values of the points associated with the function y = f(x) − 2. Of course, all the y-values should be 2 less than the y-values associated with the original equation y = f(x).

Note the result. The graph of y = f(x) − 2 in Figure 16.5.1.12(a), when compared with the graph of y = f(x) in Figure 16.5.1.10(a), is shifted downward 2 units.

Let's summarize what we've learned about vertical translations.

A Visual Summary — Vertical Translations (Shifts)

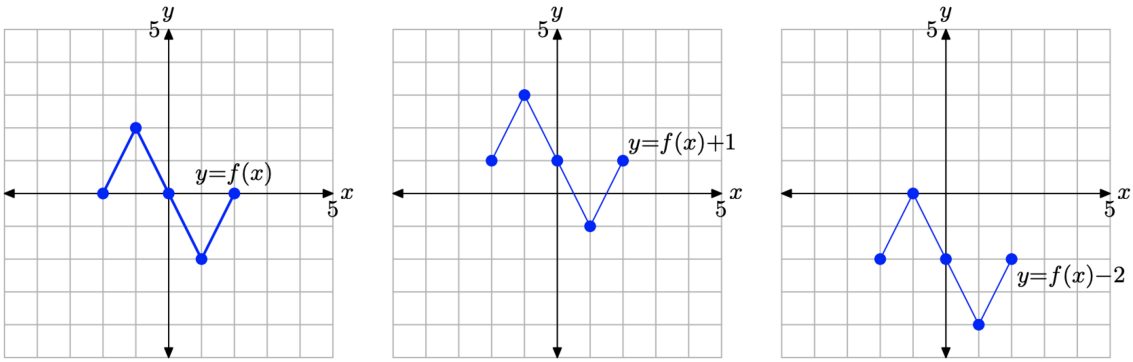
Consider the images in Figure 16.5.1.13

- In Figure 16.5.1.13(a), we see pictured the graph of the original function y = f(x).
- In Figure 16.5.1.13(b), note that each key point on the graph of y = f(x) + 1 has a y-value that is precisely 1 unit larger than the y-value of the corresponding point on the graph of y = f(x) in in Figure 16.5.1.13(a).

- In Figure 16.5.1.13(c), note that each key point on the graph of y = f(x) − 2 has a y-value that is precisely 2 units smaller than the y-value of the corresponding point on the graph of y = f(x) in in Figure 16.5.1.13(a).
- Note that the x-value of each transformed point remains the same.

The visual summary in Figure 16.5.1.13 makes sketching the graphs of y = f(x) + 1 and y = f(x) − 2 an easy task.

- Given the graph of y = f(x), to sketch the graph of y = f(x) + 1, simply take each point on the graph of y = f(x) and move it upwards 1 unit, keeping the same x-value.
- Given the graph of y = f(x), to sketch the graph of y = f(x) − 2, simply take each point on the graph of y = f(x) and move it downwards 2 units, keeping the same x-value.



(a) $y = f(x)$ (b) $y = f(x) + 1$ (c) $y = f(x) - 2$

Figure 16.5.1.13 The graph of y = f(x) + 1 is formed by shifting (vertically) the graph of y = f(x) upward 1 unit. The graph of y = f(x) − 2 is formed by shifting (vertically) the graph of y = f(x) downward 2 units.

In general, we can state the following.

summary

Suppose that we are given the graph of y = f(x) and suppose that c is any positive real number.

- The graph of y = f(x)+c is shifted c units upward from the graph of y = f(x).
- The graph of y = f(x) − c is shifted c units downward from the graph of y = f(x).

Composing Transformations

Sometimes we will want to perform one transformation, then take the result of the first transformation and apply a second transformation. Let's look at an example.

Example 16.5.1.7

Consider the graph of y = f(x) presented in Figure 16.5.1.14

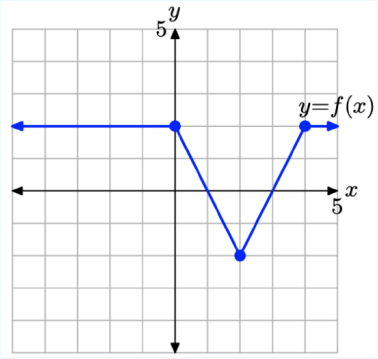


Figure 16.5.1.14 The graph of y = f(x) that will be transformed in Example 16.5.1.7.

Use the concepts discussed in the Visual Summaries to sketch the graph of y = −2f(x) without creating and referring to a table of points.

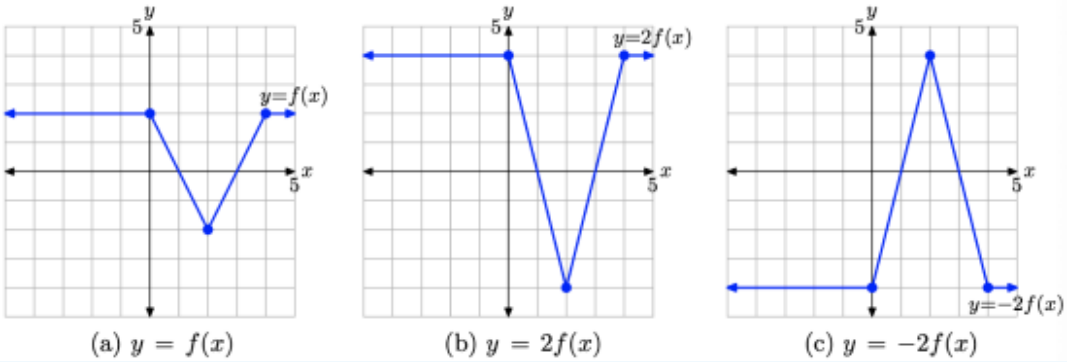
Solution

Note that the equation y = −2f(x) can be formed by a sequence of two transformations.

1. First, scale the original function y = f(x) to obtain the equation y = 2f(x).
2. Second, negate the resulting function y = 2f(x) to obtain the equation y = −2f(x).

Thus, the graph of y = −2f(x) can be formed as follows:

1. Start with the graph of y = f(x) and double the y-value of each point on the graph of y = f(x), keeping the same x-value. The result is the graph of y = 2f(x) shown in Figure 16.5.1.15(b).
2. Next, negate the y-value of each point on the graph of y = 2f(x), keeping the same x-value. The result is the graph of y = −2f(x) in Figure 16.5.1.15(c).



(a) $y = f(x)$ (b) $y = 2f(x)$ (c) $y = -2f(x)$

Figure 16.5.1.15 Transforming the graph of y = f(x) with a sequence of two transformations.

It is interesting to note that you will get the same result if you negate first, then scale the result. We will leave it to our readers to check that this is true.

Let's look at one final example.

Example 16.5.1.8

Consider the graph of y = f(x) presented in Figure 16.5.1.16

Use the concepts discussed in the Visual Summaries to sketch the graph of y = −f(x)+2 without creating and referring to a table of points.

Solution

Note that the equation y = −f(x) + 2 can be formed by a sequence of two transformations.

1. First, negate the original function y = f(x) to obtain the equation y = −f(x).
2. Second, add 2 to the resulting function y = −f(x) to obtain the equation y = −f(x) + 2.

Thus, the graph of y = −f(x) + 2 can be formed as follows.

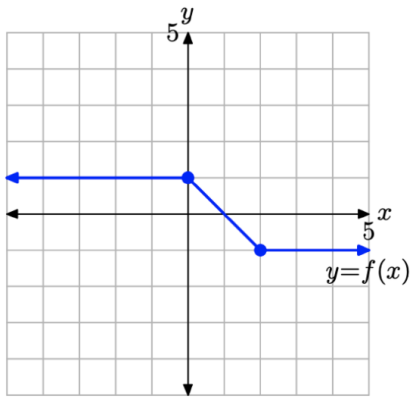
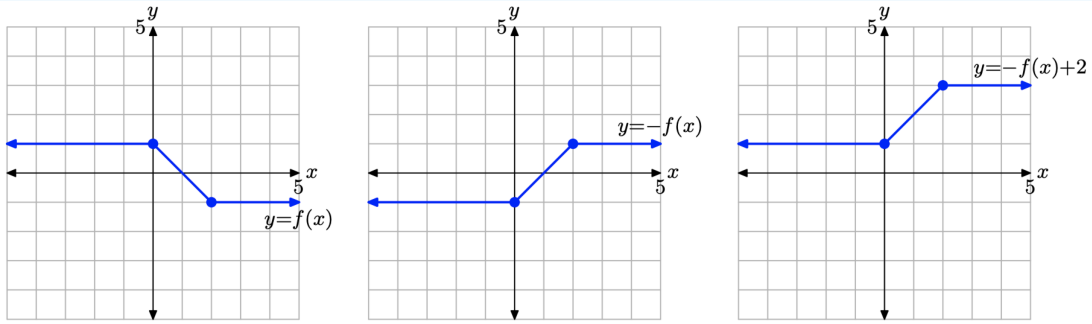


Figure 16.5.1.16 The graph of y = f(x) that will be transformed in Example 16.5.1.8

1. First, start with the graph of y = f(x) in Figure 16.5.1.17(a) and negate the y-value of each point to produce the graph of y = −f(x) Figure 16.5.1.17(b).

2. Next, add 2 to the y-value of each point on the graph of y = −f(x) in Figure 16.5.1.17(b) to produce the graph of y = −f(x) + 2 in Figure 16.5.1.17(c).



(a) $y = f(x)$ (b) $y = -f(x)$ (c) $y = -f(x) + 2$

Figure 16.5.1.17 Transforming the graph of y = f(x), first reflecting across the x-axis, then shifting 2 units upward to obtain the graph of y = −f(x) + 2.

In Example 16.5.1.7, where we started with the graph of y = f(x) and then graphed y = 2f(x), the order of the transformations did not matter. Scale by 2, then negate, or negate and scale by 2, you get the same result (readers should verify this claim). However, in this example, the order in which the transformations are applied does matter. To see this, let's do the following:

1. Add 2 to shift the graph of y = f(x) in Figure 16.5.1.18(a) two units upward to obtain the graph of y = f(x) + 2 in Figure 16.5.1.18(b).
2. Negate the y-value of each point on the graph of y = f(x) + 2 in Figure 16.5.1.18(b) to obtain the graph of y = −(f(x) + 2) in Figure 16.5.1.18(c). Note that we must negate the entire y-value. Hence the parentheses.

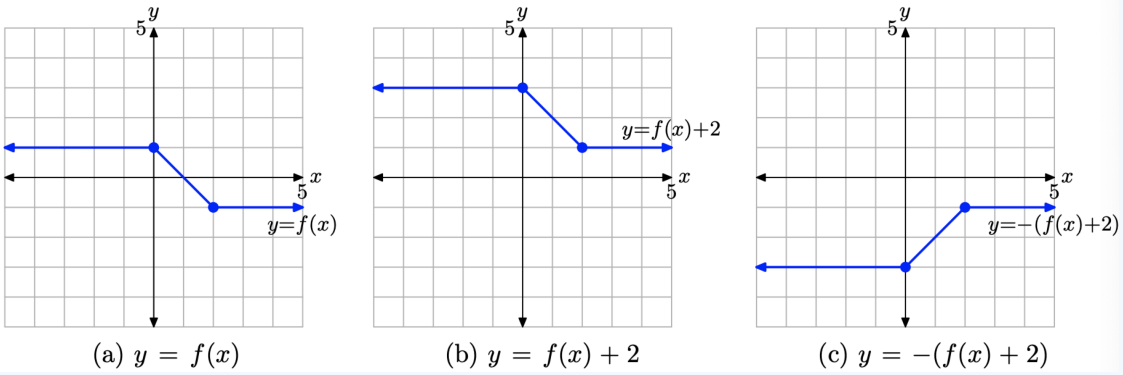
Unfortunately, the graph of y = −(f(x)+ 2) in Figure 16.5.1.18(c) is not the same as the graph of y = −f(x) + 2 in Figure 16.5.1.17(c). But of course, this makes complete sense, as the equations (in the case of Figure 16.5.1.18(c))

$$y = -(f(x) + 2) = -f(x) - 2$$

and (in the case of Figure 16.5.1.17(c))

$$y = -f(x) + 2$$

are also not the same.



(a) $y = f(x)$ (b) $y = f(x) + 2$ (c) $y = -(f(x) + 2)$

Figure 16.5.1.18 Transforming the graph of y = f(x), shifting 2 units upward to obtain the graph of y = f(x) + 2, then reflecting across the x-axis to obtain the graph of y = −(f(x) + 2).

Therefore, care must be taken when applying more than one transformation. Here is a good rule of thumb to live by.

Do Vertical Scalings and Reflections First, then Vertical Translations. When performing a sequence of vertical transformations, it is usually easier (less confusing) to apply vertical scalings and reflections before vertical translations.

However, as long as you perform the transformations correctly, you should obtain the correct result. In Example 16.5.1.8 if you want to sketch the graph of y = −f(x)+2 by doing the translation first, the correct way to proceed is as follows (though somewhat counterintuitive):

1. First, shift the graph of y = f(x) downward 2 units to obtain the graph of y = f(x) − 2.
2. Second, reflect the graph of y = f(x) − 2 across the x-axis to obtain the graph of y = −(f(x) − 2). Again, note the use of parentheses as we negate the entire y-value.

Finally, note that

$$y = -(f(x) - 2) = -f(x) + 2$$

We will leave it to our readers to show that this sequence produces the correct result, a graph identical to the correct answer shown in Figure 16.5.1.17(c).

Summary

In this section we've seen how a handful of transformations greatly enhance our graphing capability. We end this section by listing the transformations presented in this section and their effects on the graph of a function.

Vertical Transformations.

Vertical Transformations. Suppose we are given the graph of y = f(x).

- If $a > 1$, then the graph of y = af(x) is stretched vertically (away from the x-axis), both positively and negatively, by a factor of a.
- If $0 < a < 1$, then the graph of y = af(x) is compressed vertically (toward the x-axis), both positively and negatively, by a factor of 1/a.
- The graph of y = −f(x) is a reflection of the graph of y = f(x) across the x-axis.
- If $c > 0$, then the graph of y = f(x) + c is shifted c units upward from the graph of y = f(x).
- If $c > 0$, then the graph of y = f(x) − c is shifted c units downward from the graph of y = f(x).

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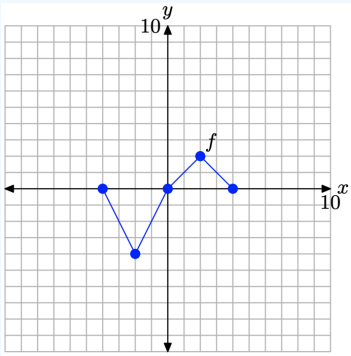
16.5.2: Horizontal Transformations

In the previous section, we introduced the concept of transformations. We made a change to the basic equation y = f(x), such as y = af(x), y = −f(x), y = f(x) − c, or y = f(x) + c, then studied how these changes affected the shape of the graph of y = f(x). In that section, we concentrated strictly on transformations that applied in th vertical direction. In this section, we will study transformations that will affect the shape of the graph in the horizontal direction.

We begin our task with an example that requires that we read the graph of a function to capture several key points that lie on the graph of the function.

Example 16.5.2.1

Consider the graph of f presented in Figure 16.5.2.1(a). Use the graph of f to complete the table in Figure 16.5.2.1(b).



(a) The graph of f.

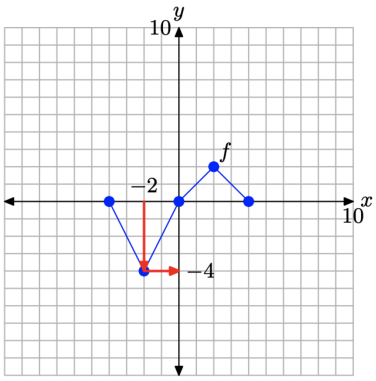
x	$f(x)$	$(x, f(x))$
-4		
-2		
0		
2		
4		

(b) The table.

Figure 16.5.2.1 Reading key values from the graph of f.

Solution

To compute f(−2), for example, we would first locate −2 on the x-axis, draw a vertical arrow to the graph of f, then a horizontal arrow to the y-axis, as shown in Figure 16.5.2.2(a). The y-value of this final destination is the value of f(−2). That is, f(−2) = −4. This allows us to complete one entry in the table, as shown in Figure 16.5.2.2(b). Continue in this manner to complete all of the entries in the table. The result is shown in Figure 16.5.2.2(c).



(a) The graph of f.

x	$f(x)$	$(x, f(x))$
-4		
-2	-4	$(-2, -4)$
0		
2		
4		

(b) Recording $f(-2) = -4$.

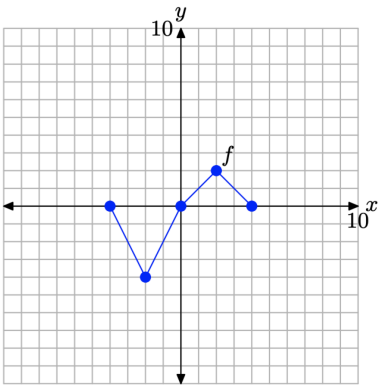
x	$f(x)$	$(x, f(x))$
-4	0	$(-4, 0)$
-2	-4	$(-2, -4)$
0	0	$(0, 0)$
2	2	$(2, 2)$
4	0	$(4, 0)$

(c) Completed table.

Figure 16.5.2.2 Recording coordinates of points on the graph of f in the tables

Horizontal Scaling

In the narrative that follows, we will have repeated need of the graph in Figure 16.5.2.2(a) and the table in Figure 16.5.2.2(c). They characterize the basic function that will be the starting point for the concepts of scaling, reflection, and translation that we develop in this section. Consequently, let's place them side-by-side for emphasis in Figure 16.5.2.3



(a)

x	$f(x)$	$(x, f(x))$
-4	0	$(-4, 0)$
-2	-4	$(-2, -4)$
0	0	$(0, 0)$
2	2	$(2, 2)$
4	0	$(4, 0)$

(b)

Figure 16.5.2.3 The original graph of f and a table of key points on the graph of f

We are now going to scale the graph of f in the horizontal direction.

Example 16.5.2.2

If y = f(x) has the graph shown in Figure 16.5.2.3(a), sketch the graph of y = f(2x).

Solution

In the previous section, we investigated the graph of y = 2f(x). The number 2 was outside the function notation and as a result we stretched the graph of y = f(x) vertically by a factor of 2. However, note that the 2 is now inside the function notation y = f(2x). Intuition would demand that this might have something to do with scaling in the x-direction (horizontal direction), but how?

Again, when we're unsure of the shape of the graph, we rely on plotting a table of points. We begin by picking these x-values: x = −2, −1, 0, 1, and 2. Note that these are precisely half of each of the x-values presented in the table in Figure 16.5.2.3(b). We will now evaluate the function y = f(2x) at each of these x-values. For example, to compute y = f(2x) at x = −2, we first insert x = −2 for x to obtain

$$y = f(2(-2)) = f(-4)$$

To complete the computation, we must now evaluate f(−4). However, this result is recorded in the table in Figure 16.5.2.3(b). There we find that f(−4) = 0, and we can complete the computation started above.

$$y = f(2(-2)) = f(-4) = 0$$

In similar fashion, to evaluate the function y = f(2x) at x = −1, first substitute x = −1 in y = f(2x) to obtain

$$y = f(2(-1)) = f(-2)$$

Now, note that f(−2) is the next recorded value in the table in Figure 16.5.2.3(b). There we find that f(−2) = −4, so we can complete the computation started above.

$$y = f(2(-1)) = f(-2) = -4$$

At this point, you might see why we chose x-values: −2, −1, 0, 1, and 2. These are precisely half of the x-values in the table of original values for the function y = f(x) in Figure 16.5.2.3(b). When the values −2, −1, 0, 1, and 2 are substituted into the function y = f(2x), they are first doubled before we go to look up the function value in the table in Figure 16.5.2.3(b).

Continuing in this manner, we evaluate the function y = f(2x) at the remaining values of x, namely, 0, 1, and 2.

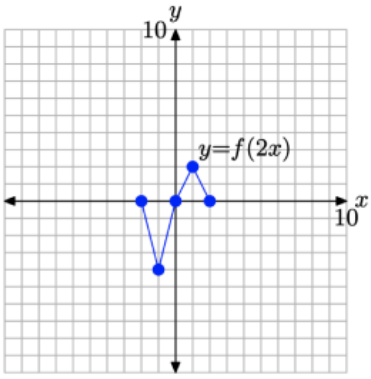
$$\begin{aligned} y &= f(2(0)) = f(0) = 0 \\ y &= f(2(1)) = f(2) = 2 \\ y &= f(2(2)) = f(4) = 0 \end{aligned}$$

We enter these values into the table in Figure 4(b) and plot them to determine the graph of y = f(2x) in Figure 16.5.2.4(a).

At this point, there are a number of comparisons you can make.

1. Compare the data in the table in Figure 16.5.2.4(b) with the original function data in the table in Figure 16.5.2.3(b). Note that the y-values in each table are identical. However, note that each x-value in the table of Figure 16.5.2.4(b) is precisely half of the corresponding x-value in the table of Figure 16.5.2.3(b).

2. Compare the graph of y = f(2x) in Figure 16.5.2.4(a) with the original graph of y = f(x) in Figure 16.5.2.3(a). Note that each x-value at each point on the graph of y = f(2x) in



(a)

x	$y = f(2x)$	$(x, f(2x))$
-2	0	$(-2, 0)$
-1	-4	$(-1, -4)$
0	0	$(0, 0)$
1	2	$(1, 2)$
2	0	$(2, 0)$

(b)

Figure 16.5.2.4 The points in the table are points on the graph of y = f(2x).

Figure 16.5.2.4(a) is precisely half the x-value of the corresponding point on the graph of y = f(x) in Figure 16.5.2.3(a).

Note the result. The graph of y = f(2x) is compressed horizontally (toward the y-axis), both positively and negatively, by a factor of 2. Note that this is exactly the opposite of what you might expect by intuition, but a careful examination of the data in the tables in Figures 16.5.2.3(b) and 16.5.2.4(b) will explain why.

Let's look at another example.

Example 16.5.2.3

If y = f(x) has the graph shown in Figure 16.5.2.3(a), sketch the graph of y = f((1/2)x).

Solution

Rather than doubling each value of x at the start, this function first halves each value of x. Thus, we will want to evaluate the function y = f((1/2)x) at x = −8, −4, 0, 4, and 8. For example, to evaluate the function y = f((1/2)x) at x = −8, first substitute x = −8 to obtain

$$y = f((1/2)(-8)) = f(-4)$$

Now, look up this value in the table in Figure 16.5.2.3(b) and note that f(−4) = 0. Thus, we can complete the computation as follows.

$$y = f((1/2)(-8)) = f(-4) = 0$$

Similarly, to evaluate the function y = f((1/2)x) at x = −4, first substitute x = −4 to obtain

$$y = f((1/2)(-4)) = f(-2)$$

Now, look up this value in the table in Figure 16.5.2.3(b) and note that f(−2) = −4. Thus, we can complete the computation as follows.

$$y = f((1/2)(-4)) = f(-2) = -4$$

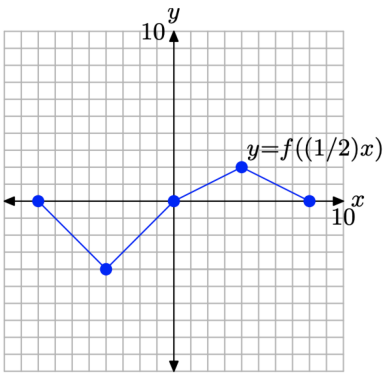
At this point, you will see why we chose x-values: −8, −4, 0, 4, and 8. These values are precisely double the x-values in the table of original values for the function y = f(x) in Figure 16.5.2.3(b). When the values −8, −4, 0, 4, and 8 are substituted into

the function y = f((1/2)x), they are first halved before we go to look up the function value in the table in Figure 16.5.2.3(b). This halving leads to the values −4, −2, 0, 2, and 4, which are precisely the values available in the table in Figure 16.5.2.3(b).

We make similar computations at the remaining values of x, namely x = 0, 4, and 8.

$$y = f((1/2)(0)) = f(0) = 0$$
$$y = f((1/2)(4)) = f(2) = 2$$
$$y = f((1/2)(8)) = f(4) = 0$$

Hopefully, these computations explain our choice of x-values above. Each of these results is recorded in the table in Figure 16.5.2.5(b) and plotted on the graph shown in Figure 16.5.2.5(a).



(a)

x	$y = f((1/2)x)$	$(x, f((1/2)x))$
-8	0	$(-8, 0)$
-4	-4	$(-4, -4)$
0	0	$(0, 0)$
4	2	$(4, 2)$
8	0	$(8, 0)$

(b)

Figure 16.5.2.5 The points in the table are points on the graph of y = f((1/2)x).

Again, note that the y-values in the table in Figure 16.5.2.5(b) are identical to the y-values in the table in Figure 16.5.2.3(b). However, each x-value in the table in Figure 16.5.2.5(b) is precisely double the corresponding x-value in the table in Figure 16.5.2.3(b).

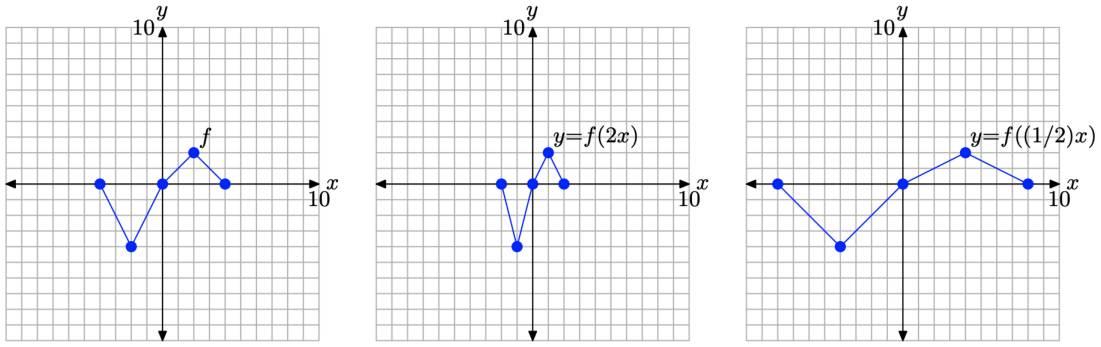
This doubling of the x-values is apparent in the graph of y = f((1/2)x) shown in Figure 16.5.2.5(a), where the graph is stretched by a factor of 2 horizontally (away from the y-axis), both positively and negatively. Note that this is exactly the opposite of what you might expect by intuition, but a careful examination of the data in the tables in Figures 16.5.2.3(b) and 16.5.2.5(b) will explain why.

Let's summarize our findings.

A Visual Summary — Horizontal Scaling

Consider the images in Figure 16.5.2.6

- In Figure 16.5.2.6(a), we see pictured the graph of the original function y = f(x).
- In Figure 16.5.2.6(b), note that each key point on the graph of y = f(2x) has an x-value that is precisely half the x-value of the corresponding point on the graph of y = f(x) in Figure 16.5.2.6(a).
- In Figure 16.5.2.6(c), note that each key point on the graph of y = f((1/2)x) has an x-value that is twice the x-value of the corresponding point on the graph of y = f(x) in Figure 16.5.2.6(a).
- Note that the y-value of each transformed point remains the same.



(a) $y = f(x)$ (b) $y = f(2x)$ (c) $y = f((1/2)x)$

Figure 16.5.2.6 The graph of y = f(2x) compresses horizontally (toward the y-axis) by a factor of 2. The graph of y = f((1/2)x) stretches horizontally (away from the y-axis) by a factor of 2.

The visual summary in Figure 16.5.2.6 makes sketching the graphs of y = f(2x) and y = f(1/2)x) an easy task.

- Given the graph of y = f(x), to sketch the graph of y = f(2x), simply take each point on the graph of y = f(x) and cut its x-value in half, keeping the same y-value.
- Given the graph of y = f(x), to sketch the graph of y = f((1/2)x), simply take each point on the graph of y = f(x) and double its x-value, keeping the same y-value.

Follow the same procedures for other scaling factors. For example, in the case of y = f(3x), take each point on the graph of y = f(x) and divide its x-value by 3, keeping the same y-value. On the other hand, to draw the graph of y = f((1/3)x), take each point on the graph of f and multiply its x-value by 3, keeping the same y-value.

In general, we can state the following.

summary

Suppose we are given the graph of y = f(x).

- If a > 1, the graph of y = f(ax) compresses horizontally (toward the y-axis) , both positively and negatively, by a factor of a.
- If 0 < a < 1, the graph of y = f(ax) stretches horizontally (away from the y-axis), both positively and negatively, by a factor of 1/a.

In the case of the first item in Summary, when we compare the general form y = f(ax) with y = f(2x), we see that a = 2. In this case, note that a > 1 and the graph of y = f(2x) compresses horizontally by a factor of 2 when compared with the graph of y = f(x) (see Figure 16.5.2.6(b)).

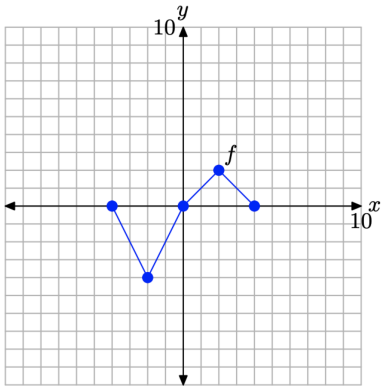
In the case of the second item in Summary, when we compare the general form y = f(ax) with the equation y = f((1/2)x), we see that a = 1/2, so

$$\frac{1}{a} = \frac{1}{1/2} = 2$$

The second item in Summary 4 says that when 0 < a < 1, the graph of y = f(ax) stretches horizontally by a factor of 1/a. Indeed, this is exactly what happened in the case of y = f((1/2)x), which stretched in the horizontal direction by a factor of 1/(1/2), or 2 (see Figure 16.5.2.6(c)).

Horizontal Reflections

For convenience, we begin by repeating the original graph of y = f(x) and its accompanying data in Figure 16.5.2.7. We are now going to reflect the graph of y = f(x) in the horizontal direction (across the y-axis).



(a)

x	$f(x)$	$(x, f(x))$
-4	0	$(-4, 0)$
-2	-4	$(-2, -4)$
0	0	$(0, 0)$
2	2	$(2, 2)$
4	0	$(4, 0)$

(b)

Figure 16.5.2.7 The original graph of f and a table of key points on the graph of f

Example 16.5.2.4

If y = f(x) has the graph shown in Figure 16.5.2.7(a), draw the graph of y = f(−x).

Solution

In the previous section, we were asked to draw the graph of y = −f(x). Note how the minus sign appears on the outside of the function. Clearly, the y-values of y = −f(x) must be opposite in sign to the y-values of y = f(x). That is why the graph of y = −f(x) was a reflection of the graph of y = f(x) across the x-axis.

However, in this example, the minus sign is inside the function, leaving one to intuit that it is the x-values, not the y-values, that are being negated. We will choose the following x-values: x = 4, 2, 0, −2, and −4. This is a bit deceptive, as it looks like we are choosing the same x-values, only in reverse order. This is not the case. We are choosing the negative of each x-value in the table in Figure 16.5.2.7(b).

To evaluate y = f(−x) at our first x-value, namely x = 4, we perform the following calculation. First substitute x = 4 to obtain

$$y = f(-(4)) = f(-4)$$

Now, look up this value in the table in Figure 16.5.2.7(b) and note that f(−4) = 0. Thus, we can complete the computation as follows.

$$y = f(-(4)) = f(-4) = 0$$

Similarly, to evaluate the function y = f(−x) at x = 2, first substitute x = 2 to obtain

$$y = f(-(2)) = f(-2)$$

Now, look up this value in the table in Figure 16.5.2.7(b) and note that f(−2) = −4. Thus, we can complete the computation as follows.

$$y = f(-(2)) = f(-2) = -4$$

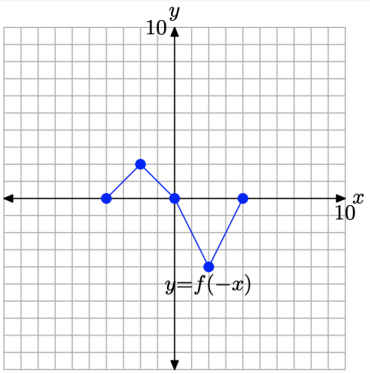
At this point, you will see why we chose x-values: 4, 2, 0, −2, and −4. These values are the negatives of the x-values in the table of original values for the function y = f(x) in Figure 16.5.2.7(b). When the values 4, 2, 0, −2, and −4 are substituted into the function y = f(−x), they are first negated before we go to look up the function value in the table in Figure 16.5.2.7(b). This negating leads to the values −4, −2, 0, 2, and 4, which are precisely the values available in the table in Figure 16.5.2.7(b).

We make similar computations at the remaining values of x, namely x = 0, −2, and −4.

$$y = f(-(0)) = f(0) = 0$$
$$y = f(-(-2)) = f(2) = 2$$
$$y = f(-(-4)) = f(4) = 0$$

We organize these points in the table in Figure 16.5.2.8(b), then plot them in Figure 16.5.2.8(a).

When you compare the entries in the table in Figure 16.5.2.8(b) with those in the table in Figure 16.5.2.7(b), note that the y-values appear in the same order, but the x-values of the table in Figure 16.5.2.7(b) have been negated in the table in Figure 16.5.2.8(b). This means that a former point such as (−2, −4) is transformed to the point (2, −4), which is a reflection of the point (−2, −4) across the y-axis.



(a)

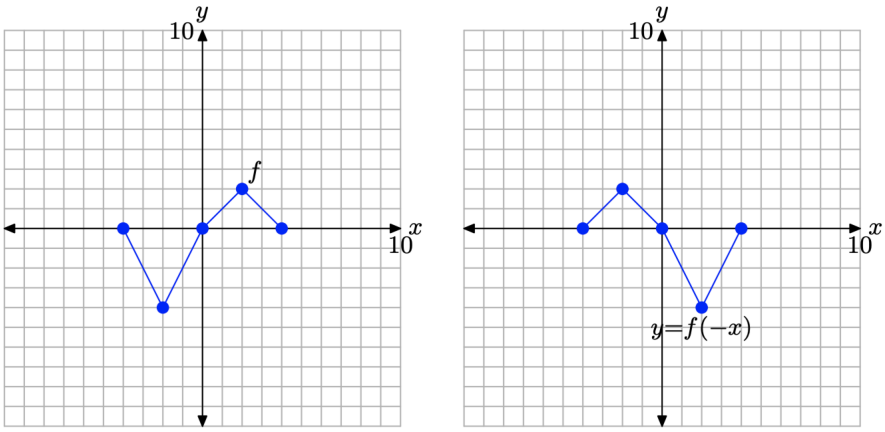
x	$y = f(-x)$	$(x, f(-x))$
4	0	$(4, 0)$
2	−4	$(2, -4)$
0	0	$(0, 0)$
−2	2	$(-2, 2)$
−4	0	$(-4, 0)$

(b)

Figure 16.5.2.8 The graph of y = f(−x) and a table of key points on the graph.

Thus, to produce the graph of y = f(−x), simply reflect the graph of y = f(x) across the y-axis.

Let's summarize what we've learned about horizontal reflections.



(a) $y = f(x)$ (b) $y = f(-x)$

Figure 16.5.2.9 The graph of y = f(−x) is a reflection of the graph of y = f(x) across the y-axis.

A Visual Summary — Horizontal Reflections

Consider the images in Figure 16.5.2.9

- In Figure 16.5.2.9(a), we see pictured the original graph of y = f(x).
- In Figure 16.5.2.9(b), the graph of y = f(−x) is a reflection of the graph of y = f(x) across the y-axis.

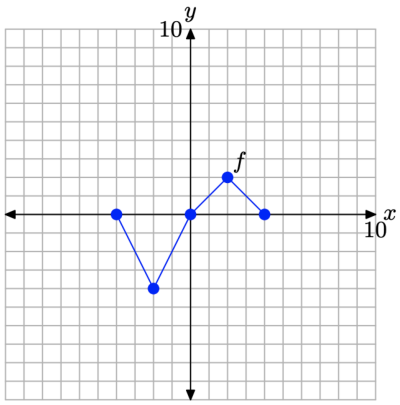
Thus, given the graph of y = f(x), it is a simple task to draw the graph of y = f(−x).

- To draw the graph of y = f(−x), take each point on the graph of y = f(x) and reflect it across the y-axis, keeping the y-value the same, but negating the x-value.

Horizontal Translations

In the previous section, we saw that the graphs of y = f(x) + c and y = f(x) − c were vertical translations of the graph of y = f(x). If c is a positive number, then the graph of y = f(x) + c shifts c units upward while the graph of y = f(x) − c shifts c units downward.

In this section, we will study horizontal translations. For convenience, we begin by repeating the original graph of y = f(x) and its accompanying data in Figure 10.



(a)

x	$f(x)$	$(x, f(x))$
-4	0	$(-4, 0)$
-2	-4	$(-2, -4)$
0	0	$(0, 0)$
2	2	$(2, 2)$
4	0	$(4, 0)$

(b)

Figure 16.5.2.10 The original graph of f and a table of key points on the graph of f

Example 16.5.2.5

If y = f(x) has the graph shown in Figure 16.5.2.10(a), sketch the graph of y = f(x + 1).

Solution

In the previous section, we drew the graph of y = f(x)+1. Note that in y = f(x)+1, the number 1 is outside the function. The result was a graph that was shifted 1 unit upwards in the y-direction.

In this case, y = f(x + 1) and the 1 is inside the function notation, leading one to intuit that the translation might be in the horizontal direction (x-direction). But how?

Again, we will set up a table of points that satisfy the equation y = f(x + 1), then plot them. Because this function requires that we first add 1 to each x-value before inserting it into the function, we will choose x-values appropriately, namely x = −5, −3, −1, 1, and 3. In a moment, it will be clear why we have chosen these particular values of x. Perhaps you already see why?

We need to evaluate the function y = f(x + 1) at each of these chosen values of x. To evaluate y = f(x+ 1) at the first value, namely x = −5, we insert x = −5 and make the calculation

$$y = f(-5+1) = f(-4)$$

To complete the calculation, we must now evaluate f(−4). However, this result is recorded in the table in Figure 16.5.2.10(b). There we find that f(−4) = 0, and we can complete the calculation started above.

$$y = f(-5+1) = f(-4) = 0$$

In similar fashion, we can evaluate the function y = f(x+1) at x = −3. First, substitute x = −3 in y = f(x + 1) to obtain

$$y = f(-3+1) = f(-2)$$

To complete the calculation, we must now evaluate f(−2). However, this result is recorded in the table in Figure 16.5.2.10(b). There we find that f(−2) = −4, and we can complete the calculation started above.

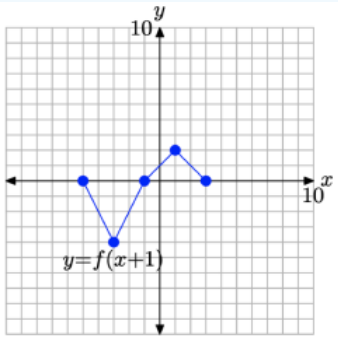
$$y = f(-3+1) = f(-2) = -4$$

At this point, you might see why we chose x-values: −5, −3, −1, 1, and 3. These are precisely one less than the x-values in the table of original values for the function y = f(x) in Figure 16.5.2.10(b). When the values −5, −3, −1, 1, and 3 are substituted into the function y = f(x + 1), we first add 1 to each value before we go to look up the function value in the table in Figure 16.5.2.10(b). This adding of 1 leads to the values −4, −2, 0, 2, and 4, which are precisely the values available in the table in Figure 16.5.2.10(b).

Continuing in this manner, we evaluate the function y = f(x + 1) at the remaining values of x, namely, −1, 1, and 3.

$$y = f(-1+1) = f(0) = 0$$
$$y = f(1+1) = f(2) = 2$$
$$y = f(3+1) = f(4) = 0$$

We assemble these results in the table in Figure 16.5.2.11(b) and plot them in Figure 16.5.2.11(a).



(a)

x	$y = f(x+1)$	$(x, f(x+1))$
-5	0	$(-5, 0)$
-3	-4	$(-3, -4)$
-1	0	$(-1, 0)$
1	2	$(1, 2)$
3	0	$(3, 0)$

(b)

Figure 16.5.2.11 The graph of y = f(x + 1) and a table of key points on the graph

When you compare the points on the graph of y = f(x+1) in the table in Figure 16.5.2.11(b) with the original points on the graph of y = f(x) in the table in Figure 16.5.2.10(b), note that the y-values are identical, but the x-values in the table in Figure 16.5.2.11(b) are all 1 unit less than the corresponding x-values in the table in Figure 16.5.2.10(b). It is no wonder that the graph of y = f(x + 1) in Figure 16.5.2.11(a) is shifted 1 unit to the left of the original graph of y = f(x) in Figure 16.5.2.10(a).

Note that this is somewhat counterintuitive, because we're seemingly adding 1 to each x-value in y = f(x + 1). Why doesn't the graph move one unit to the right? Well, a careful comparison of the x-values in the tables in Figures 16.5.2.10(b) and 16.5.2.11(b) reveals the answer. In order to use the data in the table in Figure 16.5.2.10(b), we must first subtract 1 from each x-value to produce the x-values in the table in Figure 16.5.2.11(b). This is why the graph of y = f(x + 1) moves 1 unit to the left instead of 1 unit to the right.

You might also recall that the function y = f(2x) compressed by a factor of 2, which is also the opposite of what intuition might dictate. Similarly, the function y = f((1/2)x) stretches by a factor of 2, which also goes counter to intuition. With these thoughts in mind, it is not surprising that y = f(x+ 1) shifts one unit to the left. Still, a comparison of the x-values in the tables in Figures 16.5.2.10(b) and 16.5.2.11(b) provide irrefutable evidence that the shift is 1 unit to the left.

Let's look at another example.

Example 16.5.2.6

If y = f(x) has the graph shown in Figure 16.5.2.10(a), sketch the graph of y = f(x − 2).

Solution

Again, we will set up a table of points that satisfy the equation y = f(x − 2), then plot them. Because this function requires that we first subtract 2 from each x-value before inserting it into the function, we will choose x-values: −2, 0, 2, 4, and 6. We need to evaluate the function y = f(x − 2) at each of these values of x.

To evaluate y = f(x − 2) at the first value, namely x = −2, insert x = −2 into the function y = f(x − 2) to obtain

$$y = f(-2-2) = f(-4)$$

In the table in Figure 16.5.2.10(b), we find that f(−4) = 0, which allows us to complete the calculation above.

$$y = f(-2-2) = f(-4) = 0$$

In similar fashion, we evaluate y = f(x − 2) at x = 0 to obtain

$$y = f(0-2) = f(-2)$$

In the table in Figure 16.5.2.10(b), we find that f(−2) = −4, which allows us to complete the calculation above.

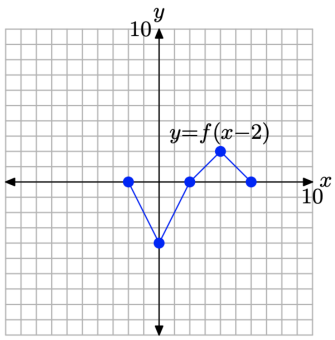
$$y = f(0-2) = f(-2) = -4$$

Hopefully, you see why we chose the x-values: −2, 0, 2, 4, and 6. These values are 2 larger than the x-values in the table of original values for the function y = f(x) in Figure 16.5.2.10(b). When the values −2, 0, 2, 4, and 6 are substituted into the function y = f(x − 2), we first subtract 2 from each value before we go to look up the function value in the table in Figure 16.5.2.10(b). This subtracting of 2 leads to −4, −2, 0, 2, and 4, precisely the values that are available in the table in Figure 16.5.2.10(b).

Continuing in this manner, we evaluate y = f(x − 2) at the remaining values of x, namely, x = 2, 4, and 6.

$$y = f(2-2) = f(0) = 0$$
$$y = f(4-2) = f(2) = 2$$
$$y = f(6-2) = f(4) = 0$$

We assemble these results in the table in Figure 16.5.2.12(b) and plot them in Figure 16.5.2.12(a).



(a)

x	$y = f(x-2)$	$(x, f(x-2))$
-2	0	$(-2, 0)$
0	-4	$(0, -4)$
2	0	$(2, 0)$
4	2	$(4, 2)$
6	0	$(6, 0)$

(b)

Figure 16.5.2.12 The graph of y = f(x − 2) and a table of key points on the graph.

When you compare the points on the graph of y = f(x−2) in the table in Figure 16.5.2.12(b) with the original points on the graph of y = f(x) in the table in Figure 16.5.2.10(b), note that the y-values are identical, but the x-values in the table in Figure 16.5.2.12(b) are all 2 larger than the corresponding x-values in the table in Figure 16.5.2.10(b). It is no wonder that the graph of y = f(x − 2) in Figure 16.5.2.12(a) is shifted 2 units to the right of the original graph of y = f(x) in Figure 16.5.2.10(a).

Again, this runs counterintuitive (why doesn't the graph of y = f(x − 2) shift 2 units to the left?), but a comparison of the x-values in the tables in Figures 16.5.2.10(b) and 16.5.2.12(b) clearly indicates a shift to the right.

Let's summarize what we've learned about horizontal translations.

Visual Summary — Horizontal Translations (Shifts)

Consider the images in Figure 16.5.2.13

- In Figure 16.5.2.13(a), we see pictured the graph of the original function y = f(x).
- In Figure 16.5.2.13(b), note that each point on the graph of y = f(x + 1) has an x-value that is 1 unit less than the x-value of the corresponding point on the graph of y = f(x) in Figure 16.5.2.13(a).
- In Figure 16.5.2.13(c), note that each point on the graph of y = f(x − 2) has an x-value that is 2 units greater than the x-value of the corresponding point on the graph of y = f(x) in Figure 16.5.2.13(a).
- Note that the y-value of each transformed point remains the same.

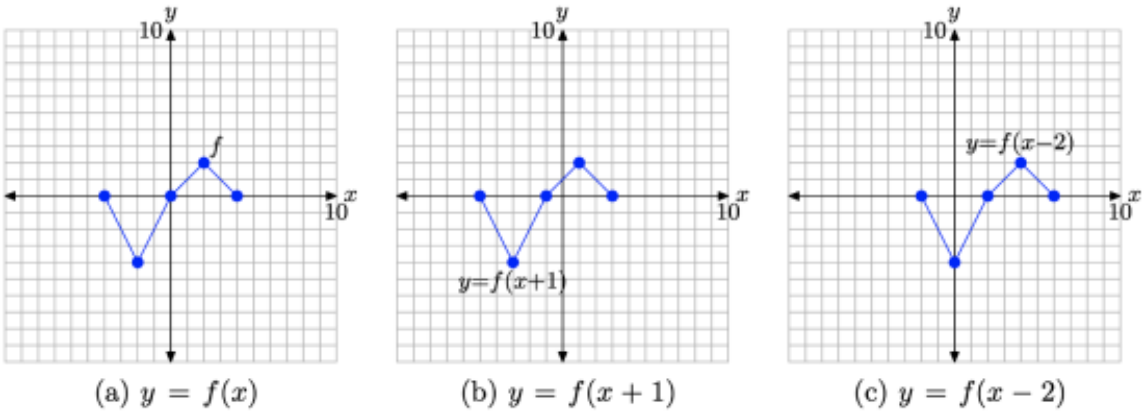


Figure 16.5.2.13 The graph of y = f(x + 1) is formed by shifting (horizontally) the graph of y = f(x) one unit to the left. The graph of y = f(x − 2) is formed by shifting (horizontally) the graph of y = f(x) two units to the right.

The visual summary in Figure 16.5.2.13makes sketching the graphs of y = f(x + 1) and y = f(x − 2) an easy task.

- Given the graph of y = f(x), to sketch the graph of y = f(x + 1), simply take each point on the graph of y = f(x) and shift it 1 unit to the left, keeping the same y-value.
- Given the graph of y = f(x), to sketch the graph of y = f(x − 2), simply take each point on the graph of y = f(x) and shift it 2 units to the right, keeping the same y-value.

In general, we can state the following.

Summary

Suppose that we are given the graph of y = f(x) and suppose that c is any positive real number.

- The graph of y = f(x+c) is shifted c units to the left of the graph of y = f(x).
- The graph of y = f(x−c) is shifted c units to the right of the graph of y = f(x).

When we looked at vertical translations in the previous section, a translation was described by first imagining a graph on a sheet of transparent plastic, then sliding the transparency (without rotating it) over a coordinate system on a sheet of graph paper. Horizontal translations can be thought of in the same way, as sliding the graph on the transparency c units to the left, or c units to the right.

Extra Practice

In this section, let's take the concepts from the Visual Summaries and put them to work on some final examples.

Example 16.5.2.7

Consider the graph of f in Figure 16.5.2.14

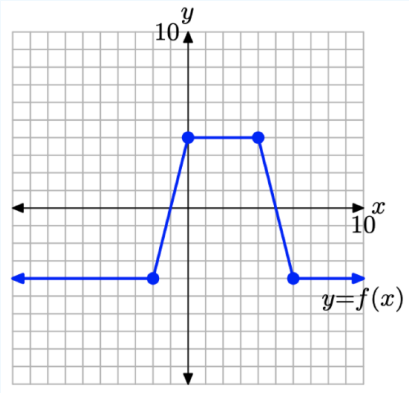


Figure 16.5.2.14 The graph of y = f(x) for Example 16.5.2.7.

Use the concepts from the Visual Summaries (scaling, reflection, and translation) to sketch the graphs of y = f(2x), y = f(−x), and y = f(x + 2) without creating and referring to tables.

Solution

To sketch the graph of y = f(2x), simply take each point on the graph of y = f(x) in Figure 16.5.2.15(a) and divide its x-value by 2, keeping the same y-value. The result is shown in Figure 16.5.2.15(b).

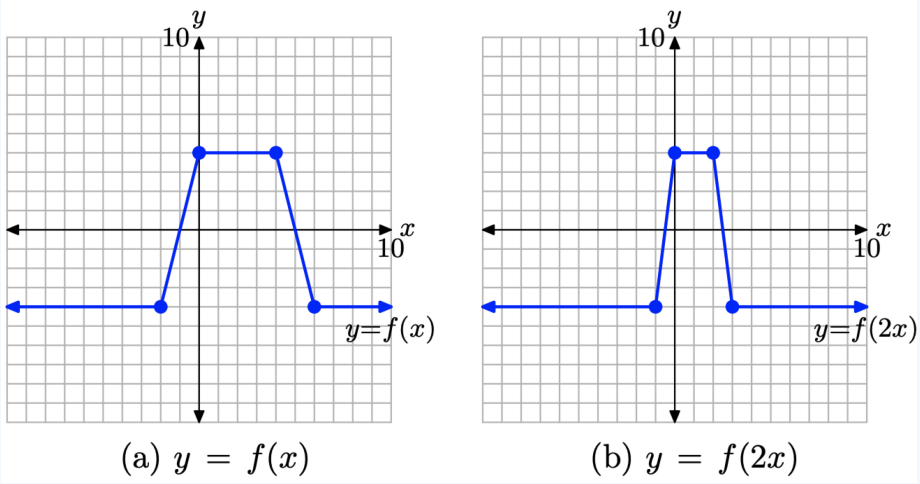


Figure 16.5.2.15 Compress the graph of y = f(x) by a factor of 2 to produce the graph of y = f(2x).

To sketch the graph of y = f(−x), simply take each point on the graph of y = f(x) in Figure 16.5.2.16(a) and negate its x-value, keeping the same y-value. The result is shown in Figure 16.5.2.16(b).

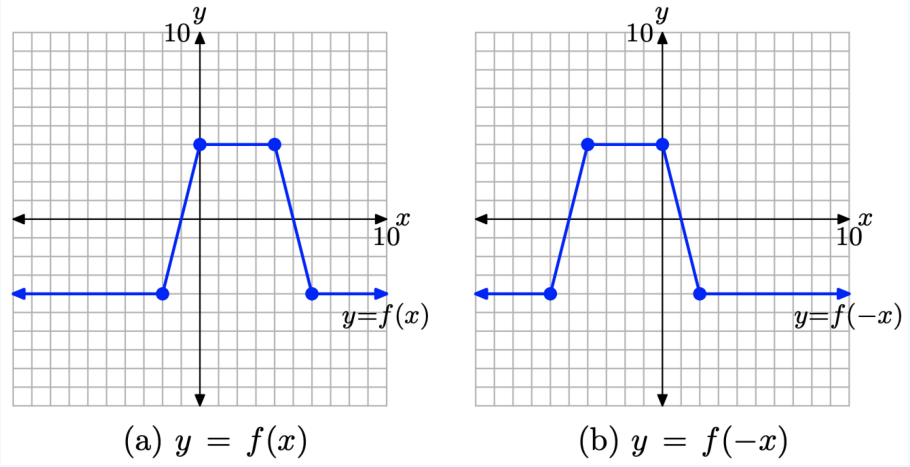


Figure 16.5.2.16 Reflect the graph of y = f(x) across the y-axis to produce the graph of y = f(−x).

To sketch the graph of y = f(x + 2), simply take each point on the graph of y = f(x) in Figure 16.5.2.17(a) and subtract 2 from its x-value, keeping the same y-value. The result is shown in Figure 16.5.2.17(b).

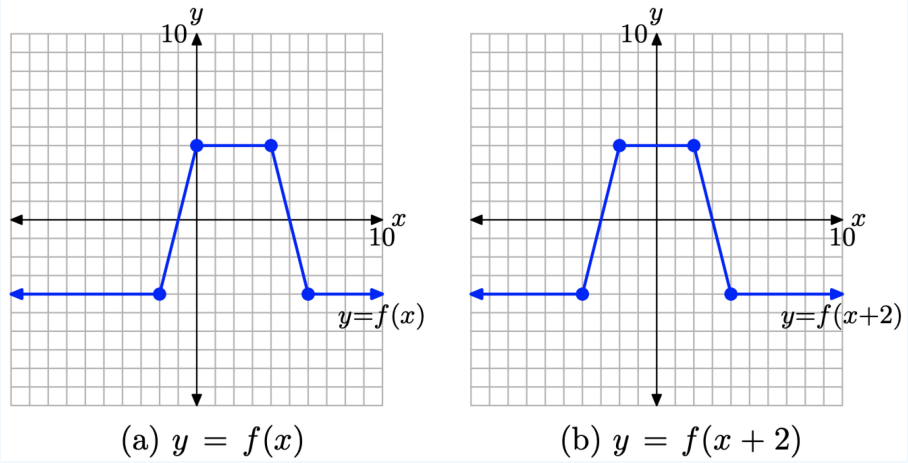


Figure 16.5.2.17 Shift the graph of y = f(x) to the left 2 units to produce the graph of y = f(x + 2).

Summary

In this section we've seen how a handful of transformations greatly enhance our graphing capability. We end this section by listing the transformations presented in this section and their effects on the graph of a function.

Vertical Transformations

Suppose we are given the graph of y = f(x).

- If a > 1, the graph of y = f(ax) compresses horizontally (toward the y-axis) , both positively and negatively, by a factor of a.
- If 0 < a < 1, the graph of y = f(ax) stretches horizontally (away from the y-axis), both positively and negatively, by a factor of 1/a.
- The graph of y = f(−x) is a reflection of the graph of y = f(x) across the y-axis.
- If c > 0, then the graph of y = f(x + c) is shifted c units to the left of the graph of y = f(x).
- If c > 0, then the graph of y = f(x − c) is shifted c units to the right of the graph of y = f(x).

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16.5.3: Graph Quadratic Functions Using Transformations

Learning Objectives

By the end of this section, you will be able to:

- Graph quadratic functions of the form $f(x) = x^2 + k$
- Graph quadratic functions of the form $f(x) = (x - h)^2$
- Graph quadratic functions of the form $f(x) = ax^2$
- Graph quadratic functions using transformations
- Find a quadratic function from its graph

Before you get started, take this readiness quiz.

1. Graph the function $f(x) = x^2$ by plotting points.
   If you missed this problem, review Example 3.54.
2. Factor completely: $y^2 - 14y + 49$.
   If you missed this problem, review Example 6.24.
3. Factor completely: $2x^2 - 16x + 32$.
   If you missed this problem, review Example 6.26.

Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the last section, we learned how to graph quadratic functions using their properties. Another method involves starting with the basic graph of $f(x) = x^2$ and 'moving' it according to information given in the function equation. We call this graphing quadratic functions using transformations.

In the first example, we will graph the quadratic function $f(x) = x^2$ by plotting points. Then we will see what effect adding a constant, k, to the equation will have on the graph of the new function $f(x) = x^2 + k$.

Example 16.5.3.1

Graph $f(x) = x^2$, $g(x) = x^2 + 2$, and $h(x) = x^2 - 2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

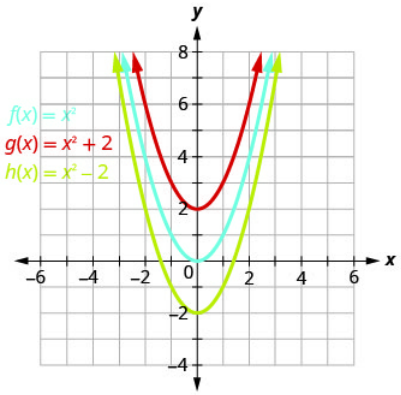
Solution:

Plotting points will help us see the effect of the constants on the basic $f(x) = x^2$ graph. We fill in the chart for all three functions.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = x^2 + 2$	$(x, g(x))$	$h(x) = x^2 - 2$	$(x, h(x))$
–3	9	(–3, 9)	9 + 2	(–3, 11)	9 – 2	(–3, 7)
–2	4	(–2, 4)	4 + 2	(–2, 6)	4 – 2	(–2, 2)
–1	1	(–1, 1)	1 + 2	(–1, 3)	1 – 2	(–1, 1)
0	0	(0, 0)	0 + 2	(0, 2)	0 – 2	(0, –2)
1	1	(1, 1)	1 + 2	(1, 3)	1 – 2	(1, –1)
2	4	(2, 4)	4 + 2	(2, 6)	4 – 2	(2, 2)
3	9	(3, 9)	9 + 2	(3, 11)	9 – 2	(3, 7)

Figure 9.7.1

The $g(x)$ values are two more than the $f(x)$ values. Also, the $h(x)$ values are two less than the $f(x)$ values. Now we will graph all three functions on the same rectangular coordinate system.



The graph of $g(x) = x^2 + 2$ is the same as the graph of $f(x) = x^2$ but shifted up 2 units.

The graph of $h(x) = x^2 - 2$ is the same as the graph of $f(x) = x^2$ but shifted down 2 units.

Figure 9.7.2

The graph of $g(x) = x^2 + 2$ is the same as the graph of $f(x) = x^2$ but shifted up 2 units.

The graph of $h(x) = x^2 - 2$ is the same as the graph of $f(x) = x^2$ but shifted down 2 units.

Exercise 16.5.3.1

a. Graph $f(x) = x^2, g(x) = x^2 + 1,$ and $h(x) = x^2 - 1$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the function has on the basic parabola.

Answer

a.

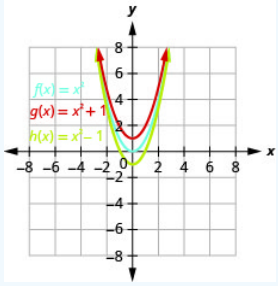


Figure 9.7.3

b. The graph of $g(x) = x^2 + 1$ is the same as the graph of $f(x) = x^2$ but shifted up 1 unit. The graph of $h(x) = x^2 - 1$ is the same as the graph of $f(x) = x^2$ but shifted down 1 unit.

Exercise 16.5.3.2

a. Graph $f(x) = x^2, g(x) = x^2 + 6,$ and $h(x) = x^2 - 6$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the function has on the basic parabola.

Answer

a.

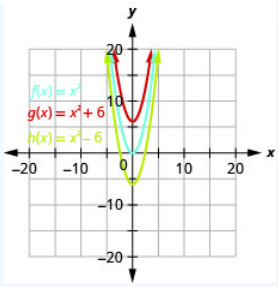


Figure 9.7.4

b. The graph of $h(x) = x^2 + 6$ is the same as the graph of $f(x) = x^2$ but shifted up 6 units. The graph of $h(x) = x^2 - 6$ is the same as the graph of $f(x) = x^2$ but shifted down 6 units.

The last example shows us that to graph a quadratic function of the form $f(x) = x^2 + k$, we take the basic parabola graph of $f(x) = x^2$ and vertically shift it up $(k > 0)$ or shift it down $(k < 0)$.

This transformation is called a vertical shift.

Graph a Quadratic Function of the Form $f(x) = x^2 + k$ Using a Vertical Shift

The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.

- If $k > 0$, shift the parabola vertically up k units.
- If $k < 0$, shift the parabola vertically down $|k|$ units.

Now that we have seen the effect of the constant, k, it is easy to graph functions of the form $f(x) = x^2 + k$. We just start with the basic parabola of $f(x) = x^2$ and then shift it up or down.

It may be helpful to practice sketching $f(x) = x^2$ quickly. We know the values and can sketch the graph from there.

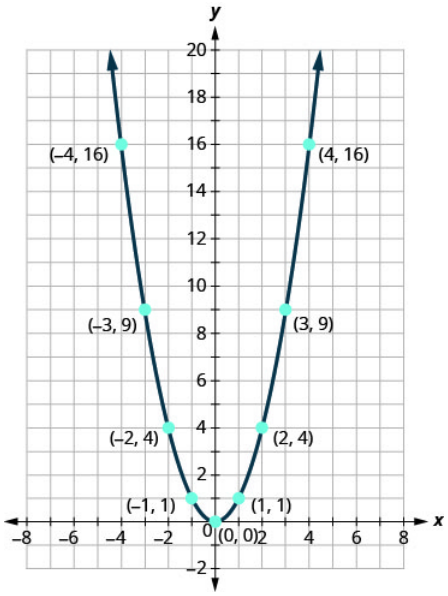


Figure 9.7.5

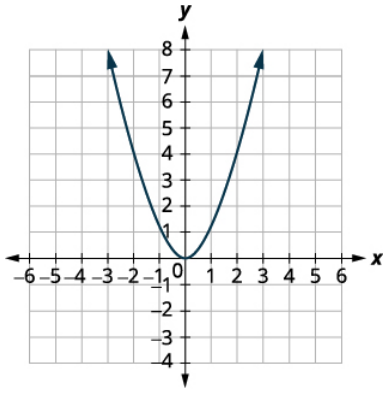
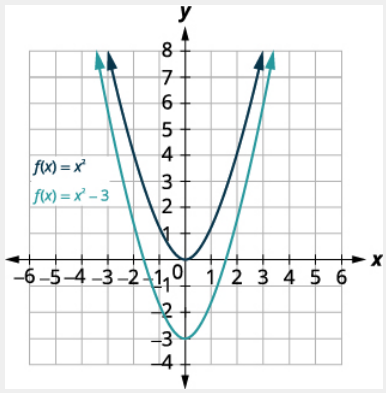
Once we know this parabola, it will be easy to apply the transformations. The next example will require a vertical shift.

Example 16.5.3.2

Graph $f(x) = x^2 - 3$ using a vertical shift.

Solution:

Table 9.7.1

We first draw the graph of $f(x) = x^2$ on the grid.	
Determine k.	$f(x) = x^2 + k$ $f(x) = x^2 - 3$
	$k = -3$
Shift the graph $f(x) = x^2$ down 3.	

Exercise 16.5.3.3

Graph $f(x) = x^2 - 5$ using a vertical shift.

Answer

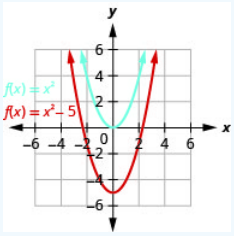


Figure 9.7.10

Exercise 16.5.3.4

Graph $f(x) = x^2 + 7$ using a vertical shift.

Answer

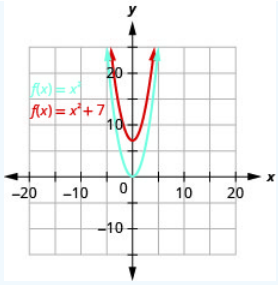


Figure 9.7.11

Graph Quadratic Functions of the Form $f(x) = (x - h)^2$

In the first example, we graphed the quadratic function $f(x) = x^2$ by plotting points and then saw the effect of adding a constant k to the function had on the resulting graph of the new function $f(x) = x^2 + k$.

We will now explore the effect of subtracting a constant, h, from x has on the resulting graph of the new function $f(x) = (x - h)^2$.

Example 16.5.3.3

Graph $f(x) = x^2, g(x) = (x - 1)^2,$ and $h(x) = (x + 1)^2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

Solution:

Plotting points will help us see the effect of the constants on the basic $f(x) = x^2$ graph. We fill in the chart for all three functions.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = (x - 1)^2$	$(x, g(x))$	$h(x) = (x + 1)^2$	$(x, h(x))$
–3	9	(–3, 9)	16	(–3, 16)	4	(–3, 4)
–2	4	(–2, 4)	9	(–2, 9)	1	(–2, 1)
–1	1	(–1, 1)	4	(–1, 4)	0	(–1, 0)
0	0	(0, 0)	1	(0, 1)	1	(0, 1)
1	1	(1, 1)	0	(1, 0)	4	(1, 4)
2	4	(2, 4)	1	(2, 1)	9	(2, 9)
3	9	(3, 9)	4	(3, 4)	16	(3, 16)

Figure 9.7.12

The $g(x)$ values and the $h(x)$ values share the common numbers $0, 1, 4, 9,$ and 16, but are shifted.

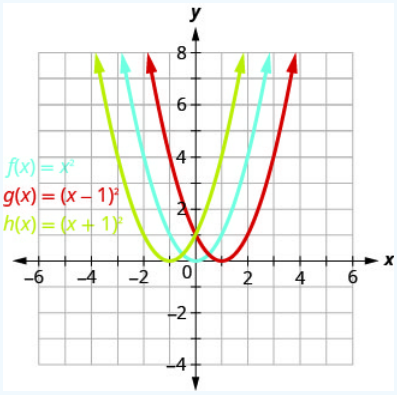


Figure 9.7.13

The graph of $g(x) = (x - 1)^2$ is the same as the graph of $f(x) = x^2$ but shifted right 1 unit.

The graph of $h(x) = (x + 1)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 1 unit.

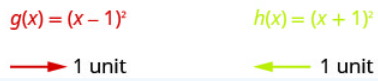


Figure 9.7.14

? Exercise 16.5.3.5

a. Graph $f(x) = x^2, g(x) = (x+2)^2,$ and $h(x) = (x-2)^2$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the function has on the basic parabola.

Answer

a.

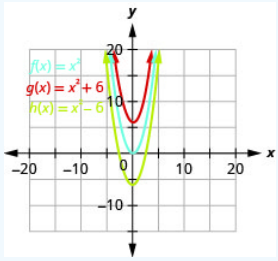


Figure 9.7.15

b. The graph of $g(x) = (x+2)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 2 units. The graph of $h(x) = (x-2)^2$ is the same as the graph of $f(x) = x^2$ but shift right 2 units.

? Exercise 16.5.3.6

a. Graph $f(x) = x^2, g(x) = x^2 + 5,$ and $h(x) = x^2 - 5$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the function has on the basic parabola.

Answer

a.

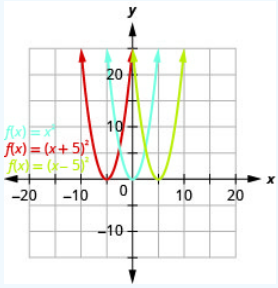


Figure 9.7.16

b. The graph of $g(x) = (x+5)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 5 units. The graph of $h(x) = (x-5)^2$ is the same as the graph of $f(x) = x^2$ but shifted right 5 units.

The last example shows us that to graph a quadratic function of the form $f(x) = (x-h)^2$, we take the basic parabola graph of $f(x) = x^2$ and shift it left $(h > 0)$ or shift it right $(h < 0)$.

This transformation is called a horizontal shift.

Graph a Quadratic Function of the Form $f(x) = (x - h)^2$ Using a Horizontal Shift

The graph of $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.

- If $h > 0$, shift the parabola horizontally left h units.
- If $h < 0$, shift the parabola horizontally right $|h|$ units.

Now that we have seen the effect of the constant, h, it is easy to graph functions of the form $f(x) = (x - h)^2$. We just start with the basic parabola of $f(x) = x^2$ and then shift it left or right.

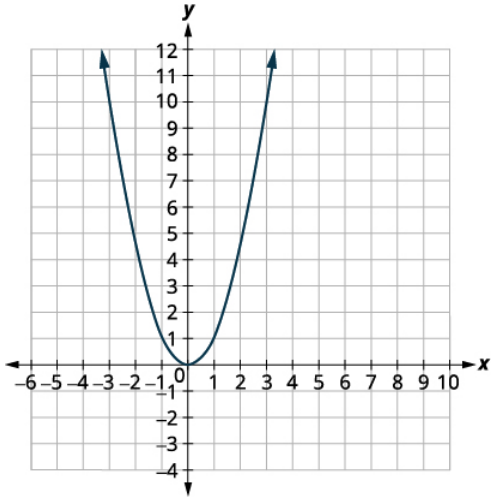
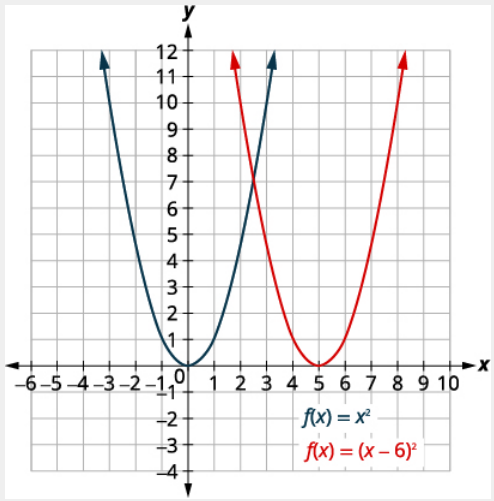
The next example will require a horizontal shift.

Example 16.5.3.4

Graph $f(x) = (x - 6)^2$ using a horizontal shift.

Solution:

Table 9.7.2

We first draw the graph of $f(x) = x^2$ on the grid.	
Determine h.	$f(x) = (x - h)^2$ $f(x) = (x - 6)^2$ $h = 6$
Shift the graph $f(x) = x^2$ to the right 6 units.	

Exercise 16.5.3.7

Graph $f(x) = (x-4)^2$ using a horizontal shift.

Answer

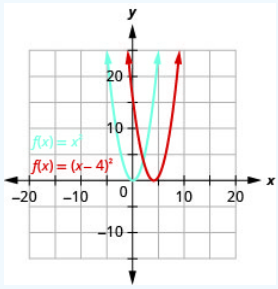


Figure 9.7.21

Exercise 16.5.3.8

Graph $f(x) = (x+6)^2$ using a horizontal shift.

Answer

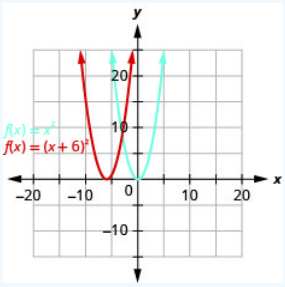


Figure 9.7.22

Now that we know the effect of the constants h and k, we will graph a quadratic function of the form $f(x) = (x-h)^2 + k$ by first drawing the basic parabola and then making a horizontal shift followed by a vertical shift. We could do the vertical shift followed by the horizontal shift, but most students prefer the horizontal shift followed by the vertical.

Example 16.5.3.5

Graph $f(x) = (x+1)^2 - 2$ using transformations.

Solution:

This function will involve two transformations and we need a plan.

Let's first identify the constants h, k.

$$f(x) = (x+1)^2 - 2$$

$$f(x) = (x-h)^2 + k$$

$$f(x) = (x-(-1))^2 + (-2)$$

$$h = -1 \quad k = -2$$

Figure 9.7.23

The h constant gives us a horizontal shift and the k gives us a vertical shift.

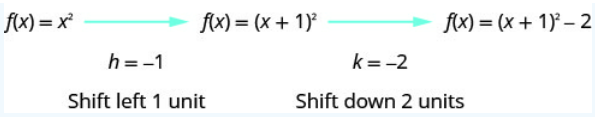


Figure 9.7.24

We first draw the graph of $f(x) = x^2$ on the grid.

To graph $f(x) = (x + 1)^2$, shift the graph $f(x) = x^2$ to the left 1 unit.

To graph $f(x) = (x + 1)^2 - 2$, shift the graph $f(x) = (x + 1)^2$ down 2 units.

Figure 9.7.25

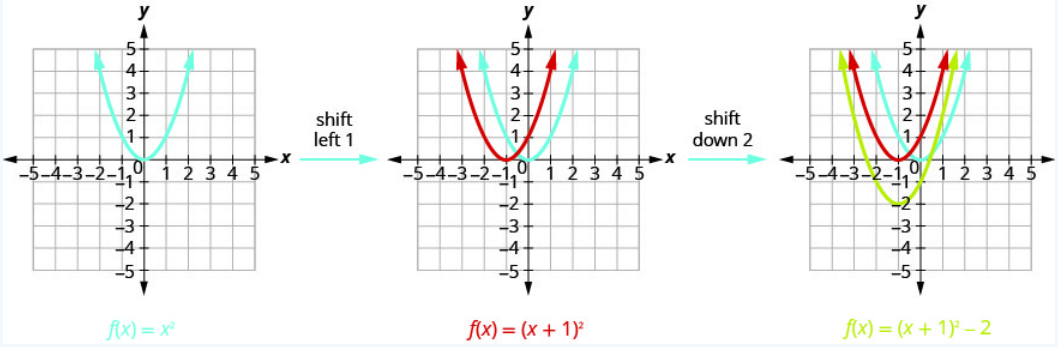


Figure 9.7.26

Exercise 16.5.3.9

Graph $f(x) = (x+2)^2 - 3$ using transformations.

Answer

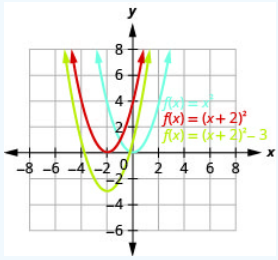


Figure 9.7.27

Exercise 16.5.3.10

Graph $f(x) = (x-3)^2 + 1$ using transformations.

Answer

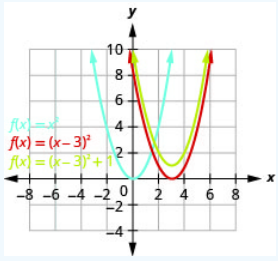


Figure 9.7.28

Graph Quadratic Functions of the Form $f(x) = ax^2$

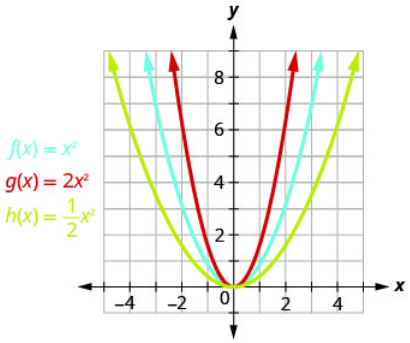
So far we graphed the quadratic function $f(x) = x^2$ and then saw the effect of including a constant h or k in the equation had on the resulting graph of the new function. We will now explore the effect of the coefficient a on the resulting graph of the new function $f(x) = ax^2$.

Let's look at the quadratic functions $f(x) = x^2$, $g(x) = 2x^2$ and $h(x) = \frac{1}{2}x^2$.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = 2x^2$	$(x, g(x))$	$h(x) = \frac{1}{2}x^2$	$(x, h(x))$
–2	4	(–2, 4)	2 • 4	(–2, 8)	$\frac{1}{2}$ • 4	(–2, 2)
–1	1	(–1, 1)	2 • 1	(–1, 2)	$\frac{1}{2}$ • 1	$\left(-1, \frac{1}{2}\right)$
0	0	(0, 0)	2 • 0	(0, 0)	$\frac{1}{2}$ • 0	(0, 0)
1	1	(1, 1)	2 • 1	(1, 2)	$\frac{1}{2}$ • 1	$\left(1, \frac{1}{2}\right)$
2	4	(2, 4)	2 • 4	(2, 8)	$\frac{1}{2}$ • 4	(2, 2)

Figure 9.7.29

If we graph these functions, we can see the effect of the constant a, assuming $a > 0$.



The graph of the function $g(x) = 2x^2$ is "skinnier" than the graph of $f(x) = x^2$.

The graph of the function $h(x) = \frac{1}{2}x^2$ is "wider" than the graph of $f(x) = x^2$.

Figure 9.7.30

To graph a function with constant a it is easiest to choose a few points on $f(x) = x^2$ and multiply the y-values by a.

Graph of a Quadratic Function of the Form $f(x) = ax^2$

The coefficient a in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.

- If $0 < |a| < 1$, the graph of $f(x) = ax^2$ will be "wider" than the graph of $f(x) = x^2$.
- If $|a| > 1$, the graph of $f(x) = ax^2$ will be "skinnier" than the graph of $f(x) = x^2$.

Example 16.5.3.6

Graph $f(x) = 3x^2$.

Solution:

We will graph the functions $f(x) = x^2$ and $g(x) = 3x^2$ on the same grid. We will choose a few points on $f(x) = x^2$ and then multiply the y-values by 3 to get the points for $g(x) = 3x^2$.

	$f(x) = x^2$	$g(x) = 3x^2$	
x	$(x, f(x))$	$(x, g(x))$	
–2	(–2, 4)	(–2, 12)	$3 \cdot 4 = 12$
–1	(–1, 1)	(–1, 3)	$3 \cdot 1 = 3$
0	(0, 0)	(0, 0)	$3 \cdot 0 = 0$
1	(1, 1)	(1, 3)	$3 \cdot 1 = 3$
2	(2, 4)	(2, 12)	$3 \cdot 4 = 12$

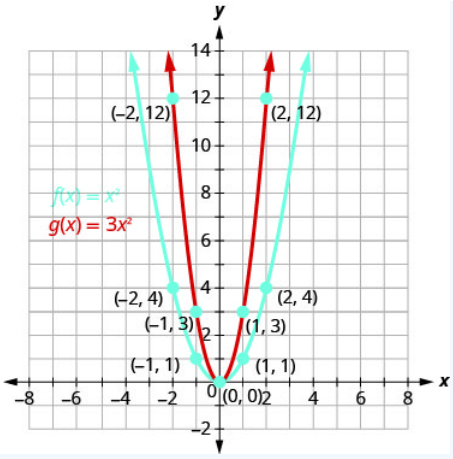


Figure 9.7.31

? Exercise 16.5.3.11

Graph $f(x) = -3x^2$.

Answer

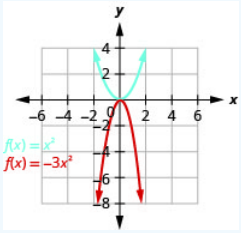


Figure 9.7.32

? Exercise 16.5.3.12

Graph $f(x) = 2x^2$.

Answer

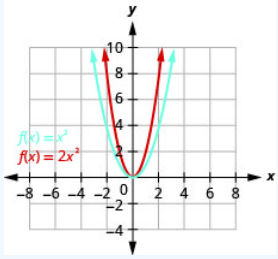


Figure 9.7.33

Graph Quadratic Functions Using Transformations

We have learned how the constants a, h, and k in the functions, $f(x) = x^2 + k, f(x) = (x - h)^2$, and $f(x) = ax^2$ affect their graphs. We can now put this together and graph quadratic functions $f(x) = ax^2 + bx + c$ by first putting them into the form $f(x) = a(x - h)^2 + k$ by completing the square. This form is sometimes known as the vertex form or standard form.

We must be careful to both add and subtract the number to the SAME side of the function to complete the square. We cannot add the number to both sides as we did when we completed the square with quadratic equations.

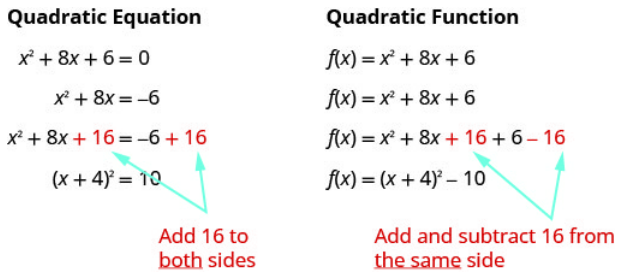


Figure 9.7.34

When we complete the square in a function with a coefficient of x^2 that is not one, we have to factor that coefficient from just the x-terms. We do not factor it from the constant term. It is often helpful to move the constant term a bit to the right to make it easier to focus only on the x-terms.

Once we get the constant we want to complete the square, we must remember to multiply it by that coefficient before we then subtract it.

Example 16.5.3.7

Rewrite $f(x) = -3x^2 - 6x - 1$ in the $f(x) = a(x-h)^2 + k$ form by completing the square.

Solution:

Table 9.7.3

	$f(x) = -3x^2 - 6x - 1$
Separate the x terms from the constant.	$f(x) = -3x^2 - 6x \quad - 1$
Factor the coefficient of x^2, -3.	$f(x) = -3(x^2 + 2x) \quad - 1$
Prepare to complete the square.	$f(x) = -3(x^2 + 2x \quad) \; - 1$
Take half of 2 and then square it to complete the square $(\frac{1}{2} \cdot 2)^2 = 1$	
The constant 1 completes the square in the parentheses, but the parentheses is multiplied by -3. So we are really adding -3. We must then add 3 to not change the value of the function.	$f(x) = -3(x^2 + 2x + 1) - 1 + 3$ $-3 \cdot 1 = -3$ so add 3
Rewrite the trinomial as a square and subtract the constants.	$f(x) = -3(x+1) + 2$
The function is now in the $f(x) = a(x-h)^2 + k$ form.	$f(x) = \quad a(x-h)^2 \quad + k$ $f(x) = -3(x+1)^2 + 2$

Exercise 16.5.3.13

Rewrite $f(x) = -4x^2 - 8x + 1$ in the $f(x) = a(x-h)^2 + k$ form by completing the square.

Answer

$f(x) = -4(x+1)^2 + 5$

Exercise 16.5.3.14

Rewrite $f(x) = 2x^2 - 8x + 3$ in the $f(x) = a(x-h)^2 + k$ form by completing the square.

Answer

$f(x) = 2(x-2)^2 - 5$

Once we put the function into the $f(x)=(x-h)^2+k$ form, we can then use the transformations as we did in the last few problems. The next example will show us how to do this.

Example 16.5.3.8

Graph $f(x)=x^2+6x+5$ by using transformations.

Solution:

Step 1: Rewrite the function in $f(x)=a(x-h)^2+k$ vertex form by completing the square.

Table 9.7.4

	$f(x)=x^2+6x+5$
Separate the x terms from the constant.	$f(x)=x^2+6x+5$
Take half of 6 and then square it to complete the square. $(\frac{1}{2}\cdot 6)^2=9$	
We both add 9 and subtract 9 to not change the value of the function.	$f(x)=x^2+6x+9+5-9$
Rewrite the trinomial as a square and subtract the constants.	$f(x)=(x+3)^2-4$
The function is now in the $f(x)=(x-h)^2+k$ form.	$f(x)=(x-h)^2+k$ $f(x)=(x+3)^2-4$

Step 2: Graph the function using transformations.

Looking at the h, k values, we see the graph will take the graph of $f(x)=x^2$ and shift it to the left 3 units and down 4 units.

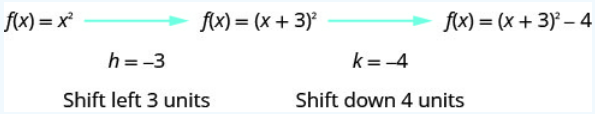


Figure 9.7.47

We first draw the graph of $f(x)=x^2$ on the grid.

To graph $f(x)=(x+3)^2$, shift the graph $f(x)=x^2$ to the left 3 units.

To graph $f(x)=(x+3)^2-4$, shift the graph $f(x)=(x+3)^2$ down 4 units.

Figure 9.7.48

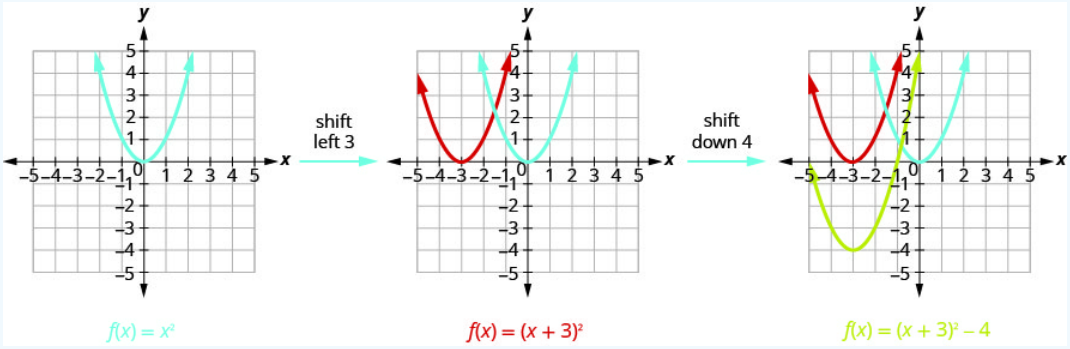


Figure 9.7.49

Exercise 16.5.3.15

Graph $f(x)=x^2+2x-3$ by using transformations.

Answer

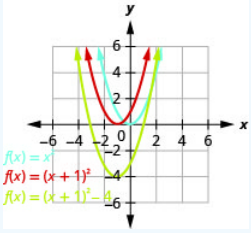


Figure 9.7.50

? Exercise 16.5.3.16

Graph $f(x) = x^2 - 8x + 12$ by using transformations.

Answer

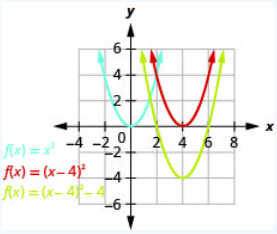


Figure 9.7.51

We list the steps to take a graph a quadratic function using transformations here.

Graph a Quadratic Function Using Transformations

1. Rewrite the function in $f(x) = a(x-h)^2 + k$ form by completing the square.
2. Graph the function using transformations.

✓ Example 16.5.3.9

Graph $f(x) = -2x^2 - 4x + 2$ by using transformations.

Solution:

Step 1: Rewrite the function in $f(x) = a(x-h)^2 + k$ vertex form by completing the square.

Table 9.7.5

	$f(x) = -2x^2 - 4x + 2$
Separate the x terms from the constant.	$f(x) = -2x^2 - 4x + 2$
We need the coefficient of x^2 to be one. We factor -2 from the x-terms.	$f(x) = -2(x^2 + 2x) + 2$
Take half of 2 and then square it to complete the square. $(\frac{1}{2} \cdot 2)^2 = 1$	
We add 1 to complete the square in the parentheses, but the parentheses is multiplied by -2. So we are really adding -2. To not change the value of the function we add 2.	$f(x) = -2(x^2 + 2x + 1) + 2 + 2$
Rewrite the trinomial as a square ad subtract the constants.	$f(x) = -2(x+1)^2 + 4$
The function is now in the $f(x) = a(x-h)^2 + k$ form.	$f(x) = a(x-h)^2 + k$ $f(x) = -2(x+1)^2 + 4$

Step 2: Graph the function using transformations.

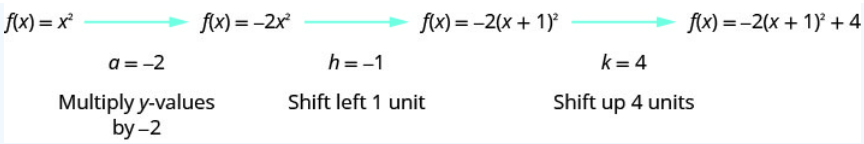


Figure 9.7.58

We first draw the graph of $f(x) = x^2$ on the grid.

To graph $f(x) = -2x^2$, multiply the y-values in parabola of $f(x) = x^2$ by -2.
To graph $f(x) = -2(x + 1)^2$, shift the graph $f(x) = -2x^2$ to the left 1 unit.
To graph $f(x) = -2(x + 1)^2 + 4$, shift the graph $f(x) = (x + 1)^2$ up 4 units.

Figure 9.7.59

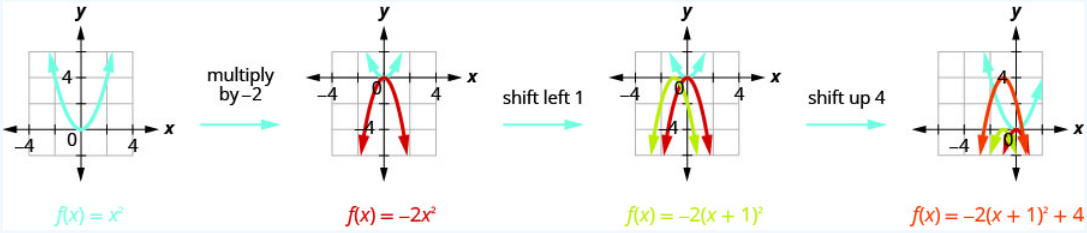


Figure 9.7.60

? Exercise 16.5.3.17

Graph $f(x) = -3x^2 + 12x - 4$ by using transformations.

Answer

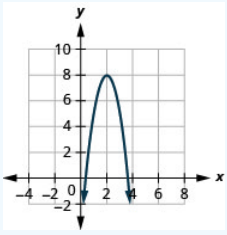


Figure 9.7.61

? Exercise 16.5.3.18

Graph $f(x) = -2x^2 + 12x - 9$ by using transformations.

Answer

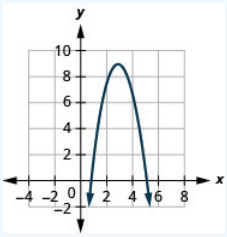


Figure 9.7.62

Now that we have completed the square to put a quadratic function into $f(x) = a(x - h)^2 + k$ form, we can also use this technique to graph the function using its properties as in the previous section.

If we look back at the last few examples, we see that the vertex is related to the constants h and k.

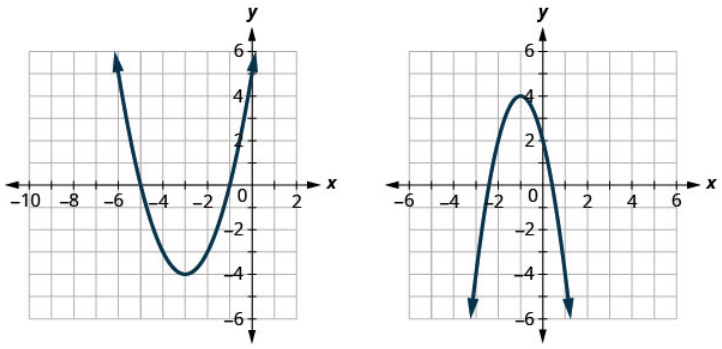


Figure 9.7.63

In each case, the vertex is (h, k). Also the **axis of symmetry** is the line $x = h$.

We rewrite our steps for graphing a quadratic function using properties for when the function is in $f(x) = a(x-h)^2 + k$ form.

Graph a Quadratic Function in the Form $f(x) = a(x-h)^2 + k$ Using Properties

1. Rewrite the function $f(x) = a(x-h)^2 + k$ form.
2. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$.
3. Find the axis of symmetry, $x = h$.
4. Find the vertex, $(h, k$.
5. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
6. Find the x-intercepts.
7. Graph the parabola.

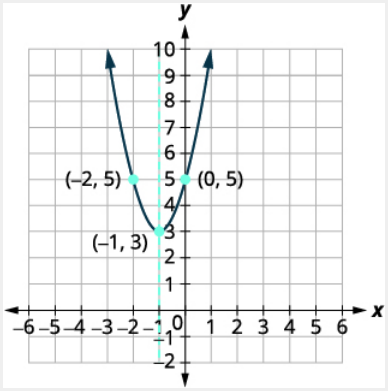
Example 16.5.3.10

a. Rewrite $f(x) = 2x^2 + 4x + 5$ in $f(x) = a(x-h)^2 + k$ form
b. Graph the function using properties

Solution:

Table 9.7.6

Rewrite the function in $f(x) = a(x-h)^2 + k$ form by completing the square.	$f(x) = 2x^2 + 4x + 5$
	$f(x) = 2\left(x^2 + 2x\right) + 5$
	$f(x) = 2\left(x^2 + 2x + 1\right) + 5 - 2$
	$f(x) = 2(x+1)^2 + 3$
Identify the constants a, h, k.	$a = 2 h = -1 k = 3$
Since $a = 2$, the parabola opens upward.	
The axis of symmetry is $x = h$.	The axis of symmetry is $x = -1$.
The vertex is (h, k).	The vertex is $(-1, 3)$.
Find the y-intercept by finding $f(0)$.	$f(0) = 2 \cdot 0^2 + 4 \cdot 0 + 5$
	$f(0) = 5$
	y-intercept $(0, 5)$

Find the point symmetric to $(0, 5)$ across the axis of symmetry.	$(-2, 5)$
Find the x-intercepts.	The discriminant is negative, so there are no x-intercepts. Graph the parabola.
	

Exercise 16.5.3.19

a. Rewrite $f(x) = 3x^2 - 6x + 5$ in $f(x) = a(x - h)^2 + k$ form
b. Graph the function using properties

Answer

a. $f(x) = 3(x - 1)^2 + 2$
b.

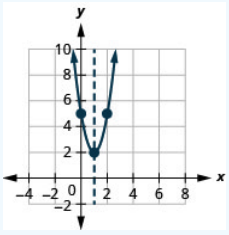


Figure 9.7.66

Exercise 16.5.3.20

a. Rewrite $f(x) = -2x^2 + 8x - 7$ in $f(x) = a(x - h)^2 + k$ form
b. Graph the function using properties

Answer

a. $f(x) = -2(x - 2)^2 + 1$
b.

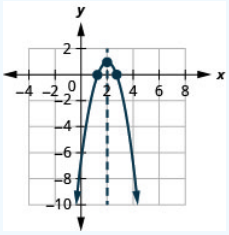


Figure 9.7.67

Find a Quadratic Function from its Graph

So far we have started with a function and then found its graph.

Now we are going to reverse the process. Starting with the graph, we will find the function.

Example 16.5.3.11

Determine the quadratic function whose graph is shown.

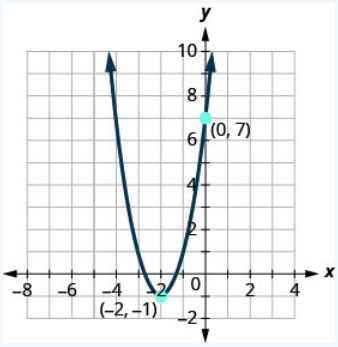


Figure 9.7.68

Solution:

Since it is quadratic, we start with the $f(x)=a(x-h)^2+k$ form.

The vertex, (h,k), is $(-2,-1)$ so $h=-2$ and $k=-1$.

$$f(x)=a(x-(-2))^2-1$$

To find a, we use the y-intercept, $(0,7)$.

So $f(0)=7$.

$$7=a(0+2)^2-1$$

Solve for a.

$$\begin{aligned} 7&=4a-1\\ 8&=4a\\ 2&=a \end{aligned}$$

Write the function.

$$f(x)=a(x-h)^2+k$$

Substitute in $h=-2, k=-1$ and $a=2$.

$$f(x)=2(x+2)^2-1$$

Exercise 16.5.3.21

Write the quadratic function in $f(x)=a(x-h)^2+k$ form whose graph is shown.

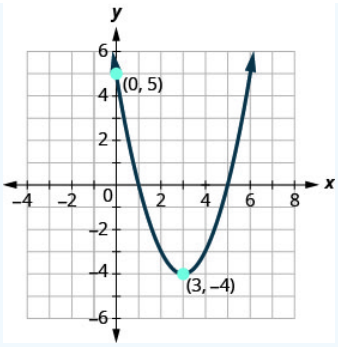


Figure 9.7.69

Answer

$f(x) = (x-3)^2 - 4$

? Exercise 16.5.3.22

Determine the quadratic function whose graph is shown.

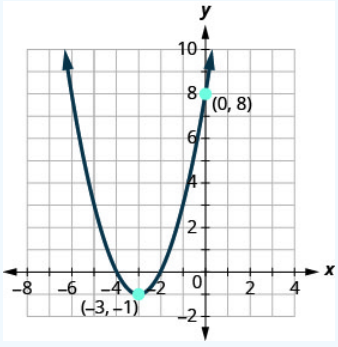


Figure 9.7.70

Answer

$f(x) = (x+3)^2 - 1$

Access these online resources for additional instruction and practice with graphing quadratic functions using transformations.

- Function Shift Rules Applied to Quadratic Functions
- Changing a Quadratic from Standard Form to Vertex Form
- Using Transformations to Graph Quadratic Functions
- Finding Quadratic Equation in Vertex Form from Graph

Key Concepts

- Graph a Quadratic Function of the form $f(x) = x^2 + k$ Using a Vertical Shift
  - The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.
    - If $k > 0$, shift the parabola vertically up k units.
    - If $k < 0$, shift the parabola vertically down $|k|$ units.
- Graph a Quadratic Function of the form $f(x) = (x-h)^2$ Using a Horizontal Shift
  - The graph of $f(x) = (x-h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.
    - If $h > 0$, shift the parabola horizontally left h units.
    - If $h < 0$, shift the parabola horizontally right $|h|$ units.
- Graph of a Quadratic Function of the form $f(x) = ax^2$

- The coefficient a in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.
  If $0 < |a| < 1$, then the graph of $f(x) = ax^2$ will be "wider" than the graph of $f(x) = x^2$.
  If $|a| > 1$, then the graph of $f(x) = ax^2$ will be "skinnier" than the graph of $f(x) = x^2$.

- How to graph a quadratic function using transformations
  1. Rewrite the function in $f(x) = a(x-h)^2 + k$ form by completing the square.
  2. Graph the function using transformations.

- Graph a quadratic function in the vertex form $f(x) = a(x-h)^2 + k$ using properties
  1. Rewrite the function in $f(x) = a(x-h)^2 + k$ form.
  2. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$.
  3. Find the axis of symmetry, $x = h$.
  4. Find the vertex, (h, k).
  5. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
  6. Find the x-intercepts, if possible.
  7. Graph the parabola.

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16.5.3E: Exercises

Practice Makes Perfect

Exercise 16.5.3E. 23 Graph Quadratic Functions of the Form $f(x) = x^2 = k$

In the following exercises,

a. Graph the quadratic functions on the same rectangular coordinate system
b. Describe what effect adding a constant, k, to the function has on the basic parabola.

1. $f(x) = x^2, g(x) = x^2 + 4, \text{ and } h(x) = x^2 - 4$
2. $f(x) = x^2, g(x) = x^2 + 7, \text{ and } h(x) = x^2 - 7$

Answer

1.

a.

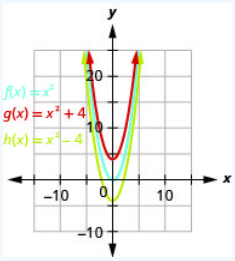

Figure 9.7.71

b. The graph of $g(x) = x^2 + 4$ is the same as the graph of $f(x) = x^2$ but shifted up 4 units. The graph of $h(x) = x^2 - 4$ is the same as the graph of $f(x) = x^2$ but shift down 4 units.

Exercise 16.5.3E. 24 Graph Quadratic Functions of the Form $f(x) = x^2 = k$

In the following exercises, graph each function using a vertical shift.

1. $f(x) = x^2 + 3$
2. $f(x) = x^2 - 7$
3. $g(x) = x^2 + 2$
4. $g(x) = x^2 + 5$
5. $h(x) = x^2 - 4$
6. $h(x) = x^2 - 5$

Answer

1.

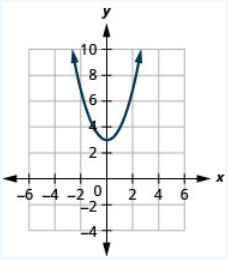

Figure 9.7.72

3.

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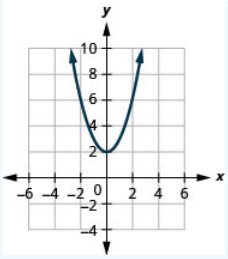

Figure 9.7.73

5.

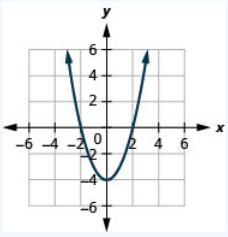

Figure 9.7.74

Exercise $16.5.3E.25$ Graph Quadratic Functions of the Form $f(x) = (x - h)^2$

In the following exercises,

a. Graph the quadratic functions on the same rectangular coordinate system
b. Describe what effect adding a constant, h, inside the parentheses has

1. $f(x) = x^2, g(x) = (x-3)^2,$ and $h(x) = (x+3)^2$
2. $f(x) = x^2, g(x) = (x+4)^2,$ and $h(x) = (x-4)^2$

Answer

1.

a.

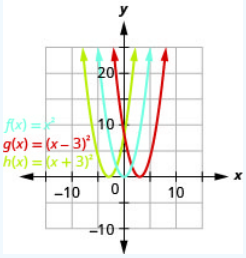

Figure 9.7.75

b. The graph of $g(x) = (x-3)^2$ is the same as the graph of $f(x) = x^2$ but shifted right 3 units. The graph of $h(x) = (x+3)^2$ is the same as the graph of $f(x) = x^2$ but shifted left 3 units.

Exercise $16.5.3E.26$ Graph Quadratic Functions of the Form $f(x) = (x - h)^2$

In the following exercises, graph each function using a horizontal shift.

1. $f(x) = (x-2)^2$
2. $f(x) = (x-1)^2$
3. $f(x) = (x+5)^2$
4. $f(x) = (x+3)^2$
5. $f(x) = (x-5)^2$

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6. $f(x) = (x+2)^2$

Answer

1.

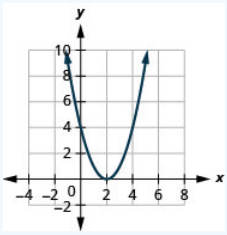

Figure 9.7.76

3.

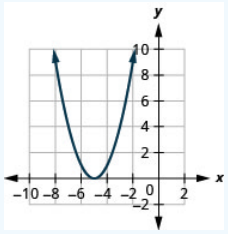

Figure 9.7.77

5.

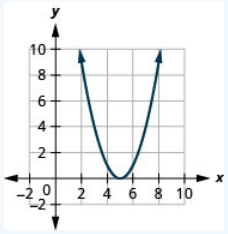

Figure 9.7.78

Exercise $16.5.3E.27$ Graph Quadratic Functions of the Form $f(x) = (x-h)^2$

In the following exercises, graph each function using transformations.

1. $f(x) = (x+2)^2 + 1$
2. $f(x) = (x+4)^2 + 2$
3. $f(x) = (x-1)^2 + 5$
4. $f(x) = (x-3)^2 + 4$
5. $f(x) = (x+3)^2 - 1$
6. $f(x) = (x+5)^2 - 2$
7. $f(x) = (x-4)^2 - 3$
8. $f(x) = (x-6)^2 - 2$

Answer

1.

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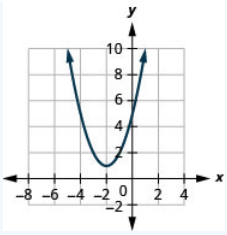

Figure 9.7.79

3.

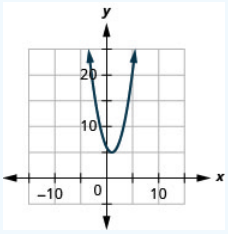

Figure 9.7.80

5.

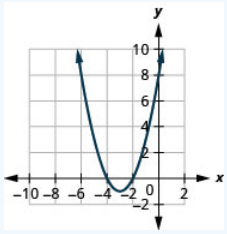

Figure 9.7.81

7.

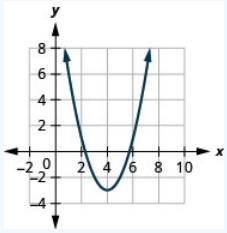

Figure 9.7.82

Exercise $16.5.3E.28$ Graph Quadratic Functions of the Form $f(x) = ax^2$

In the following exercises, graph each function.

1. $f(x) = -2x^2$
2. $f(x) = 4x^2$
3. $f(x) = -4x^2$
4. $f(x) = -x^2$
5. $f(x) = \frac{1}{2}x^2$
6. $f(x) = \frac{1}{3}x^2$
7. $f(x) = \frac{1}{4}x^2$
8. $f(x) = -\frac{1}{2}x^2$

Answer

1.

https://math.libretexts.org/@go/page/46415

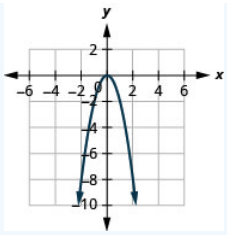

Figure 9.7.83

3.

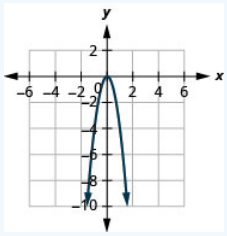

Figure 9.7.84

5.

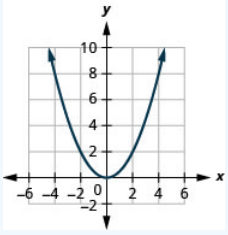

Figure 9.7.85

7.

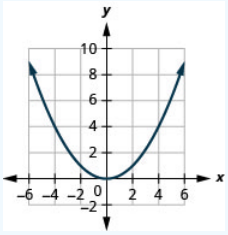

Figure 9.7.86

Exercise $16.5.3E.29$ Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x)=a(x-h)^2+k$ form by completing the square.

1. $f(x)=-3x^2-12x-5$
2. $f(x)=2x^2-12x+7$
3. $f(x)=3x^2+6x-1$
4. $f(x)=-4x^2-16x-9$

Answer

1. $f(x)=-3(x+2)^2+7$
3. $f(x)=3(x+1)^2-4$

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Exercise $16.5.3E.30$ Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x)=a(x-h)^2+k$ form
b. Graph it by using transformations

1. $f(x)=x^2+6x+5$
2. $(x)=x^2+4x-12$
3. $f(x)=x^2+4x-12$
4. $f(x)=x^2-6x+8$
5. $f(x)=x^2-6x+15$
6. $f(x)=x^2+8x+10$
7. $f(x)=-x^2+8x-16$
8. $f(x)=-x^2+2x-7$
9. $f(x)=-x^2-4x+2$
10. $f(x)=-x^2+4x-5$
11. $f(x)=5x^2-10x+8$
12. $f(x)=3x^2+18x+20$
13. $f(x)=2x^2-4x+1$
14. $f(x)=3x^2-6x-1$
15. $f(x)=-2x^2+8x-10$
16. $f(x)=-3x^2+6x+1$

Answer

1.

a. f(x)=(x+3)^{2}-4
b.

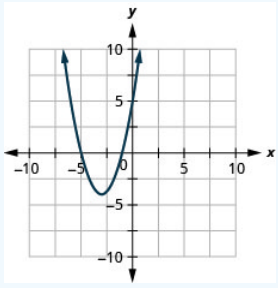

Figure 9.7.87

3.

a. $f(x)=(x+2)^2-1$
b.

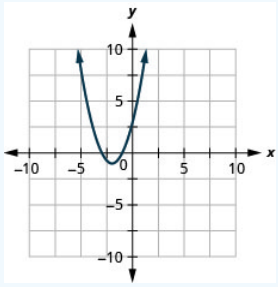

Figure 9.7.88

5.

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a. $f(x) = (x-3)^2 + 6$

b.

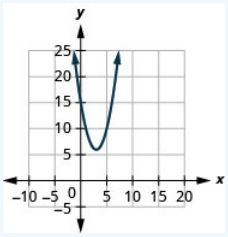

Figure 9.7.89

7.

a. $f(x) = -(x-4)^2 + 0$

b.

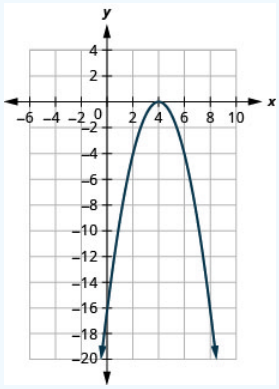

Figure 9.7.90

9.

a. $f(x) = -(x+2)^2 + 6$

b.

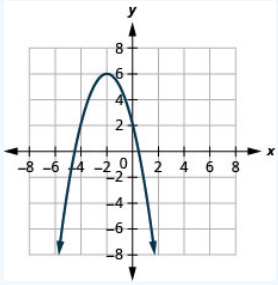

Figure 9.7.91

11.

a. $f(x) = 5(x-1)^2 + 3$

b.

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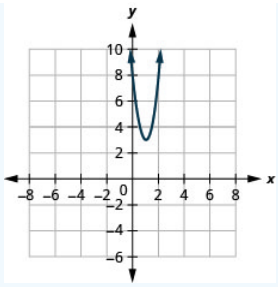

Figure 9.7.92

13.

a. $f(x) = 2(x-1)^2 - 1$

b.

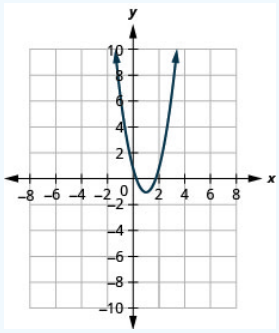

Figure 9.7.93

15.

a. $f(x) = -2(x-2)^2 - 2$

b.

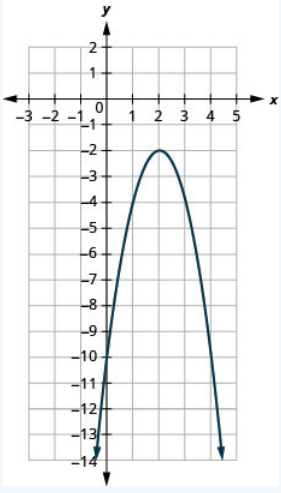

Figure 9.7.94

Exercise $16.5.3E.31$ Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x) = a(x-h)^2 + k$ form

b. Graph it using properties

1. $f(x) = 2x^2 + 4x + 6$
2. $f(x) = 3x^2 - 12x + 7$

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3. $f(x) = -x^2 + 2x - 4$
4. $f(x) = -2x^2 - 4x - 5$

Answer

1.

a. $f(x) = 2(x+1)^2 + 4$

b.

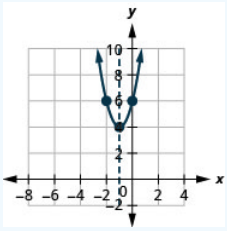

Figure 9.7.95

3.

a. $f(x) = -(x-1)^2 - 3$

b.

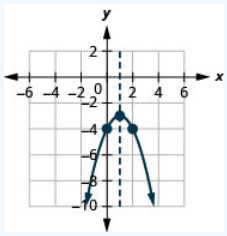

Figure 9.7.96

Exercise $16.5.3E.32$ Matching

In the following exercises, match the graphs to one of the following functions:

a. $f(x) = x^2 + 4$
b. $f(x) = x^2 - 4$
c. $f(x) = (x+4)^2$
d. $f(x) = (x-4)^2$
e. $f(x) = (x+4)^2 - 4$
f. $f(x) = (x+4)^2 + 4$
g. $f(x) = (x-4)^2 - 4$
h. $f(x) = (x-4)^2 + 4$

1.

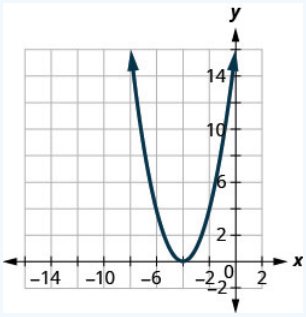

Figure 9.7.97

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2.

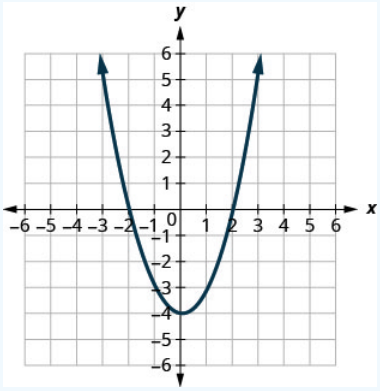

Figure 9.7.98

3.

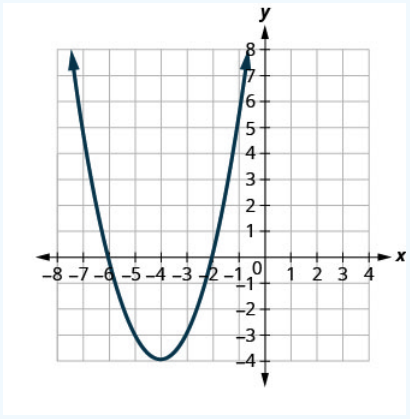

Figure 9.7.99

4.

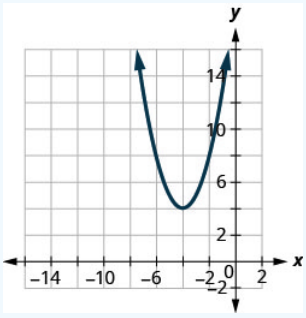

Figure 9.7.100

5.

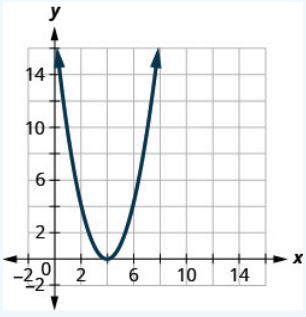

Figure 9.7.101

6.

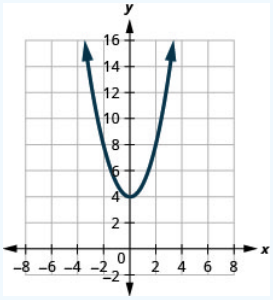

Figure 9.7.102

7.

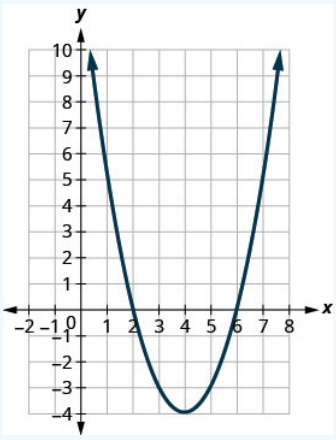

Figure 9.7.103

8.

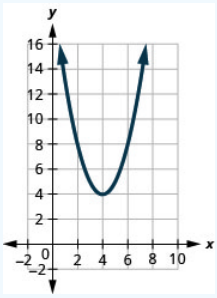

Figure 9.7.104

Answer

1. c
3. e
5. d
7. g

Exercise $16.5.3E.33$ Find a Quadratic Function from its Graph

In the following exercises, write the quadratic function in $f(x) = a(x-h)^2 + k$ form whose graph is shown.

1.

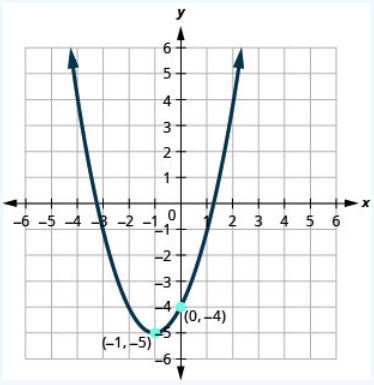

Figure 9.7.105

2.

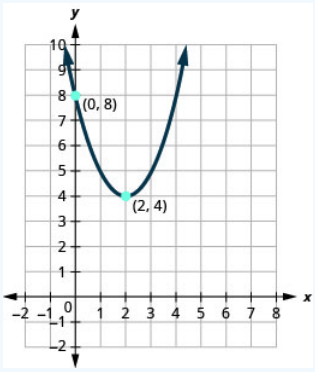

Figure 9.7.106

3.

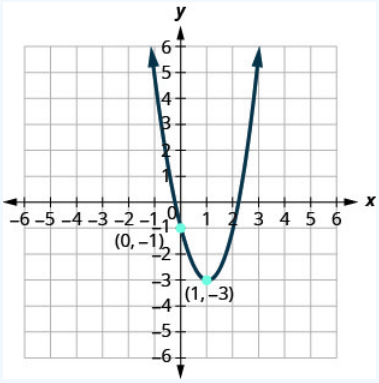

Figure 9.7.107

4.

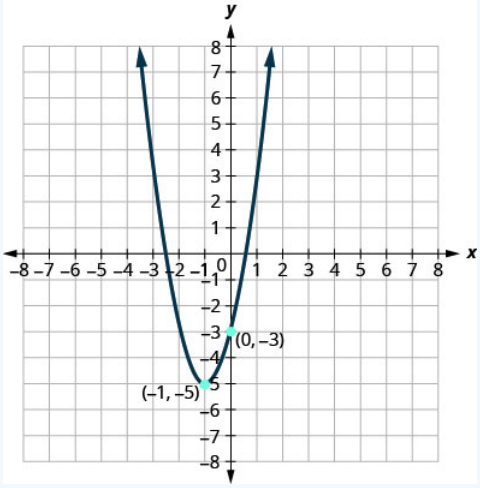

Figure 9.7.108

Answer

1. $f(x) = (x+1)^2 - 5$
3. $f(x) = 2(x-1)^2 - 3$

? Exercise $16.5.3E.34$ Writing Exercise

1. Graph the quadratic function $f(x) = x^2 + 4x + 5$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?
2. Graph the quadratic function $f(x) = 2x^2 - 4x - 3$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?

Answer

1. Answers may vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph Quadratic Functions of the form $f(x) = x^2 + k$.			
graph Quadratic Functions of the form $f(x) = (x - h)^2$.			
graph Quadratic Functions of the form $f(x) = ax^2$.			
graph Quadratic Functions Using Transformations.			
find a Quadratic Function from its Graph.			

Figure 9.7.109

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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16.5E: Exercises

2.1 Exercises

In Exercises $16.5E.1$-$16.5E.6$, state the domain and range of the given relation.

Exercise $16.5E.1$

R = {(1, 3),(2, 4),(3, 4)}

Answer

The domain is the set of all first coordinates = {1, 2, 3}. The range is the set of all second coordinates {3, 4} (note that in a set you do not list an object twice, so we only list 4 once).

Exercise $16.5E.2$

R = {(1, 3),(2, 4),(2, 5)}

Exercise $16.5E.3$

R = {(1, 4),(2, 5),(2, 6)}

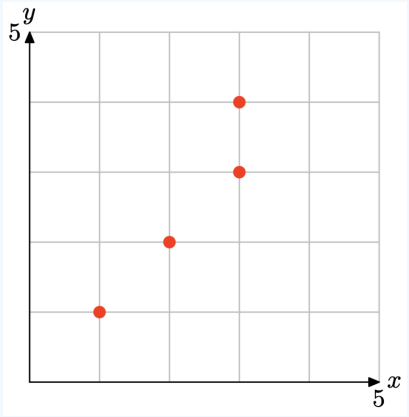
Answer

The domain is the set of all first coordinates = {1, 2} (note that in a set you do not list an object twice, so we only list 2 once). The range is the set of all second coordinates {4, 5, 6}.

Exercise $16.5E.4$

R = {(1, 5),(2, 4),(3, 6)}

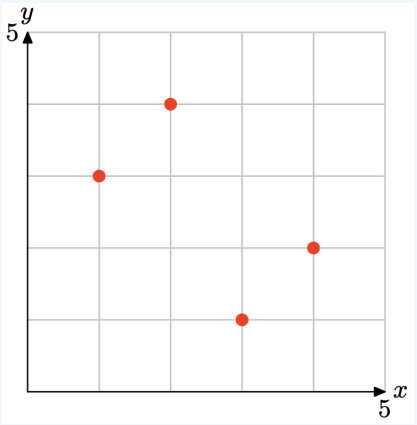
Exercise $16.5E.5$



Answer

Read off the x-coordinate of each point to get that the domain is {1, 2, 3}. Then read off the y-coordinates to get that the range is {1, 2, 3, 4}

Exercise 16.5E.6



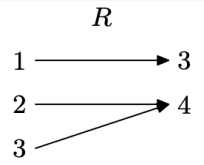
In Exercises 16.5E.7-16.5E.12, create a mapping diagram for the given relation and state whether or not it is a function.

Exercise 16.5E.7

The relation in Exercise 16.5E.1.

Answer

Create a mapping diagram for R.



Since no domain value is paired with two range values, this is a function (each x maps to a single y). Note that having two different domain values go to a single range value (2 and 3 both map to 4) is permissible for a function.

Exercise 16.5E.8

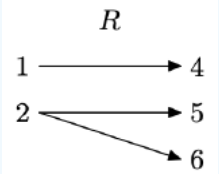
The relation in Exercise 16.5E.2.

Exercise 16.5E.9

The relation in Exercise 16.5E.3.

Answer

Create a mapping diagram for R.



The number 2 is mapped to two different range values (one x maps to two y's), so this is not a function.

Exercise 16.5E.10

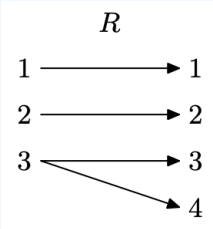
The relation in Exercise 16.5E.4.

Exercise 16.5E.11

The relation in Exercise 16.5E.5.

Answer

Create a mapping diagram for R.



The number 3 is mapped to two different range values (one x maps to two y's), so this is not a function.

Exercise 16.5E.12

The relation in Exercise 16.5E.6.

Exercise 16.5E.13

Given that g takes a real number and doubles it, then $g: x \rightarrow ?$.

Answer

Doubles means 'multiplies by 2,' so $g: x \rightarrow 2x$.

Exercise 16.5E.14

Given that f takes a real number and divides it by 3, then $f: x \rightarrow ?$.

Exercise 16.5E.15

Given that g takes a real number and adds 3 to it, then $g: x \rightarrow ?$.

Answer

$g: x \rightarrow x+3$

Exercise 16.5E.16

Given that h takes a real number and subtracts 4 from it, then $h: x \rightarrow ?$.

Exercise 16.5E.17

Given that g takes a real number, doubles it, then adds 5, then $g: x \rightarrow ?$

Answer

For an x put into g, g doubles it, giving 2x, and then adds five, resulting in 2x + 5. Therefore, $g: x \rightarrow 2x+5$

Exercise 16.5E.18

Given that h takes a real number, subtracts 3 from it, then divides the result by 4, then $h: x \rightarrow ?$

Given that the function f is defined by the rule $h: x \rightarrow 3x-5$, determine where the input number is mapped in Exercises 16.5E.19-16.5E.22.

Exercise 16.5E.19

$f: 3 \rightarrow ?$

Answer

Put 3 into f. This means, replace x with 3 and compute the output. $f: 3 \rightarrow 3(3)-5=4$, so $f: 3 \rightarrow 4$.

Exercise 16.5E.20

$f: -5 \rightarrow ?$

Exercise 16.5E.21

$f: a \rightarrow ?$

Answer

Put a into f, just like you would a number. This means, replace x with a and compute the output. $f: a \rightarrow 3(a)-5=3a-5$, so $f: 3 \rightarrow 3a-5$.

Exercise 16.5E.22

$f: 2a+3 \rightarrow ?$

Given that the function f is defined by the rule $f: x \rightarrow 4-5x$, determine where the input number is mapped in Exercises 16.5E.23-16.5E.26.

Exercise 16.5E.23

$f: 2 \rightarrow ?$

Answer

Put 2 into f by replacing x with it. $f: 2 \rightarrow 4-5(2)=-6$, so $f: 2 \rightarrow -6$.

Exercise 16.5E.24

$f: -3 \rightarrow ?$

Exercise 16.5E.25

$f: a \rightarrow ?$

Answer

Put a into f by replacing x with it, just as you would with a number. $f: a \rightarrow 4-5(a)$, so $f: 2 \rightarrow 4-5a$.

Exercise 16.5E.26

$f: 2a+11 \rightarrow ?$

Given that the function f is defined by the rule $f: x \rightarrow x^2 - 4x - 6$, determine where the input number is mapped in Exercises 16.5E.27-16.5E.30.

Exercise 16.5E.27

$f: 1 \rightarrow ?$

Answer

Put 1 into f by replacing x with it. $f: 1 \rightarrow (1)2 - 4(1) - 6 = 1 - 4 - 6 = -9$, so $f: 1 \rightarrow -9$.

Exercise 16.5E.28

$f: -2 \rightarrow ?$

Exercise 16.5E.29

$f: -1 \rightarrow ?$

Answer

Put −1 into f by replacing x with it. $f: -1 \rightarrow (-1)2 - 4(-1) - 6 = 1 + 4 - 6 = -1$, so $f: 1 \rightarrow -1$.

Exercise 16.5E.30

$f: a \rightarrow ?$

Given that the function f is defined by the rule $f: x \rightarrow 3x - 9$, determine where the input number is mapped in Exercises 16.5E.31-16.5E.34.

Exercise 16.5E.31

$f: a \rightarrow ?$

Answer

Put a into f by replacing x with it, just as you would with a number. $f: a \rightarrow 3a - 9$.

Exercise 16.5E.32

$f: a+1 \rightarrow ?$

Exercise 16.5E.33

$f: 2a - 5 \rightarrow ?$

Answer

Put 2a − 5 into f by replacing x with it, just as you would with a number. We get $f: 2a - 5 \rightarrow 3(2a - 5) - 9 = 6a - 15 - 9 = 6a - 24$, so $f: 2a - 5 \rightarrow 6a - 24$

Exercise 16.5E.34

$f: a+h \rightarrow ?$

Given that the functions f and g are defined by the rules $f: x \rightarrow 2x+3$ and $g: x \rightarrow 4-x$, determine where the input number is mapped in Exercises 16.5E.35- 16.5E.38.

Exercise 16.5E.35

$f: 2 \rightarrow ?$

Answer

Put 2 into f by replacing x with it. We get $f: 2 \rightarrow 2(2)+3=7$, so $f: 2 \rightarrow 7$.

Exercise 16.5E.36

$f: 2 \rightarrow ?$

Exercise 16.5E.37

$f: a+1 \rightarrow ?$

Answer

Put a + 1 into f by replacing x with it, just as you would with a number. We get $f: a+1 \rightarrow 2(a+1)+3=2a+2+3=2a+5$, so $f: a+1 \rightarrow 2a+5$

Exercise 16.5E.38

$f: a-3 \rightarrow ?$

Exercise 16.5E.39

Given that g takes a real number and triples it, then g(x) = ?.

Answer

Triples means 'multiplies by 3,' so g(x) = 3x

Exercise 16.5E.40

Given that f takes a real number and divides it by 5, then f(x) = ?.

Exercise 16.5E.41

Given that g takes a real number and subtracts it from 10, then g(x) = ?.

Answer

g takes an input x and subtracts it FROM 10, so g(x) = 10 − x.

Exercise 16.5E.42

Given that f takes a real number, multiplies it by 5 and then adds 4 to the result, then f(x) = ?.

Exercise 16.5E.43

Given that f takes a real number, doubles it, then subtracts the result from 11, then f(x) = ?.

Answer

f takes an input x, doubles it to get 2x, and takes this away FROM 11, getting 11 − 2x. Therefore, f(x) = 11 − 2x.

Exercise 16.5E.44

Given that h takes a real number, doubles it, adds 5, then takes the square root of the result, then h(x) = ?.

In Exercises 16.5E.45-16.5E.54, evaluate the given function at the given value b.

Exercise 16.5E.45

f(x) = 12x + 2 for b = 6.

Answer

Substitute 6 for x in 12x + 2 and simplify to get 74: f(6) = 12(6) + 2 = 74.

Exercise 16.5E.46

f(x) = −11x − 4 for b = −3.

Exercise 16.5E.47

f(x) = −9x − 1 for b = −5.

Answer

Substitute −5 for x in −9x−1 and simplify to get 44: f(−5) = −9(−5)−1 = 44.

Exercise 16.5E.48

f(x) = 11x + 4 for b = −4.

Exercise 16.5E.49

f(x) = 4 for b = −12.

Answer

f is a constant function, so f(x) = 4 for all x. Therefore, f(−12) = 4.

Exercise 16.5E.50

f(x) = 7 for b = −7.

Exercise 16.5E.51

f(x) = 0 for b = −7.

Answer

f is a constant function, so f(x) = 0 for all x. Therefore, f(−7) = 0.

Exercise 16.5E.52

f(x) = 12x + 8 for b = −3.

Exercise 16.5E.53

f(x) = −9x + 3 for b = −1.

Answer

Substitute −1 for x in −9x+3 and simplify to get 12: f(−1) = −9(−1)+3 = 12

Exercise 16.5E.54

f(x) = 6x − 3 for b = 3.

In Exercises 16.5E.55-16.5E.58, given that the function f is defined by the rule f(x) = 2x+ 7, determine where the input number is mapped.

Exercise 16.5E.55

f(a) = ?

Answer

Put a into f by replacing x with it, just as you would with a number. This yields f(a) = 2a + 7.

Exercise 16.5E.56

f(a + 1) = ?

Exercise 16.5E.57

f(3a − 2) = ?

Answer

Put 3a − 2 into f by replacing x with it, just as you would with a number. This yields f(3a − 2) = 2(3a − 2) + 7 = 6a − 4 + 7 = 6a + 3.

Exercise 16.5E.58

f(a + h) = ?

In Exercises 16.5E.59-16.5E.62, given that the function g is defined by the rule g(x) = 3 − 2x, determine where the input number is mapped.

Exercise 16.5E.59

g(a) = ?

Answer

Put a into g by replacing x with it, just as you would with a number. This yields g(a) = 3 − 2a.

Exercise 16.5E.60

g(a + 3) = ?

Exercise 16.5E.61

g(2 − 5a) = ?

Answer

Put 2 − 5a into g by replacing x with it, just as you would with a number. This yields g(2 − 5a) = 3 − 2(2 − 5a) = 3 − 4 + 10a = −1 + 10a or 10a − 1.

Exercise 16.5E.62

g(a + h) = ?

Given that the functions f and g are defined by the rules f(x) = 1 − x and g(x) = 2x + 13, determine where the input number is mapped in Exercises 16.5E.63- 16.5E.66.

Exercise 16.5E.63

f(a) = ?

Answer

Put a into f by replacing x with it, just as you would with a number. This yields f(a) = 1 − a.

Exercise 16.5E.64

g(a) = ?

Exercise 16.5E.65

f(a + 3) = ?

Answer

Put a + 3 into f by replacing x with it, just as you would with a number. This yields f(a + 3) = 1 − (a + 3) = 1 − a − 3 = −a − 2.

Exercise 16.5E.66

g(4 − a) = ?

Given that the functions f and g are defined by the rules f(x) = 3x + 4 and g(x) = 2x−5, determine where the input number is mapped in Exercises 16.5E.67-16.5E.70.

Exercise 16.5E.67

f(g(2)) = ?

Answer

First compute g(2) = 2(2) − 5 = −1. This means that f(g(2)) is really f(−1). Plugging −1 in for x into the function f, we get f(g(2)) = f(−1) = 3(−1) + 4 = −3 + 4 = 1.

Exercise 16.5E.68

g(f(2)) = ?

Exercise 16.5E.69

f(g(a)) = ?

Answer

First compute g(a) = 2a−5. This means that f(g(a)) is really f(2a−5). Plugging 2a − 5 in for x into the function f, we get f(g(a)) = f(2a − 5) = 3(2a − 5) + 4 = 6a − 15 + 4 = 6a − 11.

Exercise 16.5E.70

g(f(a)) = ?

Given that the functions f and g are defined by the rules f(x) = 2x − 9 and g(x) = 11, determine where the input number is mapped in Exercises 16.5E.71-16.5E.74.

Exercise 16.5E.71

f(g(2)) = ?

Answer

First compute g(2) = 11 (note that, no matter what you put into g, it outputs 11). This means that f(g(2)) is really f(11). Plugging 11 in for x into the function f, we get f(g(2)) = f(11) = 2(11) − 9 = 22 − 9 = 13.

Exercise 16.5E.72

g(f(2)) = ?

Exercise 16.5E.73

f(g(a)) = ?

Answer

First compute g(a) = 11 (note that, no matter what you put into g, it outputs 11). This means that f(g(a)) is really f(11). Plugging 11 in for x into the function f, we get f(g(2)) = f(11) = 2(11) − 9 = 22 − 9 = 13.

Exercise 16.5E.74

g(f(a)) = ?

Use set-builder notation to describe the domain of each of the functions defined in Exercises 16.5E.75-16.5E.78.

Exercise 16.5E.75

$$f(x) = \frac{93}{x+98}$$

Answer

An input of x = −98 would cause division by zero, so −98 is not in the domain. All other possible inputs are valid. The domain, in set-builder notation, is $\{x : x \neq -98\}$.

Exercise 16.5E.76

$$f(x) = \frac{54}{x+65}$$

Exercise 16.5E.77

$$f(x) = -\frac{87}{x-88}$$

Answer

An input of x = 88 would cause division by zero, so 88 is not in the domain. All other possible inputs are valid. The domain, in set-builder notation, is $\{x : x \neq 88\}$..

Exercise 16.5E.78

$$f(x) = -\frac{30}{x-52}$$

Use set-builder and interval notation to describe the domain of the functions defined in Exercises 16.5E.79-16.5E.82

Exercise 16.5E.79

$$f(x) = \sqrt{x+69}$$

Answer

The square root of a negative number is not defined as a real number. Thus, x + 69 must be greater than or equal to zero. Then $x+69geq0$ implies that $x \geq -69$, so the domain is the interval $[-69, \infty)$, or in set-builder notation, $\{x : x \geq -69\}$.

Exercise 16.5E.80

$$f(x) = \sqrt{x+62}$$

Exercise 16.5E.81

$$f(x) = \sqrt{x-81}$$

Answer

The square root of a negative number is not defined as a real number. Thus, x − 81 must be greater than or equal to zero. Then $x - 81 \geq 0$ implies that $x \geq 81$, so the domain is the interval $[81, \infty)$, or in set-builder notation, $\{x : x \geq 81\}$.

Exercise 16.5E.82

$f(x) = \sqrt{x - 98}$

Two integers are said to be relatively prime if their greatest common divisor is 1. For example, the greatest common divisor of 6 and 35 is 1, so 6 and 35 are relatively prime. On the other hand, the greatest common divisor of 14 and 21 is not 1 (it is 7), so 14 and 21 are not relatively prime. The Euler ϕ-function is defined as follows:

• If n = 1, then $\phi(n) = 1$.

• If n > 1, then $\phi(n)$ is the number of positive integers less than n that are relatively prime to n. In Exercises $16.5E.83$- $16.5E.84$, evaluate the Euler ϕ-function at the given input.

Exercise 16.5E.83

$\phi(12)$

Answer

1, 5, 7, and 11 are less than 12 and are each relatively prime to 12. Therefore, $\phi(12) = 4$.

Exercise 16.5E.84

$\phi(36)$

2.2 Exercises

Perform each of the following tasks for the functions defined by the equations in Exercises $16.5E.1$-$16.5E.8$.

i. Set up a table of points that satisfy the given equation. Please place this table of points next to your graph on your graph paper.

ii. Set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot each of the points from your table on your coordinate system.

iii. If you are confident that you "see" the shape of the graph, make a "leap of faith" and plot all pairs that satisfy the given equation by drawing a smooth curve (free-hand) on your coordinate system that contains all previously plotted points (use a ruler only if the graph of the equation is a line). If you are not confident that you "see" the shape of the graph, then add more points to your table, plot them on your coordinate system, and see if this helps. Continue this process until you "see" the shape of the graph and can fill in the rest of the points that satisfy the equation by drawing a smooth curve (or line) on your coordinate system.

Exercise 16.5E.1

$f(x) = 2x + 1$

Answer

Evaluate the function $f(x) = 2x + 1$ at −2, −1, 0, and 1.

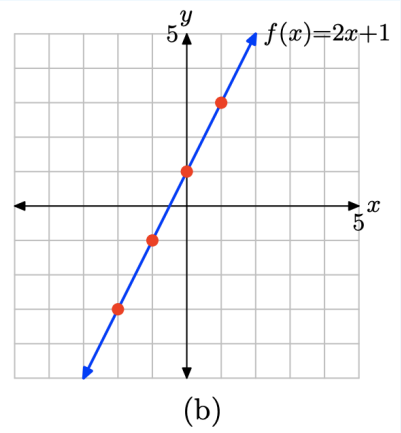
$$\begin{aligned} f(-2) &= 2(-2)+1 &&= -3 \\ f(-1) &= 2(-1)+1 &&= -1 \\ f(0) &= 2(0)+1 &&= 1 \\ f(1) &= 2(1)+1 &&= 3 \end{aligned}$$

Place these results in table (a) and plot them as shown in (b). There is enough evidence here to intuit that the graph of f is the line shown in (b).

x	$f(x) = 2x + 1$	(x, f(x))
-2	-3	(−2, −3)
-1	-1	(−1, −1)

x	$f(x)=2x+1$	(x, f(x))
0	1	(0,1)
1	3	(1,3)

(a)



(b)

? Exercise $16.5E.2$

$f(x)=1-x$

? Exercise $16.5E.3$

$f(x)=3-\frac{1}{2}x$

Answer

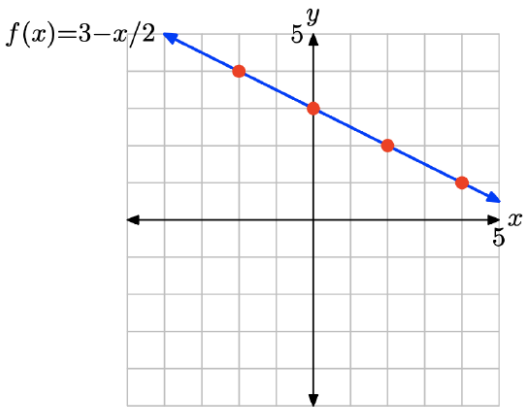
Evaluate the function f(x) = 3 − (1/2)x at x = −2, 0, 2, and 4.

$$f(-2)=3-(1/2)(-2)=4$$
$$f(0)=3-(1/2)(0)=3$$
$$f(2)=3-(1/2)(2)=2$$
$$f(4)=3-(1/2)(4)=1$$

Place these results in table (a) and plot them as shown in (b). There is enough evidence here to intuit that the graph of f is the line shown in (b).

x	$f(x)=3-x/2$	(x, f(x))
-2	4	(−2, 4)
0	3	(0,3)
2	2	(2,2)
4	1	(4,1)

(a)



(b)

Exercise 16.5E.4

$f(x) = -1 + \frac{1}{2}x$

Exercise 16.5E.5

$f(x) = x^2 - 2$

Answer

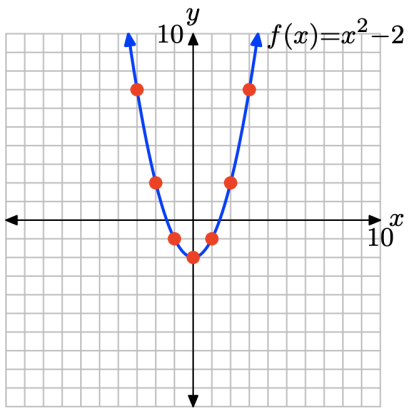
Evaluate $f(x) = x^2 - 2$ at x = −3, −2, −1, 0, 1, 2, and 3.

$$f(-3) = (-3)^2 - 2 = 7$$
$$f(-2) = (-2)^2 - 2 = 2$$
$$f(-1) = (-1)^2 - 2 = -1$$
$$f(0) = (0)^2 - 2 = -2$$
$$f(1) = (1)^2 - 2 = -1$$
$$f(2) = (2)^2 - 2 = 2$$
$$f(3) = (3)^2 - 2 = 7$$

Place these results in table (a) and plot them as shown in (b). There is enough evidence here to intuit that the graph of f is the curve shown in (b).

x	$f(x) = x^2 - 2$	(x, f(x))
-3	7	(−3, 7)
-2	2	(-2,2)
-1	-1	(-1,-1)
0	-2	(0,-2)
1	-1	(1,-1)
2	2	(2,2)
3	7	(3,7)

(a)



(b)

Exercise 16.5E.6

$f(x) = 4 - x^2$

Exercise 16.5E.7

$f(x) = \frac{1}{2}x^2 - 6$

Answer

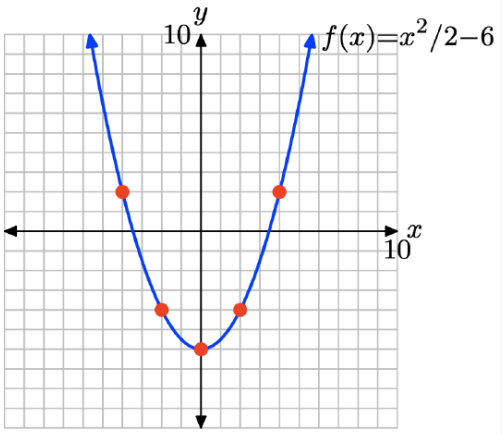
Evaluate $f(x) = x^2/2 - 6$ at x = −4, −2, 0, 2, and 4.

$$f(-4) = (-4)^2/2 - 6 = 2$$
$$f(-2) = (-2)^2/2 - 6 = -4$$
$$f(0) = (0)^2/2 - 6 = -6$$
$$f(2) = (2)^2/2 - 6 = -4$$
$$f(4) = (4)^2/2 - 6 = 2$$

Place these results in table (a) and plot them as shown in (b). There is enough evidence here to intuit that the graph of f is the curve shown in (b).

x	$f(x) = x^2 - 2$	(x, f(x))
-4	2	(−4, 2)
-2	-4	(-2,-4)
0	-6	(0,-6)
2	-4	(2,-4)
4	2	(4,2)

(a)



(b)

Exercise 16.5E. 8

$f(x) = 8 - \frac{1}{2}x^2$

Perform each of the following tasks for the functions Exercises 16.5E. 9-16.5E. 10.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.

ii. Use the table feature of your graphing calculator to evaluate the function at the given values of x. Record these results in a table next to your coordinate system on your graph paper.

iii. Plot the points in the table on your coordinate system then use them to draw the graph of the given function. Label the graph with its equation.

Exercise 16.5E. 9

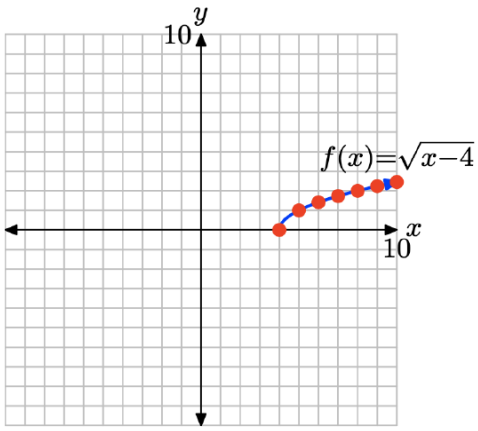
$f(x) = \sqrt{x-4}$ at x = 4, 5, 6, 7, 8, 9, and 10.

Answer

Load the function $f(x) = \sqrt{x-4}$ into Y1 as shown in (a). Select TBLSET, then highlight ASK for the independent variable and press ENTER (see (b)). It doesn't matter what is entered for TblStart or Δ Tbl. Select TABLE and enter the x-values 4, 5, 6, 7, 8, 9, and 10, as shown in (c).

(a) (b) (c)

Plot the points in table (c) in (d). This is enough to intuit that the graph of f is the curve shown in (d).



(d)

? Exercise 16.5E. 10

$f(x) = \sqrt{4-x}$ at x = −10, −8, −6, −4, −2, 0, 2, and 4.

In Exercises 16.5E. 11-16.5E. 14, the graph of the given function is a parabola, a graph that has a "U-shape." A parabola has only one turning point. For each exercise, perform the following tasks.

i. Load the equation into the Y= menu of your graphing calculator. Adjust the WINDOW parameters so that the "turning point" (actually called the vertex) is visible in the viewing window.

ii. Make a reasonable copy of the image in the viewing window on your homework paper. Draw all lines with a ruler (including the axes), but draw curves freehand. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graph with its equation.

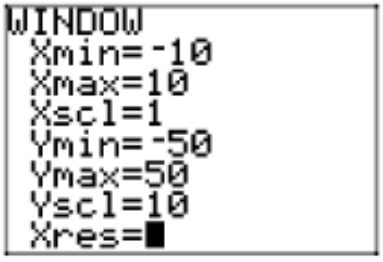
? Exercise 16.5E. 11

$f(x) = x^2 - x - 30$

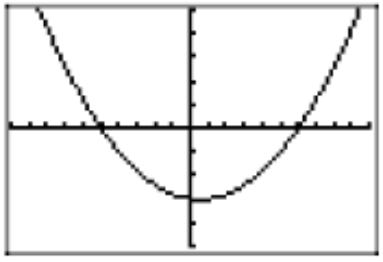
Answer

Load the function $f(x) = x^2 - x - 30$ into Y1 as shown in (a). Adjust the WINDOW parameters as shown in (b). Push the GRAPH button to obtain the graph of f in (c).

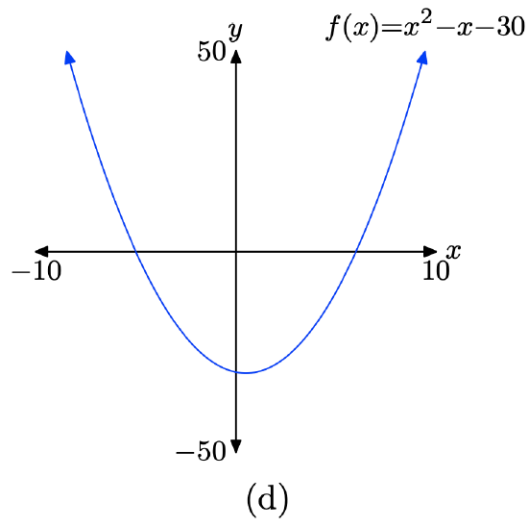
(a)



(b)


(c)

Copy the image onto your homework as shown in (d).



(d)

Exercise 16.5E.12

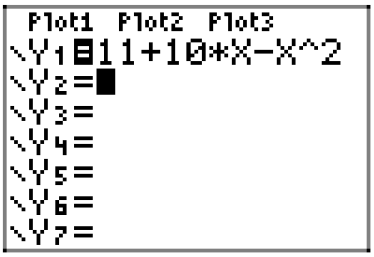
$f(x) = 24 - 2x - x^2$

Exercise 16.5E.13

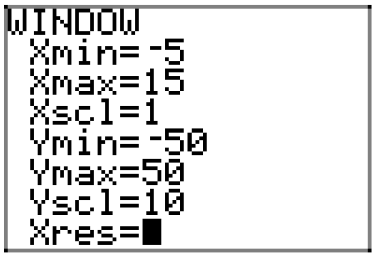
$f(x) = 11 + 10x - x^2$

Answer

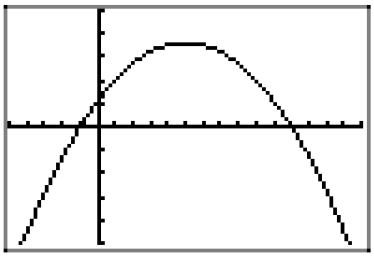
Load the function $f(x) = 11 + 10x - x^2$ into Y1 as shown in (a). Adjust the WINDOW parameters as shown in (b). Push the GRAPH button to obtain the graph of f in (c).



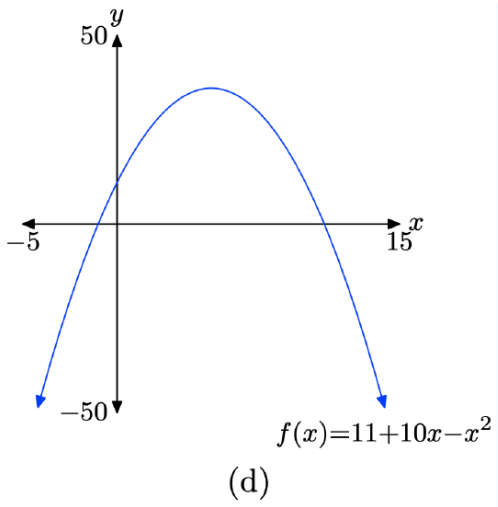
(a)



(b)


(c)

Copy the image onto your homework as shown in (d).



(d)

Exercise 16.5E. 14

$f(x) = x^2 + 11x - 12$

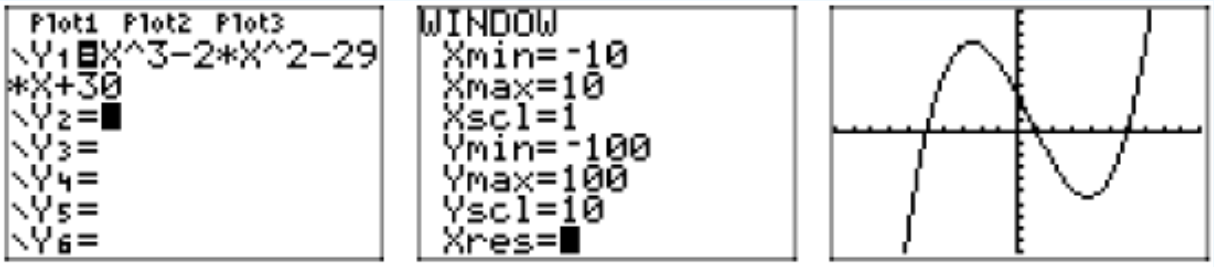
Each of the equations in Exercises $16.5E.15$ - $16.5E.18$ are called “cubic polynomials.” Each equation has been carefully chosen so that its graph has exactly two “turning points.” For each exercise, perform each of the following tasks. i. Load the equation into the Y= menu of your graphing calculator and adjust the WINDOW parameters so that both “turning points” are visible in the viewing window. ii. Make a reasonable copy of the graph in the viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, then label the graph with its equation. Remember to draw all lines with a ruler.

Exercise 16.5E. 15

$f(x) = x^3 - 2x^2 - 29x + 30$

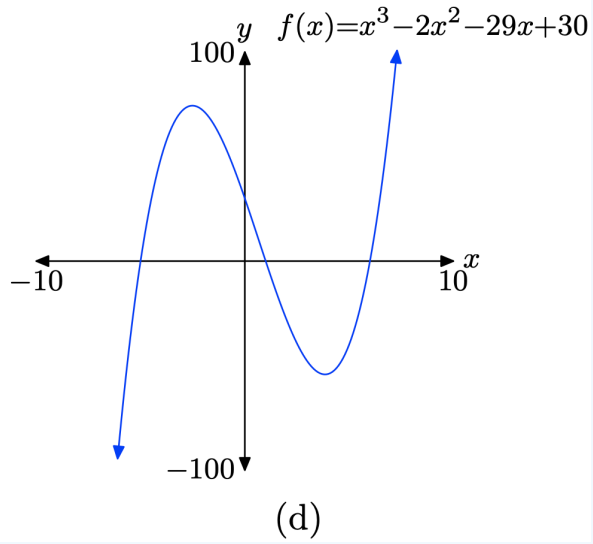
Answer

Load the function $f(x) = x^3 - 2x^2 - 29x + 30$ into Y1 as shown in (a). Adjust the WINDOW parameters as shown in (b). Push the GRAPH button to obtain the graph of f in (c).



(a) (b) (c)

Copy the image onto your homework as shown in (d).



(d)

Exercise 16.5E. 16

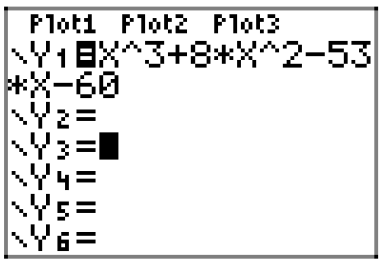
$f(x) = -x^3 + 2x^2 + 19x - 20$

Exercise 16.5E. 17

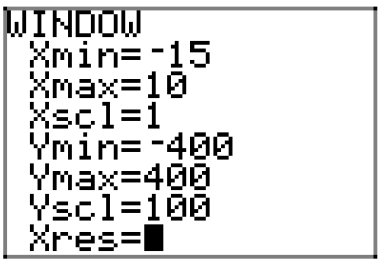
$f(x) = x^3 + 8x^2 - 53x - 60$

Answer

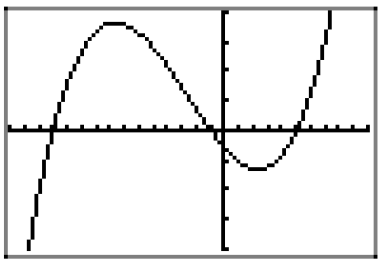
Load the function $f(x) = x^3 + 8x^2 - 53x - 60$ into Y1 as shown in (a). Adjust the WINDOW parameters as shown in (b). Push the GRAPH button to obtain the graph of f in (c).



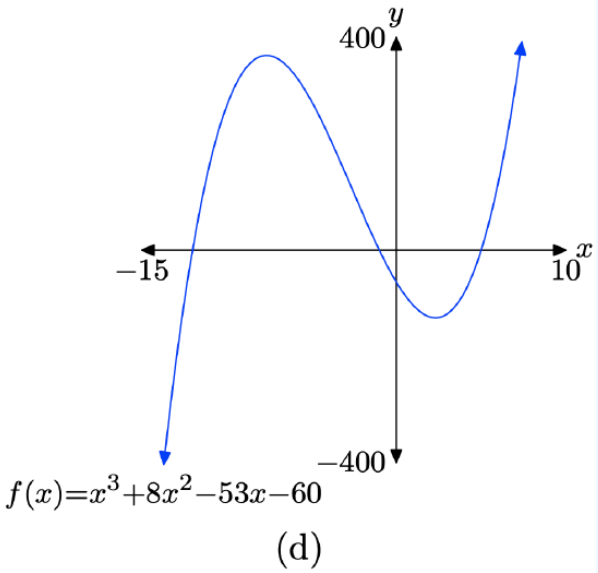
(a)



(b)


(c)

Copy the image onto your homework as shown in (d).



(d)

? Exercise 16.5E. 18

$f(x) = -x^3 + 16x^2 - 43x - 60$

Perform each of the following tasks for the equations in Exercises 16.5E. 19-16.5E. 22

i. Load the equation into the Y= menu. Adjust the WINDOW parameters until you think all important behavior ("turning points," etc.) is visible in the viewing window. Note: This is more difficult than it sounds, particularly when we have no advance notion of what the graph might look like. However, experiment with several settings until you "discover" the settings that exhibit the most important behavior.

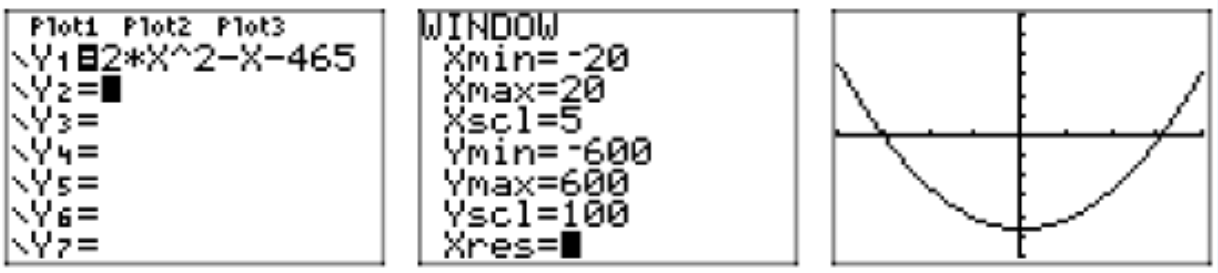
ii. Copy the image on the screen onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graph with its equation.

? Exercise 16.5E. 19

$f(x) = 2x^2 - x - 465$

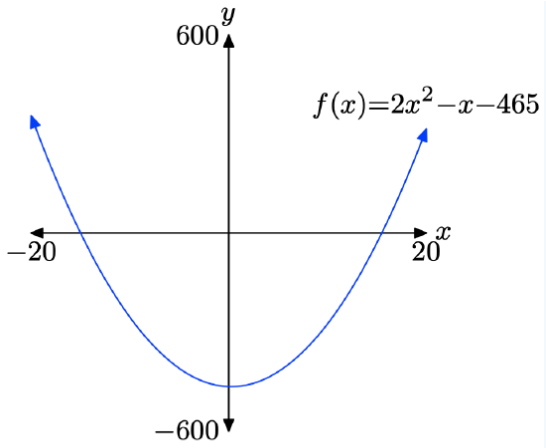
Answer

Load the function $f(x) = 2x^2 - x - 465$ into Y1 as shown in (a). Adjust the WINDOW parameters as shown in (b). Push the GRAPH button to obtain the graph of f in (c).



(a) (b) (c)

Copy the image onto your homework as shown in (d).



(d)

Exercise 16.5E.20

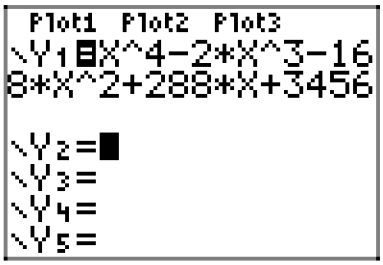
$f(x) = x^3 - 24x^2 + 65x + 1050$

Exercise 16.5E.21

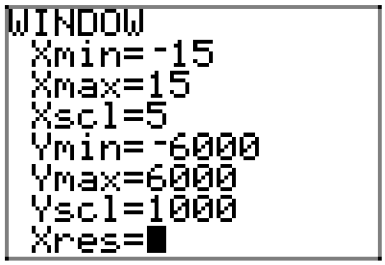
$f(x) = x^4 - 2x^3 - 168x^2 + 288x + 3456$

Answer

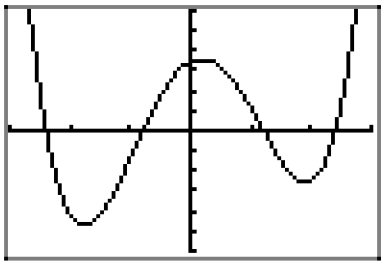
Load the function $f(x) = x^4 - 2x^3 - 168x^2 + 288x + 3456$ into Y1 as shown in (a). Adjust the WINDOW parameters as shown in (b). Push the GRAPH button to obtain the graph of f in (c).



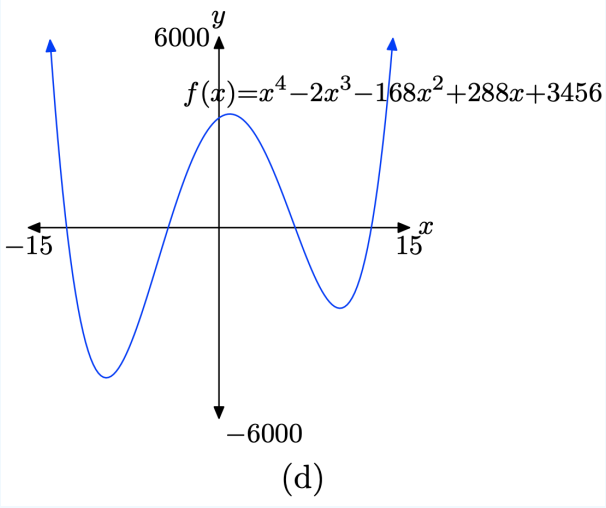
(a)



(b)


(c)

Copy the image onto your homework as shown in (d)



(d)

Exercise 16.5E. 22

$f(x) = -x^4 - 3x^3 + 141x^2 + 523x - 660$

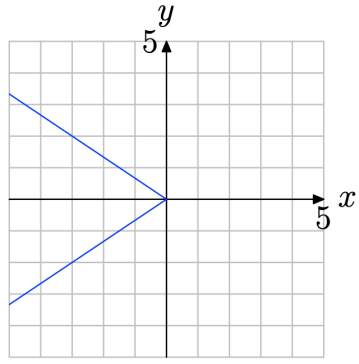
2.3 Exercises

For Exercises 16.5E. 1-16.5E. 6, perform each of the following tasks.

i. Make a copy of the graph on a sheet of graph paper and apply the vertical line test.

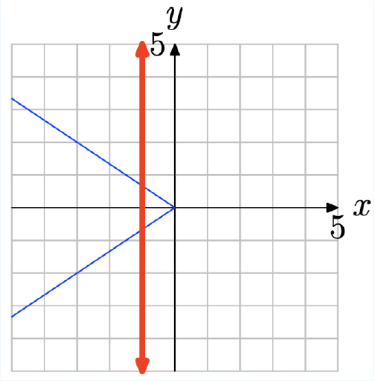
ii. Write a complete sentence stating whether or not the graph represents a function. Explain the reason for your response.

Exercise 16.5E. 1

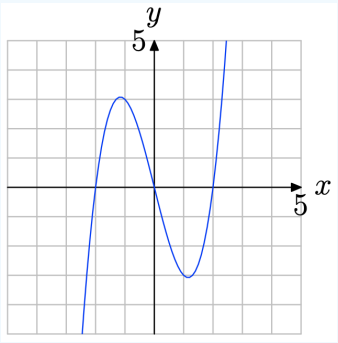


Answer

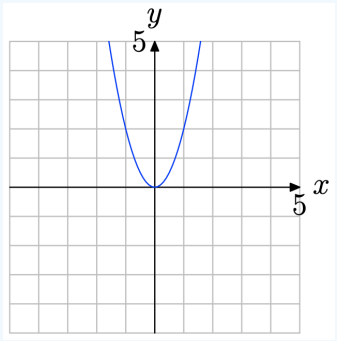
Note that in the figure below a vertical line cuts the graph more than once. Therefore, the graph does not represent the graph of a function.



? Exercise 16.5E. 2

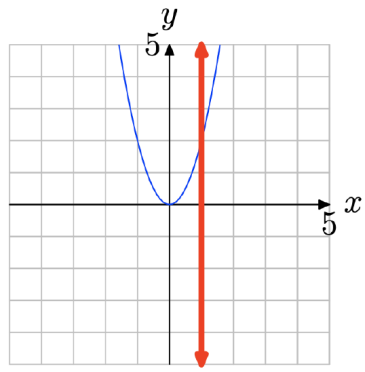


? Exercise 16.5E. 3

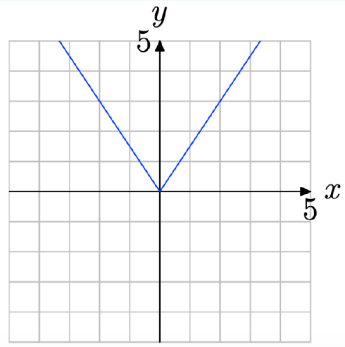


Answer

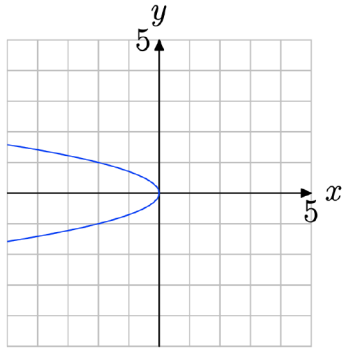
No vertical line cuts the graph more than once (see figure below). Therefore, the graph represents a function.



Exercise 16.5E.4

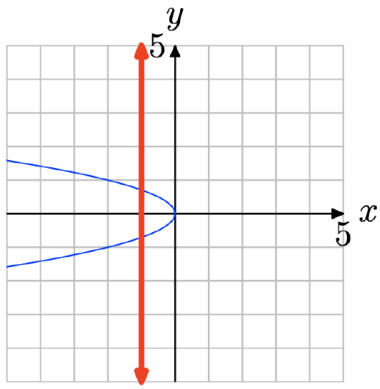


Exercise 16.5E.5

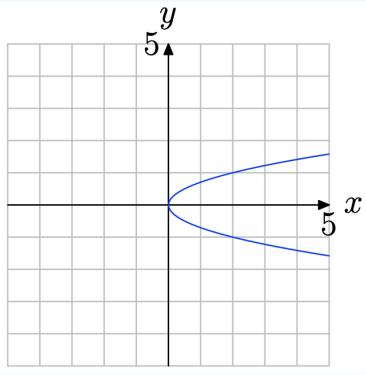


Answer

Note that in the figure below a vertical line cuts the graph more than once. Therefore, the graph does not represent the graph of a function.



Exercise 16.5E.6



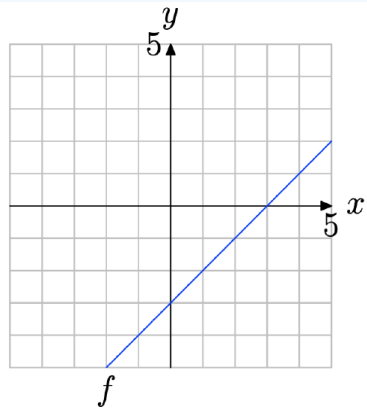
In Exercises 16.5E.7-16.5E.12, perform each of the following tasks.

i. Make an exact copy of the graph of the function f on a sheet of graph paper. Label and scale each axis. Remember to draw all lines with a ruler.

ii. Use the technique of Examples 3 and 4 in the narrative to evaluate the function at the given value. Draw and label the arrows as shown in Figures 4 and 5 in the narrative.

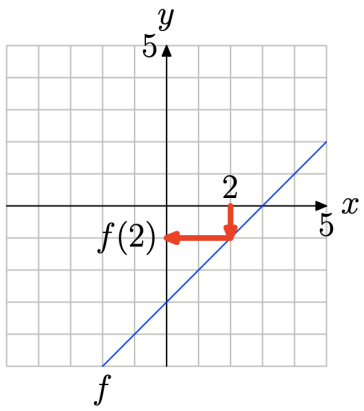
Exercise 16.5E.7

Use the graph of f to determine f(2).



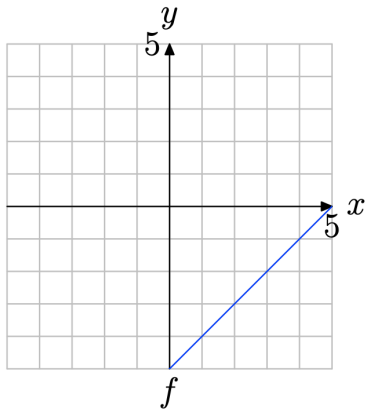
Answer

Locate x = 2 on the x-axis (see figure below), draw a vertical arrow to the graph of f, then a horizontal arrow to the y-axis. Thus, f(2) = −1.



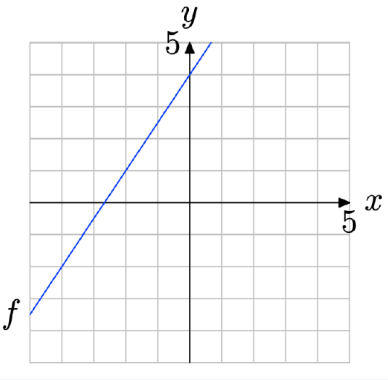
? Exercise 16.5E. 8

Use the graph of f to determine f(3).



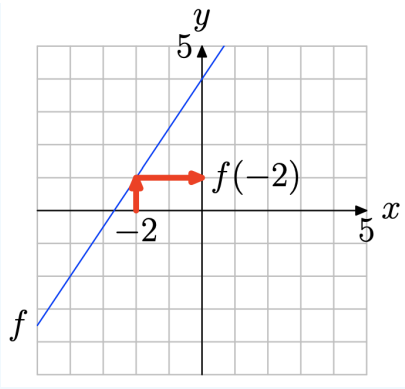
? Exercise 16.5E. 9

Use the graph of f to determine f(−2).



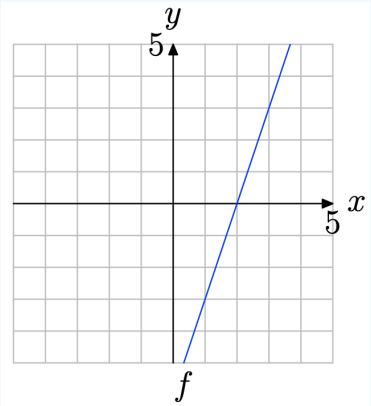
Answer

Locate x = −2 on the x-axis (see figure below), draw a vertical arrow to the graph of f, then a horizontal arrow to the y-axis. Thus, f(−2) = 1.



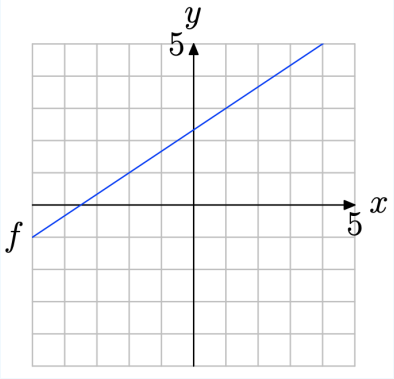
Exercise 16.5E.10

Use the graph of f to determine f(1).



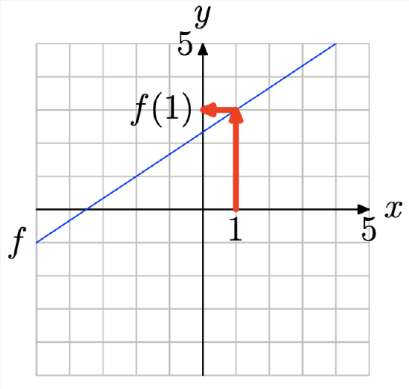
Exercise 16.5E.11

Use the graph of f to determine f(1).



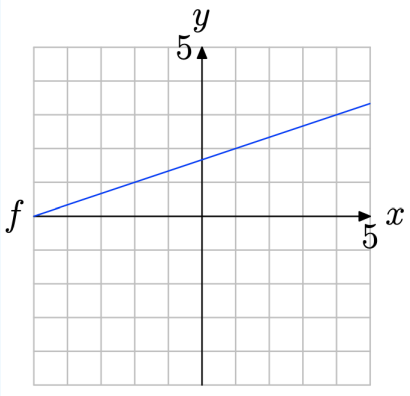
Answer

Locate x = 1 on the x-axis (see figure below), draw a vertical arrow to the graph of f, then a horizontal arrow to the y-axis. Thus, f(1) = 3.



Exercise 16.5E.12

Use the graph of f to determine f(−2).



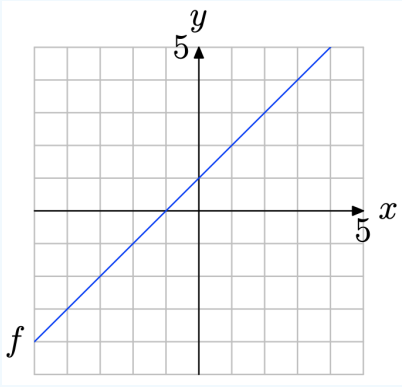
In Exercises 16.5E.13-16.5E.18, perform each of the following tasks.

i. Make an exact copy of the graph of the function f on a sheet of graph paper. Label and scale each axis. Remember to draw all lines with a ruler.

ii. Use the technique of Example 5 in the narrative to find the value of x that maps onto the given value. Draw and label the arrows as shown in Figure 6 in the narrative.

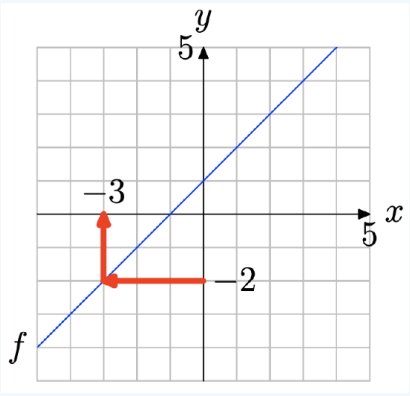
Exercise 16.5E.13

Use the graph of f to solve the equation f(x) = −2.



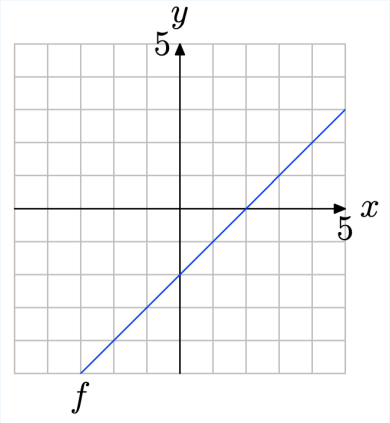
Answer

Locate y = −2 on the y-axis (see figure below), draw a horizontal arrow to the graph of f, then a vertical arrow to the y-axis. Thus, the solution of f(x) = −2 is x = −3.



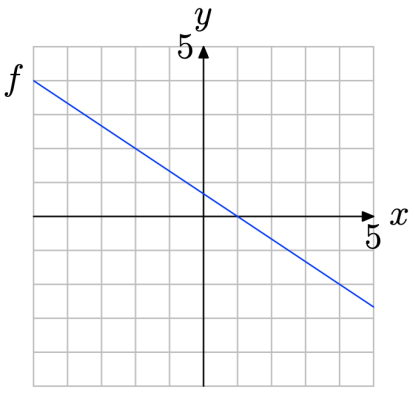
? Exercise 16.5E. 14

Use the graph of f to solve the equation f(x) = 1.



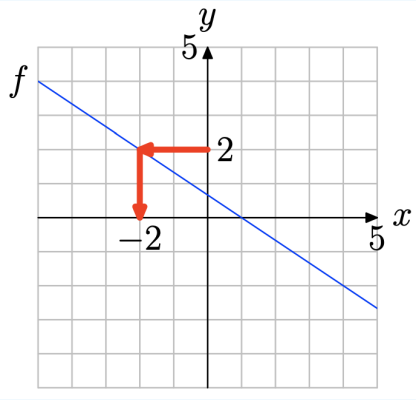
? Exercise 16.5E. 15

Use the graph of f to solve the equation f(x) = 2



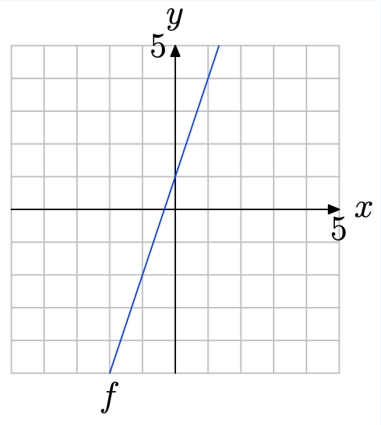
Answer

Locate y = 2 on the y-axis (see figure below), draw a horizontal arrow to the graph of f, then a vertical arrow to the y-axis. Thus, the solution of f(x) = 2 is x = −2.



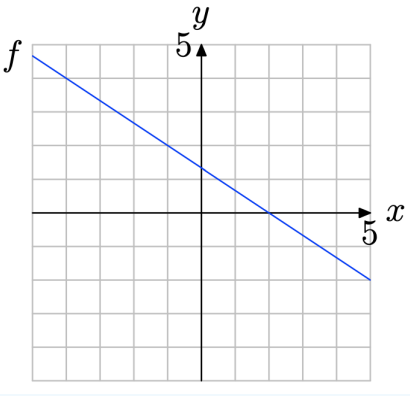
Exercise 16.5E.16

Use the graph of f to solve the equation f(x) = −2.



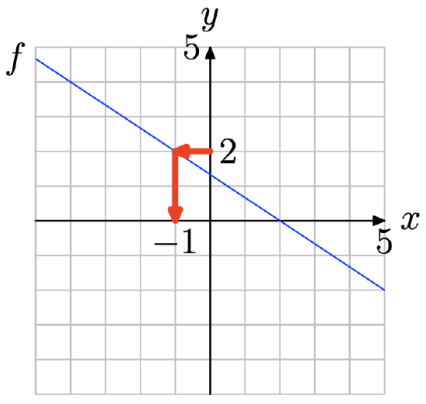
Exercise 16.5E.17

Use the graph of f to solve the equation f(x) = 2.



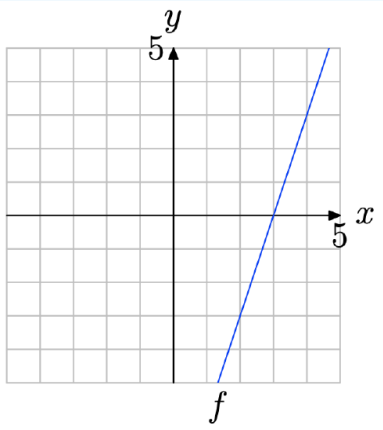
Answer

Locate y = 2 on the y-axis (see figure below), draw a horizontal arrow to the graph of f, then a vertical arrow to the y-axis. Thus, the solution of f(x) = 2 is x = −1.



Exercise 16.5E.18

Use the graph of f to solve the equation f(x) = −3.



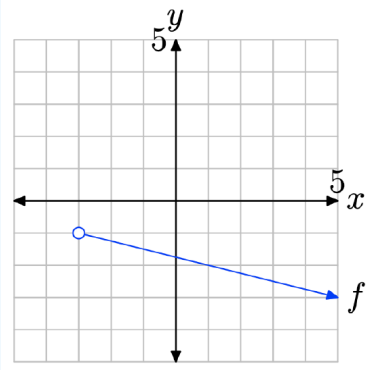
In the Exercises 16.5E.19-16.5E.22, perform each of the following tasks.

i. Make a copy of the graph of f on a sheet of graph paper. Label and scale each axis.

ii. Using a different colored pen or pencil, project each point on the graph of f onto the x-axis. Shade the resulting domain on the x-axis.

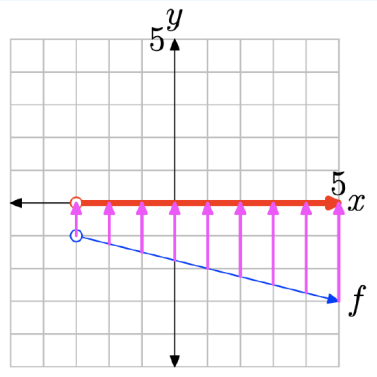
iii. Use both set-builder and interval notation to describe the domain.

Exercise 16.5E.19

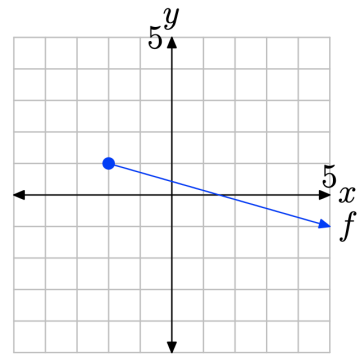


Answer

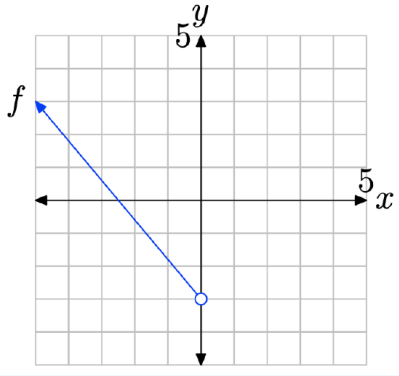
To find the domain of the function, project the graph of f onto the x-axis. Note that all values of x that lie to the right of -3 lie in shadow and are hence in the domain of f. Therefore, the domain is best described with the notation $\{x : x > -3\} = (-3, \infty)$.



Exercise $16.5E.20$

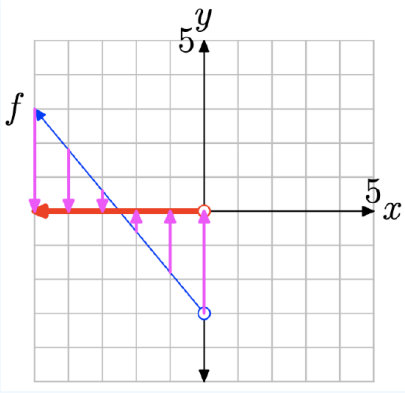


Exercise $16.5E.21$

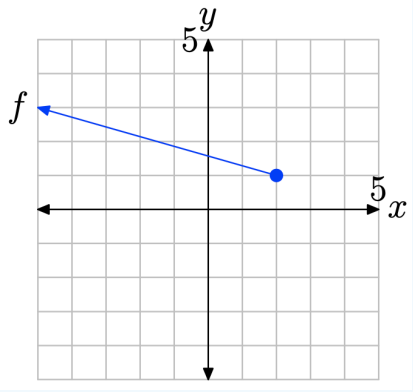


Answer

To find the domain of the function, project the graph of f onto the x-axis. Note that all values of x that lie to the left of 0 lie in shadow and are hence in the domain of f. Therefore, the domain is best described with the interval notation $\{x : x < 0\} = (-\infty, 0)$.



? Exercise 16.5E. 22

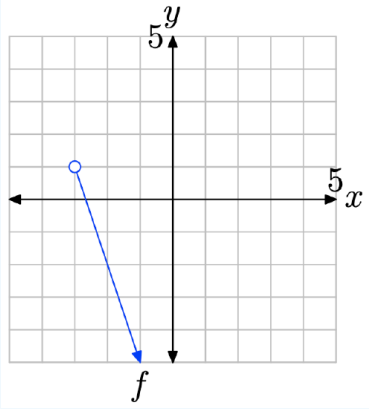


In Exercises 16.5E. 23-16.5E. 26, perform each of the following tasks.

i. Make a copy of the graph of f on a sheet of graph paper. Label and scale each axis.

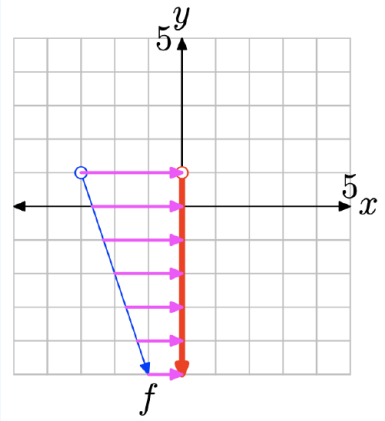
ii. Using a different colored pen or pencil, project each point on the graph of f onto the y-axis. Shade the resulting range on the y-axis. iii. Use both set-builder and interval notation to describe the range.

? Exercise 16.5E. 23

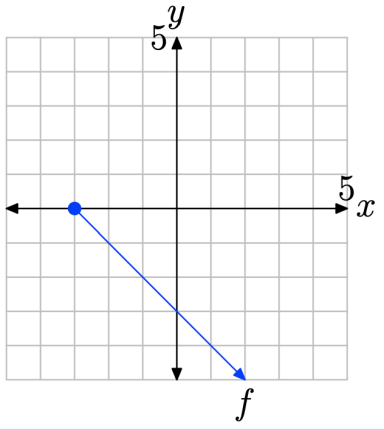


Answer

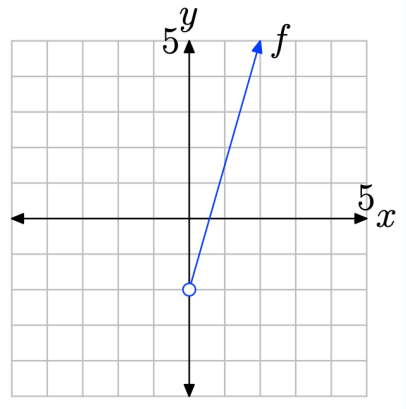
To find the range of the function, project the graph of f onto the y-axis. Note that all values of y that lie below 1 lie in shadow and are hence in the range of f. Therefore, the range is best described with the interval notation $\{y : y < 1\} = (-\infty, 1)$.



Exercise 16.5E.24

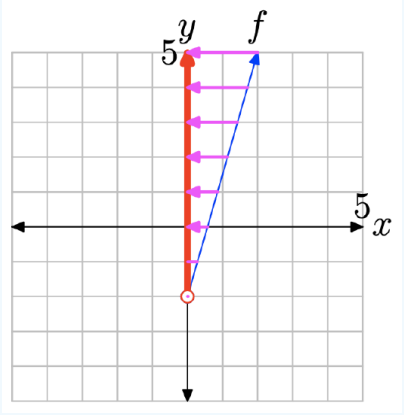


Exercise 16.5E.25

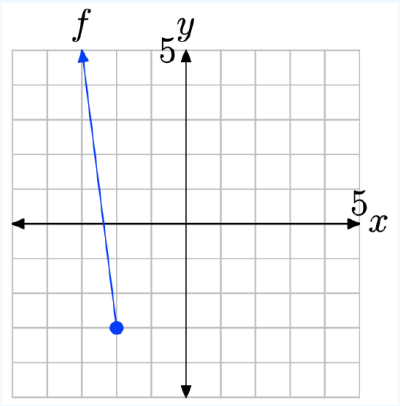


Answer

To find the range of the function, project the graph of f onto the y-axis. Note that all values of y that lie above −2 lie in shadow and are hence in the range of f. Therefore, the range is best described with the interval notation $\{y : y > -2\} = (-2, \infty)$.



Exercise 16.5E.26



In Exercises 16.5E.27-16.5E.30, perform each of the following tasks.

i. Use your graphing calculator to draw the graph of the given function. Make a reasonably accurate copy of the image in your viewing screen on your homework paper. Label and scale each axis with the WINDOW parameters xmin, xmax, ymin, and ymax. Label the graph with its equation.

ii. Using a colored pencil, project each point on the graph onto the x-axis; i.e., shade the domain on the x-axis. Use interval and set-builder notation to describe the domain.

iii. Use a purely algebraic technique, as demonstrated in Example 8 in the narrative, to find the domain. Compare this result with that found in part (ii).

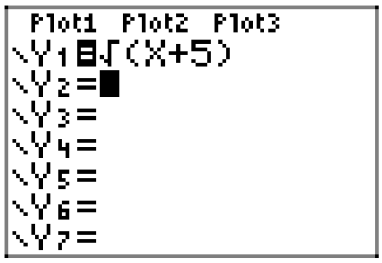
iv. Using a different colored pencil, project each point on the graph onto the y-axis; i.e., shade the range on the y-axis. Use interval and set-builder notation to describe the range.

Exercise 16.5E.27

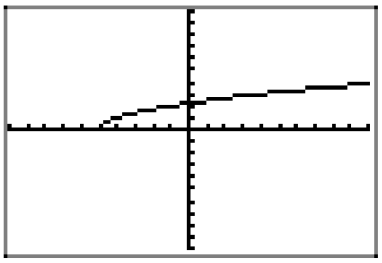
$f(x) = \sqrt{x+5}$.

Answer

Load the function $f(x) = \sqrt{x+5}$ into Y1 as shown in (a). Select 6:ZStandrd from the ZOOM menu to produce the graph in (b).

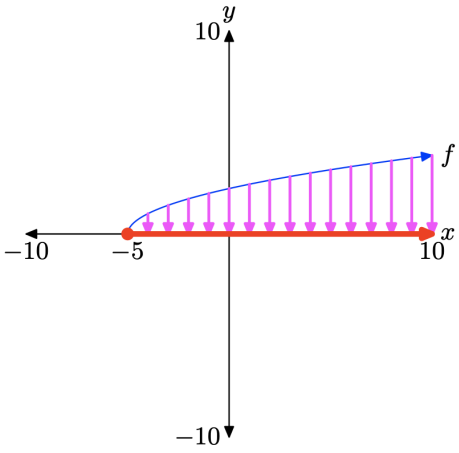


(a)

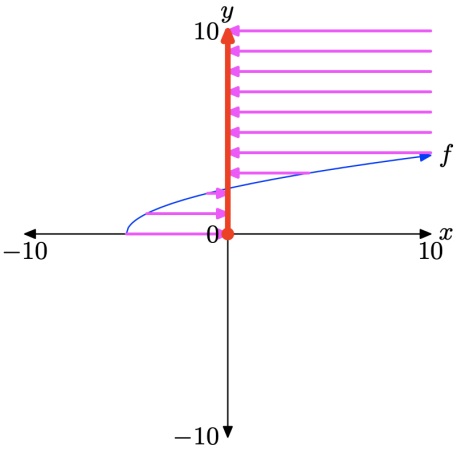


(b)

Copy the image in (b) onto your homework paper, then project the domain and range onto the x- and y-axes, as shown in (c) and (d), respectively.



(c) $\mathrm{D} = [-5, \infty) = \{x : x \geq -5\}$



(d) $\mathrm{R} = [0, \infty) = \{y : y \geq 0\}$

To find the domain algebraically, note that you cannot take the square root of a negative number, so the expression under the radical in $f(x) = \sqrt{x+5}$, namely x+5, must either be positive or zero (nonnegative). That is,

$$x + 5 \geq 0$$

or equivalently,

$$x \geq -5$$

Thus, the domain of f is Domain = $[-5, \infty)$, or in set-builder notation, Domain = $x : x \geq -5$.

? Exercise 16.5E. 28

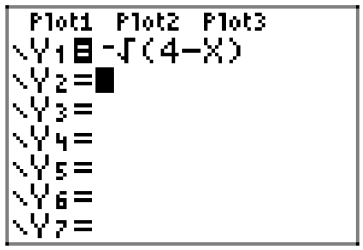
$f(x) = \sqrt{5 - x}$

? Exercise 16.5E. 29

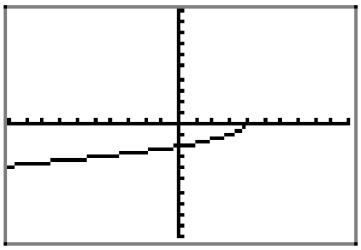
$f(x) = -\sqrt{4 - x}$.

Answer

Load the function $f(x) = -\sqrt{4 - x}$. into Y1 as shown in (a). Select 6:ZStandrd from the ZOOM menu to produce the graph in (b).

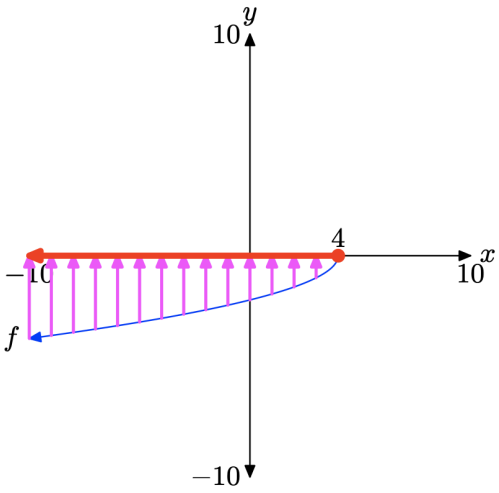


(a)

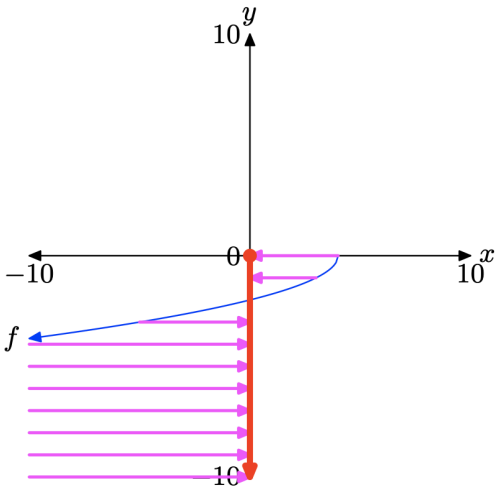


(b)

Copy the image in (b) onto your homework paper, then project the domain and range onto the x- and y-axes, as shown in (c) and (d), respectively.



(c) $\mathrm{D} = (-\infty, 4] = \{x : x \leq 4\}$



(d) $\mathrm{R} = (-\infty, 0] = \{y : y \leq 0\}$

To find the domain algebraically, note that you cannot take the square root of a negative number, so the expression under the radical in $f(x) = -\sqrt{4-x}$, namely 4−x, must either be positive or zero (nonnegative). That is,

$$4 - x \geq 0$$

or equivalently,

$$-x \geq -4$$
$$x \leq 4$$

Thus, the domain of f is Domain = $(\infty, 4]$, or in set-builder notation, Domain = $\{x : x \leq 4\}$.

? Exercise $16.5E.30$

$f(x) = -\sqrt{x+4}$

2.4 Exercises

In Exercises $16.5E.1$-$16.5E.6$, you are given the definition of two functions f and g. Compare the functions, as in Example 1 of the narrative, at the given values of x.

? Exercise $16.5E.1$

f(x) = x+2, g(x) = 4−x at x = −3, 1, and 2.

Answer

We're given that f(x) = x + 2 and g(x) = 4 − x. At x = −3,

$$f(-3) = -3 + 2 = -1$$
$$g(-3) = 4 - (-3) = 7$$

Therefore, f(−3) < g(−3). At x = 1,

$$f(1) = 1 + 2 = 3$$
$$g(1) = 4 - 1 = 3.$$

Therefore, f(1) = g(1). At x = 2,

$$f(2) = 2 + 2 = 4$$
$$g(2) = 4 - 2 = 2.$$

Therefore, f(2) > g(2).

? Exercise 16.5E. 2

f(x) = 2x − 3, g(x) = 3 − x at x = −4, 2, and 5.

? Exercise 16.5E. 3

f(x) = 3−x, g(x) = x+9 at x = −4, −3, and −2.

Answer

We're given that f(x) = 3 − x and g(x) = x + 9. At x = −4,

$$f(-4) = 3 - (-4) = 7$$
$$g(-4) = -4 + 9 = 5$$

Therefore, f(−4) > g(−4). At x = −3,

$$f(-3) = 3 - (-3) = 6$$
$$g(-3) = -3 + 9 = 6$$

Therefore, f(−3) = g(−3). At x = −2,

$$f(-2) = 3 - (-2) = 5$$
$$g(-2) = -2 + 9 = 7$$

Therefore, f(−2) < g(−2).

? Exercise 16.5E. 4

$f(x) = x^2$, g(x) = 4x + 5 at x = −2, 1, and 6.

? Exercise 16.5E. 5

$f(x) = x^2$, g(x) = −3x − 2 at x = −3, −1, and 0.

Answer

We're given that $f(x) = x^2$ and g(x) = −3x − 2. At x = −3,

$$f(-3) = (-3)2 = 9$$
$$g(-3) = -3(-3) - 2 = 7$$

Therefore, f(−3) > g(−3). At x = −1,

$$f(-1) = (-1)2 = 1$$
$$g(-1) = -3(-1) - 2 = 1$$

Therefore, f(−1) = g(−1). At x = 0,

$$f(0) = (0)2 = 0$$
$$g(0) = -3(0) - 2 = -2$$

Therefore, f(0) > g(0).

? Exercise $16.5E.6$

f(x) = |x|, g(x) = 4 − x at x = 1, 2, and 3.

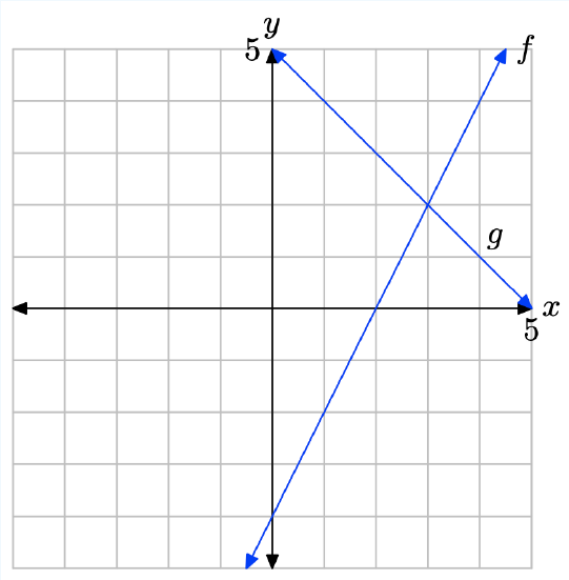
In Exercises $16.5E.7$-$16.5E.12$, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Make an accurate copy of the image on graph paper (label each equation, label and scale each axis), drop a dashed vertical line through the point of intersection, then label and shade the solution of f(x) = g(x) on the x-axis.

ii. Make a second copy of the image on graph paper, drop a dashed, vertical line through the point of intersection, then label and shade the solution of f(x) > g(x) on the x-axis. Use set-builder and interval notation to describe your solution set.

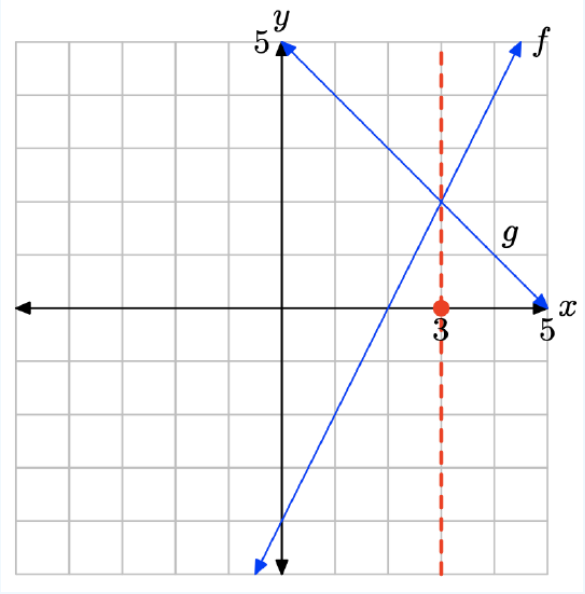
iii. Make a third copy of the image on graph paper, drop a dashed, vertical line through the point of intersection, then label and shade the solution of f(x) < g(x) on the x-axis. Use setbuilder and interval notation to describe your solution set.

? Exercise $16.5E.7$

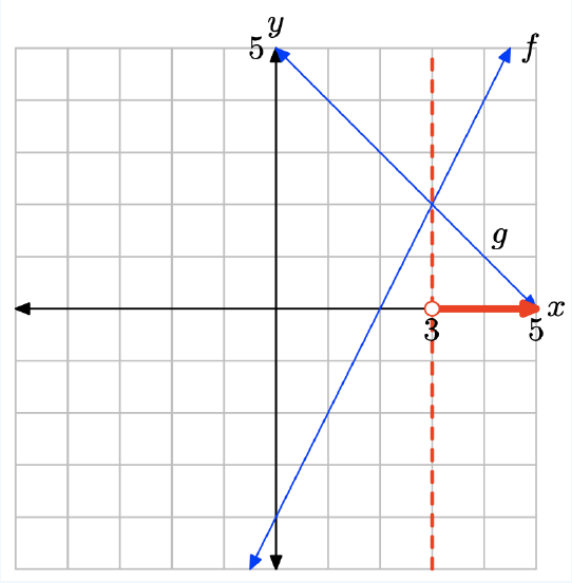


Answer

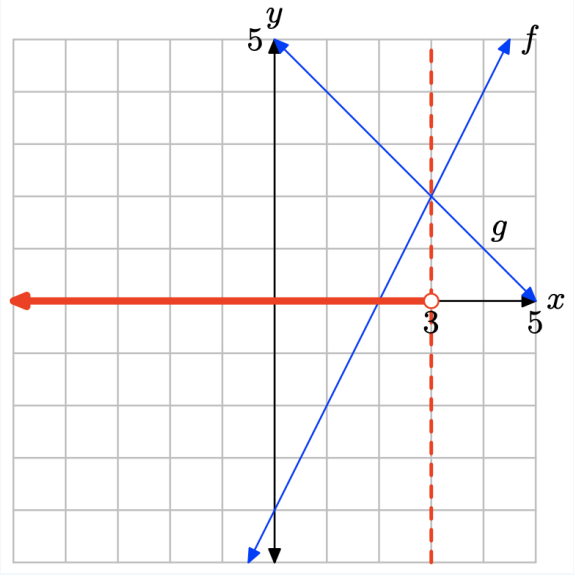
The graph of f intersects the graph of g at x = 3. The solution of f(x) = g(x) is x = 3.



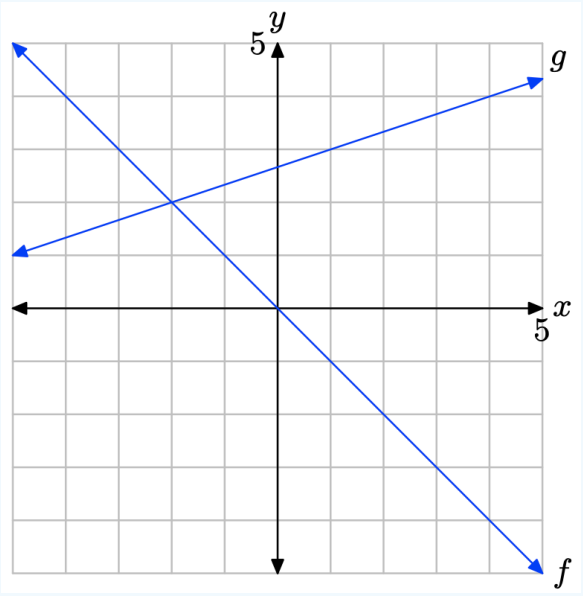
The graph of f lies above the graph of g to the right of x = 3. The solution of f(x) > g(x) is $(3, \infty) = \{x : x > 3\}$.



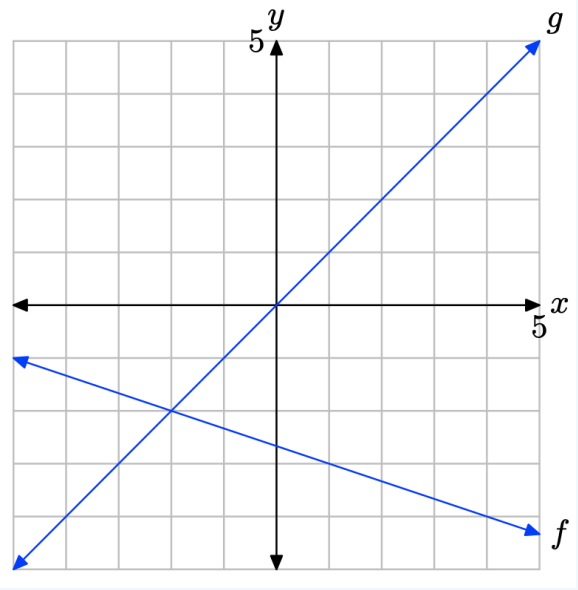
The graph of f lies below the graph of g to the left of x = 3. The solution of f(x) < g(x) is $(-\infty, 3) = \{x : x < 3\}$.



Exercise 16.5E.8

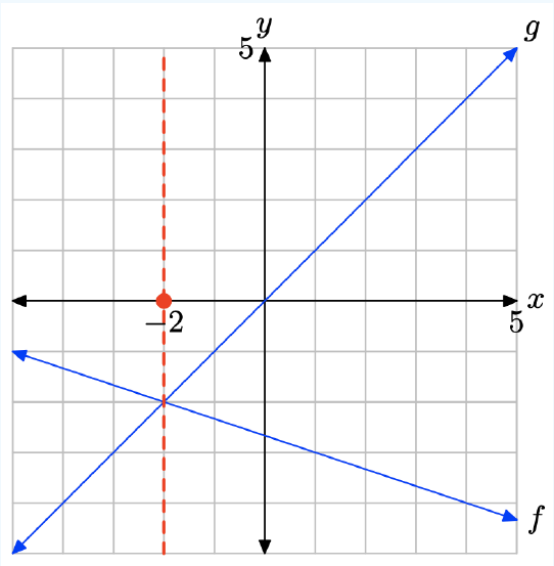


Exercise 16.5E.9

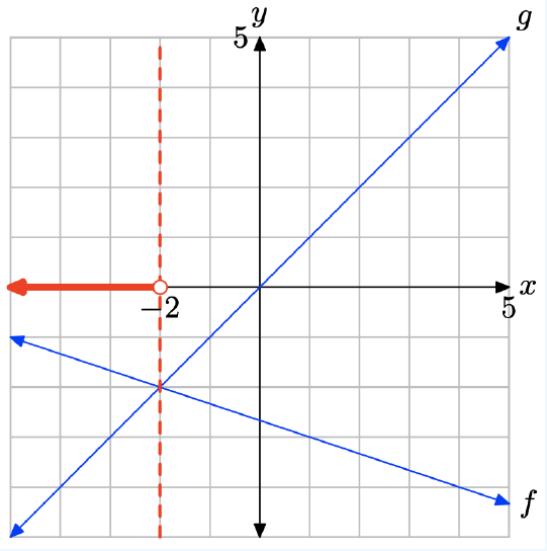


Answer

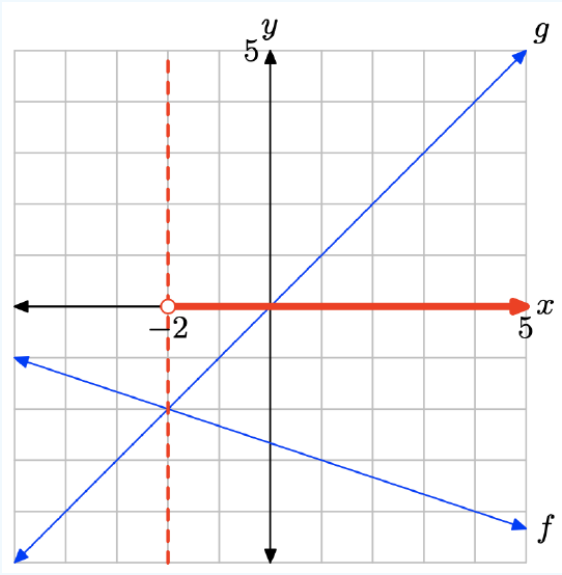
The graph of f intersects the graph of g at x = −2. The solution of f(x) = g(x) is x = −2.



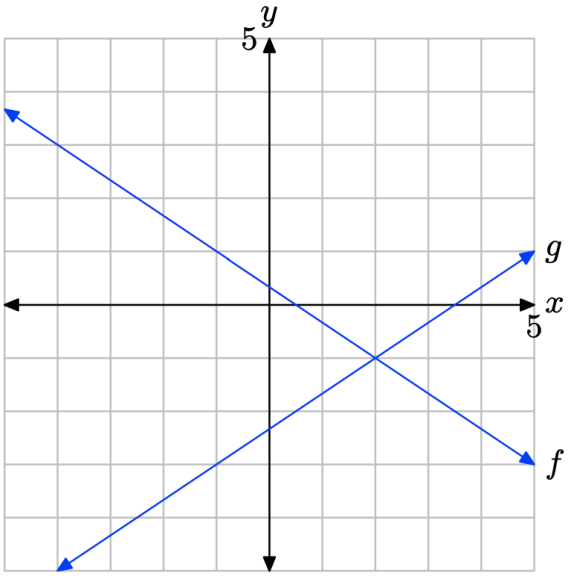
The graph of f lies above the graph of g to the left of x = −2. The solution of f(x) > g(x) is $(-\infty, -2) = \{x : x < -2\}$.



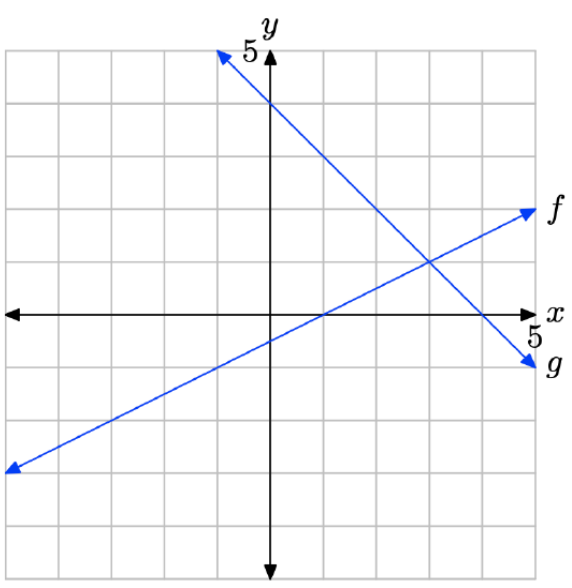
The graph of f lies below the graph of g to the right of x = −2. The solution of f(x) < g(x) is \((−2,\infty) = {\x : x > −2\}\).



? Exercise 16.5E. 10

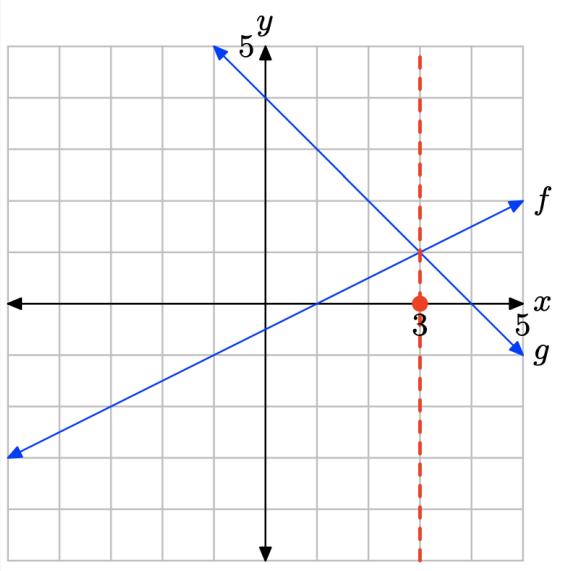


Exercise 16.5E.11

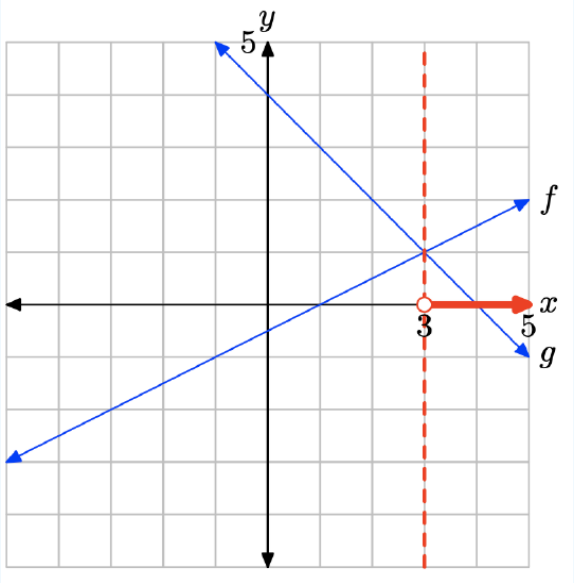


Answer

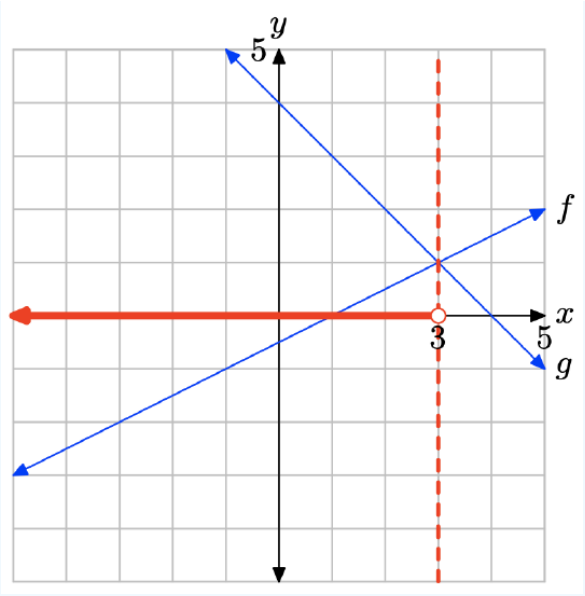
The graph of f intersects the graph of g at x = 3. The solution of f(x) = g(x) is x = 3.



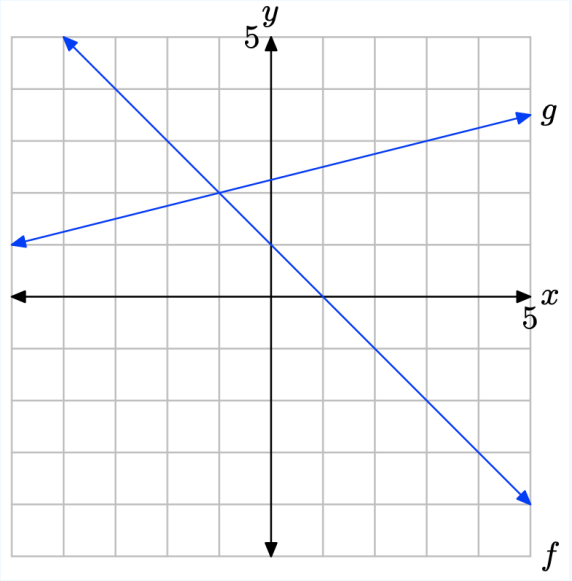
The graph of f is above the graph of g to the right of x = 3. The solution of f(x) > g(x) is $(3, \infty) = \{x : x > 3\}$.



The graph of f is below the graph of g to the left of x = 3. The solution of f(x) < g(x) is $(-\infty, 3) = \{x : x < 3\}$.



Exercise $16.5E.12$

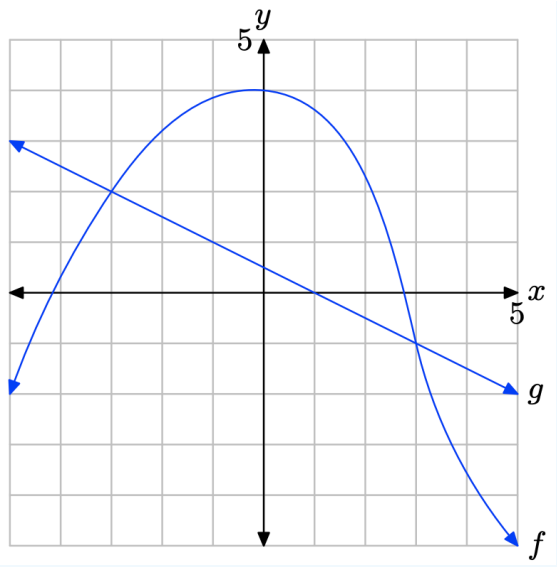


In Exercises $16.5E.13$-$16.5E.16$, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Make an accurate copy of the image on graph paper, drop dashed, vertical lines through the points of intersection, then label and shade the solution of $f(x) \geq g(x)$ on the x-axis. Use set-builder and interval notation to describe your solution set.

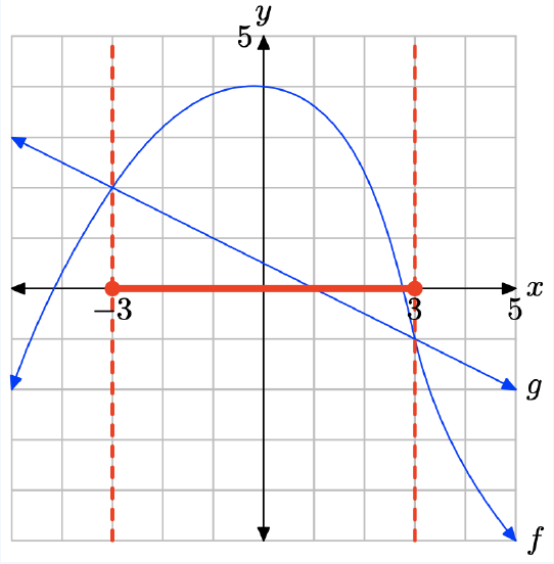
ii. Make a second copy of the image on graph paper, drop dashed, vertical lines through the points of intersection, then label and shade the solution of f(x) < g(x) on the x-axis. Use set-builder and interval notation to describe your solution set.

Exercise $16.5E.13$

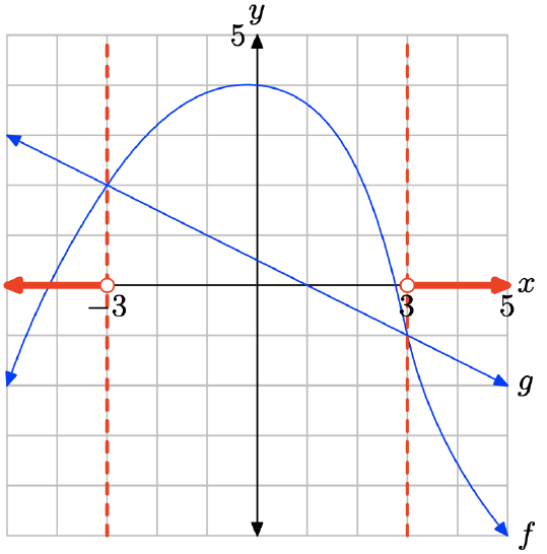


Answer

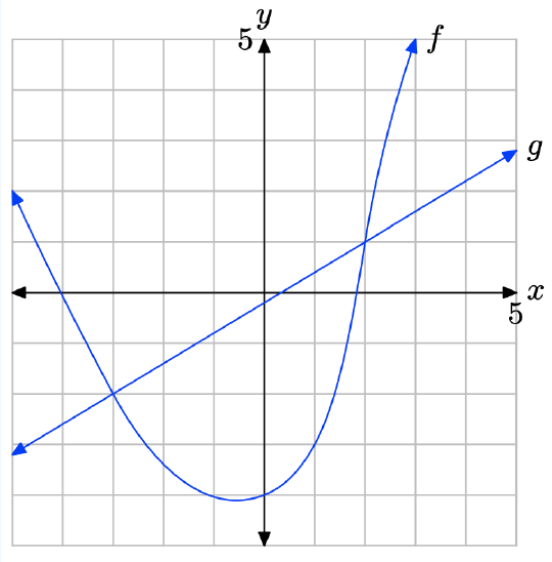
The graph of f intersects the graph of g at x = −3 and x = 3. The graph of f lies above the graph of g for values of x that lie between −3 and 3. Therefore, the solution of $f(x) \geq g(x)$ is $[-3, 3] = \{x : -3 \leq x \leq 3\}$.



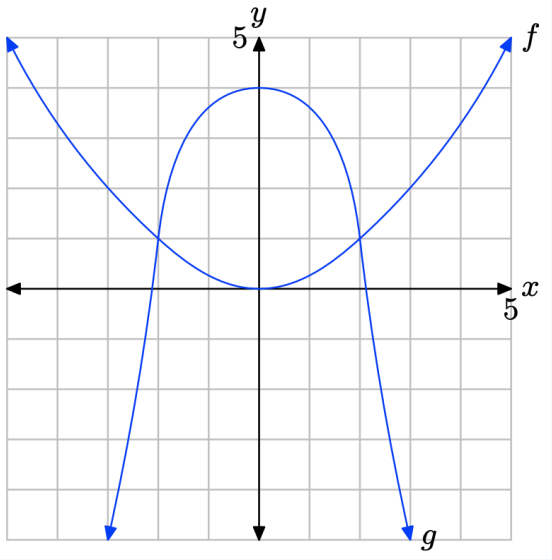
The graph of f is below the graph of g for values of x that lie to the left of −3 or to the right of 3. Therefore, the solution of f(x) < g(x) is $(-\infty, -3) \cup (3, \infty)$ or $\{x : x < -3 or x > 3\}$.



Exercise 16.5E.14

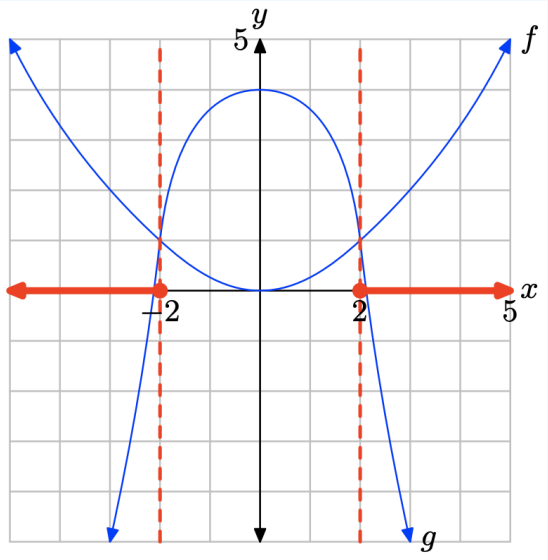


Exercise 16.5E.15

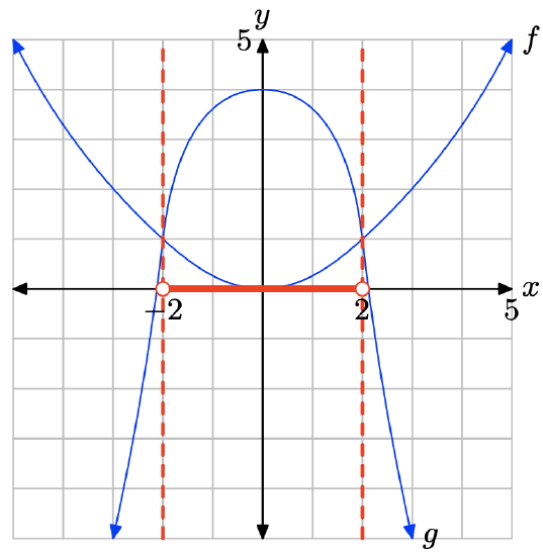


Answer

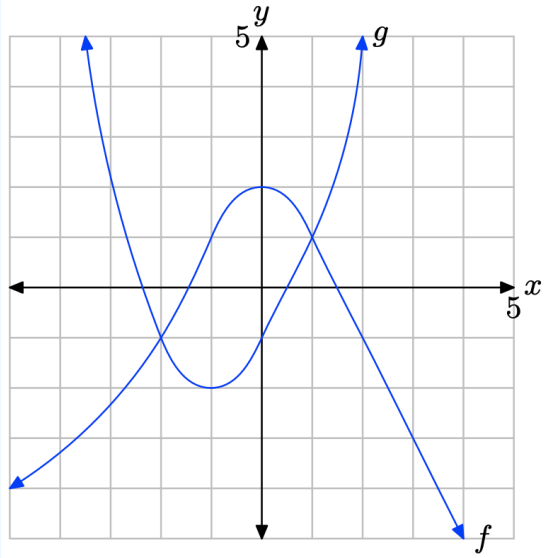
The graph of f intersects the graph of g at x = −2 and at x = 2. The graph of f lies above the graph of g for all values of x that lie to the left of −2 or to the right of 2. Therefore, the solution of $f(x) \geq g(x)$ is $(-\infty, -2] \cup [2, \infty)$ or $\{x : x \leq -2 or x \geq 2\}$.



The graph of f lies below the graph of g for values of x that lie between −2 and 2. Therefore, the solution of f(x) < g(x) is $(-2, 2) = \{x : -2 < x < 2\}$.



Exercise 16.5E. 16



In Exercises 16.5E. 17-16.5E. 20, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Load each side of the equation into the Y= menu of your calculator. Adjust the WINDOW parameters so that the point of intersection of the graphs is visible in the viewing window. Use the intersect utility in the CALC menu of your calculator to determine the x-coordinate of the point of intersection.

ii. Make an accurate copy of the image in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, and label each graph with its equation.

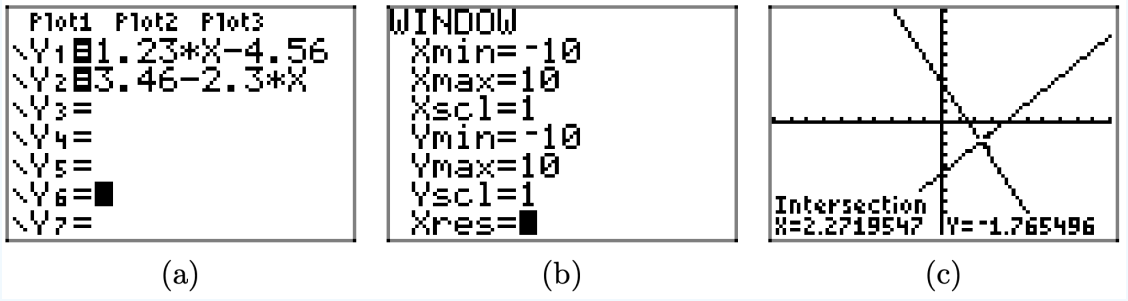
iii. Draw a dashed, vertical line through the point of intersection. Shade and label the solution of the equation on the x-axis.

Exercise 16.5E. 17

1.23x − 4.56 = 3.46 − 2.3x

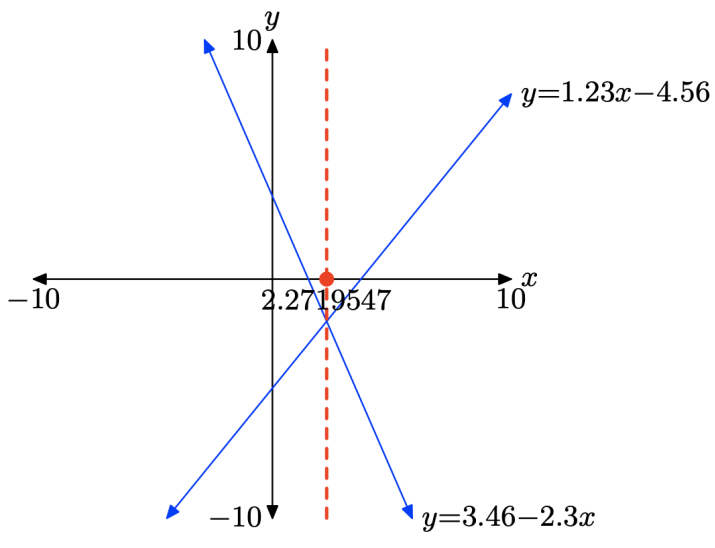
Answer

To solve the equation 1.23x − 4.56 = 3.46 − 2.3x graphically, start by loading the left- and right-hand sides of the equation into Y1 and Y2, respectively, as shown in (a). Use the intersect utility in the CALC menu to determine the point of intersection, as shown in (c).



(a) (b) (c)

Therefore, the solution of the equation is x = 2.2719547, which is shaded on the x-axis in the image that follows. Answers may vary due to roundoff error.



? Exercise 16.5E. 18

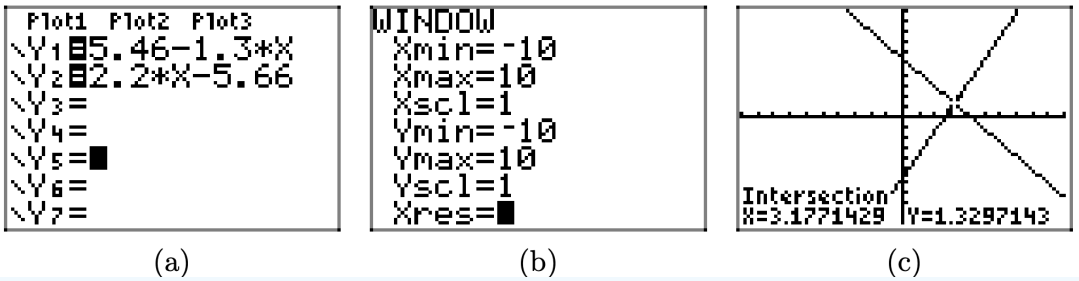
2.23x − 1.56 = 5.46 − 3.3x

? Exercise 16.5E. 19

5.46 − 1.3x = 2.2x − 5.66

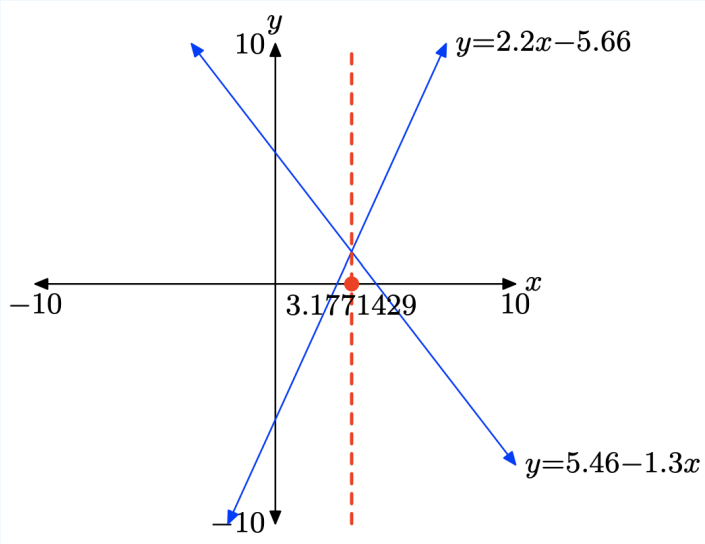
Answer

To solve the equation 5.46 − 1.3x = 2.2x − 5.66 graphically, start by loading the left- and right-hand sides of the equation into Y1 and Y2, respectively, as shown in (a). Use the intersect utility in the CALC menu to determine the point of intersection, as shown in (c).



(a) (b) (c)

Therefore, the solution of the equation is x = 3.1771429, which is shaded on the x-axis in the image that follows. Answers may vary due to roundoff error.



? Exercise 16.5E. 20

2.46 − 1.4x = 1.2x − 2.66

In Exercises 16.5E. 21-16.5E. 26, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Load each side of the inequality into the Y= menu of your calculator. Adjust the WINDOW parameters so that the point(s) of intersection of the graphs is visible in the viewing window. Use the intersect utility in the CALC menu of your calculator to determine the coordinates of the point(s) of intersection.

ii. Make an accurate copy of the image in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, and label each graph with its equation.

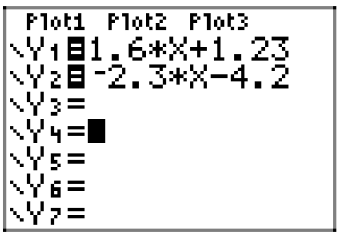
iii. Draw a dashed, vertical line through the point(s) of intersection. Shade and label the solution of the inequality on the x-axis. Use both set-builder and interval notation to describe the solution set.

? Exercise 16.5E. 21

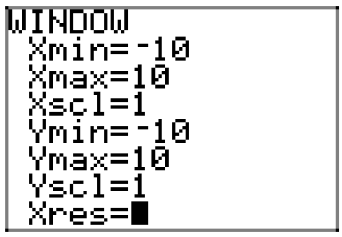
$1.6x + 1.23 \geq -2.3x - 4.2$

Answer

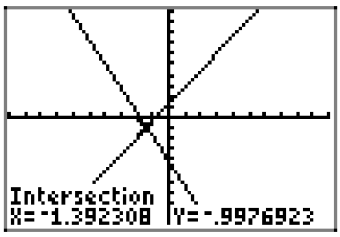
To solve the inequality $1.6x + 1.23 \geq -2.3x - 4.2$ graphically, start by loading the left- and right-hand sides of the inequality into Y1 and Y2, respectively, as shown in (a). Use the intersect utility in the CALC menu to determine the point of intersection, as shown in (c).



(a)

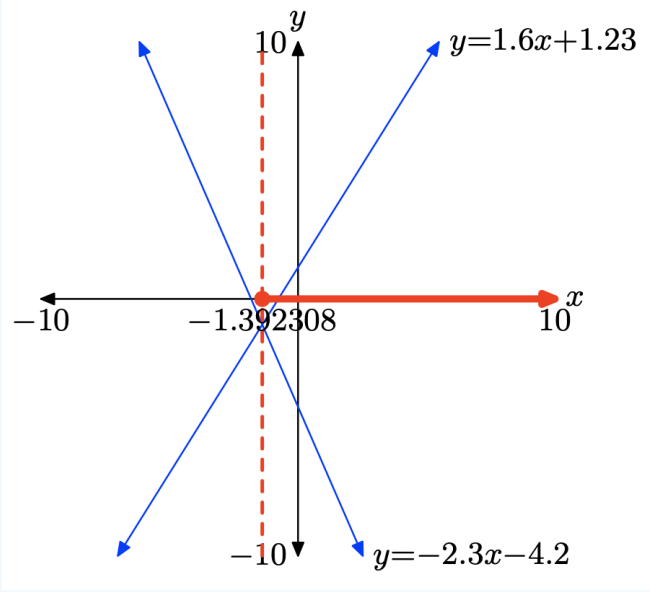


(b)



(c)

The two graphs intersect at x = −1.392308. The graph of y = 1.6x + 1.23 is above the graph of y = −2.3x−4.2 for all values of x that lie to the right of −1.392308. Therefore, the solution of $1.6x+1.23 \geq -2.3x-4.2$ is $[-1.392308, \infty) = \{x : x \geq -1.392308\}$



? Exercise $16.5E.22$

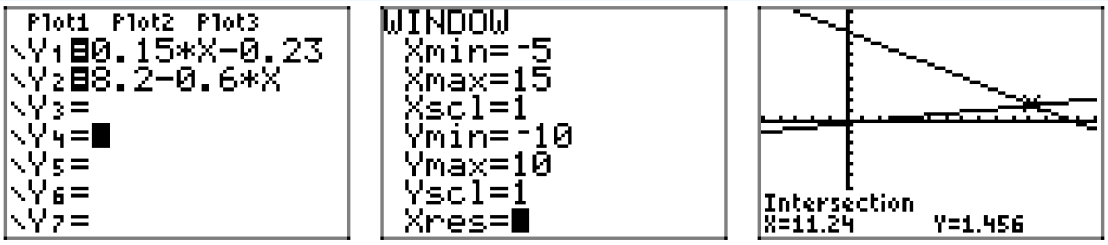
1.24x + 5.6 < 1.2 − 0.52x

? Exercise $16.5E.23$

0.15x − 0.23 > 8.2 − 0.6x

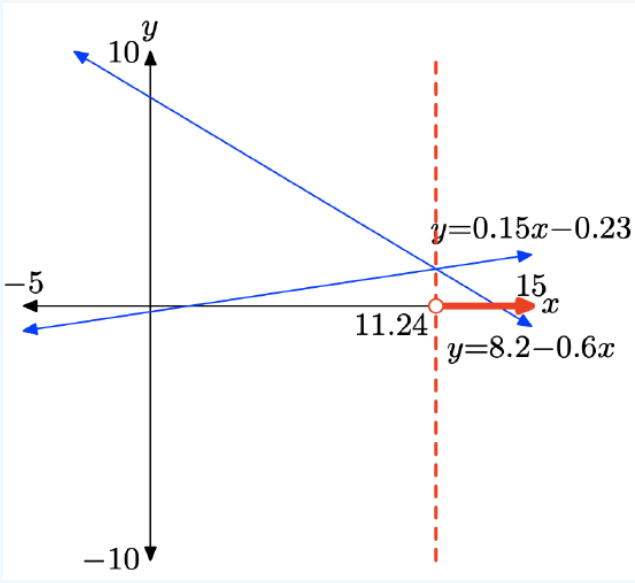
Answer

To solve the inequality 0.15x − 0.23 > 8.2 − 0.6x graphically, start by loading the left- and right-hand sides of the inequality into Y1 and Y2, respectively, as shown in (a). Adjust the viewing window as shown in (b). Use the intersect utility in the CALC menu to determine the point of intersection, as shown in (c).



(a) (b) (c)

The graph of y = 0.15x − 0.23 is above the graph of y = 8.2 − 0.6x for all values of x that lie to the right of 11.24. Therefore, the solution of 0.15x − 0.23 > 8.2 − 0.6x is $(11.24, \infty) = \{x : x > 11.24\}$



Exercise 16.5E. 24

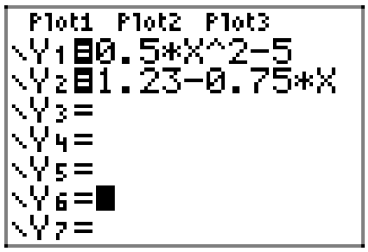
$-1.23x - 9.76 \leq 1.44x + 22.8$

Exercise 16.5E. 25

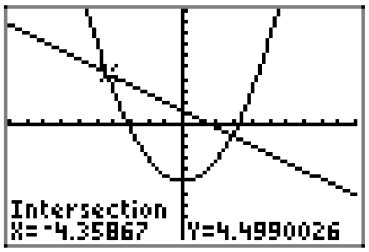
$0.5x^2 - 5 < 1.23 - 0.75x$

Answer

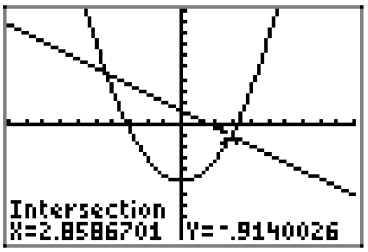
To solve the inequality $0.5x^2 - 5 < 1.23 - 0.75x$ graphically, start by loading the left- and right-hand sides of the inequality into Y1 and Y2, respectively, as shown in (a). Use the intersect utility in the CALC menu to determine the points of intersection, as shown in (b) and (c).



(a)

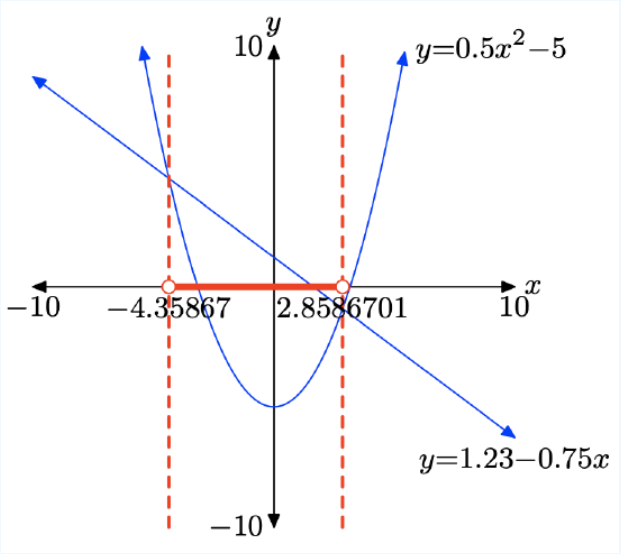


(b)



(c)

The graph of y = 0.5x2−5 is below the graph of y = 1.23−0.75x for all values of x that lie between −4.35867 and 2.8586701. Therefore, the solution of $0.5x^2 - 5 < 1.23 - 0.75x$ is (−4.35867, 2.8586701) or {x : −4.35867 < x < 2.8586701}.



? Exercise 16.5E. 26

$4 - 0.5x^2 \leq 0.72x - 1.34$

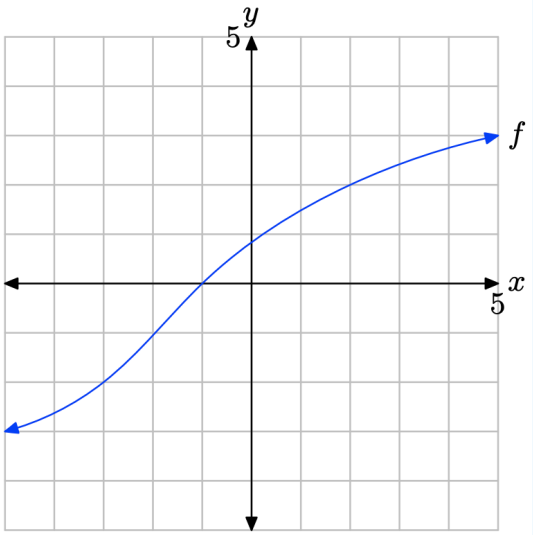
In Exercises 16.5E. 27-16.5E. 30, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Make an accurate copy of the image on graph paper (label the graph with the letter f and label and scale each axis), drop a dashed vertical line through the x-intercept of the graph of f, then label and shade the solution of f(x) = 0 on the x-axis. Use set-builder notation to describe your solution.

ii. Make a second copy of the image on graph paper, drop a dashed, vertical line through the x-intercept of the graph of f, then label and shade the solution of f(x) > 0 on the x-axis. Use set-builder and interval notation to describe your solution set.

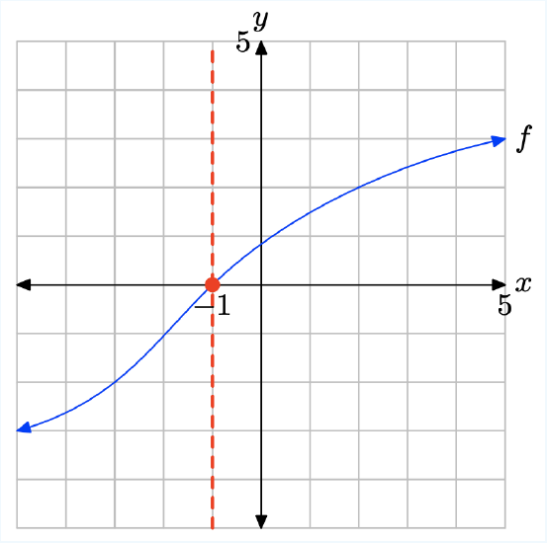
iii. Make a third copy of the image on graph paper, drop a dashed, vertical line through the x-intercept of the graph of f, then label and shade the solution of f(x) < 0 on the x-axis. Use set-builder and interval notation to describe your solution set.

? Exercise 16.5E. 27

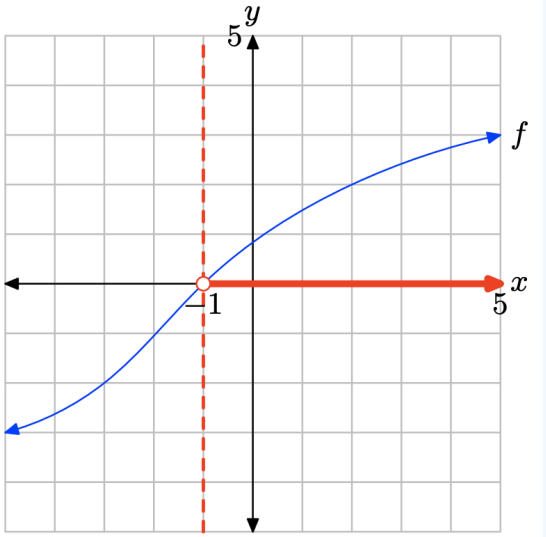


Answer

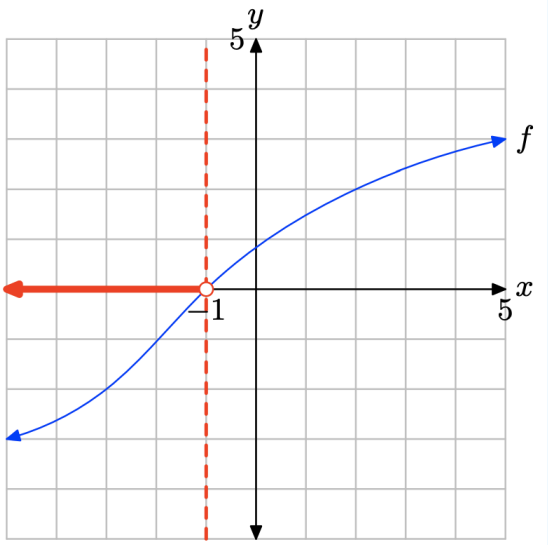
The graph of f intercepts the x-axis at x = −1. Therefore, the solution of f(x) = 0 is x = −1.



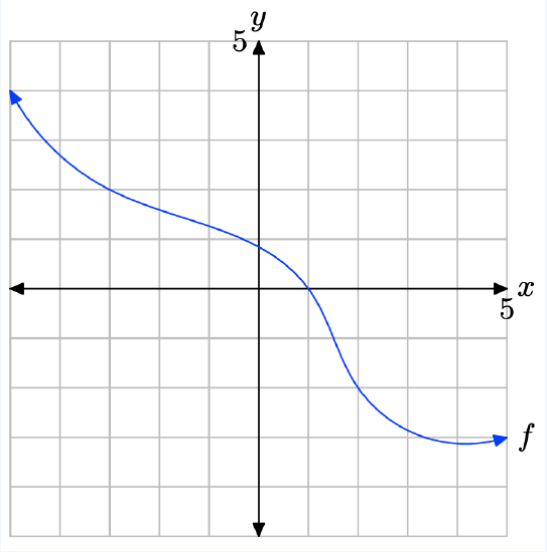
The graph of f lies above the x-axis for all values of x that lie to the right of −1. Therefore, the solution of f(x) > 0 is $(-1, \infty) = \{x : x > -1\}$.



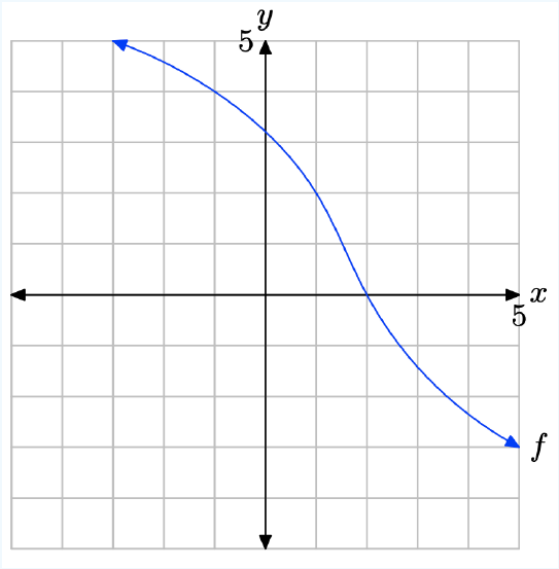
The graph of f lies below the x-axis for all values of x that lie to the left of −1. Therefore, the solution of f(x) < 0 is $(-\infty, -1) = \{x : x < -1\}$



? Exercise $16.5E.28$

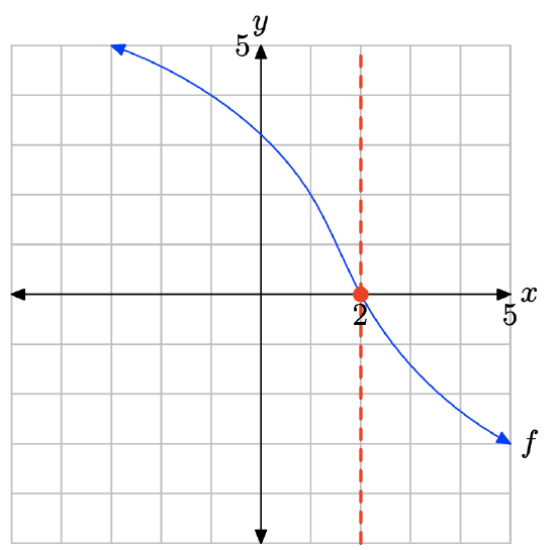


Exercise $16.5E.29$

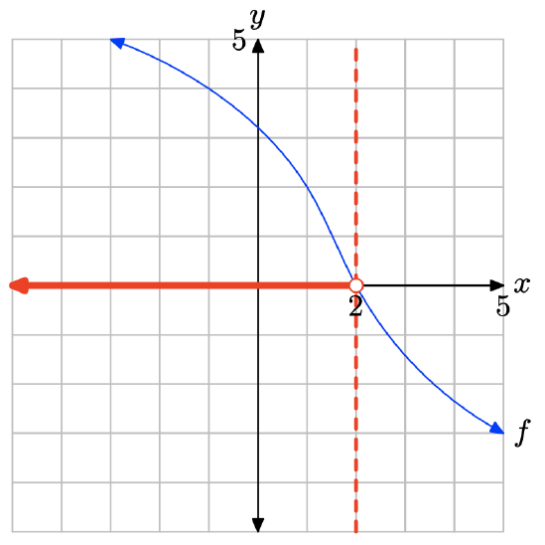


Answer

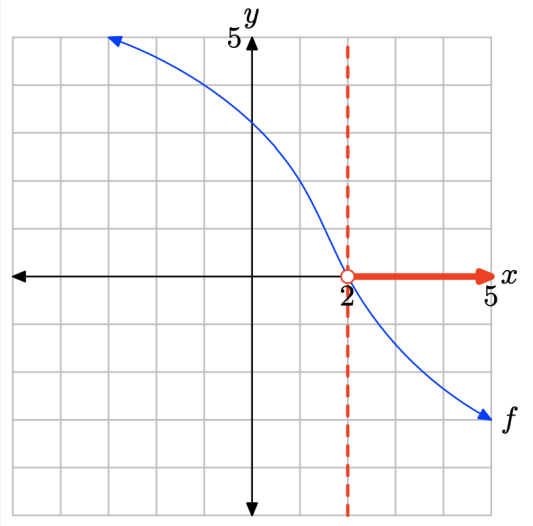
The graph of f intercepts the x-axis at x = 2. Therefore, the solution of f(x) = 0 is x = 2.



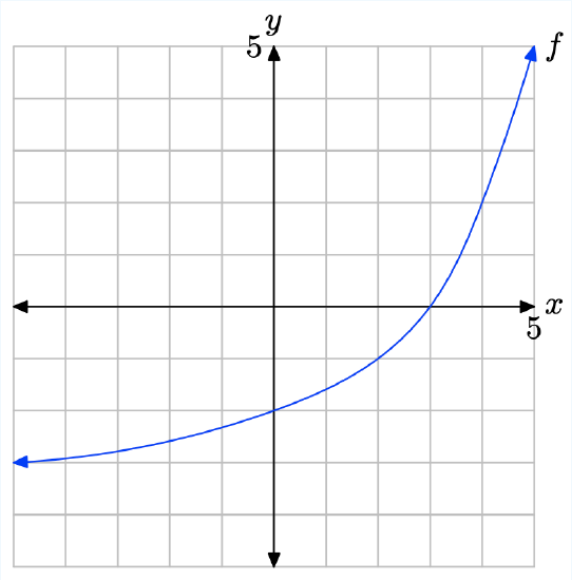
The graph of f lies above the x-axis for all values of x that lie to the left of x = 2. Therefore, the solution of f(x) > 0 is $(-\infty, 2) = \{x : x < 2\}$.



The graph of f lies below the x-axis for all values of x that lie to the right of x = 2. Therefore, the solution of f(x) < 0 is $(2, \infty) = \{x : x > 2\}$



Exercise $16.5E.30$

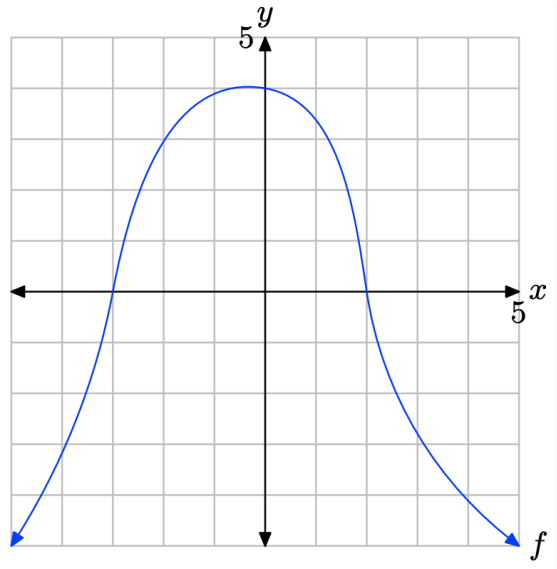


In Exercises $16.5E.31$-$16.5E.34$, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Make an accurate copy of the image on graph paper, drop dashed, vertical lines through the x-intercepts, then label and shade the solution of $f(x) \geq 0$ on the x-axis. Use set-builder and interval notation to describe your solution set.

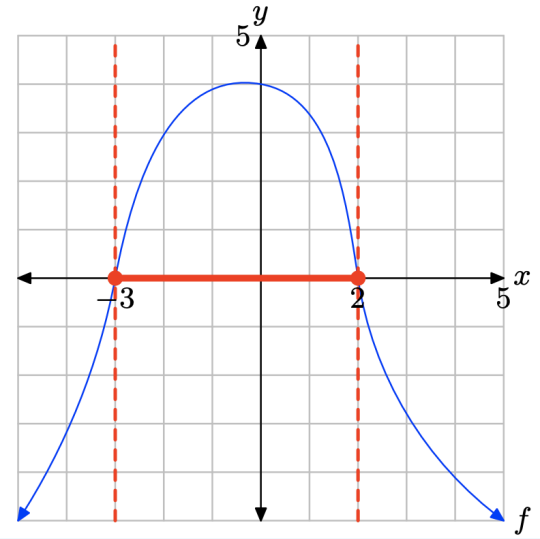
ii. Make a second copy of the image on graph paper, drop dashed, vertical lines through the x-intercepts, then label and shade the solution of f(x) < 0 on the x-axis. Use set-builder and interval notation to describe your solution set.

Exercise $16.5E.31$

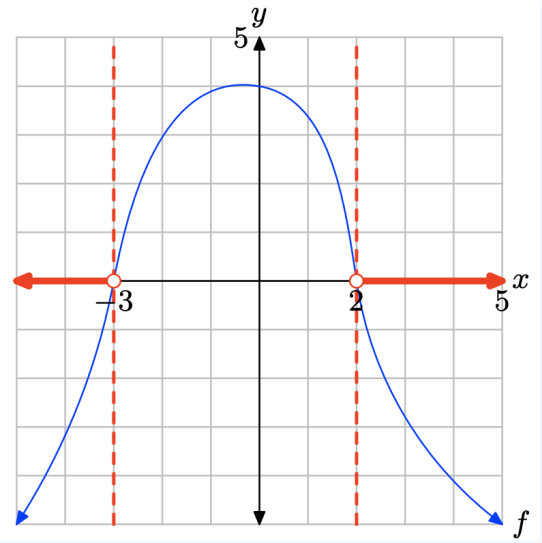


Answer

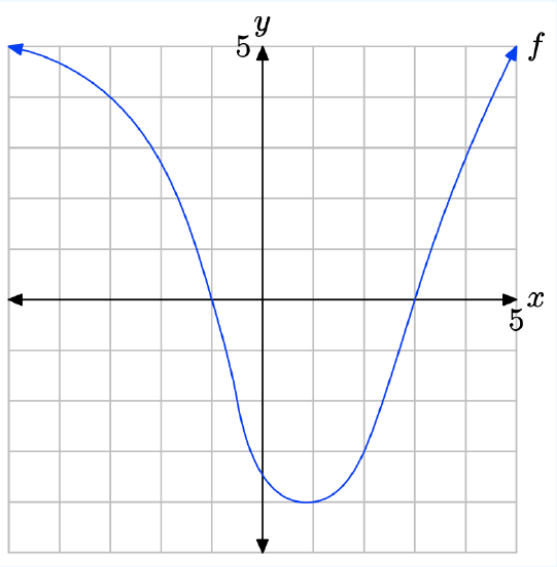
The graph of f intercepts the x-axis at x = −3 and x = 2. The graph of f lies above the x-axis for all values of x that lie between x = −3 and x = 2. Therefore, the solution of $f(x) \geq 0$ is $[-3, 2] = \{x : -3 \leq x \leq 2\}$.



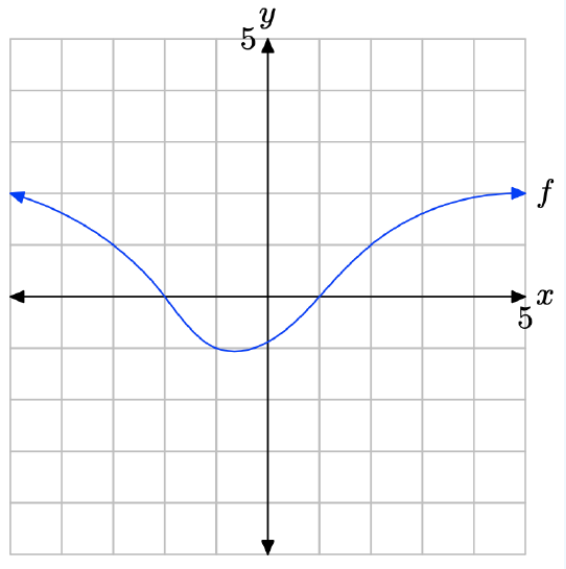
The graph of f lies below the x-axis for all values of x that lie to the left of x = −3 or to the right of x = 2. Therefore, the solution of f(x) < 0 is $(-\infty, -3) \cup (2, \infty) = \{x : x < -3 or x > 2\}$.



Exercise 16.5E.32

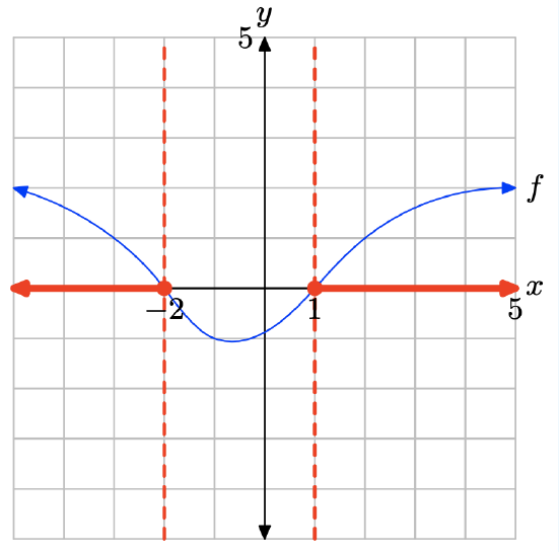


Exercise 16.5E.33

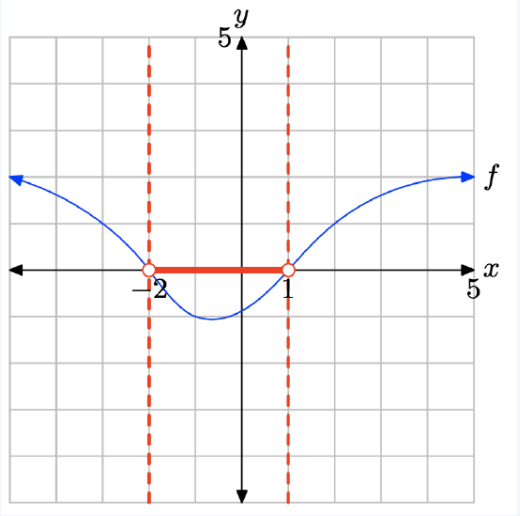


Answer

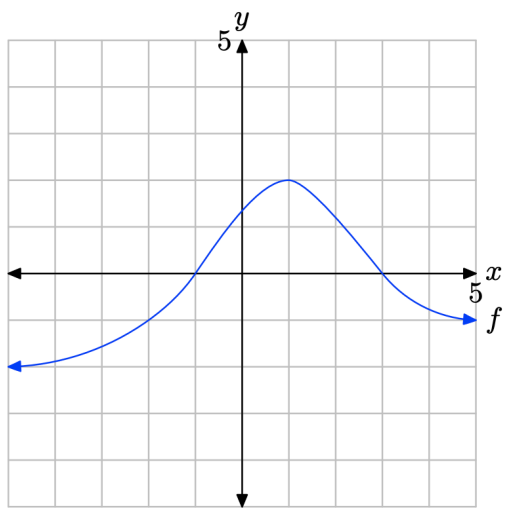
The graph of f intercepts the x-axis at x = −2 and x = 1. The graph of f lies above the x-axis for all values of x that lie to the left x = −2 or to the right of x = 1. Therefore, the solution of $f(x) \geq 0$ is $(-\infty, -2] \cup [1, \infty) = \{x : x \leq -2 or x \geq 1\}$.



The graph of f lies below the x-axis for all values of x that lie between x = −2 and x = 1. Therefore, the solution of f(x) < 0 is $(-2, 1) = \{x : -2 < x < 1\}$.



Exercise 16.5E.34



In Exercises 16.5E.35-16.5E.38, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Load the given function f into the Y= menu of your calculator. Adjust the WINDOW parameters so that the x-intercept(s) of the graph of f is visible in the viewing window. Use the zero utility in the CALC menu of your calculator to determine the coordinates of the x-intercept(s) of the graph of f.

ii. Make an accurate copy of the image in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, and label the graph with its equation.

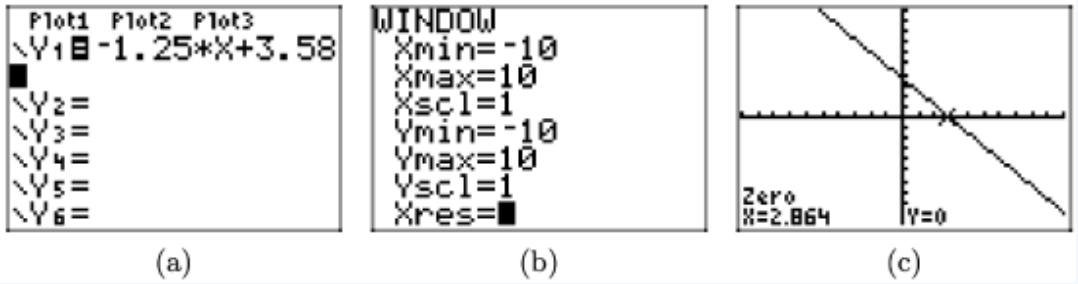
iii. Draw a dashed, vertical line through the x-intercept(s). Shade and label the solution of the inequality f(x) > 0 on the x-axis. Use both set-builder and interval notation to describe the solution set.

Exercise 16.5E.35

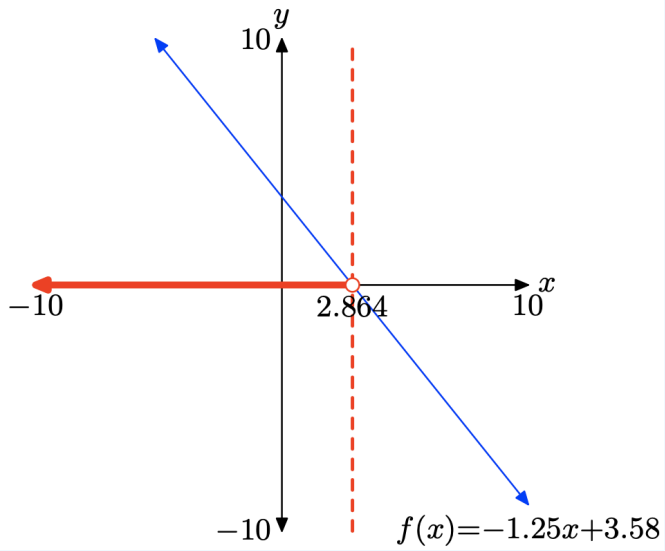
f(x) = −1.25x + 3.58

Answer

To solve the inequality f(x) > 0 graphically, start by loading f(x) = −1.25x+3.58 into Y1. Use the zero utility in the CALC menu to determine the zero of f, as shown in (c).



The graph of f lies above the x-axis for all values of x that lie to the left of x = 2.864. Therefore, the solution of f(x) > 0 is $(-\infty, 2.864) = \{x : x < 2.864\}$. Answers may vary due to round-off error.



Exercise 16.5E. 36

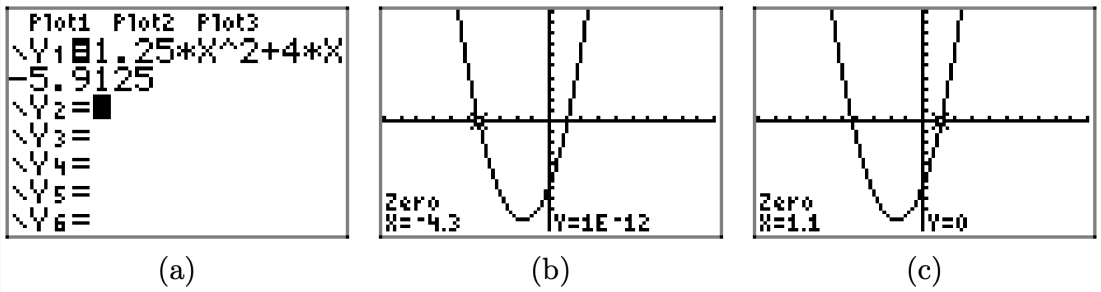
f(x) = 1.34x − 4.52

Exercise 16.5E. 37

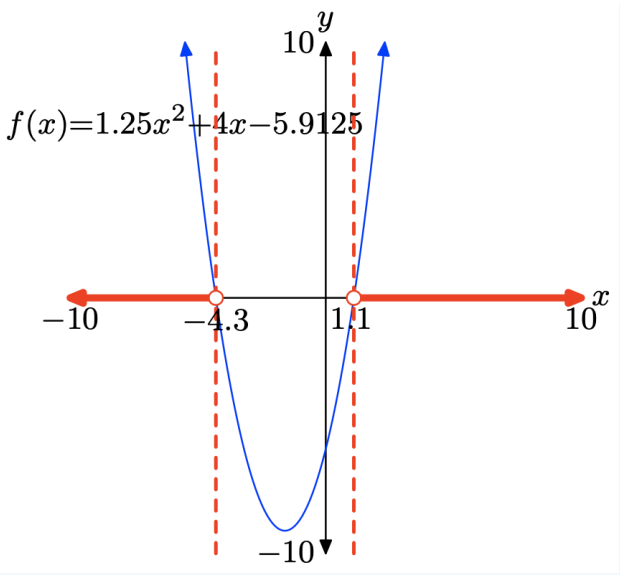
$f(x) = 1.25x^2 + 4x - 5.9125$

Answer

To solve the inequality f(x) > 0 graphically, start by loading $f(x) = 1.25x^2 + 4x - 5.9125$ into Y1. Use the zero utility in the CALC menu to determine the zeros of f, as shown in (b) and (c).



The graph of f lies above the x-axis for all values of x that lie to the left of x = −4.3 or to the right of x = 1.1. Therefore, the solution of f(x) > 0 is $(-\infty, -4.3) \cup (1.1, \infty)$ or $\{x : x < -4.3 or x > 1.1\}$. Answers may vary due to round-off error.



Exercise 16.5E. 38

$f(x) = -1.32x^2 - 3.96x + 5.9532$

In Exercises 16.5E. 39-16.5E. 42, perform each of the following tasks. Remember to use a ruler to draw all lines.

i. Load the given function f into the Y= menu of your calculator. Adjust the WINDOW parameters so that the x-intercept(s) of the graph of f is visible in the viewing window. Use the zero utility in the CALC menu of your calculator to determine the coordinates of the x-intercept(s) of the graph of f.

ii. Make an accurate copy of the image in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, and label the graph with its equation.

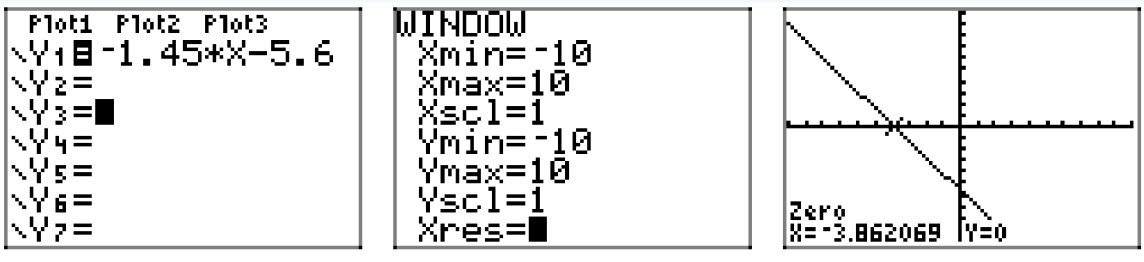
iii. Draw a dashed, vertical line through the x-intercept(s). Shade and label the solution of the inequality $f(x) \leq 0$ on the x-axis. Use both set-builder and interval notation to describe the solution set.

Exercise 16.5E. 39

f(x) = −1.45x − 5.6

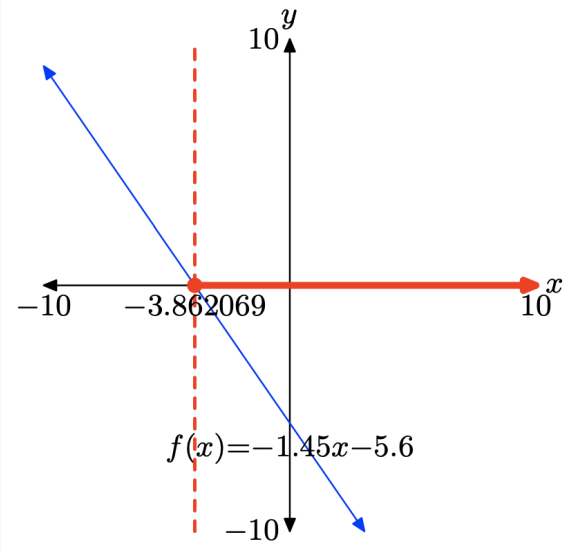
Answer

To solve the inequality $f(x) \leq 0$ graphically, start by loading f(x) = −1.45x−5.6 into Y1. Use the zero utility in the CALC menu to determine the zero of f, as shown in (c).



(a) (b) (c)

The graph of f intercepts the x-axis at x = −3.862069. The graph of f lies below the x-axis for all values of x that lie to the right of x = −3.862069. Therefore, the solution of $f(x) \leq 0$ is $[-3.862069, \infty) = \{x : x \geq -3.862069\}$ Answers may vary due to roundoff error.



Exercise 16.5E. 40

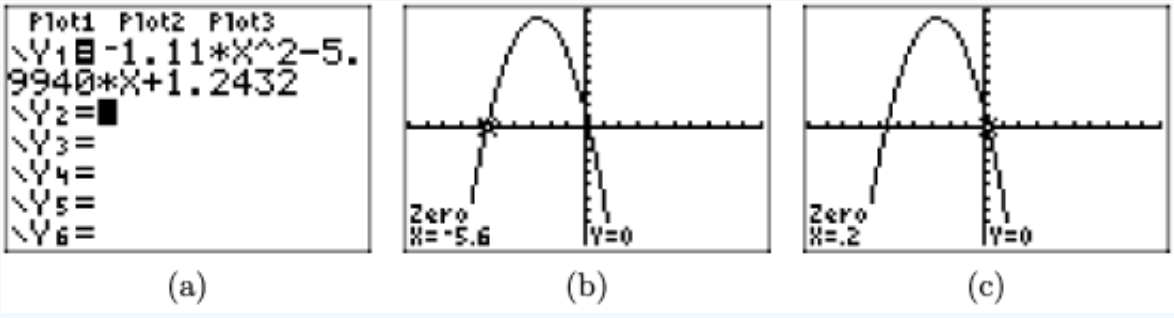
f(x) = 1.35x + 8.6

Exercise 16.5E. 41

$f(x) = -1.11x^2 - 5.9940x + 1.2432$

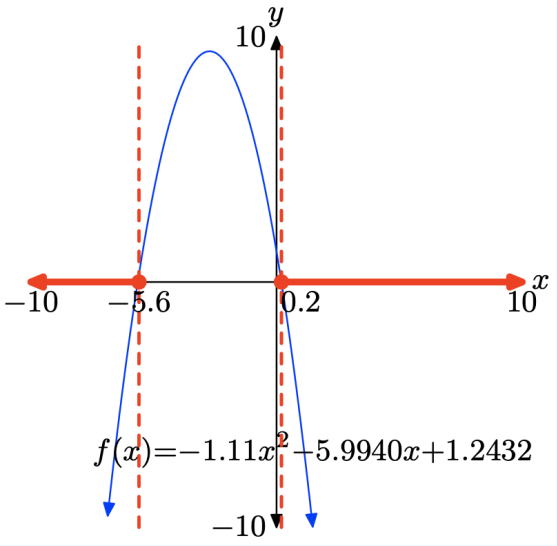
Answer

To solve the inequality $f(x) \leq 0$ graphically, start by loading $f(x) = -1.11x^2 - 5.9940x + 1.2432$into Y1. Use the zero utility in the CALC menu to determine the zeros of f, as shown in (b) and (c).



(a) (b) (c)

The graph of f intercepts the x-axis at x = −5.6 and x = 0.2. The graph of f lies below the x-axis for all values of x that lie to the left of x = −5.6 or to the right of x = 0.2. Therefore, the solution of $f(x) \leq 0$ is $(-\infty, -5.6] \cup [0.2, \infty)$ or $\{x : x \leq -5.6 or x \geq 0.2\}$. Answers may vary due to roundoff error.

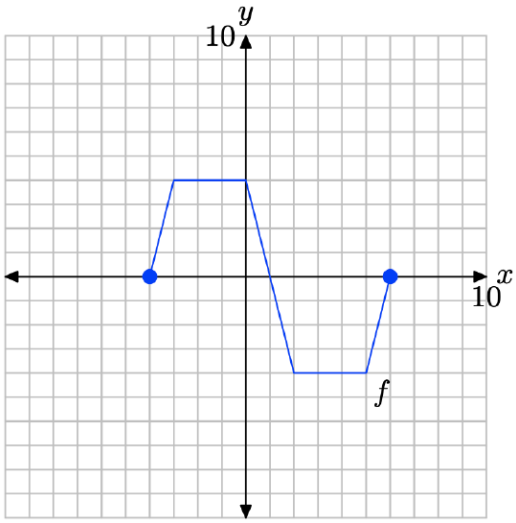


Exercise $16.5E.42$

$f(x) = 1.22x^2 - 6.3440x + 1.3176$

2.5 Exercises

Pictured below is the graph of a function f.



The table that follows evaluates the function f in the plot at key values of x. Notice the horizontal format, where the first point in the table is the ordered pair (−4, 0).

x	-4	-3	0	2	5	6
f(x)	0	4	4	-4	-4	0

Use the graph and the table to complete each of following tasks for Exercises $16.5E.1$ - $16.5E.10$.

i. Set up a coordinate system on graph paper. Label and scale each axis, then copy and label the original graph of f onto your coordinate system. Remember to draw all lines with a ruler.

ii. Use the original table to help complete the table for the given function in the exercise.

iii. Using a different colored pencil, plot the data from your completed table on the same coordinate system as the original graph of f. Use these points to help complete the graph of the given function in the exercise, then label this graph with its equation given in the exercise.

Exercise 16.5E.1

y= 2f(x).

x	-4	-3	0	2	5	6
y						

Answer

The original function table.

x	-4	-3	0	2	5	6
f(x)	0	4	4	-4	-4	0

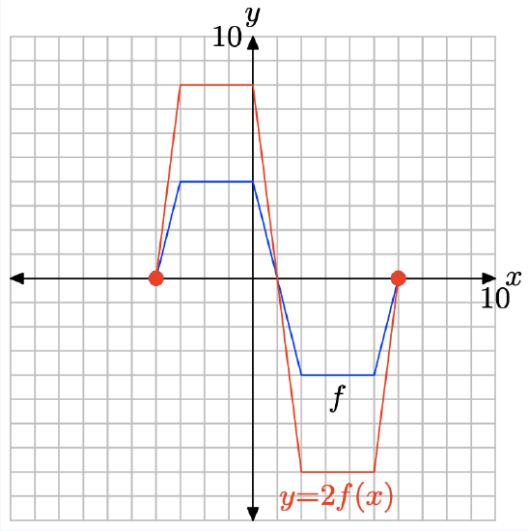
Evaluate the function y = 2f(x) at x = −4, −3, 0, 2, 5, and 6.

$$y = 2f(-4) = 2(0) = 0$$
$$y = 2f(-3) = 2(4) = 8$$
$$y = 2f(0) = 2(4) = 8$$
$$y = 2f(2) = 2(-4) = -8$$
$$y = 2f(5) = 2(-4) = -8$$
$$y = 2f(6) = 2(0) = 0$$

Points satisfying y = 2f(x).

x	-4	-3	0	2	5	6
y	0	8	8	-8	-8	0

Plot the points in the table to get the graph of y = 2f(x).



Note that multiplying by 2, as in y = 2f(x), stretches the graph of y = f(x) vertically by a factor of 2.

Exercise $16.5E.2$

y = (1/2)f(x).

x	-4	-3	0	2	5	6
y						

Exercise $16.5E.3$

y = −f(x).

x	-4	-3	0	2	5	6
y						

Answer

The original function table.

x	-4	-3	0	2	5	6
f(x)	0	4	4	-4	-4	0

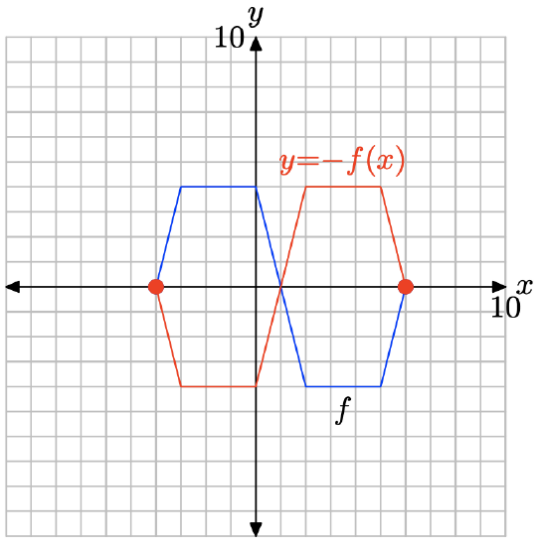
Evaluate the function y = −f(x) at x = −4, −3, 0, 2, 5, and 6.

$$y = -f(-4) = -(0) = 0$$
$$y = -f(-3) = -(4) = -4$$
$$y = -f(0) = -(4) = -4$$
$$y = -f(2) = -(-4) = 4$$
$$y = -f(5) = -(-4) = 4$$
$$y = -f(6) = -(0) = 0$$

Points satisfying y = −f(x).

x	-4	-3	0	2	5	6
y	0	-4	-4	4	4	0

Plot the points in the table to get the graph of y = −f(x).



Note that negating the function, as in y = −f(x), reflects the graph of y = f(x) across the x-axis.

Exercise 16.5E.4

y = f(x) − 2.

x	-4	-3	0	2	5	6
y						

Exercise 16.5E.5

y = f(x) + 4.

x	-4	-3	0	2	5	6
y						

Answer

The original function table.

x	-4	-3	0	2	5	6
f(x)	0	4	4	-4	-4	0

Evaluate the function y = f(x) + 4 at x = −4, −3, 0, 2, 5, and 6.

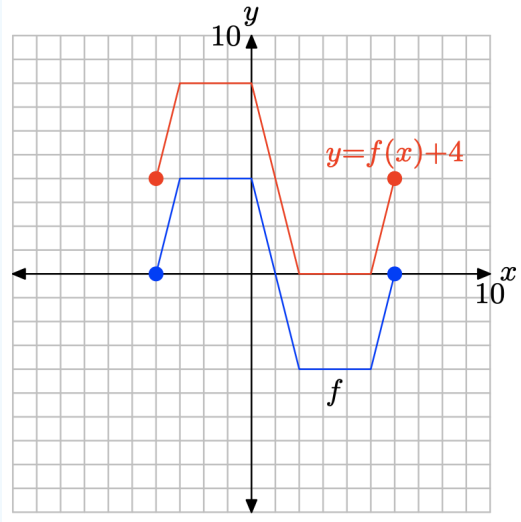
$$y = f(-4) + 4 = (0) + 4 = 4$$
$$y = f(-3) + 4 = (4) + 4 = 8$$
$$y = f(0) + 4 = (4) + 4 = 8$$
$$y = f(2) + 4 = (-4) + 4 = 0$$
$$y = f(5) + 4 = (-4) + 4 = 0$$
$$y = f(6) + 4 = (0) + 4 = 4$$

Points satisfying y = f(x) + 4.

x	-4	-3	0	2	5	6

y	4	8	8	0	0	4

Plot the points in the table to get the graph of y = f(x) + 4.



Note that adding 4, as in y = f(x) + 4, translates the graph of y = f(x) upwards 4 units.

Exercise 16.5E.6

y = −2f(x)

x	-4	-3	0	2	5	6
y						

Exercise 16.5E.7

y = (−1/2)f(x)

x	-4	-3	0	2	5	6
y						

Answer

The original function table.

x	-4	-3	0	2	5	6
f(x)	0	4	4	-4	-4	0

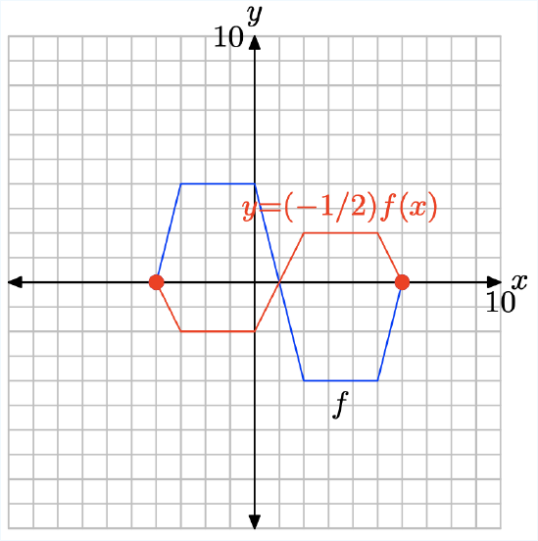
Evaluate the function y = (−1/2)f(x) at x = −4, −3, 0, 2, 5, and 6.

$$y = (-1/2)f(-4) = (-1/2)(0) = 0$$
$$y = (-1/2)f(-3) = (-1/2)(4) = -2$$
$$y = (-1/2)f(0) = (-1/2)(4) = -2$$
$$y = (-1/2)f(2) = (-1/2)(-4) = 2$$
$$y = (-1/2)f(5) = (-1/2)(-4) = 2$$
$$y = (-1/2)f(6) = (-1/2)(0) = 0$$

Points satisfying y = (−1/2)f(x).

x	-4	-3	0	2	5	6
y	0	-2	-2	2	2	0

Plot the points in the table to get the graph of y = (−1/2)f(x).



Note that multiplying by −1/2, as in y = (−1/2)f(x), compresses the graph of y = f(x) vertically by a factor of 2, then reflects the result across the x-axis.

Exercise 16.5E.8

y = −f(x) + 3.

x	-4	-3	0	2	5	6
y						

Exercise 16.5E.9

y = −f(x) − 2

x	-4	-3	0	2	5	6
y						

Answer

The original function table.

x	-4	-3	0	2	5	6
f(x)	0	4	4	-4	-4	0

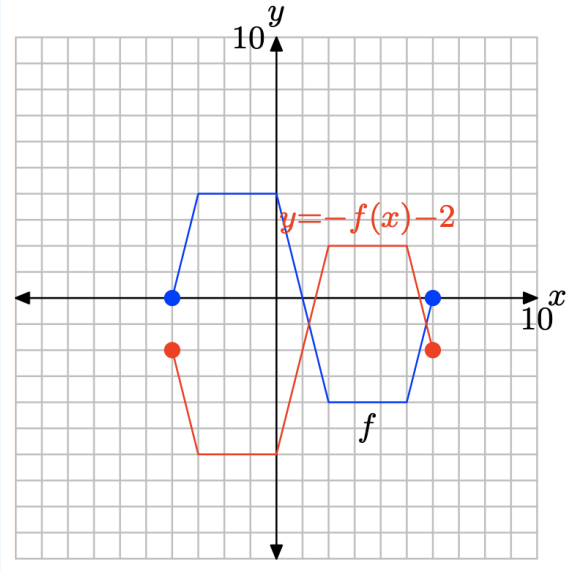
Evaluate the function y = −f(x) − 2 at x = −4, −3, 0, 2, 5, and 6.

$$y = -f(-4) - 2 = -(0) - 2 = -2$$
$$y = -f(-3) - 2 = -(4) - 2 = -6$$
$$y = -f(0) - 2 = -(4) - 2 = -6$$
$$y = -f(2) - 2 = -(-4) - 2 = 2$$
$$y = -f(5) - 2 = -(-4) - 2 = 2$$
$$y = -f(6) - 2 = -(0) - 2 = -2$$

Points satisfying y = −f(x) − 2.

x	-4	-3	0	2	5	6
y	-2	-6	-6	2	2	-2

Plot the points in the table to get the graph of y = −f(x) − 2.



Note that negating then subtracting 2, as in y = −f(x) − 2, first reflects the graph of y = f(x) across the x-axis, then translates the resulting reflection 2 units downward.

Exercise 16.5E.10

y = (−1/2)f(x) + 3

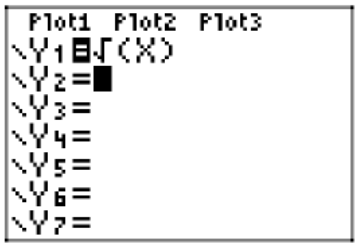
x	-4	-3	0	2	5	6
y						

Exercise 16.5E.11

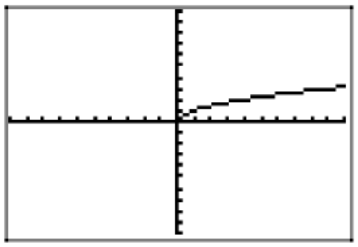
Use your graphing calculator to draw the graph of $y = \sqrt{x}$. Then, draw the graph of $y = \sqrt{x}$. In your own words, explain what you learned from this exercise.

Answer

First, draw the graph of $y = \sqrt{x}$.

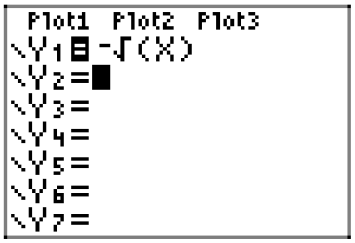


(a)

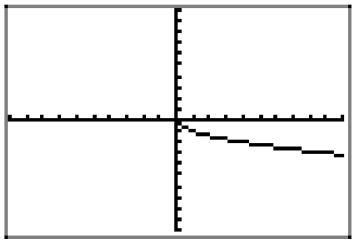


(b)

The graph of $y = -\sqrt{x}$ is a reflection of the graph of $y = \sqrt{x}$ across the x-axis.



(c)



(d)

Negating a function appears to reflect the graph of the function across the x-axis.

? Exercise 16.5E. 12

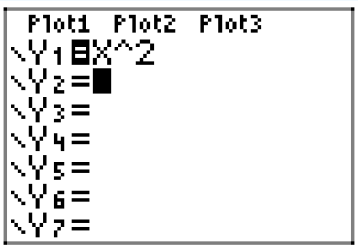
Use your graphing calculator to draw the graph of y = |x|. Then, draw the graph of y = −|x|. In your own words, explain what you learned from this exercise.

? Exercise 16.5E. 13

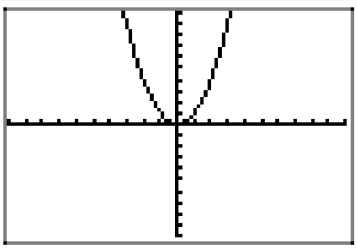
Use your graphing calculator to draw the graph of $y = x^2$. Then, in succession, draw the graphs of $y = x^2 - 2$, $y = x^2 - 4$, and $y = x^2 - 6$. In your own words, explain what you learned from this exercise.

Answer

First, draw the graph of $y = x^2$.

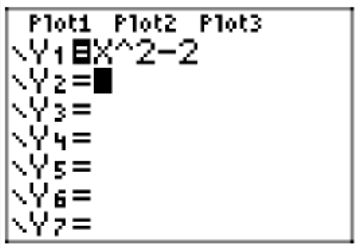


(a)

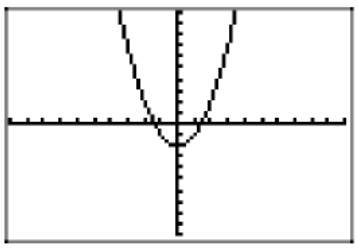


(b)

Subtracting 2 (as in $y = x^2 - 2$) translates the graph of $y = x^2$ two units downward in the y-direction.

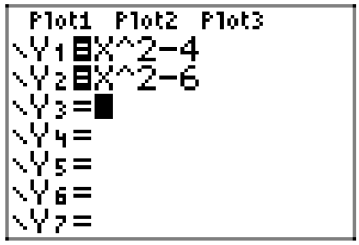


(c)

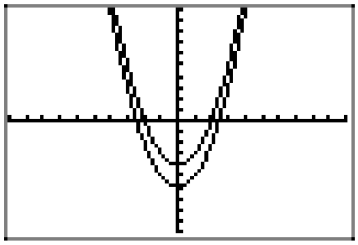


(d)

Similarly, subtracting 4 and 6 translates the graph of $y = x^2$ four units and 6 units downward, respectively



(e)



(f)

In general, if c is positive, then the graph of y = f(x) − c is obtained by translating the graph of y = f(x) downward c units.

Exercise 16.5E. 14

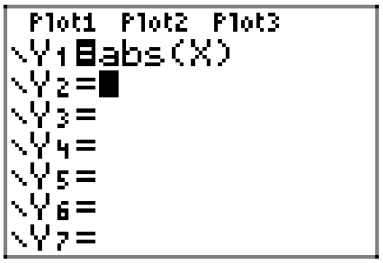
Use your graphing calculator to draw the graph of $y = x^2$. Then, in succession, draw the graphs of $y = x^2 + 2$, $y = x^2 + 4$, and $y = x^2 + 6$. In your own words, explain what you learned from this exercise.

Exercise 16.5E. 15

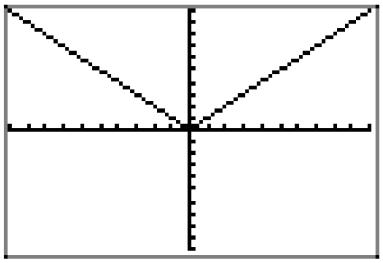
Use your graphing calculator to draw the graph of y = |x|. Then, in succession, draw the graphs of y = 2|x|, y = 3|x|, and y = 4|x|. In your own words, explain what you learned from this exercise.

Answer

First, draw the graph of y = |x|.

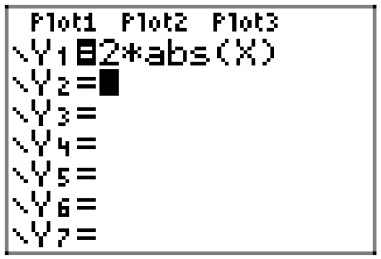


(a)

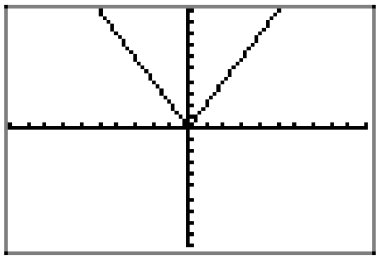


(b)

Multiplying by 2, as in y = 2|x|, stretches the graph of y = |x| vertically by a factor of 2.

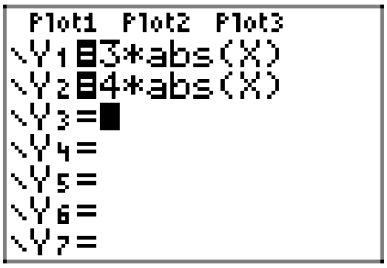


(c)

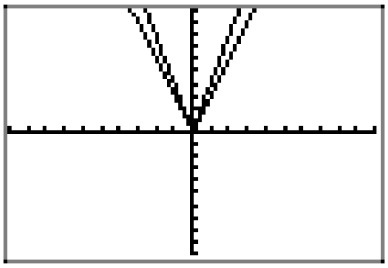


(d)

Similarly, multiplying by 3 and 4, as in y = 3|x| and y = 4|x|, stretches the graph of y = |x| vertically by factors of 3 and 4, respectively.



(e)



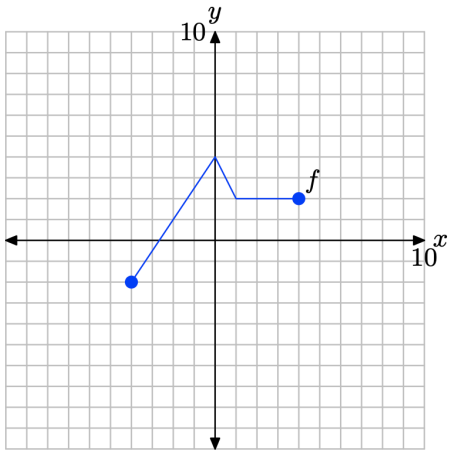
(f)

In general, if a > 1, then the graph of y = af(x) is obtained by stretching the graph of y = f(x) vertically by a factor of a.

? Exercise 16.5E. 16

Use your graphing calculator to draw the graph of y = |x|. Then, in succession, draw the graphs of y = (1/2)|x|, y = (1/3)|x|, and y = (1/4)|x|. In your own words, explain what you learned from this exercise.

Pictured below is the graph of a function f. In Exercises 16.5E. 17-16.5E. 22, use this graph to perform each of the following tasks.



i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of f on your coordinate system. Remember to draw all lines with a ruler.

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of vertical scaling, vertical reflection, and vertical translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow without using tables.

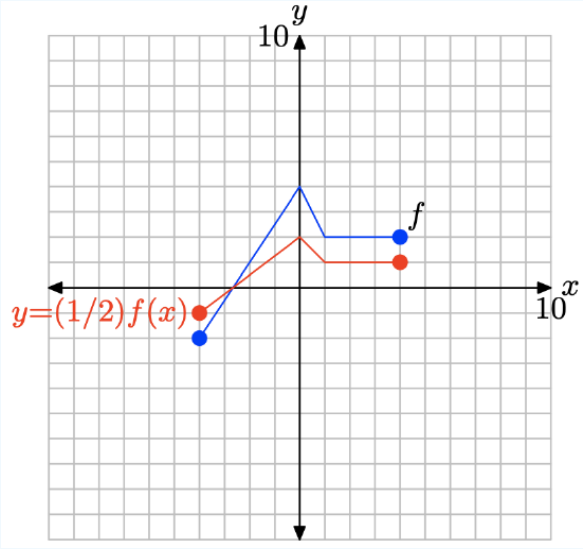
iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function f and the graph of the function in the exercise.

Exercise 16.5E.17

y = (1/2)f(x).

Answer

To obtain the plot of y = (1/2)f(x), simply multiply the y-value of each point on the graph of y = f(x) by 1/2, keeping the x-value the same.



Note that multiplying by 1/2, as in y = (1/2)f(x), compresses the graph of y = f(x) vertically by a factor of 2.

Exercise 16.5E.18

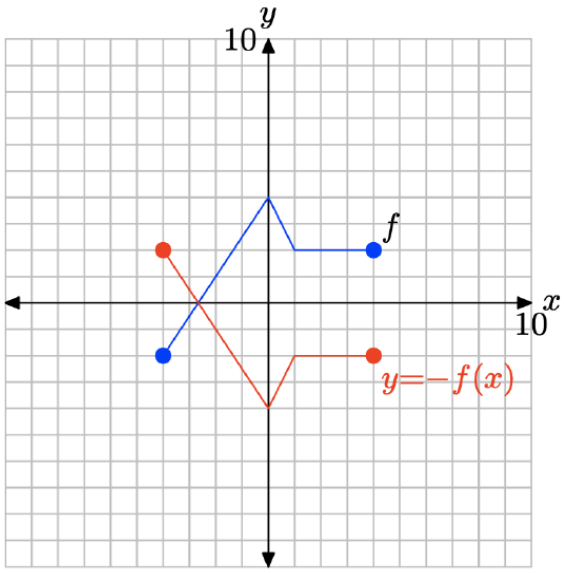
y = 2f(x).

Exercise 16.5E.19

y= −f(x).

Answer

To obtain the plot of y = −f(x), simply negate the y-value of each point on the graph of y = f(x).



Note that negating, as in y = −f(x), reflects the graph of y = f(x) across the x-axis.

Exercise 16.5E.20

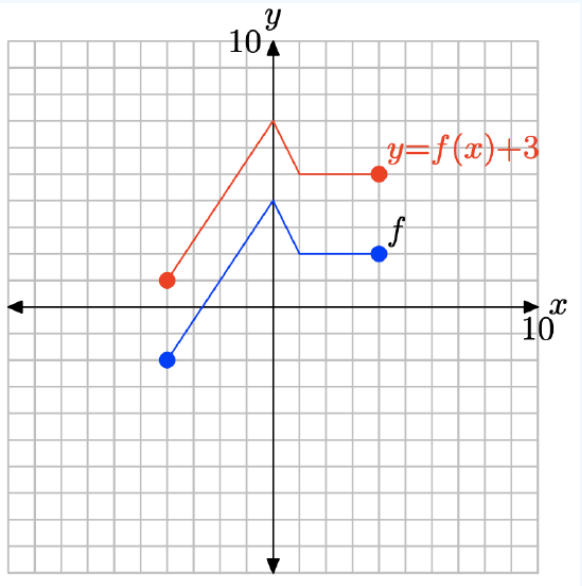
y = f(x) − 1

Exercise 16.5E.21

y = f(x) + 3.

Answer

To obtain the plot of y = f(x) + 3, simply add 3 to the y-value of each point on the graph of y = f(x).

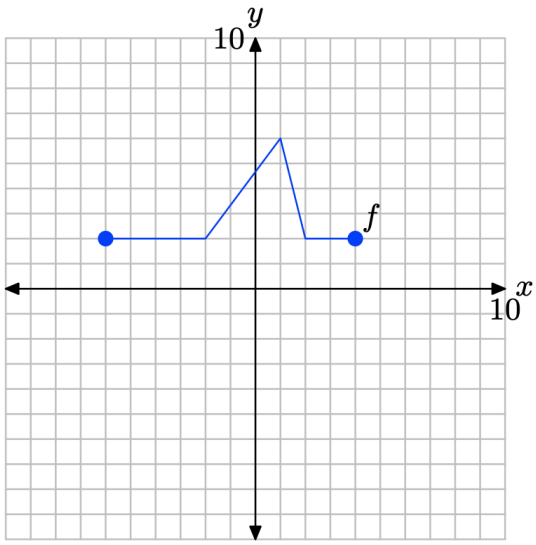


Note that adding 3, as in y = f(x) + 3, translates the graph of y = f(x) upwards 3 units.

Exercise 16.5E.22

y = f(x) − 4

Pictured below is the graph of a function f. In Exercises 16.5E.23-16.5E.28, use this graph to perform each of the following tasks.



i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of f on your coordinate system. Remember to draw all lines with a ruler.

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of vertical scaling, vertical reflection, and vertical translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow without using tables.

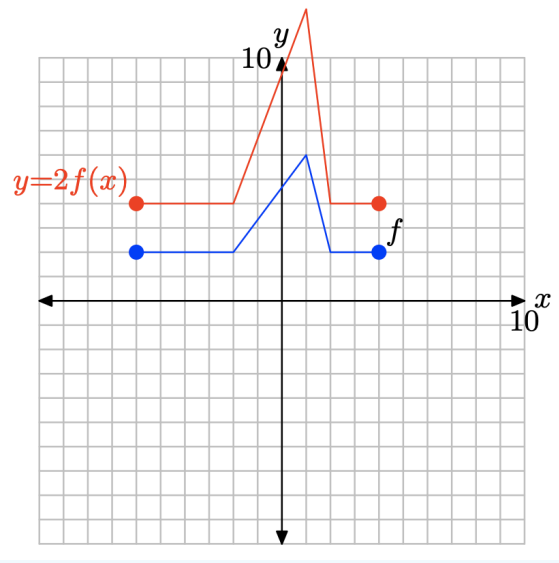
iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function f and the graph of the function in the exercise.

Exercise 16.5E.23

y = 2f(x)

Answer

To obtain the plot of y = 2f(x), simply multiply the y-value of each point of y = f(x) by 2.



Note that multiplying by 2, as in y = 2f(x), stretches the graph of y = f(x) vertically by a factor of 2.

Exercise 16.5E.24

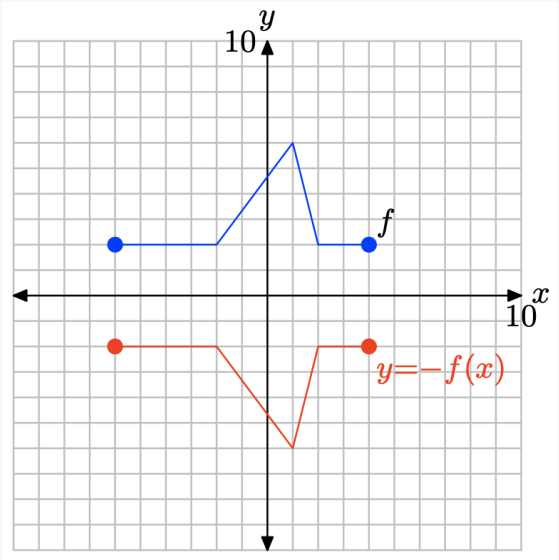
y = (1/2)f(x)

Exercise 16.5E.25

y = −f(x).

Answer

To obtain the plot of y = −f(x), simply negate the y-value of each point on the graph of y = f(x).



Note that negating a function, as in y = −f(x), reflects the graph of y = f(x) across the x-axis.

Exercise 16.5E.26

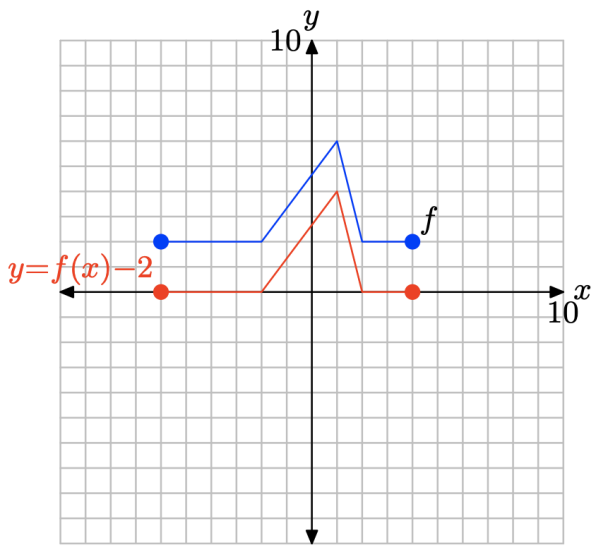
y = f(x) + 3

Exercise 16.5E.27

y = f(x) − 2

Answer

To obtain the plot of y = f(x) − 2, simply subtract 2 from the y-value of each point on the graph of y = f(x).

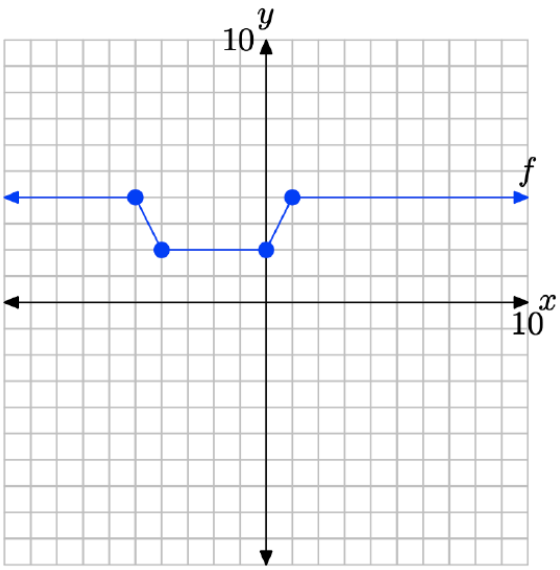


Note that subtracting 2, as in y = f(x)−2, translates the graph of y = f(x) downwards 2 units.

Exercise 16.5E.28

y = f(x) − 1

Pictured below is the graph of a function f. In Exercises 16.5E.29-16.5E.34, use this graph to perform each of the following tasks.



i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of f on your coordinate system. Remember to draw all lines with a ruler.

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of vertical scaling, vertical reflection, and vertical translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow without using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function f and the graph of the function in the exercise.

Exercise 16.5E.29

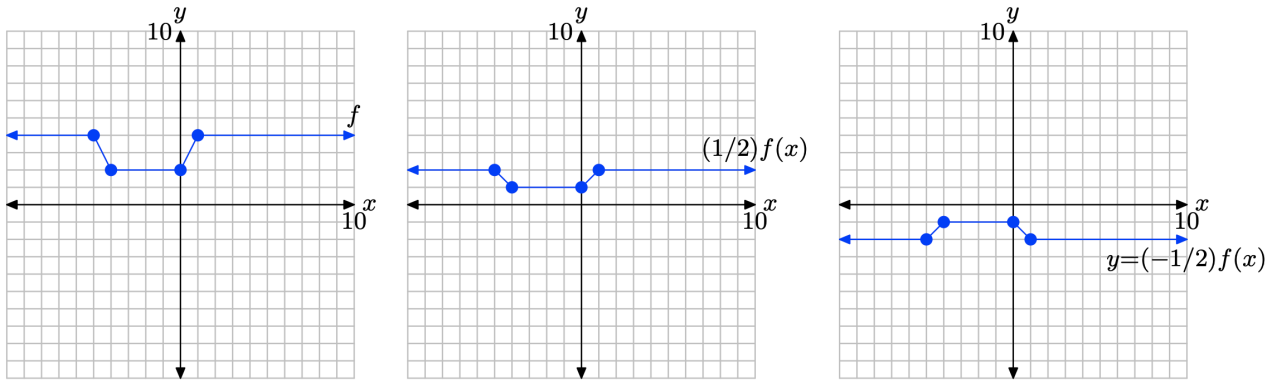
y = (−1/2)f(x)

Answer

We proceed in two steps:

1. First, multiply the y-value of each point on the graph of y = f(x) by 1/2 to produce the graph of y = (1/2)f(x) in (b). This compresses the graph of y = f(x) by a factor of 2.

2. Secondly, multiply the y-value of each point on the graph of y = (1/2)f(x) by −1 to produce the graph of y = (−1/2)f(x) in (c). This reflects the graph of y = (1/2)f(x) across the x-axis.



Exercise 16.5E.30

y = −2f(x).

Exercise 16.5E.31

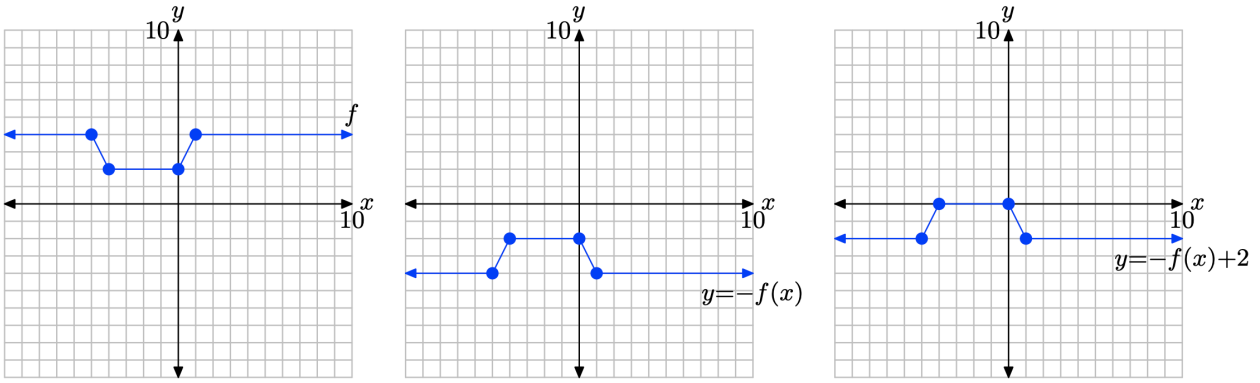
y = −f(x) + 2

Answer

We proceed in two steps:

1. First, multiply the y-value of each point on the graph of y = f(x) by −1 to produce the graph of y = −f(x) in (b). This reflects the graph of y = f(x) across the x-axis.

2. Secondly, add 2 to the y-value of each point on the graph of y = −f(x) to produce the graph of y = −f(x) + 2 in (c). This shifts the graph of y = −f(x) upward 2 units.



(a) $y = f(x)$ (b) $y = -f(x)$ (c) $y = -f(x) + 2$

Exercise 16.5E. 32

y = −f(x) − 3

Exercise 16.5E. 33

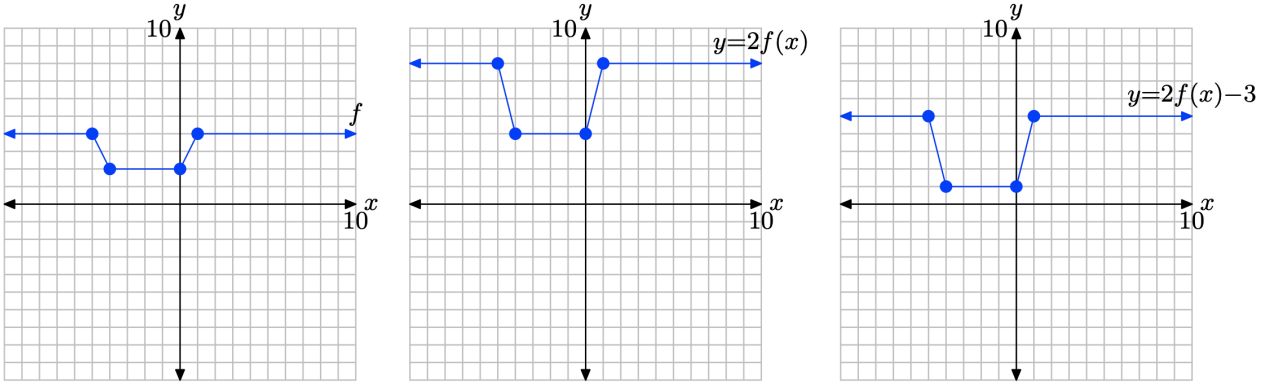
y = 2f(x) − 3.

Answer

We proceed in two steps:

1. First, multiply the y-value of each point on the graph of y = f(x) by 2 to produce the graph of y = 2f(x) in (b). This stretches the graph of y = f(x) vertically by a factor of 2.

2. Secondly, subtract 3 from the y-value of each point on the graph of y = 2f(x) to produce the graph of y = 2f(x) − 3 in (c). This shifts the graph of y = 2f(x) downward 3 units.



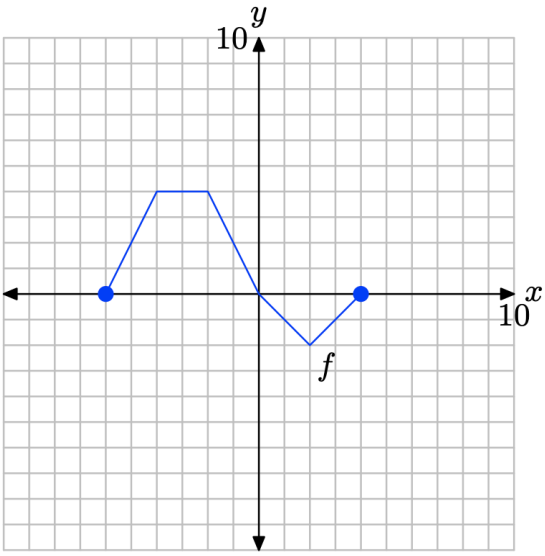
(a) $y = f(x)$ (b) $y = 2f(x)$ (c) $y = 2f(x) - 3$

Exercise 16.5E. 34

y = (−1/2)f(x) + 1

2.6 Exercises

Pictured below is the graph of a function f.



The table that follows evaluates the function f in the plot at key values of x. Notice the horizontal format, where the first point in the table is the ordered pair (−6, 0).

x	-6	-4	-2	0	2	4
f(x)	0	4	4	0	-2	0

Use the graph and the table to complete each of following tasks for Exercises $16.5E.1$ - $16.5E.10$.

i. Set up a coordinate system on graph paper. Label and scale each axis, then copy and label the original graph of f onto your coordinate system. Remember to draw all lines with a ruler.

ii. Use the original table to help complete the table for the given function in the exercise.

iii. Using a different colored pencil, plot the data from your completed table on the same coordinate system as the original graph of f. Use these points to help complete the graph of the given function in the exercise, then label this graph with its equation given in the exercise.

? Exercise $16.5E.1$

y = f(2x).

x	-3	-2	-1	0	1	2
y						

Answer

The original function table.

x	-6	-4	-2	0	2	4
f(x)	0	4	4	0	-2	0

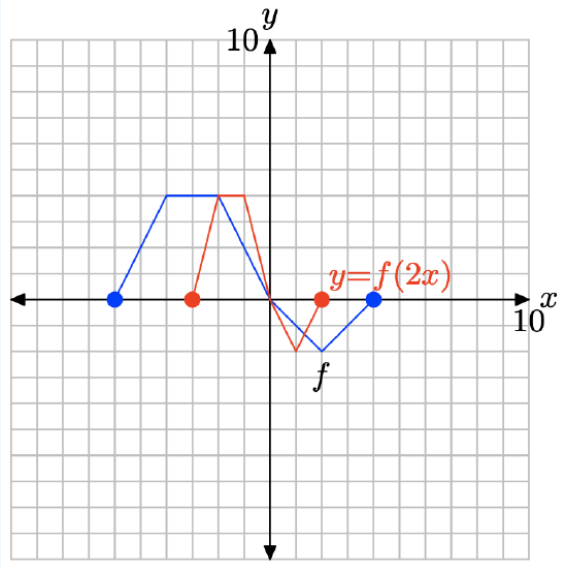
Evaluate the function y = f(2x) at x = −3, −2, −1, 0, 1, and 2.

$$y = f(2(-3)) = f(-6) = 0$$
$$y = f(2(-2)) = f(-4) = 4$$
$$y = f(2(-1)) = f(-2) = 4$$
$$y = f(2(0)) = f(0) = 0$$
$$y = f(2(1)) = f(2) = -2$$
$$y = f(2(2)) = f(4) = 0$$

Points satisfying y = f(2x).

x	-3	-2	-1	0	1	2
y	0	4	4	0	-2	0

Plot the points in the table to get the graph of y = f(2x).



Note that replacing x with 2x, as in y = f(2x), compresses the graph of y = f(x) horizontally by a factor of 2.

Exercise 16.5E.2

y = f((1/2)x).

x	-12	-8	-4	0	4	8
y						

Exercise 16.5E.3

y = f(−x).

x	-4	-2	0	2	4	6
y						

Answer

The original function table.

x	-6	-4	-2	0	2	4

f(x)	0	4	4	0	-2	0

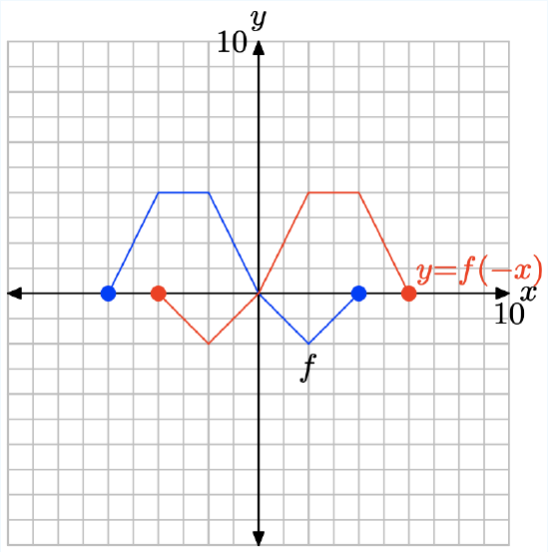
Evaluate the function y = f(−x) at x = −4, −2, 0, 2, 4, and 6.

$$y = f(-(-4)) = f(4) = 0$$
$$y = f(-(-2)) = f(2) = -2$$
$$y = f(-(0)) = f(0) = 0$$
$$y = f(-(2)) = f(-2) = 4$$
$$y = f(-(4)) = f(-4) = 4$$
$$y = f(-(6)) = f(-6) = 0$$

Points satisfying y = f(−x).

x	-4	-2	0	2	4	6
y	0	-2	0	4	4	0

Plot the points in the table to get the graph of y = f(−x).



Note that replacing x with −x, as in y = f(−x), reflects the graph of y = f(x) across the y-axis.

Exercise 16.5E.4

y = f(x + 3).

x	-9	-7	-5	-3	-1	1
y						

Exercise 16.5E.5

y = f(x − 1).

x	-5	-3	-1	1	3	5
y						

Answer

The original function table.

x	-6	-4	-2	0	2	4
f(x)	0	4	4	0	-2	0

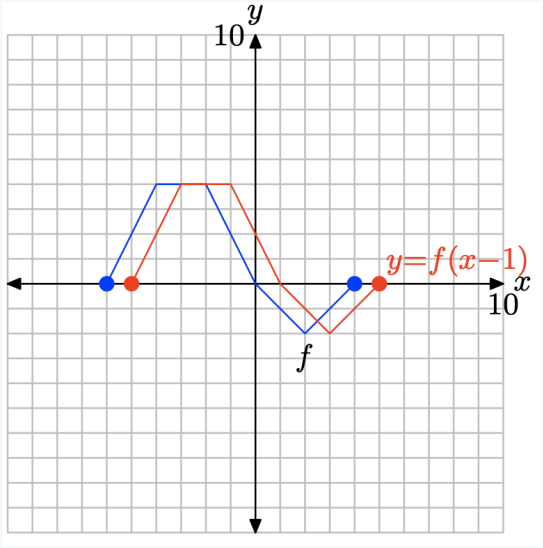
Evaluate the function y = f(x − 1) at x = −5, −3, −1, 1, 3, and 5.

$$y = f((-5) - 1) = f(-6) = 0$$
$$y = f((-3) - 1) = f(-4) = 4$$
$$y = f((-1) - 1) = f(-2) = 4$$
$$y = f((1) - 1) = f(0) = 0$$
$$y = f((3) - 1) = f(2) = -2$$
$$y = f((5) - 1) = f(4) = 0$$

Points satisfying y = f(x − 1).

x	-5	-3	-1	1	3	5
y	0	4	4	0	-2	0

Plot the points in the table to get the graph of y = f(x − 1).



Note that replacing x with x − 1, as in y = f(x − 1), translates the graph of y = f(x) horizontally 1 unit to the right.

Exercise 16.5E.6

y = f(−2x).

x	-2	-1	0	1	2	3
y						

Exercise 16.5E.7

y = f((−1/2)x).

x	-8	-4	0	4	8	12

y						

Answer

The original function table.

x	-6	-4	-2	0	2	4
f(x)	0	4	4	0	-2	0

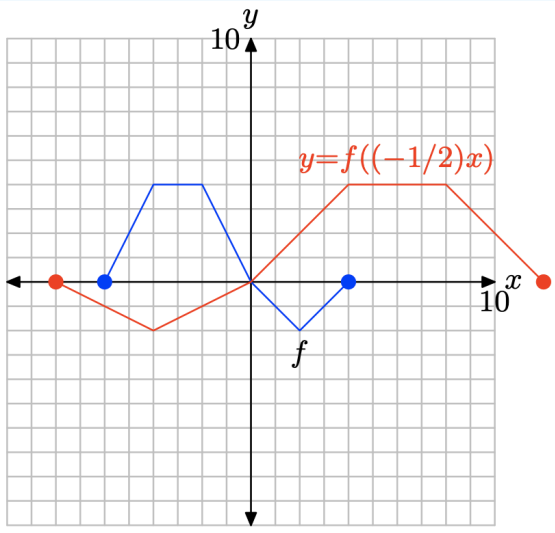
Evaluate the function y = f((−1/2)x) at x = −8, −4, 0, 4, 8, and 12.

$$y = f((-1/2)(-8)) = f(4) = 0$$
$$y = f((-1/2)(-4)) = f(2) = -2$$
$$y = f((-1/2)(0)) = f(0) = 0$$
$$y = f((-1/2)(4)) = f(-2) = 4$$
$$y = f((-1/2)(8)) = f(-4) = 4$$
$$y = f((-1/2)(12)) = f(-6) = 0$$

Points satisfying y = f((−1/2)x).

x	-8	-4	0	4	8	12
y	0	-2	0	4	4	0

Plot the points in the table to get the graph of y = f((−1/2)x).



Note that replacing x with (−1/2)x, as in y = f((−1/2)x), stretches the graph by a factor of 2, then reflects the result across the y-axis.

Exercise 16.5E. 8

y= f(−x − 2)

x	-6	-4	-2	0	2	4
y						

Exercise $16.5E.9$

y= f(−x + 1).

x	-3	-1	1	3	5	7
y						

Answer

The original function table.

x	-6	-4	-2	0	2	4
f(x)	0	4	4	0	-2	0

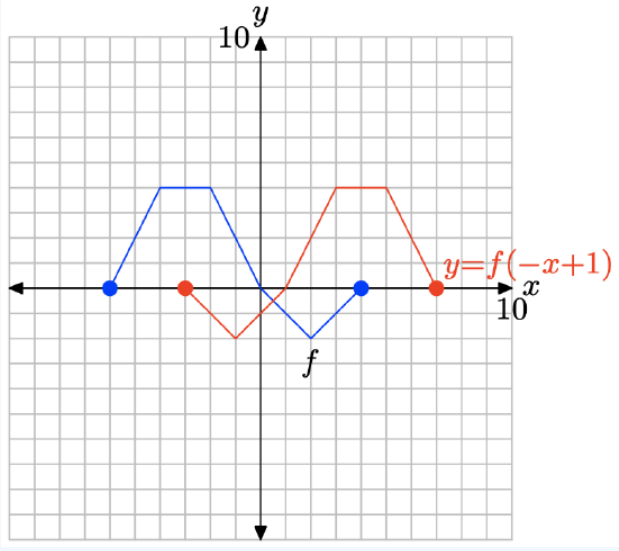
Evaluate the function y = f(−x + 1) at x = −3, −1, 1, 3, 5, and 7.

$$y = f(-(-3)+1) = f(4) = 0$$
$$y = f(-(-1)+1) = f(2) = -2$$
$$y = f(-(1)+1) = f(0) = 0$$
$$y = f(-(3)+1) = f(-2) = 4$$
$$y = f(-(5)+1) = f(-4) = 4$$
$$y = f(-(7)+1) = f(-6) = 0$$

Points satisfying y = f(−x + 1).

x	-3	-1	1	3	5	7
y	0	-2	0	4	4	0

Plot the points in the table to get the graph of y = f(−x + 1).



Note that y = f(−x + 1) is the same as y = f(−(x − 1)). If we replace x with −x to get y = f(−x), then x in this last result with x − 1 to get y = f(−(x − 1)), this has the effect of first reflecting the graph of y = f(x) across the y-axis, then shifting the result to the right 1 unit.

Exercise 16.5E. 10

y = f(−x/4).

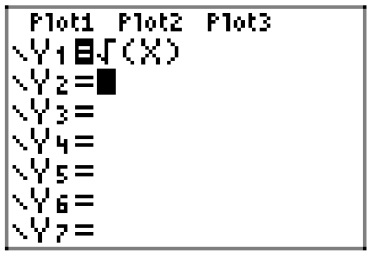
x	-16	-8	0	8	16	24
y						

Exercise 16.5E. 11

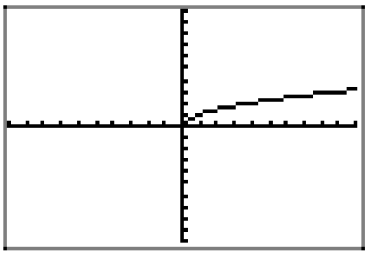
Use your graphing calculator to draw the graph of $y = \sqrt{x}$. Then, draw the graph of $y = \sqrt{-x}$. In your own words, explain what you learned from this exercise.

Answer

First, draw the graph of $y = \sqrt{x}$.

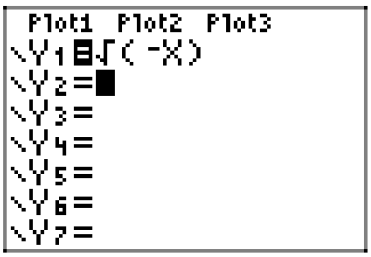


(a)

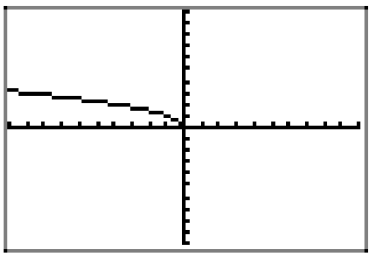


(b)

The graph of $y = \sqrt{-x}$ is a reflection of the graph of $y = \sqrt{x}$ across the y-axis.



(c)



(d)

Replacing x with −x, as in y = f(−x), reflects the graph of y = f(x) across the y-axis.

Exercise 16.5E. 12

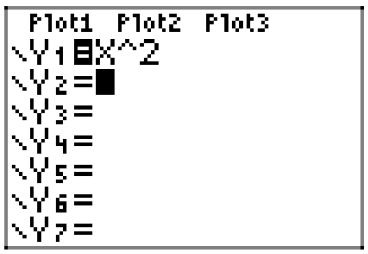
Use your graphing calculator to draw the graph of y = |x|. Then, draw the graph of y = | − x|. In your own words, explain what you learned from this exercise.

Exercise 16.5E. 13

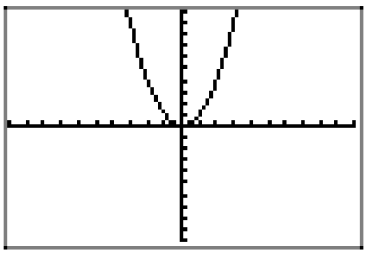
Use your graphing calculator to draw the graph of $y = x^2$. Then, in succession, draw the graphs of $y = (x-2)^2, y = (x-4)^2$, and $y = (x-6)^2$. In your own words, explain what you learned from this exercise.

Answer

First, draw the graph of $y = x^2$.

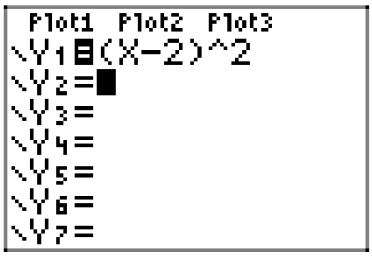


(a)

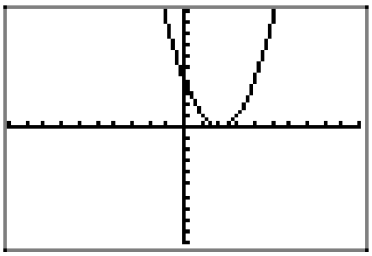


(b)

Replacing x with x − 2 translates the graph of $y = x^2$ two units to the right in the horizontal direction.

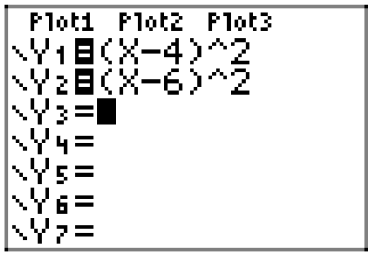


(c)

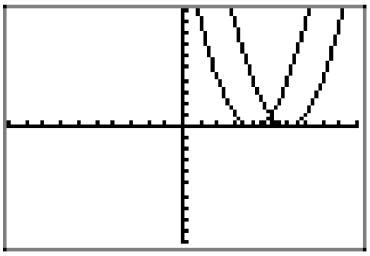


(d)

Similarly, replacing x with x − 4 and x − 6 translates the graph of $y = x^2$ four units and 6 units to the right, respectively.



(e)



(f)

In general, if c is positive, then the graph of y = f(x − c) is obtained by translating the graph of y = f(x) to the right c units.

Exercise 16.5E. 14

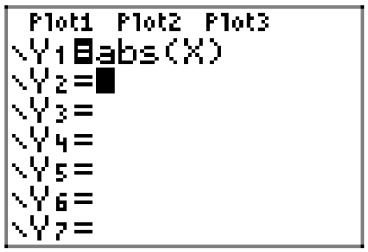
Use your graphing calculator to draw the graph of $y = x^2$. Then, in succession, draw the graphs of $y = (x+2)^2, y = (x+4)^2$, and $y = (x+6)^2$. In your own words, explain what you learned from this exercise.

Exercise 16.5E. 15

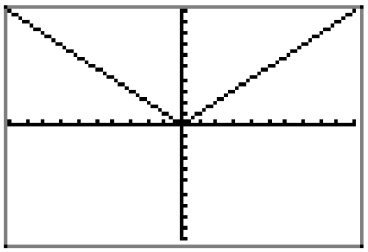
Use your graphing calculator to draw the graph of y = |x|. Then, in succession, draw the graphs of y = |2x|, y = |3x|, and y = |4x|. In your own words, explain what you learned from this exercise.

Answer

First, draw the graph of y = |x|.

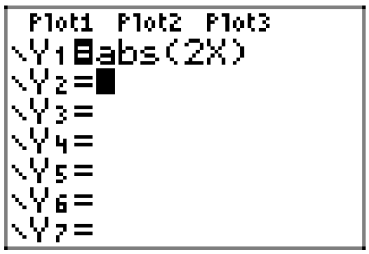


(a)



(b)

Replacing x with 2x compresses the graph of y = |x| by a factor of 2 in the horizontal direction.



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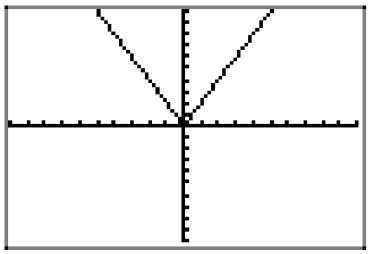
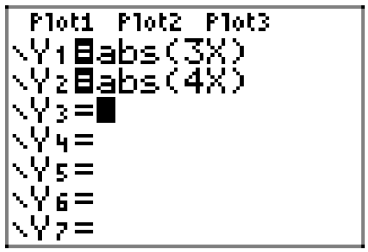
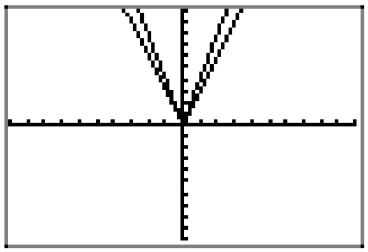


(d)

Similarly, replacing x with 3x and 4x by a factor of 3 and 4 in the horizontal direction, respectively.



(e)



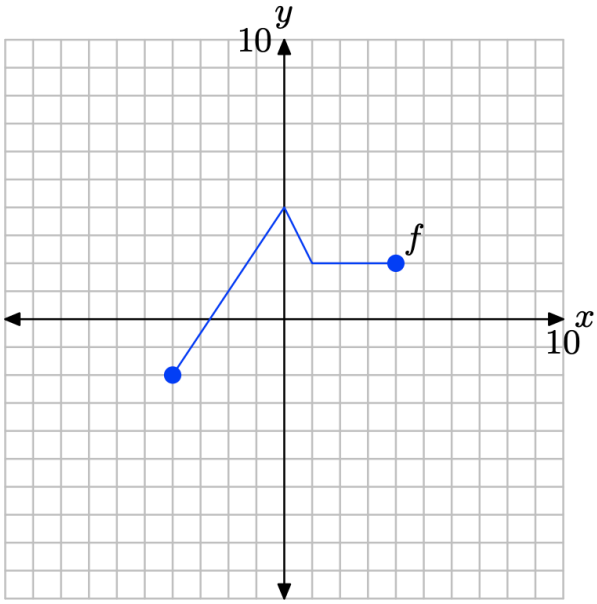
(f)

In general, if a > 1, then the graph of y = f(ax) is obtained by compressing the graph of y = f(x) by a factor of a in the horizontal direction.

? Exercise 16.5E. 16

Use your graphing calculator to draw the graph of y = |x|. Then, in succession, draw the graphs of y = |(1/2)x|, y = |(1/3)x|, and y = |(1/4)x|. In your own words, explain what you learned from this exercise.

Pictured below is the graph of a function f. In Exercises 16.5E. 17-16.5E. 22, use this graph to perform each of the following tasks.



i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of f on your coordinate system. Remember to draw all lines with a ruler.

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of horizontal scaling, horizontal reflection, and horizontal translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow without using tables.

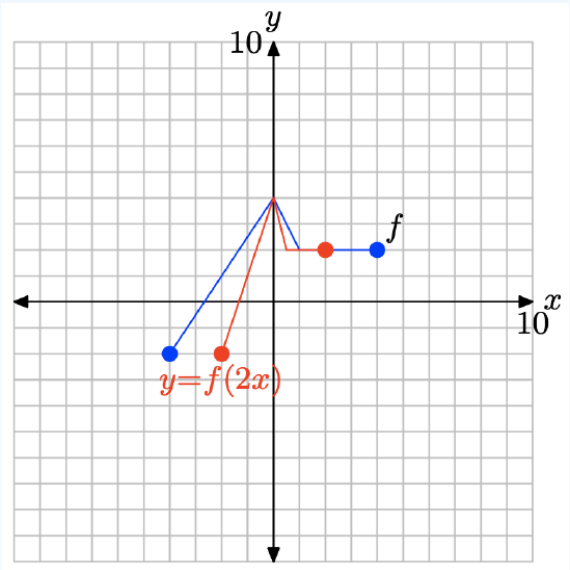
iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function f and the graph of the function in the exercise.

? Exercise 16.5E. 17

y = f(2x).

Answer

To obtain a plot for y = f(2x), take each point on the graph of y = f(x) and divide its x-value by 2, keeping the y-value the same.



Note that replacing x with 2x, as in y = f(2x), compresses the graph of y = f(x) in the horizontal direction by a factor of 2.

Exercise 16.5E.18

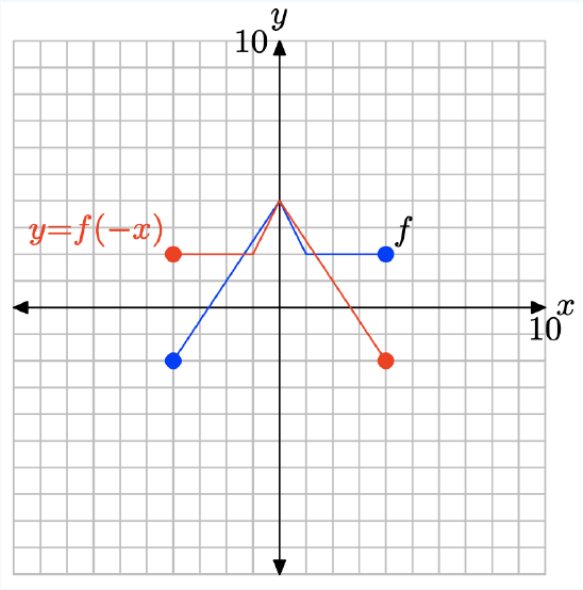
y = f((1/2)x).

Exercise 16.5E.19

y = f(−x).

Answer

To obtain a plot of y = f(−x), take each point on the graph of y = f(x) and negate its x-value, keeping the y-value the same.



Note that replacing x with −x, as in y = f(−x), reflects the graph of f across the y-axis.

Exercise 16.5E.20

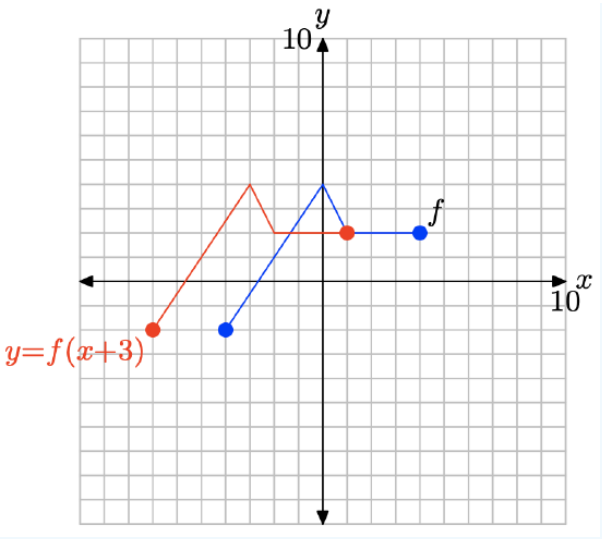
y = f(x − 1).

Exercise 16.5E.21

y = f(x + 3).

Answer

To obtain a plot of y = f(x + 3), take each point on the graph of y = f(x) and subtract 3 from its x-value, keeping the y-value the same.

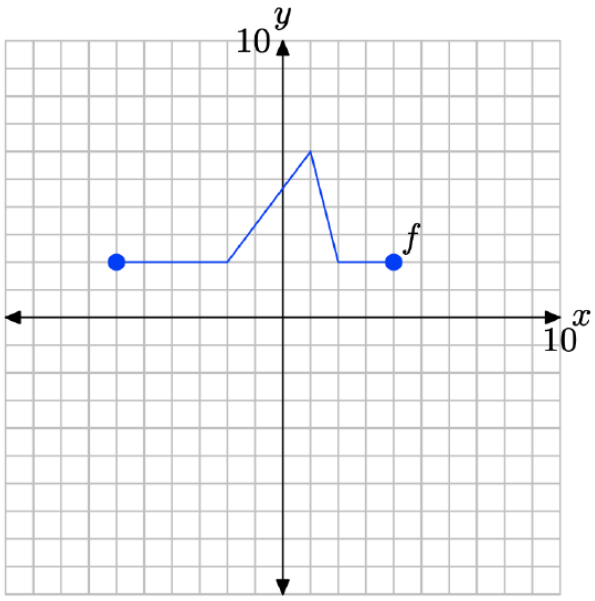


Note that replacing x with x + 3, as in y = f(x + 3), translates the graph of y = f(x) to the left 3 units.

? Exercise 16.5E.22

y = f(x − 2).

Pictured below is the graph of a function f. In Exercises 16.5E.23-16.5E.28, use this graph to perform each of the following tasks.



i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of f on your coordinate system. Remember to draw all lines with a ruler.

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of horizontal scaling, horizontal reflection, and horizontal translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow without using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original

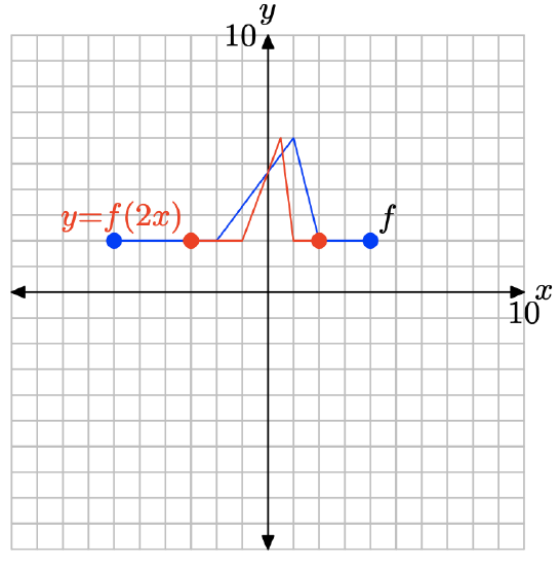
function f and the graph of the function in the exercise.

Exercise 16.5E.23

y = f(2x).

Answer

To obtain a plot of y = f(2x), take each point on the graph of y = f(x) and divide its x-value by 2, keeping the y-value the same.



Replacing x with 2x, as in y = f(2x), compresses the graph of y = f(x) horizontally by a factor of 2.

Exercise 16.5E.24

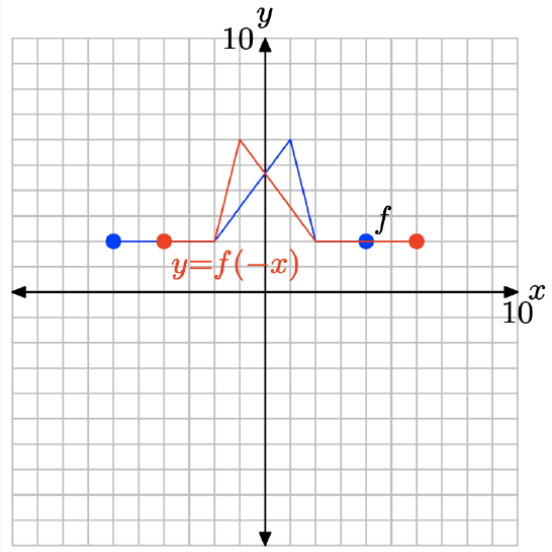
y = f((1/2)x).

Exercise 16.5E.25

y = f(−x).

Answer

To obtain a plot of y = f(−x), take each point on the graph of y = f(x) and negate its x-value, keeping the same y-value.



Replacing x with −x, as in y = f(−x), reflects the graph of y = f(x) across the y-axis.

Exercise 16.5E.26

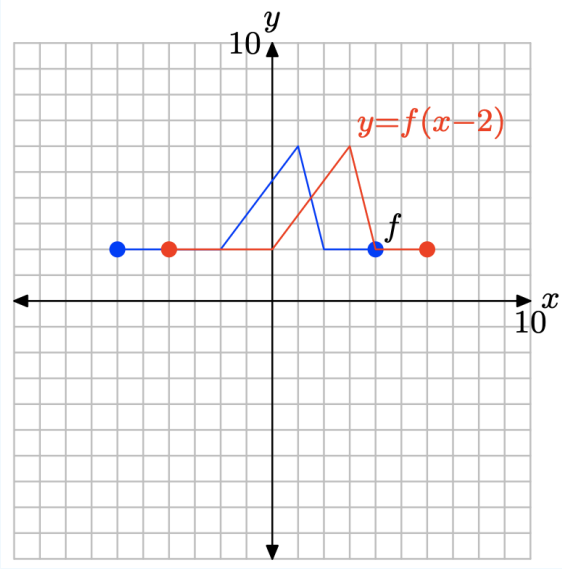
y = f(x + 3)

Exercise 16.5E.27

y= f(x − 2).

Answer

To obtain a plot of y = f(x − 2), take each point on the graph of y = f(x) and add 2 to its x-value, keeping its y-value the same.



Replacing x with x − 2, as in y = f(x − 2), shifts the graph of y = f(x) to the right 2 units.

Exercise $16.5E.28$

y = f(x + 1).

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16.6: The Algebra of Functions

Learning Objectives

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, $Cost = C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the **cost function** at that temperature. We would write $C(T(5))$.

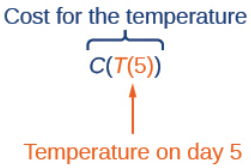

Figure 16.6.1: Explanation of $C(T(5))$, which is the cost for the temperature and $T(5)$ is the temperature on day 5.

By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y) \tag{16.6.1}$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w \tag{16.6.2}$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by the relations.

$$(f+g)(x) = f(x)+g(x)$$
$$(f-g)(x) = f(x)-g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example 16.6.1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x)=x-1$ and $g(x)=x^2-1$. Are they the same function?

Solution

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ (g-f)(x) &= x^2-1-(x-1)\\ &= x^2-x\\ &= x(x-1)\end{aligned}$$

$$\left(\frac{g}{f}\right)(x) = g(x)f(x)$$

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1}\\ &= \frac{(x+1)(x-1)}{x-1}\\ &= x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

16.6.1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.

$$f(x) = x-1$$

and

$$g(x) = x^2-1$$

Are they the same function?

Answer

$(fg)(x) = f(x)g(x) = (x-1)(x2-1) = x^3-x^2-x+1$

$(f-g)(x) = f(x)-g(x) = (x-1)-(x^2-1) = x-x^2$

No, the functions are not the same.

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of **combining functions** so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f \circ g(x) = f(g(x)) \tag{16.6.3}$$

We read the left-hand side as"f composed with g at x," and the right-hand side as"f of g of x."The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

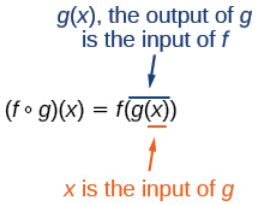

Figure 16.6.2: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)) \tag{16.6.4}$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Example 16.6.2: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is **commutative**.

$$f(x)=2x+1 \quad g(x)=3-x$$

Solution

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3-x)+1 \\ &= 6-2x+1 \\ &= 7-2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3-(2x+1) \\ &= 3-2x-1 \\ &= 2-2x \end{aligned}$$

We find that $g(f(x))\neq f(g(x))$, so the operation of function composition is not commutative.

Example 16.6.3: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example 16.6.4: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y=f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Question/Answer

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order

16.6.2

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Answer

A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G(a(F))$ does not make sense.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example16.6.5: Using a Table to Evaluate a Composite Function

Using Table 16.6.1, evaluate $f(g(3))$ and $g(f(3))$.

Table 16.6.1

x	$f(g(3))$	$g(f(3))$
1	6	3
2	8	5
3	3	2
4	1	7

Solution

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g : $g(3) = 2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2) = 8$.

$$g(3) = 2 \tag{16.6.5}$$

$$f(g(3)) = f(2) = 8 \tag{16.6.6}$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for g, we can evaluate

$$g(f(3)) = g(3) = 2 \tag{16.6.7}$$

Table 16.6.2 shows the composite functions $f \circ g$ and $g \circ f$ as tables.

Table 16.6.2

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

Exercise 16.6.3

Using Table 16.6.1, evaluate $f(g(1))$ and $g(f(4))$.

Answer

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

HowTO: Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

Example 16.6.6: Using a Graph to Evaluate a Composite Function

Using Figure 16.6.3, evaluate $f(g(1))$.

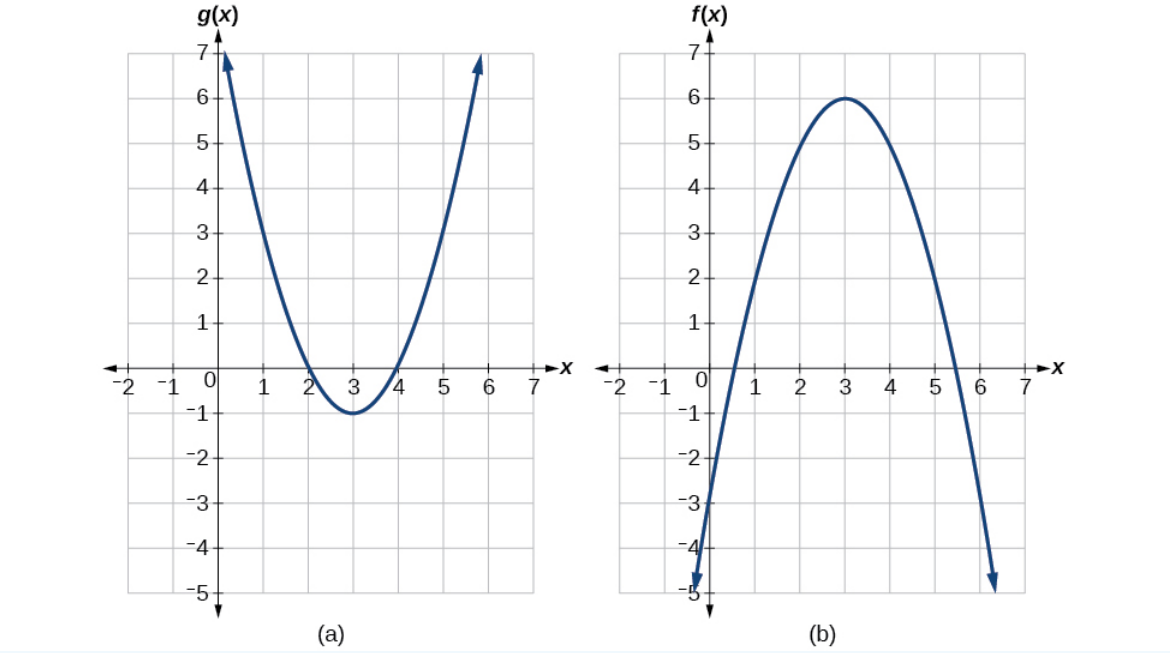

Figure 16.6.3: Two graphs of a positive and negative parabola.

Solution

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure 16.6.4

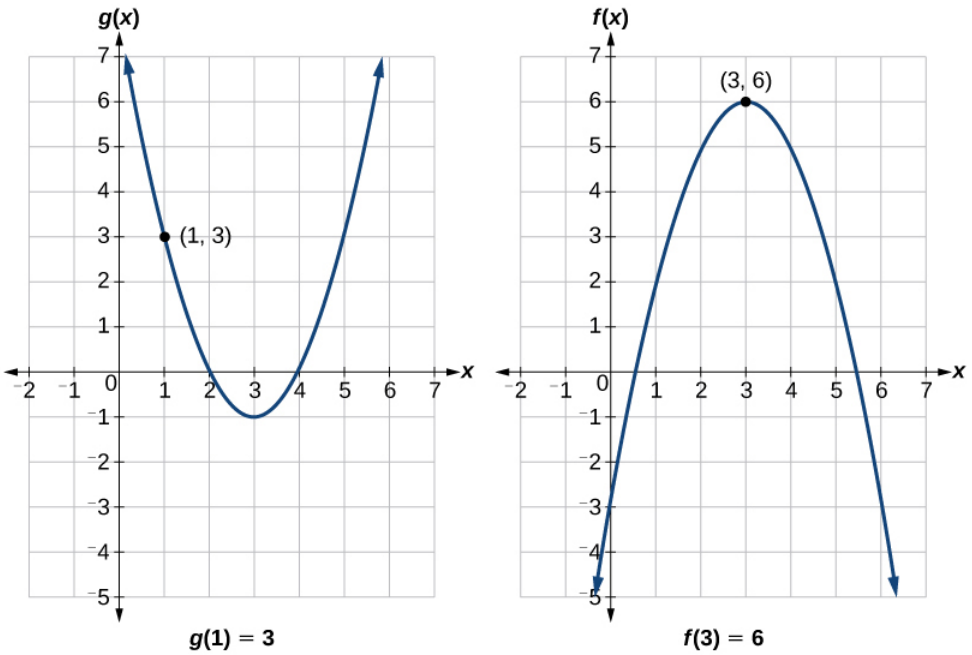

Figure 16.6.4: Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3) \tag{16.6.8}$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Analysis

Figure 16.6.5 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

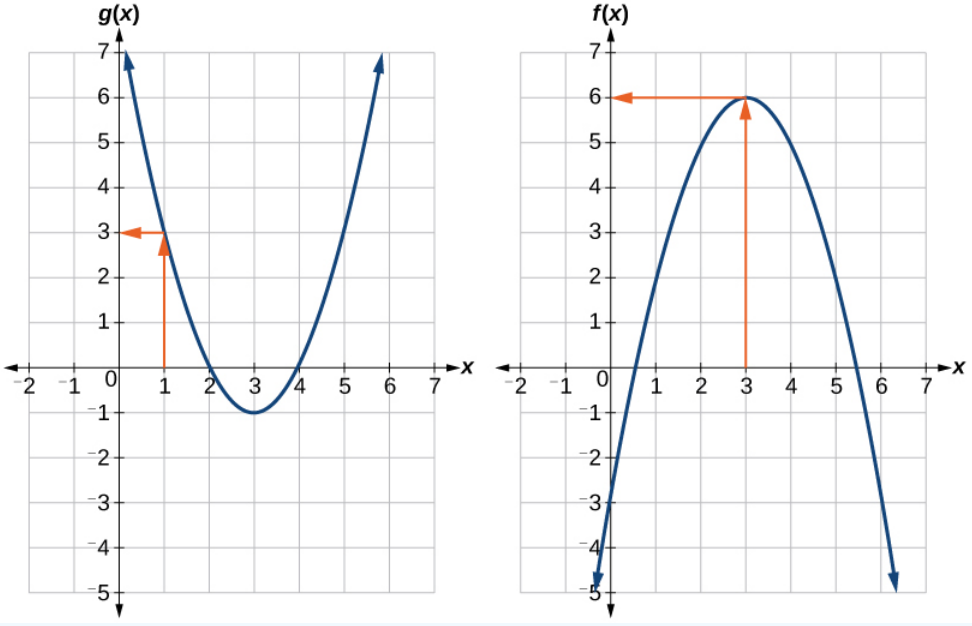

Figure 16.6.5: Two graphs of a positive and negative parabola.

Exercise 16.6.4

Using Figure 16.6.3, evaluate $g(f(2))$.

Answer

$g(f(2)) = g(5) = 3$

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

HowTo: Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Example 16.6.7: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$h(1) = 3(1) + 2$$
$$h(1) = 5$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$f(h(1)) = f(5)$$
$$f(h(1)) = 5^2 - 5$$
$$f(h(1)) = 20$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Exercise 16.6.5

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Answer a

8

Answer b

20

Finding the Domain of a Composite Function

As we discussed previously, the **domain of a composite function** such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of $gandg(x)$ is in the domain of f.

Definition: Domain of a Composite Function

The *domain of a composite function* $f(g(x))$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

HOWTO: Given a function composition f(g(x)),determine its domain.

1. Find the domain of g.
2. Find the domain of f.
3. Find those inputs x in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

Example 16.6.8A: Finding the Domain of a Composite Function

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = 5x - 1 \text{ and } g(x) = \frac{4}{3x-2} \tag{16.6.9}$$

Solution

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x) = 1$.

$$\begin{aligned} \frac{4}{3x-2} &= 1 \\ 4 &= 3x - 2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2 \tag{16.6.10}$$

We can write this in interval notation as

$$(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, 2) \cup (2, \infty) \tag{16.6.11}$$

Example 16.6.8B: Finding the Domain of a Composite Function Involving Radicals

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \sqrt{x+2} \text{ and } g(x) = \sqrt{3-x} \tag{16.6.12}$$

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f\circ g)(x) = \sqrt{\sqrt{3-x}+2} \tag{16.6.13}$$

For $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$, $\sqrt{3-x}+2 \geq 0$, since the radicand of a square root must be positive. Since square roots are positive, $\sqrt{3-x} \geq 0$,or, $3-x \geq 0$, which gives a domain of $(-\infty, 3]$.

Analysis

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f\circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

Exercise 16.6.6

Find the domain of

$$(f\circ g)(x) \text{ where } f(x) = \frac{1}{x-2} \text{ and } g(x) = \sqrt{x+4} \tag{16.6.14}$$

Answer

$[-4, 0)\cup(0, \infty)$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to **decompose a composite function**, so we may choose the decomposition that appears to be most expedient.

Example 16.6.9: Decomposing a Function

Write $f(x) = \sqrt{5-x^2}$ as the composition of two functions.

Solution

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5-x^2$ is the inside of the square root. We could then decompose the function as

$$h(x) = 5-x^2 \text{ and } g(x) = \sqrt{x} \tag{16.6.15}$$

We can check our answer by recomposing the functions.

$$g(h(x)) = g(5-x^2) = \sqrt{5-x^2} \tag{16.6.16}$$

Exercise 16.6.7

Write $f(x) = \dfrac{4}{3-\sqrt{4+x^2}}$ as the composition of two functions.

Answer

Possible answers:

$g(x) = \sqrt{4+x^2}$

$h(x) = \dfrac{4}{3-x}$

$f = h\circ g$

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions (http://openstaxcollege.org/l/compfunction)
- Composite Function Notation Application (http://openstaxcollege.org/l/compfuncnot)
- Composite Functions Using Graphs (http://openstaxcollege.org/l/compfuncgraph)
- Decompose Functions (http://openstaxcollege.org/l/decompfunction)
- Composite Function Values (http://openstaxcollege.org/l/compfuncvalue)

Key Equation

- Composite function $(f \circ g)(x) = f(g(x))$

Key Concepts

- We can perform algebraic operations on functions. See Example.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See Example and Example.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Example.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example.
- A composite function can be evaluated from a graph. See Example.
- A composite function can be evaluated from a formula. See Example.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See Example and Example.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See Example.

Glossary

composite function

the new function formed by function composition, when the output of one function is used as the input of another

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16.E: Review Exercises 1

Chapter Review Exercises

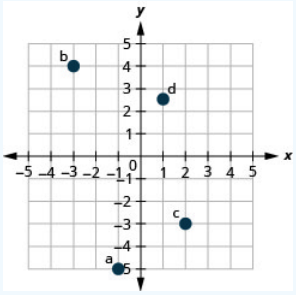
Graph Linear Equations in Two Variables

Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system.

1. ⓐ $(-1,-5)$
ⓑ $(-3,4)$
ⓒ $(2,-3)$
ⓓ $(1,\frac{5}{2})$

Answer



2. ⓐ $(-2,0)$
ⓑ $(0,-4)$
ⓒ $(0,5)$
ⓓ $(3,0)$

In the following exercises, determine which ordered pairs are solutions to the given equations.

3. $5x+y=10$;

ⓐ $(5,1)$
ⓑ $(2,0)$
ⓒ $(4,-10)$

Answer

ⓑ, ⓒ

4. $y=6x-2$;

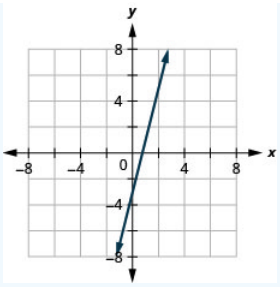
ⓐ $(1,4)$
ⓑ $(13,0)$
ⓒ $(6,-2)$

Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

5. $y=4x-3$

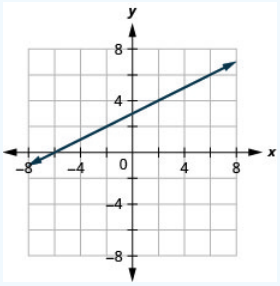
Answer



6. $y = -3x$

7. $y = \frac{1}{2}x + 3$

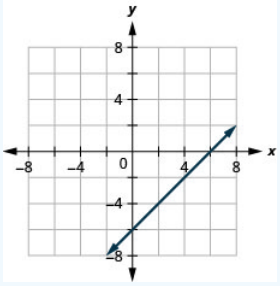
Answer



8. $y = -\frac{4}{5}|x - 1$

9. $x - y = 6$

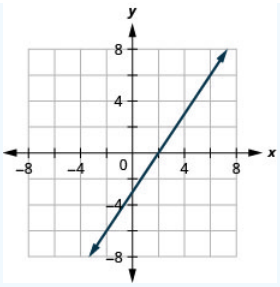
Answer



10. $2x + y = 7$

11. $3x - 2y = 6$

Answer



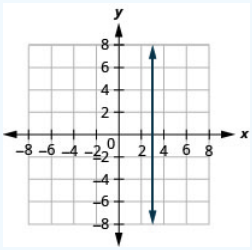
Graph Vertical and Horizontal lines

In the following exercises, graph each equation.

12. $y = -2$

13. $x = 3$

Answer

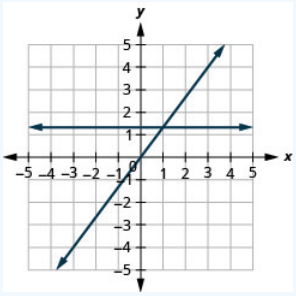


In the following exercises, graph each pair of equations in the same rectangular coordinate system.

14. $y = -2x$ and $y = -2$

15. $y = \frac{4}{3}x$ and $y = \frac{4}{3}$

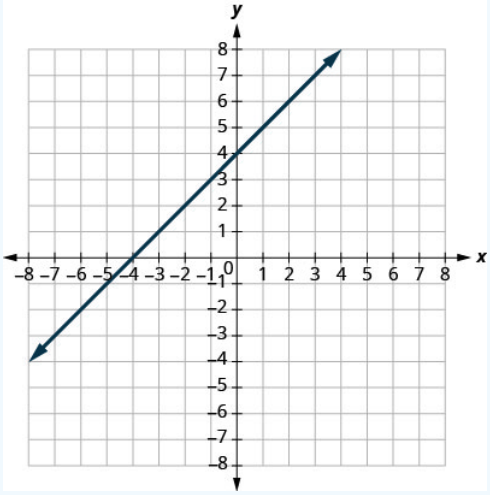
Answer



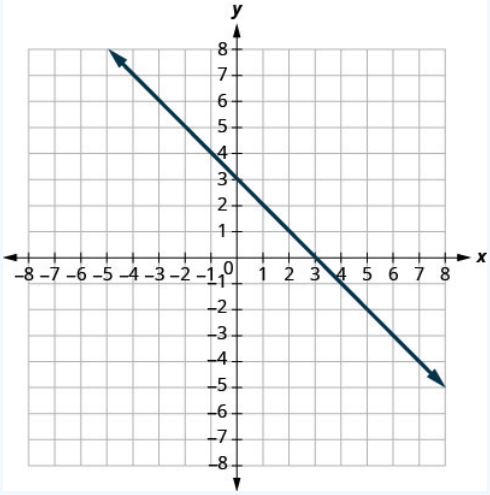
Find *x*- and *y*-Intercepts

In the following exercises, find the *x*- and *y*-intercepts.

16.



17.



Answer

$(0,3)(3,0)$

In the following exercises, find the intercepts of each equation.

18. $x-y=-1$

19. $x+2y=6$

Answer

$(6,0),\ (0,3)$

20. $2x+3y=12$

21. $y=\frac{3}{4}x-12$

Answer

$(16,0),\ (0,-12)$

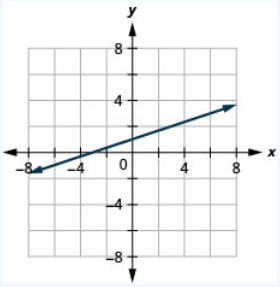
22. $y = 3x$

Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

23. $-x + 3y = 3$

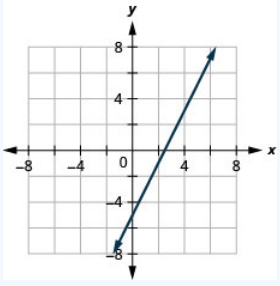
Answer



24. $x - y = 4$

25. $2x - y = 5$

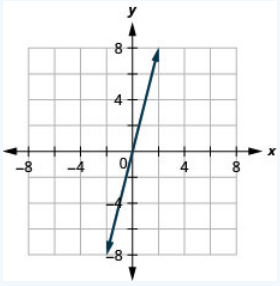
Answer



26. $2x - 4y = 8$

27. $y = 4x$

Answer

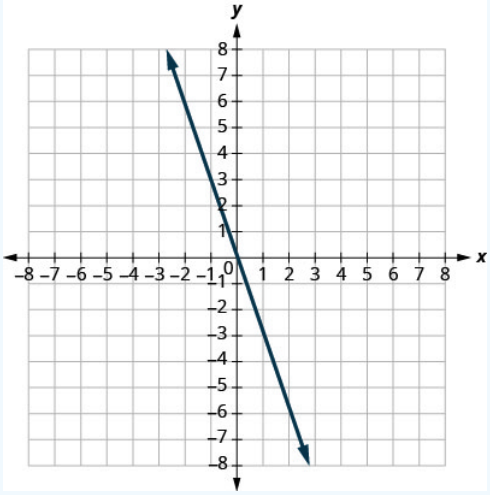


Slope of a Line

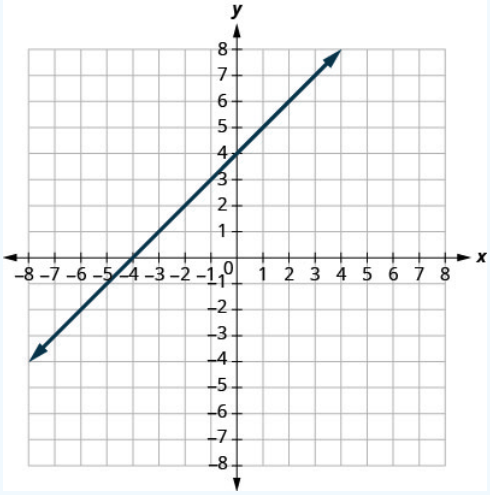
Find the Slope of a Line

In the following exercises, find the slope of each line shown.

28.



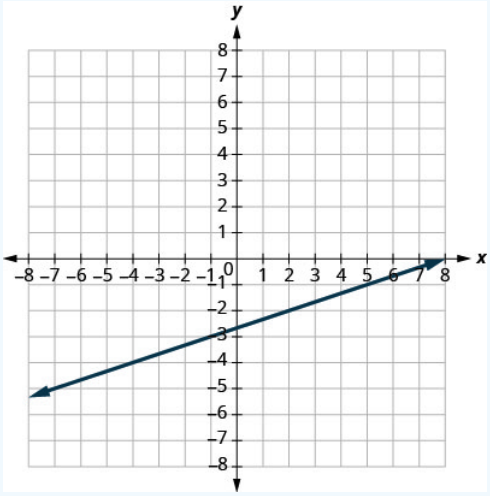
29.



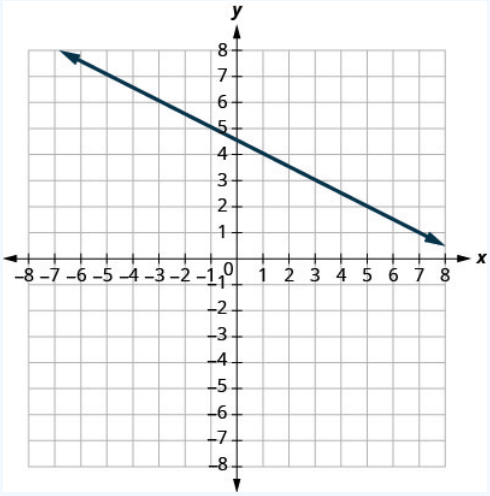
Answer

1

30.



31.



Answer

-12

In the following exercises, find the slope of each line.

32. $y = 2$

33. $x = 5$

Answer

undefined

34. $x = -3$

35. $y = -1$

Answer

0

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

36. $(-1, -1), (0, 5)$

37. $(3.5), (4, -1)$

Answer

-6

38. $(-5, -2), (3, 2)$

39. $(2, 1), (4, 6)$

Answer

52

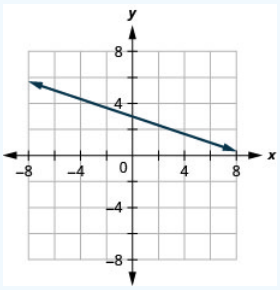
Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

40. $(2,-2);\ m=52$

41. $(-3,4);\ m=-13$

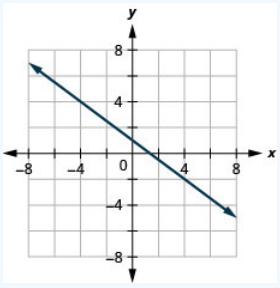
Answer



42. x-intercept $-4; m=3$

43. y-intercept $1; m=-34$

Answer



Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and y-intercept of each line.

44. $y=-4x+9$

45. $y=53x-6$

Answer

$m=53;\ (0,-6)$

46. $5x+y=10$

47. $4x-5y=8$

Answer

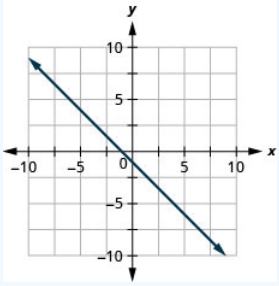
$m=\frac{4}{5};\ (0,-\frac{8}{5})$

In the following exercises, graph the line of each equation using its slope and y-intercept.

48. $y = 2x + 3$

49. $y = -x - 1$

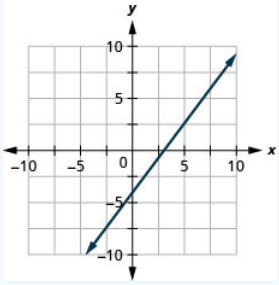
Answer



50. $y = -25x + 3$

51. $4x - 3y = 12$

Answer



In the following exercises, determine the most convenient method to graph each line.

52. $x = 5$

53. $y = -3$

Answer

horizontal line

54. $2x + y = 5$

55. $x - y = 2$

Answer

intercepts

56. $y = 22x + 2$

57. $y = 34x - 1$

Answer

plotting points

Graph and Interpret Applications of Slope-Intercept

58. Katherine is a private chef. The equation $C = 6.5m + 42$ models the relation between her weekly cost, C, in dollars and the number of meals, m, that she serves.

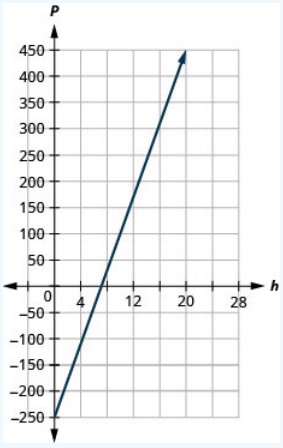
ⓐ Find Katherine's cost for a week when she serves no meals.
ⓑ Find the cost for a week when she serves 14 meals.
ⓒ Interpret the slope and C-intercept of the equation.
ⓓ Graph the equation.

59. Marjorie teaches piano. The equation $P = 35h - 250$ models the relation between her weekly profit, P, in dollars and the number of student lessons, s, that she teaches.

ⓐ Find Marjorie's profit for a week when she teaches no student lessons.
ⓑ Find the profit for a week when she teaches 20 student lessons.
ⓒ Interpret the slope and P-intercept of the equation.
ⓓ Graph the equation.

Answer

ⓐ $-\$250$
ⓑ $\$450$
ⓒ The slope, 35, means that Marjorie's weekly profit, P, increases by \$35 for each additional student lesson she teaches. The P-intercept means that when the number of lessons is 0, Marjorie loses \$250.
ⓓ



Use Slopes to Identify Parallel and Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel, perpendicular, or neither.

60. $4x - 3y = -1; \quad y = 43x - 3$

61. $y = 5x - 1; \quad 10x + 2y = 0$

Answer

neither

62. $3x - 2y = 5; \quad 2x + 3y = 6$

63. $2x - y = 8; \quad x - 2y = 4$

Answer

not parallel

Find the Equation of a Line

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

64. Slope $\frac{1}{3}$ and y-intercept $(0, -6)$

65. Slope -5 and y-intercept $(0, -3)$

Answer

$y = -5x - 3$

66. Slope 0 and y-intercept $(0, 4)$

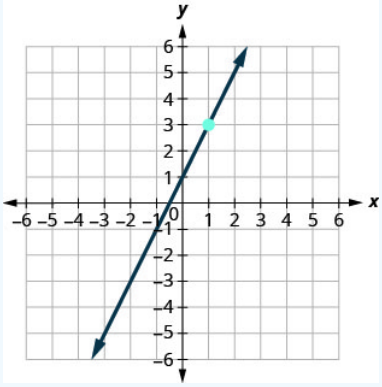
67. Slope -2 and y-intercept $(0, 0)$

Answer

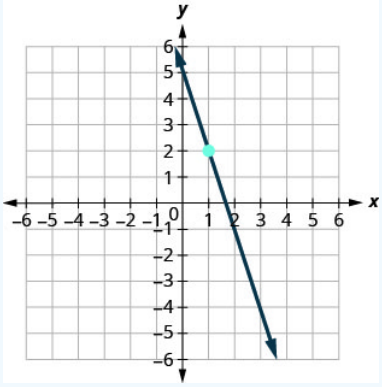
$y = -2x$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.

68.



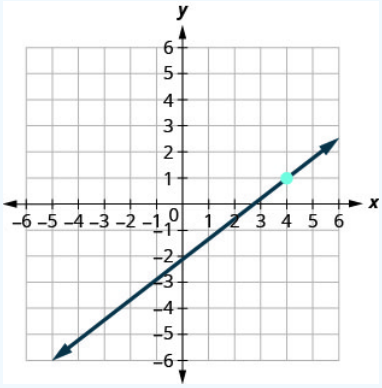
69.



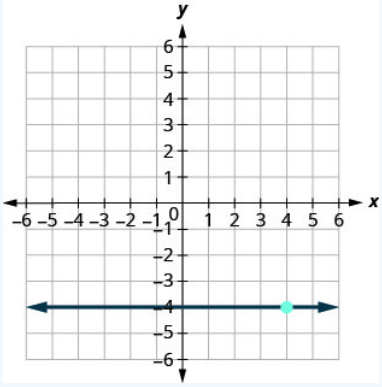
Answer

$y = -3x + 5$

70.



71.



Answer

$y = -4$

Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

72. $m = -\frac{1}{4}$, point $(-8, 3)$

73. $m = \frac{3}{5}$, point $(10, 6)$

Answer

$y = \frac{3}{5}x$

74. Horizontal line containing $(-2, 7)$

75. $m = -2$, point $(-1, -3)$

Answer

$y = -2x - 5$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

76. $(2, 10)$ and $(-2, -2)$

77. $(7, 1)$ and $(5, 0)$

Answer

$y = \frac{1}{2}x - \frac{5}{2}$

78. $(3, 8)$ and $(3, -4)$

79. $(5, 2)$ and $(-1, 2)$

Answer

$y = 2$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

80. line $y = -3x + 6$, point $(1, -5)$

81. line $2x + 5y = -10$, point $(10, 4)$

Answer

$y = -\frac{2}{5}x + 8$

82. line $x = 4$, point $(-2, -1)$

83. line $y = -5$, point $(-4, 3)$

Answer

$y = 3$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–intercept form.

84. line $y = -\frac{4}{5}x + 2$, point $(8, 9)$

85. line $2x - 3y = 9$, point $(-4, 0)$

Answer

$y = -\frac{3}{2}x - 6$

86. line $y = 3$, point $(-1, -3)$

87. line $x = -5$ point $(2, 1)$

Answer

$y = 1$

Graph Linear Inequalities in Two Variables

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.

88. Determine whether each ordered pair is a solution to the inequality $y < x - 3$:

ⓐ $(0, 1)$ ⓑ $(-2, -4)$ ⓒ $(5, 2)$ ⓓ $(3, -1)$
ⓔ $(-1, -5)$

89. Determine whether each ordered pair is a solution to the inequality $x + y > 4$:

ⓐ $(6, 1)$ ⓑ $(-3, 6)$ ⓒ $(3, 2)$ ⓓ $(-5, 10)$ ⓔ $(0, 0)$

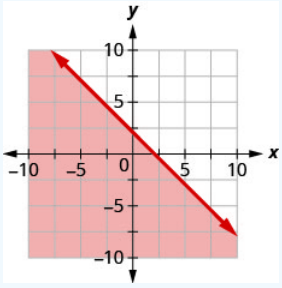
Answer

ⓐ yes ⓑ no ⓒ yes ⓓ yes; ⓔ nom

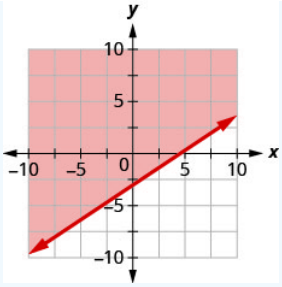
Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

90. Write the inequality shown by the graph with the boundary line $y = -x + 2$.



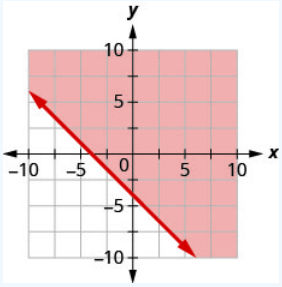
91. Write the inequality shown by the graph with the boundary line $y = \frac{2}{3}x - 3$.



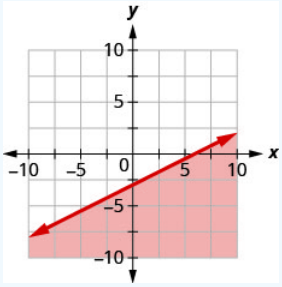
Answer

$y > \frac{2}{3}x - 3$

92. Write the inequality shown by the shaded region in the graph with the boundary line $x + y = -4$.



93. Write the inequality shown by the shaded region in the graph with the boundary line $x - 2y = 6$.



Answer

$x - 2y \geq 6$

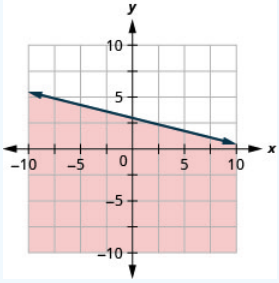
Graph Linear Inequalities in Two Variables

In the following exercises, graph each linear inequality.

94. Graph the linear inequality $y > \frac{2}{5}x - 4$.

95. Graph the linear inequality $y \leq -\frac{1}{4}x + 3$.

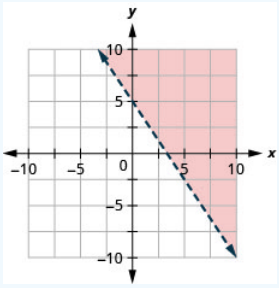
Answer



96. Graph the linear inequality $x - y \leq 5$.

97. Graph the linear inequality $3x + 2y > 10$.

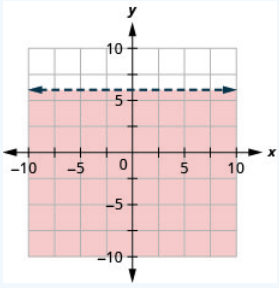
Answer



98. Graph the linear inequality $y \leq -3x$.

99. Graph the linear inequality $y < 6$.

Answer



Solve Applications using Linear Inequalities in Two Variables

100. Shanthie needs to earn at least $500 a week during her summer break to pay for college. She works two jobs. One as a swimming instructor that pays $10 an hour and the other as an intern in a law office for $25 hour. How many hours does Shanthie need to work at each job to earn at least $500 per week?

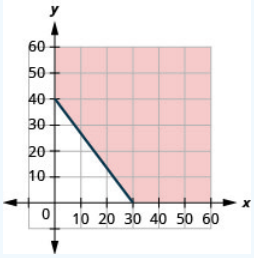
ⓐ Let *x* be the number of hours she works teaching swimming and let *y* be the number of hours she works as an intern. Write an inequality that would model this situation.
ⓑ Graph the inequality.
ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Shanthie.

101. Atsushi he needs to exercise enough to burn 600 calories each day. He prefers to either run or bike and burns 20 calories per minute while running and 15 calories a minute while biking.

ⓐ If x is the number of minutes that Atsushi runs and y is the number minutes he bikes, find the inequality that models the situation.
ⓑ Graph the inequality.
ⓒ List three solutions to the inequality. What options do the solutions provide Atsushi?

Answer

ⓐ $20x+15y\geq 60020x+15y\geq 600$
ⓑ



ⓒ Answers will vary.

Relations and Functions

Find the Domain and Range of a Relation

In the following exercises, for each relation, ⓐ find the domain of the relation ⓑ find the range of the relation.

102. $\{(5,-2),\ (5,-4),\ (7,-6),\ (8,-8),\ (9,-10)\}$

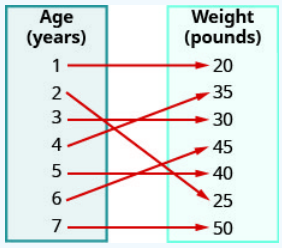
103. $\{(-3,7),\ (-2,3),\ (-1,9),\ (0,-3),\ (-1,8)\}$

Answer

ⓐ $D:-3,-2,-1,0$
ⓑ $R:7,3,9,-3,8$

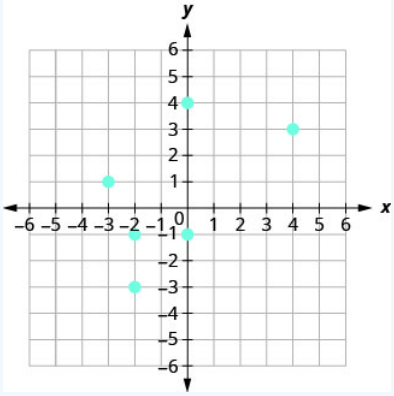
In the following exercise, use the mapping of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

104. The mapping below shows the average weight of a child according to age.



In the following exercise, use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

105.



Answer

ⓐ $(4,3),\ (-2,-3),\ (-2,-1),\ (-3,1),\ (0,-1),\ (0,4)$
ⓑ $D: -3,-2,0,4$
ⓒ $R: -3,-1,1,3,4$

Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.

106. $\{(9,-5),\ (4,-3),\ (1,-1),\ (0,0),\ (1,1),\ (4,3),\ (9,5)\}$

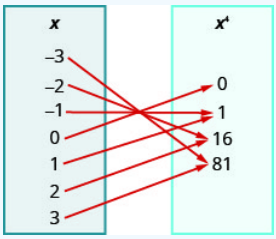
107. $\{(-3,27),\ (-2,8),\ (-1,1),\ (0,0),\ (1,1),\ (2,8),\ (3,27)\}$

Answer

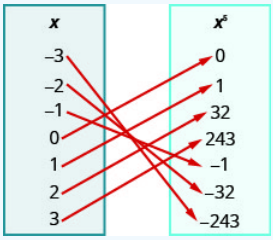
ⓐ yes ⓑ $-3,-2,-1,0,1,2,3$
ⓒ $0,1,8,27$

In the following exercises, use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the function ⓒfind the range of the function.

108.



109.



Answer

ⓐ $-3, -2, -1, 0, 1, 2, 3$
ⓑ $-3, -2, -1, 0, 1, 2, 3$
ⓒ $-243, -32, -1, 0, 1, 32, 243$

In the following exercises, determine whether each equation is a function.

110. $2x + y = -3$

111. $y = x^2$

Answer

yes

112. $y = 3x - 5$

113. $y = x^3$

Answer

yes

114. $2x + y2 = 4$

Find the Value of a Function

In the following exercises, evaluate the function:

ⓐ $f(-2)$ ⓑ $f(3)$ ⓒ $f(a)$.

115. $f(x) = 3x - 4$

Answer

ⓐ $f(-2) = -10$ ⓑ $f(3) = 5$ ⓒ $f(a) = 3a - 4$

116. $f(x) = -2x + 5$

117. $f(x) = x^2 - 5x + 6$

Answer

ⓐ $f(-2) = 20$ ⓑ $f(3) = 0$ ⓒ $f(a) = a^2 - 5a + 6$

118. $f(x) = 3x^2 - 2x + 1$

In the following exercises, evaluate the function.

119. $g(x) = 3x2 - 5x;\ g(2)$

Answer

2

120. $F(x) = 2x2 - 3x + 1;\ F(-1)$

121. $h(t) = 4|t-1|+2;\ h(t) = 4$

Answer

18

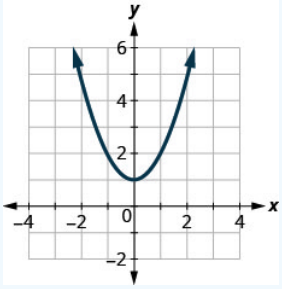
122. $f(x) = x + 2x - 1;\ f(3)$

Graphs of Functions

Use the Vertical line Test

In the following exercises, determine whether each graph is the graph of a function.

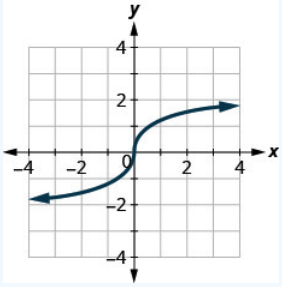
123.



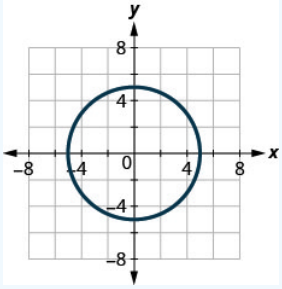
Answer

yes

124.



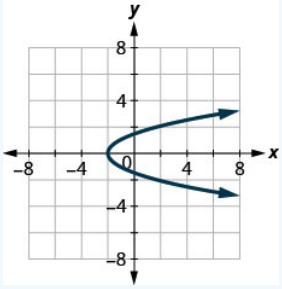
125.



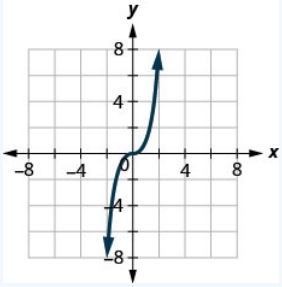
Answer

no

126.



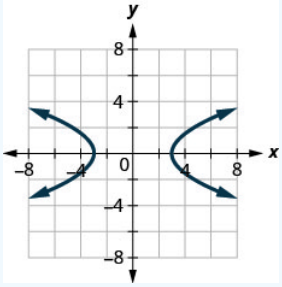
127.



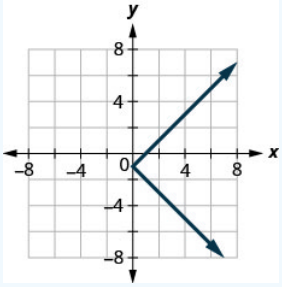
Answer

yes

128.



129.



Answer

no

Identify Graphs of Basic Functions

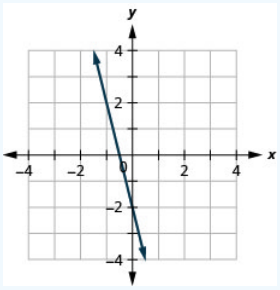
In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

130. $f(x) = 5x + 1$

131. $f(x) = -4x - 2$

Answer

ⓐ



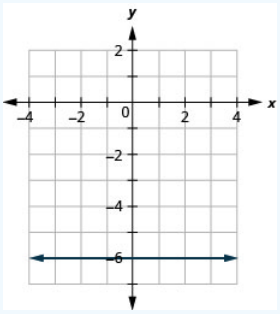
ⓑ $D : (-\text{inf}, \text{inf}), R : (-\text{inf}, \text{inf})$

132. $f(x) = \frac{2}{3}x - 1$

133. $f(x) = -6$

Answer

ⓐ



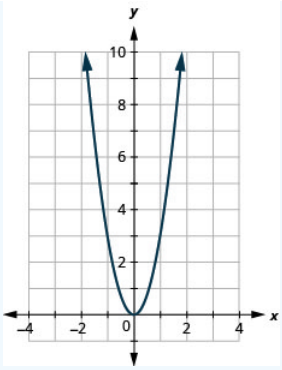
ⓑ $D : (-\text{inf}, \text{inf}), R : (-\text{inf}, \text{inf})$

134. $f(x) = 2x$

135. $f(x) = 3x^2$

Answer

ⓐ



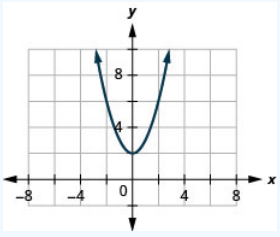
ⓑ $D: (-\text{inf}, \text{inf}), R: (-\text{inf}, 0]$

136. $f(x) = -12x^2$

137. $f(x) = x^2 + 2$

Answer

ⓐ



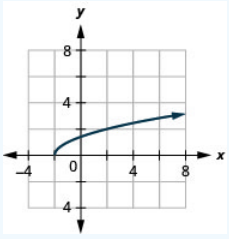
ⓑ $D: (-\text{inf}, \text{inf}), R: (-\text{inf}, \text{inf})$

138. $f(x) = x^3 - 2$

139. $f(x) = \sqrt{x+2}$

Answer

ⓐ



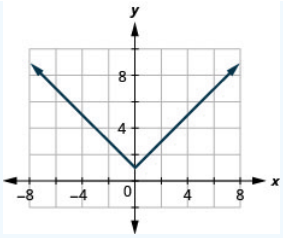
ⓑ $D: [-2, -2, \text{inf}),\ R: [0, \text{inf})$

140. $f(x) = -|x|$

141. $f(x) = |x| + 1$

Answer

ⓐ

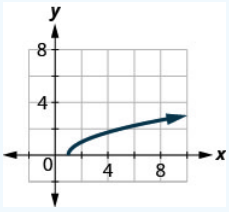


ⓑ $D : (-\text{inf}, \text{inf}),\ R : [1, \text{inf})$

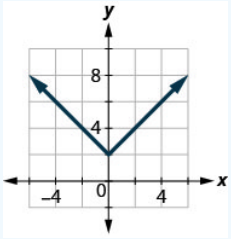
Read Information from a Graph of a Function

In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation

142.



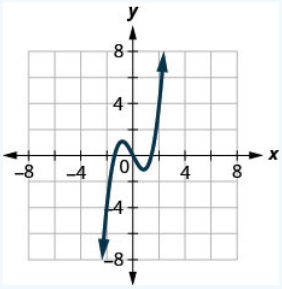
143.



Answer

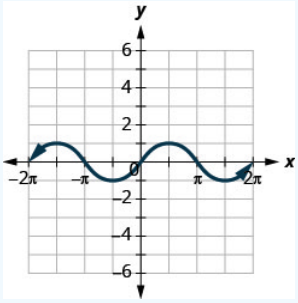
$D : (-\text{inf}, \text{inf}), R : [2, \text{inf})$

144.



In the following exercises, use the graph of the function to find the indicated values.

145.

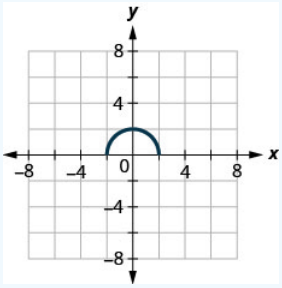


ⓐ Find $f(0)$.
ⓑ Find $f(12\pi)$.
ⓒ Find $f(-32\pi)$.
ⓓ Find the values for x when $f(x) = 0$.
ⓔ Find the x-intercepts.
ⓕ Find the y-intercept(s).
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Answer

ⓐ $f(x) = 0$ ⓑ $f(\pi/2) = 1$
ⓒ $f(-3\pi/2) = 1$ ⓓ $f(x) = 0$ for $x = -2\pi, -\pi, 0, \pi, 2\pi$
ⓔ $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$ ⓕ $(0, 0)$
ⓖ $[-2\pi, 2\pi]$ ⓗ $[-1, 1]$

146.



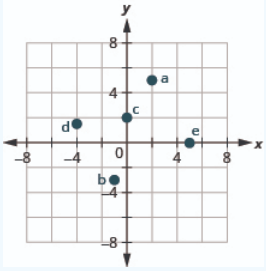
ⓐ Find $f(0)$.
ⓑ Find the values for x when $f(x) = 0$.
ⓒ Find the x-intercepts.
ⓓ Find the y-intercept(s).
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation.

Practice Test

1. Plot each point in a rectangular coordinate system.

ⓐ $(2, 5)$
ⓑ $(-1, -3)$
ⓒ $(0, 2)$
ⓓ $(-4, 32)$
ⓔ $(5, 0)$

Answer

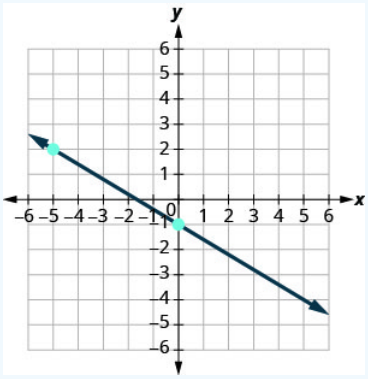


2. Which of the given ordered pairs are solutions to the equation $3x - y = 6$?

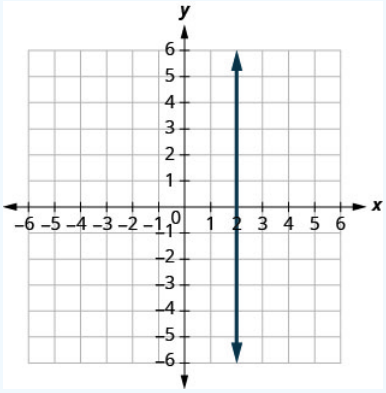
ⓐ $(3, 3)$ ⓑ $(2, 0)$ ⓒ $(4, -6)$

3. Find the slope of each line shown.

ⓐ



ⓑ



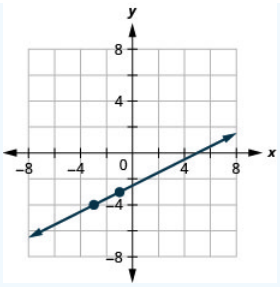
Answer

ⓐ $-\frac{3}{5}$ ⓑ undefined

4. Find the slope of the line between the points $(5, 2)$ and $(-1, -4)$.

5. Graph the line with slope $\frac{1}{2}$ containing the point $(-3, -4)$.

Answer

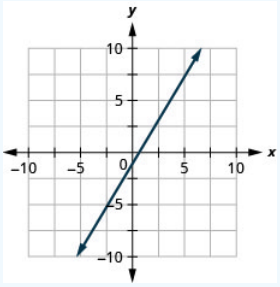


6. Find the intercepts of $4x + 2y = -8$ and graph.

Graph the line for each of the following equations.

7. $y = \frac{5}{3}x - 1$

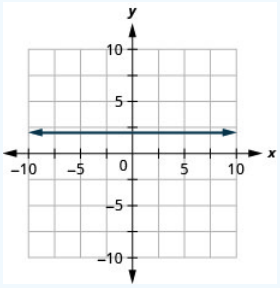
Answer



8. $y = -x$

9. $y = 2$

Answer



Find the equation of each line. Write the equation in slope-intercept form.

10. slope $-\frac{3}{4}$ and y-intercept $(0, -2)$

11. $m = 2$, point $(-3, -1)$

Answer

$y = 2x + 5$

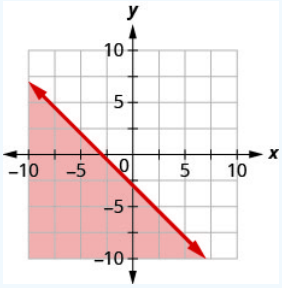
12. containing $(10, 1)$ and $(6, -1)$

13. perpendicular to the line $y = \frac{5}{4}x + 2$, containing the point $(-10, 3)$

Answer

$y = -\frac{4}{5}x - 5$

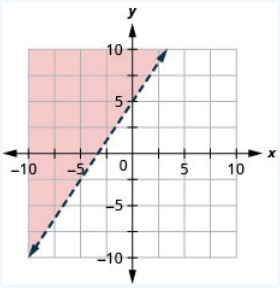
14. Write the inequality shown by the graph with the boundary line $y = -x - 3$.



Graph each linear inequality.

15. $y > \frac{3}{2}x + 5$

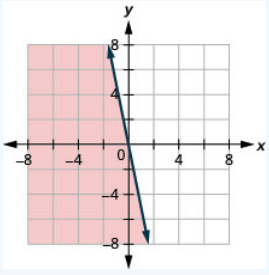
Answer



16. $x - y \geq -4$

17. $y \leq -5x$

Answer



18. Hiro works two part time jobs in order to earn enough money to meet her obligations of at least $450 a week. Her job at the mall pays $10 an hour and her administrative assistant job on campus pays $15 an hour. How many hours does Hiro need to work at each job to earn at least $450?

ⓐ Let *x* be the number of hours she works at the mall and let *y* be the number of hours she works as administrative assistant. Write an inequality that would model this situation.

ⓑ Graph the inequality .
ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then explain what that means for Hiro.

19. Use the set of ordered pairs to ⓐ determine whether the relation is a function, ⓑ find the domain of the relation, and ⓒfind the range of the relation.

$\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

Answer

ⓐ yes ⓑ $\{-3, -2, -1, 0, 1, 2, 3\}$ⓒ $\{0, 1, 8, 27\}$

20. Evaluate the function: ⓐ $f(-1)$ ⓑ $f(2)$ ⓒ $f(c)$.

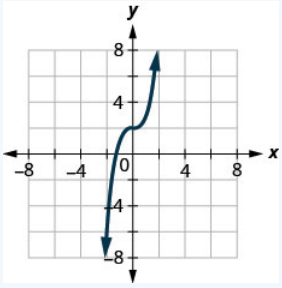
$f(x) = 4x^2 - 2x - 3$

21. For $h(y) = 3|y - 1| - 3$, evaluate $h(-4)$.

Answer

12

22. Determine whether the graph is the graph of a function. Explain your answer.

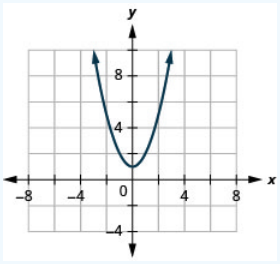


In the following exercises, ⓐ graph each function ⓑ state its domain and range.
Write the domain and range in interval notation.

23. $f(x) = x^2 + 1$

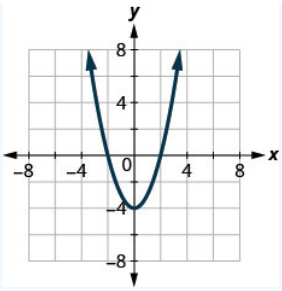
Answer

ⓐ



ⓑ $D : (-\text{inf}, \text{inf}), R : [1, \text{inf})$

24. $f(x) = \sqrt{x + 1}$



ⓑ Find the y-intercepts.
ⓒ Find $f(-1)$.
ⓓ Find $f(1)$.
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation.

Answer

ⓐ $x = -2, 2$ ⓑ $y = -4$
ⓒ $f(-1) = -3$ ⓓ $f(1) = -3$
ⓔ $D : (-\text{inf}, \text{inf})$ ⓕ $R : [-4, \text{inf})$

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16.E: Review Exercises 2

Chapter Review Exercises

Solve Quadratic Equations Using the Square Root Property

Exercise 16.E.1 Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

1. $y^2 = 144$
2. $n^2 - 80 = 0$
3. $4a^2 = 100$
4. $2b^2 = 72$
5. $r^2 + 32 = 0$
6. $t^2 + 18 = 0$
7. $\frac{2}{3}w^2 - 20 = 30$
8. $5c^2 + 3 = 19$

Answer

1. $y = \pm 12$
3. $a = \pm 5$
5. $r = \pm 4\sqrt{2}i$
7. $w = \pm 5\sqrt{3}$

Exercise 16.E.2 Solve Quadratic Equations of the Form $a(x-h)^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

1. $(p-5)^2 + 3 = 19$
2. $(u+1)^2 = 45$
3. $\left(x - \frac{1}{4}\right)^2 = \frac{3}{16}$
4. $\left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$
5. $(n-4)^2 - 50 = 150$
6. $(4c-1)^2 = -18$
7. $n^2 + 10n + 25 = 12$
8. $64a^2 + 48a + 9 = 81$

Answer

1. $p = -1, 9$
3. $x = \frac{1}{4} \pm \frac{\sqrt{3}}{4}$
5. $n = 4 \pm 10\sqrt{2}$
7. $n = -5 \pm 2\sqrt{3}$

Solve Quadratic Equations by Completing the Square

Exercise 16.E.3 Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

1. $x^2 + 22x$

2. $m^2 - 8m$
3. $a^2 - 3a$
4. $b^2 + 13b$

Answer

1. $(x+11)^2$
3. $\left(a - \frac{3}{2}\right)^2$

Exercise $16.E.4$ Solve Quadratic Equations Using Completing the Square

In the following exercises, solve by completing the square.

1. $d^2 + 14d = -13$
2. $y^2 - 6y = 36$
3. $m^2 + 6m = -109$
4. $t^2 - 12t = -40$
5. $v^2 - 14v = -31$
6. $w^2 - 20w = 100$
7. $m^2 + 10m - 4 = -13$
8. $n^2 - 6n + 11 = 34$
9. $a^2 = 3a + 8$
10. $b^2 = 11b - 5$
11. $(u+8)(u+4) = 14$
12. $(z-10)(z+2) = 28$

Answer

1. $d = -13, -1$
3. $m = -3 \pm 10i$
5. $v = 7 \pm 3\sqrt{2}$
7. $m = -9, -1$
9. $a = \frac{3}{2} \pm \frac{\sqrt{41}}{2}$
11. $u = -6 \pm 2\sqrt{2}$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

Exercise $16.E.5$ Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

1. $3p^2 - 18p + 15 = 15$
2. $5q^2 + 70q + 20 = 0$
3. $4y^2 - 6y = 4$
4. $2x^2 + 2x = 4$
5. $3c^2 + 2c = 9$
6. $4d^2 - 2d = 8$
7. $2x^2 + 6x = -5$
8. $2x^2 + 4x = -5$

Answer

1. $p = 0, 6$

3. $y = -\frac{1}{2}, 2$

5. $c = -\frac{1}{3} \pm \frac{2\sqrt{7}}{3}$

7. $x = \frac{3}{2} \pm \frac{1}{2}i$

Exercise 16.E.6 Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

1. $4x^2 - 5x + 1 = 0$
2. $7y^2 + 4y - 3 = 0$
3. $r^2 - r - 42 = 0$
4. $t^2 + 13t + 22 = 0$
5. $4v^2 + v - 5 = 0$
6. $2w^2 + 9w + 2 = 0$
7. $3m^2 + 8m + 2 = 0$
8. $5n^2 + 2n - 1 = 0$
9. $6a^2 - 5a + 2 = 0$
10. $4b^2 - b + 8 = 0$
11. $u(u - 10) + 3 = 0$
12. $5z(z - 2) = 3$
13. $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$
14. $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$
15. $4c^2 + 4c + 1 = 0$
16. $9d^2 - 12d = -4$

Answer

1. $x = \frac{1}{4}, 1$

3. $r = -6, 7$

5. $v = \frac{-1 \pm \sqrt{21}}{8}$

7. $m = \frac{-4 \pm \sqrt{10}}{3}$

9. $a = \frac{5}{12} \pm \frac{\sqrt{23}}{12}i$

11. $u = 5 \pm \sqrt{21}$

13. $p = \frac{4 \pm \sqrt{5}}{5}$

15. $c = -\frac{1}{2}$

Exercise 16.E.7 Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

1. a. $9x^2 - 6x + 1 = 0$
   b. $3y^2 - 8y + 1 = 0$
   c. $7m^2 + 12m + 4 = 0$
   d. $5n^2 - n + 1 = 0$
2. a. $5x^2 - 7x - 8 = 0$
   b. $7x^2 - 10x + 5 = 0$
   c. $25x^2 - 90x + 81 = 0$
   d. $15x^2 - 8x + 4 = 0$

Answer

1.

a. 1
b. 2
c. 2
d. 2

Exercise 16.E.8 Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

1. a. $16r^2-8r+1=0$
   b. $5t^2-8t+3=9$
   c. $3(c+2)^2=15$
2. a. $4d^2+10d-5=21$
   b. $25x^2-60x+36=0$
   c. $6(5v-7)^2=150$

Answer

1.

a. Factor
b. Quadratic Formula
c. Square Root

Solve Equations in Quadratic Form

Exercise 16.E.9 Solve Equations in Quadratic Form

In the following exercises, solve.

1. $x^4-14x^2+24=0$
2. $x^4+4x^2-32=0$
3. $4x^4-5x^2+1=0$
4. $(2y+3)^2+3(2y+3)-28=0$
5. $x+3\sqrt{x}-28=0$
6. $6x+5\sqrt{x}-6=0$
7. $x^{\frac{2}{3}}-10x^{\frac{1}{3}}+24=0$
8. $x+7x^{\frac{1}{2}}+6=0$
9. $8x^{-2}-2x^{-1}-3=0$

Answer

1. $x=\pm\sqrt{2}, x=\pm 2\sqrt{3}$
3. $x=\pm 1, x=\pm\frac{1}{2}$
5. $x=16$
7. $x=64, x=216$
9. $x=-2, x=\frac{4}{3}$

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Exercise 16.E.10 Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using the method of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth, if needed.

1. Find two consecutive odd numbers whose product is 323.
2. Find two consecutive even numbers whose product is 624.
3. A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.
4. Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.
5. A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.

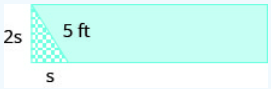

Figure 9.E.1

6. A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

7. The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

8. For Sophia's graduation party, several tables of the same width will be arranged end to end to give serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth . Round answer to the nearest tenth.

9. A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

10. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 360 miles. If the plane was flying at 150 mph, what was the speed of the wind that affected the plane?

11. Ezra kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

12. Two handymen can do a home repair in 2 hours if they work together. One of the men takes 3 hours more than the other man to finish the job by himself. How long does it take for each handyman to do the home repair individually?

Answer

2. Two consecutive even numbers whose product is 624 are 24 and 26, and -24 and -26.

4. The height is 14 inches and the width is 10 inches.

6. The length of the diagonal is 3.6 feet.

8. The width of the serving table is 4.7 feet and the length is 16.1 feet.

Figure 9.E.2

10. The speed of the wind was 30 mph.

12. One man takes 3 hours and the other man 6 hours to finish the repair alone.

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Graph Quadratic Functions Using Properties

Exercise 16.E.11 Recognize the Graph of a Quadratic Function

In the following exercises, graph by plotting point.

1. Graph $y = x^2 - 2$
2. Graph $y = -x^2 + 3$

Answer

2.

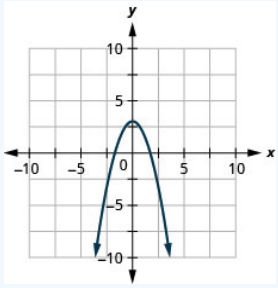

Figure 9.E.3

Exercise 16.E.12 Recognize the Graph of a Quadratic Function

In the following exercises, determine if the following parabolas open up or down.

1. a. $y = -3x^2 + 3x - 1$
   b. $y = 5x^2 + 6x + 3$
2. a. $y = x^2 + 8x - 1$
   b. $y = -4x^2 - 7x + 1$

Answer

2.

a. Up
b. Down

Exercise 16.E.13 Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find

a. The equation of the axis of symmetry
b. The vertex

1. $y = -x^2 + 6x + 8$
2. $y = 2x^2 - 8x + 1$

Answer

2. $x = 2$; $(2, -7)$

Exercise 16.E.14 Find the Intercepts of a Parabola

In the following exercises, find the x- and y-intercepts.

1. $y = x^2 - 4x + 5$

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2. $y = x^2 - 8x + 15$
3. $y = x^2 - 4x + 10$
4. $y = -5x^2 - 30x - 46$
5. $y = 16x^2 - 8x + 1$
6. $y = x^2 + 16x + 64$

Answer

2. $y : (0, 15)$
   $x : (3, 0), (5, 0)$
4. $y : (0, -46)$
   $x :$ none
6. $y : (0, -64)$
   $x : (-8, 0)$

Graph Quadratic Functions Using Properties

Exercise $16.E.15$ Graph Quadratic Functions Using Properties

In the following exercises, graph by using its properties.

1. $y = x^2 + 8x + 15$
2. $y = x^2 - 2x - 3$
3. $y = -x^2 + 8x - 16$
4. $y = 4x^2 - 4x + 1$
5. $y = x^2 + 6x + 13$
6. $y = -2x^2 - 8x - 12$

Answer

2.

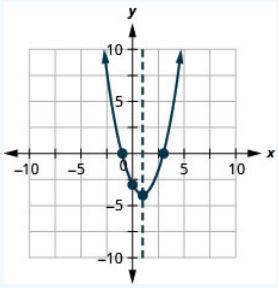

Figure 9.E.4

4.

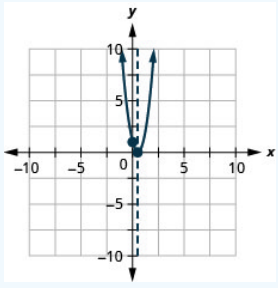

Figure 9.E.5

6.

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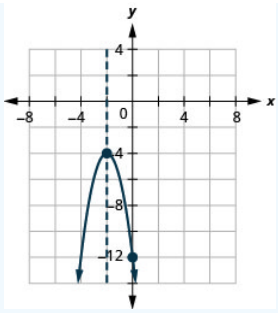

Figure 9.E.6

Exercise 16.E.16 Solve Maximum and Minimum Applications

In the following exercises, find the minimum or maximum value.

1. $y = 7x^2 + 14x + 6$
2. $y = -3x^2 + 12x - 10$

Answer

2. The maximum value is 2 when $x = 2$.

Exercise 16.E.17 Solve Maximum and Minimum Applications

In the following exercises, solve. Rounding answers to the nearest tenth.

1. A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.
2. A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.

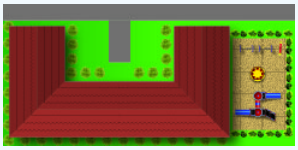
Figure 9.E.7

Answer

2. The length adjacent to the building is 90 feet giving a maximum area of $4,050$ square feet.

Graph Quadratic Functions Using Transformations

Exercise 16.E.18 Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using a vertical shift.

1. $g(x) = x^2 + 4$
2. $h(x) = x^2 - 3$

Answer

2.

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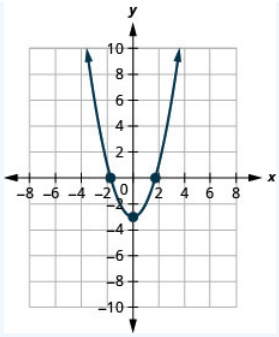

Figure 9.E.8

Exercise $16.E.19$ Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using a horizontal shift.

1. $f(x) = (x+1)^2$
2. $g(x) = (x-3)^2$

Answer

2.

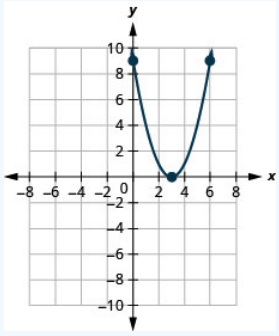

Figure 9.E.9

Exercise $16.E.20$ Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using transformations.

1. $f(x) = (x+2)^2 + 3$
2. $f(x) = (x+3)^2 - 2$
3. $f(x) = (x-1)^2 + 4$
4. $f(x) = (x-4)^2 - 3$

Answer

2.

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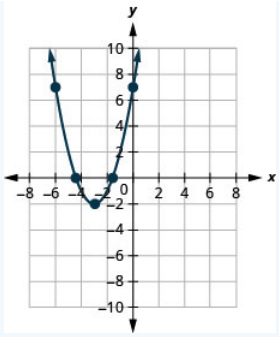

Figure 9.E.10

4.

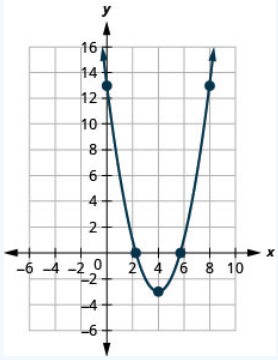

Figure 9.E.11

Exercise 16.E. 21 Graph Quadratic Functions of the Form $f(x) = ax^2$

In the following exercises, graph each function.

1. $f(x) = 2x^2$
2. $f(x) = -x^2$
3. $f(x) = \frac{1}{2}x^2$

Answer

2.

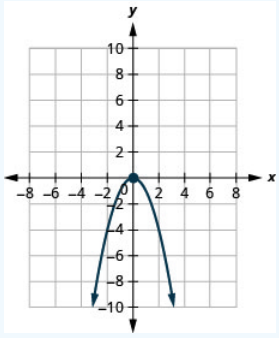

Figure 9.E.12

Exercise 16.E. 22 Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x) = a(x-h)^2 + k$ form by completing the square.

1. $f(x) = 2x^2 - 4x - 4$

2. $f(x) = 3x^2 + 12x + 8$

Answer

1. $f(x) = 2(x-1)^2 - 6$

Exercise $16.E.23$ Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x) = a(x-h)^2 + k$ form
b. Graph it by using transformations

1. $f(x) = 3x^2 - 6x - 1$
2. $f(x) = -2x^2 - 12x - 5$
3. $f(x) = 2x^2 + 4x + 6$
4. $f(x) = 3x^2 - 12x + 7$

Answer

1.

a. $f(x) = 3(x-1)^2 - 4$
b.

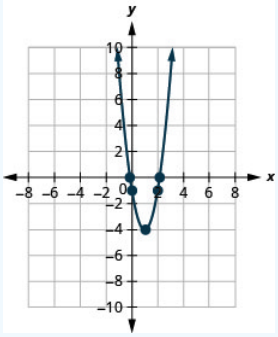

Figure 9.E.13

3.

a. $f(x) = 2(x+1)^2 + 4$
b.

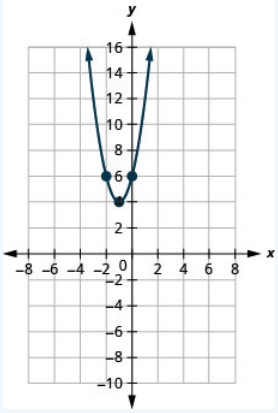

Figure 9.E.14

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Exercise 16.E.24 Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x)=a(x-h)^2+k$ form
b. Graph it using properties

1. $f(x)=-3x^2-12x-5$
2. $f(x)=2x^2-12x+7$

Answer

1.

a. $f(x)=-3(x+2)^2+7$
b.

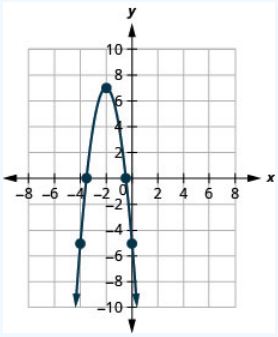

Figure 9.E.15

Exercise 16.E.25 Find a Quadratic Function From its Graph

In the following exercises, write the quadratic function in $f(x)=a(x-h)^2+k$ form.

1.

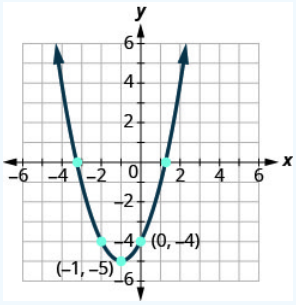

Figure 9.E.16

2.

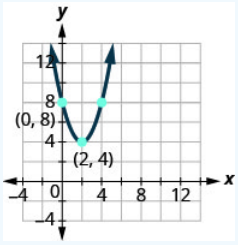

Figure 9.E.17

Answer

1. $f(x)=(x+1)^2-5$

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Exercise $16.E.26$ Solve Quadratic Inequalities Graphically

In the following exercises, solve graphically and write the solution in interval notation.

1. $x^2 - x - 6 > 0$
2. $x^2 + 4x + 3 \leq 0$
3. $-x^2 - x + 2 \geq 0$
4. $-x^2 + 2x + 3 < 0$

Answer

1.

a.

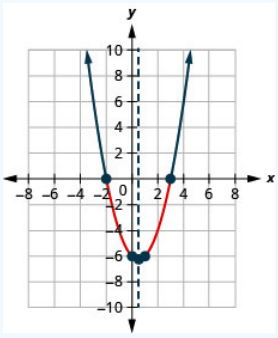

Figure 9.E.18

b. $(-\infty, -2) \cup (3, \infty)$

3.

a.

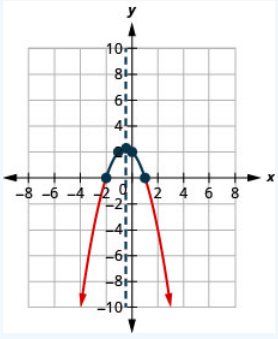

Figure 9.E.19

b. $[-2, 1]$

Exercise $16.E.27$ Solve Quadratic Inequalities Graphically

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

1. $x^2 - 6x + 8 < 0$
2. $x^2 + x > 12$
3. $x^2 - 6x + 4 \leq 0$
4. $2x^2 + 7x - 4 > 0$
5. $-x^2 + x - 6 > 0$
6. $x^2 - 2x + 4 \geq 0$

Answer

1. $(2, 4)$
3. $[3-\sqrt{5}, 3+\sqrt{5}]$
5. no solution

Practice Test

Exercise 16.E.28

1. Use the Square Root Property to solve the quadratic equation $3(w+5)^2 = 27$.
2. Use Completing the Square to solve the quadratic equation $a^2 - 8a + 7 = 23$.
3. Use the Quadratic Formula to solve the quadratic equation $2m^2 - 5m + 3 = 0$.

Answer

1. $w = -2, w = -8$
3. $m = 1, m = \frac{3}{2}$

Exercise 16.E.29

Solve the following quadratic equations. Use any method.

1. $2x(3x-2) - 1 = 0$
2. $\frac{9}{4}y^2 - 3y + 1 = 0$

Answer

2. $y = \frac{2}{3}$

Exercise 16.E.30

Use the discriminant to determine the number and type of solutions of each quadratic equation.

1. $6p^2 - 13p + 7 = 0$
2. $3q^2 - 10q + 12 = 0$

Answer

2. 2 complex

Exercise 16.E.31

Solve each equation.

1. $4x^4 - 17x^2 + 4 = 0$
2. $y^{\frac{2}{3}} + 2y^{\frac{1}{3}} - 3 = 0$

Answer

2. $y = 1, y = -27$

Exercise 16.E.32

For each parabola, find

a. Which direction it opens
b. The equation of the axis of symmetry

c. The vertex
d. The x-and y-intercepts
e. The maximum or minimum value

1. $y = 3x^2 + 6x + 8$
2. $y = -x^2 - 8x + 16$

Answer

2.

a. down
b. $x = -4$
c. $(-4, 0)$
d. $y : (0, 16); x : (-4, 0)$
e. minimum value of -4 when $x = 0$

Exercise $16.E.33$

Graph each quadratic function using intercepts, the vertex, and the equation of the axis of symmetry.

1. $f(x) = x^2 + 6x + 9$
2. $f(x) = -2x^2 + 8x + 4$

Answer

2.

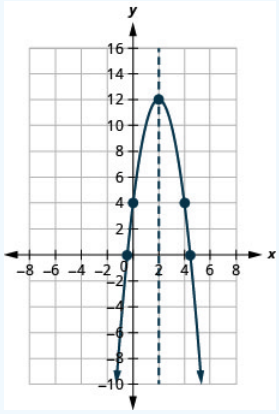

Figure 9.E.20

Exercise $16.E.34$

In the following exercises, graph each function using transformations.

1. $f(x) = (x + 3)^2 + 2$
2. $f(x) = x^2 - 4x - 1$

Answer

2.

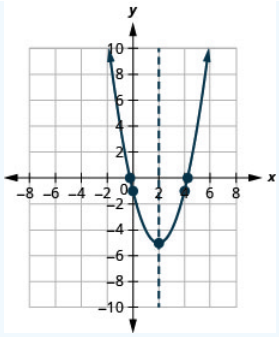

Figure 9.E.21

Exercise $16.E.35$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

1. $x^2 - 6x - 8 \leq 0$
2. $2x^2 + x - 10 > 0$

Answer

2. $\left(-\infty, -\frac{5}{2}\right) \cup (2, \infty)$

Exercise $16.E.36$

Model the situation with a quadratic equation and solve by any method.

1. Find two consecutive even numbers whose product is 360.
2. The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

Answer

2. A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$ find how long it will take the balloon to reach the maximum height, and then find the maximum height. Round to the nearest tenth.

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CHAPTER OVERVIEW

17: Radical Expressions and Functions

17.1: Prelude to Roots and Radicals
17.2: Simplify Expressions with Roots
17.2E: Exercises
17.3: Simplify Radical Expressions
17.3E: Exercises
17.4: Simplify Rational Exponents
17.4E: Exercises
17.5: Add, Subtract, and Multiply Radical Expressions
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17.6: Divide Radical Expressions
17.6E: Exercises
17.7: Solve Radical Equations
17.7E: Exercises
17.8: Use Radicals in Functions
17.8E: Exercises
17.9: Use the Complex Number System
17.9E: Exercises
17.10: Review Exercises

Thumbnail: The mathematical expression "The (principal) square root of x". Image used with permission (GPL, David Vignoni (original icon); Flamurai (SVG convertion); bayo (color)).

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17.1: Prelude to Roots and Radicals

Imagine charging your cell phone is less than five seconds. Consider cleaning radioactive waste from contaminated water. Think about filtering salt from ocean water to make an endless supply of drinking water. Ponder the idea of bionic devices that can repair spinal injuries. These are just of few of the many possible uses of a material called graphene. Materials scientists are developing a material made up of a single layer of carbon atoms that is stronger than any other material, completely flexible, and conducts electricity better than most metals. Research into this type of material requires a solid background in mathematics, including understanding roots and radicals. In this chapter, you will learn to simplify expressions containing roots and radicals, perform operations on radical expressions and equations, and evaluate radical functions.

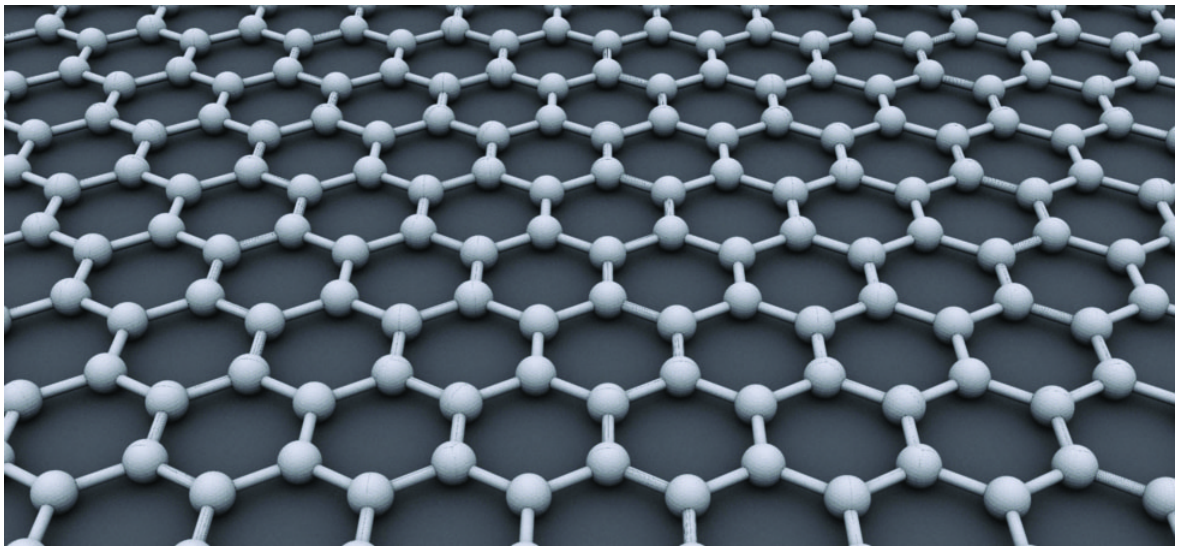

Figure 17.1.1 : Graphene is an incredible strong and flexible material made from carbon. It can also conduct electricity. Notice the hexagonal grid pattern. (credit: "AlexanderAIUS" / Wikimedia Commons)

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17.2: Simplify Expressions with Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

Before you get started, take this readiness quiz.

1. Simplify: a. $(-9)^2$ b. -9^2 c. $(-9)^3$
   If you missed this problem, review Example 2.21.
2. Round 3.846 to the nearest hundredth.
   If you missed this problem, review Example 1.34.
3. Simplify: a. $x^3 \cdot x^3$ b. $y^2 \cdot y^2 \cdot y^2$ c. $z^3 \cdot z^3 \cdot z^3 \cdot z^3$
   If you missed this problem, review Example 5.12.

Simplify Expressions with Roots

In Foundations, we briefly looked at square roots. Remember that when a real number n is multiplied by itself, we write n^2 and read it 'n^2 squared'. This number is called the **square** of n, and n is called the **square root**. For example,

13^2 is read "13 squared"

169 is called the square of 13, since $13^2 = 169$

13 is a square root of 169

Definition 17.2.1: Square and Square Root of a Number

Square

If $n^2 = m$, then m is the **square** of n.

Square Root

If $n^2 = m$, then n is a **square root** of m.

Notice $(-13)^2 = 169$ also, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, $\sqrt{m}$, which denotes the positive square root of m. The positive square root is also called the **principal square root**.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Definition 17.2.2: Square Root Notation

$\sqrt{m}$ is read "the square root of m."

If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

radical sign $\longrightarrow \sqrt{m} \longleftarrow$ radicand

Figure 8.1.1

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{169} = 13$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{169} = -13$.

Example 17.2.1

Simplify:

a. $\sqrt{144}$
b. $-\sqrt{289}$

Solution:

a.

$$\sqrt{144}$$

Since $12^2 = 144$.

$$12$$

b.

$$-\sqrt{289}$$

Since $17^2 = 289$ and the negative is in front of the radical sign.

$$-17$$

Exercise 17.2.1

Simplify:

a. $-\sqrt{64}$
b. $\sqrt{225}$

Answer

a. -8
b. 15

Exercise 17.2.2

Simplify:

a. $\sqrt{100}$
b. $-\sqrt{121}$

Answer

a. 10
b. -11

Can we simplify $-\sqrt{49}$? Is there a number whose square is -49?

$$(\_\_\_)^2 = -49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to $\sqrt{-49}$. The square root of a negative number is not a real number.

Example 17.2.2

Simplify:

a. $\sqrt{-196}$
b. $-\sqrt{64}$

Solution:

a.

$$\sqrt{-196}$$

There is no real number whose square is -196.

$\sqrt{-196}$ is not a real number.

b.

$$-\sqrt{64}$$

The negative is in front of the radical.

$$-8$$

Exercise 17.2.3

Simplify:

a. $\sqrt{-169}$
b. $-\sqrt{81}$

Answer

a. not a real number
b. -9

Exercise 17.2.4

Simplify:

a. $-\sqrt{49}$
b. $\sqrt{-121}$

Answer

a. -7
b. not a real number

So far we have only talked about squares and square roots. Let's now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

We write:	We say:
n^2	n squared
n^3	n cubed
n^4	n to the fourth power
n^5	n to the fifth power

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from -5 to 5. See *Figure 8.1.2*

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

Figure 8.1.2

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We'll copy the row with the powers of -2 to help you see this.

n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

Even power
Positive result

Odd power
Negative result

Figure 8.1.3

We will now extend the square root definition to higher roots.

Definition 17.2.3: Nth Root of a Number

If $b^n = a$, then b is an nth root of a.

The principal n^{th} root of a is written $\sqrt[n]{a}$.

The n is called the **index** of the radical.

Just like we use the word 'cubed' for b^3, we use the term 'cube root' for $\sqrt[3]{a}$.

We can refer to *Figure 8.1.2* to help find higher roots.

$$4^3 = 64 \qquad \sqrt[3]{64} = 4$$
$$3^4 = 81 \qquad \sqrt[4]{81} = 3$$
$$(-2)^5 = -32 \qquad \sqrt[5]{-32} = -2$$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

Properties of $\sqrt[n]{a}$

When n is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an odd number, $\sqrt[n]{a}$ is a real number for all the values of a.

We will apply these properties in the next two examples.

Example 17.2.3

Simplify:

a. $\sqrt[3]{64}$
b. $\sqrt[4]{81}$
c. $\sqrt[5]{32}$

Solution:

a.

$$\sqrt[3]{64}$$

Since $4^3 = 64$.

$$4$$

b.

$$\sqrt[4]{81}$$

Since $(3)^4 = 81$.

$$3$$

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c.

$$\sqrt[5]{32}$$

Since $(2)^5 = 32$.

$$2$$

Exercise 17.2.5

Simplify:

a. $\sqrt[3]{27}$
b. $\sqrt[4]{256}$
c. $\sqrt[5]{243}$

Answer

a. 3
b. 4
c. 3

Exercise 17.2.6

Simplify:

a. $\sqrt[3]{1000}$
b. $\sqrt[4]{16}$
c. $\sqrt[5]{243}$

Answer

a. 10
b. 2
c. 3

In this example be alert for the negative signs as well as even and odd powers.

Example 17.2.4

Simplify:

a. $\sqrt[3]{-125}$
b. $\sqrt[4]{16}$
c. $\sqrt[5]{-243}$

Solution:

a.

$$\sqrt[3]{-125}$$

Since $(-5)^3 = -125$.

$$-5$$

b.

$$\sqrt[4]{16}$$

Think, $(?)^4 = -16$. No real number raised to the fourth power is negative.

Not a real number.

c.

$$\sqrt[5]{-243}$$

Since $(-3)^5 = -243$.

$$-3$$

Exercise 17.2.7

Simplify:

a. $\sqrt[3]{-27}$
b. $\sqrt[4]{-256}$
c. $\sqrt[5]{-32}$

Answer

a. -3
b. not real
c. -2

Exercise 17.2.8

Simplify:

a. $\sqrt[3]{-216}$
b. $\sqrt[4]{-81}$
c. $\sqrt[5]{-1024}$

Answer

a. -6
b. not real
c. -4

Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the $\sqrt{4} = 2$, what is the value of $\sqrt{21}$ or $\sqrt[3]{50}$? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

To get a numerical estimate of a square root, we look for **perfect square** numbers closest to the radicand. To find an estimate of $\sqrt{11}$, we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.

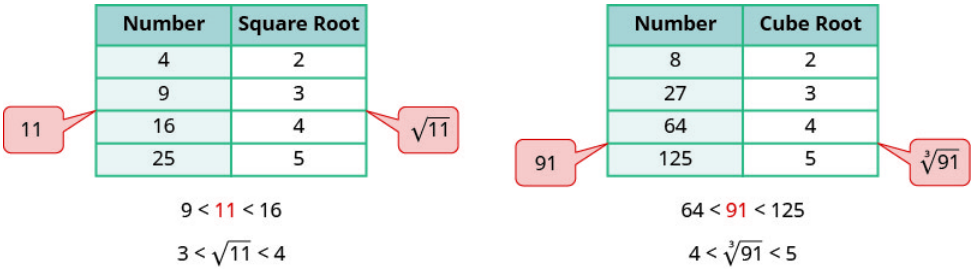


Figure 8.1.4

Similarly, to estimate $\sqrt[3]{91}$, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

Example 17.2.5

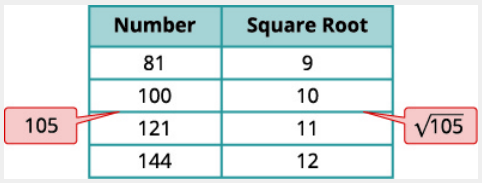
Estimate each root between two consecutive whole numbers:

a. $\sqrt{105}$
b. $\sqrt[3]{43}$

Solution:

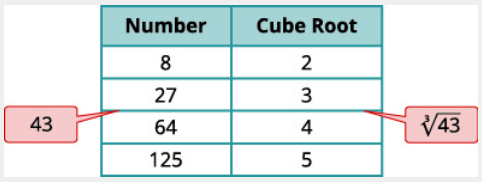
a. Think of the perfect square numbers closest to 105. Make a small table of these perfect squares and their squares roots.

Table 8.1.1

	$\sqrt{105}$
	
Locate 105 between two consecutive perfect squares.	$100 < 105 < 121$
$\sqrt{105}$ is between their square roots.	$10 < \sqrt{105} < 11$

b. Similarly we locate 43 between two perfect cube numbers.

Table 8.1.2

	$\sqrt[3]{43}$
	
Locate 43 between two consecutive perfect cubes.	$27 < 43 < 64$
$\sqrt[3]{43}$ is between their cube roots.	$3 < \sqrt[3]{43} < 4$

Exercise 17.2.9

Estimate each root between two consecutive whole numbers:

a. $\sqrt{38}$
b. $\sqrt[3]{93}$

Answer

a. $6 < \sqrt{38} < 7$
b. $4 < \sqrt[3]{93} < 5$

Exercise 17.2.10

Estimate each root between two consecutive whole numbers:

a. $\sqrt{84}$
b. $\sqrt[3]{152}$

Answer

a. $9 < \sqrt{84} < 10$
b. $5 < \sqrt[3]{152} < 6$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the $\sqrt{x}$ key on your calculator. To find a cube root, or any root with higher index, you will use the $\sqrt[y]{x}$ key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that

$$\sqrt{5} \approx 2.236067978 \text{rounded to two decimal places is } \sqrt{5} \approx 2.24$$

$$\sqrt[4]{93} \approx 3.105422799 \text{rounded to two decimal places is } \sqrt[4]{93} \approx 3.11$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$(2.236067978)^2 = 5.000000002 \qquad (3.105422799)^4 = 92.999999991$$
$$(2.24)^2 = 5.0176 \qquad (3.11)^4 = 93.54951841$$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

Example 17.2.6

Round to two decimal places:

a. $\sqrt{17}$
b. $\sqrt[3]{49}$
c. $\sqrt[4]{51}$

Solution:

a.

$$\sqrt{17}$$

Use the calculator square root key.

$$4.123105626\ldots$$

Round to two decimal places.

$$4.12$$

$$\sqrt{17} \approx 4.12$$

b.

$$\sqrt[3]{49}$$

Use the calculator $\sqrt[y]{x}$ key.

$$3.659305710\ldots$$

Round to two decimal places.

$$3.66$$

$$\sqrt[3]{49} \approx 3.66$$

c.

$$\sqrt[4]{51}$$

Use the calculator $\sqrt[y]{x}$ key.

$$2.6723451177\ldots$$

Round to two decimal places.

$$2.67$$

$$\sqrt[4]{51} \approx 2.67$$

Exercise 17.2.11

Round to two decimal places:

1. $\sqrt{11}$
2. $\sqrt[3]{71}$
3. $\sqrt[4]{127}$

Answer

a. ≈ 3.32
b. ≈ 4.14
c. ≈ 3.36

Exercise 17.2.12

Round to two decimal places:

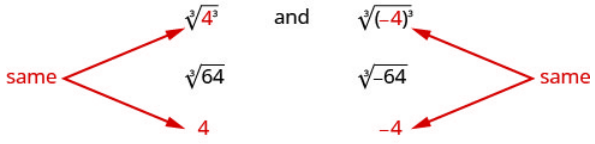
1. $\sqrt{13}$
2. $\sqrt[3]{84}$
3. $\sqrt[4]{98}$

Answer

a. ≈ 3.61
b. ≈ 4.38
c. ≈ 3.15

Simplify Variable Expressions with Roots

The odd root of a number can be either positive or negative. For example,

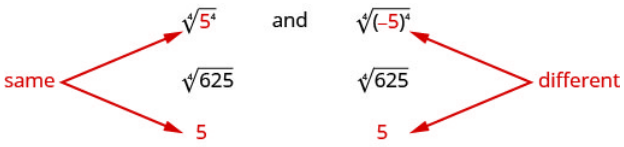


In either case, when n is odd, $\sqrt[n]{a^n} = a$.

Figure 8.1.13

But what about an even root? We want the principal root, so $\sqrt[4]{625} = 5$.

But notice,



Here we see that sometimes when n is even, $\sqrt[n]{a^n} \neq a$.

Figure 8.1.14

How can we make sure the fourth root of -5 raised to the fourth power is 5? We can use the absolute value. $|-5| = 5$. So we say that when n is even $\sqrt[n]{a^n} = |a|$. This guarantees the principal root is positive.

Definition 17.2.4: Simplifying Odd and Even Roots

For any integer $n \geq 2$,

$$\text{when the index } n \text{ is odd } \sqrt[n]{a^n} = a$$

$$\text{when the index } n \text{ is even } \sqrt[n]{a^n} = |a|$$

We must use the **absolute value** signs when we take an even root of an expression with a variable in the radical.

Example 17.2.7

Simplify:

a. $\sqrt{x^2}$
b. $\sqrt[3]{n^3}$
c. $\sqrt[4]{p^4}$
d. $\sqrt[5]{y^5}$

Solution:

a. We use the absolute value to be sure to get the positive root.

$$\sqrt{x^2}$$

Since the index n is even, $\sqrt[n]{a^n} = |a|$.

b. This is an odd indexed root so there is no need for an absolute value sign.

$$\sqrt[3]{m^3}$$

Since the index is n is odd, $\sqrt[n]{a^n} = a$.

$$m$$

c.

$$\sqrt[4]{p^4}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$|p|$$

d.

$$\sqrt[5]{y^5}$$

Since the index n is odd, $\sqrt[n]{a^n} = a$.

$$y$$

Exercise 17.2.13

Simplify:

a. $\sqrt{b^2}$
b. $\sqrt[3]{w^3}$
c. $\sqrt[4]{m^4}$
d. $\sqrt[5]{q^5}$

Answer

a. $|b|$
b. w
c. $|m|$
d. q

Exercise 17.2.14

Simplify:

a. $\sqrt{y^2}$
b. $\sqrt[3]{p^3}$
c. $\sqrt[4]{z^4}$
d. $\sqrt[5]{q^5}$

Answer

a. $|y|$
b. p
c. $|z|$
d. q

What about square roots of higher powers of variables? The Power Property of Exponents says $(a^m)^n = a^{m \cdot n}$. So if we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}$$

Looking now at the square root.

$$\sqrt{a^{2m}}$$

Since $(a^m)^2 = a^{2m}$.

$$\sqrt{(a^m)^2}$$

Since n is even $\sqrt[n]{a^n} = |a|$.

$$|a^m|$$

So $\sqrt{a^{2m}} = |a^m|$.

We apply this concept in the next example.

Example 17.2.8

Simplify:

a. $\sqrt{x^6}$
b. $\sqrt{y^{16}}$

Solution:

a.

$$\sqrt{x^6}$$

Since $\left(x^3\right)^2 = x^6$.

$$\sqrt{(x^3)^2}$$

Since the index n is even $\sqrt{a^n} = |a|$.

$$|x^3|$$

b.

$$\sqrt{y^{16}}$$

Since $\left(y^8\right)^2 = y^{16}$.

$$\sqrt{(y^8)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$y^8$$

In this case the absolute value sign is not needed as y^8 is positive.

Exercise 17.2.15

Simplify:

a. $\sqrt{y^{18}}$
b. $\sqrt{z^{12}}$

Answer

a. $|y^9|$
b. z^6

Exercise 17.2.16

Simplify:

a. $\sqrt{m^4}$
b. $\sqrt{b^{10}}$

Answer

a. m^2
b. $|b^5|$

The next example uses the same idea for higher roots.

Example 17.2.9

Simplify:

a. $\sqrt[3]{y^{18}}$
b. $\sqrt[4]{z^8}$

Solution:

a.

$$\sqrt[3]{y^{18}}$$

Since $\left(y^6\right)^3 = y^{18}$.

$$\sqrt[3]{\left(y^6\right)^3}$$

Since n is odd, $\sqrt[n]{a^n} = a$.

$$y^6$$

b.

$$\sqrt[4]{z^8}$$

Since $\left(z^2\right)^4 = z^8$.

$$\sqrt[4]{\left(z^2\right)^4}$$

Since z^2 is positive, we do not need an absolute value sign.

$$z^2$$

Exercise 17.2.17

Simplify:

a. $\sqrt[4]{u^{12}}$
b. $\sqrt[3]{v^{15}}$

Answer

a. $|u^3|$
b. v^5

Exercise 17.2.18

Simplify:

a. $\sqrt[5]{c^{20}}$
b. $\sqrt[6]{d^{24}}$

Answer

a. c^4
b. d^4

In the next example, we now have a coefficient in front of the variable. The concept $\sqrt{a^{2m}} = |a^m|$ works in much the same way.

$$\sqrt{16r^{22}} = 4\left|r^{11}\right| \text{ because } \left(4r^{11}\right)^2 = 16r^{22}.$$

But notice $\sqrt{25u^8} = 5u^4$ and no absolute value sign is needed as u^4 is always positive.

Example 17.2.10

Simplify:

a. $\sqrt{16n^2}$
b. $-\sqrt{81c^2}$

Solution:

a.

$$\sqrt{16n^2}$$

Since $(4n)^2 = 16n^2$.

$$\sqrt{(4n)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$4|n|$$

b.

$$-\sqrt{81c^2}$$

Since $(9c)^2 = 81c^2$.

$$-\sqrt{(9c)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$-9|c|$$

Exercise 17.2.19

Simplify:

a. $\sqrt{64x^2}$
b. $-\sqrt{100p^2}$

Answer

a. $8|x|$
b. $-10|p|$

Exercise 17.2.20

Simplify:

a. $\sqrt{169y^2}$
b. $-\sqrt{121y^2}$

Answer

a. $13|y|$
b. $-11|y|$

This example just takes the idea farther as it has roots of higher index.

Example 17.2.11

Simplify:

a. $\sqrt[3]{64p^6}$
b. $\sqrt[4]{16q^{12}}$

Solution:

a.

$$\sqrt[3]{64p^6}$$

Rewrite $64p^6$ as $\left(4p^2\right)^3$.

$$\sqrt[3]{\left(4p^2\right)^3}$$

Take the cube root.

$$4p^2$$

b.

$$\sqrt[4]{16q^{12}}$$

Rewrite the radicand as a fourth power.

$$\sqrt[4]{\left(2q^3\right)^4}$$

Take the fourth root.

$$2|q^3|$$

Exercise 17.2.21

Simplify:

a. $\sqrt[3]{27x^{27}}$
b. $\sqrt[4]{81q^{28}}$

Answer

a. $3x^9$
b. $3|q^7|$

Exercise 17.2.22

Simplify:

a. $\sqrt[3]{125q^9}$
b. $\sqrt[5]{243q^{25}}$

Answer

a. $5p^3$
b. $3q^5$

The next examples have two variables.

Example 17.2.12

Simplify:

a. $\sqrt{36x^2y^2}$
b. $\sqrt{121a^6b^8}$
c. $\sqrt[3]{64p^{63}q^9}$

Solution:

a.

$$\sqrt{36x^2y^2}$$

Since $(6xy)^2 = 36x^2y^2$

$$\sqrt{(6xy)^2}$$

Take the square root.

$$6|xy|$$

b.

$$\sqrt{121a^6b^8}$$

Since $\left(11a^3b^4\right)^2 = 121a^6b^8$

$$\sqrt{\left(11a^3b^4\right)^2}$$

Take the square root.

$$11\left|a^3\right|b^4$$

c.

$$\sqrt[3]{64p^{63}q^9}$$

Since $\left(4p^{21}q^3\right)^3 = 64p^{63}q^9$

$$\sqrt[3]{(4p^{21}q^3)^3}$$

Take the cube root.

$$4p^{21}q^3$$

Exercise 17.2.23

Simplify:

a. $\sqrt{100a^2b^2}$
b. $\sqrt{144p^{12}q^{20}}$
c. $\sqrt[3]{8x^{30}y^{12}}$

Answer

a. $10|ab|$
b. $12p^6q^{10}$
c. $2x^{10}y^4$

Exercise 17.2.24

Simplify:

a. $\sqrt{225m^2n^2}$
b. $\sqrt{169x^{10}y^{14}}$
c. $\sqrt[3]{27w^{36}z^{15}}$

Answer

a. $15|mn|$
b. $13\,|x^5y^7|$
c. $3w^{12}z^5$

Access this online resource for additional instruction and practice with simplifying expressions with roots.

- Simplifying Variables Exponents with Roots using Absolute Values

Key Concepts

- **Square Root Notation**
  - $\sqrt{m}$ is read 'the square root of m'
  - If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

    Figure 8.1.1
  - The square root of m, $\sqrt{m}$, is a positive number whose square is m.
- **n^{th} Root of a Number**
  - If $b^n = a$, then b is an nth root of a.
  - The principal n^{th} root of a is written $\sqrt[n]{a}$.
  - n is called the *index* of the radical.
- **Properties of $\sqrt[n]{a}$**
  - When n is an even number and
    - $a \geq 0$, then $\sqrt[n]{a}$ is a real number
    - $a < 0$, then $\sqrt[n]{a}$ is not a real number
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Simplifying Odd and Even Roots**

- For any integer $n \geq 2$,
  - when n is odd $\sqrt[n]{a^n} = a$
  - when n is even $\sqrt[n]{a^n} = |a|$
- We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Glossary

square of a number

If $n^2 = m$, then m is the square of n.

square root of a number

If $n^2 = m$, then n is a square root of m.

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17.2E: Exercises

Simplifying Expressions with Roots

In the following exercises, simplify.

1. a. $\sqrt{64}$ b. $-\sqrt{81}$

Answer

a. 8 b. -9

2. a. $\sqrt{169}$ b. $-\sqrt{100}$

3. a. $\sqrt{196}$ b. $-\sqrt{1}$

Answer

a. 14 b. -1

4. a. $\sqrt{144}$ b. $-\sqrt{121}$

5. a. $\sqrt{\frac{4}{9}}$ b. $-\sqrt{0.01}$

Answer

a. $\frac{2}{3}$ b. -0.1

6. a. $\sqrt{\frac{64}{121}}$ b. $-\sqrt{0.16}$

7. a. $\sqrt{-121}$ b. $-\sqrt{289}$

Answer

a. not a real number b. -17

8. a. $-\sqrt{400}$ b. $\sqrt{-36}$

9. a. $-\sqrt{225}$ b. $\sqrt{-9}$

Answer

a. -15 b. not a real number

10. a. $\sqrt{-49}$ b. $-\sqrt{256}$

11. a. $\sqrt[3]{216}$ b. $\sqrt[4]{256}$

Answer

a. 6 b. 4

12. a. $\sqrt[3]{27}$ b. $\sqrt[4]{16}$ c. $\sqrt[5]{243}$

13. a. $\sqrt[3]{512}$ b. $\sqrt[4]{81}$ c. $\sqrt[5]{1}$

Answer

a. 8 b. 3 b. 1

14. a. $\sqrt[3]{125}$ b. $\sqrt[4]{1296}$ c. $\sqrt[5]{1024}$

15. a. $\sqrt[3]{-8}$ b. $\sqrt[4]{-81}$ c. $\sqrt[5]{-32}$

Answer

a. -2 b. not a real number c. -2

16. a. $\sqrt[3]{-64}$ b. $\sqrt[4]{-16}$ c. $\sqrt[5]{-243}$

17. a. $\sqrt[3]{-125}$ b. $\sqrt[4]{-1296}$ c. $\sqrt[5]{-1024}$

Answer

a. -5 b. not a real number c. -4

18. a. $\sqrt[3]{-512}$ b. $\sqrt[4]{-81}$ c. $\sqrt[5]{-1}$

In the following exercises, estimate each root by giving the interval of two consecutive whole numbers in which the root lies.

19. a. $\sqrt{70}$ b. $\sqrt[3]{71}$

Answer

a. $8 < \sqrt{70} < 9$ b. $4 < \sqrt[3]{71} < 5$

20. a. $\sqrt{55}$ b. $\sqrt[3]{119}$

21. a. $\sqrt{200}$ b. $\sqrt[3]{137}$

Answer

a. $14 < \sqrt{200} < 15$ b. $5 < \sqrt[3]{137} < 6$

22. a. $\sqrt{172}$ b. $\sqrt[3]{200}$

In the following exercises, approximate each root and round to two decimal places.

23. a. $\sqrt{19}$ b. $\sqrt[3]{89}$ c. $\sqrt[4]{97}$

Answer

a. ≈ 4.36 b. ≈ 4.46 c. ≈ 3.14

24. a. $\sqrt{21}$ b. $\sqrt[3]{93}$ c. $\sqrt[4]{101}$

25. a. $\sqrt{53}$ b. $\sqrt[3]{147}$ c. $\sqrt[4]{452}$

Answer

a. ≈ 7.28 b. ≈ 5.28 c. ≈ 4.61

26. a. $\sqrt{47}$ b. $\sqrt[3]{163}$ c. $\sqrt[4]{527}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

27. a. $\sqrt[5]{u^5}$ b. $\sqrt[8]{v^8}$

Answer

a. u b. $|v|$

28. a. $\sqrt[3]{a^3}$ b. $\sqrt[9]{b^9}$

29. a. $\sqrt[4]{y^4}$ b. $\sqrt[7]{m^7}$

Answer

a. $|y|$ b. m

30. a. $\sqrt[8]{k^8}$ b. $\sqrt[6]{p^6}$

31. a. $\sqrt{x^6}$ b. $\sqrt{y^{16}}$

Answer

a. $|x^3|$ b. y^8

32. a. $\sqrt{a^{14}}$ b. $\sqrt{w^{24}}$

33. a. $\sqrt{x^{24}}$ b. $\sqrt{y^{22}}$

Answer

a. x^{12} b. $|y^{11}|$

34. a. $\sqrt{a^{12}}$ b. $\sqrt{b^{26}}$

35. a. $\sqrt[3]{x^9}$ b. $\sqrt[4]{y^{12}}$

Answer

a. x^3 b. $|y^3|$

36. a. $\sqrt[5]{a^{10}}$ b. $\sqrt[3]{b^{27}}$

37. a. $\sqrt[4]{m^8}$ b. $\sqrt[5]{n^{20}}$

Answer

a. m^2 b. n^4

38. a. $\sqrt[6]{r^{12}}$ b. $\sqrt[3]{s^{30}}$

39. a. $\sqrt{49x^2}$ b. $-\sqrt{81x^{18}}$

Answer

a. $7|x|$ b. $-9|x^9|$

40. a. $\sqrt{100y^2}$ b. $-\sqrt{100m^{32}}$

41. a. $\sqrt{121m^{20}}$ b. $-\sqrt{64a^2}$

Answer

a. $11m^{10}$ b. $-8|a|$

42. a. $\sqrt{81x^{36}}$ b. $-\sqrt{25x^2}$

43. a. $\sqrt[4]{16x^8}$ b. $\sqrt[6]{64y^{12}}$

Answer

a. $2x^2$ b. $2y^2$

44. a. $\sqrt[3]{-8c^9}$ b. $\sqrt[3]{125d^{15}}$

45. a. $\sqrt[3]{216a^6}$ b. $\sqrt[5]{32b^{20}}$

Answer

a. $6a^2$ b. $2b^4$

46. a. $\sqrt[7]{128r^{14}}$ b. $\sqrt[4]{81s^{24}}$

47. a. $\sqrt{144x^2y^2}$ b. $\sqrt{169w^8y^{10}}$ c. $\sqrt[3]{8a^{51}b^6}$

Answer

a. $12|xy|$ b. $13w^4\,|y^5|$ c. $2a^{17}b^2$

48. a. $\sqrt{196a^2b^2}$ b. $\sqrt{81p^{24}q^6}$ c. $\sqrt[3]{27p^{45}q^9}$

49. a. $\sqrt{121a^2b^2}$ b. $\sqrt{9c^8d^{12}}$ c. $\sqrt[3]{64x^{15}y^{66}}$

Answer

a. $11|ab|$ b. $3c^4d^6$ c. $4x^5y^{22}$

50. a. $\sqrt{225x^2y^2z^2}$ b. $\sqrt{36r^6s^{20}}$ c. $\sqrt[3]{125y^{18}z^{27}}$

Writing Exercises

51. Why is there no real number equal to $\sqrt{-64}$?

Answer

Since the square of any real number is positive, it's not possible for a real number to square to -64.

52. What is the difference between 9^2 and $\sqrt{9}$?

53. Explain what is meant by the nth root of a number.

Answer

If you raise this root to the nth power, it will give you back the original number (under the radical).

54. Explain the difference of finding the nth root of a number when the index is even compared to when the index is odd.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with roots.			
estimate and approximate roots.			
simplify variable expressions with roots.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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17.3: Simplify Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

Before you get started, take this readiness quiz.

1. Simplify: $\frac{x^9}{x^4}$.
   If you missed this problem, review Example 5.13.
2. Simplify: $\frac{y^3}{y^{11}}$.
   If you missed this problem, review Example 5.13.
3. Simplify: $\left(n^2\right)^6$.
   If you missed this problem, review Example 5.17.

Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A **radical expression**, $\sqrt[n]{a}$, is considered simplified if it has no factors of m^n. So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

Definition 17.3.1: Simplified Radical Expression

For real numbers a and m, and $n \geq 2$,

$\sqrt[n]{a}$ is considered simplified if a has no factors of m^n

For example, $\sqrt{5}$ is considered simplified because there are no perfect square factors in 5. But $\sqrt{12}$ is not simplified because 12 has a perfect square factor of 4.

Similarly, $\sqrt[3]{4}$ is simplified because there are no perfect cube factors in 4. But $\sqrt[3]{24}$ is not simplified because 24 has a perfect cube factor of 8.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that

$$(ab)^n = a^n b^n. \tag{17.3.1}$$

The corresponding of **Product Property of Roots** says that

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}. \tag{17.3.2}$$

Definition 17.3.2: Product Property of nth Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, and $n \geq 2$ is an integer, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

We use the Product Property of Roots to remove all perfect square factors from a square root.

Example 17.3.1: Simplify square roots using the product property of roots

Simplify: $\sqrt{98}$.

Solution:

Step 1: Find the largest factor in the radicand that is a perfect power of the index.	We see that 49 is the largest factor of 98 that has a power of 2.	$\sqrt{98}$
Rewrite the radicand as a product of two factors, using that factor.	In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first.	$\sqrt{49 \cdot 2}$
Step 2: Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{49} \cdot \sqrt{2}$
Step 3: Simplify the root of the perfect power.		$7\sqrt{2}$

Try It 17.3.1

Simplify: $\sqrt{48}$

Answer

$4\sqrt{3}$

Try It 17.3.2

Simplify: $\sqrt{45}$.

Answer

$3\sqrt{5}$

Notice in the previous example that the simplified form of $\sqrt{98}$ is $7\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index. The expression $7\sqrt{2}$ is very different from $\sqrt[7]{2}$.

Simplify a Radical Expression Using the Product Property

1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
2. Use the product rule to rewrite the radical as the product of two radicals.
3. Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares, cubes, and fourth powers.

Example 17.3.2

Simplify:

a. $\sqrt{500}$
b. $\sqrt[3]{16}$

c. $\sqrt[4]{243}$

Solution:

a.

$$\sqrt{500}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{100 \cdot 5}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{100} \cdot \sqrt{5}$$

Simplify.

$$10\sqrt{5}$$

b.

$$\sqrt[3]{16}$$

Rewrite the radicand as a product using the greatest perfect cube factor. $2^3 = 8$

$$\sqrt[3]{8 \cdot 2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{8} \cdot \sqrt[3]{2}$$

Simplify.

$$2\sqrt[3]{2}$$

c.

$$\sqrt[4]{243}$$

Rewrite the radicand as a product using the greatest perfect fourth power factor. $3^4 = 81$

$$\sqrt[4]{81 \cdot 3}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{81} \cdot \sqrt[4]{3}$$

Simplify.

$$3\sqrt[4]{3}$$

? Try It 17.3.3

Simplify: a. $\sqrt{288}$ b. $\sqrt[3]{81}$ c. $\sqrt[4]{64}$

Answer

a. $12\sqrt{2}$ b. $3\sqrt[3]{3}$ c. $2\sqrt[4]{4}$

? Try It 17.3.4

Simplify: a. $\sqrt{432}$ b. $\sqrt[3]{625}$ c. $\sqrt[4]{729}$

Answer

a. $12\sqrt{3}$ b. $5\sqrt[3]{5}$ c. $3\sqrt[4]{9}$

The next example is much like the previous examples, but with variables. Don't forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

Example 17.3.3

Simplify:

a. $\sqrt{x^3}$
b. $\sqrt[3]{x^4}$
c. $\sqrt[4]{x^7}$

Solution:

a.

$$\sqrt{x^3}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{x^2 \cdot x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{x^2} \cdot \sqrt{x}$$

Simplify.

$$|x|\sqrt{x}$$

b.

$$\sqrt[3]{x^4}$$

Rewrite the radicand as a product using the largest perfect cube factor.

$$\sqrt[3]{x^3 \cdot x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$$

Simplify.

$$x\sqrt[3]{x}$$

c.

$$\sqrt[4]{x^7}$$

Rewrite the radicand as a product using the greatest perfect fourth power factor.

$$\sqrt[4]{x^4 \cdot x^3}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$$

Simplify.

$$|x|\sqrt[4]{x^3}$$

Try It 17.3.5

Simplify: a. $\sqrt{b^5}$ b. $\sqrt[4]{y^6}$ c. $\sqrt[3]{z^5}$

Answer

a. $b^2\sqrt{b}$ b. $|y|\sqrt[4]{y^2}$ c. $z\sqrt[3]{z^2}$

Try It 17.3.6

Simplify: a. $\sqrt{p^9}$ b. $\sqrt[5]{y^8}$ c. $\sqrt[6]{q^{13}}$

Answer

a. $p^4\sqrt{p}$ b. $p\sqrt[5]{p^3}$ c. $q^2\sqrt[6]{q}$

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

Example 17.3.4

Simplify:

a. $\sqrt{72n^7}$
b. $\sqrt[3]{24x^7}$
c. $\sqrt[4]{80y^{14}}$

Solution:

a.

$$\sqrt{72n^7}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{36n^6 \cdot 2n}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{36n^6} \cdot \sqrt{2n}$$

Simplify.

$$6\left|n^3\right|\sqrt{2n}$$

b.

$$\sqrt[3]{24x^7}$$

Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{8x^6 \cdot 3x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{8x^6} \cdot \sqrt[3]{3x}$$

Rewrite the first radicand as $\left(2x^2\right)^3$.

$$\sqrt[3]{\left(2x^2\right)^3} \cdot \sqrt[3]{3x}$$

Simplify.

$$2x^2\sqrt[3]{3x}$$

c.

$$\sqrt[4]{80y^{14}}$$

Rewrite the radicand as a product using perfect fourth power factors.

$$\sqrt[4]{16y^{12} \cdot 5y^2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{16y^{12}} \cdot \sqrt[4]{5y^2}$$

Rewrite the first radicand as $\left(2y^3\right)^4$.

$$\sqrt[4]{(2y^3)^4}\cdot\sqrt[4]{5y^2}$$

Simplify.

$$2\left|y^3\right|\sqrt[4]{5y^2}$$

Try It 17.3.7

Simplify: a. $\sqrt{32y^5}$ b. $\sqrt[3]{54p^{10}}$ c. $\sqrt[4]{64q^{10}}$

Answer

a. $4y^2\sqrt{2y}$ b. $3p^3\sqrt[3]{2p}$ c. $2q^2\sqrt[4]{4q^2}$

Try It 17.3.8

Simplify: a. $\sqrt{75a^9}$ b. $\sqrt[3]{128m^{11}}$ c. $\sqrt[4]{162n^7}$

Answer

a. $5a^4\sqrt{3a}$ b. $4m^3\sqrt[3]{2m^2}$ c. $3|n|\sqrt[4]{2n^3}$

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

Example 17.3.5

Simplify:

a. $\sqrt{63u^3v^5}$
b. $\sqrt[3]{40x^4y^5}$
c. $\sqrt[4]{48x^4y^7}$

Solution:

a.

$$\sqrt{63u^3v^5}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{9u^2v^4\cdot 7uv}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{9u^2v^4}\cdot\sqrt{7uv}$$

Rewrite the first radicand as $\left(3uv^2\right)^2$.

$$\sqrt{(3uv^2)^2}\cdot\sqrt{7uv}$$

Simplify.

$$3|u|v^2\sqrt{7uv}$$

b.

$$\sqrt[3]{40x^4y^5}$$

Rewrite the radicand as a product using the largest perfect cube factor.

$$\sqrt[3]{8x^3y^3\cdot 5xy^2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{8x^3y^3} \cdot \sqrt[3]{5xy^2}$$

Rewrite the first radicand as $(2xy)^3$.

$$\sqrt[3]{(2xy)^3} \cdot \sqrt[3]{5xy^2}$$

Simplify.

$$2xy\sqrt[3]{5xy^2}$$

c.

$$\sqrt[4]{48x^4y^7}$$

Rewrite the radicand as a product using the largest perfect fourth power factor.

$$\sqrt[4]{16x^4y^4 \cdot 3y^3}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{16x^4y^4} \cdot \sqrt[4]{3y^3}$$

Rewrite the first radicand as $(2xy)^4$.

$$\sqrt[4]{(2xy)^4} \cdot \sqrt[4]{3y^3}$$

Simplify.

$$2|xy|\sqrt[4]{3y^3}$$

Try It 17.3.9

Simplify:

a. $\sqrt{98a^7b^5}$
b. $\sqrt[3]{56x^5y^4}$
c. $\sqrt[4]{32x^5y^8}$

Answer

a. $7|a^3|b^2\sqrt{2ab}$
b. $2xy\sqrt[3]{7x^2y}$
c. $2|x|y^2\sqrt[4]{2x}$

Try It 17.3.10

Simplify:

a. $\sqrt{180m^9n^{11}}$
b. $\sqrt[3]{72x^6y^5}$
c. $\sqrt[4]{80x^7y^4}$

Answer

a. $6m^4|n^5|\sqrt{5mn}$
b. $2x^2y\sqrt[3]{9y^2}$
c. $2|xy|\sqrt[4]{5x^3}$

Example 17.3.6

Simplify:

a. $\sqrt[3]{-27}$
b. $\sqrt[4]{-16}$

Solution:

a.

$$\sqrt[3]{-27}$$

Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{(-3)^3}$$

Take the cube root.

$$-3$$

b.

$$\sqrt[4]{-16}$$

There is no real number n where $n^4 = -16$.

Not a real number

Try It 17.3.11

Simplify:

a. $\sqrt[3]{-64}$
b. $\sqrt[4]{-81}$

Answer

a. -4
b. no real number

Try It 17.3.12

Simplify:

a. $\sqrt[3]{-625}$
b. $\sqrt[4]{-324}$

Answer

a. $-5\sqrt[3]{5}$
b. no real number

We have seen how to use the **order of operations** to simplify some expressions with radicals. In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer since one term contains a radical and the other does not. The next example also includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

Example 17.3.7

Simplify:

a. $3+\sqrt{32}$
b. $\dfrac{4-\sqrt{48}}{2}$

Solution:

a.

$$3+\sqrt{32}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$3+\sqrt{16\cdot 2}$$

Rewrite the radical as the product of two radicals.

$$3+\sqrt{16}\cdot\sqrt{2}$$

Simplify.

$$3+4\sqrt{2}$$

The terms cannot be added as one has a radical and the other does not. Trying to add an integer and a radical is like trying to add an integer and a variable. They are not like terms!

b.

$$\frac{4-\sqrt{48}}{2}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\frac{4-\sqrt{16\cdot 3}}{2}$$

Rewrite the radical as the product of two radicals.

$$\frac{4-\sqrt{16}\cdot\sqrt{3}}{2}$$

Simplify.

$$\frac{4-4\sqrt{3}}{2}$$

Factor the common factor from the numerator.

$$\frac{4(1-\sqrt{3})}{2}$$

Remove the common factor, 2, from the numerator and denominator.

$$\frac{\cancel{2}\cdot 2(1-\sqrt{3})}{\cancel{2}}$$

Simplify.

$$2(1-\sqrt{3})$$

? Try It 17.3.13

Simplify:

a. $5+\sqrt{75}$

b. $\dfrac{10-\sqrt{75}}{5}$

Answer

a. $5+5\sqrt{3}$

b. $2-\sqrt{3}$

? Try It 17.3.14

Simplify:

a. $2+\sqrt{98}$

b. $\dfrac{6-\sqrt{45}}{3}$

Answer

a. $2+7\sqrt{2}$
b. $2-\sqrt{5}$

Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

Example 17.3.8

Simplify:

a. $\sqrt{\dfrac{45}{80}}$
b. $\sqrt[3]{\dfrac{16}{54}}$
c. $\sqrt[4]{\dfrac{5}{80}}$

Solution:

a.

$$\sqrt{\frac{45}{80}}$$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.

$$\sqrt{\frac{5\cdot 9}{5\cdot 16}}$$

Simplify the fraction by removing common factors.

$$\sqrt{\frac{9}{16}}$$

Simplify. Note $\left(\dfrac{3}{4}\right)^2=\dfrac{9}{16}$.

$$\frac{3}{4}$$

b.

$$\sqrt[3]{\frac{16}{54}}$$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.

$$\sqrt[3]{\frac{2\cdot 8}{2\cdot 27}}$$

Simplify the fraction by removing common factors.

$$\sqrt[3]{\frac{8}{27}}$$

Simplify. Note $\left(\dfrac{2}{3}\right)^3=\dfrac{8}{27}$.

$$\frac{2}{3}$$

c.

$$\sqrt[4]{\frac{5}{80}}$$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.

$$\sqrt[4]{\frac{5 \cdot 1}{5 \cdot 16}}$$

Simplify the fraction by removing common factors.

$$\sqrt[4]{\frac{1}{16}}$$

Simplify. Note $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

$$\frac{1}{2}$$

? Try It 17.3.15

Simplify:

a. $\sqrt{\frac{75}{48}}$

b. $\sqrt[3]{\frac{54}{250}}$

c. $\sqrt[4]{\frac{32}{162}}$

Answer

a. $\frac{5}{4}$

b. $\frac{3}{5}$

c. $\frac{2}{3}$

? Try It 17.3.16

Simplify:

a. $\sqrt{\frac{98}{162}}$

b. $\sqrt[3]{\frac{24}{375}}$

c. $\sqrt[4]{\frac{4}{324}}$

Answer

a. $\frac{7}{9}$

b. $\frac{2}{5}$

c. $\frac{1}{3}$

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the **Quotient Property** to simplify under the radical. We divide the like bases by subtracting their exponents,

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

Example 17.3.9

Simplify:

a. $\sqrt{\frac{m^6}{m^4}}$

b. $\sqrt[3]{\frac{a^8}{a^5}}$

c. $\sqrt[4]{\frac{a^{10}}{a^2}}$

Solution:

a.

$$\sqrt{\frac{m^6}{m^4}}$$

Simplify the fraction inside the radical first. Divide the like bases by subtracting the exponents.

$$\sqrt{m^2}$$

Simplify.

$$|m|$$

b.

$$\sqrt[3]{\frac{a^8}{a^5}}$$

Use the Quotient Property of exponents to simplify the fraction under the radical first.

$$\sqrt[3]{a^3}$$

Simplify.

$$a$$

c.

$$\sqrt[4]{\frac{a^{10}}{a^2}}$$

Use the Quotient Property of exponents to simplify the fraction under the radical first.

$$\sqrt[4]{a^8}$$

Rewrite the radicand using perfect fourth power factors.

$$\sqrt[4]{(a^2)^4}$$

Simplify.

$$a^2$$

Try It 17.3.17

Simplify:

a. $\sqrt{\frac{a^8}{a^6}}$
b. $\sqrt[4]{\frac{x^7}{x^3}}$
c. $\sqrt[4]{\frac{y^{17}}{y^5}}$

Answer

a. $|a|$
b. $|x|$
c. y^3

Try It 17.3.18

Simplify:

a. $\sqrt{\frac{x^{14}}{x^{10}}}$
b. $\sqrt[3]{\frac{m^{13}}{m^7}}$
c. $\sqrt[5]{\frac{n^{12}}{n^2}}$

Answer

a. x^2
b. m^2
c. n^2

Remember the **Quotient to a Power Property**? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

Definition 17.3.3

Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \text{ and } \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Example 17.3.10 how to simplify the quotient of radical expressions

Simplify: $\sqrt{\frac{27m^3}{196}}$

Solution:

Step 1: Simplify the fraction in the radicand, if possible.

$\frac{27m^3}{196}$ cannot be simplified.

$$\sqrt{\frac{27m^3}{196}}$$

Step 2: Use the Quotient Property to rewrite the radical as the quotient of two radicals.

We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.

$$\frac{\sqrt{27m^3}}{\sqrt{196}}$$

Step 3: Simplify the radicals in the numerator and the denominator.

$9m^2$ and 196 are perfect squares.

$$\frac{\sqrt{9m^2}\cdot\sqrt{3m}}{\sqrt{196}}$$

$$\frac{3m\sqrt{3m}}{14}$$

? Try It 17.3.19

Simplify: $\sqrt{\frac{24p^3}{49}}$.

Answer

$$\frac{2|p|\sqrt{6p}}{7}$$

? Try It 17.3.20

Simplify: $\sqrt{\frac{48x^5}{100}}$.

Answer

$$\frac{2x^2\sqrt{3x}}{5}$$

Simplify a Square Root Using the Quotient Property

1. Simplify the fraction in the radicand, if possible.
2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
3. Simplify the radicals in the numerator and the denominator.

✓ Example 17.3.11

Simplify:

a. $\sqrt{\frac{45x^5}{y^4}}$

b. $\sqrt[3]{\frac{24x^7}{y^3}}$

c. $\sqrt[4]{\frac{48x^{10}}{y^8}}$

Solution:

a.

$$\sqrt{\frac{45x^5}{y^4}}$$

We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.

$$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9x^4}\cdot\sqrt{5x}}{y^2}$$

Simplify.

$$\frac{3x^2\sqrt{5x}}{y^2}$$

b.

$$\sqrt[3]{\frac{24x^7}{y^3}}$$

The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.

$$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$$

Rewrite each radicand as a product using perfect cube factors.

$$\frac{\sqrt[3]{8x^6\cdot 3x}}{\sqrt[3]{y^3}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[3]{(2x^2)^3}\cdot\sqrt[3]{3x}}{\sqrt[3]{y^3}}$$

Simplify.

$$\frac{2x^2\sqrt[3]{3x}}{y}$$

c.

$$\sqrt[4]{\frac{48x^{10}}{y^8}}$$

The fraction in the radicand cannot be simplified.

$$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$$

Use the Quotient Property to write as two radicals. Rewrite each radicand as a product using perfect fourth power factors.

$$\frac{\sqrt[4]{16x^8\cdot 3x^2}}{\sqrt[4]{y^8}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[4]{(2x^2)^4}\cdot\sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$$

Simplify.

$$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$$

Try It 17.3.21

Simplify:

a. $\sqrt{\frac{80m^3}{n^6}}$

b. $\sqrt[3]{\frac{108c^{10}}{d^6}}$

c. $\sqrt[4]{\frac{80x^{10}}{y^4}}$

Answer

a. $\frac{4|m|\sqrt{5m}}{|n^3|}$

b. $\frac{3c^3\sqrt[3]{4c}}{d^2}$

c. $\frac{2x^2\sqrt[4]{5x^2}}{|y|}$

Try It 17.3.22

Simplify:

a. $\sqrt{\frac{54u^7}{v^8}}$

b. $\sqrt[3]{\frac{40r^3}{s^6}}$

c. $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$

Answer

a. $\frac{3u^3\sqrt{6u}}{v^4}$

b. $\frac{2r\sqrt[3]{5}}{s^2}$

c. $\frac{3\left|m^3\right|\sqrt[4]{2m^2}}{|n^3|}$

Be sure to simplify the fraction in the radicand first, if possible.

Example 17.3.12

Simplify:

a. $\sqrt{\frac{18p^5q^7}{32pq^2}}$

b. $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$

c. $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$

Solution:

a.

$$\sqrt{\frac{18p^5q^7}{32pq^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt{\frac{9p^4q^5}{16}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$$

Simplify.

$$\frac{3p^2q^2\sqrt{q}}{4}$$

b.

$$\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt[3]{\frac{8x^3y^5}{27}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt[3]{8x^3y^5}}{\sqrt[3]{27}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt[3]{8x^3y^3}\cdot\sqrt[3]{y^2}}{\sqrt[3]{27}}$$

Simplify.

$$\frac{2xy\sqrt[3]{y^2}}{3}$$

c.

$$\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt[4]{\frac{a^5b^4}{16}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt[4]{a^5b^4}}{\sqrt[4]{16}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt[4]{a^4b^4}\cdot\sqrt[4]{a}}{\sqrt[4]{16}}$$

Simplify.

$$\frac{|ab|\sqrt[4]{a}}{2}$$

? Try It 17.3.23

Simplify:

a. $\sqrt{\frac{50x^5y^3}{72x^4y}}$

b. $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$

c. $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$

Answer

a. $\frac{5|y|\sqrt{x}}{6}$

b. $\frac{2xy\sqrt[3]{y^2}}{3}$

c. $\frac{|ab|\sqrt[4]{a}}{2}$

? Try It 17.3.24

Simplify:

a. $\sqrt{\frac{48m^7n^2}{100m^5n^8}}$

b. $\sqrt[3]{\frac{54x^7y^5}{250x^2y^2}}$

c. $\sqrt[4]{\frac{32a^9b^7}{162a^3b^3}}$

Answer

a. $\frac{2|m|\sqrt{3}}{5\,|n^3|}$

b. $\frac{3xy\sqrt[3]{x^2}}{5}$

c. $\frac{2|ab|\sqrt[4]{a^2}}{3}$

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the **Quotient Property** again, to combine them into one radical. We will then look to see if we can simplify the expression.

Example 17.3.13

Simplify:

a. $\dfrac{\sqrt{48a^7}}{\sqrt{3a}}$

b. $\dfrac{\sqrt[3]{-108}}{\sqrt[3]{2}}$

c. $\dfrac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$

Solution:

a.

$$\frac{\sqrt{48a^7}}{\sqrt{3a}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt{\frac{48a^7}{3a}}$$

Simplify the fraction under the radical.

$$\sqrt{16a^6}$$

Simplify.

$$4\left|a^3\right|$$

b.

$$\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt[3]{\frac{-108}{2}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{-54}$$

Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{(-3)^3\cdot 2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{(-3)^3}\cdot\sqrt[3]{2}$$

Simplify.

$$-3\sqrt[3]{2}$$

c.

$$\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt[4]{\frac{96x^7}{3x^2}}$$

Simplify the fraction under the radical.

$$\sqrt[4]{32x^5}$$

Rewrite the radicand as a product using perfect fourth power factors.

$$\sqrt[4]{16x^4}\cdot\sqrt[4]{2x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{(2x)^4}\cdot\sqrt[4]{2x}$$

Simplify.

$$2|x|\sqrt[4]{2x}$$

? Try It 17.3.25

Simplify:

a. $\dfrac{\sqrt{98z^5}}{\sqrt{2z}}$

b. $\dfrac{\sqrt[3]{-500}}{\sqrt[3]{2}}$

c. $\dfrac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$

Answer

a. $7z^2$

b. $-5\sqrt[3]{2}$

c. $3|m|\sqrt[4]{2m^2}$

? Try It 17.3.26

Simplify:

a. $\dfrac{\sqrt{128m^9}}{\sqrt{2m}}$

b. $\dfrac{\sqrt[3]{-192}}{\sqrt[3]{3}}$

c. $\dfrac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$

Answer

a. $8m^4$

b. -4

c. $3|n|\sqrt[4]{2}$

Access these online resources for additional instruction and practice with simplifying radical expressions.

- Simplifying Square Root and Cube Root with Variables
- Express a Radical in Simplified Form-Square and Cube Roots with Variables and Exponents
- Simplifying Cube Roots

Key Concepts

- **Simplified Radical Expression**
  - For real numbers a, m and $n \geq 2$
    $\sqrt[n]{a}$ is considered simplified if a has no factors of m^n
- **Product Property of nth Roots**

- For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$
  $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

- **How to simplify a radical expression using the Product Property**
  1. Find the largest factor in the radicand that is a perfect power of the index.
     Rewrite the radicand as a product of two factors, using that factor.
  2. Use the product rule to rewrite the radical as the product of two radicals.
  3. Simplify the root of the perfect power.

- **Quotient Property of Radical Expressions**
  - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

- **How to simplify a radical expression using the Quotient Property.**
  1. Simplify the fraction in the radicand, if possible.
  2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
  3. Simplify the radicals in the numerator and the denominator.

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17.3E: Exercises

Practice Makes Perfect

Exercise SET A: use the product property to simplify radical expressions

In the following exercises, use the Product Property to simplify radical expressions.

1. $\sqrt{27}$
2. $\sqrt{80}$
3. $\sqrt{125}$
4. $\sqrt{96}$
5. $\sqrt{147}$
6. $\sqrt{450}$
7. $\sqrt{800}$
8. $\sqrt{675}$
9. a. $\sqrt[4]{32}$
   b. $\sqrt[5]{64}$
10. a. $\sqrt[3]{625}$
    b. $\sqrt[6]{128}$
11. a. $\sqrt[5]{64}$
    b. $\sqrt[3]{256}$
12. a. $\sqrt[4]{3125}$
    b. $\sqrt[3]{81}$

Answer

1. $3\sqrt{3}$
3. $5\sqrt{5}$
5. $7\sqrt{3}$
7. $20\sqrt{2}$
9. a. $2\sqrt[4]{2}$
   b. $2\sqrt[5]{2}$
11. a. $2\sqrt[5]{2}$
    b. $4\sqrt[3]{4}$

Exercise SET B: use the product property to simplify radical expressions

In the following exercises, simplify using absolute value signs as needed.

1. a. $\sqrt{y^{11}}$
   b. $\sqrt[3]{r^5}$
   c. $\sqrt[4]{s^{10}}$
2. a. $\sqrt{m^{13}}$
   b. $\sqrt[5]{u^7}$
   c. $\sqrt[6]{v^{11}}$
3. a. $\sqrt{n^{21}}$
   b. $\sqrt[3]{q^8}$

c. $\sqrt[8]{n^{10}}$

4. a. $\sqrt{r^{25}}$
   b. $\sqrt[5]{p^8}$
   c. $\sqrt[4]{m^5}$
5. a. $\sqrt{125r^{13}}$
   b. $\sqrt[3]{108x^5}$
   c. $\sqrt[4]{48y^6}$
6. a. $\sqrt{80s^{15}}$
   b. $\sqrt[5]{96a^7}$
   c. $\sqrt[6]{128b^7}$
7. a. $\sqrt{242m^{23}}$
   b. $\sqrt[4]{405m10}$
   c. $\sqrt[5]{160n^8}$
8. a. $\sqrt{175n^{13}}$
   b. $\sqrt[5]{512p^5}$
   c. $\sqrt[4]{324q^7}$
9. a. $\sqrt{147m^7n^{11}}$
   b. $\sqrt[3]{48x^6y^7}$
   c. $\sqrt[4]{32x^5y^4}$
10. a. $\sqrt{96r^3s^3}$
    b. $\sqrt[3]{80x^7y^6}$
    c. $\sqrt[4]{80x^8y^9}$
11. a. $\sqrt{192q^3r^7}$
    b. $\sqrt[3]{54m^9n^{10}}$
    c. $\sqrt[4]{81a^9b^8}$
12. a. $\sqrt{150m^9n^3}$
    b. $\sqrt[3]{81p^7q^8}$
    c. $\sqrt[4]{162c^{11}d^{12}}$
13. a. $\sqrt[3]{-864}$
    b. $\sqrt[4]{-256}$
14. a. $\sqrt[5]{-486}$
    b. $\sqrt[6]{-64}$
15. a. $\sqrt[5]{-32}$
    b. $\sqrt[8]{-1}$
16. a. $\sqrt[3]{-8}$
    b. $\sqrt[4]{-16}$
17. a. $5+\sqrt{12}$
    b. $\dfrac{10-\sqrt{24}}{2}$
18. a. $8+\sqrt{96}$
    b. $\dfrac{8-\sqrt{80}}{4}$
19. a. $1+\sqrt{45}$
    b. $\dfrac{3+\sqrt{90}}{3}$
20. a. $3+\sqrt{125}$
    b. $\dfrac{15+\sqrt{75}}{5}$

Answer

1.

a. $|y^5|\sqrt{y}$
b. $r\sqrt[3]{r^2}$
c. $s^2\sqrt[4]{s^2}$

3.

a. $n^{10}\sqrt{n}$
b. $q^2\sqrt[3]{q^2}$
c. $|n|\sqrt[8]{n^2}$

5.

a. $5r^6\sqrt{5r}$
b. $3x\sqrt[3]{4x^2}$
c. $2|y|\sqrt[4]{3y^2}$

7.

a. $11\,|m^{11}|\sqrt{2m}$
b. $3m^2\sqrt[4]{5m^2}$
c. $2n\sqrt[5]{5n^3}$

9.

a. $7\,|m^3n^5|\sqrt{3mn}$
b. $2x^2y^2\sqrt[3]{6y}$
c. $2|xy|\sqrt[4]{2x}$

11.

a. $8\,|qr^3|\sqrt{3qr}$
b. $3m^3n^3\sqrt[3]{2n}$
c. $3a^2b^2\sqrt[4]{a}$

13.

a. $-6\sqrt[3]{4}$
b. not real

15.

a. -2
b. not real

17.

a. $5+2\sqrt{3}$
b. $5-\sqrt{6}$

19.

a. $1+3\sqrt{5}$
b. $1+\sqrt{10}$

Exercise Set C: use the quotient property to simplify radical expressions

In the following exercises, use the Quotient Property to simplify square roots.

1. a. $\sqrt{\frac{45}{80}}$

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b. $\sqrt[3]{\frac{8}{27}}$

c. $\sqrt[4]{\frac{1}{81}}$

2. a. $\sqrt{\frac{72}{98}}$

b. $\sqrt[3]{\frac{24}{81}}$

c. $\sqrt[4]{\frac{6}{96}}$

3. a. $\sqrt{\frac{100}{36}}$

b. $\sqrt[3]{\frac{81}{375}}$

c. $\sqrt[4]{\frac{1}{256}}$

4. a. $\sqrt{\frac{121}{16}}$

b. $\sqrt[3]{\frac{16}{250}}$

c. $\sqrt[4]{\frac{32}{162}}$

5. a. $\sqrt{\frac{x^{10}}{x^6}}$

b. $\sqrt[3]{\frac{p^{11}}{p^2}}$

c. $\sqrt[4]{\frac{q^{17}}{q^{13}}}$

6. a. $\sqrt{\frac{p^{20}}{p^{10}}}$

b. $\sqrt[5]{\frac{d^{12}}{d^7}}$

c. $\sqrt[8]{\frac{m^{12}}{m^4}}$

7. a. $\sqrt{\frac{y^4}{y^8}}$

b. $\sqrt[5]{\frac{u^{21}}{u^{11}}}$

c. $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

8. a. $\sqrt{\frac{q^8}{q^{14}}}$

b. $\sqrt[3]{\frac{r^{14}}{r^5}}$

c. $\sqrt[4]{\frac{c^{21}}{c^9}}$

9. $\sqrt{\frac{96x^7}{121}}$

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10. $\sqrt{\frac{108y^4}{49}}$
11. $\sqrt{\frac{300m^5}{64}}$
12. $\sqrt{\frac{125n^7}{169}}$
13. $\sqrt{\frac{98r^5}{100}}$
14. $\sqrt{\frac{180s^{10}}{144}}$
15. $\sqrt{\frac{28q^6}{225}}$
16. $\sqrt{\frac{150r^3}{256}}$
17. a. $\sqrt{\frac{75r^9}{s^8}}$
    b. $\sqrt[3]{\frac{54a^8}{b^3}}$
    c. $\sqrt[4]{\frac{64c^5}{d^4}}$
18. a. $\sqrt{\frac{72x^5}{y^6}}$
    b. $\sqrt[5]{\frac{96r^{11}}{s^5}}$
    c. $\sqrt[6]{\frac{128u^7}{v^{12}}}$
19. a. $\sqrt{\frac{28p^7}{q^2}}$
    b. $\sqrt[3]{\frac{81s^8}{t^3}}$
    c. $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$
20. a. $\sqrt{\frac{45r^3}{s^{10}}}$
    b. $\sqrt[3]{\frac{625u^{10}}{v^3}}$
    c. $\sqrt[4]{\frac{729c^{21}}{d^8}}$
21. a. $\sqrt{\frac{32x^5y^3}{18x^3y}}$
    b. $\sqrt[3]{\frac{5x^6y^9}{40x^5y^3}}$
    c. $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$
22. a. $\sqrt{\frac{75r^6s^8}{48rs^4}}$
    b. $\sqrt[3]{\frac{24x^8y^4}{81x^2y}}$

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c. $\sqrt[4]{\frac{32m^9n^2}{162mn^2}}$

23. a. $\sqrt{\frac{27p^2q}{108p^4q^3}}$

b. $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$

c. $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

24. a. $\sqrt{\frac{50r^5s^2}{128r^2s^6}}$

b. $\sqrt[3]{\frac{24m^9n^7}{375m^4n}}$

c. $\sqrt[4]{\frac{81m^2n^8}{256m^1n^2}}$

25. a. $\frac{\sqrt{45p^9}}{\sqrt{5q^2}}$

b. $\frac{\sqrt[4]{64}}{\sqrt[4]{2}}$

c. $\frac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

26. a. $\frac{\sqrt{80q^5}}{\sqrt{5q}}$

b. $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

c. $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

27. a. $\frac{\sqrt{50m^7}}{\sqrt{2m}}$

b. $\sqrt[3]{\frac{1250}{2}}$

c. $\sqrt[4]{\frac{486y^9}{2y^3}}$

28. a. $\frac{\sqrt{72n^{11}}}{\sqrt{2n}}$

b. $\sqrt[3]{\frac{162}{6}}$

c. $\sqrt[4]{\frac{160r^{10}}{5r^3}}$

Answer

1.

a. $\frac{3}{4}$

b. $\frac{2}{3}$

c. $\frac{1}{3}$

3.

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a. $\frac{5}{3}$
b. $\frac{3}{5}$
c. $\frac{1}{4}$

5.

a. x^2
b. p^3
c. $|q|$

7.

a. $\frac{1}{y^2}$
b. u^2
c. $|v^3|$

9. $\frac{4\left|x^3\right|\sqrt{6x}}{11}$

11. $\frac{10m^2\sqrt{3m}}{8}$

13. $\frac{7r^2\sqrt{2r}}{10}$

15. $\frac{2\left|q^3\right|\sqrt{7}}{15}$

17.

a. $\frac{5r^4\sqrt{3r}}{s^4}$
b. $\frac{3a^2\sqrt[3]{2a^2}}{|b|}$
c. $\frac{2|c|\sqrt[4]{4c}}{|d|}$

19.

a. $\frac{2\left|p^3\right|\sqrt{7p}}{|q|}$
b. $\frac{3s^2\sqrt[3]{3s^2}}{t}$
c. $\frac{2\left|p^3\right|\sqrt[4]{4p^3}}{|q^3|}$

21.

a. $\frac{4|xy|}{3}$
b. $\frac{y^2\sqrt[3]{x}}{2}$
c. $\frac{|ab|\sqrt[4]{a}}{4}$

23.

a. $\frac{1}{2|pq|}$

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b. $\frac{2cd\sqrt[5]{2d^2}}{5}$

c. $\frac{|mn|\sqrt[6]{2}}{2}$

25.

a. $\frac{3p^4\sqrt{p}}{|q|}$

b. $2\sqrt[4]{2}$

c. $2x\sqrt[5]{2x}$

27.

a. $5\left|m^3\right|$

b. $5\sqrt[3]{5}$

c. $3|y|\sqrt[4]{3y^2}$

Exercise SET D: writing exercises

1. Explain why $\sqrt{x^4}=x^2$. Then explain why $\sqrt{x^{16}}=x^8$.
2. Explain why $7+\sqrt{9}$ is not equal to $\sqrt{7+9}$.
3. Explain how you know that $\sqrt[5]{x^{10}}=x^2$.
4. Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

Answer

1. Answers may vary

3. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify radical expressions.			
use the Quotient Property to simplify radical expressions.			

Figure 8.2.1

b. After reviewing this checklist, what will you do to become confident for all objectives?

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17.4: Simplify Rational Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{15}+\frac{5}{12}$.
   If you missed this problem, review Example 1.28.
2. Simplify: $(4x^2y^5)^3$.
   If you missed this problem, review Example 5.18.
3. Simplify: 5^{-3}.
   If you missed this problem, review Example 5.14.

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n=a^{m\cdot n}$ when m and n are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number p such that $(8^p)^3=8$. We will use the Power Property of Exponents to find the value of p.

$$(8^p)^3=8$$

Multiple the exponents on the left.

$$8^{3p}=8$$

Write the exponent 1 on the right.

$$8^{3p}=8^1$$

Since the bases are the same, the exponents must be equal.

$$3p=1$$

Solve for p.

$$p=\frac{1}{3}$$

So $\left(8^{\frac{1}{3}}\right)^3=8$. But we know also $(\sqrt[3]{8})^3=8$. Then it must be that $8^{\frac{1}{3}}=\sqrt[3]{8}$.

This same logic can be used for any positive integer exponent n to show that $a^{\frac{1}{n}}=\sqrt[n]{a}$.

Definition 17.4.1: Rational Exponent $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number and $n\geq 2$, then

$$a^{\frac{1}{n}}=\sqrt[n]{a}$$

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

Example 17.4.1

Write as a radical expression:

a. $x^{\frac{1}{2}}$
b. $y^{\frac{1}{3}}$
c. $z^{\frac{1}{4}}$

Solution:

We want to write each expression in the form $\sqrt[n]{a}$.

a.

$$x^{\frac{1}{2}}$$

The denominator of the rational exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.

$$\sqrt{x}$$

b.

$$y^{\frac{1}{3}}$$

The denominator of the exponent is 3, so the index is 3.

$$\sqrt[3]{y}$$

c.

$$z^{\frac{1}{4}}$$

The denominator of the exponent is \4, so the index is 4.

$$\sqrt[4]{z}$$

Exercise 17.4.1

Write as a radical expression:

a. $t^{\frac{1}{2}}$
b. $m^{\frac{1}{3}}$
c. $r^{\frac{1}{4}}$

Answer

a. $\sqrt{t}$
b. $\sqrt[3]{m}$
c. $\sqrt[4]{r}$

Exercise 17.4.2

Write as a radical expression:

a. $b^{\frac{1}{6}}$
b. $z^{\frac{1}{5}}$
c. $p^{\frac{1}{4}}$

Answer

a. $\sqrt[6]{b}$
b. $\sqrt[5]{z}$
c. $\sqrt[4]{p}$

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

Example 17.4.2

Write with a rational exponent:

a. $\sqrt{5y}$
b. $\sqrt[3]{4x}$
c. $3\sqrt[4]{5z}$

Solution:

We want to write each radical in the form $a^{\frac{1}{n}}$

a.

$$\sqrt{5y}$$

No index is shown, so it is 2.

The denominator of the exponent will be 2.

Put parentheses around the entire expression $5y$.

$$(5y)^{\frac{1}{2}}$$

b.

$$\sqrt[3]{4x}$$

The index is 3, so the denominator of the exponent is 3. Include parentheses $(4x)$.

$$(4x)^{\frac{1}{3}}$$

c.

$$3\sqrt[4]{5z}$$

The index is 4, so the denominator of the exponent is 4. Put parentheses only around the $5z$ since 3 is not under the radical sign.

$$3(5z)^{\frac{1}{4}}$$

Exercise 17.4.3

Write with a rational exponent:

a. $\sqrt{10m}$
b. $\sqrt[5]{3n}$
c. $3\sqrt[4]{6y}$

Answer

a. $(10m)^{\frac{1}{2}}$
b. $(3n)^{\frac{1}{5}}$
c. $3(6y)^{\frac{1}{4}}$

Exercise 17.4.4

Write with a rational exponent:

a. $\sqrt[7]{3k}$
b. $\sqrt[4]{5j}$
c. $8\sqrt[3]{2a}$

Answer

a. $(3k)^{\frac{1}{7}}$
b. $(5j)^{\frac{1}{4}}$
c. $8(2a)^{\frac{1}{3}}$

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

Example 17.4.3

Simplify:

a. $25^{\frac{1}{2}}$
b. $64^{\frac{1}{3}}$
c. $256^{\frac{1}{4}}$

Solution:

a.

$$25^{\frac{1}{2}}$$

Rewrite as a square root.

$$\sqrt{25}$$

Simplify.

$$5$$

b.

$$64^{\frac{1}{3}}$$

Rewrite as a cube root.

$$\sqrt[3]{64}$$

Recognize 64 is a perfect cube.

$$\sqrt[3]{4^3}$$

Simplify.

$$4$$

c.

$$256^{\frac{1}{4}}$$

Rewrite as a fourth root.

$$\sqrt[4]{256}$$

Recognize 256 is a perfect fourth power.

$$\sqrt[4]{4^4}$$

Simplify.

4

Exercise 17.4.5

Simplify:

a. $36^{\frac{1}{2}}$
b. $8^{\frac{1}{3}}$
c. $16^{\frac{1}{4}}$

Answer

a. 6
b. 2
c. 2

Exercise 17.4.6

Simplify:

a. $100^{\frac{1}{2}}$
b. $27^{\frac{1}{3}}$
c. $81^{\frac{1}{4}}$

Answer

a. 10
b. 3
c. 3

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n} = \frac{1}{a^n}$ in one case.

Example 17.4.4

Simplify:

a. $(-16)^{\frac{1}{4}}$
b. $-16^{\frac{1}{4}}$
c. $(16)^{-\frac{1}{4}}$

Solution:

a.

$$(-16)^{\frac{1}{4}}$$

Rewrite as a fourth root.

$$\sqrt[4]{-16}$$

$$\sqrt[4]{(-2)^4}$$

Simplify.

No real solution

b.

$$-16^{\frac{1}{4}}$$

The exponent only applies to the 16. Rewrite as a fourth root.

$$-\sqrt[4]{16}$$

Rewrite 16 as 2^4

$$-\sqrt[4]{2^4}$$

Simplify.

$$-2$$

c.

$$(16)^{-\frac{1}{4}}$$

Rewrite using the property $a^{-n}=\frac{1}{a^n}$.

$$\frac{1}{(16)^{\frac{1}{4}}}$$

Rewrite as a fourth root.

$$\frac{1}{\sqrt[4]{16}}$$

Rewrite 16 as 2^4.

$$\frac{1}{\sqrt[4]{2^4}}$$

Simplify.

$$\frac{1}{2}$$

Exercise 17.4.7

Simplify:

a. $(-64)^{-\frac{1}{2}}$
b. $-64^{\frac{1}{2}}$
c. $(64)^{-\frac{1}{2}}$

Answer

a. No real solution
b. -8
c. $\frac{1}{8}$

Exercise 17.4.8

Simplify:

a. $(-256)^{\frac{1}{4}}$
b. $-256^{\frac{1}{4}}$
c. $(256)^{-\frac{1}{4}}$

Answer

a. No real solution
b. -4
c. $\frac{1}{4}$

Simplify Expressions with $a^{\frac{m}{n}}$

We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property tells us to multiply the exponents and so $\left(a^{\frac{1}{n}}\right)^m$ and $(a^m)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

This leads us to the following defintion.

Definition 17.4.2: Rational Exponent $a^{\frac{m}{n}}$

For any positive integers m and n,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

Example 17.4.5

Write with a rational exponent:

a. $\sqrt{y^3}$
b. $(\sqrt[3]{2x})^4$
c. $\sqrt{\left(\frac{3a}{4b}\right)^3}$

Solution:

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$

a.

Let

$$\sqrt[2]{y^3} \tag{17.4.1}$$

The numerator of the exponent is the exponent 3.
The denominator of the exponent is the index of the radical, 2.

$$y^{\frac{3}{2}} \tag{17.4.2}$$

b.

Let

$$(\sqrt[3]{2x})^4 \tag{17.4.3}$$

The numerator of the exponent is the exponent 4.
The denominator of the exponent is the index of the radical, 3.

$$(2x)^{\frac{4}{3}} \tag{17.4.4}$$

c.

Let

$$\sqrt{\left(\frac{3a}{4b}\right)^3} \tag{17.4.5}$$

The numerator of the exponent is the exponent 3.
The denominator of the exponent is the index of the radical, 2.

$$\left(\frac{3a}{4b}\right)^{\frac{3}{2}} \tag{17.4.6}$$

Exercise 17.4.9

Write with a rational exponent:

a. $\sqrt{x^5}$
b. $(\sqrt[4]{3y})^3$
c. $\sqrt{\left(\frac{2m}{3n}\right)^5}$

Answer

a. $x^{\frac{5}{2}}$
b. $(3y)^{\frac{3}{4}}$
c. $\left(\frac{2m}{3n}\right)^{\frac{5}{2}}$

Exercise 17.4.10

Write with a rational exponent:

a. $\sqrt[5]{a^2}$
b. $(\sqrt[3]{5ab})^5$
c. $\sqrt{\left(\frac{7xy}{z}\right)^3}$

Answer

a. $a^{\frac{2}{5}}$
b. $(5ab)^{\frac{5}{3}}$
c. $\left(\frac{7xy}{z}\right)^{\frac{3}{2}}$

Remember that $a^{-n} = \frac{1}{a^n}$. The negative sign in the exponent does not change the sign of the expression.

Example 17.4.6

Simplify:

a. $125^{\frac{2}{3}}$
b. $16^{-\frac{3}{2}}$
c. $32^{-\frac{2}{5}}$

Solution:

We will rewrite the expression as a radical first using the defintion, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

a.

$$125^{\frac{2}{3}}$$

The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.

$$(\sqrt[3]{125})^2$$

Simplify.

$$(5)^2$$

$$25$$

b. We will rewrite each expression first using $a^{-n} = \frac{1}{a^n}$ and then change to radical form.

$$16^{-\frac{3}{2}}$$

Rewrite using $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{16^{\frac{3}{2}}}$$

Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.

$$\frac{1}{(\sqrt{16})^3}$$

Simplify.

$$\frac{1}{4^3}$$

$$\frac{1}{64}$$

c.

$$32^{-\frac{2}{5}}$$

Rewrite using $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{32^{\frac{2}{5}}}$$

Change to radical form.

$$\frac{1}{(\sqrt[5]{32})^2}$$

Rewrite the radicand as a power.

$$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$$

Simplify.

$$\frac{1}{2^2}$$

$$\frac{1}{4}$$

? Exercise 17.4.11

Simplify:

a. $27^{\frac{2}{3}}$
b. $81^{-\frac{3}{2}}$
c. $16^{-\frac{3}{4}}$

Answer

a. 9
b. $\frac{1}{729}$
c. $\frac{1}{8}$

Exercise 17.4.12

Simplify:

a. $4^{\frac{3}{2}}$

b. $27^{-\frac{2}{3}}$

c. $625^{-\frac{3}{4}}$

Answer

a. 8

b. $\frac{1}{9}$

c. $\frac{1}{125}$

Example 17.4.7

Simplify:

a. $-25^{\frac{3}{2}}$

b. $-25^{-\frac{3}{2}}$

c. $(-25)^{\frac{3}{2}}$

Solution:

a.

$$-25^{\frac{3}{2}}$$

Rewrite in radical form.

$$-(\sqrt{25})^3$$

Simplify the radical.

$$-(5)^3$$

Simplify.

$$-125$$

b.

$$-25^{-\frac{3}{2}}$$

Rewrite using $a^{-n} = \frac{1}{a^n}$.

$$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$$

Rewrite in radical form.

$$-\left(\frac{1}{(\sqrt{25})^3}\right)$$

Simplify the radical.

$$-\left(\frac{1}{(5)^3}\right)$$

Simplify.

$$-\frac{1}{125}$$

c.

$$(-25)^{\frac{3}{2}}$$

Rewrite in radical form.

$$(\sqrt{-25})^3$$

There is no real number whose square root is -25.

Not a real number.

Exercise 17.4.13

Simplify:

a. $-16^{\frac{3}{2}}$
b. $-16^{-\frac{3}{2}}$
c. $(-16)^{-\frac{3}{2}}$

Answer

a. -64
b. $-\frac{1}{64}$
c. Not a real number

Exercise 17.4.14

Simplify:

a. $-81^{\frac{3}{2}}$
b. $-81^{-\frac{3}{2}}$
c. $(-81)^{-\frac{3}{2}}$

Answer

a. -729
b. $-\frac{1}{729}$
c. Not a real number

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponents here to have them for reference as we simplify expressions.

Properties of Exponents

If a and b are real numbers and m and n are rational numbers, then

Product Property

$$a^m \cdot a^n = a^{m+n}$$

Power Property

$$(a^m)^n = a^{m \cdot n}$$

Product to a Power

$$(ab)^m = a^m b^m$$

Quotient Property

$$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$$

Zero Exponent Definition

$$a^0 = 1, a \neq 0$$

Quotient to a Power Property

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

Negative Exponent Property

$$a^{-n} = \frac{1}{a^n}, a \neq 0$$

We will apply these properties in the next example.

Example 17.4.8

Simplify:

a. $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$

b. $\left(z^9\right)^{\frac{2}{3}}$

c. $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$

Solution

a. The Product Property tells us that when we multiple the same base, we add the exponents.

$$x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$$

The bases are the same, so we add the exponents.

$$x^{\frac{1}{2}+\frac{5}{6}}$$

Add the fractions.

$$x^{\frac{8}{6}}$$

Simplify the exponent.

$$x^{\frac{4}{3}}$$

b. The Power Property tells us that when we raise a power to a power, we multiple the exponents.

$$\left(z^9\right)^{\frac{2}{3}}$$

To raise a power to a power, we multiple the exponents.

$$z^{9 \cdot \frac{2}{3}}$$

Simplify.

$$z^6$$

c. The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

$$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$$

To divide with the same base, we subtract the exponents.

$$\frac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$$

Simplify.

$$\frac{1}{x^{\frac{4}{3}}}$$

Exercise 17.4.15

Simplify:

a. $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$

b. $\left(x^{6}\right)^{\frac{4}{3}}$

c. $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$

Answer

a. $x^{\frac{3}{2}}$

b. x^{8}

c. $\frac{1}{x}$

Exercise 17.4.16

Simplify:

a. $y^{\frac{3}{4}} \cdot y^{\frac{5}{8}}$

b. $\left(m^{9}\right)^{\frac{2}{9}}$

c. $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$

Answer

a. $y^{\frac{11}{8}}$

b. m^{2}

c. $\frac{1}{d}$

Sometimes we need to use more than one property. In the next example, we will use both the **Product to a Power Property** and then the **Power Property**.

Example 17.4.9

Simplify:

a. $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$

b. $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$

Solution:

a.

$$\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

First we use the Product to a Power Property.

$$(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

Rewrite 27 as a power of 3.

$$\left(3^{3}\right)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

To raise a power to a power, we multiple the exponents.

$$\left(3^2\right)\left(u^{\frac{1}{3}}\right)$$

Simplify.

$$9u^{\frac{1}{3}}$$

b.

$$\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$$

First we use the Product to a Power Property.

$$\left(m^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$$

To raise a power to a power, we multiply the exponents.

$$mn^{\frac{3}{4}}$$

? Exercise 17.4.17

Simplify:

a. $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$

b. $\left(x^{\frac{3}{4}}y^{\frac{1}{2}}\right)^{\frac{2}{3}}$

Answer

a. $8x^{\frac{1}{5}}$

b. $x^{\frac{1}{2}}y^{\frac{1}{3}}$

? Exercise 17.4.18

Simplify:

a. $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$

b. $\left(a^{\frac{3}{2}}b^{\frac{1}{2}}\right)^{\frac{4}{3}}$

Answer

a. $729n^{\frac{3}{5}}$

b. $a^2b^{\frac{2}{3}}$

We will use both the **Product Property** and the **Quotient Property** in the next example.

✓ Example 17.4.10

Simplify:

a. $\frac{x^{\frac{3}{4}}\cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$

b. $\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$

Solution:

a.

$$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$$

Use the Product Property in the numerator, add the exponents.

$$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$$

Use the Quotient Property, subtract the exponents.

$$x^{\frac{8}{4}}$$

Simplify.

$$x^2$$

b.

$$\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$$

Use the Quotient Property, subtract the exponents.

$$\left(\frac{16x^{\frac{6}{3}}}{y^{\frac{6}{6}}}\right)^{\frac{1}{2}}$$

Simplify.

$$\left(\frac{16x^2}{y}\right)^{\frac{1}{2}}$$

Use the Product to a Power Property, multiply the exponents.

$$\frac{4x}{y^{\frac{1}{2}}}$$

Exercise 17.4.19

Simplify:

a. $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$

b. $\left(\frac{25m^{\frac{1}{6}}n^{\frac{11}{6}}}{m^{\frac{2}{3}}n^{-\frac{1}{6}}}\right)^{\frac{1}{2}}$

Answer

a. m^2

b. $\frac{5n}{m^{\frac{1}{4}}}$

Exercise 17.4.20

Simplify:

a. $\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$

b. $\left(\frac{27x^{\frac{4}{5}}y^{\frac{1}{6}}}{x^{\frac{1}{5}}y^{-\frac{5}{6}}}\right)^{\frac{1}{3}}$

Answer

a. u^3

b. $3x^{\frac{1}{5}}y^{\frac{1}{3}}$

Access these online resources for additional instruction and practice with simplifying rational exponents.

- Review-Rational Exponents
- Using Laws of Exponents on Radicals: Properties of Rational Exponents

Key Concepts

- **Rational Exponent $a^{\frac{1}{n}}$**
  - If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- **Rational Exponent $a^{\frac{m}{n}}$**
  - For any positive integers m and n, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
- **Properties of Exponents**
  - If a, b are real numbers and m, n are rational numbers, then
    - **Product Property** $a^m \cdot a^n = a^{m+n}$
    - **Power Property** $(a^m)^n = a^{m \cdot n}$
    - **Product to a Power** $(ab)^m = a^m b^m$
    - **Quotient Property** $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
    - **Zero Exponent Definition** $a^0 = 1, a \neq 0$
    - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
    - **Negative Exponent Property** $a^{-n} = \frac{1}{a^n}, a \neq 0$

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17.4E: Exercises

Practice Makes Perfect

Exercise SET A: simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

1. a. $x^{\frac{1}{2}}$ b. $y^{\frac{1}{3}}$ c. $z^{\frac{1}{4}}$
2. a. $r^{\frac{1}{2}}$ b. $s^{\frac{1}{3}}$ c. $t^{\frac{1}{4}}$
3. a. $u^{\frac{1}{5}}$ b. $v^{\frac{1}{9}}$ c. $w^{\frac{1}{20}}$
4. a. $g^{\frac{1}{7}}$ b. $h^{\frac{1}{5}}$ c. $j^{\frac{1}{25}}$

Answer

1. a. $\sqrt{x}$ b. $\sqrt[3]{y}$ c. $\sqrt[4]{z}$
3. a. $\sqrt[5]{u}$ b. $\sqrt[9]{v}$ c. $\sqrt[20]{w}$

Exercise Set B: simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt[7]{x}$ b. $\sqrt[9]{y}$ c. $\sqrt[5]{f}$
2. a. $\sqrt[8]{4}$ b. $\sqrt[10]{s}$ c. $\sqrt[4]{t}$
3. a. $\sqrt[3]{7c}$ b. $\sqrt[7]{12d}$ c. $2\sqrt[4]{6b}$
4. a. $\sqrt[4]{5x}$ b. $\sqrt[8]{9y}$ c. $7\sqrt[5]{3z}$
5. a. $\sqrt{21p}$ b. $\sqrt[4]{8q}$ c. $4\sqrt[6]{36r}$
6. a. $\sqrt[3]{25a}$ b. $\sqrt{3b}$ c. $\sqrt[8]{40c}$

Answer

1. a. $x^{\frac{1}{7}}$ b. $y^{\frac{1}{9}}$ c. $f^{\frac{1}{5}}$
3. a. $(7c)^{\frac{1}{4}}$ b. $(12d)^{\frac{1}{7}}$ c. $2(6b)^{\frac{1}{4}}$
5. a. $(21p)^{\frac{1}{2}}$ b. $(8q)^{\frac{1}{4}}$ c. $4(36r)^{\frac{1}{6}}$

Exercise SET C: simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, simplify.

1. a. $81^{\frac{1}{2}}$ b. $125^{\frac{1}{3}}$ c. $64^{\frac{1}{2}}$
2. a. $625^{\frac{1}{4}}$ b. $243^{\frac{1}{5}}$ c. $32^{\frac{1}{5}}$
3. a. $16^{\frac{1}{4}}$ b. $16^{\frac{1}{2}}$ c. $625^{\frac{1}{4}}$
4. a. $64^{\frac{1}{3}}$ b. $32^{\frac{1}{5}}$ c. $81^{\frac{1}{4}}$
5. a. $(-216)^{\frac{1}{3}}$ b. $-216^{\frac{1}{3}}$ c. $(216)^{-\frac{1}{3}}$
6. a. $(-1000)^{\frac{1}{3}}$ b. $-1000^{\frac{1}{3}}$ c. $(1000)^{-\frac{1}{3}}$
7. a. $(-81)^{\frac{1}{4}}$ b. $-81^{\frac{1}{4}}$ c. $(81)^{-\frac{1}{4}}$
8. a. $(-49)^{\frac{1}{2}}$ b. $-49^{\frac{1}{2}}$ c. $(49)^{-\frac{1}{2}}$
9. a. $(-36)^{\frac{1}{2}}$ b. $-36^{\frac{1}{2}}$ c. $(36)^{-\frac{1}{2}}$
10. a. $(-16)^{\frac{1}{4}}$ b. $-16^{\frac{1}{4}}$ c. $16^{-\frac{1}{4}}$
11. a. $(-100)^{\frac{1}{2}}$ b. $-100^{\frac{1}{2}}$ c. $(100)^{-\frac{1}{2}}$

12. a. $(-32)^{\frac{1}{5}}$ b. $(243)^{-\frac{1}{5}}$ c. $-125^{\frac{1}{3}}$

Answer

1. a. 9 b. 5 c. 8
3. a. 2 b. 4 c. 5
5. a. -6 b. -6 c. $\frac{1}{6}$
7. a. not real b. -3 c. $\frac{1}{3}$
9. a. not real b. -6 c. $\frac{1}{6}$
11. a. not real b. -10 c. $\frac{1}{10}$

Exercise SET D: simplify expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt{m^5}$ b. $(\sqrt[3]{3y})^7$ c. $\sqrt[5]{\left(\frac{4x}{5y}\right)^3}$
2. a. $\sqrt[4]{r^7}$ b. $(\sqrt[5]{2pq})^3$ c. $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$
3. a. $\sqrt[5]{u^2}$ b. $(\sqrt[3]{6x})^5$ c. $\sqrt[4]{\left(\frac{18a}{5b}\right)^7}$
4. a. $\sqrt[3]{a}$ b. $(\sqrt[4]{21v})^3$ c. $\sqrt[4]{\left(\frac{2xy}{5z}\right)^2}$

Answer

1. a. $m^{\frac{5}{2}}$ b. $(3y)^{\frac{7}{3}}$ c. $\left(\frac{4x}{5y}\right)^{\frac{3}{5}}$
3. a. $u^{\frac{2}{5}}$ b. $(6x)^{\frac{5}{3}}$ c. $\left(\frac{18a}{5b}\right)^{\frac{7}{4}}$

Exercise SET E: simplify expressions with $a^{\frac{m}{n}}$

In the following exercises, simplify.

1. a. $64^{\frac{5}{2}}$ b. $81^{\frac{-3}{2}}$ c. $(-27)^{\frac{2}{3}}$
2. a. $25^{\frac{3}{2}}$ b. $9^{-\frac{3}{2}}$ c. $(-64)^{\frac{2}{3}}$
3. a. $32^{\frac{2}{5}}$ b. $27^{-\frac{2}{3}}$ c. $(-25)^{\frac{1}{2}}$
4. a. $100^{\frac{3}{2}}$ b. $49^{-\frac{5}{2}}$ c. $(-100)^{\frac{3}{2}}$
5. a. $-9^{\frac{3}{2}}$ b. $-9^{-\frac{3}{2}}$ c. $(-9)^{\frac{3}{2}}$
6. a. $-64^{\frac{3}{2}}$ b. $-64^{-\frac{3}{2}}$ c. $(-64)^{\frac{3}{2}}$

Answer

1. a. $32,768$ b. $\frac{1}{729}$ c. 9
3. a. 4 b. $\frac{1}{9}$ c. not real
5. a. -27 b. $-\frac{1}{27}$ c. not real

Exercise SET F: use the laws of exponents to simplify expressions with rational exponents

In the following exercises, simplify. Assume all variables are positive.

1. a. $c^{\frac{1}{4}} \cdot c^{\frac{5}{8}}$ b. $\left(p^{12}\right)^{\frac{3}{4}}$ c. $\dfrac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$
2. a. $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$ b. $\left(b^{15}\right)^{\frac{3}{5}}$ c. $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$
3. a. $y^{\frac{1}{2}} \cdot y^{\frac{3}{4}}$ b. $\left(x^{12}\right)^{\frac{2}{3}}$ c. $\dfrac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$
4. a. $q^{\frac{2}{3}} \cdot q^{\frac{5}{6}}$ b. $\left(h^{6}\right)^{\frac{4}{3}}$ c. $\dfrac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$
5. a. $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$ b. $\left(a^{\frac{1}{3}}b^{\frac{2}{3}}\right)^{\frac{3}{2}}$
6. a. $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$ b. $\left(m^{\frac{4}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{4}}$
7. a. $\left(16u^{\frac{1}{3}}\right)^{\frac{3}{4}}$ b. $\left(4p^{\frac{1}{3}}q^{\frac{1}{2}}\right)^{\frac{3}{2}}$
8. a. $\left(625n^{\frac{8}{3}}\right)^{\frac{3}{4}}$ b. $\left(9x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}}$
9. a. $\dfrac{r^{\frac{5}{2}} \cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$ b. $\left(\dfrac{36s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$
10. a. $\dfrac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$ b. $\left(\dfrac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$
11. a. $\dfrac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$ b. $\left(\dfrac{8x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$
12. a. $\dfrac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$ b. $\left(\dfrac{16m^{\frac{1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{4}}$

Answer

1. a. $c^{\frac{7}{8}}$ b. p^9 c. $\frac{1}{r}$
3. a. $y^{\frac{5}{4}}$ b. x^8 c. $\dfrac{1}{m}$
5. a. $81q^2$ b. $a^{\frac{1}{2}}b$
7. a. $8u^{\frac{1}{4}}$ b. $8p^{\frac{1}{2}}q^{\frac{3}{4}}$
9. a. $r^{\frac{7}{2}}$ b. $\dfrac{6s}{t}$
11. a. c^2 b. $\dfrac{2x}{3y}$

Exercise SET G: writing exercises

1. Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.
2. Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

Answer

1. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with $a^{\frac{1}{n}}$.			
simplify expressions with $a^{\frac{m}{n}}$.			
use the Laws of Exponents to simply expressions with rational exponents.			

Figure 8.3.4

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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17.5: Add, Subtract, and Multiply Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

Before you get started, take this readiness quiz.

1. Add: $3x^2+9x-5-(x^2-2x+3)$.
   If you missed this problem, review Example 5.5.
2. Simplify: $(2+a)(4-a)$.
   If you missed this problem, review Example 5.28.
3. Simplify: $(9-5y)^2$.
   If you missed this problem, review Example 5.31.

Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

Definition 17.5.1: Like Radicals

Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that $3x+8x$ is $11x$.Similarly we add $3\sqrt{x}+8\sqrt{x}$ and the result is $11\sqrt{x}$.

Think about adding like terms with variables as you do the next few examples. When you have like radicals, you just add or subtract the coefficients. When the radicals are not like, you cannot combine the terms.

Example 17.5.1

Simplify:

a. $2\sqrt{2}-7\sqrt{2}$
b. $5\sqrt[3]{y}+4\sqrt[3]{y}$
c. $7\sqrt[4]{x}-2\sqrt[4]{y}$

Solution:

a.

$$2\sqrt{2}-7\sqrt{2}$$

Since the radicals are like, we subtract the coefficients.

$$-5\sqrt{2}$$

b.

$$5\sqrt[3]{y}+4\sqrt[3]{y}$$

Since the radicals are like, we add the coefficients.

$$9\sqrt[3]{y}$$

c.

$$7\sqrt[4]{x}-2\sqrt[4]{y}$$

The indices are the same but the radicals are different. These are not like radicals. Since the radicals are not like, we cannot subtract them.

Exercise 17.5.1

Simplify:

a. $8\sqrt{2}-9\sqrt{2}$
b. $4\sqrt[3]{x}+7\sqrt[3]{x}$
c. $3\sqrt[4]{x}-5\sqrt[4]{y}$

Answer

a. $-\sqrt{2}$
b. $11\sqrt[3]{x}$
c. $3\sqrt[4]{x}-5\sqrt[4]{y}$

Exercise 17.5.2

Simplify:

a. $5\sqrt{3}-9\sqrt{3}$
b. $5\sqrt[3]{y}+3\sqrt[3]{y}$
c. $5\sqrt[4]{m}-2\sqrt[3]{m}$

Answer

a. $-4\sqrt{3}$
b. $8\sqrt[3]{y}$
c. $5\sqrt[4]{m}-2\sqrt[3]{m}$

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

Example 17.5.2

Simplify:

a. $2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$
b. $\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$

Solution:

a.

$$2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$$

Since the radicals are like, we combine them.

$$0\sqrt{5n}$$

Simplify.

$$0$$

b.

$$\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$$

Since the radicals are like, we combine them.

$$2\sqrt[4]{3xy}$$

Exercise 17.5.3

Simplify:

a. $\sqrt{7x}-7\sqrt{7x}+4\sqrt{7x}$
b. $4\sqrt[4]{5xy}+2\sqrt[4]{5xy}-7\sqrt[4]{5xy}$

Answer

a. $-2\sqrt{7x}$
b. $-\sqrt[4]{5xy}$

Exercise 17.5.4

Simplify:

a. $4\sqrt{3y}-7\sqrt{3y}+2\sqrt{3y}$
b. $6\sqrt[3]{7mn}+\sqrt[3]{7mn}-4\sqrt[3]{7mn}$

Answer

a. $-\sqrt{3y}$
b. $3\sqrt[3]{7mn}$

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

Example 17.5.3

Simplify:

a. $\sqrt{20}+3\sqrt{5}$
b. $\sqrt[3]{24}-\sqrt[3]{375}$
c. $\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$

Solution:

a.

$$\sqrt{20}+3\sqrt{5}$$

Simplify the radicals, when possible.

$$\sqrt{4}\cdot\sqrt{5}+3\sqrt{5}$$
$$2\sqrt{5}+3\sqrt{5}$$

Combine the like radicals.

$$5\sqrt{5}$$

b.

$$\sqrt[3]{24}-\sqrt[3]{375}$$

Simplify the radicals.

$$\sqrt[3]{8}\cdot\sqrt[3]{3}-\sqrt[3]{125}\cdot\sqrt[3]{3}$$
$$2\sqrt[3]{3}-5\sqrt[3]{3}$$

Combine the like radicals.

$$-3\sqrt[3]{3}$$

c.

$$\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$$

Simplify the radicals.

$$\frac{1}{2}\sqrt[4]{16}\cdot\sqrt[4]{3}-\frac{2}{3}\sqrt[4]{81}\cdot\sqrt[4]{3}$$
$$\frac{1}{2}\cdot 2\cdot\sqrt[4]{3}-\frac{2}{3}\cdot 3\cdot\sqrt[4]{3}$$
$$\sqrt[4]{3}-2\sqrt[4]{3}$$

Combine the like radicals.

$$-\sqrt[4]{3}$$

Exercise 17.5.5

Simplify:

a. $\sqrt{18}+6\sqrt{2}$
b. $6\sqrt[3]{16}-2\sqrt[3]{250}$
c. $\frac{2}{3}\sqrt[3]{81}-\frac{1}{2}\sqrt[3]{24}$

Answer

a. $9\sqrt{2}$
b. $2\sqrt[3]{2}$
c. $\sqrt[3]{3}$

Exercise 17.5.6

Simplify:

a. $\sqrt{27}+4\sqrt{3}$
b. $4\sqrt[3]{5}-7\sqrt[3]{40}$
c. $\frac{1}{2}\sqrt[3]{128}-\frac{5}{3}\sqrt[3]{54}$

Answer

a. $7\sqrt{3}$
b. $-10\sqrt[3]{5}$
c. $-3\sqrt[3]{2}$

In the next example, we will remove both constant and variable factors from the radicals. Now that we have practiced taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption thoughout the rest of this chapter.

Example 17.5.4

Simplify:

a. $9\sqrt{50m^2}-6\sqrt{48m^2}$
b. $\sqrt[3]{54n^5}-\sqrt[3]{16n^5}$

Solution:

a.

$$9\sqrt{50m^2}-6\sqrt{48m^2}$$

Simplify the radicals.

$$9\sqrt{25m^2}\cdot\sqrt{2}-6\sqrt{16m^2}\cdot\sqrt{3}$$

$$9 \cdot 5m \cdot \sqrt{2} - 6 \cdot 4m \cdot \sqrt{3}$$

$$45m\sqrt{2} - 24m\sqrt{3}$$

The radicals are not like and so cannot be combined.

b.

$$\sqrt[3]{54n^5} - \sqrt[3]{16n^5}$$

Simplify the radicals.

$$\sqrt[3]{27n^3} \cdot \sqrt[3]{2n^2} - \sqrt[3]{8n^3} \cdot \sqrt[3]{2n^2}$$

$$3n\sqrt[3]{2n^2} - 2n\sqrt[3]{2n^2}$$

Combine the like radicals.

$$n\sqrt[3]{2n^2}$$

? Exercise 17.5.7

Simplify:

a. $\sqrt{32m^7} - \sqrt{50m^7}$
b. $\sqrt[3]{135x^7} - \sqrt[3]{40x^7}$

Answer

a. $-m^3\sqrt{2m}$
b. $x^2\sqrt[3]{5x}$

? Exercise 17.5.8

Simplify:

a. $\sqrt{27p^3} - \sqrt{48p^3}$
b. $\sqrt[3]{256y^5} - \sqrt[3]{32n^5}$

Answer

a. $-p\sqrt{3p}$
b. $4y\sqrt[3]{4y^2} - 2n\sqrt[3]{4n^2}$

Multiply Radical Expressions

We have used the **Product Property of Roots** to simplify square roots by removing the perfect square factors. We can use the Product Property of Roots 'in reverse' to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

Definition 17.5.2: Product Property of Roots

For any real numbers, $\sqrt[n]{a}$ and $\sqrt[b]{n}$, and for any integer $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

When we multiply two radicals they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables. The result is $12xy$. Keep this in mind as you do these examples.

Example 17.5.5

Simplify:

a. $(6\sqrt{2})(3\sqrt{10})$
b. $(-5\sqrt[3]{4})(-4\sqrt[3]{6})$

Solution:

a.

$$(6\sqrt{2})(3\sqrt{10})$$

Multiply using the Product Property.

$$18\sqrt{20}$$

Simplify the radical.

$$18\sqrt{4}\cdot\sqrt{5}$$

Simplify.

$$18\cdot 2\cdot\sqrt{5}$$
$$36\sqrt{5}$$

b.

$$(-5\sqrt[3]{4})(-4\sqrt[3]{6})$$

Multiply using the Product Property.

$$20\sqrt[3]{24}$$

Simplify the radical.

$$20\sqrt[3]{8}\cdot\sqrt[3]{3}$$

Simplify.

$$20\cdot 2\cdot\sqrt[3]{3}$$
$$40\sqrt[3]{3}$$

Exercise 17.5.9

Simplify:

a. $(3\sqrt{2})(2\sqrt{30})$
b. $(2\sqrt[3]{18})(-3\sqrt[3]{6})$

Answer

a. $12\sqrt{15}$
b. $-18\sqrt[3]{2}$

Exercise 17.5.10

Simplify:

a. $(3\sqrt{3})(3\sqrt{6})$
b. $(-4\sqrt[3]{9})(3\sqrt[3]{6})$

Answer

a. $27\sqrt{2}$
b. $-36\sqrt[3]{2}$

We follow the same procedures when there are variables in the radicands.

Example 17.5.6

Simplify:

a. $\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$
b. $\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$

Solution:

a.

$$\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$$

Multiply.

$$40\sqrt{18p^4}$$

Simplify the radical.

$$40\sqrt{9p^4}\cdot\sqrt{2}$$

Simplify.

$$40\cdot 3p^2\cdot\sqrt{3}$$
$$120p^2\sqrt{3}$$

b. When the radicands involve large numbers, it is often advantageous to factor them in order to find the perfect powers.

$$\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$$

Multiply.

$$6\sqrt[4]{4\cdot 5\cdot 4\cdot 7y^5}$$

Simplify the radical.

$$6\sqrt[4]{16y^4}\cdot\sqrt[4]{35y}$$

Simplify.

$$6\cdot 2y\sqrt[4]{35y}$$

Multiply.

$$12y\sqrt[4]{35y}$$

Exercise 17.5.11

Simplify:

a. $\left(6\sqrt{6x^2}\right)\left(8\sqrt{30x^4}\right)$
b. $\left(-4\sqrt[4]{12y^3}\right)\left(-\sqrt[4]{8y^3}\right)$

Answer

a. $36x^3\sqrt{5}$
b. $8y\sqrt[4]{3y^2}$

Exercise 17.5.12

Simplify:

a. $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$
b. $\left(-4\sqrt[4]{9a^3}\right)\left(3\sqrt[4]{27a^2}\right)$

Answer

a. $144y^2\sqrt{5y}$
b. $-36\sqrt[4]{3a}$

Use Polynomial Multiplication to Multiply Radical Expressions

In the next a few examples, we will use the **Distributive Property** to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

Example 17.5.7

Simplify:

a. $\sqrt{6}(\sqrt{2}+\sqrt{18})$
b. $\sqrt[3]{9}(5-\sqrt[3]{18})$

Solution:

a.

$$\sqrt{6}(\sqrt{2}+\sqrt{18})$$

Multiply.

$$\sqrt{12}+\sqrt{108}$$

Simplify.

$$\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$$

Simplify.

$$2\sqrt{3}+6\sqrt{3}$$

Combine like radicals.

$$8\sqrt{3}$$

b.

$$\sqrt[3]{9}(5-\sqrt[3]{18})$$

Distribute.

$$5\sqrt[3]{9}-\sqrt[3]{162}$$

Simplify.

$$5\sqrt[3]{9}-\sqrt[3]{27}\cdot\sqrt[3]{6}$$

Simplify.

$$5\sqrt[3]{9}-3\sqrt[3]{6}$$

Exercise 17.5.13

Simplify:

a. $\sqrt{6}(1+3\sqrt{6})$
b. $\sqrt[3]{4}(-2-\sqrt[3]{6})$

Answer

a. $18+\sqrt{6}$
b. $-2\sqrt[3]{4}-2\sqrt[3]{3}$

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Exercise 17.5.14

Simplify:

a. $\sqrt{8}(2-5\sqrt{8})$
b. $\sqrt[3]{3}(-\sqrt[3]{9}-\sqrt[3]{6})$

Answer

a. $-40+4\sqrt{2}$
b. $-3-\sqrt[3]{18}$

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

Example 17.5.8

Simplify:

a. $(3-2\sqrt{7})(4-2\sqrt{7})$
b. $(\sqrt[3]{x}-2)(\sqrt[3]{x}+4)$

Solution:

a.

$$(3-2\sqrt{7})(4-2\sqrt{7})$$

Multiply.

$$12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$$

Simplify.

$$12-6\sqrt{7}-8\sqrt{7}+28$$

Combine like terms.

$$40-14\sqrt{7}$$

b.

$$(\sqrt[3]{x}-2)(\sqrt[3]{x}+4)$$

Multiply.

$$\sqrt[3]{x^2}+4\sqrt[3]{x}-2\sqrt[3]{x}-8$$

Combine like terms.

$$\sqrt[3]{x^2}+2\sqrt[3]{x}-8$$

Exercise 17.5.15

Simplify:

a. $(6-3\sqrt{7})(3+4\sqrt{7})$
b. $(\sqrt[3]{x}-2)(\sqrt[3]{x}-3)$

Answer

a. $-66+15\sqrt{7}$
b. $\sqrt[3]{x^2}-5\sqrt[3]{x}+6$

Exercise 17.5.16

Simplify:

a. $(2-3\sqrt{11})(4-\sqrt{11})$
b. $(\sqrt[3]{x}+1)(\sqrt[3]{x}+3)$

Answer

a. $41-14\sqrt{11}$
b. $\sqrt[3]{x^2}+4\sqrt[3]{x}+3$

Example 17.5.9

Simplify: $(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$

Solution:

$$(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$$

Multiply.

$$3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$$

Simplify.

$$6+12\sqrt{10}-\sqrt{10}-20$$

Combine like terms.

$$-14+11\sqrt{10}$$

Exercise 17.5.17

Simplify: $(5\sqrt{3}-\sqrt{7})(\sqrt{3}+2\sqrt{7})$

Answer

$1+9\sqrt{21}$

Exercise 17.5.18

Simplify: $(\sqrt{6}-3\sqrt{8})(2\sqrt{6}+\sqrt{8})$

Answer

$-12-20\sqrt{3}$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

Special Products

Binomial Squares

$$(a+b)^2=a^2+2ab+b^2$$
$$(a-b)^2=a^2-2ab+b^2$$

Product of Conjugates

$$(a+b)(a-b)=a^2-b^2$$

We will use the special product formulas in the next few examples. We will start with the **Product of Binomial Squares Pattern**.

Example 17.5.10

Simplify:

a. $(2+\sqrt{3})^2$
b. $(4-2\sqrt{5})^2$

Solution:

a.

Table 8.4.1

	$(a+b)^2$ $(2+\sqrt{3})^2$
Multiply using the Product of Binomial Squares Pattern.	$a^2+2ab+b^2$ $2^2+2\cdot 2\cdot\sqrt{3}+(\sqrt{3})^2$
Simplify.	$4+4\sqrt{3}+3$
Combine like terms.	$7+4\sqrt{3}$

b.

Table 8.4.2

	$(a-b)^2$ $(4-2\sqrt{5})^2$
Multiple, using the Product of Binomial Squares Pattern.	$a^2-2ab+b^2$ $4^2-2\cdot 4\cdot 2\sqrt{5}+(2\sqrt{5})^2$
Simplify.	$16-16\sqrt{5}+4\cdot 5$
	$16-16\sqrt{5}+20$
Combine like terms.	$36-16\sqrt{5}$

Exercise 17.5.19

Simplify:

a. $(10+\sqrt{2})^2$
b. $(1+3\sqrt{6})^2$

Answer

a. $102+20\sqrt{2}$
b. $55+6\sqrt{6}$

Exercise 17.5.20

Simplify:

a. $(6-\sqrt{5})^2$
b. $(9-2\sqrt{10})^2$

Answer

a. $41-12\sqrt{5}$
b. $121-36\sqrt{10}$

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

Example 17.5.11

Simplify: $(5-2\sqrt{3})(5+2\sqrt{3})$

Solution:

Table 8.4.3

	$(a-b)(a+b)$ $(5-2\sqrt{3})(5+2\sqrt{3})$
Multiply using the Product of Conjugates Pattern.	a^2-b^2 $5^2-(2\sqrt{3})^2$
Simplify.	$25-4\cdot 3$
	13

Exercise 17.5.21

Simplify: $(3-2\sqrt{5})(3+2\sqrt{5})$

Answer

-11

Exercise 17.5.22

Simplify: $(4+5\sqrt{7})(4-5\sqrt{7})$

Answer

-159

Access these online resources for additional instruction and practice with adding, subtracting, and multiplying radical expressions.

- Multiplying Adding Subtracting Radicals
- Multiplying Special Products: Square Binomials Containing Square Roots
- Multiplying Conjugates

Key Concepts

- **Product Property of Roots**
  - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$ $\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}$ and $\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}$
- **Special Products**

Binomial Squares	Product of Conjugates
$(a+b)^2=a^2+2ab+b^2$	$(a+b)(a-b)=a^2-b^2$
$(a-b)^2=a^2-2ab+b^2$	

Glossary

like radicals

Like radicals are radical expressions with the same index and the same radicand.

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17.5E: Exercises

Practice Makes Perfect

Exercise A: add and subtract radical expressions

In the following exercises, simplify. Assume all variables are greater than or equal to zero so that absolute values are not needed.

1. a. $8\sqrt{2}-5\sqrt{2}$ b. $5\sqrt[3]{m}+2\sqrt[3]{m}$ c. $8\sqrt[4]{m}-2\sqrt[4]{n}$
2. a. $7\sqrt{2}-3\sqrt{2}$ b. $7\sqrt[3]{p}+2\sqrt[3]{p}$ c. $5\sqrt[3]{x}-3\sqrt[3]{x}$
3. a. $3\sqrt{5}+6\sqrt{5}$ b. $9\sqrt[3]{a}+3\sqrt[3]{a}$ c. $5\sqrt[4]{2z}+\sqrt[4]{2z}$
4. a. $4\sqrt{5}+8\sqrt{5}$ b. $\sqrt[3]{m}-4\sqrt[3]{m}$ c. $\sqrt{n}+3\sqrt{n}$
5. a. $3\sqrt{2a}-4\sqrt{2a}+5\sqrt{2a}$ b. $5\sqrt[4]{3ab}-3\sqrt[4]{3ab}-2\sqrt[4]{3ab}$
6. a. $\sqrt{11b}-5\sqrt{11b}+3\sqrt{11b}$ b. $8\sqrt[4]{11cd}+5\sqrt[4]{11cd}-9\sqrt[4]{11cd}$
7. a. $8\sqrt{3c}+2\sqrt{3c}-9\sqrt{3c}$ b. $2\sqrt[3]{4pq}-5\sqrt[3]{4pq}+4\sqrt[3]{4pq}$
8. a. $3\sqrt{5d}+8\sqrt{5d}-11\sqrt{5d}$ b. $11\sqrt[3]{2rs}-9\sqrt[3]{2rs}+3\sqrt[3]{2rs}$
9. a. $\sqrt{27}-\sqrt{75}$ b. $\sqrt[3]{40}-\sqrt[3]{320}$ c. $\frac{1}{2}\sqrt[4]{32}+\frac{2}{3}\sqrt[4]{162}$
10. a. $\sqrt{72}-\sqrt{98}$ b. $\sqrt[3]{24}+\sqrt[3]{81}$ c. $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$
11. a. $\sqrt{48}+\sqrt{27}$ b. $\sqrt[3]{54}+\sqrt[3]{128}$ c. $6\sqrt[4]{5}-\frac{3}{2}\sqrt[4]{320}$
12. a. $\sqrt{45}+\sqrt{80}$ b. $\sqrt[3]{81}-\sqrt[3]{192}$ c. $\frac{5}{2}\sqrt[4]{80}+\frac{7}{3}\sqrt[4]{405}$
13. a. $\sqrt{72a^5}-\sqrt{50a^5}$ b. $9\sqrt[4]{80p^4}-6\sqrt[4]{405p^4}$
14. a. $\sqrt{48b^5}-\sqrt{75b^5}$ b. $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$
15. a. $\sqrt{80c^7}-\sqrt{20c^7}$ b. $2\sqrt[4]{162r^{10}}+4\sqrt[4]{32r^{10}}$
16. a. $\sqrt{96d^9}-\sqrt{24d^9}$ b. $5\sqrt[4]{243s^6}+2\sqrt[4]{3s^6}$
17. $3\sqrt{128y^2}+4y\sqrt{162}-8\sqrt{98y^2}$
18. $3\sqrt{75y^2}+8y\sqrt{48}-\sqrt{300y^2}$

Answer

1. a. $3\sqrt{2}$ b. $7\sqrt[3]{m}$ c. $6\sqrt[4]{m}$
3. a. $9\sqrt{5}$ b. $12\sqrt[3]{a}$ c. $6\sqrt[4]{2z}$
5. a. $4\sqrt{2a}$ b. 0
7. a. $\sqrt{3c}$ b. $\sqrt[3]{4pq}$
9. a. $-2\sqrt{3}$ b. $-2\sqrt[3]{5}$ c. $3\sqrt[4]{2}$

11. a. $7\sqrt{3}$ b. $7\sqrt[3]{2}$ c. $3\sqrt[4]{5}$

13. a. $a^2\sqrt{2a}$ b. 0

15. a. $2c^3\sqrt{5c}$ b. $14r^2\sqrt[4]{2r^2}$

17. $4y\sqrt{2}$

Exercise B: multiply radical expressions

In the following exercises, simplify.

1. a. $(-2\sqrt{3})(3\sqrt{18})$

   b. $(8\sqrt[3]{4})(-4\sqrt[3]{18})$

   c.

2. a. $(-4\sqrt{5})(5\sqrt{10})$

   b. $(-2\sqrt[3]{9})(7\sqrt[3]{9})$

   c.

3. a. $(5\sqrt{6})(-\sqrt{12})$

   b. $(-2\sqrt[4]{18})(-\sqrt[4]{9})$

   c.

4. a. $(-2\sqrt{7})(-2\sqrt{14})$

   b. $(-3\sqrt[4]{8})(-5\sqrt[4]{6})$

   c.

5. a. $\left(4\sqrt{12z^3}\right)\left(3\sqrt{9z}\right)$

   b. $\left(5\sqrt[3]{3x^3}\right)\left(3\sqrt[3]{18x^3}\right)$

c.

6. a. $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$

   b. $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

   c.

7. a. $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$

   b. $\left(2\sqrt[4]{8y^2}\right)\left(-2\sqrt[4]{12y^3}\right)$

   c.

8. a. $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$

   b. $\left(-\sqrt[4]{6b^3}\right)\left(3\sqrt[4]{8b^3}\right)$

   c.

Answer

1.

   a. $-18\sqrt{6}$

   b. $-64\sqrt[3]{9}$

   c.

3.

   a. $-30\sqrt{2}$

   b. $6\sqrt[4]{2}$

c.

5.

a. $72z^2\sqrt{3}$

b. $45x^2\sqrt[3]{2}$

c.

7.

a. $-42z^5\sqrt{2z}$

b. $-8y\sqrt[4]{6y}$

Exercise C: use polynomial multiplication to multiply radical expressions

In the following exercises, multiply.

1. a. $\sqrt{7}(5+2\sqrt{7})$

   b. $\sqrt[3]{6}(4+\sqrt[3]{18})$

   c.

2. a. $\sqrt{11}(8+4\sqrt{11})$

   b. $\sqrt[3]{3}(\sqrt[3]{9}+\sqrt[3]{18})$

   c.

3. a. $\sqrt{11}(-3+4\sqrt{11})$

   b. $\sqrt[4]{3}(\sqrt[4]{54}+\sqrt[4]{18})$

   c.

4. a. $\sqrt{2}(-5+9\sqrt{2})$

   b. $\sqrt[4]{2}(\sqrt[4]{12}+\sqrt[4]{24})$

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c.

5. $(7+\sqrt{3})(9-\sqrt{3})$

6. $(8-\sqrt{2})(3+\sqrt{2})$

7. a. $(9-3\sqrt{2})(6+4\sqrt{2})$

   b. $(\sqrt[3]{x}-3)(\sqrt[3]{x}+1)$

   c.

8. a. $(3-2\sqrt{7})(5-4\sqrt{7})$

   b. $(\sqrt[3]{x}-5)(\sqrt[3]{x}-3)$

   c.

9. a. $(1+3\sqrt{10})(5-2\sqrt{10})$

   b. $(2\sqrt[3]{x}+6)(\sqrt[3]{x}+1)$

   c.

10. a. $(7-2\sqrt{5})(4+9\sqrt{5})$

    b. $(3\sqrt[3]{x}+2)(\sqrt[3]{x}-2)$

    c.

11. $(\sqrt{3}+\sqrt{10})(\sqrt{3}+2\sqrt{10})$

12. $(\sqrt{11}+\sqrt{5})(\sqrt{11}+6\sqrt{5})$

13. $(2\sqrt{7}-5\sqrt{11})(4\sqrt{7}+9\sqrt{11})$

14. $(4\sqrt{6}+7\sqrt{13})(8\sqrt{6}-3\sqrt{13})$

15. a. $(3+\sqrt{5})^2$

    b. $(2-5\sqrt{3})^2$

    c.

16. a. $(4+\sqrt{11})^2$

    b. $(3-2\sqrt{5})^2$

    c.

17. a. $(9-\sqrt{6})^2$

    b. $(10+3\sqrt{7})^2$

    c.

18. a. $(5-\sqrt{10})^2$

    b. $(8+3\sqrt{2})^2$

    c.

19. $(4+\sqrt{2})(4-\sqrt{2})$

20. $(7+\sqrt{10})(7-\sqrt{10})$

21. $(4+9\sqrt{3})(4-9\sqrt{3})$

22. $(1+8\sqrt{2})(1-8\sqrt{2})$

23. $(12-5\sqrt{5})(12+5\sqrt{5})$

24. $(9-4\sqrt{3})(9+4\sqrt{3})$

25. $(\sqrt[3]{3x}+2)(\sqrt[3]{3x}-2)$

26. $(\sqrt[3]{4x}+3)(\sqrt[3]{4x}-3)$

Answer

1.

a. $14+5\sqrt{7}$

b. $4\sqrt[3]{6}+3\sqrt[3]{4}$

c.

3.

a. $44-3\sqrt{11}$

b. $3\sqrt[4]{2}+\sqrt[4]{54}$

c.

5. $60+2\sqrt{3}$

7.

a. $30+18\sqrt{2}$

b. $\sqrt[3]{x^2}-2\sqrt[3]{x}-3$

c.

9.

a. $-54+13\sqrt{10}$

b. $2\sqrt[3]{x^2}+8\sqrt[3]{x}+6$

c.

11. $23+3\sqrt{30}$

13. $-439-2\sqrt{77}$

15.

a. $14+6\sqrt{5}$

b. $79-20\sqrt{3}$

17.

a. $87-18\sqrt{6}$

b. $163+60\sqrt{7}$

19. 14

21. -227

23. 19

25. $\sqrt[3]{9x^2}-4$

Exercise D: mixed practice

1. $\frac{2}{3}\sqrt{27}+\frac{3}{4}\sqrt{48}$
2. $\sqrt{175k^4}-\sqrt{63k^4}$
3. $\frac{5}{6}\sqrt{162}+\frac{3}{16}\sqrt{128}$
4. $\sqrt[3]{24}+\sqrt[3]{81}$
5. $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$
6. $8\sqrt[4]{13}-4\sqrt[4]{13}-3\sqrt[4]{13}$
7. $5\sqrt{12c^4}-3\sqrt{27c^6}$
8. $\sqrt{80a^5}-\sqrt{45a^5}$
9. $\frac{3}{5}\sqrt{75}-\frac{1}{4}\sqrt{48}$
10. $21\sqrt[3]{9}-2\sqrt[3]{9}$
11. $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$
12. $11\sqrt{11}-10\sqrt{11}$
13. $\sqrt{3}\cdot\sqrt{21}$
14. $(4\sqrt{6})(-\sqrt{18})$
15. $(7\sqrt[3]{4})(-3\sqrt[3]{18})$
16. $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$
17. $(\sqrt{29})^2$
18. $(-4\sqrt{17})(-3\sqrt{17})$
19. $(-4+\sqrt{17})(-3+\sqrt{17})$
20. $\left(3\sqrt[4]{8a^2}\right)\left(\sqrt[4]{12a^3}\right)$

21. $(6-3\sqrt{2})^2$

22. $\sqrt{3}(4-3\sqrt{3})$

23. $\sqrt[3]{3}(2\sqrt[3]{9}+\sqrt[3]{18})$

24. $(\sqrt{6}+\sqrt{3})(\sqrt{6}+6\sqrt{3})$

Answer

1. $5\sqrt{3}$
3. $9\sqrt{2}$
5. $-\sqrt[4]{5}$
7. $10c^2\sqrt{3}-9c^3\sqrt{3}$
9. $2\sqrt{3}$
11. $17q^2$
13. $3\sqrt{7}$
15. $-42\sqrt[3]{9}$
17. 29
19. $29-7\sqrt{17}$
21. $72-36\sqrt{2}$
23. $6+3\sqrt[3]{2}$

Exercise E: writing exercises

1. Explain when a radical expression is in simplest form.
2. Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.
3. a. Explain why $(-\sqrt{n})^2$ is always non-negative, for $n \geq 0$.
   b. Explain why $-(\sqrt{n})^2$ is always non-positive, for $n \geq 0$.
4. Use the binomial square pattern to simplify $(3+\sqrt{2})^2$. Explain all your steps.

Answer

1. Answers will vary
3. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract radical expressions.			
multiply radical expressions.			
use polynomial multiplication to multiply radical expressions.			

Figure 8.4.14

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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17.6: Divide Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

Before you get started, take this readiness quiz.

1. Simplify: $\frac{30}{48}$.
   If you missed this problem, review Example 1.24.
2. Simplify: $x^2 \cdot x^4$.
   If you missed this problem, review Example 5.12.
3. Multiply: $(7+3x)(7-3x)$.
   If you missed this problem, review Example 5.32.

Divide Radical Expressions

We have used the **Quotient Property of Radical Expressions** to simplify roots of fractions. We will need to use this property 'in reverse' to simplify a fraction with radicals. We give the Quotient Property of Radical Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars re needed.

Definition 17.6.1: Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

Example 17.6.1

Simplify:

a. $\frac{\sqrt{72x^3}}{\sqrt{162x}}$

b. $\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$

Solution:

a.

$$\frac{\sqrt{72x^3}}{\sqrt{162x}}$$

Rewrite using the quotient property,

$$\sqrt{\frac{72x^3}{162x}}$$

Remove common factors.

$$\sqrt{\frac{\cancel{18}\cdot 4\cdot x^2\cdot \cancel{x}}{\cancel{18}\cdot 9\cdot \cancel{x}}}$$

Simplify.

$$\sqrt{\frac{4x^2}{9}}$$

Simplify the radical.

$$\frac{2x}{3}$$

b.

$$\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$$

Rewrite using the quotient property, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$.

$$\sqrt[3]{\frac{32x^2}{4x^5}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{\frac{8}{x^3}}$$

Simplify the radical.

$$\frac{2}{x}$$

Exercise 17.6.1

Simplify:

a. $\frac{\sqrt{50s^3}}{\sqrt{128s}}$

b. $\frac{\sqrt[3]{56a}}{\sqrt[3]{7a^4}}$

Answer

a. $\frac{5s}{8}$

b. $\frac{2}{a}$

Exercise 17.6.2

Simplify:

a. $\frac{\sqrt{75q^5}}{\sqrt{108q}}$

b. $\frac{\sqrt[3]{72b^2}}{\sqrt[3]{9b^5}}$

Answer

a. $\frac{5q^2}{6}$

b. $\frac{2}{b}$

Example 17.6.2

Simplify:

a. $\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$

b. $\frac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$

Solution:

a.

$$\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$$

Rewrite using the quotient property.

$$\sqrt{\frac{147ab^8}{3a^3b^4}}$$

Remove common factors in the fraction.

$$\sqrt{\frac{49b^4}{a^2}}$$

Simplify the radical.

$$\frac{7b^2}{a}$$

b.

$$\frac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$$

Rewrite using the quotient property.

$$\sqrt[3]{\frac{-250mn^{-2}}{2m^{-2}n^4}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{\frac{-125m^3}{n^6}}$$

Simplify the radical.

$$-\frac{5m}{n^2}$$

Exercise 17.6.3

Simplify:

a. $\frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$

b. $\frac{\sqrt[3]{-128x^2y^{-1}}}{\sqrt[3]{2x^{-1}y^2}}$

Answer

a. $\frac{9x^2}{y^2}$
b. $\frac{-4x}{y}$

Exercise 17.6.4

Simplify:

a. $\frac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$
b. $\frac{\sqrt[3]{-81pq^{-1}}}{\sqrt[3]{3p^{-2}q^5}}$

Answer

a. $\frac{10n^3}{m}$
b. $\frac{-3p}{q^2}$

Example 17.6.3

Simplify: $\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$

Solution:

$$\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$$

Rewrite using the quotient property.

$$\sqrt{\frac{54x^5y^3}{3x^2y}}$$

Remove common factors in the fraction.

$$\sqrt{18x^3y^2}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{9x^2y^2\cdot 2x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{9x^2y^2}\cdot\sqrt{2x}$$

Simplify.

$$3xy\sqrt{2x}$$

Exercise 17.6.5

Simplify: $\frac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$

Answer

$4xy\sqrt{2x}$

Exercise 17.6.6

Simplify: $\frac{\sqrt{96a^5b^4}}{\sqrt{2a^3b}}$

Answer

$4ab\sqrt{3b}$

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process!

For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator. This process is still used today, and is useful in other areas of mathematics, too.

Definition 17.6.2: Rationalizing the Denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a **radical expression** is not considered simplified if the radicand contains a fraction.

Simplified Radical Expressions

A radical expression is considered simplified if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that $(\sqrt{a})^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

Example 17.6.4

Simplify:

a. $\frac{4}{\sqrt{3}}$

b. $\sqrt{\frac{3}{20}}$

c. $\frac{3}{\sqrt{6x}}$

Solution:

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

a.

Table 8.5.1

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	$\frac{4\sqrt{3}}{3}$

b. We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

Table 8.5.2

	$\sqrt{\frac{3}{20}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$\frac{\sqrt{3}}{\sqrt{20}}$
Simplify the denominator.	$\frac{\sqrt{3}}{2\sqrt{5}}$
Multiply the numerator and denominator by $\sqrt{5}$.	$\frac{\sqrt{3} \cdot \sqrt{5}}{2\sqrt{5} \cdot \sqrt{5}}$
Simplify.	$\frac{\sqrt{15}}{2 \cdot 5}$
Simplify.	$\frac{\sqrt{15}}{10}$

c.

Table 8.5.3

	$\frac{3}{\sqrt{6x}}$
Multiply the numerator and denominator by $\sqrt{6x}$.	$\frac{3 \cdot \sqrt{6x}}{\sqrt{6x} \cdot \sqrt{6x}}$
Simplify.	$\frac{3\sqrt{6x}}{6x}$
Simplify.	$\frac{\sqrt{6x}}{2x}$

? Exercise 17.6.7

Simplify:

a. $\frac{5}{\sqrt{3}}$

b. $\sqrt{\frac{3}{32}}$

c. $\frac{2}{\sqrt{2x}}$

Answer

a. $\frac{5\sqrt{3}}{3}$

b. $\frac{\sqrt{6}}{8}$

c. $\dfrac{\sqrt{2x}}{x}$

Exercise 17.6.8

Simplify:

a. $\dfrac{6}{\sqrt{5}}$

b. $\sqrt{\dfrac{7}{18}}$

c. $\dfrac{5}{\sqrt{5x}}$

Answer

a. $\dfrac{6\sqrt{5}}{5}$

b. $\dfrac{\sqrt{14}}{6}$

c. $\dfrac{\sqrt{5x}}{x}$

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the radical in the denominator. When we took the square root, the denominator no longer had a radical.

We will follow a similar process to rationalize higher roots. To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index. When we simplify the new radical, the denominator will no longer have a radical.

For example,

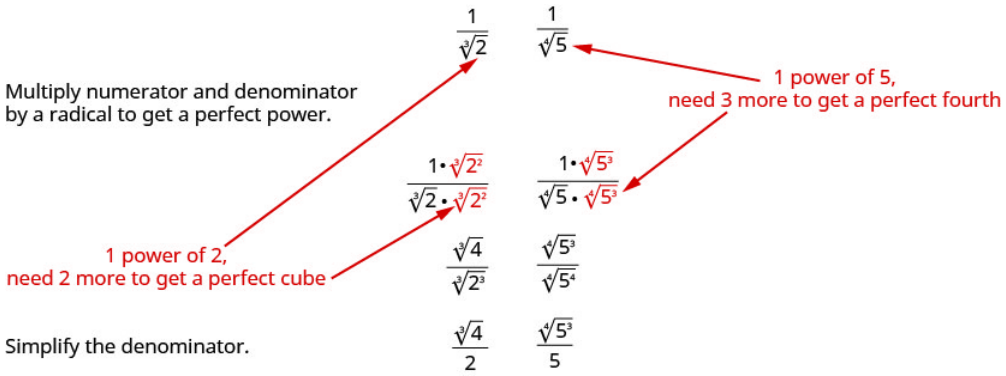

Figure 8.5.14

We will use this technique in the next examples.

Example 17.6.5

Simplify:

a. $\dfrac{1}{\sqrt[3]{6}}$

b. $\sqrt[3]{\dfrac{7}{24}}$

c. $\dfrac{3}{\sqrt[3]{4x}}$

Solution:

To rationalize a denominator with a cube root, we can multiply by a cube root that will give us a perfect cube in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

a.

Table 8.5.4

	$\frac{1}{\sqrt[3]{6}}$
The radical in the denominator has one factor of 6. Multiply both the numerator and denominator by $\sqrt[3]{6^2}$, which gives us 2 more factors of 6.	$\frac{1 \cdot \sqrt[3]{6^2}}{\sqrt[3]{6} \cdot \sqrt[3]{6^2}}$
Multiply. Notice the radicand in the denominator has 3 powers of 6.	$\frac{\sqrt[3]{6^2}}{\sqrt[3]{6^3}}$
Simplify the cube root in the denominator.	$\frac{\sqrt[3]{36}}{6}$

b. We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

Table 8.5.5

	$\sqrt[3]{\frac{7}{24}}$
The fraction is not a perfect cube, so rewrite using the Quotient Property.	$\frac{\sqrt[3]{7}}{\sqrt[3]{24}}$
Simplify the denominator.	$\frac{\sqrt[3]{7}}{2\sqrt[3]{3}}$
Multiply the numerator and denominator by $\sqrt[3]{3^2}$. This will give us 3 factors of 3.	$\frac{\sqrt[3]{7} \cdot \sqrt[3]{3^2}}{2\sqrt[3]{3} \cdot \sqrt[3]{3^2}}$
Simplify.	$\frac{\sqrt[3]{63}}{2\sqrt[3]{3^3}}$
Remember, $\sqrt[3]{3^3} = 3$.	$\frac{\sqrt[3]{63}}{2 \cdot 3}$
Simplify.	$\frac{\sqrt[3]{63}}{6}$

c.

Table 8.5.6

	$\frac{3}{\sqrt[3]{4x}}$
Rewrite the radicand to show the factors.	$\frac{3}{\sqrt[3]{2^2 \cdot x}}$
Multiple the numerator and denominator by $\sqrt[3]{2 \cdot x^2}$. This will get us 3 factors of 2 and 3 factors of x.	$\frac{3 \cdot \sqrt[3]{2 \cdot x^2}}{\sqrt[3]{2^2 x} \cdot \sqrt[3]{2 \cdot x^2}}$
Simplify.	$\frac{3\sqrt[3]{2x^2}}{\sqrt[3]{2^3 x^3}}$
Simplify the radical in the denominator.	$\frac{3\sqrt[3]{2x^2}}{2x}$

Exercise 17.6.9

Simplify:

a. $\frac{1}{\sqrt[3]{7}}$

b. $\sqrt[3]{\frac{5}{12}}$

c. $\frac{5}{\sqrt[3]{9y}}$

Answer

a. $\frac{\sqrt[3]{49}}{7}$

b. $\frac{\sqrt[3]{90}}{6}$

c. $\frac{5\sqrt[3]{3y^2}}{3y}$

Exercise 17.6.10

Simplify:

a. $\frac{1}{\sqrt[3]{2}}$

b. $\sqrt[3]{\frac{3}{20}}$

c. $\frac{2}{\sqrt[3]{25n}}$

Answer

a. $\frac{\sqrt[3]{4}}{2}$

b. $\frac{\sqrt[3]{150}}{10}$

c. $\frac{2\sqrt[3]{5n^2}}{5n}$

Example 17.6.6

Simplify:

a. $\frac{1}{\sqrt[4]{2}}$

b. $\sqrt[4]{\frac{5}{64}}$

c. $\frac{2}{\sqrt[4]{8x}}$

Solution:

To rationalize a denominator with a fourth root, we can multiply by a fourth root that will give us a perfect fourth power in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

a.

Table 8.5.7

	$\frac{1}{\sqrt[4]{2}}$
The radical in the denominator has one factor of 2. Multiply both the numerator and denominator by $\sqrt[4]{2^3}$, which gives us 3 more factors of 2.	$\frac{1 \cdot \sqrt[4]{2^3}}{\sqrt[4]{2} \cdot \sqrt[4]{2^3}}$
Multiply. Notice the radicand in the denominator has 4 powers of 2.	$\frac{\sqrt[4]{8}}{\sqrt[4]{2^4}}$
Simplify the fourth root in the denominator.	$\frac{\sqrt[4]{8}}{2}$

b. We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

Table 8.5.8

	$\sqrt[4]{\frac{5}{64}}$
The fraction is not a perfect fourth power, so rewrite using the Quotient Property.	$\frac{\sqrt[4]{5}}{\sqrt[4]{64}}$
Rewrite the radicand in the denominator to show the factors.	$\frac{\sqrt[4]{5}}{\sqrt[4]{2^6}}$
Simplify the denominator.	$\frac{\sqrt[4]{5}}{2\sqrt[4]{2^2}}$
Multiply the numerator and denominator by $\sqrt[4]{2^2}$. This will give us 4 factors of 2.	$\frac{\sqrt[4]{5} \cdot \sqrt[4]{2^2}}{2\sqrt[4]{2^2} \cdot \sqrt[4]{2^2}}$
Simplify.	$\frac{\sqrt[4]{5} \cdot \sqrt[4]{4}}{2\sqrt[4]{2^4}}$
Remember, $\sqrt[4]{2^4} = 2$.	$\frac{\sqrt[4]{20}}{2 \cdot 2}$
Simplify.	$\frac{\sqrt[4]{20}}{4}$

c.

Table 8.5.9

	$\frac{2}{\sqrt[4]{8x}}$
Rewrite the radicand to show the factors.	$\frac{2}{\sqrt[4]{2^3 \cdot x}}$
Multiply the numerator and denominator by $\sqrt[4]{2 \cdot x^3}$. This will get us 4 factors of 2 and 4 factors of x.	$\frac{2 \cdot \sqrt[4]{2 \cdot x^3}}{\sqrt[4]{2^3 x} \cdot \sqrt[4]{2 \cdot x^3}}$
Simplify.	$\frac{2\sqrt[4]{2x^3}}{\sqrt[4]{2^4 x^4}}$
Simplify the radical in the denominator.	$\frac{2\sqrt[4]{2x^3}}{2x}$
Simplify the fraction.	$\frac{\sqrt[4]{2x^3}}{x}$

Exercise 17.6.11

Simplify:

a. $\frac{1}{\sqrt[4]{3}}$

b. $\sqrt[4]{\frac{3}{64}}$

c. $\frac{3}{\sqrt[4]{125x}}$

Answer

a. $\frac{\sqrt[4]{27}}{3}$

b. $\frac{\sqrt[4]{12}}{4}$

c. $\frac{3\sqrt[4]{5x^3}}{5x}$

Exercise 17.6.12

Simplify:

a. $\frac{1}{\sqrt[4]{5}}$

b. $\sqrt[4]{\frac{7}{128}}$

c. $\frac{4}{\sqrt[4]{4x}}$

Answer

a. $\frac{\sqrt[4]{125}}{5}$

b. $\frac{\sqrt[4]{224}}{8}$

c. $\frac{\sqrt[4]{64x^3}}{x}$

Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the **Product of Conjugates Pattern** to **rationalize the denominator**.

$$(a-b)(a+b) \qquad (2-\sqrt{5})(2+\sqrt{5})$$
$$a^2-b^2 \qquad 2^2-(\sqrt{5})^2$$
$$4-5$$
$$-1$$

When we multiple a binomial that includes a square root by its conjugate, the product has no square roots.

Example 17.6.7

Simplify: $\frac{5}{2-\sqrt{3}}$

Solution:

Table 8.5.10

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$
Multiply the conjugates in the denominator.	$\frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{1}$
Simplify.	$5(2+\sqrt{3})$

? Exercise 17.6.13

Simplify: $\frac{3}{1-\sqrt{5}}$.

Answer

$-\frac{3(1+\sqrt{5})}{4}$

? Exercise 17.6.14

Simplify: $\frac{2}{4-\sqrt{6}}$.

Answer

$\frac{4+\sqrt{6}}{5}$

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

✓ Example 17.6.8

Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution:

Table 8.5.11

	$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u}-\sqrt{6})(\sqrt{u}+\sqrt{6})}$
Multiply the conjugates in the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u})^2-(\sqrt{6})^2}$
Simplify the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}$

Exercise 17.6.15

Simplify: $\dfrac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

Answer

$$\frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{x-2}$$

Exercise 17.6.16

Simplify: $\dfrac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$

Answer

$$\frac{\sqrt{10}(\sqrt{y}+\sqrt{3})}{y-3}$$

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

Example 17.6.9

Simplify: $\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution:

Table 8.5.12

	$\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$
Multiply the conjugates in the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$\frac{(\sqrt{x}+\sqrt{7})^2}{x-7}$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

Exercise 17.6.17

Simplify: $\dfrac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

Answer

$$\frac{(\sqrt{p}+\sqrt{2})^2}{p-2}$$

Exercise 17.6.18

Simplify: $\dfrac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$

Answer

$$\frac{(\sqrt{q}-\sqrt{10})^2}{q-10}$$

Key Concepts

- **Quotient Property of Radical Expressions**
  - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then, $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$
- **Simplified Radical Expressions**
  - A radical expression is considered simplified if there are:
    - no factors in the radicand that have perfect powers of the index
    - no fractions in the radicand
    - no radicals in the denominator of a fraction

Glossary

rationalizing the denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

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17.6E: Exercises

Practice Makes Perfect

Exercise SET A: divide square roots

In the following exercises, simplify.

1. a. $\frac{\sqrt{128}}{\sqrt{72}}$ b. $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$
2. a. $\frac{\sqrt{48}}{\sqrt{75}}$ b. $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$
3. a. $\frac{\sqrt{200m^5}}{\sqrt{98m}}$ b. $\frac{\sqrt[3]{54y^2}}{\sqrt[3]{2y^5}}$
4. a. $\frac{\sqrt{108n^7}}{\sqrt{243n^3}}$ b. $\frac{\sqrt[3]{54y}}{\sqrt[3]{16y^4}}$
5. a. $\frac{\sqrt{75r^3}}{\sqrt{108r^7}}$ b. $\frac{\sqrt[3]{24x^7}}{\sqrt[3]{81x^4}}$
6. a. $\frac{\sqrt{196q}}{\sqrt{484q^5}}$ b. $\frac{\sqrt[3]{16m^4}}{\sqrt[3]{54m}}$
7. a. $\frac{\sqrt{108p^5q^2}}{\sqrt{3p^3q^6}}$ b. $\frac{\sqrt[3]{-16a^4b^{-2}}}{\sqrt[3]{2a^{-2}b}}$
8. a. $\frac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$ b. $\frac{\sqrt[3]{-375y^4z^2}}{\sqrt[3]{3y^{-2}z^4}}$
9. a. $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$ b. $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$
10. a. $\frac{\sqrt{810c^{-3}d^7}}{\sqrt{1000cd}}$ b. $\frac{\sqrt[3]{24a^7b^{-1}}}{\sqrt[3]{-81a^{-2}b^2}}$
11. $\frac{\sqrt{56x^5y^4}}{\sqrt{2xy^3}}$
12. $\frac{\sqrt{72a^3b^6}}{\sqrt{3ab^3}}$
13. $\frac{\sqrt[3]{48a^3b^6}}{\sqrt[3]{3a^{-1}b^3}}$
14. $\frac{\sqrt[3]{162x^{-3}y^6}}{\sqrt[3]{2x^3y^{-2}}}$

Answer

1. a. $\frac{4}{3}$ b. $\frac{4}{3}$
3. a. $\frac{10m^2}{7}$ b. $\frac{3}{y}$
5. a. $\frac{5}{6r^2}$ b. $\frac{2x}{3}$

7. a. $\frac{6p}{q^2}$ b. $-\frac{2a^2}{b}$

9. a. $\frac{8m^4}{3n^4}$ b. $-\frac{2x^2}{3y^2}$

11. $4x^4\sqrt{7y}$

13. $2ab\sqrt[3]{2a}$

Exercise SET B: Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

15. a. $\frac{10}{\sqrt{6}}$ b. $\sqrt{\frac{4}{27}}$ c. $\frac{10}{\sqrt{5x}}$

16. a. $\frac{8}{\sqrt{3}}$ b. $\sqrt{\frac{7}{40}}$ c. $\frac{8}{\sqrt{2y}}$

17. a. $\frac{6}{\sqrt{7}}$ b. $\sqrt{\frac{8}{45}}$ c. $\frac{12}{\sqrt{3p}}$

18. a. $\frac{4}{\sqrt{5}}$ b. $\sqrt{\frac{27}{80}}$ c. $\frac{18}{\sqrt{6q}}$

19. a. $\frac{1}{\sqrt[3]{5}}$ b. $\sqrt[3]{\frac{5}{24}}$ c. $\frac{4}{\sqrt[3]{36a}}$

20. a. $\frac{1}{\sqrt[3]{3}}$ b. $\sqrt[3]{\frac{5}{32}}$ c. $\frac{7}{\sqrt[3]{49b}}$

21. a. $\frac{1}{\sqrt[3]{11}}$ b. $\sqrt[3]{\frac{7}{54}}$ c. $\frac{3}{\sqrt[3]{3x^2}}$

22. a. $\frac{1}{\sqrt[3]{13}}$ b. $\sqrt[3]{\frac{3}{128}}$ c. $\frac{3}{\sqrt[3]{6y^2}}$

23. a. $\frac{1}{\sqrt[4]{7}}$ b. $\sqrt[4]{\frac{5}{32}}$ c. $\frac{4}{\sqrt[4]{4x^2}}$

24. a. $\frac{1}{\sqrt[4]{4}}$ b. $\sqrt[4]{\frac{9}{32}}$ c. $\frac{6}{\sqrt[4]{9x^3}}$

25. a. $\frac{1}{\sqrt[4]{9}}$ b. $\sqrt[4]{\frac{25}{128}}$ c. $\frac{6}{\sqrt[4]{27a}}$

26. a. $\frac{1}{\sqrt[4]{8}}$ b. $\sqrt[4]{\frac{27}{128}}$ c. $\frac{16}{\sqrt[4]{64b^2}}$

Answer

15. a. $\frac{5\sqrt{6}}{3}$ b. $\frac{2\sqrt{3}}{9}$ c. $\frac{2\sqrt{5x}}{x}$

17. a. $\frac{6\sqrt{7}}{7}$ b. $\frac{2\sqrt{10}}{15}$ c. $\frac{4\sqrt{3p}}{p}$

19. a. $\frac{\sqrt[3]{25}}{5}$ b. $\frac{\sqrt[3]{45}}{6}$ c. $\frac{2\sqrt[3]{6a^2}}{3a}$

21. a. $\frac{\sqrt[3]{121}}{11}$ b. $\frac{\sqrt[3]{28}}{6}$ c. $\frac{\sqrt[3]{9x}}{x}$

23. a. $\frac{\sqrt[4]{343}}{7}$ b. $\frac{\sqrt[4]{40}}{4}$ c. $\frac{2\sqrt[4]{4x^2}}{x}$

25. a. $\frac{\sqrt[4]{9}}{3}$ b. $\frac{\sqrt[4]{50}}{4}$ c. $\frac{2\sqrt[4]{3a^2}}{a}$

Exercise SET C: Rationalize a Two Term Denominator

In the following exercises, simplify.

27. $\frac{8}{1-\sqrt{5}}$

28. $\frac{7}{2-\sqrt{6}}$

29. $\frac{6}{3-\sqrt{7}}$

30. $\frac{5}{4-\sqrt{11}}$

31. $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

32. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

33. $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

34. $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

35. $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

36. $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

37. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

38. $\frac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

Answer

27. $-2(1+\sqrt{5})$

29. $3(3+\sqrt{7})$

31. $\frac{\sqrt{3}(\sqrt{m}+\sqrt{5})}{m-5}$

33. $\frac{\sqrt{2}(\sqrt{x}+\sqrt{6})}{x-6}$

35. $\frac{(\sqrt{r}+\sqrt{5})^2}{r-5}$

37. $\frac{(\sqrt{x}+2\sqrt{2})^2}{x-8}$

Exercise SET D: writing exercises

1. a. Simplify $\sqrt{\frac{27}{3}}$ and explain all your steps.

   b. Simplify $\sqrt{\frac{27}{5}}$ and explain all your steps.

   c. Why are the two methods of simplifying square roots different?

2. Explain what is meant by the word rationalize in the phrase, "rationalize a denominator."
3. Explain why multiplying $\sqrt{2x}-3$ by its conjugate results in an epxression with no radicals.
4. Explain why multiplying $\frac{7}{\sqrt[3]{x}}$ by $\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$ does not rationalize the denominator.

Answer

1. Answers will vary
3. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
divide radical expressions.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

Figure 8.5.63

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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17.7: Solve Radical Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Before you get started, take this readiness quiz.

1. Simplify: $(y-3)^2$.
   If you missed this problem, review Example 5.31.
2. Solve: $2x-5=0$.
   If you missed this problem, review Example 2.2.
3. Solve $n^2-6n+8=0$.
   If you missed this problem, review Example 6.45.

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Definition: Radical Equation

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the nth power. This will eliminate the radical.

$$\text{For } a \geq 0, (\sqrt[n]{a})^n = a \text{ .}$$

Example 17.7.1 how to solve a radical equation

Solve: $\sqrt{5n-4}-9=0$.

Solution:

Table 8.6.1

Step 1: Isolate the radical on one side of the equation.	To isolate the radical, add 9 to both sides. Simplify.	$\sqrt{5n-4}-9=0$ $\sqrt{5n-4}-9+9=0+9$ $\sqrt{5n-4}=9$
Step 2: Raise both sides of the equation to the power of the index.	Since the index of a square root is 2, we square both sides.	$(\sqrt{5n-4})^2=(9)^2$
Step 3: Solve the new equation.	Remember, $(\sqrt{a})^2=a$.	$5n-4=81$ $5n=85$ $n=17$

Step 4: Check the answer in the original equation.		Check the answer. $\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9=0$ $0=0$ The solution is $n=17$.

Exercise 17.7.1

Solve: $\sqrt{3m+2}-5=0$.

Answer

$m=\frac{23}{3}$

Exercise 17.7.2

Solve: $\sqrt{10z+1}-2=0$.

Answer

$z=\frac{3}{10}$

Solve a Radical Equation With One Radical

1. Isolate the radical on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Solve the new equation.
4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

Example 17.7.2

Solve: $\sqrt{9k-2}+1=0$.

Solution:

Table 8.6.2

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 to both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$

Because the square root is equal to a negative number, the equation has no solution.

Exercise 17.7.3

Solve: $\sqrt{2r-3}+5=0$.

Answer

no solution

Exercise 17.7.4

Solve: $\sqrt{7s-3}+2=0$.

Answer

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Definition: Binomial Squares

$(a+b)^2=a^2+2ab+b^2$
$(a-b)^2=a^2-2ab+b^2$

Don't forget the middle term!

Example 17.7.3

Solve: $\sqrt{p-1}+1=p$.

Solution:

Table 8.6.3

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right, Then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$
Use the Zero Product Property.	$0=p-1 \quad 0=p-2$
Solve each equation.	$p=1 \quad p=2$
Check the answers.	
$p=1$ $\sqrt{p-1}+1=p$ $\sqrt{1-1}+1\stackrel{?}{=}1$ $\sqrt{0}+1\stackrel{?}{=}1$ $1=1$ ✓ $p=2$ $\sqrt{p-1}+1=p$ $\sqrt{2-1}+1\stackrel{?}{=}2$ $\sqrt{1}+1\stackrel{?}{=}2$ $2=2$ ✓	

The solutions are $p=1, p=2$.

Exercise 17.7.5

Solve: $\sqrt{x-2}+2=x$.

Answer

$x = 2, x = 3$

Exercise 17.7.6

Solve: $\sqrt{y-5}+5=y$.

Answer

$y = 5, y = 6$

When the index of the radical is 3, we cube both sides to remove the radical.

$$(\sqrt[3]{a})^3 = a$$

Example 17.7.4

Solve: $\sqrt[3]{5x+1}+8=4$.

Solution:

Table 8.6.4

	$\sqrt[3]{5x+1}+8=4$
To isolate the radical, subtract 8 from both sides.	$\sqrt[3]{5x+1}=-4$
Cube both sides of the equation.	$(\sqrt[3]{5x+1})^3=(-4)^3$
Simplify.	$5x+1=-64$
Solve the equation.	$5x=-65$
	$x=-13$
Check the answer.	
$x=-13$ $\quad \sqrt[3]{5x+1}+8=4$ $\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$ $\sqrt[3]{-64}+8\stackrel{?}{=}4$ $-4+8\stackrel{?}{=}4$ $4=4\checkmark$	
	The solution is $x=-13$.

Exercise 17.7.7

Solve: $\sqrt[3]{4x-3}+8=5$

Answer

$x = -6$

Exercise 17.7.8

Solve: $\sqrt[3]{6x-10}+1=-3$

Answer

$x = -9$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(a^m)^n = a^{m \cdot n}$, we have for example,

$$\left(x^{\frac{1}{2}}\right)^2 = x, \left(x^{\frac{1}{3}}\right)^3 = x$$

Remember, $x^{\frac{1}{2}} = \sqrt{x}$ and $x^{\frac{1}{3}} = \sqrt[3]{x}$.

Example 17.7.5

Solve: $(3x-2)^{\frac{1}{4}} + 3 = 5$.

Solution:

Table 8.6.5

	$(3x-2)^{\frac{1}{4}} + 3 = 5$
To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x-2)^{\frac{1}{4}} = 2$
Raise each side of the equation to the fourth power.	$\left((3x-2)^{\frac{1}{4}}\right)^4 = (2)^4$
Simplify.	$3x-2 = 16$
Solve the equation.	$3x = 18$
	$x = 6$
Check the answer.	
$x=6 \quad (3x-2)^{\frac{1}{4}}+3=5$ $(3\cdot 6-2)^{\frac{1}{4}}+3 \stackrel{?}{=} 5$ $(16)^{\frac{1}{4}}+3 \stackrel{?}{=} 5$ $2+3 \stackrel{?}{=} 5$ $5=5$ ✓	
	The solution is $x = 6$.

Exercise 17.7.9

Solve: $(9x+9)^{\frac{1}{4}} - 2 = 1$

Answer

$x = 8$

Exercise 17.7.10

Solve: $(4x-8)^{\frac{1}{4}} + 5 = 7$

Answer

$x = 6$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an **extraneous solution**!

Example 17.7.6

Solve: $\sqrt{r+4}-r+2=0$.

Solution:

Table 8.6.6

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$(\sqrt{r+4})^2=(r-2)^2$
Simplify and then solve the equation.	$r+4=r^2-4r+4$
If it is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the Zero Product Property.	$0=r \quad 0=r-5$
Solve the equation.	$r=0 \quad r=5$
Check your answer.	
$r=0$, $\sqrt{r+4}-r+2=0$ $\sqrt{0+4}-0+2\stackrel{?}{=}0$ $\sqrt{4}+2\stackrel{?}{=}0$ $4\neq 0$ $r=5$, $\sqrt{r+4}-r+2=0$ $\sqrt{5+4}-5+2\stackrel{?}{=}0$ $\sqrt{9}-3\stackrel{?}{=}0$ $0=0$ ✓	The solution is $r=5$.
	$r=0$ is an extreneous solution.

Exercise 17.7.11

Solve: $\sqrt{m+9}-m+3=0$

Answer

$m=7$

Exercise 17.7.12

Solve: $\sqrt{n+1}-n+1=0$.

Answer

$n=3$

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

Example 17.7.7

Solve: $3\sqrt{3x-5}-8=4$.

Solution:

Table 8.6.7

	$3\sqrt{3x-5}-8=4$
Isolate the radical term.	$3\sqrt{3x-5}=12$

Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5}=4$
Square both sides of the equation.	$(\sqrt{3x-5})^2=(4)^2$
Simplify, then solve the new equation.	$3x-5=16$
	$3x=21$
Solve the equation.	$x=7$
Check the answer.	
$x=7$ $3\sqrt{3x-5}-8=4$ $3\sqrt{3(7)-5}-8\stackrel{?}{=}4$ $3\sqrt{21-5}-8\stackrel{?}{=}4$ $3\sqrt{16}-8\stackrel{?}{=}4$ $3(4)-8\stackrel{?}{=}4$ $4=4$ ✓	
	The solution is $x=7$.

Exercise 17.7.13

Solve: $2\sqrt{4a+4}-16=16$.

Answer

$a=63$

Exercise 17.7.14

Solve: $3\sqrt{2b+3}-25=50$

Answer

$b=311$

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

Example 17.7.8

Solve: $\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$.

Solution:

The radical terms are isolated.

$$\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$$

Since the index is 3, cube both sides of the equation.

$$(\sqrt[3]{4x-3})^3=(\sqrt[3]{3x+2})^3$$

Simplify, then solve the new equation.

$$\begin{aligned} 4x-3 &= 3x+2 \\ x-3 &= 2 \\ x &= 5 \end{aligned}$$

The solution is $x=5$.

Check the answer.

We leave it to you to show that 5 checks!

? Exercise 17.7.15

Solve: $\sqrt[3]{5x-4} = \sqrt[3]{2x+5}$.

Answer

$x=3$

? Exercise 17.7.16

Solve: $\sqrt[3]{7x+1} = \sqrt[3]{2x-5}$.

Answer

$x=-\frac{6}{5}$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

✓ Example 17.7.9 how to solve a radical equation

Solve: $\sqrt{m}+1 = \sqrt{m+9}$.

Solution:

Table 8.6.8

Step 1: Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Step 2: Raise both sides of the equation to the power of the index.	We square both sides. Simplify--be very careful as you multiply!	$(\sqrt{m}+1)^2=(\sqrt{m+9})^2$
Step 3: Are there any more radicals? If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term. Here, we can easily isolate the radical by dividing both sides by 2. Square both sides.	$\begin{aligned} m+2\sqrt{m}+1 &= m+9 \\ 2\sqrt{m} &= 8 \\ \sqrt{m} &= 4 \\ (\sqrt{m})^2 &= (4)^2 \\ m &= 16 \end{aligned}$
Step 4: Check the answer in the original equation.		$\begin{aligned} \sqrt{m}+1 &= \sqrt{m+9} \\ \sqrt{16}+1 &\stackrel{?}{=} \sqrt{16+9} \\ 4+1 &\stackrel{?}{=} 5 \\ 5 &= 5 \end{aligned}$ The solution is $m=16$.

Exercise 17.7.17

Solve: $3-\sqrt{x}=\sqrt{x-3}$.

Answer

$x=4$

Exercise 17.7.18

Solve: $\sqrt{x}+2=\sqrt{x+16}$.

Answer

$x=9$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
   If yes, repeat Step 1 and Step 2 again.
   If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern in $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Example 17.7.10

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution:

Table 8.6.9

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right is isolated. Square both sides.	$(\sqrt{q-2}+3)^2=(\sqrt{4q+1})^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation so we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the $\sqrt{q-2}$.	$(6\sqrt{q-2})^2=\left(\begin{matrix}a-b\\3q-6\end{matrix}\right)^2$ $\begin{matrix}a^2 & - & 2ab & + b^2\end{matrix}$ $6^2(\sqrt{q-2})^2=(3q)^2-2\cdot 3q\cdot 6+6^2$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$
Distribute.	$36q-72=9q^2-36q+36$
It is a quadratic equation, so get zero on one side.	$0=9q^2-72q+108$
Factor the right side.	$0=9(q^2-8q+12)$ $0=9(q-6)(q-2)$

Use the Zero Product Property.	$q-6=0 \quad q-2=0$ $q=6 \quad q=2$
The checks are left to you.	The solutions are $q=6$ and $q=2$.

Exercise 17.7.19

Solve: $\sqrt{x-1}+2=\sqrt{2x+6}$

Answer

$x=5$

Exercise 17.7.20

Solve: $\sqrt{x}+2=\sqrt{3x+4}$

Answer

$x=0x=4$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Use a Problem Solving Strategy for Applications With Formulas

1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of **gravity** on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Definition: Falling Objects

On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula

$$t=\frac{\sqrt{h}}{4}$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h=64$ into the formula.

Table 8.6.10

	$t=\frac{\sqrt{h}}{4}$
	$t=\frac{\sqrt{64}}{4}$

Take the square root of 64.	$t=\frac{8}{4}$
Simplify the fraction.	$t=2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example 17.7.11

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution:

Table 8.6.11

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking.	The time it takes for the sunglasses to reach the river.
Step 3: **Name** what we are looking.	Let (t=\) time.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$t=\frac{\sqrt{h}}{4}$, and $h=400$ $t=\frac{\sqrt{400}}{4}$
Step 5: **Solve the equation**.	$t=\frac{20}{4}$
	$t=5$
Step 6: **Check** the answer in the problem and make sure it makes sense.	$5\stackrel{?}{=}\frac{\sqrt{400}}{4}$ $5\stackrel{?}{=}\frac{20}{4}$ $5=5\checkmark$
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7: **Answer** the equation.	It will take 5 seconds for the sunglasses to reach the river.

Exercise 17.7.21

A helicopter dropped a rescue package from a height of $1,296$ feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Answer

9 seconds

Exercise 17.7.22

A window washer dropped a squeegee from a platform 196 feet above the sidewalk. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Answer

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the **speed**, in miles per hour, a car was going before applying the brakes.

Definition: Skid Marks and Speed of a Car

If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

Example 17.7.12

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

Table 8.6.12

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking for.	The speed of a car.
Step 3: **Name** what we are looking for.	Let $s =$ the speed.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$s = \sqrt{24d}$, and $d = 190$ $s = \sqrt{24(190)}$
Step 5: **Solve the equation**.	$s = \sqrt{4{,}560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
	$67.5 \stackrel{?}{\approx} \sqrt{24(190)}$ $67.5 \stackrel{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277...$ ✓
	The speed of the car before the brakes were applied was 67.5 miles per hour.

Exercise 17.7.23

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

42.7mph

Exercise 17.7.24

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

54.1mph

Access these online resources for additional instruction and practice with solving radical equations.

- Solving an Equation Involving a Single Radical
- Solving Equations with Radicals and Rational Exponents
- Solving Radical Equations
- Solve Radical Equations
- Radical Equation Application

Key Concepts

- **Binomial Squares**
  $(a+b)^2 = a^2 + 2ab + b^2$
  $(a-b)^2 = a^2 - 2ab + b^2$
- **Solve a Radical Equation**
  1. Isolate one of the radical terms on one side of the equation.
  2. Raise both sides of the equation to the power of the index.
  3. Are there any more radicals?
     If yes, repeat Step 1 and Step 2 again.
     If no, solve the new equation.
  4. Check the answer in the original equation.
- **Problem Solving Strategy for Applications with Formulas**
  1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  2. Identify what we are looking for.
  3. Name what we are looking for by choosing a variable to represent it.
  4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.
- **Falling Objects**
  - On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
  - If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Glossary

radical equation

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

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17.7E: Exercises

Practice Makes Perfect

Exercise SET A: Solve Radical Equations

In the following exercises, solve.

1. $\sqrt{5x-6}=8$
2. $\sqrt{4x-3}=7$
3. $\sqrt{5x+1}=-3$
4. $\sqrt{3y-4}=-2$
5. $\sqrt[3]{2x}=-2$
6. $\sqrt[3]{4x-1}=3$
7. $\sqrt{2m-3}-5=0$
8. $\sqrt{2n-1}-3=0$
9. $\sqrt{6v-2}-10=0$
10. $\sqrt{12u+1}-11=0$
11. $\sqrt{4m+2}+2=6$
12. $\sqrt{6n+1}+4=8$
13. $\sqrt{2u-3}+2=0$
14. $\sqrt{5v-2}+5=0$
15. $\sqrt{u-3}+3=u$
16. $\sqrt{v-10}+10=v$
17. $\sqrt{r-1}=r-1$
18. $\sqrt{s-8}=s-8$
19. $\sqrt[3]{6x+4}=4$
20. $\sqrt[3]{11x+4}=5$
21. $\sqrt[3]{4x+5}-2=-5$
22. $\sqrt[3]{9x-1}-1=-5$
23. $(6x+1)^{\frac{1}{2}}-3=4$
24. $(3x-2)^{\frac{1}{2}}+1=6$
25. $(8x+5)^{\frac{1}{3}}+2=-1$
26. $(12x-5)^{\frac{1}{3}}+8=3$
27. $(12x-3)^{\frac{1}{4}}-5=-2$
28. $(5x-4)^{\frac{1}{4}}+7=9$
29. $\sqrt{x+1}-x+1=0$
30. $\sqrt{y+4}-y+2=0$
31. $\sqrt{z+100}-z=-10$
32. $\sqrt{w+25}-w=-5$

33. $3\sqrt{2x-3}-20=7$

34. $2\sqrt{5x+1}-8=0$

35. $2\sqrt{8r+1}-8=2$

36. $3\sqrt{7y+1}-10=8$

Answer

1. $m=14$
3. no solution
5. $x=-4$
7. $m=14$
9. $v=17$
11. $m=\frac{7}{2}$
13. no solution
15. $u=3, u=4$
17. $r=1, r=2$
19. $x=10$
21. $x=-8$
23. $x=8$
25. $x=-4$
27. $x=7$
29. $x=3$
31. $z=21$
33. $x=42$
35. $r=3$

Exercise SET B: Solve Radical Equations with Two Radicals

In the following exercises, solve.

37. $\sqrt{3u+7}=\sqrt{5u+1}$

38. $\sqrt{4v+1}=\sqrt{3v+3}$

39. $\sqrt{8+2r}=\sqrt{3r+10}$

40. $\sqrt{10+2c}=\sqrt{4c+16}$

41. $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$

42. $\sqrt[3]{8x-5}=\sqrt[3]{3x+5}$

43. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

44. $\sqrt[3]{x^2-x+18}=\sqrt[3]{2x^2-3x-6}$

45. $\sqrt{a}+2=\sqrt{a+4}$

46. $\sqrt{r}+6=\sqrt{r+8}$

47. $\sqrt{u}+1=\sqrt{u+4}$

48. $\sqrt{x}+1=\sqrt{x+2}$

49. $\sqrt{a+5}-\sqrt{a}=1$
50. $-2=\sqrt{d-20}-\sqrt{d}$
51. $\sqrt{2x+1}=1+\sqrt{x}$
52. $\sqrt{3x+1}=1+\sqrt{2x-1}$
53. $\sqrt{2x-1}-\sqrt{x-1}=1$
54. $\sqrt{x+1}-\sqrt{x-2}=1$
55. $\sqrt{x+7}-\sqrt{x-5}=2$
56. $\sqrt{x+5}-\sqrt{x-3}=2$

Answer

37. $u=3$
39. $r=-2$
41. $x=1$
43. $x=-8, x=2$
45. $a=0$
47. $u=\frac{9}{4}$
49. $a=4$
51. $x=0\ x=4$
53. $x=1\ x=5$
55. $x=9$

Exercise SET C: Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

57. **Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.
58. **Landscaping** Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.
59. **Gravity** A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.
60. **Gravity** A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.
61. **Accident investigation** The skid marks for a car involved in an accident measured 216 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.
62. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

57. 8.7 feet
59. 4.7 seconds
61. 72 feet

Exercise SET D: Writing Exercises

63. Explain why an equation of the form $\sqrt{x}+1=0$ has no solution.

64. a. Solve the equations $\sqrt{r+4}-r+2=0$.
    b. Explain why one of the "solutions" that was found was not actually a solution to the equation.

Answer

63. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve radical equations.			
solve radical equations with two radicals.			
use radicals in applications.			

Figure 8.6.42

b. After reviewing this checklist, what will you do to become confident for all objectives?

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17.8: Use Radicals in Functions

Learning Objectives

By the end of this section, you will be able to:

- Evaluate a radical function
- Find the domain of a radical function
- Graph radical functions

Before you get started, take this readiness quiz.

1. Solve: $1-2x \geq 0$.
   If you missed this problem, review Example 2.50.
2. For $f(x)=3x-4$, evaluate $f(2), f(-1), f(0)$
   If you missed this problem, review Example 3.48.
3. Graph $f(x)=\sqrt{x}$. State the domain and range of the function in interval notation.
   If you missed this problem, review Example 3.56.

Evaluate a Radical Function

In this section we will extend our previous work with functions to include radicals. If a function is defined by a radical expression, we call it a **radical function**.

- The square root function is $f(x)=\sqrt{x}$.
- The cube root function is $f(x)=\sqrt[3]{x}$.

Definition 17.8.1: radical function

A **radical function** is a function that is defined by a radical expression.

To evaluate a radical function, we find the value of $f(x)$ for a given value of x just as we did in our previous work with functions.

Example 17.8.1

For the function $f(x)=\sqrt{2x-1}$, find

a. $f(5)$
b. $f(-2)$

Solution:

a.

$$f(x)=\sqrt{2x-1}$$

To evaluate $f(5)$, substitute 5 for x.

$$f(5)=\sqrt{2\cdot 5-1}$$

Simplify.

$$f(5)=\sqrt{9}$$

Take the square root.

$$f(5)=3$$

b.

$$f(x)=\sqrt{2x-1}$$

To evaluate $f(-2)$, substitute -2 for x.

$$f(-2)=\sqrt{2(-2)-1}$$

Simplify.

$$f(-2)=\sqrt{-5}$$

Since the square root of a negative number is not a real number, the function does not have a value at $x=-2$.

Exercise 17.8.1

For the function $f(x)=\sqrt{3x-2}$, find

a. $f(6)$
b. $f(0)$

Answer

a. $f(6)=4$
b. no value at $x=0$

Exercise 17.8.2

For the function $g(x)=\sqrt{5x+5}$, find

a. $g(4)$
b. $g(-3)$

Answer

a. $g(4)=5$
b. no value at $f(-3)$

We follow the same procedure to evaluate cube roots.

Example 17.8.2

For the function $g(x)=\sqrt[3]{x-6}$, find

a. $g(14)$
b. $g(-2)$

Solution:

a.

$$g(x)=\sqrt[3]{x-6}$$

To evaluate $g(14)$, substitute 14 for x.

$$g(14)=\sqrt[3]{14-6}$$

Simplify.

$$g(14)=\sqrt[3]{8}$$

Take the cube root.

$$g(14)=2$$

b.

$$g(x)=\sqrt[3]{x-6}$$

To evaluate $g(-2)$, substitute -2 for x.

$$g(-2)=\sqrt[3]{-2-6}$$

Simplify.

$$g(-2) = \sqrt[3]{-8}$$

Take the cube root.

$$g(-2) = -2$$

Exercise 17.8.3

For the function $g(x) = \sqrt[3]{3x-4}$, find

a. $g(4)$
b. $g(1)$

Answer

a. $g(4) = 2$
b. $g(1) = -1$

Exercise 17.8.4

For the function $h(x) = \sqrt[3]{5x-2}$, find

a. $h(2)$
b. $h(-5)$

Answer

a. $h(2) = 2$
b. $h(-5) = -3$

The next example has fourth roots.

Example 17.8.3

For the function $f(x) = \sqrt[4]{5x-4}$, find

a. $f(4)$
b. $f(-12)$

Solution:

a.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(4)$, substitute 4 for x.

$$f(4) = \sqrt[4]{5 \cdot 4 - 4}$$

Simplify.

$$f(4) = \sqrt[4]{16}$$

Take the fourth root.

$$f(4) = 2$$

b.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(-12)$, substitute -12 for x.

$$f(-12) = \sqrt[4]{5(-12)-4}$$

Simplify.

$$f(-12) = \sqrt[4]{-64}$$

Since the fourth root of a negative number is not a real number, the function does not have a value at $x = -12$.

Exercise 17.8.5

For the function $f(x) = \sqrt[4]{3x+4}$, find

a. $f(4)$
b. $f(-1)$

Answer

a. $f(4) = 2$
b. $f(-1) = 1$

Exercise 17.8.6

For the function $g(x) = \sqrt[4]{5x+1}$, find

a. $g(16)$
b. $g(3)$

Answer

a. $g(16) = 3$
b. $g(3) = 2$

Find the Domain of a Radical Function

To find the **domain** and **range** of radical functions, we use our properties of radicals. For a radical with an even index, we said the radicand had to be greater than or equal to zero as even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Properties of $\sqrt[n]{a}$

When n is an **even** number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.

So, to find the domain of a radical function with even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number.

Domain of a Radical Function

When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.

When the **index** of the radical is **odd**, the radicand can be any real number.

Example 17.8.4

Find the domain of the function, $f(x) = \sqrt{3x-4}$. Write the domain in interval notation.

Solution:

Since the function, $f(x) = \sqrt{3x-4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0. We set the radicand to be greater than or equal to 0 and then solve to find the domain.

Solve.

$$\begin{aligned} 3x-4 &\geq 0 \\ 3x &\geq 4 \\ x &\geq \frac{4}{3} \end{aligned}$$

The domain of $f(x)=\sqrt{3x-4}$ is all values $x \geq \frac{4}{3}$ and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

Exercise 17.8.7

Find the domain of the function, $f(x)=\sqrt{6x-5}$. Write the domain in interval notation.

Answer

$\left[\frac{5}{6}, \infty\right)$

Exercise 17.8.8

Find the domain of the function, $f(x)=\sqrt{4-5x}$. Write the domain in interval notation.

Answer

$\left(-\infty, \frac{4}{5}\right]$

Example 17.8.5

Find the domain of the function, $g(x)=\sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

Solution:

Solve the function, $g(x)=\sqrt{\frac{6}{x-1}}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

The radicand cannot be zero since the numerator is not zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive since the numerator is positive. We know a positive divided by a positive is positive.

We set $x-1>0$ and solve.

$$x-1>0$$

Solve.

$$x>1$$

Also, since the radicand is a fraction, we must realize that the denominator cannot be zero.

We solve $x-1=0$ to find the value that must be eliminated from the domain.

$$x-1=0$$

Solve.

$$x=1 \text{ so } x/neq1 \text{ in the domain.}$$

Putting this together we get the domain is $x>1$ and we write it as $(1, \infty)$.

Exercise 17.8.9

Find the domain of the function, $f(x)=\sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

Answer

$(-3,\infty)$

Exercise 17.8.10

Find the domain of the function, $h(x)=\sqrt{\frac{9}{x-5}}$. Write the domain in interval notation.

Answer

$(5,\infty)$

The next example involves a cube root and so will require different thinking.

Example 17.8.6

Find the domain of the function, $f(x)=\sqrt[3]{2x^2+3}$. Write the domain in interval notation.

Solution:

Since the function, $f(x)=\sqrt[3]{2x^2+3}$ has a radical with an index of 3, which is odd, we know the radicand can be any real number. This tells us the domain is any real number. In interval notation, we write $(-\infty,\infty)$.

The domain of $f(x)=\sqrt[3]{2x^2+3}$ is all real numbers and we write it in interval notation as $(-\infty,\infty)$.

Exercise 17.8.11

Find the domain of the function, $f(x)=\sqrt[3]{3x^2-1}$. Write the domain in interval notation.

Answer

$(-\infty,\infty)$

Exercise 17.8.12

Find the domain of the function, $g(x)=\sqrt[3]{5x-4}$. Write the domain in interval notation.

Answer

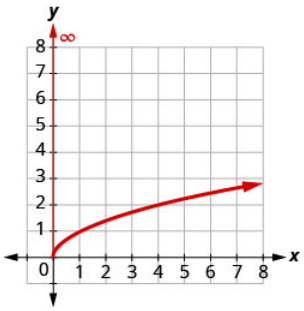
$(-\infty,\infty)$

Graph Radical Functions

Before we graph any radical function, we first find the **domain** of the function. For the function, $f(x)=\sqrt{x}$, the index is even, and so the radicand must be greater than or equal to 0.

This tells us the domain is $x\geq 0$ and we write this in interval notation as $[0,\infty)$.

Previously we used point plotting to graph the function, $f(x)=\sqrt{x}$. We chose x-values, substituted them in and then created a chart. Notice we chose points that are perfect squares in order to make taking the square root easier.



x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Figure 8.7.1

Once we see the graph, we can find the range of the function. The y-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

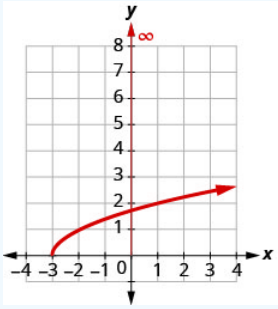
Example 17.8.7

For the function $f(x)=\sqrt{x+3}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 2, we know the radicand must be greater than or equal to zero. If $x+3 \geq 0$, then $x \geq -3$. This tells us the domain is all values $x \geq -3$ and written in interval notation as $[-3, \infty)$.
b. To graph the function, we choose points in the interval $[-3, \infty)$ that will also give us a radicand which will be easy to take the square root.



x	$f(x)=\sqrt{x+3}$	$(x, f(x))$
–3	0	(–3, 0)
–2	1	(–2, 1)
1	2	(1, 2)
6	3	(6, 3)

Figure 8.7.2

c. Looking at the graph, we see the y-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

Exercise 17.8.13

For the function $f(x)=\sqrt{x+2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[-2, \infty)$

b.

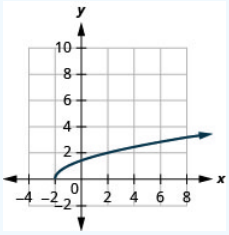


Figure 8.7.3

c. range: $[0, \infty)$

Exercise 17.8.14

For the function $f(x) = \sqrt{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[2, \infty)$
b.

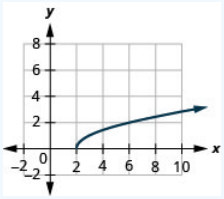


Figure 8.7.4

c. range: $[0, \infty)$

In our previous work graphing functions, we graphed $f(x) = x^3$ but we did not graph the function $f(x) = \sqrt[3]{x}$. We will do this now in the next example.

Example 17.8.8

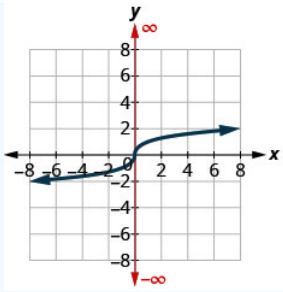
For the function, $f(x) = \sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 3, we know the radicand can be any real number. This tells us the domain is all real numbers and written in interval notation as $(-\infty, \infty)$

b. To graph the function, we choose points in the interval $(-\infty, \infty)$ that will also give us a radicand which will be easy to take the cube root.



x	$f(x)=\sqrt[3]{x}$	$(x, f(x))$
–8	–2	(–8, –2)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
8	2	(8, 2)

Figure 8.7.5

c. Looking at the graph, we see the y-values of the function are all real numbers. The range then is $(-\infty, \infty)$.

Exercise 17.8.15

For the function $f(x)=-\sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$
b.

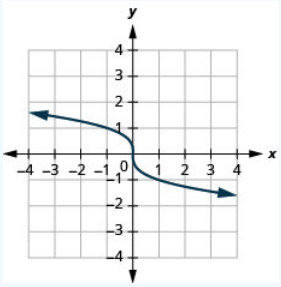


Figure 8.7.6

c. range: $(-\infty, \infty)$

Exercise 17.8.16

For the function $f(x)=\sqrt[3]{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$
b.

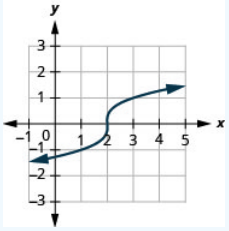


Figure 8.7.7

c. range: $(-\infty, \infty)$

Access these online resources for additional instruction and practice with radical functions.

- Domain of a Radical Function
- Domain of a Radical Function 2
- Finding Domain of a Radical Function

Key Concepts

- **Properties of $\sqrt[n]{a}$**
  - When n is an **even** number and:
    $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
    $a < 0$, then $\sqrt[n]{a}$ is not a real number.
  - When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Domain of a Radical Function**
  - When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
  - When the **index** of the radical is **odd**, the radicand can be any real number.

Glossary

radical function

A radical function is a function that is defined by a radical expression.

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17.8E: Exercises

Practice Makes Perfect

Exercise $17.8E.17$ Evaluate a Radical Function

In the following exercises, evaluate each function.

1. $f(x)=\sqrt{4x-4}$, find
   a. $f(5)$
   b. $f(0)$
2. $f(x)=\sqrt{6x-5}$, find
   a. $f(5)$
   b. $f(-1)$
3. $g(x)=\sqrt{6x+1}$, find
   a. $g(4)$
   b. $g(8)$
4. $g(x)=\sqrt{3x+1}$, find
   a. $g(8)$
   b. $g(5)$
5. $F(x)=\sqrt{3-2x}$, find
   a. $F(1)$
   b. $F(-11)$
6. $F(x)=\sqrt{8-4x}$, find
   a. $F(1)$
   b. $F(-2)$
7. $G(x)=\sqrt{5x-1}$, find
   a. $G(5)$
   b. $G(2)$
8. $G(x)=\sqrt{4x+1}$, find
   a. $G(11)$
   b. $G(2)$
9. $g(x)=\sqrt[3]{2x-4}$, find
   a. $g(6)$
   b. $g(-2)$
10. $g(x)=\sqrt[3]{7x-1}$, find
    a. $g(4)$
    b. $g(-1)$
11. $h(x)=\sqrt[3]{x^2-4}$, find
    a. $h(-2)$
    b. $h(6)$
12. $h(x)=\sqrt[3]{x^2+4}$, find
    a. $h(-2)$
    b. $h(6)$
13. For the function $f(x)=\sqrt[4]{2x^3}$, find
    a. $f(0)$
    b. $f(2)$
14. For the function $f(x)=\sqrt[4]{3x^3}$, find

a. $f(0)$
b. $f(3)$

15. For the function $g(x)=\sqrt[4]{4-4x}$, find
    a. $g(1)$
    b. $g(-3)$
16. For the function $g(x)=\sqrt[4]{8-4x}$, find
    a. $g(-6)$
    b. $g(2)$

Answer

1.
    a. $f(5)=4$
    b. no value at $x=0$
3.
    a. $g(4)=5$
    b. $g(8)=7$
5.
    a. $F(1)=1$
    b. $F(-11)=5$
7.
    a. $G(5)=2\sqrt{6}$
    b. $G(2)=3$
9.
    a. $g(6)=2$
    b. $g(-2)=-2$
11.
    a. $h(-2)=0$
    b. $h(6)=2\sqrt[3]{4}$
13.
    a. $f(0)=0$
    b. $f(2)=2$
15.
    a. $g(1)=0$
    b. $g(-3)=2$

? Exercise $17.8E.18$ Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

1. $f(x)=\sqrt{3x-1}$
2. $f(x)=\sqrt{4x-2}$
3. $g(x)=\sqrt{2-3x}$
4. $g(x)=\sqrt{8-x}$
5. $h(x)=\sqrt{\frac{5}{x-2}}$
6. $h(x)=\sqrt{\frac{6}{x+3}}$

7. $f(x)=\sqrt{\frac{x+3}{x-2}}$
8. $f(x)=\sqrt{\frac{x-1}{x+4}}$
9. $g(x)=\sqrt[3]{8x-1}$
10. $g(x)=\sqrt[3]{6x+5}$
11. $f(x)=\sqrt[3]{4x^2-16}$
12. $f(x)=\sqrt[3]{6x^2-25}$
13. $F(x)=\sqrt[4]{8x+3}$
14. $F(x)=\sqrt[4]{10-7x}$
15. $G(x)=\sqrt[5]{2x-1}$
16. $G(x)=\sqrt[5]{6x-3}$

Answer

1. $\left[\frac{1}{3},\infty\right)$
3. $\left(-\infty,\frac{2}{3}\right]$
5. $(2,\infty)$
7. $(-\infty,-3]\cup(2,\infty)$
9. $(-\infty,\infty)$
11. $(-\infty,\infty)$
13. $\left[-\frac{3}{8},\infty\right)$
15. $(-\infty,\infty)$

? Exercise $17.8E.19$ graph radical functions

In the following exercises,

a. find the domain of the function
b. graph the function
c. use the graph to determine the range

1. $f(x)=\sqrt{x+1}$
2. $f(x)=\sqrt{x-1}$
3. $g(x)=\sqrt{x+4}$
4. $g(x)=\sqrt{x-4}$
5. $f(x)=\sqrt{x}+2$
6. $f(x)=\sqrt{x}-2$
7. $g(x)=2\sqrt{x}$
8. $g(x)=3\sqrt{x}$
9. $f(x)=\sqrt{3-x}$
10. $f(x)=\sqrt{4-x}$
11. $g(x)=-\sqrt{x}$
12. $g(x)=-\sqrt{x}+1$
13. $f(x)=\sqrt[3]{x+1}$
14. $f(x)=\sqrt[3]{x-1}$
15. $g(x)=\sqrt[3]{x+2}$
16. $g(x)=\sqrt[3]{x-2}$
17. $f(x)=\sqrt[3]{x}+3$
18. $f(x)=\sqrt[3]{x}-3$
19. $g(x)=\sqrt[3]{x}$
20. $g(x)=-\sqrt[3]{x}$

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21. $f(x) = 2\sqrt[3]{x}$
22. $f(x) = -2\sqrt[3]{x}$

Answer

1.

a. domain: $[-1, \infty)$

b.

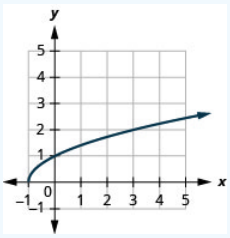

Figure 8.7.8

c. $[0, \infty)$

3.

a. domain: $[-4, \infty)$

b.

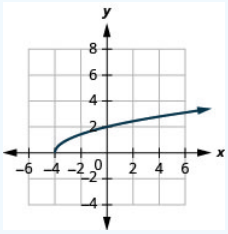

Figure 8.7.9

c. $[0, \infty)$

5.

a. domain: $[0, \infty)$

b.

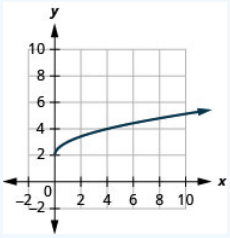

Figure 8.7.10

c. $[2, \infty)$

7.

a. domain: $[0, \infty)$

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b.

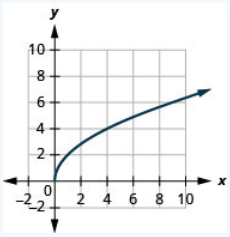

Figure 8.7.11

c. $[0, \infty)$

9.

a. domain: $(-\infty, 3]$

b.

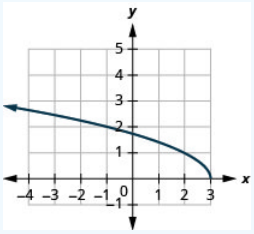

Figure 8.7.12

c. $[0, \infty)$

11.

a. domain: $[0, \infty)$

b.

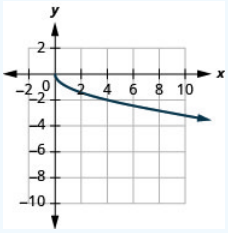

Figure 8.7.13

c. $(-\infty, 0]$

13.

a. domain: $(-\infty, \infty)$

b.

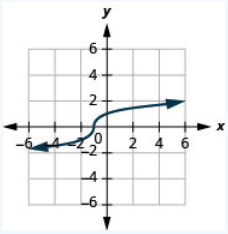

Figure 8.7.14

c. $(-\infty, \infty)$

15.

a. domain: $(-\infty, \infty)$

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b.

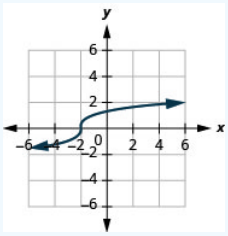

Figure 8.7.15

c. $(-\infty, \infty)$

17.

a. domain: $(-\infty, \infty)$

b.

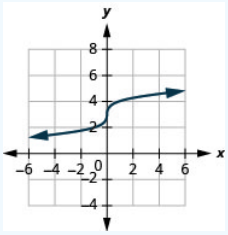

Figure 8.7.16

c. $(-\infty, \infty)$

19.

a. domain: $(-\infty, \infty)$

b.

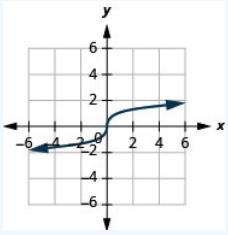

Figure 8.7.17

c. $(-\infty, \infty)$

21.

a. domain: $(-\infty, \infty)$

b.

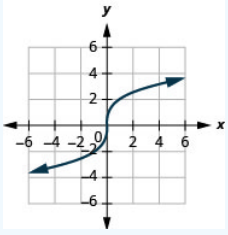

Figure 8.7.18

c. $(-\infty, \infty)$

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Exercise $17.8E.20$ writing exercises

1. Explain how to find the domain of a fourth root function.
2. Explain how to find the domain of a fifth root function.
3. Explain why $y = \sqrt[3]{x}$ is a function.
4. Explain why the process of finding the domain of a radical function with an even index is different from the process when the index is odd.

Answer

1. Answers may vary

3. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate a radical function.			
find the domain of a radical function.			
graph a radical function.			

Figure 8.7.19

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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17.9: Use the Complex Number System

Learning Objectives

By the end of this section, you will be able to:

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of i

Before you get started, take this readiness quiz.

1. Given the numbers $-4, -\sqrt{7}, 0.\overline{5}, \frac{7}{3}, 3, \sqrt{81}$, list the
   a. rational numbers
   b. irrational numbers
   c. real numbers

   If you missed this problem, review Example 1.42.
2. Multiply: $(x-3)(2x+5)$.
   If you missed this problem, review Example 5.28.
3. Rationalize the denominator: $\frac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$
   If you missed this problem, review Example 5.32.

Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify $\sqrt{-1}$, we are looking for a real number x so that $x^2 = -1$. Since all real numbers squared are positive numbers, there is no real number that equals -1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like $\sqrt{5}$. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** i as the number whose square is -1.

Definition 17.9.1

The **imaginary unit** i is the number whose square is -1.

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

We will use the imaginary unit to simplify the square roots of negative numbers.

Definition 17.9.2

Square Root of a Negative Number

If b is a positive real number, then

$$\sqrt{-b} = \sqrt{b}i$$

We will use this definition in the next example. Be careful that it is clear that the i is not under the radical. Sometimes you will see this written as $\sqrt{-b} = i\sqrt{b}$ to emphasize the i is not under the radical. But the $\sqrt{-b} = \sqrt{b}i$ is considered standard form.

Example 17.9.1

Write each expression in terms of i and simplify is possible:

a. $\sqrt{-25}$
b. $\sqrt{-7}$
c. $\sqrt{-12}$

Solution:

a.

$$\sqrt{-25}$$

Use the definition of the square root of negative numbers.

$$\sqrt{25}i$$

Simplify.

$$5i$$

b.

$$\sqrt{-7}$$

Use the definition of the square root of negative numbers.

$$\sqrt{7}i$$

Simplify.

Be careful that it is clear that i is not under the radical sign.

c.

$$\sqrt{-12}$$

Use the definition of the square root of negative numbers.

$$\sqrt{12}i$$

Simplify $\sqrt{12}$.

$$2\sqrt{3}i$$

Exercise 17.9.1

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-81}$
b. $\sqrt{-5}$
c. $\sqrt{-18}$

Answer

a. $9i$
b. $\sqrt{5}i$
c. $3\sqrt{2}i$

Exercise 17.9.2

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-36}$
b. $\sqrt{-3}$
c. $\sqrt{-27}$

Answer

a. $6i$
b. $\sqrt{3}i$
c. $3\sqrt{3}i$

Now that we are familiar with the imaginary number i, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form $a+bi$, where a, b are real numbers. We call a the real part and b the imaginary part.

Definition 17.9.3

A **complex number** is of the form $a+bi$, where a and b are real numbers.

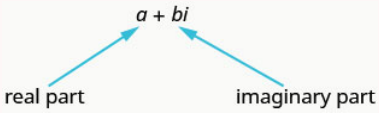

Figure 8.8.1

A complex number is in standard form when written as $a+bi$, where a and b are real numbers.

If $b=0$, then $a+bi$ becomes $a+0\cdot i=a$, and is a real number.

If $b\neq 0$, then $a+bi$ is an imaginary number.

If $a=0$, then $a+bi$ becomes $0+bi=bi$, and is called a pure imaginary number.

We summarize this here.

Table 8.8.1

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$R	$0+bi$ bi	Pure imaginary numbe4

The standard form of a complex number is $a+bi$, so this explains why the preferred form is $\sqrt{-b}=\sqrt{b}i$ when $b>0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.

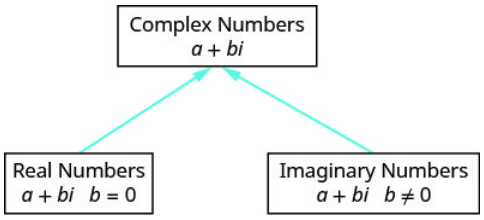

Figure 8.8.2

Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

Example 17.9.2

Add: $\sqrt{-12}+\sqrt{-27}$.

Solution:

$$\sqrt{-12}+\sqrt{-27}$$

Use the definition of the square root of negative numbers.

$$\sqrt{12}i+\sqrt{27}i$$

Simplify the square roots.

$$2\sqrt{3}i+3\sqrt{3}i$$

Add.

$$5\sqrt{3}i$$

Exercise 17.9.3

Add: $\sqrt{-8}+\sqrt{-32}$.

Answer

$6\sqrt{2}i$

Exercise 17.9.4

Add: $\sqrt{-27}+\sqrt{-48}$

Answer

$7\sqrt{3}i$

Remember to add both the real parts and the imaginary parts in this next example.

Example 17.9.3

Simplify:

a. $(4-3i)+(5+6i)$
b. $(2-5i)-(5-2i)$

Solution:

a.

$$(4-3i)+(5+6i)$$

Use the Associative Property to put the real parts and the imaginary parts together.

$$(4+5)+(-3i+6i)$$

Simplify.

$$9+3i$$

b.

$$(2-5i)-(5-2i)$$

Distribute.

$$2-5i-5+2i$$

Use the Associative Property to put the real parts and the imaginary parts together.

$$2-5-5i+2i$$

Simplify.

$$-3-3i$$

Exercise 17.9.5

Simplify:

a. $(2+7i)+(4-2i)$
b. $(8-4i)-(2-i)$

Answer

a. $6+5i$
b. $6-3i$

Exercise 17.9.6

Simplify:

a. $(3-2i)+(-5-4i)$
b. $(4+3i)-(2-6i)$

Answer

a. $-2-6i$
b. $2+9i$

Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

Example 17.9.4

Multiply: $2i(7-5i)$

Solution:

$$2i(7-5i)$$

Distribute.

$$14i-10i^2$$

Simplify i^2.

$$14i-10(-1)$$

Multiply.

$$14i+10$$

Write in standard form.

$$10+14i$$

Exercise 17.9.7

Multiply: $4i(5-3i)$.

Answer

$12+20i$

Exercise 17.9.8

Multiply: $-3i(2+4i)$.

Answer

$12-6i$

In the next example, we multiply the binomials using the **Distributive Property** or **FOIL**.

Example 17.9.5

Multiply: $(3+2i)(4-3i)$.

Solution:

$$(3+2i)(4-3i)$$

Use FOIL.

$$12-9i+8i-6i^2$$

Simplify i^2 and combine like terms.

$$12-i-6(-1)$$

Multiply.

$$12-i+6$$

Combine the real parts.

$$18-i$$

Exercise 17.9.9

Multiple: $(5-3i)(-1-2i)$.

Answer

$-11-7i$

Exercise 17.9.10

Multiple: $(-4-3i)(2+i)$.

Answer

$-5-10i$

In the next example, we could use FOIL or the **Product of Binomial Squares Pattern**.

Example 17.9.6

Multiply: $(3+2i)^2$

Solution:

Table 8.8.2

	$(a+b)^2$ $3+2i$
Use the Product of Binomial Squares Pattern, $(a+b)^2=a^2+2ab+b^2$.	$a^2+2 \quad a \quad b+ \quad b^2$ $3^2+2 \cdot 3 \cdot 2i+(2i)^2$
Simplify.	$9+12i+4i^2$
Simplify i^2.	$9+12i+4(-1)$
Simplify.	$5+12i$

Exercise 17.9.11

Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

Answer

$-21+20i$

Exercise 17.9.12

Multiply using the Binomial Squares pattern: $(-5+4i)^2$.

Answer

$9-40i$

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $\sqrt{-b}=\sqrt{b}i$.This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

Example 17.9.7

Multiply: $\sqrt{-36}\cdot\sqrt{-4}$.

Solution:

To multiply square roots of negative numbers, we first write them as complex numbers.

$$\sqrt{-36}\cdot\sqrt{-4}$$

Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.

$$\sqrt{36}i\cdot\sqrt{4}i$$

Simplify.

$$6i\cdot 2i$$

Multiply.

$$12i^2$$

Simplify i^2 and multiply.

$$-12$$

Exercise 17.9.13

Multiply: $\sqrt{-49} \cdot \sqrt{-4}$.

Answer

-14

Exercise 17.9.14

Multiply: $\sqrt{-36} \cdot \sqrt{-81}$.

Answer

-54

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

Example 17.9.8

Multiply: $(3-\sqrt{-12})(5+\sqrt{-27})$.

Solution:

To multiply square roots of negative numbers, we first write them as complex numbers.

$$(3-\sqrt{-12})(5+\sqrt{-27})$$

Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.

$$(3-2\sqrt{3}i)(5+3\sqrt{3}i)$$

Use FOIL.

$$15+9\sqrt{3}i-10\sqrt{3}i-6\cdot 3i^2$$

Combine like terms and simplify i^2.

$$15-\sqrt{3}i-6\cdot(-3)$$

Multiply and combine like terms.

$$33-\sqrt{3}i$$

Exercise 17.9.15

Multiply: $(4-\sqrt{-12})(3-\sqrt{-48})$.

Answer

$-12-22\sqrt{3}i$

Exercise 17.9.16

Multiply: $(-2+\sqrt{-8})(3-\sqrt{-18})$.

Answer

$6+12\sqrt{2}i$

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a-b),(a+b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a+bi$, its conjugate is $a-bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

Definition 17.9.4

A **complex conjugate pair** is of the form $a+bi, a-bi$.

We will multiply a complex conjugate pair in the next example.

Example 17.9.9

Multiply: $(3-2i)(3+2i)$.

Solution:

$$(3-2i)(3+2i)$$

Use FOIL

$$9+6i-6i-4i^2$$

Combine like terms and simplify i^2.

$$9-4(-1)$$

Multiply and combine like terms.

$$13$$

Exercise 17.9.17

Multiply: $(4-3i)\cdot(4+3i)$.

Answer

25

Exercise 17.9.18

Multiply: $(-2+5i)\cdot(-2-5i)$.

Answer

29

From our study of polynomials, we know the product of conjugates is always of the form $(a-b)(a+b)=a^2-b^2$ .The result is called a **difference of squares**. We can multiply a complex conjugate pair using this pattern.

The last example we used FOIL. Now we will use the **Product of Conjugates Pattern**.

$$(a-b)(a+b)$$

$$(3-2i)(3+2i)$$

$$a^2-b^2$$

$$(3)^2-(2i)^2$$

$$9-4i^2$$

$$9-4(-1)$$

$$13$$

Figure 8.8.8

Notice this is the same result we found in Example 8.8.9.

When we multiply complex conjugates, the product of the last terms will always have an i^2 which simplifies to -1.

$$(a-bi)(a+bi)$$
$$a^2-(bi)^2$$
$$a^2-b^2i^2$$
$$a^2-b^2(-1)$$
$$a^2+b^2$$

This leads us to the Product of Complex Conjugates Pattern: $(a-bi)(a+bi)=a^2+b^2$

Definition 17.9.5

Product of Complex Conjugates

If a and b are real numbers, then

$$(a-bi)(a+bi)=a^2+b^2$$

Example 17.9.10

Multiply using the Product of Complex Conjugates Pattern: $(8-2i)(8+2i)$.

Solution:

Table 8.8.3

	$(a-b)(a+b)$ $(8-2i)(8+2i)$
Use the Product of Complex Conjugates Pattern, $(a-bi)(a+bi)=a^2+b^2$.	a^2+b^2 8^2+2^2
Simplify the squares.	$64+4$
Add.	68

Exercise 17.9.19

Multiply using the Product of Complex Conjugates Pattern: $(3-10i)(3+10i)$.

Answer

109

Exercise 17.9.20

Multiply using the Product of Complex Conjugates Pattern: $(-5+4i)(-5-4i)$.

Answer

41

Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

Example 17.9.11 how to divide complex numbers

Divide: $\frac{4+3i}{3-4i}$.

Solution:

Table 8.8.4

Step 1: Write both the numerator and denominator in standard form.	They are both in standard form.	$\frac{4+3i}{3-4i}$
Step 2: Multiply the numerator and denominator by the complex conjugate of the denominator.	The complex conjugate of $3-4i$ is $3+4i$.	$\frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$
Step 3: Simplify and write the result in standard form.	Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator. Combine like terms. Simplify. Write the result in standard form.	$\frac{12+16i+9i+12i^2}{9+16}$ $\frac{12+25i-12}{25}$ $\frac{25i}{25}$ i

Exercise 17.9.21

Divide: $\frac{2+5i}{5-2i}$.

Answer

i

Exercise 17.9.22

Divide: $\frac{1+6i}{6-i}$.

Answer

i

We summarize the steps here.

How to Divide Complex Numbers

1. Write both the numerator and denominator in standard form.
2. Multiply both the numerator and denominator by the complex conjugate of the denominator.
3. Simplify and write the result in standard form.

Example 17.9.12

Divide, writing the answers in standard form: $\frac{-3}{5+2i}$.

Solution:

$$\frac{-3}{5+2i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{-3(5-2i)}{(5+2i)(5-2i)}$$

Multiply in the numerator and use the Product of Complex Conjugates Pattern in the denominator.

$$\frac{-15+6i}{5^2+2^2}$$

Simplify.

$$\frac{-15+6i}{29}$$

Write in standard form.

$$-\frac{15}{29}+\frac{6}{29}i$$

Exercise 17.9.23

Divide, writing the answer in standard form: $\frac{4}{1-4i}$.

Answer

$\frac{4}{17}+\frac{16}{17}i$

Exercise 17.9.24

Divide, writing the answer in standard form: $\frac{-2}{-1+2i}$.

Answer

$\frac{2}{5}+\frac{4}{5}i$

Be careful as you find the conjugate of the denominator.

Example 17.9.13

Divide: $\frac{5+3i}{4i}$.

Solution:

$$\frac{5+3i}{4i}$$

Write the denominator in standard form.

$$\frac{5+3i}{0+4i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(5+3i)(0-4i)}{(0+4i)(0-4i)}$$

Simplify.

$$\frac{(5+3i)(-4i)}{(4i)(-4i)}$$

Multiply.

$$\frac{-20i-12i^2}{-16i^2}.$$

Simplify the i^2.

$$\frac{-20i+12}{16}$$

Rewrite in standard form.

$$\frac{12}{16}-\frac{20}{16}i$$

Simplify the fractions.

$$\frac{3}{4}-\frac{5}{4}i$$

Exercise 17.9.25

Divide: $\frac{3+3i}{2i}$.

Answer

$\frac{3}{2}-\frac{3}{2}i$

Exercise 17.9.26

Divide: $\frac{2+4i}{5i}$.

Answer

$\frac{4}{5}-\frac{2}{5}i$

Simplify Powers of i

The powers of i make an interesting pattern that will help us simplify higher powers of i. Let's evaluate the powers of i to see the pattern.

$$\begin{array}{cccc} i^1 & i^2 & i^3 & i^4 \\ i & -1 & i^2\cdot i & i^2\cdot i^2 \\ & & -1\cdot i & (-1)(-1) \\ & & -i & 1 \end{array}$$

$$\begin{array}{cccc} i^5 & i^6 & i^7 & i^8 \\ i^4\cdot i & i^4\cdot i^2 & i^4\cdot i^3 & i^4\cdot i^4 \\ 1\cdot i & 1\cdot i^2 & 1\cdot i^3 & 1\cdot 1 \\ i & i^2 & i^3 & 1 \\ & -1 & -i & \end{array}$$

We summarize this now.

$$\begin{array}{ll} i^1=i & i^5=i \\ i^2=-1 & i^6=-1 \\ i^3=-i & i^7=-i \\ i^4=1 & i^8=1 \end{array}$$

If we continued, the pattern would keep repeating in blocks of four. We can use this pattern to help us simplify powers of i. Since $i^4=1$, we rewrite each power, i^n, as a product using i^4 to a power and another power of i.

We rewrite it in the form $i^n=\left(i^4\right)^q\cdot i^r$, where the exponent, q, is the quotient of n divided by 4 and the exponent, r, is the remainder from this division. For example, to simplify i^{57}, we divide 57 by 4 and we get 14 with a remainder of 1. In other words, $57=4\cdot 14+1$. So we write $i^{57}=\left(1^4\right)^{14}\cdot i^1$ and then simplify from there.

$$\begin{array}{r} 14 \\ 4\overline{)57} \\ \underline{4} \\ 17 \\ \underline{16} \\ 1 \end{array} \qquad \begin{array}{l} i^{57} \\ \left(1^4\right)^{14}\cdot i^1 \\ 1\cdot i \\ i \end{array}$$

Figure 8.8.13

Example 17.9.14

Simplify: i^{86}.

Solution:

$$i^{86}$$

Divide 86 by 4 and rewrite i^{86} in the $i^n=\left(i^4\right)^q\cdot i^r$ form.

$$\left(1^4\right)^{21}\cdot i^2$$

$$\begin{array}{rl} 21 & \text{(divide the exponent by the number of repeats in the imaginary number i)} \\ 4\overline{)86} & \\ \underline{8} & \text{(four goes into eight twice with zero remainders)} \\ 06 & \\ \underline{4} & \text{(four goes into 6 one time with 2 remainders)} \\ 2 & \text{(remainder is 2 so calculate the imaginary number with an exponent of two)} \end{array} \tag{17.9.1}$$

Simplify.

$$(1)^{21} \cdot (-1)$$

Simplify.

$$-1$$

? Exercise 17.9.27

Simplify: i^{74}.

Answer

-1

? Exercise 17.9.28

Simplify: i^{92}.

Answer

1

Access these online resources for additional instruction and practice with the complex number system.

- Expressing Square Roots of Negative Numbers with i
- Subtract and Multiply Complex Numbers
- Dividing Complex Numbers
- Rewriting Powers of i

Key Concepts

- **Square Root of a Negative Number**
  - If b is a positive real number, then \(\sqrt{-b}=\sqrt{b} i\

Table 8.8.1

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$	$0+bi$ bi	Pure imaginary number

- A complex number is in **standard form** when written as $a + bi$, where a, b are real numbers.

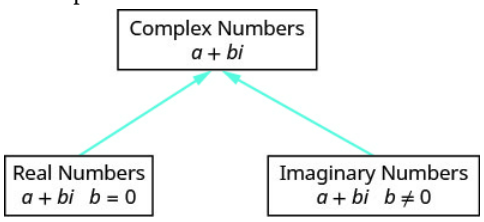

Figure 8.8.2

- **Product of Complex Conjugates**
  - If a, b are real numbers, then $(a-bi)(a+bi) = a^2+b^2$
- **How to Divide Complex Numbers**
  1. Write both the numerator and denominator in standard form.
  2. Multiply the numerator and denominator by the complex conjugate of the denominator.
  3. Simplify and write the result in standard form.

Glossary

complex conjugate pair

A complex conjugate pair is of the form $a+bi, a-bi$.

complex number

A complex number is of the form $a+bi$, where a and b are real numbers. We call a the real part and b the imaginary part.

complex number system

The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit

The imaginary unit i is the number whose square is -1. $i^2 = -1$ or $i = \sqrt{-1}$.

standard form

A complex number is in standard form when written as $a+bi$, where a, b are real numbers.

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17.9E: Exercises

Practice Makes Perfect

Exercises 1 - 4: Evaluate a Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

1. a. $\sqrt{-16}$ b. $\sqrt{-11}$ c. $\sqrt{-8}$
2. a. $\sqrt{-121}$ b. $\sqrt{-1}$ c. $\sqrt{-20}$
3. a. $\sqrt{-100}$ b. $\sqrt{-13}$ c. $\sqrt{-45}$
4. a. $\sqrt{-49}$ b. $\sqrt{-15}$ c. $\sqrt{-75}$

Answer

1. a. $4i$ b. $i\sqrt{11}$ c. $2i\sqrt{2}$
3. a. $10i$ b. $i\sqrt{13}$ c. $3i\sqrt{5}$

ExerciseS 5 - 21: Add or Subtract Complex Numbers

In the following exercises, add or subtract, putting the answer in $a+bi$ form.

5. $\sqrt{-75}+\sqrt{-48}$
6. $\sqrt{-12}+\sqrt{-75}$
7. $\sqrt{-50}+\sqrt{-18}$
8. $\sqrt{-72}+\sqrt{-8}$
9. $(1+3i)+(7+4i)$
10. $(6+2i)+(3-4i)$
11. $(8-i)+(6+3i)$
12. $(7-4i)+(-2-6i)$
13. $(1-4i)-(3-6i)$
14. $(8-4i)-(3+7i)$
15. $(6+i)-(-2-4i)$
16. $(-2+5i)-(-5+6i)$
17. $(5-\sqrt{-36})+(2-\sqrt{-49})$
18. $(-3+\sqrt{-64})+(5-\sqrt{-16})$
19. $(-7-\sqrt{-50})-(-32-\sqrt{-18})$
20. $(-5+\sqrt{-27})-(-4-\sqrt{-48})$

Answer

5. $0+\left(9\sqrt{3}\right)i$
7. $0+\left(8\sqrt{2}\right)i$
9. $8+7i$
11. $14+2i$
13. $-2+2i$
15. $8+5i$

17. $7-13i$

19. $25-\left(2\sqrt{2}\right)i$

? ExerciseS 21 - 28: Multiply Complex Numbers

In the following exercises, multiply, putting the answer in $a+bi$ form.

21. $4i(5-3i)$

22. $2i(-3+4i)$

23. $-6i(-3-2i)$

24. $-i(6+5i)$

25. $(4+3i)(-5+6i)$

26. $(-2-5i)(-4+3i)$

27. $(-3+3i)(-2-7i)$

28. $(-6-2i)(-3-5i)$

Answer

21. $12+20i$

23. $-12+18i$

25. $-38+9i$

27. $27+15i$

? ExerciseS 29 - 32: Multiply Complex Numbers

In the following exercises, multiply using the Product of Binomial Squares Pattern, putting the answer in $a+bi$ form.

29. $(3+4i)^2$

30. $(-1+5i)^2$

31. $(-2-3i)^2$

32. $(-6-5i)^2$

Answer

29. $-7+24i$

31. $-5-12i$

? Exercises 33 - 46: Multiply Complex Numbers

In the following exercises, multiply, putting the answer in $a+bi$ form.

33. $\sqrt{-25}\cdot\sqrt{-36}$

34. $\sqrt{-4}\cdot\sqrt{-16}$

35. $\sqrt{-9}\cdot\sqrt{-100}$

36. $\sqrt{-64}\cdot\sqrt{-9}$

37. $(-2-\sqrt{-27})(4-\sqrt{-48})$

38. $(5-\sqrt{-12})(-3+\sqrt{-75})$

39. $(2+\sqrt{-8})(-4+\sqrt{-18})$

40. $(5+\sqrt{-18})(-2-\sqrt{-50})$

41. $(2-i)(2+i)$

42. $(4-5i)(4+5i)$

43. $(7-2i)(7+2i)$

44. $(-3-8i)(-3+8i)$

Answer

33. $30i = 0+30i$

35. $-30 = -30+0i$

37. $-44+\left(4\sqrt{3}\right)i$

39. $-20-\left(2\sqrt{2}\right)i$

41. $5 = 5+0i$

43. $53 = 53+0i$

? ExerciseS 45 - 49: Multiply Complex Numbers

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

45. $(7-i)(7+i)$

46. $(6-5i)(6+5i)$

47. $(9-2i)(9+2i)$

48. $(-3-4i)(-3+4i)$

Answer

45. 50

47. 85

? ExerciseS 49 - 60: Divide Complex Numbers

In the following exercises, divide, putting the answer in $a+bi$ form.

49. $\dfrac{3+4i}{4-3i}$

50. $\dfrac{5-2i}{2+5i}$

51. $\dfrac{2+i}{3-4i}$

52. $\dfrac{3-2i}{6+i}$

53. $\dfrac{3}{2-3i}$

54. $\dfrac{2}{4-5i}$

55. $\dfrac{-4}{3-2i}$

56. $\dfrac{-1}{3+2i}$

57. $\dfrac{1+4i}{3i}$

58. $\dfrac{4+3i}{7i}$

59. $\dfrac{-2-3i}{4i}$

60. $\dfrac{-3-5i}{2i}$

Answer

49. $i=0+i$

51. $\frac{2}{25}+\frac{11}{25}i$

53. $\frac{6}{13}+\frac{9}{13}i$

55. $-\frac{12}{13}-\frac{8}{13}i$

57. $\frac{4}{3}-\frac{1}{3}i$

59. $-\frac{3}{4}+\frac{1}{2}i$

ExerciseS 61 - 68: Simplify Powers of i

In the following exercises, simplify.

61. i^{41}

62. i^{39}

63. i^{66}

64. i^{48}

65. i^{128}

66. i^{162}

67. i^{137}

68. i^{255}

Answer

61. $i^{41}=i^{40}\cdot i=\left(i^4\right)^{10}\cdot i=i$

63. $i^{66}=i^{64}\cdot i^2=\left(i^4\right)^{16}\cdot(-1)=-1$

65. $i^{128}=\left(i^4\right)^{32}=1$

67. $i^{137}=i^{136}\cdot i=\left(i^4\right)^{34}\cdot i=1\cdot i=i$

ExerciseS 69 - 72: Writing Exercises

69. Explain the relationship between real numbers and complex numbers.

70. Aniket multiplied as follows and he got the wrong answer. What is wrong with his reasoning?

$\sqrt{-7}\cdot\sqrt{-7}$

$\sqrt{49}$

7

71. Why is $\sqrt{-64} = 8i$ but $\sqrt[3]{-64} = -4$.

72. Explain how dividing complex numbers is similar to rationalizing a denominator.

Answer

69. Answers may vary

71. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate the square root of a negative number.			
add or subtract complex numbers.			
multiply complex numbers.			
divide complex numbers.			
simplify powers of *i*.			

Figure 8.8.15

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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17.10: Review Exercises

Chapter Review Exercises

Simplify Expressions with Roots

Exercise 17.10.1 Simplify Expressions with Roots

In the following exercises, simplify.

1. a. $\sqrt{225}$
   b. $-\sqrt{16}$
2. a. $-\sqrt{169}$
   b. $\sqrt{-8}$
3. a. $\sqrt[3]{8}$
   b. $\sqrt[4]{81}$
   c. $\sqrt[5]{243}$
4. a. $\sqrt[3]{-512}$
   b. $\sqrt[4]{-81}$
   c. $\sqrt[5]{-1}$

Answer

1.
   a. 15
   b. -4
3.
   a. 2
   b. 3
   c. 3

Exercise 17.10.2 Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

1. a. $\sqrt{68}$
   b. $\sqrt[3]{84}$

Answer

1.
   a. $8 < \sqrt{68} < 9$
   b. $4 < \sqrt[3]{84} < 5$

Exercise 17.10.3 Estimate and Approximate Roots

In the following exercises, approximate each root and round to two decimal places.

1. a. $\sqrt{37}$
   b. $\sqrt[3]{84}$
   c. $\sqrt[4]{125}$

Answer

1. Solve for yourself

Exercise 17.10.4 Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

1. a. $\sqrt[3]{a^3}$
   b. $\sqrt[7]{b^7}$
2. a. $\sqrt{a^{14}}$
   b. $\sqrt{w^{24}}$
3. a. $\sqrt[4]{m^8}$
   b. $\sqrt[5]{n^{20}}$
4. a. $\sqrt{121m^{20}}$
   b. $-\sqrt{64a^2}$
5. a. $\sqrt[3]{216a^6}$
   b. $\sqrt[5]{32b^{20}}$
6. a. $\sqrt{144x^2y^2}$
   b. $\sqrt{169w^8y^{10}}$
   c. $\sqrt[3]{8a^{51}b^6}$

Answer

1.
   a. a
   b. $|b|$
3.
   a. m^2
   b. n^4
5.
   a. $6a^2$
   b. $2b^4$

Simplify Radical Expressions

Exercise 17.10.5 Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

1. $\sqrt{125}$
2. $\sqrt{675}$
3. a. $\sqrt[3]{625}$
   b. $\sqrt[6]{128}$

Answer

1. $5\sqrt{5}$
3.
   a. $5\sqrt[3]{5}$
   b. $2\sqrt[6]{2}$

Exercise 17.10.6 Use the Product Property to Simplify Radical Expressions

In the following exercises, simplify using absolute value signs as needed.

1. a. $\sqrt{a^{23}}$
   b. $\sqrt[3]{b^8}$
   c. $\sqrt[8]{c^{13}}$
2. a. $\sqrt{80s^{15}}$
   b. $\sqrt[5]{96a^7}$
   c. $\sqrt[6]{128b^7}$
3. a. $\sqrt{96r^3s^3}$
   b. $\sqrt[3]{80x^7y^6}$
   c. $\sqrt[4]{80x^8y^9}$
4. a. $\sqrt[5]{-32}$
   b. $\sqrt[8]{-1}$
5. a. $8+\sqrt{96}$
   b. $\frac{2+\sqrt{40}}{2}$

Answer

2.

a. $4\left|s^7\right|\sqrt{5s}$
b. $2a\sqrt[5]{3a^2}$
c. $2|b|\sqrt[6]{2b}$

4.

a. -2
b. not real

Exercise 17.10.7 Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

1. a. $\sqrt{\frac{72}{98}}$
   b. $\sqrt[3]{\frac{24}{81}}$
   c. $\sqrt[4]{\frac{6}{96}}$
2. a. $\sqrt{\frac{y^4}{y^8}}$
   b. $\sqrt[5]{\frac{u^{21}}{u^{11}}}$
   c. $\sqrt[6]{\frac{v^{30}}{v^{12}}}$
3. $\sqrt{\frac{300m^5}{64}}$
4. a. $\sqrt{\frac{28p^7}{q^2}}$
   b. $\sqrt[3]{\frac{81s^8}{t^3}}$
   c. $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$
5. a. $\sqrt{\frac{27p^2q}{108p^4q^3}}$

b. $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$

c. $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

6. a. $\frac{\sqrt{80q^5}}{\sqrt{5q}}$

b. $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

c. $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

Answer

1.

a. $\frac{6}{7}$

b. $\frac{2}{3}$

c. $\frac{1}{2}$

3. $\frac{10m^2\sqrt{3m}}{8}$

5.

a. $\frac{1}{2|pq|}$

b. $\frac{2cd\sqrt[5]{2d^2}}{5}$

c. $\frac{|mn|\sqrt[6]{2}}{2}$

Simplify Rational Exponents

Exercise 17.10.8 Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

1. a. $r^{\frac{1}{2}}$

b. $s^{\frac{1}{3}}$

c. $t^{\frac{1}{4}}$

Answer

1.

a. $\sqrt{r}$

b. $\sqrt[3]{s}$

c. $\sqrt[4]{t}$

Exercise 17.10.9 Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt{21p}$

b. $\sqrt[4]{8q}$

c. $4\sqrt[6]{36r}$

Answer

1. Solve for yourself

Exercise 17.10.10 Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, simplify.

1. a. $625^{\frac{1}{4}}$
   b. $243^{\frac{1}{5}}$
   c. $32^{\frac{1}{5}}$
2. a. $(-1,000)^{\frac{1}{3}}$
   b. $-1,000^{\frac{1}{3}}$
   c. $(1,000)^{-\frac{1}{3}}$
3. a. $(-32)^{\frac{1}{5}}$
   b. $(243)^{-\frac{1}{5}}$
   c. $-125^{\frac{1}{3}}$

Answer

1.
   a. 5
   b. 3
   c. 2
3.
   a. -2
   b. $\frac{1}{3}$
   c. -5

Exercise 17.10.11 Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt[4]{r^7}$
   b. $(\sqrt[5]{2pq})^3$
   c. $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

Answer

1. Solve for yourself

Exercise 17.10.12 Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, simplify.

1. a. $25^{\frac{3}{2}}$
   b. $9^{-\frac{3}{2}}$
   c. $(-64)^{\frac{2}{3}}$
2. a. $-64^{\frac{3}{2}}$
   b. $-64^{-\frac{3}{2}}$
   c. $(-64)^{\frac{3}{2}}$

Answer

1.

a. 125
b. $\frac{1}{27}$
c. 16

Exercise 17.10.13 Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

1. a. $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$
   b. $\left(b^{15}\right)^{\frac{3}{5}}$
   c. $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$
2. a. $\frac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$
   b. $\left(\frac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

Answer

1.
a. 6^3
b. b^9
c. $\frac{1}{w}$

Add, Subtract and Multiply Radical Expressions

Exercise 17.10.14 add and Subtract Radical Expressions

In the following exercises, simplify.

1. a. $7\sqrt{2}-3\sqrt{2}$
   b. $7\sqrt[3]{p}+2\sqrt[3]{p}$
   c. $5\sqrt[3]{x}-3\sqrt[3]{x}$
2. a. $\sqrt{11b}-5\sqrt{11b}+3\sqrt{11b}$
   b. $8\sqrt[4]{11cd}+5\sqrt[4]{11cd}-9\sqrt[4]{11cd}$
3. a. $\sqrt{48}+\sqrt{27}$
   b. $\sqrt[3]{54}+\sqrt[3]{128}$
   c. $6\sqrt[4]{5}-\frac{3}{2}\sqrt[4]{320}$
4. a. $\sqrt{80c^7}-\sqrt{20c^7}$
   b. $2\sqrt[4]{162r^{10}}+4\sqrt[4]{32r^{10}}$
5. $3\sqrt{75y^2}+8y\sqrt{48}-\sqrt{300y^2}$

Answer

1.
a. $4\sqrt{2}$
b. $9\sqrt[3]{p}$
c. $2\sqrt[3]{x}$

3.
a. $7\sqrt{3}$
b. $7\sqrt[3]{2}$

c. $3\sqrt[4]{5}$

5. $37y\sqrt{3}$

Exercise 17.10.15 Multiply Radical Expressions

In the following exercises, simplify.

1. a. $(5\sqrt{6})(-\sqrt{12})$
   b. $(-2\sqrt[4]{18})(-\sqrt[4]{9})$
2. a. $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$
   b. $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

Answer

2.

a. $126x^2\sqrt{2}$
b. $48a\sqrt[3]{a^2}$

Exercise 17.10.16 Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

1. a. $\sqrt{11}(8+4\sqrt{11})$
   b. $\sqrt[3]{3}(\sqrt[3]{9}+\sqrt[3]{18})$
2. a. $(3-2\sqrt{7})(5-4\sqrt{7})$
   b. $(\sqrt[3]{x}-5)(\sqrt[3]{x}-3)$
3. $(2\sqrt{7}-5\sqrt{11})(4\sqrt{7}+9\sqrt{11})$
4. a. $(4+\sqrt{11})^2$
   b. $(3-2\sqrt{5})^2$
5. $(7+\sqrt{10})(7-\sqrt{10})$
6. $(\sqrt[3]{3x}+2)(\sqrt[3]{3x}-2)$

Answer

2.

a. $71-22\sqrt{7}$
b. $\sqrt[3]{x^2}-8\sqrt[3]{x}+15$

4.

a. $27+8\sqrt{11}$
b. $29-12\sqrt{5}$

6. $\sqrt[3]{9x^2}-4$

Divide Radical Expressions

Exercise 17.10.17 Divide Square Roots

In the following exercises, simplify.

1. a. $\frac{\sqrt{48}}{\sqrt{75}}$
   b. $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$
2. a. $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$

b. $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

Answer

2.

a. $\frac{8m^4}{3n^4}$

b. $-\frac{x^2}{2y^2}$

Exercise 17.10.18 rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

1. a. $\frac{8}{\sqrt{3}}$
   b. $\sqrt{\frac{7}{40}}$
   c. $\frac{8}{\sqrt{2y}}$
2. a. $\frac{1}{\sqrt[3]{11}}$
   b. $\sqrt[3]{\frac{7}{54}}$
   c. $\frac{3}{\sqrt[3]{3x^2}}$
3. a. $\frac{1}{\sqrt[4]{4}}$
   b. $\sqrt[4]{\frac{9}{32}}$
   c. $\frac{6}{\sqrt[4]{9x^3}}$

Answer

2.

a. $\frac{\sqrt[3]{121}}{11}$

b. $\frac{\sqrt[3]{28}}{6}$

c. $\frac{\sqrt[3]{9x}}{x}$

Exercise 17.10.19 Rationalize a Two Term Denominator

In the following exercises, simplify.

1. $\frac{7}{2-\sqrt{6}}$
2. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$
3. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

Answer

1. $-\frac{7(2+\sqrt{6})}{2}$
3. $\frac{(\sqrt{x}+2\sqrt{2})^2}{x-8}$

Solve Radical Equations

Exercise 17.10.20 Solve Radical Equations

In the following exercises, solve.

1. $\sqrt{4x-3}=7$
2. $\sqrt{5x+1}=-3$
3. $\sqrt[3]{4x-1}=3$
4. $\sqrt{u-3}+3=u$
5. $\sqrt[3]{4x+5}-2=-5$
6. $(8x+5)^{\frac{1}{3}}+2=-1$
7. $\sqrt{y+4}-y+2=0$
8. $2\sqrt{8r+1}-8=2$

Answer

2. no solution

4. $u=3, u=4$

6. $x=-4$

8. $r=3$

Exercise 17.10.21 Solve Radical Equations with Two Radicals

In the following exercises, solve.

1. $\sqrt{10+2c}=\sqrt{4c+16}$
2. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$
3. $\sqrt{r}+6=\sqrt{r+8}$
4. $\sqrt{x+1}-\sqrt{x-2}=1$

Answer

2. $x=-8, x=2$

4. $x=3$

Exercise 17.10.22 Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

1. **Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answers to th nearest tenth of a foot.
2. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

2. 64.8 feet

Use Radicals in Functions

Exercise 17.10.23 Evaluate a Radical Function

In the following exercises, evaluate each function.

1. $g(x) = \sqrt{6x+1}$, find
   a. $g(4)$
   b. $g(8)$
2. $G(x) = \sqrt{5x-1}$, find
   a. $G(5)$
   b. $G(2)$
3. $h(x) = \sqrt[3]{x^2-4}$, find
   a. $h(-2)$
   b. $h(6)$
4. For the function $g(x) = \sqrt[4]{4-4x}$, find
   a. $g(1)$
   b. $g(-3)$

Answer

2.

a. $G(5) = 2\sqrt{6}$
b. $G(2) = 3$

4.

a. $g(1) = 0$
b. $g(-3) = 2$

Exercise 17.10.24 Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

1. $g(x) = \sqrt{2-3x}$
2. $F(x) = \sqrt{\frac{x+3}{x-2}}$
3. $f(x) = \sqrt[3]{4x^2-16}$
4. $F(x) = \sqrt[4]{10-7x}$

Answer

2. $(2, \infty)$
4. $\left[\frac{7}{10}, \infty\right)$

Exercise 17.10.25 graph Radical Functions

In the following exercises,

a. find the domain of the function
b. graph the function
c. use the graph to determine the range

1. $g(x) = \sqrt{x+4}$
2. $g(x) = 2\sqrt{x}$
3. $f(x) = \sqrt[3]{x-1}$
4. $f(x) = \sqrt[3]{x}+3$

Answer

2.

a. domain: $[0, \infty)$

b.

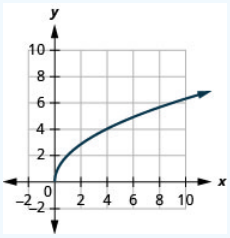

Figure 8.E.1

c. range: $[0, \infty)$

4.

a. domain: $(-\infty, \infty)$

b.

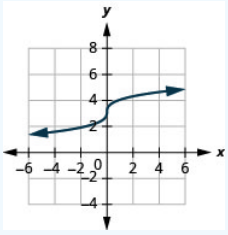

Figure 8.E.2

c. range: $(-\infty, \infty)$

Use the Complex Number System

Exercise 17.10.26 evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

1. a. $\sqrt{-100}$
   b. $\sqrt{-13}$
   c. $\sqrt{-45}$

Answer

Solve for yourself

Exercise 17.10.27 Add or Subtract Complex Numbers

In the following exercises, add or subtract.

1. $\sqrt{-50} + \sqrt{-18}$
2. $(8 - i) + (6 + 3i)$
3. $(6 + i) - (-2 - 4i)$
4. $(-7 - \sqrt{-50}) - (-32 - \sqrt{-18})$

Answer

1. $8\sqrt{2}i$
3. $8 + 5i$

Exercise 17.10.28 Multiply Complex Numbers

In the following exercises, multiply.

1. $(-2-5i)(-4+3i)$
2. $-6i(-3-2i)$
3. $\sqrt{-4}\cdot\sqrt{-16}$
4. $(5-\sqrt{-12})(-3+\sqrt{-75})$

Answer

1. $23+14i$
3. -6

Exercise 17.10.29 Multiply Complex Numbers

In the following exercises, multiply using the Product of Binomial Squares Pattern.

1. $(-2-3i)^2$

Answer

1. $-5-12i$

Exercise 17.10.30 Multiply Complex Numbers

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

1. $(9-2i)(9+2i)$

Answer

Solve for yourself

Exercise 17.10.31 divide Complex Numbers

In the following exercises, divide.

1. $\frac{2+i}{3-4i}$
2. $\frac{-4}{3-2i}$

Answer

1. $\frac{2}{25}+\frac{11}{25}i$

Exercise 17.10.32 Simplify Powers of i

In the following exercises, simplify.

1. i^{48}
2. i^{255}

Answer

1. 1

Exercise 17.10.33

In the following exercises, simplify using absolute values as necessary.

1. $\sqrt[3]{125x^9}$
2. $\sqrt{169x^8y^6}$
3. $\sqrt[3]{72x^8y^4}$
4. $\sqrt{\frac{45x^3y^4}{180x^5y^2}}$

Answer

1. $5x^3$
3. $2x^2y\sqrt[3]{9x^2y}$

Exercise 17.10.34

In the following exercises, simplify. Assume all variables are positive.

1. a. $216^{-\frac{1}{4}}$
   b. $-49^{\frac{3}{2}}$
2. $\sqrt{-45}$
3. $\frac{x^{-\frac{1}{4}}\cdot x^{\frac{5}{4}}}{x^{-\frac{3}{4}}}$
4. $\left(\frac{8x^{\frac{2}{3}}y^{-\frac{5}{2}}}{x^{-\frac{7}{3}}y^{\frac{1}{2}}}\right)^{\frac{1}{3}}$
5. $\sqrt{48x^5}-\sqrt{75x^5}$
6. $\sqrt{27x^2}-4x\sqrt{12}+\sqrt{108x^2}$
7. $2\sqrt{12x^5}\cdot 3\sqrt{6x^3}$
8. $\sqrt[3]{4}(\sqrt[3]{16}-\sqrt[3]{6})$
9. $(4-3\sqrt{3})(5+2\sqrt{3})$
10. $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$
11. $\frac{\sqrt{245xy^{-4}}}{\sqrt{45x^4y^3}}$
12. $\frac{1}{\sqrt[3]{5}}$
13. $\frac{3}{2+\sqrt{3}}$
14. $\sqrt{-4}\cdot\sqrt{-9}$
15. $-4i(-2-3i)$
16. $\frac{4+i}{3-2i}$
17. i^{172}

Answer

1.
   a. $\frac{1}{4}$
   b. -343
3. $x^{\frac{7}{4}}$
5. $-x^2\sqrt{3x}$
7. $36x^4\sqrt{2}$

9. $2-7\sqrt{3}$

11. $\frac{7x^5}{3y^7}$

13. $3(2-\sqrt{3})$

15. $-12+8i$

17. $-i$

Exercise 17.10.35

In the following exercises, solve.

1. $\sqrt{2x+5}+8=6$
2. $\sqrt{x+5}+1=x$
3. $\sqrt[3]{2x^2-6x-23}=\sqrt[3]{x^2-3x+5}$

Answer

2. $x=4$

Exercise 17.10.36

In the following exercise,

a. find the domain of the function
b. graph the function
c. use the graph to determine the range

1. $g(x)=\sqrt{x+2}$

Answer

1.

a. domain: $[-2,\infty)$
b.

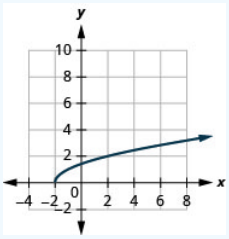

Figure 8.E.3

c. range: $[0,\infty)$

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CHAPTER OVERVIEW

18: Quadratic Equations and Functions

18.1: Prelude to Quadratic Equations and Functions
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Thumbnail: Plot of the quadratic function. Image used with permission (Public Domain; N.Mori).

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18.1: Prelude to Quadratic Equations and Functions

Blink your eyes. You've taken a photo. That's what will happen if you are wearing a contact lens with a built-in camera. Some of the same technology used to help doctors see inside the eye may someday be used to make cameras and other devices. These technologies are being developed by biomedical engineers using many mathematical principles, including an understanding of quadratic equations and functions. In this chapter, you will explore these kinds of equations and learn to solve them in different ways. Then you will solve applications modeled by quadratics, graph them, and extend your understanding to quadratic inequalities.

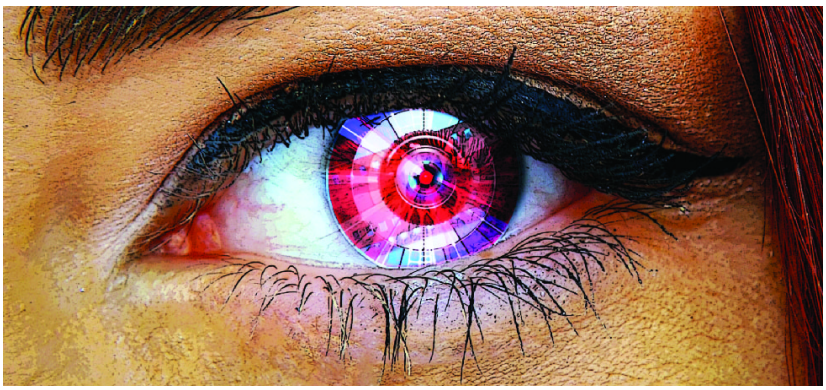

Figure 9.0.1: Several companies have patented contact lenses equipped with cameras, suggesting that they may be the future of wearable camera technology. (credit: "intographics"/Pixabay)

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18.2: Solve Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

Before you get started, take this readiness quiz.

1. Simplify: $\sqrt{128}$.
   If you missed this problem, review Example 8.13.
2. Simplify: $\sqrt{\frac{32}{5}}$.
   If you missed this problem, review Example 8.50.
3. Factor: $9x^2 - 12x + 4$.
   If you missed this problem, review Example 6.23.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form ax^2. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

Solve Quadratic Equations of the Form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

$$x^2 = 9$$

Put the equation in standard form.

$$x^2 - 9 = 0$$

Factor the difference of squares.

$$(x-3)(x+3) = 0$$

Use the Zero Produce Property.

$$x - 3 = 0 \quad x - 3 = 0$$

Solve each equation.

$$x = 3 \quad x = -3$$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x = 4, x = -4$ and $x = 5, x = -5$

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

If $n^2 = m$, then n is a square root of m.

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root Property**.

Definition 18.2.1

Square Root Property

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

$$\begin{aligned} & & x^2 &= 9 \\ &\text{Use the Square Root Property.} & x &= \pm\sqrt{9} \\ & & x &= \pm 3 \end{aligned}$$

So $x = 3$ or $x = -3$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

$$x^2 = 7$$

Use the Square Root Property. $x = \sqrt{7}, \quad x = -\sqrt{7}$

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

Example 18.2.1 How to Solve a Quadratic Equation of the form $ax^2 - k$ Using the Square Root Property

Solve: $x^2 - 50 = 0$.

Solution:

Table 9.1.1

Step 1: Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 - 50 = 0$ $x^2 = 50$
Step 2: Use the Square Root Property.	Remember to write the $\pm$ symbol.	$x = \pm\sqrt{50}$
Step 3: Simplify the radical.	Rewrite to show two solutions.	$x = \pm\sqrt{25}\cdot\sqrt{2}$ $x = \pm 5\sqrt{2}$ $x = 5\sqrt{2},\ x = -5\sqrt{2}$
Step 4: Check the solutions.	Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$	$x^2 - 50 = 0$ $(5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25\cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$ $x^2 - 50 = 0$ $(-5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25\cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$

Exercise 18.2.1

Solve: $x^2 - 48 = 0$.

Answer

$x = 4\sqrt{3}, x = -4\sqrt{3}$

Exercise 18.2.2

Solve: $y^2 - 27 = 0$.

Answer

$y = 3\sqrt{3}, y = -3\sqrt{3}$

The steps to take to use the **Square Root Property** to solve a quadratic equation are listed here.

Solve a Quadratic Equation Using the Square Root Property

1. Isolate the quadratic term and make its coefficient one.
2. Use Square Root Property.
3. Simplify the radical.
4. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

Example 18.2.2

Solve: $3z^2 = 108$.

Solution:

Table 9.1.2

	$3z^2 = 108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$\frac{3z^2}{3} = \frac{108}{3}$
Simplify.	$z^2 = 36$
Use the Square Root Property.	$z = \pm\sqrt{36}$
Simplify the radical.	$z = \pm 6$
Rewrite to show two solutions.	$z = 6, \quad z = -6$
Check the solutions: $3z^2 = 108$ $\quad$ $3z^2 = 108$ $3(6)^2 \stackrel{?}{=} 108$ $\quad$ $3(-6)^2 \stackrel{?}{=} 108$ $3(36) \stackrel{?}{=} 108$ $\quad$ $3(36) \stackrel{?}{=} 108$ $108 = 108$ ✓ $\quad$ $108 = 108$ ✓ Figure 9.1.1	

Exercise 18.2.3

Solve: $2x^2 = 98$.

Answer

$x = 7, x = -7$

Exercise 18.2.4

Solve: $5m^2 = 80$.

Answer

$m = 4, m = -4$

The Square Root Property states 'If $x^2 = k$,' What will happen if $k < 0$? This will be the case in the next example.

Example 18.2.3

Solve: $x^2 + 72 = 0$.

Solution:

Table 9.1.3

	$x^2 + 72 = 0$
Isolate the quadratic term.	$x^2 = -72$
Use the Square Root Property.	$x = \pm\sqrt{-72}$
Simplify using complex numbers.	$x = \pm\sqrt{72}i$
Simplify the radical.	$x = \pm 6\sqrt{2}i$
Rewrite to show two solutions	$x = 6\sqrt{2}i, x = -6\sqrt{2}i$
Check the solutions: $x^2 + 72 = 0$ $(6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0$ $(6)^2 \cdot (\sqrt{2})^2 \cdot i^2 + 72 \stackrel{?}{=} 0$ $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ $0 = 0$ ✓	$x^2 + 72 = 0$ $(-6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0$ $(-6)^2 \cdot (\sqrt{2})^2 \cdot i^2 + 72 \stackrel{?}{=} 0$ $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ $0 = 0$ ✓

Exercise 18.2.5

Solve: $c^2 + 12 = 0$.

Answer

$c = 2\sqrt{3}i, \quad c = -2\sqrt{3}i$

Exercise 18.2.6

Solve: $q^2 + 24 = 0$.

Answer

$c = 2\sqrt{6}i, \quad c = -2\sqrt{6}i$

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

Example 18.2.4

Solve: $\frac{2}{3}u^2 + 5 = 17$.

Solution:

Table 9.1.4

	$\frac{2}{3}u^2+5=17$
Isolate the quadratic term.	$\frac{2}{3}u^2=12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2}\cdot\frac{2}{3}u^2=\frac{3}{2}\cdot 12$
Simplify.	$u^2=18$
Use the Square Root Property.	$u=\pm\sqrt{18}$
Simplify the radical.	$u=\pm\sqrt{9\cdot 2}$
Simplify.	$u=\pm 3\sqrt{2}$
Rewrite to show two solutions.	$u=3\sqrt{2},\quad u=-3\sqrt{2}$
Check: $\frac{2}{3}u^2+5=17$ $\frac{2}{3}(3\sqrt{2})^2+5\stackrel{?}{=}17$ $\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$ $12+5\stackrel{?}{=}17$ $17=17\checkmark$	$\frac{2}{3}u^2+5=17$ $\frac{2}{3}(-3\sqrt{2})^2+5\stackrel{?}{=}17$ $\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$ $12+5\stackrel{?}{=}17$ $17=17\checkmark$

? Exercise 18.2.7

Solve: $\frac{1}{2}x^2+4=24$.

Answer

$x=2\sqrt{10}, x=-2\sqrt{10}$

? Exercise 18.2.8

Solve: $\frac{3}{4}y^2-3=18$.

Answer

$y=2\sqrt{7}, y=-2\sqrt{7}$

The solutions to some equations may have fractions inside the radicals. When this happens, we must **rationalize the denominator**.

✓ Example 18.2.5

Solve: $2x^2-8=41$.

Solution:

Table 9.1.5

	$2x^2-8=41$
Isolate the quadratic term.	$2x^2=49$
Divide by 2 to make the coefficient 1.	$\frac{2x^2}{2}=\frac{49}{2}$
Simplify.	$x^2=\frac{49}{2}$

Use the Square Root Property.	$x = \pm\sqrt{\frac{49}{2}}$
Rewrite the radical as a fraction of square roots.	$x = \pm\frac{\sqrt{49}}{\sqrt{2}}$
Rationalize the denominator.	$x = \pm\frac{\sqrt{49}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$
Simplify.	$x = \pm\frac{7\sqrt{2}}{2}$
Rewrite to show two solutions.	$x = \frac{7\sqrt{2}}{2}, \quad x = -\frac{7\sqrt{2}}{2}$
Check: We leave te check for you.	

? Exercise 18.2.9

Solve: $5r^2 - 2 = 34$.

Answer

$r = \frac{6\sqrt{5}}{5}, \quad r = -\frac{6\sqrt{5}}{5}$

? Exercise 18.2.10

Solve: $3t^2 + 6 = 70$.

Answer

$t = \frac{8\sqrt{3}}{3}, \quad t = -\frac{8\sqrt{3}}{3}$

Solve Quadratic Equation of the Form $a(x-h)^2 = k$ Using the Square Root Property

We can use the **Square Root Property** to solve an equation of the form $a(x-h)^2 = k$ as well. Notice that the quadratic term, x, in the original form $ax^2 = k$ is replaced with $(x-h)$.

$ax^2 = k \qquad a(x-h)^2 = k$

Figure 9.1.20

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the Square Root Property can be used on $(x-h)^2$.

✓ Example 18.2.6

Solve: $4(y-7)^2 = 48$.

Solution:

Table 9.1.6

	$4(y-7)^2 = 48$
Divide both sides by the coefficient 4.	$(y-7)^2 = 12$
Use the Square Root Property on the binomial.	$y-7 = \pm\sqrt{12}$
Simplify the radical.	$y-7 = \pm 2\sqrt{3}$
Solve for y.	$y = 7 \pm 2\sqrt{3}$

Rewrite to show two solutions.	$y = 7 + 2\sqrt{3}$ $y = 7 - 2\sqrt{3}$

Check:

$4(y-7)^2 = 48$ | $4(y-7)^2 = 48$

$4(7+2\sqrt{3}-7)^2 \stackrel{?}{=} 48$ | $4(7-2\sqrt{3}-7)^2 \stackrel{?}{=} 48$

$4(2\sqrt{3})^2 \stackrel{?}{=} 48$ | $4(-2\sqrt{3})^2 \stackrel{?}{=} 48$

$4(12) \stackrel{?}{=} 48$ | $4(12) \stackrel{?}{=} 48$

$48 = 48$ ✓ | $48 = 48$ ✓

Figure 9.1.21

? Exercise 18.2.11

Solve: $3(a-3)^2 = 54$.

Answer

$a = 3 + 3\sqrt{2}, \quad a = 3 - 3\sqrt{2}$

? Exercise 18.2.12

Solve: $2(b+2)^2 = 80$.

Answer

$b = -2 + 2\sqrt{10}, \quad b = -2 - 2\sqrt{10}$

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

✓ Example 18.2.7

Solve: $\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}$.

Solution:

$$\left(x - \tfrac{1}{3}\right)^2 = \tfrac{5}{9}$$

Use the Square Root Property.

$$x - \tfrac{1}{3} = \pm\sqrt{\tfrac{5}{9}}$$

Rewrite the radical as a fraction of square roots.

$$x - \tfrac{1}{3} = \pm\tfrac{\sqrt{5}}{\sqrt{9}}$$

Simplify the radical.

$$x - \tfrac{1}{3} = \pm\tfrac{\sqrt{5}}{3}$$

Solve for x.

$$x = \tfrac{1}{3} \pm \tfrac{\sqrt{5}}{3}$$

Rewrite to show two solutions.

$$x = \tfrac{1}{3} + \tfrac{\sqrt{5}}{3}, x = \tfrac{1}{3} - \tfrac{\sqrt{5}}{3}$$

Check:

We leave the check for you.

Exercise 18.2.13

Solve: $\left(x-\frac{1}{2}\right)^2=\frac{5}{4}$.

Answer

$x=\frac{1}{2}+\frac{\sqrt{5}}{2}, x=\frac{1}{2}-\frac{\sqrt{5}}{2}$

Exercise 18.2.14

Solve: $\left(y+\frac{3}{4}\right)^2=\frac{7}{16}$.

Answer

$y=-\frac{3}{4}+\frac{\sqrt{7}}{4}, y=-\frac{3}{4}-\frac{\sqrt{7}}{4}$

We will start the solution to the next example by isolating the binomial term.

Example 18.2.8

Solve: $2(x-2)^2+3=57$.

Solution:

$$2(x-2)^2+3=57$$

Subtract 3 from both sides to isolate the binomial term.

$$2(x-2)^2=54$$

Divide both sides by 2.

$$(x-2)^2=27$$

Use the Square Root Property.

$$x-2=\pm\sqrt{27}$$

Simplify the radical.

$$x-2=\pm 3\sqrt{3}$$

Solve for x.

$$x=2\pm 3\sqrt{3}$$

Rewrite to show two solutions.

$$x=2+3\sqrt{3}, x=2-3\sqrt{3}$$

Check:

We leave the check for you.

Exercise 18.2.15

Solve: $5(a-5)^2+4=104$.

Answer

$a=5+2\sqrt{5}, a=5-2\sqrt{5}$

Exercise 18.2.16

Solve: $3(b+3)^2-8=88$.

Answer

$b=-3+4\sqrt{2}, \quad b=-3-4\sqrt{2}$

Sometimes the solutions are complex numbers.

Example 18.2.9

Solve: $(2x-3)^2=-12$.

Solution:

$$(2x-3)^2=-12$$

Use the Square Root Property.

$$2x-3=\pm\sqrt{-12}$$

Simplify the radical.

$$2x-3=\pm 2\sqrt{3}i$$

Add 3 to both sides.

$$2x=3\pm 2\sqrt{3}i$$

Divide both sides by 2.

$$x=\frac{3\pm 2\sqrt{3}i}{2}$$

Rewrite in standard form.

$$x=\frac{3}{2}\pm\frac{2\sqrt{3}i}{2}$$

Simplify.

$$x=\frac{3}{2}\pm\sqrt{3}i$$

Rewrite to show two solutions.

$$x=\frac{3}{2}+\sqrt{3}i, x=\frac{3}{2}-\sqrt{3}i$$

Check:

We leave the check for you.

Exercise 18.2.17

Solve: $(3r+4)^2=-8$.

Answer

$r=-\frac{4}{3}+\frac{2\sqrt{2}i}{3}, r=-\frac{4}{3}-\frac{2\sqrt{2}i}{3}$

Exercise 18.2.18

Solve: $(2t-8)^2=-10$.

Answer

$t=4+\frac{\sqrt{10}i}{2}, t=4-\frac{\sqrt{10}i}{2}$

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Example 18.2.10

Solve: $4n^2+4n+1=16$.

Solution:

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

Table 9.1.7

	$4n^2+4n+1=16$
Factor the perfect square trinomial.	$(2n+1)^2=16$
Use the Square Root Property.	$2n+1=\pm\sqrt{16}$
Simplify the radical.	$2n+1=\pm 4$
Solve for n.	$2n=-1\pm 4$
Divide each side by 2.	$\frac{2n}{2}=\frac{-1\pm 4}{2}$ $n=\frac{-1\pm 4}{2}$
Rewrite to show two solutions.	$n=\frac{-1+4}{2}, n=\frac{-1-4}{2}$
Simplify each equation.	$n=\frac{3}{2}, \quad n=-\frac{5}{2}$
Check: $4n^2+4n+1=16$ $4\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $9+6+1\stackrel{?}{=}16$ $16=16$ ✓ $4n^2+4n+1=16$ $4\left(-\frac{5}{2}\right)^2+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{25}{4}\right)+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $25-10+1\stackrel{?}{=}16$ $16=16$ ✓	

Figure 9.1.22

Exercise 18.2.19

Solve: $9m^2-12m+4=25$.

Answer

$m=\frac{7}{3}, \quad m=-1$

Exercise 18.2.20

Solve: $16n^2+40n+25=4$.

Answer

$n=-\frac{3}{4}, \quad n=-\frac{7}{4}$

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- Solving Quadratic Equations: The Square Root Property

- Using the Square Root Property to Solve Quadratic Equations

Key Concepts

- Square Root Property
  - If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$or $x = \pm\sqrt{k}$

  How to solve a quadratic equation using the square root property.

  1. Isolate the quadratic term and make its coefficient one.
  2. Use Square Root Property.
  3. Simplify the radical.
  4. Check the solutions.

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18.2E: Exercises

Practice Makes Perfect

ExerciseS 1 - 22: Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

1. $a^2 = 49$
2. $b^2 = 144$
3. $r^2 - 24 = 0$
4. $t^2 - 75 = 0$
5. $u^2 - 300 = 0$
6. $v^2 - 80 = 0$
7. $4m^2 = 36$
8. $3n^2 = 48$
9. $\frac{4}{3}x^2 = 48$
10. $\frac{5}{3}y^2 = 60$
11. $x^2 + 25 = 0$
12. $y^2 + 64 = 0$
13. $x^2 + 63 = 0$
14. $y^2 + 45 = 0$
15. $\frac{4}{3}x^2 + 2 = 110$
16. $\frac{2}{3}y^2 - 8 = -2$
17. $\frac{2}{5}a^2 + 3 = 11$
18. $\frac{3}{2}b^2 - 7 = 41$
19. $7p^2 + 10 = 26$
20. $2q^2 + 5 = 30$
21. $5y^2 - 7 = 25$
22. $3x^2 - 8 = 46$

Answer

1. $a = \pm 7$
3. $r = \pm 2\sqrt{6}$
5. $u = \pm 10\sqrt{3}$
7. $m = \pm 3$
9. $x = \pm 6$
11. $x = \pm 5i$
13. $x = \pm 3\sqrt{7}i$
15. $x = \pm 9$
17. $a = \pm 2\sqrt{5}$

19. $p = \pm\frac{4\sqrt{7}}{7}$

21. $y = \pm\frac{4\sqrt{10}}{5}$

? ExerciseS 23 - 46: Solve Quadratic Equations of the Form $a(x-h)^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

23. $(u-6)^2 = 64$

24. $(v+10)^2 = 121$

25. $(m-6)^2 = 20$

26. $(n+5)^2 = 32$

27. $\left(r-\frac{1}{2}\right)^2 = \frac{3}{4}$

28. $\left(x+\frac{1}{5}\right)^2 = \frac{7}{25}$

29. $\left(y+\frac{2}{3}\right)^2 = \frac{8}{81}$

30. $\left(t-\frac{5}{6}\right)^2 = \frac{11}{25}$

31. $(a-7)^2+5 = 55$

32. $(b-1)^2-9 = 39$

33. $4(x+3)^2-5 = 27$

34. $5(x+3)^2-7 = 68$

35. $(5c+1)^2 = -27$

36. $(8d-6)^2 = -24$

37. $(4x-3)^2+11 = -17$

38. $(2y+1)^2-5 = -23$

39. $m^2-4m+4 = 8$

40. $n^2+8n+16 = 27$

41. $x^2-6x+9 = 12$

42. $y^2+12y+36 = 32$

43. $25x^2-30x+9 = 36$

44. $9y^2+12y+4 = 9$

45. $36x^2-24x+4 = 81$

46. $64x^2+144x+81 = 25$

Answer

23. $u = 14, u = -2$

25. $m = 6 \pm 2\sqrt{5}$

27. $r = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$

29. $y = -\frac{2}{3} \pm \frac{2\sqrt{2}}{9}$

31. $a = 7 \pm 5\sqrt{2}$

33. $x = -3 \pm 2\sqrt{2}$

35. $c = -\frac{1}{5} \pm \frac{3\sqrt{3}}{5}i$

37. $x = \frac{3}{4} \pm \frac{\sqrt{7}}{2}i$

39. $m = 2 \pm 2\sqrt{2}$

41. $x = 3 + 2\sqrt{3}, x = 3 - 2\sqrt{3}$

43. $x = -\frac{3}{5}, x = \frac{9}{5}$

45. $x = -\frac{7}{6}, x = \frac{11}{6}$

ExerciseS 47 - 68: Mixed Practice

In the following exercises, solve using the Square Root Property.

47. $2r^2 = 32$

48. $4t^2 = 16$

49. $(a-4)^2 = 28$

50. $(b+7)^2 = 8$

51. $9w^2 - 24w + 16 = 1$

52. $4z^2 + 4z + 1 = 49$

53. $a^2 - 18 = 0$

54. $b^2 - 108 = 0$

55. $\left(p - \frac{1}{3}\right)^2 = \frac{7}{9}$

56. $\left(q - \frac{3}{5}\right)^2 = \frac{3}{4}$

57. $m^2 + 12 = 0$

58. $n^2 + 48 = 0$

59. $u^2 - 14u + 49 = 72$

60. $v^2 + 18v + 81 = 50$

61. $(m-4)^2 + 3 = 15$

62. $(n-7)^2 - 8 = 64$

63. $(x+5)^2 = 4$

64. $(y-4)^2 = 64$

65. $6c^2 + 4 = 29$

66. $2d^2 - 4 = 77$

67. $(x-6)^2 + 7 = 3$

68. $(y-4)^2 + 10 = 9$

Answer

47. $r = \pm 4$

49. $a = 4 \pm 2\sqrt{7}$

51. $w = 1, w = \frac{5}{3}$

53. $a = \pm 3\sqrt{2}$

https://math.libretexts.org/@go/page/46443

55. $p = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$

57. $m = \pm 2\sqrt{2i}$

59. $u = 7 \pm 6\sqrt{2}$

61. $m = 4 \pm 2\sqrt{3}$

63. $x = -3, x = -7$

65. $c = \pm \frac{5\sqrt{6}}{6}$

67. $x = 6 \pm 2i$

? ExerciseS 69 - 70: Writing exercises

69. In your own words, explain the Square Root Property.

70. In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x+2)^2 = 16$.

Answer

69. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x-h)^2 = k$ using the square root property.			

Figure 9.1.23

Choose how would you respond to the statement "I can solve quadratic equations of the form a times the square of x minus h equals k using the Square Root Property." "Confidently," "with some help," or "No, I don't get it."

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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18.3: Solve Quadratic Equations by Completing the Square

Learning Objectives

By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2+bx+c=0$ by completing the square
- Solve quadratic equations of the form $ax^2+bx+c=0$ by completing the square

Before you get started, take this readiness quiz.

1. Expand: $(x+9)^2$.
   If you missed this problem, review Example 5.32.
2. Factor $y^2-14y+49$.
   If you missed this problem, review Example 6.9.
3. Factor $5n^2+40n+80$.
   If you missed this problem, review Example 6.14.

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called **completing the square**, which is important for our work on conics later.

Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y-7)^2=12$ because the left side was a perfect square.

$$\begin{aligned}(y-7)^2&=12\\ y-7&=\pm\sqrt{12}\\ y-7&=\pm 2\sqrt{3}\\ y&=7\pm 2\sqrt{3}\end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x-k)^2$ in order to use the Square Root Property.

$$\begin{aligned}x^2-10x+25&=18\\ (x-5)^2&=18\end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let's look at two examples to help us recognize the patterns.

$$\begin{array}{cc}(x+9)^2 & (y-7)^2\\ (x+9)(x+9) & (y-7)(y-7)\\ x^2+9x+9x+81 & y^2-7y-7y+49\\ x^2+18x+81 & y^2-14y+49\end{array}$$

We restate the patterns here for reference.

Definition 18.3.1: Binomial Squares Pattern

If a and b are real numbers,

$$(a+b)^2=a^2+2ab+b^2$$

$$\underbrace{(a+b)^2}_{\text{(binomial)}^2}=\underbrace{a^2}_{\text{(first term)}^2}+\underbrace{2ab}_{2\cdot\text{(product of terms)}}+\underbrace{b^2}_{\text{(second term)}^2}$$

$$(a-b)^2=a^2-2ab+b^2$$

$$\underbrace{(a-b)^2}_{\text{(binomial)}^2}=\underbrace{a^2}_{\text{(first term)}^2}-\underbrace{2ab}_{2\cdot\text{(product of terms)}}+\underbrace{b^2}_{\text{(second term)}^2}$$

We can use this pattern to “make” a perfect square.

We will start with the expression x^2+6x. Since there is a plus sign between the two terms, we will use the $(a+b)^2$ pattern, $a^2+2ab+b^2=(a+b)^2$.

$$\underset{a^2+2ab+b^2}{x^2+6x+\_}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find b. But first we start with determining a. Notice that the first term of x^2+6x is a square, x^2. This tells us that $a=x$.

$$\underset{a^2+\quad 2ab\quad +\ b^2}{x^2+2\cdot x\cdot b+b^2}$$

What number, b, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b=3$.

$$\underset{a^2+\quad 2ab\quad +\ b^2}{x^2+2\cdot 3\cdot x+\_}$$

Now to complete the perfect square trinomial, we will find the last term by squaring b, which is $3^2=9$.

$$\underset{a^2+2ab+b^2}{x^2+6x+9}$$

We can now factor.

$$\underset{(a+b)^2}{(x+3)^2}$$

So we found that adding 9 to x^2+6x ‘completes the square’, and we write it as $(x+3)^2$.

Howto: Complete a Square of x^2+bx

1. Identify b, the coefficient of x.
2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
3. Add the $\left(\frac{1}{2}b\right)^2$ to x^2+bx.
4. Factor the perfect square trinomial, writing it as a binomial squared.

Example 18.3.1

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. x^2-26x
b. y^2-9y
c. $n^2+\frac{1}{2}n$

Solution:

a.

	$\underset{x^2-\ bx}{x^2-26x}$
The coefficient of x is -26.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-26)\right)^2$ $(13)^2$ 169	
Add 169 to the binomial to complete the square.	$x^2-26x+169$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x-13)^2$

b.

	$x^2 - bx$ $y^2 - 9y$
The coefficient of y is -9.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-9)\right)^2$ $\left(-\frac{9}{2}\right)^2$ $\frac{81}{4}$	
Add $\frac{81}{4}$ to the binomial to complete the square.	$y^2 - 9y + \frac{81}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(y - \frac{9}{2}\right)^2$

c.

	$x^2 + bx$ $n^2 + \frac{1}{2}n$
The coefficient of n is $\frac{1}{2}$.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2$ $\left(\frac{1}{4}\right)^2$ $\frac{1}{16}$	
Add $\frac{1}{16}$ to the binomial to complete the square.	$n^2 + \frac{1}{2}n + \frac{1}{16}$
Rewrite as a binomial square.	$\left(n + \frac{1}{4}\right)^2$

Exercise 18.3.1

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. $a^2 - 20a$
b. $m^2 - 5m$
c. $p^2 + \frac{1}{4}p$

Answer

a. $(a-10)^2$
b. $\left(b - \frac{5}{2}\right)^2$
c. $\left(p + \frac{1}{8}\right)^2$

Exercise 18.3.2

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. $b^2 - 4b$
b. $n^2 + 13n$
c. $q^2 - \frac{2}{3}q$

Answer

a. $(b-2)^2$

b. $\left(n+\frac{13}{2}\right)^2$

c. $\left(q-\frac{1}{3}\right)^2$

Solve Quadratic Equations of the Form $x^2+bx+c=0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a **quadratic equation** by **completing the square** too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2+6x=40$, and we want to complete the square on the left, we will add 9 to both sides of the equation.

	$x^2+6x=40$
	$x^2+6x+\_\_=40+\_\_$
	$x^2+6x+9=40+9$
Add 9 to both sides to complete the square.	$(x+3)^2=49$

Now the equation is in the form to solve using the **Square Root Property**! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

Example 18.3.2 How to Solve a Quadratic Equation of the Form $x^2+bx+x=0$ by Completing the Square

Solve by completing the square: $x^2+8x=48$.

Solution:

Step 1: Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$x^2+bx \quad c$ $x^2+8x=48$
Step 2: Find $\left(\frac{1}{2}\cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	Take half of 8 and square it. $4^2=16$ Add 16 to BOTH sides of the equation.	$x^2+8x+\underline{\quad\quad}=48$ $\left(\frac{1}{2}\cdot 8\right)^2$ $x^2+8x+16=48+16$
Step 3: Factor the perfect square trinomial as a binomial square.	$x^2+8x+16=(x+4)^2$ Add the terms on the right.	$(x+4)^2=64$
Step 4: Use the Square Root Property.		$x+4=\pm\sqrt{64}$
Step 5: Simplify the radical and then solve the two resulting equations.		$x+4=\pm 8$ $x+4=8 \quad x+4=-8$ $x=4 \quad\quad x=-12$
Step 6: Check the solutions.	Put each answer in the original equation to check. Substitute $x=4$ and $x=-12$.	$x^2+8x=48$ $(4)^2+8(4)\stackrel{?}{=}48$ $16+32\stackrel{?}{=}48$ $48=48$ $x^2+8x=48$ $(-12)^2+8(-12)\stackrel{?}{=}48$ $144-96\stackrel{?}{=}48$ $48=48$

Exercise 18.3.3

Solve by completing the square: $x^2 + 4x = 5$.

Answer

$x = -5, x = -1$

Exercise 18.3.4

Solve by completing the square: $y^2 - 10y = -9$.

Answer

$y = 1, y = 9$

The steps to solve a quadratic equation by completing the square are listed here.

Solve a Quadratic Equation of the Form $x^2 + bx + c = 0$ by Completing the Square

1. Isolate the variable terms on one side and the constant terms on the other.
2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
3. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
4. Use the Square Root Property.
5. Simplify the radical and then solve the two resulting equations.
6. Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

Example 18.3.3

Solve by completing the square: $x^2 + 4x = -21$.

Solution:

	$\overset{x^2}{x^2} + \overset{bx}{4x} = \overset{c}{-21}$
The variable terms are on the left side. Take half of 4 and square it.	$x^2 + 4x + \underset{\left(\frac{1}{2} \cdot 4\right)^2}{\_\_\_\_\_} = -21$
$\left(\frac{1}{2}(4)\right)^2 = 4$	
Add 4 to both sides.	$x^2 + 4x + 4 = -21 + 4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x + 2)^2 = -17$
Use the Square Root Property.	$x + 2 = \pm\sqrt{-17}$
Simplifying using complex numbers.	$x + 2 = \pm\sqrt{17}\, i$
Subtract 2 from each side.	$x = -2 \pm \sqrt{17}\, i$
Rewrite to show two solutions.	$x = -2 + \sqrt{17}\, i, \quad x = -2 - \sqrt{17}\, i$
We leave the check to you.	

Exercise 18.3.5

Solve by completing the square: $y^2 - 10y = -35$.

Answer

$y = 5 + \sqrt{15}i, y = 5 - \sqrt{15}i$

Exercise 18.3.6

Solve by completing the square: $z^2 + 8z = -19$.

Answer

$z = -4 + \sqrt{3}i, z = -4 - \sqrt{3}i$

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

Example 18.3.4

Solve by completing the square: $y^2 - 18y = -6$.

Solution:

	$\underset{x^2}{y^2} - \underset{bx}{18y} = \underset{c}{-6}$
The variable terms are on the left side. Take half of -18 and square it.	
$\left(\frac{1}{2}(-18)\right)^2 = 81$	$y^2 - 18y + \underset{\left(\frac{1}{2}\cdot(-18)\right)^2}{\_\_\_\_} = -6$
Add 81 to both sides.	$y^2 - 18y + 81 = -6 + 81$
Factor the perfect square trinomial, writing it as a binomial squared.	$(y-9)^2 = 75$
Use the Square Root Property.	$y - 9 = \pm\sqrt{75}$
Simplify the radical.	$y - 9 = \pm 5\sqrt{3}$
Solve for y.	$y = 9 \pm 5\sqrt{3}$
Check. $y^2 - 18y = -6$ $(9+5\sqrt{3})^2 - 18(9+5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓ $y^2 - 18y = -6$ $(9-5\sqrt{3})^2 - 18(9-5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓ Figure 9.2.35	$y^2 - 18y = -6$ $(9+5\sqrt{3})^2 - 18(9+5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓ $y^2 - 18y = -6$ $(9-5\sqrt{3})^2 - 18(9-5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓

Another way to check this would be to use a calculator. Evaluate $y^2 - 18y$ for both of the solutions. The answer should be -6.

Exercise 18.3.7

Solve by completing the square: $x^2 - 16x = -16$.

Answer

$x = 8 + 4\sqrt{3}, x = 8 - 4\sqrt{3}$

Exercise 18.3.8

Solve by completing the square: $y^2+8y=11$.

Answer

$y=-4+3\sqrt{3}, y=-4-3\sqrt{3}$

We will start the next example by isolating the variable terms on the left side of the equation.

Example 18.3.5

Solve by completing the square: $x^2+10x+4=15$.

Solution:

	$x^2+10x+4=15$
Isolate the variable terms on the left side. Subtract 4 to get the constant terms on the right side.	$x^2+10x=11$
Take half of 10 and square it.	
$\left(\frac{1}{2}(10)\right)^2=25$	$x^2-10x+\underset{\left(\frac{1}{2}\cdot(10)\right)^2}{\_\_\_\_\_}=11$
Add 25 to both sides.	$x^2+10x+25=11+25$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+5)^2=36$
Use the Square Root Property.	$x+5=\pm\sqrt{36}$
Simplify the radical.	$x+5=\pm 6$
Solve for x.	$x=-5\pm 6$
Rewrite to show two solutions.	$x=-5+6, \quad x=-5-6$
Solve the equations.	$x=1, \quad x=-11$
Check: $x^2+10x+4=15$ $(1)^2+10(1)+4\stackrel{?}{=}15$ $1+10+4\stackrel{?}{=}15$ $15=15$ ✓ $x^2+10x+4=15$ $(-11)^2+10(-11)+4\stackrel{?}{=}15$ $121+110+4\stackrel{?}{=}15$ $15=15$ ✓ Figure 9.2.47	

Exercise 18.3.9

Solve by completing the square: $a^2+4a+9=30$.

Answer

$a=-7, a=3$

Exercise 18.3.10

Solve by completing the square: $b^2+8b-4=16$.

Answer

$b=-10, b=2$

To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

Example 18.3.6

Solve by completing the square: $n^2=3n+11$.

Solution:

	$n^2=3n+11$
Subtract $3n$ to get the variable terms on the left side.	$n^2-3n=11$
Take half of -3 and square it.	
$\left(\frac{1}{2}(-3)\right)^2=\frac{9}{4}$	$n^2-3n+\underset{\left(\frac{1}{2}\cdot(-3)\right)^2}{\_\_\_\_}=11$
Add $\frac{9}{4}$ to both sides.	$n^2-3n+\frac{9}{4}=11+\frac{9}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(n-\frac{3}{2}\right)^2=\frac{44}{4}+\frac{9}{4}$
Add the fractions on the right side.	$\left(n-\frac{3}{2}\right)^2=\frac{53}{4}$
Use the Square Root Property.	$n-\frac{3}{2}=\pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n-\frac{3}{2}=\pm\frac{\sqrt{53}}{2}$
Solve for n.	$n=\frac{3}{2}\pm\frac{\sqrt{53}}{2}$
Rewrite to show two solutions.	$n=\frac{3}{2}+\frac{\sqrt{53}}{2}, \quad n=\frac{3}{2}-\frac{\sqrt{53}}{2}$
Check: We leave the check for you!	

Exercise 18.3.11

Solve by completing the square: $p^2=5p+9$.

Answer

$p=\frac{5}{2}+\frac{\sqrt{61}}{2}, p=\frac{5}{2}-\frac{\sqrt{61}}{2}$

Exercise 18.3.12

Solve by completing the square: $q^2=7q-3$.

Answer

$q = \frac{7}{2} + \frac{\sqrt{37}}{2}, q = \frac{7}{2} - \frac{\sqrt{37}}{2}$

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the **Zero Product Property** since it says "If $a \cdot b = 0$, then $a = 0$ or $b = 0$." Instead, we multiply the factors and then put the equation into standard form to solve by completing the square.

Example 18.3.7

Solve by completing the square: $(x-3)(x+5) = 9$.

Solution:

	$(x-3)(x+5) = 9$
We multiple the binomials on the left.	$x^2 + 2x - 15 = 9$
Add 15 to isolate the constant terms on the right.	$x^2 + 2x \quad = 24$
Take half of 2 and square it.	
$\left(\frac{1}{2} \cdot (2)\right)^2 = 1$	$x^2 + 2x + \underset{\left(\frac{1}{2} \cdot (2)\right)^2}{\_\_\_\_} = 24$
Add 1 to both sides.	$x^2 + 2x + 1 = 24 + 1$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+1)^2 = 25$
Use the Square Root Property.	$x + 1 = \pm\sqrt{25}$
Solve for x.	$x = -1 \pm 5$
Rewrite to show two solutions.	$x = -1 + 5, x = -1 - 5$
Simplify.	$x = 4, \quad x = -6$
Check: We leave the check for you!	

Exercise 18.3.13

Solve by completing the square: $(c-2)(c+8) = 11$.

Answer

$c = -9, c = 3$

Exercise 18.3.14

Solve by completing the square: $(d-7)(d+3) = 56$.

Answer

$d = 11, d = -7$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

The process of **completing the square** works best when the coefficient of x^2 is 1, so the left side of the equation is of the form $x^2 + bx + c$. If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

Example 18.3.8

Solve by completing the square: $3x^2 - 12x - 15 = 0$.

Solution:

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2 - 12x - 15 = 0$
Factor out the greatest common factor.	$3(x^2 - 4x - 5) = 0$
Divide both sides by 3 to isolate the trinomial with coefficient 1.	$\frac{3(x^2 - 4x - 5)}{3} = \frac{0}{3}$
Simplify.	$x^2 - 4x - 5 = 0$
Add 5 to get the constant terms on the right side.	$x^2 - 4x = 5$
Take half of 4 and square it.	
$\left(\frac{1}{2}(-4)\right)^2 = 4$	$x^2 - 4x + \underset{\left(\frac{1}{2}\cdot(4)\right)^2}{\_\_\_\_} = 5$
Add 4 to both sides.	$x^2 - 4x + 4 = 5 + 4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x-2)^2 = 9$
Use the Square Root Property.	$x - 2 = \pm\sqrt{9}$
Solve for x.	$x - 2 = \pm 3$
Rewrite to show two solutions.	$x = 2 + 3,\ x = 2 - 3$
Simplify.	$x = 5, \quad x = -1$
Check: $x = 5$: $3x^2 - 12x - 15 = 0$; $3(5)^2 - 12(5) - 15 \stackrel{?}{=} 0$; $75 - 60 - 15 \stackrel{?}{=} 0$; $0 = 0$ ✓ — $x = -1$: $3x^2 - 12x - 15 = 0$; $3(-1)^2 - 12(-1) - 15 \stackrel{?}{=} 0$; $3 + 12 - 15 \stackrel{?}{=} 0$; $0 = 0$ ✓	

Figure 9.2.80

Exercise 18.3.15

Solve by completing the square: $2m^2 + 16m + 14 = 0$.

Answer

$m = -7, m = -1$

Exercise 18.3.16

Solve by completing the square: $4n^2 - 24n - 56 = 8$.

Answer

$n = -2, n = 8$

To complete the square, the coefficient of the x^2 must be 1. When the **leading coefficient** is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

Example 18.3.9

Solve by completing the square: $2x^2 - 3x = 20$.

Solution:

To complete the square we need the coefficient of x^2 to be one. We will divide both sides of the equation by the coefficient of x^2. Then we can continue with solving the equation by completing the square.

	$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.	$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.	$x^2 - \frac{3}{2}x = 10$
Take half of $-\frac{3}{2}$ and square it.	
$\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$	$x^2 - \frac{3}{2}x + \underline{\quad} = 10$ $\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2$
Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$
Use the Square Root Property.	$x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x - \frac{3}{4} = \pm\frac{13}{4}$
Solve for x.	$x = \frac{3}{4} \pm \frac{13}{4}$
Rewrite to show two solutions.	$x = \frac{3}{4} + \frac{13}{4}, \quad x = \frac{3}{4} - \frac{13}{4}$
Simplify.	$x = 4, \qquad x = -\frac{5}{2}$
Check: We leave the check for you!	

Exercise 18.3.17

Solve by completing the square: $3r^2 - 2r = 21$.

Answer

$r = -\frac{7}{3}, r = 3$

Exercise 18.3.18

Solve by completing the square: $4t^2+2t=20$.

Answer

$t=-\frac{5}{2}, t=2$

Now that we have seen that the coefficient of x^2 must be 1 for us to complete the square, we update our procedure for solving a **quadratic equation** by completing the square to include equations of the form $ax^2+bx+c=0$.

Howto: Solve a Quadratic Equation of the Form $ax^2+bx+c=0$ by Completing the Square

1. Divide by aa to make the coefficient of x^2 term 1.
2. Isolate the variable terms on one side and the constant terms on the other.
3. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
5. Use the Square Root Property.
6. Simplify the radical and then solve the two resulting equations.
7. Check the solutions.

Example 18.3.10

Solve by completing the square: $3x^2+2x=4$.

Solution:

Again, our first step will be to make the coefficient of x^2 one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2+2x=4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2+2x}{3}=\frac{4}{3}$
Simplify.	$x^2+\frac{2}{3}x=\frac{4}{3}$
Take half of $\frac{2}{3}$ and square it.	
$\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2=\frac{1}{9}$	$x^2+\frac{2}{3}x+\underset{\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2}{\_\_\_\_}=\frac{4}{3}$
Add $\frac{1}{9}$ to both sides.	$x^2+\frac{2}{3}x+\frac{1}{9}=\frac{4}{3}+\frac{1}{9}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x+\frac{1}{3}\right)^2=\frac{12}{9}+\frac{1}{9}$
Use the Square Root Property.	$x+\frac{1}{3}=\pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x+\frac{1}{3}=\pm\frac{\sqrt{13}}{3}$
Solve for x.	$x=-\frac{1}{3}\pm\frac{\sqrt{13}}{3}$
Rewrite to show two solutions.	$x=-\frac{1}{3}+\frac{\sqrt{13}}{3},\quad x=-\frac{1}{3}-\frac{\sqrt{13}}{3}$
Check: We leave the check for you!	

Exercise 18.3.19

Solve by completing the square: $4x^2+3x=2$.

Answer

$x=-\frac{3}{8}+\frac{\sqrt{41}}{8}, x=-\frac{3}{8}-\frac{\sqrt{41}}{8}$

Exercise 18.3.20

Solve by completing the square: $3y^2-10y=-5$.

Answer

$y=\frac{5}{3}+\frac{\sqrt{10}}{3}, y=\frac{5}{3}-\frac{\sqrt{10}}{3}$

Access these online resources for additional instruction and practice with completing the square.

- Completing Perfect Square Trinomials
- Completing the Square 1
- Completing the Square to Solve Quadratic Equations
- Completing the Square to Solve Quadratic Equations: More Examples
- Completing the Square 4

Key Concepts

- Binomial Squares Pattern
  If a and b are real numbers,

$$(a+b)^2=a^2+2ab+b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

$$(a-b)^2=a^2-2ab+b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

- How to Complete a Square
  1. Identify b, the coefficient of x.
  2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
  3. Add the $\left(\frac{1}{2}b\right)^2$ to x^2+bx
  4. Rewrite the trinomial as a binomial square
- How to solve a quadratic equation of the form $ax^2+bx+c=0$ by completing the square.
  1. Divide by a to make the coefficient of x^2 term 1.
  2. Isolate the variable terms on one side and the constant terms on the other.
  3. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
  4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.
  5. Use the Square Root Property.
  6. Simplify the radical and then solve the two resulting equations.
  7. Check the solutions.

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18.3E: Exercises

Practice Makes Perfect

? ExerciseS 1 - 4: Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

1. a. m^2-24m
   b. x^2-11x
   c. $p^2-\frac{1}{3}p$
2. a. n^2-16n
   b. y^2+15y
   c. $q^2+\frac{3}{4}q$
3. a. p^2-22p
   b. y^2+5y
   c. $m^2+\frac{2}{5}m$
4. a. q^2-6q
   b. x^2-7x
   c. $n^2-\frac{2}{3}n$

Answer

1. a. $(m-12)^2$ b. $\left(x-\frac{11}{2}\right)^2$ c. $\left(p-\frac{1}{6}\right)^2$

3. a. $(p-11)^2$ b. $\left(y+\frac{5}{2}\right)^2$ c. $\left(m+\frac{1}{5}\right)^2$

? ExerciseS 5 - 28: Solve Quadratic Equations of the Form $x^2+bx+c=0$ by Completing the Square

In the following exercises, solve by completing the square.

5. $u^2+2u=3$
6. $z^2+12z=-11$
7. $x^2-20x=21$
8. $y^2-2y=8$
9. $m^2+4m=-44$
10. $n^2-2n=-3$
11. $r^2+6r=-11$
12. $t^2-14t=-50$
13. $a^2-10a=-5$
14. $b^2+6b=41$
15. $x^2+5x=2$
16. $y^2-3y=2$
17. $u^2-14u+12=-1$
18. $z^2+2z-5=2$
19. $r^2-4r-3=9$
20. $t^2-10t-6=5$

21. $v^2 = 9v + 2$

22. $w^2 = 5w - 1$

23. $x^2 - 5 = 10x$

24. $y^2 - 14 = 6y$

25. $(x + 6)(x - 2) = 9$

26. $(y + 9)(y + 7) = 80$

27. $(x + 2)(x + 4) = 3$

28. $(x - 2)(x - 6) = 5$

Answer

5. $u = -3, u = 1$

7. $x = -1, x = 21$

9. $m = -2 \pm 2\sqrt{10}i$

11. $r = -3 \pm \sqrt{2}i$

13. $a = 5 \pm 2\sqrt{5}$

15. $x = -\frac{5}{2} \pm \frac{\sqrt{33}}{2}$

17. $u = 1, u = 13$

19. $r = -2, r = 6$

21. $v = \frac{9}{2} \pm \frac{\sqrt{89}}{2}$

23. $x = 5 \pm \sqrt{30}$

25. $x = -7, x = 3$

27. $x = -5, x = -1$

? ExerciseS 29 - 40: Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

29. $3m^2 + 30m - 27 = 6$

30. $2x^2 - 14x + 12 = 0$

31. $2n^2 + 4n = 26$

32. $5x^2 + 20x = 15$

33. $2c^2 + c = 6$

34. $3d^2 - 4d = 15$

35. $2x^2 + 7x - 15 = 0$

36. $3x^2 - 14x + 8 = 0$

37. $2p^2 + 7p = 14$

38. $3q^2 - 5q = 9$

39. $5x^2 - 3x = -10$

40. $7x^2 + 4x = -3$

Answer

29. $m = -11, m = 1$

31. $n = 1 \pm \sqrt{14}$

33. $c = -2, c = \frac{3}{2}$

35. $x = -5, x = \frac{3}{2}$

37. $p = -\frac{7}{4} \pm \frac{\sqrt{161}}{4}$

39. $x = \frac{3}{10} \pm \frac{\sqrt{191}}{10} i$

? ExerciseS 41 - 42: writing exercises

41. Solve the equation $x^2 + 10x = -25$

a. by using the Square Root Property
b. by Completing the Square
c. Which method do you prefer? Why?

42. Solve the equation $y^2 + 8y = 48$ by completing the square and explain all your steps.

Answer

41. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.			
solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.			

Figure 9.2.103

b. After reviewing this checklist, what will you do to become confident for all objectives?

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18.4: Solve Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Before you get started, take this readiness quiz.

1. Evaluate b^2-4ab when $a=3$ and $b=-2$.
2. Simplify $\sqrt{108}$.
3. Simplify $\sqrt{50}$.

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes'. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for x.

We start with the standard form of a quadratic equation and solve it for x by completing the square.

	$ax^2+bx+c=0, \quad a\neq 0$
Isolate the variable terms on one side.	$ax^2+bx \quad =-c$
Make the coefficient of x^2 equal to 1, by dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x \quad =-\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x \quad =-\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\frac{b}{a}\right)^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c\cdot 4a}{a\cdot 4a}$
Simplify.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The final equation is called the "Quadratic Formula."

Definition 18.4.1: Quadratic Formula

The solutions to a **quadratic equation** of the form $ax^2+bx+c=0$, where $a\neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{18.4.1}$$

To use the **Quadratic Formula**, we substitute the values of a, b, and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the Quadratic Formula (Equation 18.4.1) is an equation. Make sure you use both sides of the equation.

Example 18.4.1 How to Solve a Quadratic Equation Using the Quadratic Formula

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

Solution:

Step 1: Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+9x-5=0$ $a=2, b=9, c=-5$
Step 2: Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $a=2, b=9, c=-5$	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$
Step 3: Simplify the fraction, and solve for x.		$x=\frac{-9\pm\sqrt{81-(-40)}}{4}$ $x=\frac{-9\pm\sqrt{121}}{4}$ $x=\frac{-9\pm 11}{4}$ $x=\frac{-9+11}{4}$ $x=\frac{-9-11}{4}$ $x=\frac{2}{4}$ $x=\frac{-20}{4}$ $x=\frac{1}{2}$ $x=-5$

Step 4: Check the solutions.	Put each answer in the original equation to check. Substitute $x=\frac{1}{2}$ and $x=-5$.	$2x^2+9x-5=0$ $2\left(\frac{1}{2}\right)^2+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+0\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $\frac{1}{2}+\frac{9}{2}-5\stackrel{?}{=}0$ $\frac{10}{2}-5\stackrel{?}{=}0$ $5-5\stackrel{?}{=}0$ $0=0$ $2x^2+9x-5=0$ $2(-5)^2+9(-5)-5\stackrel{?}{=}0$ $2\cdot25-45-5\stackrel{?}{=}0$ $50-45-5\stackrel{?}{=}0$ $0=0$

Exercise 18.4.1

Solve by using the Quadratic Formula: $3y^2-5y+2=0$.

Answer

$y=1, y=\frac{2}{3}$

Exercise 18.4.2

Solve by using the Quadratic Formula: $4z^2+2z-6=0$.

Answer

$z=1, z=-\frac{3}{2}$

HowTo: Solve a Quadratic Equation Using the Quadratic Formula

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, and c.
2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.
3. Simplify.
4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic Formula is an EQUATION. Be sure you start with "$x=$".

Example 18.4.2

Solve by using the Quadratic Formula: $x^2-6x=-5$.

Solution:

	$x^2-6x=-5$

Write the equation in standard form by adding 5 to each side.	$x^2-6x+5=0$
This equation is now in standard form.	$ax^2+bx+c=0$ $x^2-6x+5=0$
Identify the values of a, b, c.	$a=1, b=-6, c=5$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(-6)\pm\sqrt{(-6)^2-4\cdot 1\cdot(5)}}{2\cdot 1}$
Simplify.	$x=\frac{6\pm\sqrt{36-20}}{2}$ $x=\frac{6\pm\sqrt{16}}{2}$ $x=\frac{6\pm 4}{2}$
Rewrite to show two solutions.	$x=\frac{6+4}{2}, \quad x=\frac{6-4}{2}$
Simplify.	$x=\frac{10}{2}, \quad x=\frac{2}{2}$
	$x=5, \quad x=1$
Check: $x^2-6x+5=0$ $\quad$ $x^2-6x+5=0$ $5^2-6\cdot 5+5\stackrel{?}{=}0$ $\quad$ $1^2-6\cdot 1+5\stackrel{?}{=}0$ $25-30+5\stackrel{?}{=}0$ $\quad$ $1-6+5\stackrel{?}{=}0$ $0=0$ ✓ $\quad$ $0=0$ ✓	

? Exercise 18.4.3

Solve by using the Quadratic Formula: $a^2-2a=15$.

Answer

$a=-3, a=5$

? Exercise 18.4.4

Solve by using the Quadratic Formula: $b^2+24=-10b$.

Answer

$b=-6, b=-4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the **Quadratic Formula**. If we get a **radical** as a solution, the final answer must have the radical in its simplified form.

✓ Example 18.4.3

Solve by using the Quadratic Formula: $2x^2+10x+11=0$.

Solution:

	$2x^2+10x+11=0$
This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+10x+11=0$

Identify the values of a, b and c.	$a = 2,\ b = 10,\ c = 11$
Write the Quadratic Formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \dfrac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$
Simplify.	$x = \dfrac{-10 \pm \sqrt{100 - 88}}{4}$
	$x = \dfrac{-10 \pm \sqrt{12}}{4}$
Simplify the radical.	$x = \dfrac{-10 \pm 2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x = \dfrac{2(-5 \pm \sqrt{3})}{4}$ $x = \dfrac{2(-5 \pm \sqrt{3})}{4}$
Remove the common factors.	$x = \dfrac{-5 \pm \sqrt{3}}{2}$
Rewrite to show two solutions.	$x = \dfrac{-5 + \sqrt{3}}{2}, \quad x = \dfrac{-5 - \sqrt{3}}{2}$
Check: We leave the check for you!	

? Exercise 18.4.5

Solve by using the Quadratic Formula: $3m^2 + 12m + 7 = 0$.

Answer

$$m = \frac{-6 + \sqrt{15}}{3}, m = \frac{-6 - \sqrt{15}}{3}$$

? Exercise 18.4.6

Solve by using the Quadratic Formula: $5n^2 + 4n - 4 = 0$.

Answer

$$n = \frac{-2 + 2\sqrt{6}}{5}, n = \frac{-2 - 2\sqrt{6}}{5}$$

When we substitute a, b, and c into the Quadratic Formula and the **radicand** is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

✓ Example 18.4.4

Solve by using the Quadratic Formula: $3p^2 + 2p + 9 = 0$.

Solution:

Table 9.3.5

	$3p^2 + 2p + 9 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $3p^2 + 2p + 9 = 0$

Identify the values of a, b, c.	$a = 3, b = 2, c = 9$
Write the Quadratic Formula.	$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$p = \frac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 3 \cdot (9)}}{2 \cdot 3}$
Simplify.	$p = \frac{-2 \pm \sqrt{4 - 108}}{6}$
	$p = \frac{-2 \pm \sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p = \frac{-2 \pm \sqrt{104}\, i}{6}$
Simplify the radical.	$p = \frac{-2 \pm 2\sqrt{26}\, i}{6}$
Factor the common factor in the numerator.	$p = \frac{2(-1 \pm \sqrt{26}\, i)}{6}$
Remove the common factors.	$p = \frac{-1 \pm \sqrt{26}\, i}{3}$
Rewrite in standard $a + bi$ form.	$p = -\frac{1}{3} \pm \frac{\sqrt{26}\, i}{3}$
Write as two solutions.	$p = -\frac{1}{3} + \frac{\sqrt{26}\, i}{3}, \quad p = -\frac{1}{3} - \frac{\sqrt{26}\, i}{3}$

? Exercise 18.4.7

Solve by using the Quadratic Formula: $4a^2 - 2a + 8 = 0$.

Answer

$$a = \frac{1}{4} + \frac{\sqrt{31}}{4}i, \quad a = \frac{1}{4} - \frac{\sqrt{31}}{4}i$$

? Exercise 18.4.8

Solve by using the Quadratic Formula: $5b^2 + 2b + 4 = 0$.

Answer

$$b = -\frac{1}{5} + \frac{\sqrt{19}}{5}i, \quad b = -\frac{1}{5} - \frac{\sqrt{19}}{5}i$$

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2 + bx + c = 0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

✓ Example 18.4.5

Solve by using the Quadratic Formula: $x(x + 6) + 4 = 0$.

Solution:

Our first step is to get the equation in standard form.

Table 9.3.6

	$x(x + 6) + 4 = 0$
Distribute to get the equation in standard form.	$x^2 + 6x + 4 = 0$

This equation is now in standard form.	$ax^2+bx+c=0$ $x^2+6x+4=0$
Identify the values of a, b, c.	$a=1,\ b=6,\ c=4$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(6)\pm\sqrt{(6)^2-4\cdot 1\cdot(4)}}{2\cdot 1}$
Simplify.	$x=\frac{-6\pm\sqrt{36-16}}{2}$
	$x=\frac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\frac{-6\pm 2\sqrt{5}}{2}$
Factor the common factor in the numerator.	$x=\frac{2(-3\pm 2\sqrt{5})}{2}$
Remove the common factors.	$x=-3\pm 2\sqrt{5}$
Write as two solutions.	$x=-3+2\sqrt{5},\ x=-3-2\sqrt{5}$
Check: We leave the check for you!	

? Exercise 18.4.9

Solve by using the Quadratic Formula: $x(x+2)-5=0$.

Answer

$x=-1+\sqrt{6}, x=-1-\sqrt{6}$

? Exercise 18.4.10

Solve by using the Quadratic Formula: $3y(y-2)-3=0$.

Answer

$y=1+\sqrt{2}, y=1-\sqrt{2}$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

✓ Example 18.4.6

Solve by using the Quadratic Formula: $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$.

Solution:

Our first step is to clear the fractions.

Table 9.3.7

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$

Multiply.	$3u^2 + 4u = 2$
Subtract 2 to get the equation in standard form.	$ax^2 + bx + c = 0$ $3u^2 + 4u - 2 = 0$
Identify the values of a, b, and c.	$a = 3,\ b = 4,\ c = -2$
Write the Quadratic Formula.	$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$u = \frac{-(4) \pm \sqrt{(4)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3}$
Simplify.	$u = \frac{-4 \pm \sqrt{16 + 24}}{6}$
	$u = \frac{-4 \pm \sqrt{40}}{6}$
Simplify the radical.	$u = \frac{-4 \pm 2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u = \frac{2(-2 \pm \sqrt{10})}{6}$
Remove the common factors.	$u = \frac{-2 \pm \sqrt{10}}{3}$
Rewrite to show two solutions.	$u = \frac{-2 + \sqrt{10}}{3}, \quad u = \frac{-2 - \sqrt{10}}{3}$
Check: We leave the check for you!	

Exercise 18.4.11

Solve by using the Quadratic Formula: $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$.

Answer

$c = \frac{2+\sqrt{7}}{3}, \quad c = \frac{2-\sqrt{7}}{3}$

Exercise 18.4.12

Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Answer

$d = \frac{9+\sqrt{33}}{4}, d = \frac{9-\sqrt{33}}{4}$

Think about the equation $(x-3)^2 = 0$. We know from the **Zero Product Property** that this equation has only one solution, $x = 3$.

We will see in the next example how using the **Quadratic Formula** to solve an equation whose standard form is a perfect square **trinomial** equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0, which leads to only one solution.

Example 18.4.7

Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

Solution:

Table 9.3.8

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$ax^2 + bx + c = 0$ $4x^2 - 20x + 25 = 0$
Identify the values of a, b, and c.	$a = 4,\ b = -20,\ c = 25$
Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$
	$x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check: We leave the check for you!	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x - 5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

Exercise 18.4.13

Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

Answer

$r = -5$

Exercise 18.4.14

Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

Answer

$t = \frac{4}{5}$

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Definition 18.4.2

Discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

Figure 9.3.85

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Table 9.3.9

Quadratic Equation (in standard form)	Discriminat $b^2 - 4ac$	Value of the Discriminant	Number and Type of Solutions
$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5)$ 121	$+$	2 real
$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25$ 0	0	1 real
$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9$ -104	$-$	2 complex

When the discriminant is positive, the quadratic equation has **2 real solutions**. $x = \frac{-b \pm \sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution**. $x = \frac{-b \pm \sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions**. $x = \frac{-b \pm \sqrt{-}}{2a}$

Figure 9.3.86

Using the Discriminant $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

Example 18.4.8

Determine the number of solutions to each quadratic equation.

a. $3x^2 + 7x - 9 = 0$
b. $5n^2 + n + 4 = 0$
c. $9y^2 - 6y + 1 = 0$

Solution:

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

a.

$$3x^2 + 7x - 9 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 3, \quad b = 7, \quad c = -9$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(7)^2-4\cdot 3\cdot(-9)$$

Simplify.

$$\begin{array}{c} 49+108 \\ 157 \end{array}$$

Since the discriminant is positive, there are 2 real solutions to the equation.

b.

$$5n^2+n+4=0$$

The equation is in standard form, identify a, b, and c.

$$a=5,\quad b=1,\quad c=4$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(1)^2-4\cdot 5\cdot 4$$

Simplify.

$$\begin{array}{c} 1-80 \\ -79 \end{array}$$

Since the discriminant is negative, there are 2 complex solutions to the equation.

c.

$$9y^2-6y+1=0$$

The equation is in standard form, identify a, b, and c.

$$a=9,\quad b=-6,\quad c=1$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(-6)^2-4\cdot 9\cdot 1$$

Simplify.

$$\begin{array}{c} 36-36 \\ 0 \end{array}$$

Since the discriminant is 0, there is 1 real solution to the equation.

? Exercise 18.4.15

Determine the number and type of solutions to each quadratic equation.

a. $8m^2-3m+6=0$
b. $5z^2+6z-2=0$
c. $9w^2+24w+16=0$

Answer

a. 2 complex solutions
b. 2 real solutions

c. 1 real solution

Exercise 18.4.16

Determine the number and type of solutions to each quadratic equation.

a. $b^2+7b-13=0$
b. $5a^2-6a+10=0$
c. $4r^2-20r+25=0$

Answer

a. 2 real solutions
b. 2 complex solutions
c. 1 real solution

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2=k$ or $a(x-h)^2=k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

Identify the Most Appropriate Method to Solve a Quadratic Equation

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form $ax^2=k$ or $a(x-h)^2=k$, it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

Example 18.4.9

Identify the most appropriate method to use to solve each quadratic equation.

a. $5z^2=17$
b. $4x^2-12x+9=0$
c. $8u^2+6u=11$

Solution:

a.

$$5z^2=17$$

Since the equation is in the $ax^2=k$, the most appropriate method is to use the Square Root Property.

b.

$$4x^2 - 12x + 9 = 0$$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

c.

$$8u^2 + 6u = 11$$

Put the equation in standard form.

$$8u^2 + 6u - 11 = 0$$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

Exercise 18.4.17

Identify the most appropriate method to use to solve each quadratic equation.

a. $x^2 + 6x + 8 = 0$
b. $(n-3)^2 = 16$
c. $5p^2 - 6p = 9$

Answer

a. Factoring
b. Square Root Property
c. Quadratic Formula

Exercise 18.4.18

Identify the most appropriate method to use to solve each quadratic equation.

a. $8a^2 + 3a - 9 = 0$
b. $4b^2 + 4b + 1 = 0$
c. $5c^2 = 125$

Answer

a. Quadratic Formula
b. Factoring or Square Root Property
c. Square Root Property

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- Using the Quadratic Formula
- Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions
- Discriminant in Quadratic Formula

Key Concepts

- Quadratic Formula
  - The solutions to a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.
  1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, c.
  2. Write the Quadratic Formula. Then substitute in the values of a, b, c.
  3. Simplify.

4. Check the solutions.

- Using the Discriminant, b^2-4ac, to Determine the Number and Type of Solutions of a Quadratic Equation
  - For a quadratic equation of the form $ax^2+bx+c=0, a\neq 0$,
    - If $b^2-4ac>0$, the equation has 2 real solutions.
    - If $b^2-4ac=0$, the equation has 1 real solution.
    - If $b^2-4ac<0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to identify the most appropriate method to solve a quadratic equation.
  1. Try Factoring first. If the quadratic factors easily, this method is very quick.
  2. Try the **Square Root Property** next. If the equation fits the form $ax^2=k$ or $a(x-h)^2=k$, it can easily be solved by using the Square Root Property.
  3. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

Glossary

discriminant

In the Quadratic Formula, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, the quantity b^2-4ac is called the discriminant.

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18.4E: Exercises

Practice Makes Perfect

? ExerciseS 1 - 32: Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

1. $4m^2+m-3=0$
2. $4n^2-9n+5=0$
3. $2p^2-7p+3=0$
4. $3q^2+8q-3=0$
5. $p^2+7p+12=0$
6. $q^2+3q-18=0$
7. $r^2-8r=33$
8. $t^2+13t=-40$
9. $3u^2+7u-2=0$
10. $2p^2+8p+5=0$
11. $2a^2-6a+3=0$
12. $5b^2+2b-4=0$
13. $x^2+8x-4=0$
14. $y^2+4y-4=0$
15. $3y^2+5y-2=0$
16. $6x^2+2x-20=0$
17. $2x^2+3x+3=0$
18. $2x^2-x+1=0$
19. $8x^2-6x+2=0$
20. $8x^2-4x+1=0$
21. $(v+1)(v-5)-4=0$
22. $(x+1)(x-3)=2$
23. $(y+4)(y-7)=18$
24. $(x+2)(x+6)=21$
25. $\frac{1}{4}m^2+\frac{1}{12}m=\frac{1}{3}$
26. $\frac{1}{3}n^2+n=-\frac{1}{2}$
27. $\frac{3}{4}b^2+\frac{1}{2}b=\frac{3}{8}$
28. $\frac{1}{9}c^2+\frac{2}{3}c=3$
29. $16c^2+24c+9=0$
30. $25d^2-60d+36=0$
31. $25q^2+30q+9=0$

32. $16y^2+8y+1=0$

Answer

1. $m=-1, m=\frac{3}{4}$
3. $p=\frac{1}{3}, p=2$
5. $p=-4, p=-3$
7. $r=-3, r=11$
9. $u=\frac{-7\pm\sqrt{73}}{6}$
11. $a=\frac{3\pm\sqrt{3}}{2}$
13. $x=-4\pm2\sqrt{5}$
15. $y=-\frac{2}{3}, y=-1$
17. $x=-\frac{3}{4}\pm\frac{\sqrt{15}}{4}i$
19. $x=\frac{3}{8}\pm\frac{\sqrt{7}}{8}i$
21. $v=2\pm2\sqrt{2}$
23. $y=-4, y=7$
25. $m=1, m=\frac{-4}{3}$
27. $b=\frac{-2\pm\sqrt{22}}{6}$
29. $c=-\frac{3}{4}$
31. $q=-\frac{3}{5}$

? ExerciseS 33 - 36 Use the Discriminant to Predict the Number of Real Solutions of a Quadratic Equation

In the following exercises, determine the number of real solutions for each quadratic equation.

33. a. $4x^2-5x+16=0$
    b. $36y^2+36y+9=0$
    c. $6m^2+3m-5=0$
34. a. $9v^2-15v+25=0$
    b. $100w^2+60w+9=0$
    c. $5c^2+7c-10=0$
35. a. $r^2+12r+36=0$
    b. $8t^2-11t+5=0$
    c. $3v^2-5v-1=0$
36. a. $25p^2+10p+1=0$
    b. $7q^2-3q-6=0$
    c. $7y^2+2y+8=0$

Answer

33. a. no real solutions b. 1 c. 2

35. a. 1 b. no real solutions c. 2

? ExerciseS 37 - 40: Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

37. a. $x^2-5x-24=0$
    b. $(y+5)^2=12$
    c. $14m^2+3m=11$
38. a. $(8v+3)^2=81$
    b. $w^2-9w-22=0$
    c. $4n^2-10=6$
39. a. $6a^2+14=20$
    b. $\left(x-\frac{1}{4}\right)^2=\frac{5}{16}$
    c. $y^2-2y=8$
40. a. $8b^2+15b=4$
    b. $\frac{5}{9}v^2-\frac{2}{3}v=1$
    c. $\left(w+\frac{4}{3}\right)^2=\frac{2}{9}$

Answer

37. a. Factor b. Square Root c. Quadratic Formula

39. a. Quadratic Formula b. Square Root c. Factor

? ExerciseS 41 - 42: Writing Exercises

41. Solve the equation $x^2+10x=120$
    a. by completing the square
    b. using the Quadratic Formula
    c. Which method do you prefer? Why?
42. Solve the equation $12y^2+23y=24$
    a. by completing the square
    b. using the Quadratic Formula
    c. Which method do you prefer? Why?

Answer

41. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

Figure 9.3.87

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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18.5: Solve Quadratic Equations in Quadratic Form

Learning Objectives

By the end of this section, you will be able to:

- Solve equations in quadratic form

Before you get started

... take this readiness quiz.

1. Factor by substitution: $y^4 - y^2 - 20$.
2. Factor by substitution: $(y-4)^2 + 8(y-4) + 15$.
3. Simplify
   a. $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$
   b. $\left(x^{\frac{1}{3}}\right)^2$
   c. $\left(x^{-1}\right)^2$

Solve Equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the $ax^2 + bx + c$ form. So we factored by substitution allowing us to make it fit the $ax^2 + bx + c$ form. We used the standard u for the substitution.

To factor the expression $x^4 - 4x^2 - 5$, we noticed the variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $\left(x^2\right)^2 = x^4$.) So we let $u = x^2$ and factored.

	$x^4 - 4x^2 - 5$
	$\left(x^2\right)^2 - 4\left(x^2\right) - 5$
Let $u = x^2$ and substitute.	$u^2 - 4u - 5$
Factor the trinomial.	$(u+1)(u-5)$
Replace u with x^2.	$\left(x^2+1\right)\left(x^2-5\right)$

Similarly, sometimes an equation is not in the $ax^2 + bx + c = 0$ form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c = 0$ form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations.

Notice that in the quadratic equation $ax^2 + bx + c = 0$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Again, we will use the standard u to make a substitution that will put the equation in quadratic form. If the substitution gives us an equation of the form $ax^2 + bx + c = 0$, we say the original equation was of **quadratic form**.

The next example shows the steps for solving an equation in quadratic form.

Example 18.5.1 How to Solve Equations in Quadratic Form

Solve: $6x^4 - 7x^2 + 2 = 0$

Solution:

Step 1: Identify a substitution that will put the equation in quadratic form.	Since $\left(x^2\right)^2 = x^4$, we let $u = x^2$.	$6x^4 - 7x^2 + 2 = 0$

Step 2: Rewrite the equation with the substitution to put it in quadratic form.	Rewrite to prepare for the substitution. Substitute $u = x^2$.	$6\left(x^2\right)^2 - 7x^2 + 2 = 0$ $6u^2 - 7u + 2 = 0$
Step 3: Solve the quadratic equation for u.	We can solve by factoring. Use the Zero Product Property.	$(2u-1)(3u-2) = 0$ $2u - 1 = 0, 3u - 2 = 0$ $2u = 1, 3u = 2$ $u = \frac{1}{2} u = \frac{2}{3}$
Step 4: Substitute the original variable back into the results, using the substitution.	Replace u with x^2.	$x^2 = \frac{1}{2} \quad x^2 = \frac{2}{3}$
Step 5: Solve for the original variable.	Solve for x, using the Square Root Property.	$x = \pm\sqrt{\frac{1}{2}} \quad x = \pm\sqrt{\frac{2}{3}}$ $x = \pm\frac{\sqrt{2}}{2} \quad x = \pm\frac{\sqrt{6}}{3}$ There are four solutions. $x = \frac{\sqrt{2}}{2} \quad x = \frac{\sqrt{6}}{3}$ $x = -\frac{\sqrt{2}}{2} \quad x = -\frac{\sqrt{6}}{3}$
Step 6: Check the solutions.	Check all four solutions. We will show one check here.	$x = \frac{\sqrt{2}}{2}$ $6x^4 - 7x^2 + 2 = 0$ $6\left(\frac{\sqrt{2}}{2}\right)^4 - 7\left(\frac{\sqrt{2}}{2}\right)^2 + 2 \stackrel{?}{=} 0$ $6\left(\frac{4}{16}\right) - 7\left(\frac{2}{4}\right)^2 + 2 \stackrel{?}{=} 0$ $\frac{3}{2} - \frac{7}{2} + \frac{4}{2} \stackrel{?}{=} 0$ $0 = 0$ We leave the other checks to you!

? Exercise 18.5.1

Solve: $x^4 - 6x^2 + 8 = 0$.

Answer

$x = \sqrt{2}, x = -\sqrt{2}, x = 2, x = -2$

? Exercise 18.5.2

Solve: $x^4 - 11x^2 + 28 = 0$.

Answer

$x = \sqrt{7}, x = -\sqrt{7}, x = 2, x = -2$

We summarize the steps to solve an equation in quadratic form.

Solve Equations in Quadratic Form

1. Identify a substitution that will put the equation in quadratic form.
2. Rewrite the equation with the substitution to put it in quadratic form.
3. Solve the quadratic equation for u.
4. Substitute the original variable back into the results, using the substitution.

5. Solve for the original variable.
6. Check the solutions.

In the next example, the binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u=x-2$ and substitute, our trinomial will be in ax^2+bx+c form.

Example 18.5.2

Solve: $(x-2)^2+7(x-2)+12=0$.

Solution:

	$(x-2)^2+7(x-2)+12=0$
Prepare for the substitution.	$(x-2)^2+7(x-2)+12=0$
Let $u=x-2$ and substitute.	$u^2+7u+12=0$
Solve by factoring.	$(u+3)(u+4)=0$ $u+3=0, \quad u+4=0$ $u=-3, \quad u=-4$
Replace u with $x-2$.	$x-2=-3, \quad x-2=-4$
Solve for x.	$x=-1, \quad x=-2$
Check: $x=-1$ $(x-2)^2+7(x-2)+12=0$ $(-1-2)^2+7(-1-2)+12 \stackrel{?}{=} 0$ $(-3)^2+7(-3)+12 \stackrel{?}{=} 0$ $9-21+12 \stackrel{?}{=} 0$ $0=0$ ✓ $x=-2$ $(x-2)^2+7(x-2)+12=0$ $(-2-2)^2+7(-2-2)+12 \stackrel{?}{=} 0$ $(-4)^2+7(-4)+12 \stackrel{?}{=} 0$ $16-28+12 \stackrel{?}{=} 0$ $0=0$ ✓	

Figure 9.4.14

Exercise 18.5.3

Solve: $(x-5)^2+6(x-5)+8=0$.

Answer

$x=3, x=1$

Exercise 18.5.4

Solve: $(y-4)^2+8(y-4)+15=0$.

Answer

$y=-1, y=1$

In the next example, we notice that $(\sqrt{x})^2=x$. Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

Example 18.5.3

Solve: $x-3\sqrt{x}+2=0$.

Solution:

The $\sqrt{x}$ in the middle term, is squared in the first term $(\sqrt{x})^2=x$. If we let $u=\sqrt{x}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

	$x-3\sqrt{x}+2=0$
Rewrite the trinomial to prepare for the substitution.	$(\sqrt{x})^2-3\sqrt{x}+2=0$
Let $u=\sqrt{x}$ and substitute.	$u^2-3u+2=0$
Solve by factoring.	$(u-2)(u-1)=0$ $u-2=0, \quad u-1=0$ $u=2, \quad u=1$ \)
Replace u with $\sqrt{x}$.	$\sqrt{x}=2, \quad \sqrt{x}=1$
Solve for x, by squaring both sides.	$x=4, \quad x=1$
Check: $x=4$: $x-3\sqrt{x}+2=0$; $4-3\sqrt{4}+2\stackrel{?}{=}0$; $4-6+2\stackrel{?}{=}0$; $0=0$ ✓ $x=1$: $x-3\sqrt{x}+2=0$; $1-3\sqrt{1}+2\stackrel{?}{=}0$; $1-3+2\stackrel{?}{=}0$; $0=0$ ✓	

Exercise 18.5.5

Solve: $x-7\sqrt{x}+12=0$.

Answer

$x=9, x=16$

Exercise 18.5.6

Solve: $x-6\sqrt{x}+8=0$.

Answer

$x=4, x=16$

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

Example 18.5.4

Solve: $x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$.

Solution:

The $x^{\frac{1}{3}}$ in the middle term is squared in the first term $\left(x^{\frac{1}{3}}\right)^2=x^{\frac{2}{3}}$. If we let $u=x^{\frac{1}{3}}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

	$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$

Rewrite the trinomial to prepare for the substitution.	$\left(x^{\frac{1}{3}}\right)^2 - 2\left(x^{\frac{1}{3}}\right) - 24 = 0$
Let $u = x^{\frac{1}{3}}$	$u^2 - 2u - 24 = 0$
Solve by factoring.	$(u-6)(u+4) = 0$ $u-6=0, \quad u+4=0$ $u=6, \quad u=-4$
Replace u with $x^{\frac{1}{3}}$.	$x^{\frac{1}{3}} = 6, \quad x^{\frac{1}{3}} = -4$
Solve for x by cubing both sides.	$\left(x^{\frac{1}{3}}\right)^3 = (6)^3, \quad \left(x^{\frac{1}{3}}\right)^3 = (-4)^3$ $x = 216, \quad x = -64$
Check: $x = 216$ $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$ $(216)^{\frac{2}{3}} - 2(216)^{\frac{1}{3}} - 24 \stackrel{?}{=} 0$ $36 - 12 - 24 \stackrel{?}{=} 0$ $0 = 0$ ✓ $x = -64$ $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$ $(-64)^{\frac{2}{3}} - 2(-64)^{\frac{1}{3}} - 24 \stackrel{?}{=} 0$ $16 + 8 - 24 \stackrel{?}{=} 0$ $0 = 0$ ✓ Figure 9.4.33	

? Exercise 18.5.7

Solve: $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} - 14 = 0$.

Answer

$x = -8, x = 343$

? Exercise 18.5.8

Solve: $x^{\frac{1}{2}} + 8x^{\frac{1}{4}} + 15 = 0$.

Answer

$x = 81, x = 625$

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents.

✓ Example 18.5.5

Solve: $3x^{-2} - 7x^{-1} + 2 = 0$.

Solution:

The x^{-1} in the middle term is squared in the first term $\left(x^{-1}\right)^2 = x^{-2}$. If we let $u = x^{-1}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

	$3x^{-2} - 7x^{-1} + 2 = 0$
Rewrite the trinomial to prepare for the substitution.	$3(x^{-1})^2 - 7(x^{-1}) + 2 = 0$
Let $u = x^{-1}$ and substitute.	$3u^2 - 7u + 2 = 0$
Solve by factoring.	$(3u-1)(u-2) = 0$
	$3u-1=0, \quad u-2=0$

	$u=\frac{1}{3}, \quad u=2$
Replace u with x^{-1}.	$x^{-1}=\frac{1}{3}, \quad x^{-1}=2$
Solve for x by taking the reciprocal since $x^{-1}=\frac{1}{x}$.	$x=3, \quad x=\frac{1}{2}$
Check: $x=3$ $3x^{-2}-7x^{-1}+2=0$ $3(3)^{-2}-7(3)^{-1}+2\stackrel{?}{=}0$ $3\left(\frac{1}{9}\right)-7\left(\frac{1}{3}\right)+2\stackrel{?}{=}0$ $\frac{1}{3}-\left(\frac{7}{3}\right)+\frac{6}{3}\stackrel{?}{=}0$ $0=0$ ✓ $x=\frac{1}{2}$ $3x^{-2}-7x^{-1}+2=0$ $3\left(\frac{1}{2}\right)^{-2}-7\left(\frac{1}{2}\right)^{-1}+2\stackrel{?}{=}0$ $3(4)-7(2)+2\stackrel{?}{=}0$ $12-14+2\stackrel{?}{=}0$ $0=0$ ✓ Figure 9.4.42	

? Exercise 18.5.9

Solve: $8x^{-2}-10x^{-1}+3=0$.

Answer

$x=\frac{4}{3}, x=2$

? Exercise 18.5.10

Solve: $6x^{-2}-23x^{-1}+20=0$.

Answer

$x=\frac{2}{5}, x=\frac{3}{4}$

Access this online video for additional instruction and practice with solving quadratic equations: https://www.youtube.com/watch?v=7X-CZMbpxuw

Key Concepts

- How to solve equations in quadratic form.
  1. Identify a substitution that will put the equation in quadratic form.
  2. Rewrite the equation with the substitution to put it in quadratic form.
  3. Solve the quadratic equation for u.
  4. Substitute the original variable back into the results, using the substitution.
  5. Solve for the original variable.
  6. Check the solutions.

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18.5E: Exercises

Practice Makes Perfect

? Exercise 18.5E.11 Solve equations in quadratic form

In the following exercises, solve.

1. $x^4-7x^2+12=0$
2. $x^4-9x^2+18=0$
3. $x^4-13x^2-30=0$
4. $x^4+5x^2-36=0$
5. $2x^4-5x^2+3=0$
6. $4x^4-5x^2+1=0$
7. $2x^4-7x^2+3=0$
8. $3x^4-14x^2+8=0$
9. $(x-3)^2-5(x-3)-36=0$
10. $(x+2)^2-3(x+2)-54=0$
11. $(3y+2)^2+(3y+2)-6=0$
12. $(5y-1)^2+3(5y-1)-28=0$
13. $\left(x^2+1\right)^2-5\left(x^2+1\right)+4=0$
14. $\left(x^2-4\right)^2-4\left(x^2-4\right)+3=0$
15. $2\left(x^2-5\right)^2-5\left(x^2-5\right)+2=0$
16. $2\left(x^2-5\right)^2-7\left(x^2-5\right)+6=0$
17. $x-\sqrt{x}-20=0$
18. $x-8\sqrt{x}+15=0$
19. $x+6\sqrt{x}-16=0$
20. $x+4\sqrt{x}-21=0$
21. $6x+\sqrt{x}-2=0$
22. $6x+\sqrt{x}-1=0$
23. $10x-17\sqrt{x}+3=0$
24. $12x+5\sqrt{x}-3=0$
25. $x^{\frac{2}{3}}+9x^{\frac{1}{3}}+8=0$
26. $x^{\frac{2}{3}}-3x^{\frac{1}{3}}=28$
27. $x^{\frac{2}{3}}+4x^{\frac{1}{3}}=12$
28. $x^{\frac{2}{3}}-11x^{\frac{1}{3}}+30=0$
29. $6x^{\frac{2}{3}}-x^{\frac{1}{3}}=12$
30. $3x^{\frac{2}{3}}-10x^{\frac{1}{3}}=8$
31. $8x^{\frac{2}{3}}-43x^{\frac{1}{3}}+15=0$
32. $20x^{\frac{2}{3}}-23x^{\frac{1}{3}}+6=0$
33. $x-8x^{\frac{1}{2}}+7=0$
34. $2x-7x^{\frac{1}{2}}=15$
35. $6x^{-2}+13x^{-1}+5=0$
36. $15x^{-2}-26x^{-1}+8=0$
37. $8x^{-2}-2x^{-1}-3=0$
38. $15x^{-2}-4x^{-1}-4=0$

Answer

1. $x=\pm\sqrt{3}, x=\pm 2$
3. $x=\pm\sqrt{15}, x=\pm\sqrt{2}i$

5. $x = \pm 1, x = \frac{\pm\sqrt{6}}{2}$

7. $x = \pm\sqrt{3}, x = \pm\frac{\sqrt{2}}{2}$

9. $x = -1, x = 12$

11. $x = -\frac{5}{3}, x = 0$

13. $x = 0, x = \pm\sqrt{3}$

15. $x = \pm\frac{11}{2}, x = \pm\frac{\sqrt{22}}{2}$

17. $x = 25$

19. $x = 4$

21. $x = \frac{1}{4}$

23. $x = \frac{1}{25}, x = \frac{9}{4}$

25. $x = -1, x = -512$

27. $x = 8, x = -216$

29. $x = \frac{27}{8}, x = -\frac{64}{27}$

31. $x = 27, x = 64,000$

33. $x = 1, x = 49$

35. $x = -2, x = -\frac{3}{5}$

37. $x = -2, x = \frac{4}{3}$

Exercise $18.5E.12$ writing exercises

1. Explain how to recognize an equation in quadratic form.
2. Explain the procedure for solving an equation in quadratic form.

Answer

1. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve equations in quadratic form.			

Figure 9.4.43

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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18.6: Solve Applications of Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve applications modeled by quadratic equations

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is -100. Find the numbers.
   If you missed this problem, review Example 2.18.
2. Solve: $\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$.
   If you missed this problem, review Example 7.35.
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
   If you missed this problem, review Example 2.34.

Solve Applications Modeled by Quadratic Equations

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications.

Let's first summarize the methods we now have to solve quadratic equations.

Methods to Solve Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

As you solve each equation, choose the method that is most convenient for you to work the problem. As a reminder, we will copy our usual Problem-Solving Strategy here so we can follow the steps.

Use a Problem-Solving Strategy

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We have solved number applications that involved consecutive even and odd integers, by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

	Consecutive even integers
	$64, 66, 68$
n	1st even integer
$n+2$	2nd consecutive even integer
$n+4$	3rd consecutive even integer

	Consecutive odd integers
	$77, 79, 81$
n	1st odd integer
$n+2$	2nd consecutive odd integer
$n+4$	3rd consecutive odd integer

Some applications of odd or even consecutive integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

Example 18.6.1

The product of two consecutive odd integers is 195. Find the integers.

Solution:

Step 1: **Read** the problem

Step 2: **Identify** what we are looking for.

We are looking for two consecutive odd integers.

Step 3: **Name** what we are looking for.

Let $n =$ the first odd integer.

$n+2 =$ the next odd integer.

Step 4: **Translate** into an equation. State the problem in one sentence.

"The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195.

Translate into an equation.

$$n(n+2) = 195$$

Step 5: **Solve** the equation. Distribute.

$$n^2 + 2n = 195$$

Write the equation in standard form.

$$n^2 + 2n - 195 = 0$$

Factor.

$$(n+15)(n-13) = 0$$

Use the Zero Product Property.

$$n+15 = 0 \quad n-13 = 0$$

Solve each equation.

$$n = -15, \quad n = 13$$

There are two values of n that are solutions. This will give us two pairs of consecutive odd integers for our solution.

First odd integer $n = 13$	First odd integer $n = -15$
next odd integer $n+2$	next odd integer $n+2$
$13+2$	$-15+2$
15	-13

Step 6: **Check** the answer.

Do these pairs work? Are they consecutive odd integers?

$13, 15$ yes
$-13, -15$ yes

Is their product 195?

$13 \cdot 15 = 195$ yes
$-13(-15) = 195$ yes

Step 7: **Answer** the question.

Two consecutive odd integers whose product is 195 are $13, 15$ and $-13, -15$.

Exercise 18.6.1

The product of two consecutive odd integers is 99. Find the integers.

Answer

The two consecutive odd integers whose product is 99 are $9, 11$, and $-9, -11$.

Exercise 18.6.2

The product of two consecutive even integers is 168. Find the integers.

Answer

The two consecutive even integers whose product is 128 are $12, 14$ and $-12, -14$.

We will use the formula for the area of a triangle to solve the next example.

Definition 18.6.1

Area of a Triangle

For a triangle with base, b, and height, h, the area, A, is given by the formula $A = \frac{1}{2}bh$.

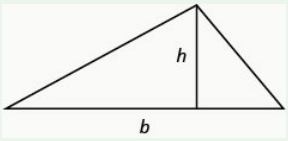


Figure 9.5.1

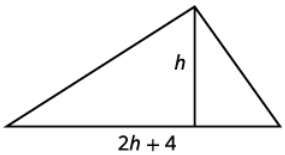
Recall that when we solve geometric applications, it is helpful to draw the figure.

Example 18.6.2

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can only have an area of 120 square feet and the architect wants the base to be 4 feet more than twice the height. Find the base and height of the window.

Solution:

Table 9.5.1

Step 1: **Read** the problem. Draw a picture.	
Step 2: **Identify** what we are looking for.	We are looking for the base and height.
Step 3: **Name** what we are looking for.	Let $h =$ the height of the triangle. $2h + 4 =$ the base of the triangle.

Step 4: **Translate** into an equation. We know the area. Write the formula for the area of a triangle.	$A=\frac{1}{2}bh$
Step 5: **Solve** the equation. Substitute in the values.	$120=\frac{1}{2}(2h+4)h$
Distribute.	$120=h^2+2h$
This is a quadratic equation, rewrite it in standard form.	$h^2+2h-120=0$
Factor.	$(h-10)(h+12)=0$
Use the Zero Product Property.	$h-10=0 \quad h+12=0$
Simplify.	$h=10, \quad h=\cancel{-12}$
Since h is the height of a window, a value of $h=-12$ does not make sense.	
The height of the triangle $h=10$.	
The base of the triangle $2h+4$. $2\cdot 10+4$ 24	
Step 6: **Check** the answer. Does a triangle with height 10 and base 24 have area 120? Yes.	
Step 7: **Answer** the question.	The height of the triangular window is 10 feet and the base is 24 feet.

? Exercise 18.6.3

Find the base and height of a triangle whose base is four inches more than six times its height and has an area of 456 square inches.

Answer

The height of the triangle is 12 inches and the base is 76 inches.

? Exercise 18.6.4

If a triangle that has an area of 110 square feet has a base that is two feet less than twice the height, what is the length of its base and height?

Answer

The height of the triangle is 11 feet and the base is 20 feet.

In the two preceding examples, the number in the radical in the **Quadratic Formula** was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

We will use the formula for the area of a rectangle to solve the next example.

Definition 18.6.2

Area of a Rectangle

For a rectangle with length, L, and width, W, the area, A, is given by the formula $A=LW$.

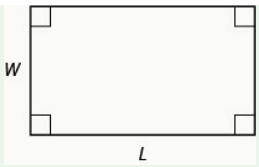


Figure 9.5.3

Example 18.6.3

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than 3 times the width. Find the length and width. Round to the nearest tenth of a foot.

Solution:

Table 9.5.2

Step 1: **Read** the problem. Draw a picture.	
Step 2: **Identify** what we are looking for.	We are looking for the length and width.
Step 3: **Name** what we are looking for.	Let $w =$ the width of the rectangle. $3w - 1 =$ the length of the rectangle
Step 4: **Translate** into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Step 5: **Solve** the equation. Substitute in the values.	$150 = (3w - 1)w$
Distribute.	$150 = 3w^2 - w$
This is a quadratic equation; rewrite it in standard form. Solve the equation using the Quadratic Formula.	$ax^2 + bx + c = 0$ $3w^2 - w - 150 = 0$
Identify the a, b, c values.	$a = 3, b = -1, c = -150$
Write the Quadratic Formula.	$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-150)}}{2 \cdot 3}$
Simplify.	$w = \frac{1 \pm \sqrt{1 + 1800}}{6}$ $w = \frac{1 \pm \sqrt{1801}}{6}$ Figure 9.5.13
Rewrite to show two solutions.	$w = \frac{1 + \sqrt{1801}}{6}, \quad w = \frac{1 - \sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w \approx 7.2, \quad$ ~~$w \approx -6.9$~~ Width $w \approx 7.2$ Length $\approx 3w - 1$ $\approx 3(7.2) - 1$ ≈ 20.6
Step 6: **Check** the answer. Make sure that the answers make sense. Since the answers are approximate, the area will not come out exactly to 150.	
Step 7: **Answer** the question.	The width of the rectangle is approximately 7.2 feet and the length is approximately 20.6 feet.

Exercise 18.6.5

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

Answer

The length of the garden is approximately 18 feet and the width 11 feet.

Exercise 18.6.6

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length.What are the length and width of the tablecloth to the nearest tenth of a foot?

Answer

The length of the tablecloth is approximately 11.8 feet and the width 6.8 feet.

The **Pythagorean Theorem** gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Definition 18.6.3

Pythagorean Theorem

In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2 + b^2 = c^2$.

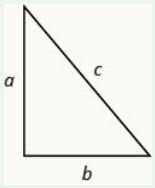


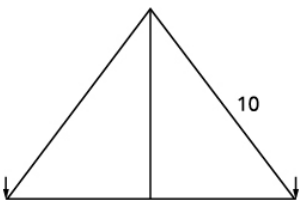
Figure 9.5.16

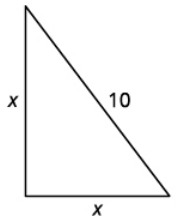
Example 18.6.4

Rene is setting up a holiday light display. He wants to make a 'tree' in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Solution:

Table 9.5.3

Step 1: **Read** the problem. Draw a picture.	
Step 2: **Identify** what we are looking for.	We are looking for the height of the pole.

Step 3: **Name** what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let $x =$ the height of the pole. $x =$ the distance from pole to stake Each side is a right triangle. We draw a picture of one of them.  Figure 9.5.18
Step 4: **Translate** into an equation. We can use the Pythagorean Theorem to solve for x. Write the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Step 5: **Solve** the equation. Substitute.	$x^2 + x^2 = 10^2$
Simplify.	$2x^2 = 100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2} = \frac{100}{2}$
Simplify.	$x^2 = 50$
Use the Square Root Property.	$x = \pm\sqrt{50}$
Simplify the radical.	$x = \pm 5\sqrt{2}$
Rewrite to show two solutions.	$x = 5\sqrt{2}, \quad x = -5\sqrt{2}$
	If we approximate this number to the nearest tenth with a calculator, we find $x \approx 7.1$.
Step 6: **Check** the answer. Check on your own in the Pythagorean Theorem.	
Step 7: **Answer** the question.	The pole should be about 7.1 feet tall.

? Exercise 18.6.7

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth.

Answer

The length of the flag pole's shadow is approximately 6.3 feet and the height of the flag pole is 18.9 feet.

? Exercise 18.6.8

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

Answer

The distance between the opposite corners is approximately 7.2 feet.

The height of a projectile shot upward from the ground is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Definition 18.6.4

The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula

$$h = -16t^2 + v_0 t$$

We can use this formula to find how many seconds it will take for a firework to reach a specific height.

Example 18.6.5

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Solution:

Table 9.5.4

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking for.	We are looking for the number of seconds, which is time.
Step 3: **Name** what we are looking for.	Let $t =$ the number of seconds.
Step 4: **Translate** into an equation. Use the formula.	$h = -16t^2 + v_0 t$
Step 5: **Solve** the equation. We know the velocity v_0 is 130 feet per second. The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$
This is a quadratic equation, rewrite it in standard form. Solve the equation using the Quadratic Formula.	$\underset{a}{16t^2} \underset{b}{-130t} \underset{c}{+260} = 0$
Identify the values of a, b, c.	$a = 16,\ b = -130,\ c = 260$
Write the Quadratic Formula.	$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot (260)}}{2 \cdot 16}$
Simplify.	$t = \frac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ $t = \frac{130 \pm \sqrt{260}}{32}$ Figure 9.5.26
Rewrite to show two solutions.	$t = \frac{130 + \sqrt{260}}{32}, \quad t = \frac{130 - \sqrt{260}}{32}$
Approximate the answer with a calculator.	$t \approx 4.6$ seconds, $\quad t \approx 3.6$ seconds
Step 6: **Check** the answer. The check is left to you.	
Step 7: **Answer** the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

Exercise 18.6.9

An arrow is shot from the ground into the air at an initial speed of 108 ft/s. Use the formula $h = -16t^2 + v_0 t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth.

Answer

The arrow will reach 180 feet on its way up after 3 seconds and again on its way down after approximately 3.8 seconds.

Exercise 18.6.10

A man throws a ball into the air with a velocity of 96 ft/s. Use the formula $h = -16t^2 + v_0 t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth.

Answer

The ball will reach 48 feet on its way up after approximately $.6$ second and again on its way down after approximately 5.4 seconds.

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate	•	Time	=	Distance

Figure 9.5.29

The formula $D = rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula $t = \frac{D}{r}$.

Some uniform motion problems are also modeled by quadratic equations.

Example 18.6.6

Professor Smith just returned from a conference that was $2,000$ miles east of his home. His total time in the airplane for the round trip was 9 hours. If the plane was flying at a rate of 450 miles per hour, what was the speed of the jet stream?

Solution:

This is a uniform motion situation. A diagram will help us visualize the situation.

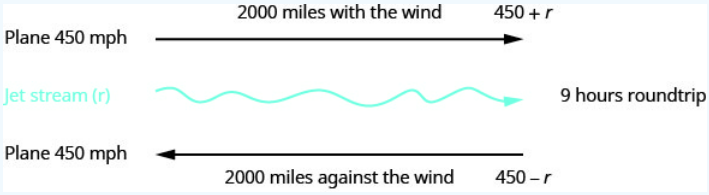


Figure 9.5.30

We fill in the chart to organize the information.

We are looking for the speed of the jet stream. Let $r =$ the speed of the jet stream.

When the plane flies with the wind, the wind increases its speed and so the rate is $450 + r$.

When the plane flies against the wind, the wind decreases its speed and the rate is $450 - r$.

Table 9.5.5

Write in the rates. Write in the distances. Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th>Type</th><th>Rate</th><th>·</th><th>Time</th><th>=</th><th>Distance</th></tr><tr><td>Headwind</td><td>$450 - r$</td><td></td><td>$\frac{2000}{450-r}$</td><td></td><td>2000</td></tr><tr><td>Tailwind</td><td>$450 + r$</td><td></td><td>$\frac{2000}{450+r}$</td><td></td><td>2000</td></tr><tr><td></td><td></td><td></td><td>9</td><td></td><td></td></tr></table>
We know the times add to 9 and so we write our equation.	$\frac{2000}{450-r} + \frac{2000}{450+r} = 9$
We multiply both sides by the LCD.	$(450-r)(450+r)\left(\frac{2000}{450-r} + \frac{2000}{450+r}\right) = 9(450-r)(450+r)$
Simplify.	$2000(450+r) + 2000(450-r) = 9(450-r)(450+r)$
Factor the 2, 000.	$2000(450+r+450-r) = 9\left(450^2 - r^2\right)$
Solve.	$2000(900) = 9\left(450^2 - r^2\right)$
Divide by 9.	$2000(100) = 450^2 - r^2$
Simplify.	$200000 = 202500 - r^2$ $-2500 = -r^2$ \ $50 = r$ The speed of the jet stream is 50 mph.
Check: Is 50 mph a reasonable speed for the jet stream? Yes. If the plane is traveling 450 mph and the wind is 50 mph, Tailwind $450 + 50 = 500\text{mph} \quad \frac{2000}{500} = 4$ hours Headwind $450 - 50 = 400\text{mph} \quad \frac{2000}{400} = 5$ hours The times add to 9 hours, so it checks.	

The speed of the jet stream was 50 mph.

Exercise 18.6.11

MaryAnne just returned from a visit with her grandchildren back east. The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?

Answer

The speed of the jet stream was 100 mph.

Exercise 18.6.12

Gerry just returned from a cross country trip. The trip was 3000 miles from his home and his total time in the airplane for the round trip was 11 hours. If the plane was flying at a rate of 550 miles per hour, what was the speed of the jet stream?

Answer

The speed of the jet stream was 50 mph.

Work applications can also be modeled by quadratic equations. We will set them up using the same methods we used when we solved them with rational equations.We'll use a similar scenario now.

Example 18.6.7

The weekly gossip magazine has a big story about the presidential election and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 12 hours more than Press #2 to do the job and when both presses are running they can print the job in 8 hours. How long does it take for each press to print the job alone?

Solution:

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take each press separately to complete the job.

Table 9.5.6

Let $x =$ the number of hours for Press #2 to complete the job. Enter the hours per job for Press #1, Press #2, and when they work together.	<table><tr><th></th><th>Number of hours needed to complete the job.</th><th>Part of job completed/hour</th></tr><tr><td>Press #1</td><td>$x+12$</td><td>$\frac{1}{x+12}$</td></tr><tr><td>Press #2</td><td>x</td><td>$\frac{1}{x}$</td></tr><tr><td>Together</td><td>8</td><td>$\frac{1}{8}$</td></tr></table>
The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together. Translate to an equation.	Work completed by Press #1 + Press #2 = Together $\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$
Solve.	$\frac{1}{x+12}+\frac{1}{x}=\frac{1}{8}$
Multiply by the LCD, $8x(x+12)$.	$8x(x+12)\left(\frac{1}{x+12}+\frac{1}{x}\right)=\left(\frac{1}{8}\right)8x(x+12)$
Simplify.	$8x+8(x+12)=x(x+12)$ $8x+8x+96=x^2+12x$ Figure 9.5.37 $0=x^2-4x-96$ Figure 9.5.38
Solve.	$0=(x-12)(x+8)$ $x-12=0, x+8=0$ Figure 9.5.40 $x=12, x=-8$ hours Figure 9.5.41
Since the idea of negative hours does not make sense, we use the values $x=12$.	$12+12$ 24 hours 12 12 hours Figure 9.5.43
Write our sentence answer.	Press #1 would take 24 hours and Press #2 would take 12 hours to do the job alone.

Exercise 18.6.13

The weekly news magazine has a big story naming the Person of the Year and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours more than Press #2 to do the job and when both presses are running they can print the job in 4 hours. How long does it take for each press to print the job alone?

Answer

Press #1 would take 12 hours, and Press #2 would take 6 hours to do the job alone.

Exercise 18.6.14

Erlinda is having a party and wants to fill her hot tub. If she only uses the red hose it takes 3 hours more than if she only uses the green hose. If she uses both hoses together, the hot tub fills in 2 hours. How long does it take for each hose to fill the hot tub?

Answer

The red hose take 6 hours and the green hose take 3 hours alone.

Access these online resources for additional instruction and practice with solving applications modeled by quadratic equations.

- Word Problems Involving Quadratic Equations
- Quadratic Equation Word Problems
- Applying the Quadratic Formula

Key Concepts

- Methods to Solve Quadratic Equations
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to use a Problem-Solving Strategy.
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- Area of a Triangle
  - For a triangle with base, b, and height, h, the area, A, is given by the formula $A = \frac{1}{2}bh$.

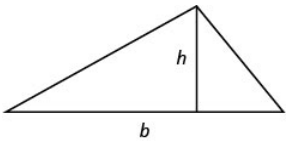


Figure 9.5.1

- Area of a Rectangle
  - For a rectangle with length,L, and width, W, the area, A, is given by the formula $A = LW$.

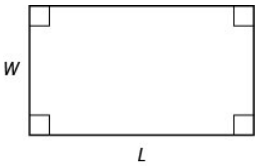


Figure 9.5.3

- Pythagorean Theorem
  - In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2 + b^2 = c^2$.

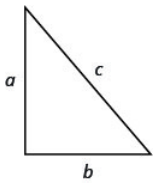


Figure 9.5.16

- Projectile motion
  - The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula $h = -16t^2 + v_0 t$.

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18.6E: Exercises

Practice Makes Perfect

Exercise $18.6E.15$ Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method.

1. The product of two consecutive odd numbers is 255. Find the numbers.
2. The product of two consecutive even numbers is 360. Find the numbers.
3. The product of two consecutive even numbers is 624. Find the numbers.
4. The product of two consecutive odd numbers is $1,023$. Find the numbers.
5. The product of two consecutive odd numbers is 483. Find the numbers.
6. The product of two consecutive even numbers is 528. Find the numbers.

Answer

1. Two consecutive odd numbers whose product is 255 are 15 and 17, and -15 and -17.

3. The first and second consecutive odd numbers are 24 and 26, and -26 and -24.

5. Two consecutive odd numbers whose product is 483 are 21 and 23, and -21 and -23.

Exercise $18.6E.16$ Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method. Round your answers to the nearest tenth, if needed.

1. A triangle with area 45 square inches has a height that is two less than four times the base Find the base and height of the triangle.
2. The base of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the base and height of the triangle.
3. The area of a triangular flower bed in the park has an area of 120 square feet. The base is 4 feet longer that twice the height. What are the base and height of the triangle?
4. A triangular banner for the basketball championship hangs in the gym. It has an area of 75 square feet. What is the length of the base and height , if the base is two-thirds of the height?
5. The length of a rectangular driveway is five feet more than three times the width. The area is 50 square feet. Find the length and width of the driveway.
6. A rectangular lawn has area 140 square yards. Its length is six yards less than twice its width. What are the length and width of the lawn?
7. A rectangular table for the dining room has a surface area of 24 square feet. The length is two more feet than twice the width of the table. Find the length and width of the table.
8. The new computer has a surface area of 168 square inches. If the the width is 5.5 inches less that the length, what are the dimensions of the computer?
9. The hypotenuse of a right triangle is twice the length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.
10. The hypotenuse of a right triangle is 10 cm long. One of the triangle's legs is three times as the length of the other leg . Round to the nearest tenth. Find the lengths of the three sides of the triangle.
11. A rectangular garden will be divided into two plots by fencing it on the diagonal. The diagonal distance from one corner of the garden to the opposite corner is five yards longer than the width of the garden. The length of the garden is three times the width. Find the length of the diagonal of the garden.

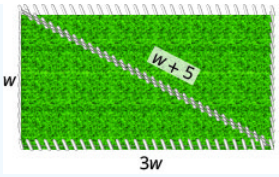

Figure 9.5.44

12. Nautical flags are used to represent letters of the alphabet. The flag for the letter, O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The hypotenuse of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.

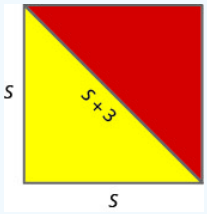

Figure 9.5.45

13. Gerry plans to place a 25-foot ladder against the side of his house to clean his gutters. The bottom of the ladder will be 5 feet from the house.How for up the side of the house will the ladder reach?

14. John has a 10-foot piece of rope that he wants to use to support his 8-foot tree. How far from the base of the tree should he secure the rope?

15. An arrow is shot vertically upward at a rate of $v_0 = 220$ feet per second. Use the projectile formula $h = -16t^2 + v_0 t$, to determine when the height of the arrow will be 400 feet.

16. A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when the height of the firework rocket will be 1200 feet.

17. A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h = -16t^2 + v_0 t + 8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h = 0$?)

18. A stone is dropped from a 196-foot platform. Use the formula $h = -16t^2 + v_0 t + 196$ to determine how many seconds it will take for the stone to hit the ground. (Since the stone is dropped, $v_0 = 0$.)

19. The businessman took a small airplane for a quick flight up the coast for a lunch meeting and then returned home. The plane flew a total of 4 hours and each way the trip was 200 miles. What was the speed of the wind that affected the plane which was flying at a speed of 120 mph?

20. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 300 miles. If the plane was flying at 125 mph, what was the speed of the wind that affected the plane?

21. Roy kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

22. Rick paddled up the river, spent the night camping, and then paddled back. He spent 10 hours paddling and the campground was 24 miles away. If Rick kayaked at a speed of 5 mph, what was the speed of the current?

23. Two painters can paint a room in 2 hours if they work together. The less experienced painter takes 3 hours more than the more experienced painter to finish the job. How long does it take for each painter to paint the room individually?

24. Two gardeners can do the weekly yard maintenance in 8 minutes if they work together. The older gardener takes 12 minutes more than the younger gardener to finish the job by himself. How long does it take for each gardener to do the weekly yard maintenance individually?

25. It takes two hours for two machines to manufacture $10,000$ parts. If Machine #1 can do the job alone in one hour less than Machine #2 can do the job, how long does it take for each machine to manufacture $10,000$ parts alone?

26. Sully is having a party and wants to fill his swimming pool. If he only uses his hose it takes 2 hours more than if he only uses his neighbor's hose. If he uses both hoses together, the pool fills in 4 hours. How long does it take for each hose to fill the pool?

Answer

1. The width of the triangle is 5 inches and the height is 18 inches.
3. The base is 24 feet and the height of the triangle is 10 feet.
5. The length of the driveway is 15.0 feet and the width is 3.3 feet.
7. The length of table is 8 feet and the width is 3 feet.
9. The length of the legs of the right triangle are 3.2 and 9.6 cm.
11. The length of the diagonal fencing is 7.3 yards.
13. The ladder will reach 24.5 feet on the side of the house.
15. The arrow will reach 400 feet on its way up in 2.2 seconds and on the way down in 11.6 seconds.
17. The bullet will take 70 seconds to hit the ground.
19. The speed of the wind was 49 mph.
21. The speed of the current was 4.3 mph.
23. The less experienced painter takes 6 hours and the experienced painter takes 3 hours to do the job alone.
25. Machine #1 takes 3.6 hours and Machine #2 takes 4.6 hours to do the job alone.

? Exercise 18.6E. 17 writing exercises

1. Make up a problem involving the product of two consecutive odd integers.
   a. Start by choosing two consecutive odd integers. What are your integers?
   b. What is the product of your integers?
   c. Solve the equation $n(n+2)=p$, where p is the product you found in part (b).
   d. Did you get the numbers you started with?
2. Make up a problem involving the product of two consecutive even integers.
   a. Start by choosing two consecutive even integers. What are your integers?
   b. What is the product of your integers?
   c. Solve the equation $n(n+2)=p$, where p is the product you found in part (b).
   d. Did you get the numbers you started with?

Answer

1. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve applications of the quadratic formula.			

Figure 9.5.46

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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18.7: Review Exercises

Chapter Review Exercises

Solve Quadratic Equations Using the Square Root Property

Exercise 18.7.1 Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

1. $y^2 = 144$
2. $n^2 - 80 = 0$
3. $4a^2 = 100$
4. $2b^2 = 72$
5. $r^2 + 32 = 0$
6. $t^2 + 18 = 0$
7. $\frac{2}{3}w^2 - 20 = 30$
8. $5c^2 + 3 = 19$

Answer

1. $y = \pm 12$
3. $a = \pm 5$
5. $r = \pm 4\sqrt{2}i$
7. $w = \pm 5\sqrt{3}$

Exercise 18.7.2 Solve Quadratic Equations of the Form $a(x-h)^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

1. $(p-5)^2 + 3 = 19$
2. $(u+1)^2 = 45$
3. $\left(x - \frac{1}{4}\right)^2 = \frac{3}{16}$
4. $\left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$
5. $(n-4)^2 - 50 = 150$
6. $(4c-1)^2 = -18$
7. $n^2 + 10n + 25 = 12$
8. $64a^2 + 48a + 9 = 81$

Answer

1. $p = -1, 9$
3. $x = \frac{1}{4} \pm \frac{\sqrt{3}}{4}$
5. $n = 4 \pm 10\sqrt{2}$
7. $n = -5 \pm 2\sqrt{3}$

Solve Quadratic Equations by Completing the Square

Exercise 18.7.3 Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

1. $x^2 + 22x$

2. $m^2 - 8m$
3. $a^2 - 3a$
4. $b^2 + 13b$

Answer

1. $(x+11)^2$
3. $\left(a - \frac{3}{2}\right)^2$

Exercise 18.7.4 Solve Quadratic Equations Using Completing the Square

In the following exercises, solve by completing the square.

1. $d^2 + 14d = -13$
2. $y^2 - 6y = 36$
3. $m^2 + 6m = -109$
4. $t^2 - 12t = -40$
5. $v^2 - 14v = -31$
6. $w^2 - 20w = 100$
7. $m^2 + 10m - 4 = -13$
8. $n^2 - 6n + 11 = 34$
9. $a^2 = 3a + 8$
10. $b^2 = 11b - 5$
11. $(u+8)(u+4) = 14$
12. $(z-10)(z+2) = 28$

Answer

1. $d = -13, -1$
3. $m = -3 \pm 10i$
5. $v = 7 \pm 3\sqrt{2}$
7. $m = -9, -1$
9. $a = \frac{3}{2} \pm \frac{\sqrt{41}}{2}$
11. $u = -6 \pm 2\sqrt{2}$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

Exercise 18.7.5 Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

1. $3p^2 - 18p + 15 = 15$
2. $5q^2 + 70q + 20 = 0$
3. $4y^2 - 6y = 4$
4. $2x^2 + 2x = 4$
5. $3c^2 + 2c = 9$
6. $4d^2 - 2d = 8$
7. $2x^2 + 6x = -5$
8. $2x^2 + 4x = -5$

Answer

1. $p = 0, 6$

3. $y = -\frac{1}{2}, 2$

5. $c = -\frac{1}{3} \pm \frac{2\sqrt{7}}{3}$

7. $x = \frac{3}{2} \pm \frac{1}{2}i$

Exercise 18.7.6 Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

1. $4x^2 - 5x + 1 = 0$
2. $7y^2 + 4y - 3 = 0$
3. $r^2 - r - 42 = 0$
4. $t^2 + 13t + 22 = 0$
5. $4v^2 + v - 5 = 0$
6. $2w^2 + 9w + 2 = 0$
7. $3m^2 + 8m + 2 = 0$
8. $5n^2 + 2n - 1 = 0$
9. $6a^2 - 5a + 2 = 0$
10. $4b^2 - b + 8 = 0$
11. $u(u - 10) + 3 = 0$
12. $5z(z - 2) = 3$
13. $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$
14. $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$
15. $4c^2 + 4c + 1 = 0$
16. $9d^2 - 12d = -4$

Answer

1. $x = \frac{1}{4}, 1$

3. $r = -6, 7$

5. $v = \frac{-1 \pm \sqrt{21}}{8}$

7. $m = \frac{-4 \pm \sqrt{10}}{3}$

9. $a = \frac{5}{12} \pm \frac{\sqrt{23}}{12}i$

11. $u = 5 \pm \sqrt{21}$

13. $p = \frac{4 \pm \sqrt{5}}{5}$

15. $c = -\frac{1}{2}$

Exercise 18.7.7 Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

1. a. $9x^2 - 6x + 1 = 0$
   b. $3y^2 - 8y + 1 = 0$
   c. $7m^2 + 12m + 4 = 0$
   d. $5n^2 - n + 1 = 0$
2. a. $5x^2 - 7x - 8 = 0$
   b. $7x^2 - 10x + 5 = 0$
   c. $25x^2 - 90x + 81 = 0$
   d. $15x^2 - 8x + 4 = 0$

Answer

1.

a. 1
b. 2
c. 2
d. 2

Exercise 18.7.8 Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

1. a. $16r^2-8r+1=0$
   b. $5t^2-8t+3=9$
   c. $3(c+2)^2=15$
2. a. $4d^2+10d-5=21$
   b. $25x^2-60x+36=0$
   c. $6(5v-7)^2=150$

Answer

1.

a. Factor
b. Quadratic Formula
c. Square Root

Solve Equations in Quadratic Form

Exercise 18.7.9 Solve Equations in Quadratic Form

In the following exercises, solve.

1. $x^4-14x^2+24=0$
2. $x^4+4x^2-32=0$
3. $4x^4-5x^2+1=0$
4. $(2y+3)^2+3(2y+3)-28=0$
5. $x+3\sqrt{x}-28=0$
6. $6x+5\sqrt{x}-6=0$
7. $x^{\frac{2}{3}}-10x^{\frac{1}{3}}+24=0$
8. $x+7x^{\frac{1}{2}}+6=0$
9. $8x^{-2}-2x^{-1}-3=0$

Answer

1. $x=\pm\sqrt{2}, x=\pm 2\sqrt{3}$
3. $x=\pm 1, x=\pm\frac{1}{2}$
5. $x=16$
7. $x=64, x=216$
9. $x=-2, x=\frac{4}{3}$

Exercise 18.7.10 Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using the method of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth, if needed.

1. Find two consecutive odd numbers whose product is 323.
2. Find two consecutive even numbers whose product is 624.
3. A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.
4. Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.
5. A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.

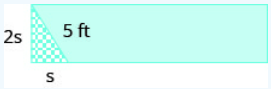

Figure 9.E.1

6. A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

7. The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

8. For Sophia's graduation party, several tables of the same width will be arranged end to end to give serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth . Round answer to the nearest tenth.

9. A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

10. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 360 miles. If the plane was flying at 150 mph, what was the speed of the wind that affected the plane?

11. Ezra kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

12. Two handymen can do a home repair in 2 hours if they work together. One of the men takes 3 hours more than the other man to finish the job by himself. How long does it take for each handyman to do the home repair individually?

Answer

2. Two consecutive even numbers whose product is 624 are 24 and 26, and -24 and -26.

4. The height is 14 inches and the width is 10 inches.

6. The length of the diagonal is 3.6 feet.

8. The width of the serving table is 4.7 feet and the length is 16.1 feet.

Figure 9.E.2

10. The speed of the wind was 30 mph.

12. One man takes 3 hours and the other man 6 hours to finish the repair alone.

Graph Quadratic Functions Using Properties

Exercise 18.7.11 Recognize the Graph of a Quadratic Function

In the following exercises, graph by plotting point.

1. Graph $y = x^2 - 2$
2. Graph $y = -x^2 + 3$

Answer

2.

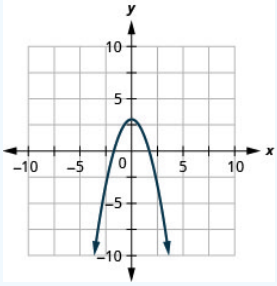

Figure 9.E.3

Exercise 18.7.12 Recognize the Graph of a Quadratic Function

In the following exercises, determine if the following parabolas open up or down.

1. a. $y = -3x^2 + 3x - 1$
   b. $y = 5x^2 + 6x + 3$
2. a. $y = x^2 + 8x - 1$
   b. $y = -4x^2 - 7x + 1$

Answer

2.

a. Up
b. Down

Exercise 18.7.13 Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find

a. The equation of the axis of symmetry
b. The vertex

1. $y = -x^2 + 6x + 8$
2. $y = 2x^2 - 8x + 1$

Answer

2. $x = 2$; $(2, -7)$

Exercise 18.7.14 Find the Intercepts of a Parabola

In the following exercises, find the x- and y-intercepts.

1. $y = x^2 - 4x + 5$

2. $y = x^2 - 8x + 15$
3. $y = x^2 - 4x + 10$
4. $y = -5x^2 - 30x - 46$
5. $y = 16x^2 - 8x + 1$
6. $y = x^2 + 16x + 64$

Answer

2. $y : (0, 15)$
   $x : (3, 0), (5, 0)$
4. $y : (0, -46)$
   $x :$ none
6. $y : (0, -64)$
   $x : (-8, 0)$

Graph Quadratic Functions Using Properties

Exercise 18.7.15 Graph Quadratic Functions Using Properties

In the following exercises, graph by using its properties.

1. $y = x^2 + 8x + 15$
2. $y = x^2 - 2x - 3$
3. $y = -x^2 + 8x - 16$
4. $y = 4x^2 - 4x + 1$
5. $y = x^2 + 6x + 13$
6. $y = -2x^2 - 8x - 12$

Answer

2.

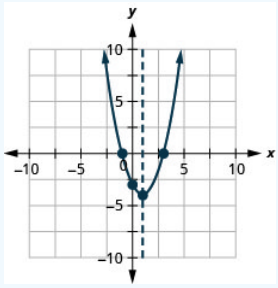

Figure 9.E.4

4.

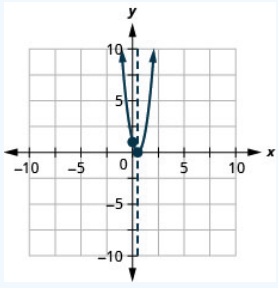

Figure 9.E.5

6.

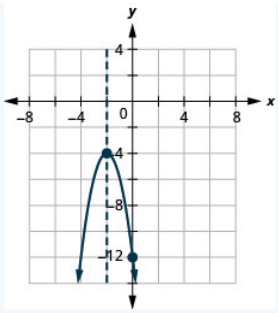

Figure 9.E.6

Exercise 18.7.16 Solve Maximum and Minimum Applications

In the following exercises, find the minimum or maximum value.

1. $y = 7x^2 + 14x + 6$
2. $y = -3x^2 + 12x - 10$

Answer

2. The maximum value is 2 when $x = 2$.

Exercise 18.7.17 Solve Maximum and Minimum Applications

In the following exercises, solve. Rounding answers to the nearest tenth.

1. A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.
2. A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.

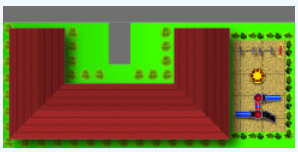
Figure 9.E.7

Answer

2. The length adjacent to the building is 90 feet giving a maximum area of $4,050$ square feet.

Graph Quadratic Functions Using Transformations

Exercise 18.7.18 Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using a vertical shift.

1. $g(x) = x^2 + 4$
2. $h(x) = x^2 - 3$

Answer

2.

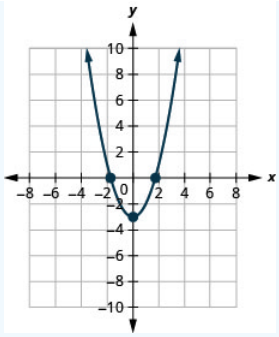

Figure 9.E.8

Exercise 18.7.19 Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using a horizontal shift.

1. $f(x) = (x+1)^2$
2. $g(x) = (x-3)^2$

Answer

2.

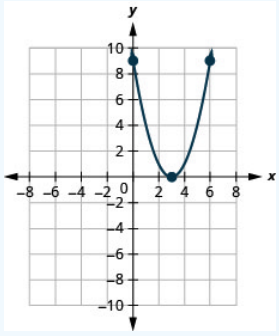

Figure 9.E.9

Exercise 18.7.20 Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using transformations.

1. $f(x) = (x+2)^2 + 3$
2. $f(x) = (x+3)^2 - 2$
3. $f(x) = (x-1)^2 + 4$
4. $f(x) = (x-4)^2 - 3$

Answer

2.

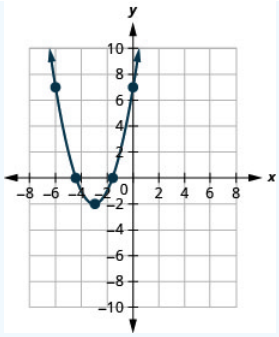

Figure 9.E.10

4.

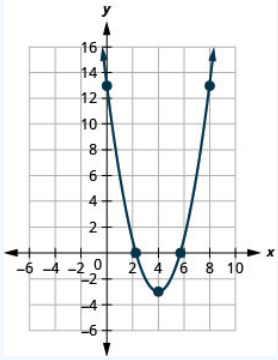

Figure 9.E.11

Exercise 18.7.21 Graph Quadratic Functions of the Form $f(x) = ax^2$

In the following exercises, graph each function.

1. $f(x) = 2x^2$
2. $f(x) = -x^2$
3. $f(x) = \frac{1}{2}x^2$

Answer

2.

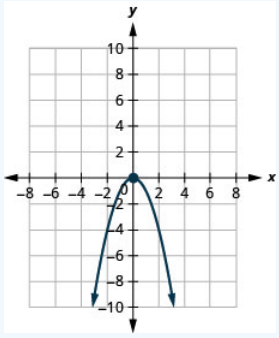

Figure 9.E.12

Exercise 18.7.22 Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x) = a(x-h)^2 + k$ form by completing the square.

1. $f(x) = 2x^2 - 4x - 4$

2. $f(x) = 3x^2 + 12x + 8$

Answer

1. $f(x) = 2(x-1)^2 - 6$

Exercise 18.7.23 Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x) = a(x-h)^2 + k$ form
b. Graph it by using transformations

1. $f(x) = 3x^2 - 6x - 1$
2. $f(x) = -2x^2 - 12x - 5$
3. $f(x) = 2x^2 + 4x + 6$
4. $f(x) = 3x^2 - 12x + 7$

Answer

1.

a. $f(x) = 3(x-1)^2 - 4$
b.

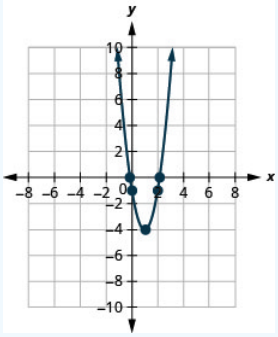

Figure 9.E.13

3.

a. $f(x) = 2(x+1)^2 + 4$
b.

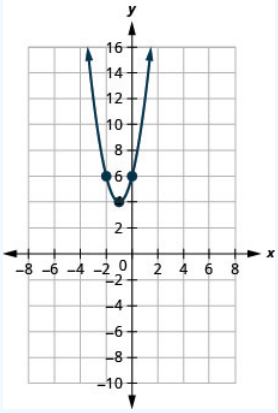

Figure 9.E.14

Exercise 18.7.24 Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x) = a(x-h)^2 + k$ form
b. Graph it using properties
   1. $f(x) = -3x^2 - 12x - 5$
   2. $f(x) = 2x^2 - 12x + 7$

Answer

1.

a. $f(x) = -3(x+2)^2 + 7$
b.

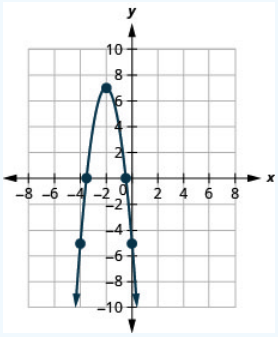

Figure 9.E.15

Exercise 18.7.25 Find a Quadratic Function From its Graph

In the following exercises, write the quadratic function in $f(x) = a(x-h)^2 + k$ form.

1.

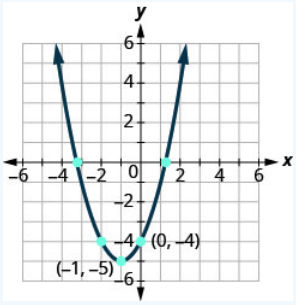

Figure 9.E.16

2.

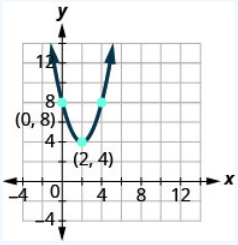

Figure 9.E.17

Answer

1. $f(x) = (x+1)^2 - 5$

Exercise 18.7.26 Solve Quadratic Inequalities Graphically

In the following exercises, solve graphically and write the solution in interval notation.

1. $x^2 - x - 6 > 0$
2. $x^2 + 4x + 3 \leq 0$
3. $-x^2 - x + 2 \geq 0$
4. $-x^2 + 2x + 3 < 0$

Answer

1.

a.

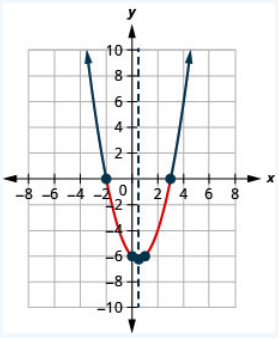

Figure 9.E.18

b. $(-\infty, -2) \cup (3, \infty)$

3.

a.

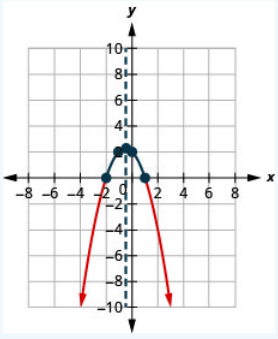

Figure 9.E.19

b. $[-2, 1]$

Exercise 18.7.27 Solve Quadratic Inequalities Graphically

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

1. $x^2 - 6x + 8 < 0$
2. $x^2 + x > 12$
3. $x^2 - 6x + 4 \leq 0$
4. $2x^2 + 7x - 4 > 0$
5. $-x^2 + x - 6 > 0$
6. $x^2 - 2x + 4 \geq 0$

Answer

1. $(2, 4)$
3. $[3-\sqrt{5}, 3+\sqrt{5}]$
5. no solution

Practice Test

Exercise 18.7.28

1. Use the Square Root Property to solve the quadratic equation $3(w+5)^2 = 27$.
2. Use Completing the Square to solve the quadratic equation $a^2 - 8a + 7 = 23$.
3. Use the Quadratic Formula to solve the quadratic equation $2m^2 - 5m + 3 = 0$.

Answer

1. $w = -2, w = -8$
3. $m = 1, m = \frac{3}{2}$

Exercise 18.7.29

Solve the following quadratic equations. Use any method.

1. $2x(3x-2) - 1 = 0$
2. $\frac{9}{4}y^2 - 3y + 1 = 0$

Answer

2. $y = \frac{2}{3}$

Exercise 18.7.30

Use the discriminant to determine the number and type of solutions of each quadratic equation.

1. $6p^2 - 13p + 7 = 0$
2. $3q^2 - 10q + 12 = 0$

Answer

2. 2 complex

Exercise 18.7.31

Solve each equation.

1. $4x^4 - 17x^2 + 4 = 0$
2. $y^{\frac{2}{3}} + 2y^{\frac{1}{3}} - 3 = 0$

Answer

2. $y = 1, y = -27$

Exercise 18.7.32

For each parabola, find

a. Which direction it opens
b. The equation of the axis of symmetry

c. The vertex
d. The x-and y-intercepts
e. The maximum or minimum value

1. $y = 3x^2 + 6x + 8$
2. $y = -x^2 - 8x + 16$

Answer

2.

a. down
b. $x = -4$
c. $(-4, 0)$
d. $y : (0, 16); x : (-4, 0)$
e. minimum value of -4 when $x = 0$

Exercise 18.7.33

Graph each quadratic function using intercepts, the vertex, and the equation of the axis of symmetry.

1. $f(x) = x^2 + 6x + 9$
2. $f(x) = -2x^2 + 8x + 4$

Answer

2.

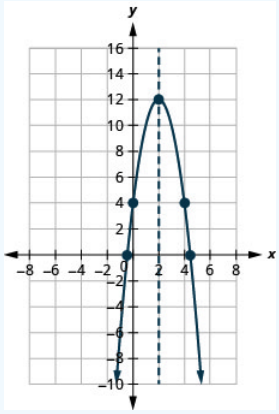

Figure 9.E.20

Exercise 18.7.34

In the following exercises, graph each function using transformations.

1. $f(x) = (x + 3)^2 + 2$
2. $f(x) = x^2 - 4x - 1$

Answer

2.

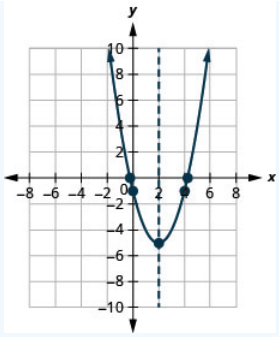

Figure 9.E.21

Exercise 18.7.35

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

1. $x^2 - 6x - 8 \leq 0$
2. $2x^2 + x - 10 > 0$

Answer

2. $\left(-\infty, -\frac{5}{2}\right) \cup (2, \infty)$

Exercise 18.7.36

Model the situation with a quadratic equation and solve by any method.

1. Find two consecutive even numbers whose product is 360.
2. The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

Answer

2. A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$ find how long it will take the balloon to reach the maximum height, and then find the maximum height. Round to the nearest tenth.

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CHAPTER OVERVIEW

19: Equations and Inequalities

In this chapter, you will explore linear equations, develop a strategy for solving them, and relate them to real-world situations.

19.1: Review- Solve Linear Inequalities

19.1E: Exercises

19.2: Solve Compound Inequalities

19.2E: Exercises

19.3: Solve Absolute Value Equations and Inequalities

19.3E: Exercises

19.4: Solve Rational Inequalities

19.4E: Exercises

19.5: Solve Quadratic Inequalities

19.5E: Exercises

19.E: Review Exercises 1

19.E: Review Exercises 2

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19.1: Review- Solve Linear Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Graph inequalities on the number line
- Solve linear inequalities
- Translate words to an inequality and solve
- Solve applications with linear inequalities

Before you get started, take this readiness quiz.

1. Translate from algebra to English: $15 > x$.
   If you missed this problem, review [link].
2. Translate to an algebraic expression: 15 is less than x.
   If you missed this problem, review [link].

Graph Inequalities on the Number Line

What number would make the inequality $x > 3$ true? Are you thinking, "x could be four"? That's correct, but x could be 6, too, or 37, or even 3.001. Any number greater than three is a solution to the inequality $x > 3$. We show all the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of three, to show that all numbers greater than three are solutions. Because the number three itself is not a solution, we put an open parenthesis at three.

We can also represent inequalities using **interval notation**. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as "**infinity**." It is not an actual number. Figure $19.1.1$shows both the number line and the interval notation.

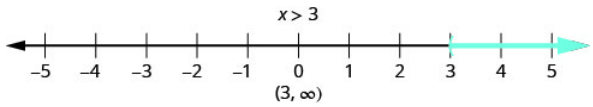


Figure 19.1.1: The inequality $x > 3$ is graphed on this number line and written in interval notation.

We use the left parenthesis symbol, (, to show that the endpoint of the inequality is not included. The left bracket symbol, [, shows that the endpoint is included.

The inequality $x \leq 1$ means all numbers less than or equal to one. Here we need to show that one is a solution, too. We do that by putting a bracket at $x = 1$. We then shade in all the numbers to the left of one, to show that all numbers less than one are solutions (Figure $19.1.2$). There is no lower end to those numbers. We write $x \leq 1 x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as "negative infinity."

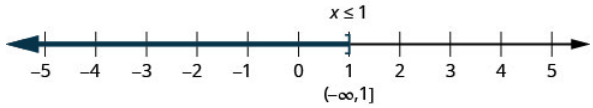
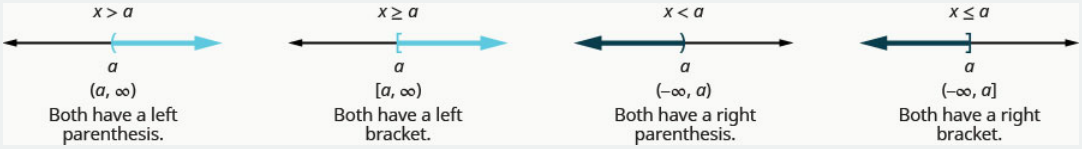


Figure 19.1.2: The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

Figure $19.1.3$shows both the number line and interval notation.

INEQUALITIES, NUMBER LINES, AND INTERVAL NOTATION



The notation for inequalities on a number line and in interval notation use the same symbols to express the endpoints of intervals.

Example 19.1.1

Graph each inequality on the number line and write in interval notation.

a. $x \geq -3$
b. $x < 2.5$
c. $x \leq -\frac{3}{5}$

Answer

ⓐ

	$x \geq -3$
Shade to the right of -3, and put a bracket at -3.	
Write in interval notation.	$[-3, \infty)$

ⓑ

	$x < 2.5$
Shade to the left of 2.5 and put a parenthesis at 2.5.	
Write in interval notation.	$(-\infty, 2.5)$

ⓒ

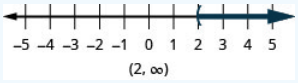
	$x \leq -\frac{3}{5}$
Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	
Write in interval notation.	$\left(-\infty, \frac{3}{5}\right]$

Graph each inequality on the number line and write in interval notation:

a. $x > 2$
b. $x \leq -1.5$
c. $x \geq \frac{3}{4}$.

Answer

ⓐ



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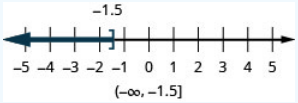

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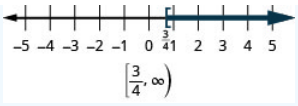


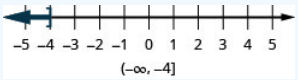
Example 19.1.3

Graph each inequality on the number line and write in interval notation:

a. $x \leq -4$
b. $x \geq 0.5$
c. $x < -\frac{2}{3}$.

Answer

ⓐ



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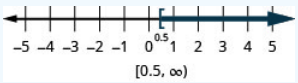

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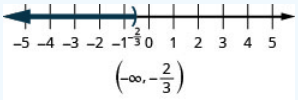


What numbers are greater than two but less than five? Are you thinking say, 2.5, 3, $3\frac{2}{3}$, 4, 4, 99? We can represent all the numbers between two and five with the inequality $2 < x < 5$. We can show $2 < x < 5$ on the number line by shading all the numbers between two and five. Again, we use the parentheses to show the numbers two and five are not included. See Figure.

The interval notation would be (2, 5).

Figure 19.1.3

Graph each inequality on the number line and write in interval notation.

ⓐ $-3 < x < 4$ ⓑ $-6 \leq x < -1$ ⓒ $0 \leq x \leq 2.5$

Answer

ⓐ

		$-3 < x < 4$
Shade between -3 and 4. Put a parentheses at -3 and 4.		
Write in interval notation.		$(-3, 4)$

ⓑ

			$-6 \leq x < -1$

Shade between -6 and -1. Put a bracket at -6, and a parenthesis at -1.			
Write in interval notation.			$[-6, 1)$

ⓒ

			$0 \leq x \leq 2.5$
Shade between 0 and 2.5. Put a bracket at 0 and at 2.5.			
Write in interval notation.			$[0, 2.5]$

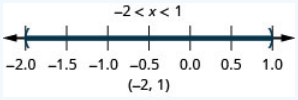
Example 19.1.5

Graph each inequality on the number line and write in interval notation:

ⓐ $-2 < x < 1$ ⓑ $-5 \leq x < -4$ ⓒ $1 \leq x \leq 4.25$

Answer

ⓐ



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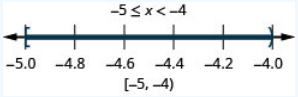

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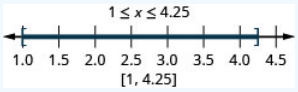


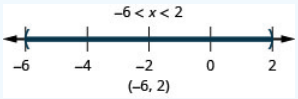
Example 19.1.6

Graph each inequality on the number line and write in interval notation:

ⓐ $-6 < x < 2$ ⓑ $-3 \leq x < -1$ ⓒ $2.5 \leq x \leq 6$

Answer

ⓐ



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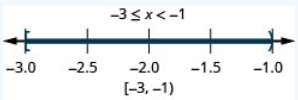

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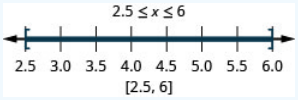


Solve Linear Inequalities

A linear inequality is much like a linear equation—but the equal sign is replaced with an inequality sign. A **linear inequality** is an inequality in one variable that can be written in one of the forms, $ax+b<c$, $ax+b\leq c$, $ax+b>c$, or $ax+b\geq c$.

LINEAR INEQUALITY

A linear inequality is an inequality in one variable that can be written in one of the following forms where a, b, and c are real numbers and $a\neq 0$:

$$ax+b<c,\quad ax+b\leq c,\quad ax+b>c,\quad ax+b\geq c.$$

When we solved linear equations, we were able to use the properties of equality to add, subtract, multiply, or divide both sides and still keep the equality. Similar properties hold true for inequalities.

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality. For example:

$$\begin{array}{ll} -4<2 & -4<2 \\ -4-5<2-5 & -4+7<2+7 \\ -9<-3 \text{ True} & 3<9 \text{ True} \end{array}$$

Notice that the inequality sign stayed the same.

This leads us to the Addition and Subtraction Properties of Inequality.

ADDITION AND SUBTRACTION PROPERTY OF INEQUALITY

For any numbers a, b, and c, if $a<b$, then

$$\begin{array}{ll} a+c<b+c & a-c<b-c \\ a+c>b+c & a-c>b-c \end{array}$$

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality

What happens to an inequality when we divide or multiply both sides by a constant?

Let's first multiply and divide both sides by a positive number.

$$\begin{array}{ll} 10<15 & 10<15 \\ 10(5)<15(5) & \frac{10}{5}<\frac{15}{5} \\ 50<75 \text{ True} & 2<3 \text{ True} \end{array}$$

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

$$\begin{array}{ll} 10<15 & 10<15 \\ 10(-5)\ ?\ 15(-5) & \frac{10}{-5}\ ?\ \frac{15}{-5} \\ -50\ ?\ -75 & -2\ ?\ -3 \\ -50>-75 & -2>-3 \end{array}$$

Notice that when we filled in the inequality signs, the inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

This gives us the Multiplication and Division Property of Inequality.

MULTIPLICATION AND DIVISION PROPERTY OF INEQUALITY

For any numbers a, b, and c,

multiply or divide by a positive

if $a<b$ and $c>0$, then $ac<bc$ and $\frac{a}{c}<\frac{b}{c}$.
if $a>b$ and $c>0$, then $ac>bc$ and $\frac{a}{c}>\frac{b}{c}$.

multiply or divide by a negative

if $a<b$ and $c<0$, then $ac>bc$ and $\frac{a}{c}>\frac{b}{c}$.
if $a>b$ and $c<0$, then $ac<bc$ and $\frac{a}{c}<\frac{b}{c}$.

When we divide or multiply an inequality by a:

- positive number, the inequality stays the same.
- negative number, the inequality reverses.

Sometimes when solving an inequality, as in the next example, the variable ends upon the right. We can rewrite the inequality in reverse to get the variable to the left.

$$x>a \text{ has the same meaning as } a<x$$

Think about it as "If Xander is taller than Andy, then Andy is shorter than Xander."

Example 19.1.7

Solve each inequality. Graph the solution on the number line, and write the solution in interval notation.

ⓐ $x-\frac{3}{8}\leq\frac{3}{4}$ ⓑ $9y<54$ ⓒ $-15<\frac{3}{5}z$

Answer

ⓐ

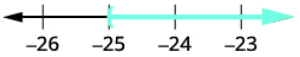
	$x-\frac{3}{8}\leq\frac{3}{4}$
Add 3838 to both sides of the inequality.	$x-\frac{3}{8}+\frac{3}{8}\leq\frac{3}{4}+\frac{3}{8}$
Simplify.	$x\leq\frac{9}{8}$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty,\frac{9}{8}]$

ⓑ

	$9y<54$
Divide both sides of the inequality by 9; since 9 is positive, the inequality stays the same.	$\frac{9y}{9}<\frac{54}{9}$
Simplify.	$y<6$

Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, 6)$

ⓒ

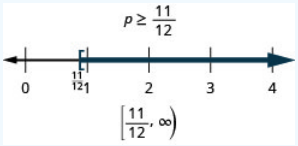
	$-15 < \frac{3}{5}z$
Multiply both sides of the inequality by $\frac{5}{3}$. Since $\frac{5}{3}$ is positive, the inequality stays the same.	$\frac{5}{3}(-15) < \frac{5}{3}\left(\frac{3}{5}z\right)$
Simplify.	$-25 < z$
Rewrite with the variable on the left.	$z > -25$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-25, \infty)$

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

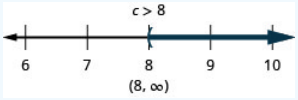
ⓐ $p - \frac{3}{4} \geq \frac{1}{6}$ ⓑ $9c > 72$ ⓒ $24 \leq \frac{3}{8}m$

Answer

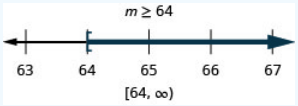
ⓐ



ⓑ



ⓒ



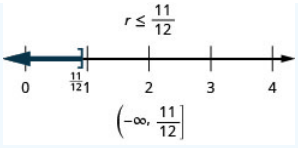
Example 19.1.9

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

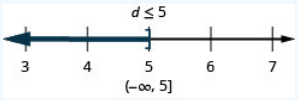
ⓐ $r - \frac{1}{3} \leq \frac{7}{12}$ ⓑ $12d \leq 60$ ⓒ $-24 < \frac{4}{3}n$

Answer

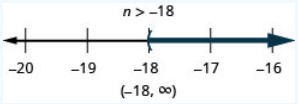
ⓐ



ⓑ



ⓒ



Be careful when you multiply or divide by a negative number—remember to reverse the inequality sign.

Example 19.1.10

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

ⓐ $-\frac{1}{3}m \ge \frac{6}{5}$ ⓑ $\frac{n}{-2} \ge 8$

Answer

ⓐ

	$-13m \ge 65$
Divide both sides of the inequality by $-\frac{1}{3}$. Since $-\frac{1}{3}$ is a negative, the inequality reverses.	$\frac{-13m}{-13} \le \frac{65}{-13}$
Simplify.	$m \le -5$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -5]$

ⓑ

	$\frac{n}{-2} \ge 8$
Multiply both sides of the inequality by -2. Since -2 is a negative, the inequality reverses.	$-2\left(\frac{n}{-2}\right) \le -2(8)$
Simplify.	$n \le -16$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -16]$

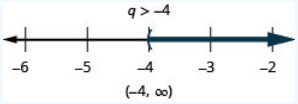
Example 19.1.11

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

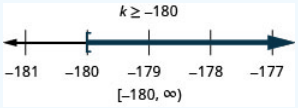
ⓐ $-8q < 32$ ⓑ $\frac{k}{-12} \le 15$.

Answer

ⓐ



ⓑ



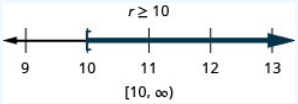
Example 19.1.12

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

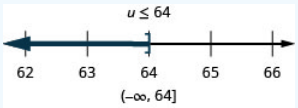
ⓐ $-7r \leq -70$ ⓑ $\frac{u}{-4} \geq -16$.

Answer

ⓐ



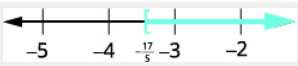
ⓑ



Most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but make sure to pay close attention when we multiply or divide to isolate the variable.

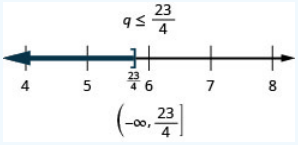
Solve the inequality $6y \leq 11y + 17$, graph the solution on the number line, and write the solution in interval notation.

Answer

	$6y \leq 11y + 17$
Subtract 11y11y from both sides to collect the variables on the left.	$6y - 11y \leq 11y - 11y + 17$
Simplify.	$-5y \leq 17$
Divide both sides of the inequality by $-5,-5$, and reverse the inequality.	$\frac{-5y}{-5} \geq \frac{17}{-5}$
Simplify.	$y \geq \frac{17}{-5}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left[-\frac{17}{5}, \infty\right)$

Solve the inequality, graph the solution on the number line, and write the solution in interval notation: $3q \geq 7q - 23$.

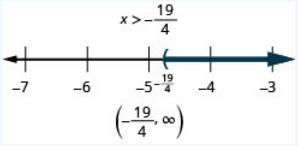
Answer



Example 19.1.15

Solve the inequality, graph the solution on the number line, and write the solution in interval notation: $6x < 10x + 19$.

Answer

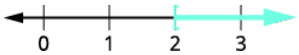


When solving inequalities, it is usually easiest to collect the variables on the side where the coefficient of the variable is largest. This eliminates negative coefficients and so we don't have to multiply or divide by a negative—which means we don't have to remember to reverse the inequality sign.

Example 19.1.16

Solve the inequality $8p + 3(p - 12) > 7p - 28$, graph the solution on the number line, and write the solution in interval notation.

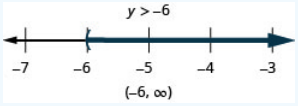
Answer

	$8p + 3(p - 12) > 7p - 28$
Simplify each side as much as possible.	
Distribute.	$8p + 3p - 36 > 7p - 28$
Combine like terms.	$11p - 36 > 7p - 28$
Subtract $7p$ from both sides to collect the variables on the left, since $11 > 7$.	$11p - 36 - 7p > 7p - 28 - 7p$
Simplify.	$4p - 36 > -28$
Add 36 to both sides to collect the constants on the right.	$4p - 36 + 36 > -28 + 36$
Simplify.	$4p > 8$
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4} > \frac{8}{4}$
Simplify.	$p > 2$
Graph the solution on the number line.	
Write the solution in interval notation.	$(2, \infty)$

Example 19.1.17

Solve the inequality $9y+2(y+6)>5y-24$, graph the solution on the number line, and write the solution in interval notation.

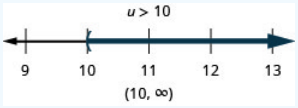
Answer



Example 19.1.18

Solve the inequality $6u+8(u-1)>10u+32$, graph the solution on the number line, and write the solution in interval notation.

Answer



Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

Example 19.1.19

Solve the inequality $8x-2(5-x)<4(x+9)+6x$, graph the solution on the number line, and write the solution in interval notation.

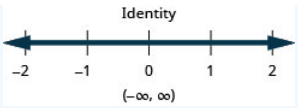
Answer

Simplify each side as much as possible.		$8x-2(5-x)<4(x+9)+6x$
Distribute.		$8x-10+2x<4x+36+6x$
Combine like terms.		$10x-10<10x+36$
Subtract $10x$ from both sides to collect the variables on the left.		$10x-10-10x<10x+36-10x$
Simplify.		$-10<36$
The x's are gone, and we have a true statement.		The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.		
Write the solution in interval notation.		$(-\infty,\infty)$

Example 19.1.20

Solve the inequality $4b-3(3-b)>5(b-6)+2b$, graph the solution on the number line, and write the solution in interval notation.

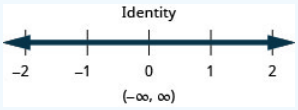
Answer



Example 19.1.21

Solve the inequality $9h-7(2-h)<8(h+11)+8h$, graph the solution on the number line, and write the solution in interval notation.

Answer

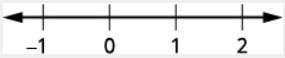


We can clear fractions in inequalities much as we did in equations. Again, be careful with the signs when multiplying or dividing by a negative.

Example 19.1.22

Solve the inequality $\frac{1}{3}a-\frac{1}{8}a>\frac{5}{24}a+\frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

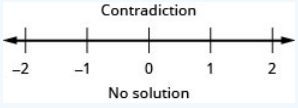
Answer

	$\frac{1}{3}a-\frac{1}{8}a>\frac{5}{24}a+\frac{3}{4}$
Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a-\frac{1}{8}a\right)>24\left(\frac{5}{24}a+\frac{3}{4}\right)$
Simplify.	$8a-3a>5a+18$
Combine like terms.	$5a>5a+18$
Subtract $5a$ from both sides to collect the variables on the left.	$5a-5a>5a-5a+18$
Simplify.	$0>18$
The statement is false.	The inequality is a contradiction. There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

Example 19.1.23

Solve the inequality $\frac{1}{4}x-\frac{1}{12}x>\frac{1}{6}x+\frac{7}{8}$, graph the solution on the number line, and write the solution in interval notation.

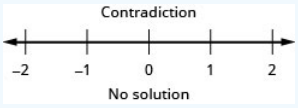
Answer



Example 19.1.24

Solve the inequality $\frac{2}{5}z - \frac{1}{3}z < \frac{1}{15}z - \frac{3}{5}$, graph the solution on the number line, and write the solution in interval notation.

Answer



Translate to an Inequality and Solve

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like “more than” and “less than.” But others are not as obvious. Table shows some common phrases that indicate inequalities.

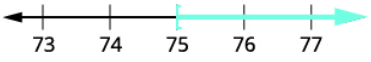
$>$	$\geq$	$<$	$\leq$
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

Example 19.1.25

Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.

Twenty-seven less than x is at least 48.

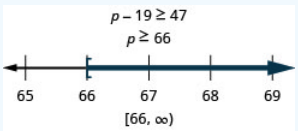
Answer

	Twenty-seven less than x is at least 48.
Translate.	$x - 27 \geq 48$
Solve—add 27 to both sides.	$x - 27 + 27 \geq 48 + 27$
Simplify.	$x \geq 75$
Graph on the number line.	
Write in interval notation.	$[75, \infty)$

Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.

Nineteen less than p is no less than 47.

Answer

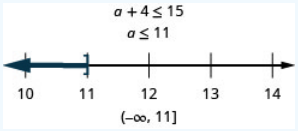


Example 19.1.27

Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.

Four more than a is at most 15.

Answer



Solve Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. The method we will use to solve applications with linear inequalities is very much like the one we used when we solved applications with equations.

We will read the problem and make sure all the words are understood. Next, we will identify what we are looking for and assign a variable to represent it. We will restate the problem in one sentence to make it easy to translate into an inequality. Then, we will solve the inequality.

Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round the algebraic solution to a whole number. The context of the application will determine whether we round up or down.

Example 19.1.28

Dawn won a mini-grant of $4,000 to buy tablet computers for her classroom. The tablets she would like to buy cost $254.12 each, including tax and delivery. What is the maximum number of tablets Dawn can buy?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the maximum number of tablets Dawn can buy
Step 3. Name what you are looking for. Choose a variable to represent that quantity.	Let n = the number of tablets.
Step 4. Translate Write a sentence that gives the information to find it.	$254.12 times the number of tablets is no more than $4,000.
Translate into an inequality.	$254.12n \leq 4000$
Step 5. Solve the inequality.	$n \leq 15.74$ but n must be a whole number of tablets, so round to 15. $n \leq 15$ tablets
Step 6. Check the answer in the problem and make sure it makes sense.	Rounding down the price to $250, 15 tablets would cost $3,750, while 16 tablets would be $4,000. So a maximum of 15 tablets at $254.12 seems reasonable.
Step 7. Write a sentence that answers the question.	Dawn can buy a maximum of 15 tablets.

Try It 19.1.29

Angie has $20 to spend on juice boxes for her son's preschool picnic. Each pack of juice boxes costs $2.63. What is the maximum number of packs she can buy?

Answer

Angie can buy 7 packs of juice.

Try It 19.1.30

Daniel wants to surprise his girlfriend with a birthday party at her favorite restaurant. It will cost $42.75 per person for dinner, including tip and tax. His budget for the party is $500. What is the maximum number of people Daniel can have at the party?

Answer

Daniel can have 11 people at the party.

Example 19.1.31

Taleisha's phone plan costs her $28.80 a month plus $0.20 per text message. How many text messages can she send/receive and keep her monthly phone bill no more than $50?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the number of text messages Taleisha can make
Step 3. Name what you are looking for. Choose a variable to represent that quantity.	Let $t =$ the number of text messages.
Step 4. Translate Write a sentence that gives the information to find it.	$28.80 plus $0.20 times the number of text messages is less than or equal to $50.
Translate into an inequality.	$28.80 + 0.20t \leq 50$
Step 5. Solve the inequality.	$0.2t \leq 21.2$ $t \leq 106$ text messages
Step 6. Check the answer in the problem and make sure it makes sense.	Yes, $28.80 + 0.20(106) = 50$.
Step 7. Write a sentence that answers the question.	Taleisha can send/receive no more than 106 text messages to keep her bill no more than $50.

Try It 19.1.32

Sergio and Lizeth have a very tight vacation budget. They plan to rent a car from a company that charges $75 a week plus $0.25 a mile. How many miles can they travel during the week and still keep within their $200 budget?

Answer

Sergio and Lizeth can travel no more than 500 miles.

Try It 19.1.33

Rameen's heating bill is $5.42 per month plus $1.08 per therm. How many therms can Rameen use if he wants his heating bill to be a maximum of $87.50.

Answer

Rameen's heating bill is $5.42 per month plus $1.08 per therm. How many therms can Rameen use if he wants his heating bill to be a maximum of $87.50.

Profit is the money that remains when the costs have been subtracted from the revenue. In the next example, we will find the number of jobs a small businesswoman needs to do every month in order to make a certain amount of profit.

Example 19.1.34

Felicity has a calligraphy business. She charges $2.50 per wedding invitation. Her monthly expenses are $650. How many invitations must she write to earn a profit of at least $2,800 per month?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the number of invitations Felicity needs to write
Step 3. Name what you are looking for. Choose a variable to represent that quantity.	Let $j =$ the number of invitations.
Step 4. Translate Write a sentence that gives the information to find it.	$2.50 times the number of invitations minus $650 is at least $2,800.
Translate into an inequality.	$2.50j - 650 \geq 2,800$
Step 5. Solve the inequality.	$2.5j \geq 3,450$ $j \geq 1,380$ invitations
Step 6. Check the answer in the problem and make sure it makes sense.	If Felicity wrote 1400 invitations, her profit would be $2.50(1400) - 650$, or $2,850. This is more than $2800.
Step 7. Write a sentence that answers the question.	Felicity must write at least 1,380 invitations.

Try It 19.1.35

Caleb has a pet sitting business. He charges $32 per hour. His monthly expenses are $2,272. How many hours must he work in order to earn a profit of at least $800 per month?

Answer

Caleb must work at least 96 hours.

Try It 19.1.36

Elliot has a landscape maintenance business. His monthly expenses are $1,100. If he charges $60 per job, how many jobs must he do to earn a profit of at least $4,000 a month?

Answer

Elliot must work at least 85 jobs.

There are many situations in which several quantities contribute to the total expense. We must make sure to account for all the individual expenses when we solve problems like this.

Example 19.1.37

Malik is planning a six-day summer vacation trip. He has $840 in savings, and he earns $45 per hour for tutoring. The trip will cost him $525 for airfare, $780 for food and sightseeing, and $95 per night for the hotel. How many hours must he tutor to have enough money to pay for the trip?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the number of hours Malik must tutor
Step 3. Name what you are looking for. Choose a variable to represent that quantity.	Let h = the number of hours.
Step 4. Translate Write a sentence that gives the information to find it.	The expenses must be less than or equal to the income. The cost of airfare plus the cost of food and sightseeing and the hotel bill must be less than the savings plus the amount earned tutoring.
Translate into an inequality.	$525 + 780 + 95(6) \leq 840 + 45h$
Step 5. Solve the inequality.	$1,875 \leq 840 + 45h$ $1,035 \leq 45h$ $h \geq 23$ hours
Step 6. Check the answer in the problem and make sure it makes sense.	We substitute 23 into the inequality. $1,875 \leq 840 + 45h$ $1,875 \leq 840 + 45(23)$ $1,875 \leq 1875$
Step 7. Write a sentence that answers the question.	Malik must tutor at least 23 hours.

Try It 19.1.38

Brenda's best friend is having a destination wedding and the event will last three days. Brenda has $500 in savings and can earn $15 an hour babysitting. She expects to pay $350 airfare, $375 for food and entertainment and $60 a night for her share of a hotel room. How many hours must she babysit to have enough money to pay for the trip?

Answer

Brenda must babysit at least 27 hours.

Try It 19.1.39

Josue wants to go on a 10-night road trip with friends next spring. It will cost him $180 for gas, $450 for food, and $49 per night to share a motel room. He has $520 in savings and can earn $30 per driveway shoveling snow. How many driveways must he shovel to have enough money to pay for the trip?

Answer

Josue must shovel at least 20 driveways.

Key Concepts

- **Inequalities, Number Lines, and Interval Notation**

  $x > a \quad x \geq a \quad x < a \quad x \leq a$

- **Linear Inequality**
  - A **linear inequality** is an inequality in one variable that can be written in one of the following forms where a, b, and c are real numbers and $a \neq 0$:

$$ax + b < c, \qquad ax + b \leq c, \qquad ax + b > c, \qquad ax + b \geq c.$$

- **Addition and Subtraction Property of Inequality**
  - For any numbers *a*, *b*, and *c*, if a<b,thena<b,then

$$a+c<b+c \quad a-c<b-c$$
$$a+c>b+c \quad a-c>b-c$$

  - We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.
- **Multiplication and Division Property of Inequality**
  - For any numbers *a*, *b*, and *c*,
    multiply or divide by a **positive**

$$\text{if } a<b \text{ and } c>0, \text{ then } ac<bc \text{ and } \tfrac{a}{c}<\tfrac{b}{c}.$$
$$\text{if } a>b \text{ and } c>0, \text{ then } ac>bc \text{ and } \tfrac{a}{c}>\tfrac{b}{c}.$$

    multiply or divide by a **negative**

$$\text{if } a<b \text{ and } c<0, \text{ then } ac>bc \text{ and } \tfrac{a}{c}>\tfrac{b}{c}.$$
$$\text{if } a>b \text{ and } c<0, \text{ then } ac<bc \text{ and } \tfrac{a}{c}<\tfrac{b}{c}.$$

- **Phrases that indicate inequalities**

$>$	$\geq$	$<$	$\leq$
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

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19.1E: Exercises

Practice Makes Perfect

Graph Inequalities on the Number Line

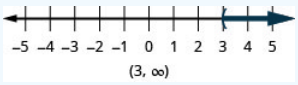
In the following exercises, graph each inequality on the number line and write in interval notation.

1. ⓐ $x < -2$
ⓑ $x \geq -3.5$
ⓒ $x \leq \frac{2}{3}$

2. ⓐ $x > 3$
ⓑ $x \leq -0.5$
ⓒ $x \geq \frac{1}{3}$

Answer

ⓐ



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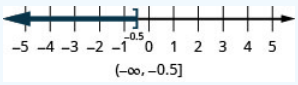

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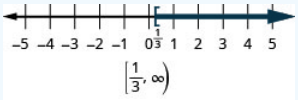
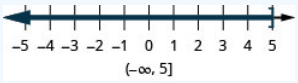


3. ⓐ $x \geq -4$
ⓑ $x < 2.5$
ⓒ $x > -\frac{3}{2}$

4. ⓐ $x \leq 5$
ⓑ $x \geq -1.5$
ⓒ $x < -\frac{7}{3}$

Answer

ⓐ



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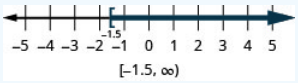

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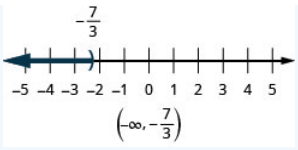
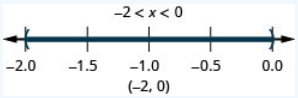


5. ⓐ $-5 < x < 2$
ⓑ $-3 \leq x < -1$
ⓒ $0 \leq x \leq 1.5$

6. ⓐ $-2 < x < 0$
ⓑ $-5 \leq x < -3$
ⓒ $0 \leq x \leq 3.5$

Answer

ⓐ



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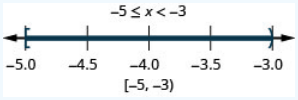

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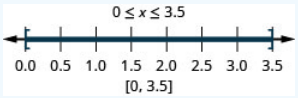
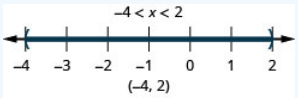


7. ⓐ $-1 < x < 3$
ⓑ $-3 < x \leq -2$
ⓒ $-1.25 \leq x \leq 0$

8. ⓐ $-4 < x < 2$
ⓑ $-5 < x \leq -2$
ⓒ $-3.75 \leq x \leq 0$

Answer

ⓐ



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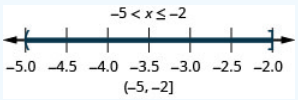

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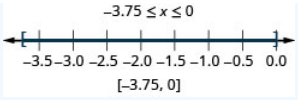


Solve Linear Inequalities

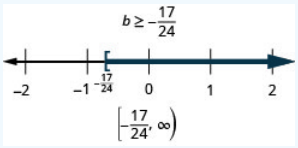
In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

9. ⓐ $a+\frac{3}{4}\geq\frac{7}{10}$
ⓑ $8x>72$
ⓒ $20>\frac{2}{5}h$

10. ⓐ $b+\frac{7}{8}\geq\frac{1}{6}$
ⓑ $6y<48$
ⓒ $40<\frac{5}{8}k$

Answer

ⓐ



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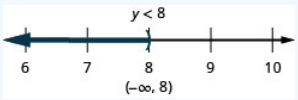

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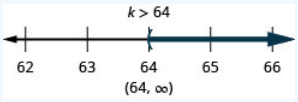
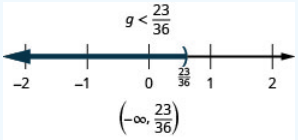


11. ⓐ $f-\frac{13}{20}<-\frac{5}{12}$
ⓑ $9t\geq-27$
ⓒ $\frac{7}{6}j\geq42$

12. ⓐ $g-\frac{11}{12}<-\frac{5}{18}$
ⓑ $7s<-28$
ⓒ $\frac{9}{4}g\leq36$

Answer

ⓐ



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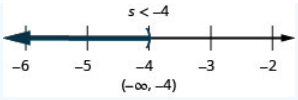

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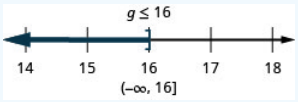
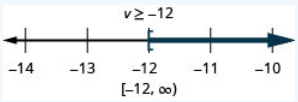


13. ⓐ $-5u \geq 65$
ⓑ $\frac{a}{-3} \leq 9$

14. ⓐ $-8v \leq 96$
ⓑ $\frac{b}{-10} \geq 30$

Answer

ⓐ



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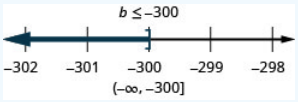
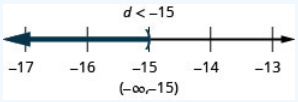


15. ⓐ $-9c < 126$
ⓑ $-25 < \frac{p}{-5}$

16. ⓐ $-7d > 105$
ⓑ $-18 > \frac{q}{-6}$

Answer

ⓐ



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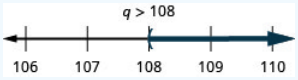

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

17. $4v \geq 9v - 40$

18. $5u \leq 8u - 21$

Answer

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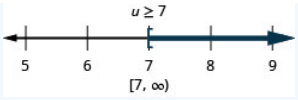
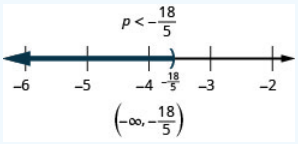


19. $13q < 7q - 29$

20. $9p > 14p - 18$

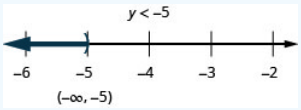
Answer



21. $12x + 3(x + 7) > 10x - 24$

22. $9y + 5(y + 3) < 4y - 35$

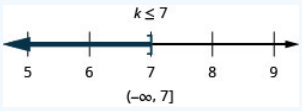
Answer



23. $6h - 4(h - 1) \leq 7h - 11$

24. $4k - (k - 2) \geq 7k - 26$

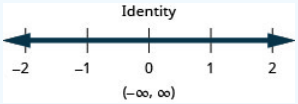
Answer



25. $8m - 2(14 - m) \geq 7(m - 4) + 3m$

26. $6n - 12(3 - n) \leq 9(n - 4) + 9n$

Answer



27. $\frac{3}{4}b - \frac{1}{3}b < \frac{5}{12}b - \frac{1}{2}$

28. $9u + 5(2u - 5) \geq 12(u - 1) + 7u$

Answer

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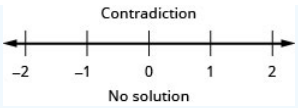
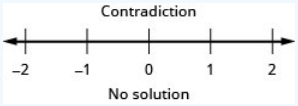


29. $\frac{2}{3}g-\frac{1}{2}(g-14)\leq\frac{1}{6}(g+42)$

30. $\frac{4}{5}h-\frac{2}{3}(h-9)\geq\frac{1}{15}(2h+90)$

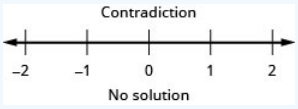
Answer



31. $\frac{5}{6}a-\frac{1}{4}a>\frac{7}{12}a+\frac{2}{3}$

32. $12v+3(4v-1)\leq19(v-2)+5v$

Answer

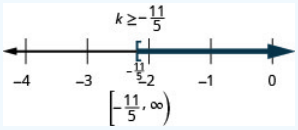


In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

33. $15k\leq-40$

34. $35k\geq-77$

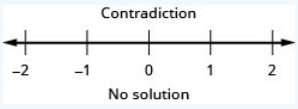
Answer



35. $23p-2(6-5p)>3(11p-4)$

36. $18q-4(10-3q)<5(6q-8)$

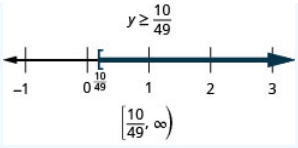
Answer



37. $-\frac{9}{4}x\geq-\frac{5}{12}$

38. $-\frac{21}{8}y\leq-\frac{15}{28}$

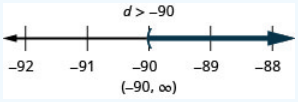
Answer



39. $c + 34 < -99$

40. $d + 29 > -61$

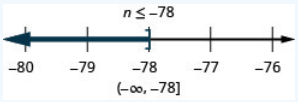
Answer



41. $\frac{m}{18} \geq -4$

42. $\frac{n}{13} \leq -6$

Answer



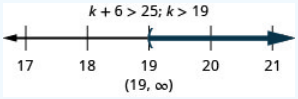
Translate to an Inequality and Solve

In the following exercises, translate and solve. Then graph the solution on the number line and write the solution in interval notation.

43. Three more than h is no less than 25.

44. Six more than k exceeds 25.

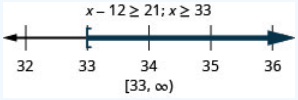
Answer



45. Ten less than w is at least 39.

46. Twelve less than x is no less than 21.

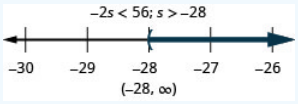
Answer



47. Negative five times r is no more than 95.

48. Negative two times s is lower than 56.

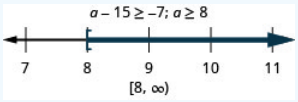
Answer



49. Nineteen less than b is at most -22.

50. Fifteen less than a is at least -7.

Answer



Solve Applications with Linear Inequalities

In the following exercises, solve.

51. Alan is loading a pallet with boxes that each weighs 45 pounds. The pallet can safely support no more than 900 pounds. How many boxes can he safely load onto the pallet?

52. The elevator in Yehire's apartment building has a sign that says the maximum weight is 2100 pounds. If the average weight of one person is 150 pounds, how many people can safely ride the elevator?

Answer

A maximum of 14 people can safely ride in the elevator.

53. Andre is looking at apartments with three of his friends. They want the monthly rent to be no more than $2,360. If the roommates split the rent evenly among the four of them, what is the maximum rent each will pay?

54. Arleen got a $20 gift card for the coffee shop. Her favorite iced drink costs $3.79. What is the maximum number of drinks she can buy with the gift card?

Answer

five drinks

55. Teegan likes to play golf. He has budgeted $60 next month for the driving range. It costs him $10.55 for a bucket of balls each time he goes. What is the maximum number of times he can go to the driving range next month?

56. Ryan charges his neighbors $17.50 to wash their car. How many cars must he wash next summer if his goal is to earn at least $1,500?

Answer

86 cars

57. Keshad gets paid $2,400 per month plus 6% of his sales. His brother earns $3,300 per month. For what amount of total sales will Keshad's monthly pay be higher than his brother's monthly pay?

58. Kimuyen needs to earn $4,150 per month in order to pay all her expenses. Her job pays her $3,475 per month plus 4% of her total sales. What is the minimum Kimuyen's total sales must be in order for her to pay all her expenses?

Answer

$16,875

59. Andre has been offered an entry-level job. The company offered him $48,000 per year plus 3.5% of his total sales. Andre knows that the average pay for this job is $62,000. What would Andre's total sales need to be for his pay to be at least as high as the average pay for this job?

60. Nataly is considering two job offers. The first job would pay her $83,000 per year. The second would pay her $66,500 plus 15% of her total sales. What would her total sales need to be for her salary on the second offer be higher than the first?

Answer

$110,000

61. Jake's water bill is $24.80 per month plus $2.20 per ccf (hundred cubic feet) of water. What is the maximum number of ccf Jake can use if he wants his bill to be no more than $60?

62. Kiyoshi's phone plan costs $17.50 per month plus $0.15 per text message. What is the maximum number of text messages Kiyoshi can use so the phone bill is no more than $56.60?

Answer

260 messages

63. Marlon's TV plan costs $49.99 per month plus $5.49 per first-run movie. How many first-run movies can he watch if he wants to keep his monthly bill to be a maximum of $100?

64. Kellen wants to rent a banquet room in a restaurant for her cousin's baby shower. The restaurant charges $350 for the banquet room plus $32.50 per person for lunch. How many people can Kellen have at the shower if she wants the maximum cost to be $1,500?

Answer

35 people

65. Moshde runs a hairstyling business from her house. She charges $45 for a haircut and style. Her monthly expenses are $960. She wants to be able to put at least $1,200 per month into her savings account order to open her own salon. How many "cut & styles" must she do to save at least $1,200 per month?

66. Noe installs and configures software on home computers. He charges $125 per job. His monthly expenses are $1,600. How many jobs must he work in order to make a profit of at least $2,400?

Answer

32 jobs

67. Katherine is a personal chef. She charges $115 per four-person meal. Her monthly expenses are $3,150. How many four-person meals must she sell in order to make a profit of at least $1,900?

68. Melissa makes necklaces and sells them online. She charges $88 per necklace. Her monthly expenses are $3,745. How many necklaces must she sell if she wants to make a profit of at least $1,650?

Answer

62 necklaces

69. Five student government officers want to go to the state convention. It will cost them $110 for registration, $375 for transportation and food, and $42 per person for the hotel. There is $450 budgeted for the convention in the student government savings account. They can earn the rest of the money they need by having a car wash. If they charge $5 per car, how many cars must they wash in order to have enough money to pay for the trip?

70. Cesar is planning a four-day trip to visit his friend at a college in another state. It will cost him $198 for airfare, $56 for local transportation, and $45 per day for food. He has $189 in savings and can earn $35 for each lawn he mows. How many lawns must he mow to have enough money to pay for the trip?

Answer

seven lawns

71. Alonzo works as a car detailer. He charges $175 per car. He is planning to move out of his parents' house and rent his first apartment. He will need to pay $120 for application fees, $950 for security deposit, and first and last months' rent at $1,140 per month. He has $1,810 in savings. How many cars must he detail to have enough money to rent the apartment?

72. Eun-Kyung works as a tutor and earns $60 per hour. She has $792 in savings. She is planning an anniversary party for her parents. She would like to invite 40 guests. The party will cost her $1,520 for food and drinks and $150 for the photographer. She will also have a favor for each of the guests, and each favor will cost $7.50. How many hours must she tutor to have enough money for the party?

Answer

20 hours

Everyday Math

73. Maximum load on a stage In 2014, a high school stage collapsed in Fullerton, California, when 250 students got on stage for the finale of a musical production. Two dozen students were injured. The stage could support a maximum of 12,750 pounds. If the average weight of a student is assumed to be 140 pounds, what is the maximum number of students who could safely be on the stage?

74. Maximum weight on a boat In 2004, a water taxi sank in Baltimore harbor and five people drowned. The water taxi had a maximum capacity of 3,500 pounds (25 people with average weight 140 pounds). The average weight of the 25 people on the water taxi when it sank was 168 pounds per person. What should the maximum number of people of this weight have been?

Answer

20 people

75. Wedding budget Adele and Walter found the perfect venue for their wedding reception. The cost is $9850 for up to 100 guests, plus $38 for each additional guest. How many guests can attend if Adele and Walter want the total cost to be no more than $12,500?

76. Shower budget Penny is planning a baby shower for her daughter-in-law. The restaurant charges $950 for up to 25 guests, plus $31.95 for each additional guest. How many guests can attend if Penny wants the total cost to be no more than $1,500?

Answer

42 guests

Writing Exercises

77. Explain why it is necessary to reverse the inequality when solving $-5x > 10$.

78. Explain why it is necessary to reverse the inequality when solving $n - 3 < 12$.

Answer

Answers will vary.

79. Find your last month's phone bill and the hourly salary you are paid at your job. Calculate the number of hours of work it would take you to earn at least enough money to pay your phone bill by writing an appropriate inequality and then solving it. Do you feel this is an appropriate number of hours? Is this the appropriate phone plan for you?

80. Find out how many units you have left, after this term, to achieve your college goal and estimate the number of units you can take each term in college. Calculate the number of terms it will take you to achieve your college goal by writing an appropriate inequality and then solving it. Is this an acceptable number of terms until you meet your goal? What are some ways you could accelerate this process?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph inequalities on the number line.			
solve linear inequalities.			
translate words to an inequality and solve.			
solve applications with linear inequalities.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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19.2: Solve Compound Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve compound inequalities with “and”
- Solve compound inequalities with “or”
- Solve applications with compound inequalities

Before you get started, take this readiness quiz.

1. Simplify: $\frac{2}{5}(x+10)$.
   If you missed this problem, review [link].
2. Simplify: $-(x-4)$.
   If you missed this problem, review [link].

Solve Compound Inequalities with “and”

Now that we know how to solve linear inequalities, the next step is to look at compound inequalities. A **compound inequality** is made up of two inequalities connected by the word “and” or the word “or.” For example, the following are compound inequalities.

$$\begin{array}{lll} x+3>-4 & \text{and} & 4x-5\leq 3 \\ 2(y+1)<0 & \text{or} & y-5\geq -2 \end{array}$$

COMPOUND INEQUALITY

A **compound inequality** is made up of two inequalities connected by the word “and” or the word “or.”

To solve a compound inequality means to find all values of the variable that make the compound inequality a true statement. We solve compound inequalities using the same techniques we used to solve linear inequalities. We solve each inequality separately and then consider the two solutions.

To solve a compound inequality with the word “and,” we look for all numbers that make *both* inequalities true. To solve a compound inequality with the word “or,” we look for all numbers that make *either* inequality true.

Let’s start with the compound inequalities with “and.” Our solution will be the numbers that are solutions to *both* inequalities known as the intersection of the two inequalities. Consider the intersection of two streets—the part where the streets overlap—belongs to both streets.

To find the solution of an "and" compound inequality, we look at the graphs of each inequality and then find the numbers that belong to both graphs—where the graphs overlap.

For the compound inequality $x>-3$ and $x\leq 2$, we graph each inequality. We then look for where the graphs “overlap”. The numbers that are shaded on both graphs, will be shaded on the graph of the solution of the compound inequality. See Figure 19.2.1.

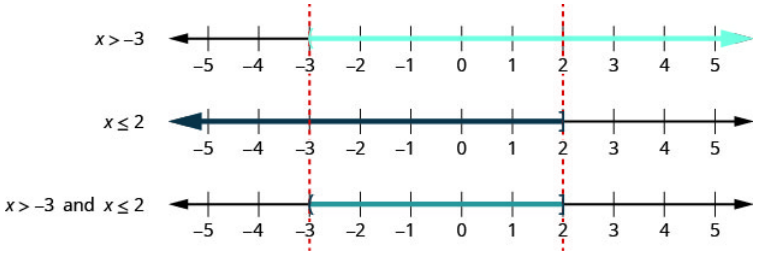


Figure 19.2.1

We can see that the numbers between -3 and 2 are shaded on both of the first two graphs. They will then be shaded on the solution graph.

The number -3 is not shaded on the first graph and so since it is not shaded on both graphs, it is not included on the solution graph.

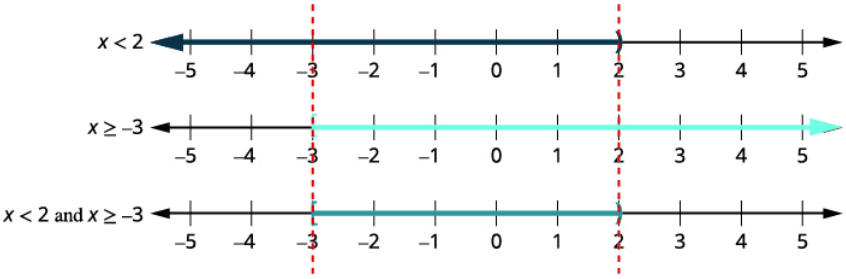
The number two is shaded on both the first and second graphs. Therefore, it is be shaded on the solution graph.

This is how we will show our solution in the next examples.

Example 19.2.1

Solve $6x-3<9$ and $2x+7\geq 3$. Graph the solution and write the solution in interval notation.

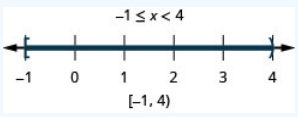
Solution

	$6x-3<9$	and	$2x+9\geq 3$
Step 1. Solve each inequality.	$6x-3<9$		$2x+9\geq 3$
	$6x<12$		$2x\geq -6$
	$x<2$	and	$x\geq -3$
Step 2. Graph each solution. Then graph the numbers that make both inequalities true. The final graph will show all the numbers that make both inequalities true—the numbers shaded on *both* of the first two graphs.			
Step 3. Write the solution in interval notation.	$[-3, 2)$		
All the numbers that make both inequalities true are the solution to the compound inequality.			

Try It 19.2.2

Solve the compound inequality. Graph the solution and write the solution in interval notation: $4x-7<9$ and $5x+8\geq 3$.

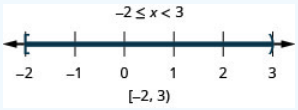
Answer



Try It 19.2.3

Solve the compound inequality. Graph the solution and write the solution in interval notation: $3x-4<5$ and $4x+9\geq 1$.

Answer



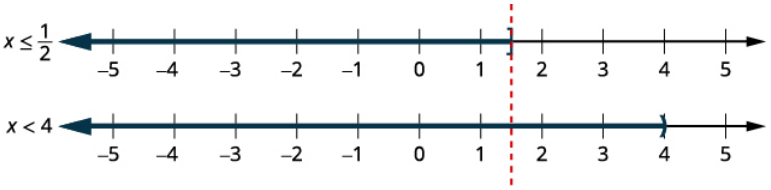
SOLVE A COMPOUND INEQUALITY WITH "AND."

1. Solve each inequality.
2. Graph each solution. Then graph the numbers that make *both* inequalities true.
   This graph shows the solution to the compound inequality.
3. Write the solution in interval notation.

Example 19.2.4

Solve $3(2x+5)\leq 18$ and $2(x-7)<-6$. Graph the solution and write the solution in interval notation.

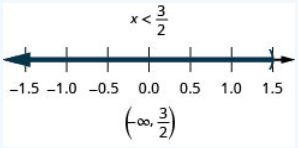
Solution

	$3(2x+5)\leq 18$	and	$2(x-7)<-6$
Solve each inequality.	$6x+15\leq 18$		$2x-14<-6$
	$6x\leq 3$		$2x<8$
	$x\leq \frac{1}{2}$	and	$x<4$
Graph each solution.			
Graph the numbers that make both inequalities true.			
Write the solution in interval notation.	$(-\infty, \frac{1}{2}]$		

Try It 19.2.5

Solve the compound inequality. Graph the solution and write the solution in interval notation: $2(3x+1)\leq 20$ and $4(x-1)<2$.

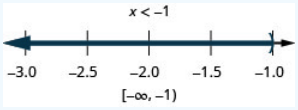
Answer



Try It 19.2.6

Solve the compound inequality. Graph the solution and write the solution in interval notation: $5(3x-1) \leq 10$ and $4(x+3) < 8$.

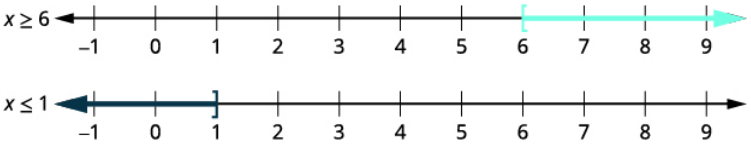
Answer



Example 19.2.7

Solve $\frac{1}{3}x - 4 \geq -2$ and $-2(x-3) \geq 4$. Graph the solution and write the solution in interval notation.

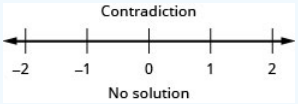
Solution

	$\frac{1}{3}x - 4 \geq -2$	and	$-2(x-3) \geq 4$
Solve each inequality.	$\frac{1}{3}x - 4 \geq -2$		$-2x + 6 \geq 4$
	$\frac{1}{3}x \geq 2$		$-2x \geq -2$
	$x \geq 6$	and	$x \leq 1$
Graph each solution.			
Graph the numbers that make both inequalities true.			
	This is a contradiction. There are no numbers that make both inequalities true, so there is no solution.		

Try It 19.2.8

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{1}{4}x - 3 \geq -1$ and $-3(x-2) \geq 2$.

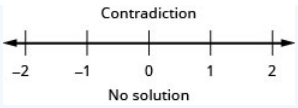
Answer



Try It 19.2.9

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{1}{5}x - 5 \geq -3$ and $-4(x-1) \geq -2$.

Answer



Sometimes we have a compound inequality that can be written more concisely. For example, $a < x$ and $x < b$ can be written simply as $a < x < b$ and then we call it a **double inequality**. The two forms are equivalent.

DOUBLE INEQUALITY

A double inequality is a compound inequality such as $a < x < b$. It is equivalent to $a < x$ and $x < b$.

Other forms:	$a < x < b$	is equivalent to	$a < x$	and	$x < b$
	$a \le x \le b$	is equivalent to	$a \le x$	and	$x \le b$
	$a > x > b$	is equivalent to	$a > x$	and	$x > b$
	$a \ge x \ge b$	is equivalent to	$a \ge x$	and	$x \ge b$

To solve a double inequality we perform the same operation on all three "parts" of the double inequality with the goal of isolating the variable in the center.

Example 19.2.10

Solve $-4 \le 3x - 7 < 8$. Graph the solution and write the solution in interval notation.

Solution

	$-4 \le 3x - 7 < 8$
Add 7 to all three parts.	$-4+7 \le 3x - 7+7 < 8+7$
Simplify.	$3 \le 3x < 15$
Divide each part by three.	$\frac{3}{3} \le \frac{3x}{3} < \frac{15}{3}$
Simplify.	$1 \le x < 5$
Graph the solution.	
Write the solution in interval notation.	$[1, 5)$

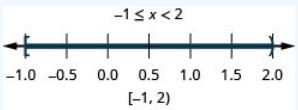
When written as a double inequality, $1 \le x < 5$, it is easy to see that the solutions are the numbers caught between one and five, including one, but not five. We can then graph the solution immediately as we did above.

Another way to graph the solution of $1 \le x < 5$ is to graph both the solution of $x \ge 1$ and the solution of $x < 5$. We would then find the numbers that make both inequalities true as we did in previous examples.

Try It 19.2.11

Solve the compound inequality. Graph the solution and write the solution in interval notation: $-5 \le 4x - 1 < 7$.

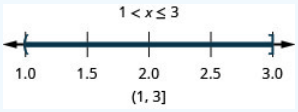
Answer



Try It 19.2.12

Solve the compound inequality. Graph the solution and write the solution in interval notation: $-3 < 2x - 5 \leq 1$.

Answer



Solve Compound Inequalities with "or"

To solve a **compound inequality** with "or", we start out just as we did with the compound inequalities with "and"—we solve the two inequalities. Then we find all the numbers that make *either* inequality true.

Just as the United States is the union of all of the 50 states, the solution will be the union of all the numbers that make either inequality true. To find the solution of the compound inequality, we look at the graphs of each inequality, find the numbers that belong to either graph and put all those numbers together.

To write the solution in **interval notation**, we will often use the **union symbol**, $\cup$, to show the union of the solutions shown in the graphs.

SOLVE A COMPOUND INEQUALITY WITH "OR."

1. Solve each inequality.
2. Graph each solution. Then graph the numbers that make either inequality true.
3. Write the solution in interval notation.

Example 19.2.13

Solve $5 - 3x \leq -1$ or $8 + 2x \leq 5$. Graph the solution and write the solution in interval notation.

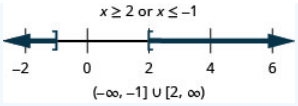
Solution

	$5 - 3x \leq -1$	or	$8 + 2x \leq 5$
Solve each inequality.	$5 - 3x \leq -1$		$8 + 2x \leq 5$
	$-3x \leq -6$		$2x \leq -3$
	$x \geq 2$	or	$x \leq -\frac{3}{2}$
Graph each solution.			
Graph numbers that make either inequality true.			
	$(-\infty, -\frac{3}{2}] \cup [2, \infty)$		

Try It 19.2.14

Solve the compound inequality. Graph the solution and write the solution in interval notation: $1 - 2x \leq -3$ or $7 + 3x \leq 4$.

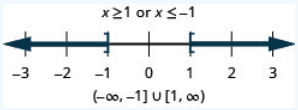
Answer



Try It 19.2.15

Solve the compound inequality. Graph the solution and write the solution in interval notation: $2-5x \leq -3$ or $5+2x \leq 3$.

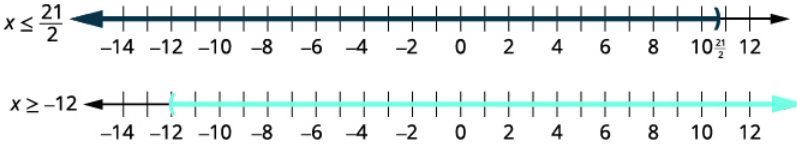
Answer



Example 19.2.16

Solve $\frac{2}{3}x-4 \leq 3$ or $\frac{1}{4}(x+8) \geq -1$. Graph the solution and write the solution in interval notation.

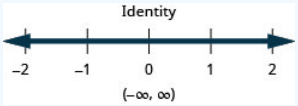
Solution

	$\frac{2}{3}x-4 \leq 3$	or	$\frac{1}{4}(x+8) \geq -1$
Solve each inequality.	$3(\frac{2}{3}x-4) \leq 3(3)$		$4 \cdot \frac{1}{4}(x+8) \geq 4 \cdot (-1)$
	$2x-12 \leq 9$		$x+8 \geq -4$
	$2x \leq 21$		$x \geq -12$
	$x \leq \frac{21}{2}$		
	$x \leq \frac{21}{2}$	or	$x \geq -12$
Graph each solution.			
Graph numbers that make either inequality true.			
	The solution covers all real numbers.		
	$(-\infty, \infty)$		

Try It 19.2.17

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{3}{5}x-7 \leq -1$ or $\frac{1}{3}(x+6) \geq -2$.

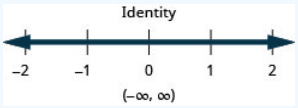
Answer



Try It 19.2.18

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{3}{4}x-3\leq 3$ or $\frac{2}{5}(x+10)\geq 0$.

Answer



Solve Applications with Compound Inequalities

Situations in the real world also involve compound inequalities. We will use the same problem solving strategy that we used to solve linear equation and inequality applications.

Recall the problem solving strategies are to first read the problem and make sure all the words are understood. Then, identify what we are looking for and assign a variable to represent it. Next, restate the problem in one sentence to make it easy to translate into a **compound inequality**. Last, we will solve the compound inequality.

Example 19.2.19

Due to the drought in California, many communities have tiered water rates. There are different rates for Conservation Usage, Normal Usage and Excessive Usage. The usage is measured in the number of hundred cubic feet (hcf) the property owner uses.

During the summer, a property owner will pay $24.72 plus $1.54 per hcf for Normal Usage. The bill for Normal Usage would be between or equal to $57.06 and $171.02. How many hcf can the owner use if he wants his usage to stay in the normal range?

Solution

Identify what we are looking for.	The number of hcf he can use and stay in the “normal usage” billing range.
Name what we are looking for.	Let x= the number of hcf he can use.
Translate to an inequality.	Bill is $24.72 plus $1.54 times the number of hcf he uses or $24.72+1.54x$.
	$\underbrace{\text{His bill will be between or equal to \$57.06 and \$171.02.}}$ $57.06\leq 24.72+1.54x\leq 171.02$
Solve the inequality.	$57.06\leq 24.72+1.54x\leq 171.02$ $57.06-24.72\leq 24.72-24.72+1.54x\leq 171.02-24.72$ $32.34\leq 1.54x\leq 146.3$ $\frac{32.34}{1.54}\leq \frac{1.54x}{1.54}\leq \frac{146.3}{1.54}$ $21\leq x\leq 95$
Answer the question.	The property owner can use $21-95$ hcf and still fall within the “normal usage” billing range.

Try It 19.2.20

Due to the drought in California, many communities now have tiered water rates. There are different rates for Conservation Usage, Normal Usage and Excessive Usage. The usage is measured in the number of hundred cubic feet (hcf) the property owner uses.

During the summer, a property owner will pay $24.72 plus $1.32 per hcf for Conservation Usage. The bill for Conservation Usage would be between or equal to $31.32 and $51.12. How many hcf can the owner use if she wants her usage to stay in the

conservation range?

Answer

The homeowner can use $5-20$ hcf and still fall within the "conservation usage" billing range.

? Try It 19.2.21

Due to the drought in California, many communities have tiered water rates. There are different rates for Conservation Usage, Normal Usage and Excessive Usage. The usage is measured in the number of hundred cubic feet (hcf) the property owner uses.

During the winter, a property owner will pay \$24.72 plus \$1.54 per hcf for Normal Usage. The bill for Normal Usage would be between or equal to \$49.36 and \$86.32. How many hcf will he be allowed to use if he wants his usage to stay in the normal range?

Answer

The homeowner can use $16-40$ hcf and still fall within the "normal usage" billing range.

Access this online resource for additional instruction and practice with solving compound inequalities.

- Compound inequalities

Key Concepts

- **How to solve a compound inequality with "and"**
  1. Solve each inequality.
  2. Graph each solution. Then graph the numbers that make *both* inequalities true. This graph shows the solution to the compound inequality.
  3. Write the solution in interval notation.
- **Double Inequality**
  - A **double inequality** is a compound inequality such as $a < x < b$. It is equivalent to $a < x$ and $x < b$.

    Other forms:

$$\begin{array}{lll} a < x < b & \text{is equivalent to} & a < x \text{ and } x < b \\ a \leq x \leq b & \text{is equivalent to} & a \leq x \text{ and } x \leq b \\ a > x > b & \text{is equivalent to} & a > x \text{ and } x > b \\ a \geq x \geq b & \text{is equivalent to} & a \geq x \text{ and } x \geq b \end{array}$$

- **How to solve a compound inequality with "or"**
  1. Solve each inequality.
  2. Graph each solution. Then graph the numbers that make either inequality true.
  3. Write the solution in interval notation.

Glossary

compound inequality

A compound inequality is made up of two inequalities connected by the word "and" or the word "or."

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19.2E: Exercises

Practice Makes Perfect

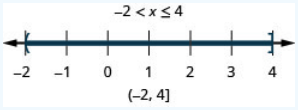
Solve Compound Inequalities with "and"

In the following exercises, solve each inequality, graph the solution, and write the solution in interval notation.

1. $x < 3$ and $x \geq 1$

2. $x \leq 4$ and $x > -2$

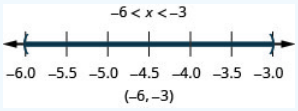
Answer



3. $x \geq -4$ and $x \leq -1$

4. $x > -6$ and $x < -3$

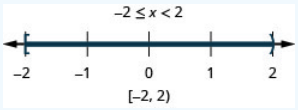
Answer



5. $5x - 2 < 8$ and $6x + 9 \geq 3$

6. $4x - 1 < 7$ and $2x + 8 \geq 4$

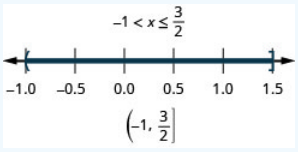
Answer



7. $4x + 6 \leq 2$ and $2x + 1 \geq -5$

8. $4x - 2 \leq 4$ and $7x - 1 > -8$

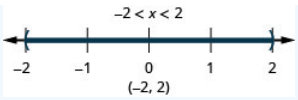
Answer



9. $2x - 11 < 5$ and $3x - 8 > -5$

10. $7x - 8 < 6$ and $5x + 7 > -3$

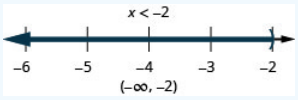
Answer



11. $4(2x-1) \leq 12$ and $2(x+1) < 4$

12. $5(3x-2) \leq 5$ and $3(x+3) < 3$

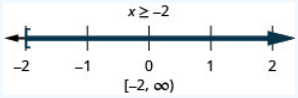
Answer



13. $3(2x-3) > 3$ and $4(x+5) \geq 4$

14. $-3(x+4) < 0$ and $-1(3x-1) \leq 7$

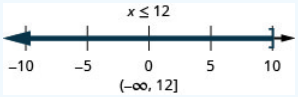
Answer



15. $\frac{1}{2}(3x-4) \leq 1$ and $\frac{1}{3}(x+6) \leq 4$

16. $\frac{3}{4}(x-8) \leq 3$ and $\frac{1}{5}(x-5) \leq 3$

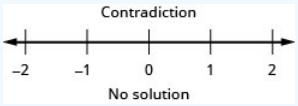
Answer



17. $5x-2 \leq 3x+4$ and $3x-4 \geq 2x+1$

18. $\frac{3}{4}x-5 \geq -2$ and $-3(x+1) \geq 6$

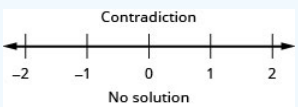
Answer



19. $\frac{2}{3}x-6 \geq -4$ and $-4(x+2) \geq 0$

20. $\frac{1}{2}(x-6)+2 < -5$ and $4-\frac{2}{3}x < 6$

Answer

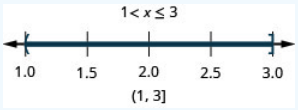


https://math.libretexts.org/@go/page/46457

21. $-5 \leq 4x - 1 < 7$

22. $-3 < 2x - 5 \leq 1$

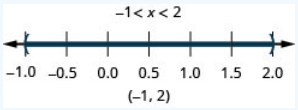
Answer



23. $5 < 4x + 1 < 9$

24. $-1 < 3x + 2 < 8$

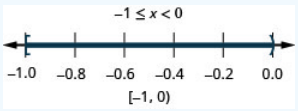
Answer



25. $-8 < 5x + 2 \leq -3$

26. $-6 \leq 4x - 2 < -2$

Answer



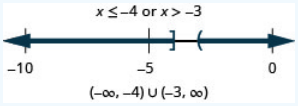
Solve Compound Inequalities with "or"

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

27. $x \leq -2$ or $x > 3$

28. $x \leq -4$ or $x > -3$

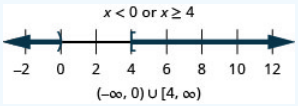
Answer



29. $x < 2$ or $x \geq 5$

30. $x < 0$ or $x \geq 4$

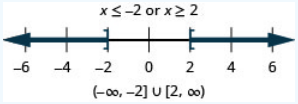
Answer



31. $2+3x \leq 4$ or $5-2x \leq -1$

32. $4-3x \leq -2$ or $2x-1 \leq -5$

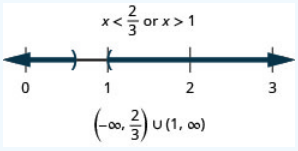
Answer



33. $2(3x-1) < 4$ or $3x-5 > 1$

34. $3(2x-3) < -5$ or
$4x-1 > 3$

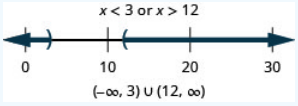
Answer



35. $\frac{3}{4}x-2 > 4$ or $4(2-x) > 0$

36. $\frac{2}{3}x-3 > 5$ or $3(5-x) > 6$

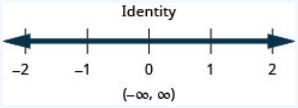
Answer



37. $3x-2 > 4$ or $5x-3 \leq 7$

38. $2(x+3) \geq 0$ or $3(x+4) \leq 6$

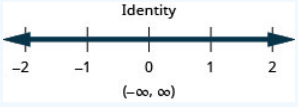
Answer



39. $\frac{1}{2}x-3 \leq 4$ or $\frac{1}{3}(x-6) \geq -2$

40. $\frac{3}{4}x+2 \leq -1$ or $\frac{1}{2}(x+8) \geq -3$

Answer



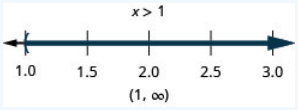
Mixed practice

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

41. $3x+7 \leq 1$ and $2x+3 \geq -5$

42. $6(2x-1) > 6$ and $5(x+2) \geq 0$

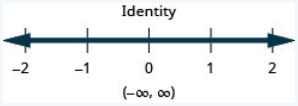
Answer



43. $4-7x \geq -3$ or $5(x-3)+8 > 3$

44. $\frac{1}{2}x-5 \leq 3$ or $\frac{1}{4}(x-8) \geq -3$

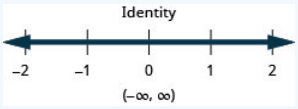
Answer



45. $-5 \leq 2x-1 < 7$

46. $\frac{1}{5}(x-5)+6 < 4$ and $3-\frac{2}{3}x < 5$

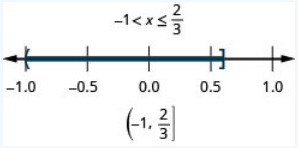
Answer



47. $4x-2 > 6$ or $3x-1 \leq -2$

48. $6x-3 \leq 1$ and $5x-1 > -6$

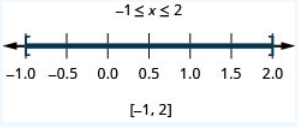
Answer



49. $-2(3x-4) \leq 2$ and $-4(x-1) < 2$

50. $-5 \leq 3x-2 \leq 4$

Answer



Solve Applications with Compound Inequalities

In the following exercises, solve.

51. Penelope is playing a number game with her sister June. Penelope is thinking of a number and wants June to guess it. Five more than three times her number is between 2 and 32. Write a compound inequality that shows the range of numbers that Penelope might be thinking of.

52. Gregory is thinking of a number and he wants his sister Lauren to guess the number. His first clue is that six less than twice his number is between four and forty-two. Write a compound inequality that shows the range of numbers that Gregory might be thinking of.

Answer

$5 \leq n \leq 24$

53. Andrew is creating a rectangular dog run in his back yard. The length of the dog run is 18 feet. The perimeter of the dog run must be at least 42 feet and no more than 72 feet. Use a compound inequality to find the range of values for the width of the dog run.

54. Elouise is creating a rectangular garden in her back yard. The length of the garden is 12 feet. The perimeter of the garden must be at least 36 feet and no more than 48 feet. Use a compound inequality to find the range of values for the width of the garden.

Answer

$6 \leq w \leq 12$

Everyday Math

55. Blood Pressure A person's blood pressure is measured with two numbers. The systolic blood pressure measures the pressure of the blood on the arteries as the heart beats. The diastolic blood pressure measures the pressure while the heart is resting.

ⓐ Let x be your systolic blood pressure. Research and then write the compound inequality that shows you what a normal systolic blood pressure should be for someone your age.

ⓑ Let y be your diastolic blood pressure. Research and then write the compound inequality that shows you what a normal diastolic blood pressure should be for someone your age.

56. Body Mass Index (BMI) is a measure of body fat is determined using your height and weight.

ⓐ Let x be your BMI. Research and then write the compound inequality to show the BMI range for you to be considered normal weight.

ⓑ Research a BMI calculator and determine your BMI. Is it a solution to the inequality in part (a)?

Answer

ⓐ answers vary ⓑ answers vary

Writing Exercises

57. In your own words, explain the difference between the properties of equality and the properties of inequality.

58. Explain the steps for solving the compound inequality $2 - 7x \geq -5$ or $4(x - 3) + 7 > 3$.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve compound inequalities with "and".			
solve compound inequalities with "or".			
solve applications with compound inequalities.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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19.3: Solve Absolute Value Equations and Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve absolute value equations
- Solve absolute value inequalities with “less than”
- Solve absolute value inequalities with “greater than”
- Solve applications with absolute value

Before you get started, take this readiness quiz.

1. Evaluate: $-|7|$.
   If you missed this problem, review [link].
2. Fill in $<, >, <, >$, or $=$ for each of the following pairs of numbers.
   ⓐ $|-8|\_\_\_-|-8|$ ⓑ $12\_\_\_-|-12|$ ⓒ $|-6|\_\_\_-6$ ⓓ $-(-15)\_\_\_-|-15|$
   If you missed this problem, review [link].
3. Simplify: $14-2|8-3(4-1)|$.
   If you missed this problem, review [link].

Solve Absolute Value Equations

As we prepare to solve absolute value equations, we review our definition of **absolute value**.

ABSOLUTE VALUE

The absolute value of a number is its distance from zero on the number line.

The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.

Absolute values are always greater than or equal to zero.

We learned that both a number and its opposite are the same distance from zero on the number line. Since they have the same distance from zero, they have the same absolute value. For example:

- -5 is 5 units away from 0, so $|-5| = 5$.
- 5 is 5 units away from 0, so $|5| = 5$.

Figure $19.3.1$ illustrates this idea.

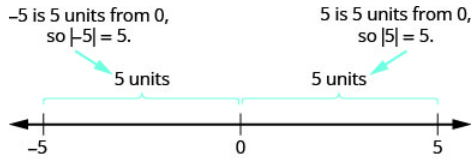


Figure 19.3.1: The numbers 5 and -5 are both five units away from zero.

For the equation |x|=5,|x|=5, we are looking for all numbers that make this a true statement. We are looking for the numbers whose distance from zero is 5. We just saw that both 5 and −5−5 are five units from zero on the number line. They are the solutions to the equation.

$$\begin{array}{ll} \text{If} & |x| = 5 \\ \text{then} & x = -5 \text{ or } x = 5 \end{array}$$

The solution can be simplified to a single statement by writing $x = \pm 5$. This is read, “x is equal to positive or negative 5”.

We can generalize this to the following property for absolute value equations.

ABSOLUTE VALUE EQUATIONS

For any algebraic expression, u, and any positive real number, a,

$$\begin{aligned} &\text{if} && |u| = a \\ &\text{then} && u = -a \text{ or } u = a \end{aligned}$$

Remember that an absolute value cannot be a negative number.

Example 19.3.1

Solve:

a. $|x| = 8$
b. $|y| = -6$
c. $|z| = 0$

Solution a

$$\begin{aligned} & && |x| = 8 \\ &\text{Write the equivalent equations.} && x = -8 \text{ or } x = 8 \\ & && x = \pm 8 \end{aligned}$$

Solution b

$$|y| = -6$$

No solution

Since an absolute value is always positive, there are no solutions to this equation.

Solution c

$$\begin{aligned} & && |z| = 0 \\ &\text{Write the equivalent equations.} && z = -0 \text{ or } z = 0 \\ &\text{Since } -0 = 0, && z = 0 \end{aligned}$$

Both equations tell us that z=0z=0 and so there is only one solution.

EXERCISE 19.3.2

Solve:

a. $|x| = 2$
b. $|y| = -4$
c. $|z| = 0$

Answer a

± 2

Answer b

no solution

Answer c

0

EXERCISE 19.3.3

Solve:

a. $|x| = 11$
b. $|y| = -5$

c. $|z| = 0$

Answer a

± 11

Answer b

no solution

Answer c

0

To solve an **absolute value equation**, we first isolate the absolute value expression using the same procedures we used to solve linear equations. Once we isolate the absolute value expression we rewrite it as the two equivalent equations.

How to Solve Absolute Value Equations

? Example 19.3.4

Solve $|5x - 4| - 3 = 8$.

Solution

Step 1. Isolate the absolute value expression.	Add 3 to both sides.	$\|5x-4\|-3=8$ $\|5x-4\|=11$
Step 2. Write the equivalent equations.		$5x-4=-11$ or $5x-4=11$
Step 3. Solve each equation.	Add 4 to each side. Divide each side by 5.	$5x=-7$ $\quad 5x=15$ $x=-\frac{7}{5}$ or $x=3$
Step 4. Check each solution.	Substitute 3 and $-\frac{7}{5}$ into the original equation.	$\|5x-4\|-3=8$ $x=3$ $\|5\cdot 3-4\|-3\stackrel{?}{=}8$ $\|15-4\|-3\stackrel{?}{=}8$ $\|11\|-3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8$ ✓ $\|5x-4\|-3=8$ $x=-\frac{7}{5}$ $\left\|5\left(-\frac{7}{5}\right)-4\right\|-3\stackrel{?}{=}8$ $\|-7-4\|-3\stackrel{?}{=}8$ $\|-11\|-3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8$ ✓

? EXERCISE 19.3.5

Solve: $|3x - 5| - 1 = 6$.

Answer

$x = 4,\ x = -\frac{2}{3}$

EXERCISE 19.3.6

Solve: $|4x - 3| - 5 = 2$.

Answer

$x = -1,\ x = \frac{5}{2}$

The steps for solving an absolute value equation are summarized here.

SOLVE ABSOLUTE VALUE EQUATIONS.

1. Isolate the absolute value expression.
2. Write the equivalent equations.
3. Solve each equation.
4. Check each solution.

Example 19.3.7

Solve $2|x - 7| + 5 = 9$.

Solution

	$2\|x - 7\| + 5 = 9$
Isolate the absolute value expression.	$2\|x - 7\| = 4$
	$\|x - 7\| = 2$
Write the equivalent equations.	$x - 7 = -2$ or $x - 7 = 2$
Solve each equation.	$x = 5$ or $x = 9$
Check: $2\|x-7\|+5=9$ $2\|5-7\|+5\stackrel{?}{=}9$ $2\|-2\|+5\stackrel{?}{=}9$ $2\cdot2+5\stackrel{?}{=}9$ $4+5\stackrel{?}{=}9$ $9=9$ ✓	$2\|x-7\|+5=9$ $2\|9-7\|+5\stackrel{?}{=}9$ $2\|2\|+5\stackrel{?}{=}9$ $2\cdot2+5\stackrel{?}{=}9$ $4+5\stackrel{?}{=}9$ $9=9$ ✓

Exercise 19.3.8

Solve: $3|x - 4| - 4 = 8$.

Answer

$x = 8,\ x = 0$

Exercise 19.3.9

Solve: $2|x - 5| + 3 = 9$.

Answer

$x = 8,\ x = 2$

Remember, an absolute value is always positive!

Example 19.3.10

Solve: $|\frac{2}{3}x - 4| + 11 = 3$.

Solution

	$|\frac{2}{3}x - 4| = -8$
Isolate the absolute value term.	$|\frac{2}{3}x - 4| = -8$
An absolute value cannot be negative.	No solution

Exercise 19.3.11

Solve: $|\frac{3}{4}x - 5| + 9 = 4$.

Answer

No solution

Exercise 19.3.12

Solve: $|\frac{5}{6}x + 3| + 8 = 6$.

Answer

No solution

Some of our absolute value equations could be of the form $|u| = |v|$ where u and v are algebraic expressions. For example, $|x - 3| = |2x + 1|$.

How would we solve them? If two algebraic expressions are equal in absolute value, then they are either equal to each other or negatives of each other. The property for absolute value equations says that for any algebraic expression, u, and a positive real number, a, if $|u| = a$, then $u = -a$ or $u = a$.

This tell us that

if	$|u| = |v|$		
then	$|u| = v$	or	$|u| = -v$
and so	$u = v$ or $u = -v$	or	$u = -v$ or $u = -(-v)$

This leads us to the following property for equations with two absolute values.

EQUATIONS WITH TWO ABSOLUTE VALUES

For any algebraic expressions, u and v,

if	$|u| = |v|$
then	$u = -v$ or $u = v$

When we take the opposite of a quantity, we must be careful with the signs and to add parentheses where needed.

Example 19.3.13

Solve: $|5x-1|=|2x+3|$.

Solution

$$|5x-1|=|2x+3|$$

Write the equivalent equations.	$5x-1=-(2x+3)$	or	$5x-1=2x+3$
	$5x-1=-2x-3$	or	$3x-1=3$
Solve each equation.	$7x-1=-3$		$3x=4$
	$7x=-2$		$x=43$
	$x=-27$	or	$x=43$

Check.

We leave the check to you.

Exercise 19.3.14

Solve: $|7x-3|=|3x+7|$.

Answer

$x=-\frac{2}{5},\ x=\frac{5}{2}$

Exercise 19.3.15

Solve: $|6x-5|=|3x+4|$.

Answer

$x=3, x=19$

Solve Absolute Value Inequalities with "Less Than"

Let's look now at what happens when we have an **absolute value inequality**. Everything we've learned about solving inequalities still holds, but we must consider how the absolute value impacts our work. Again we will look at our definition of absolute value. The absolute value of a number is its distance from zero on the number line. For the equation $|x|=5$, we saw that both 5 and -5 are five units from zero on the number line. They are the solutions to the equation.

$$|x|=5$$
$$x=-5 \quad \text{or} \quad x=5$$

What about the inequality $|x|\leq 5$? Where are the numbers whose distance is less than or equal to 5? We know -5 and 5 are both five units from zero. All the numbers between -5 and 5 are less than five units from zero (Figure 19.3.2).

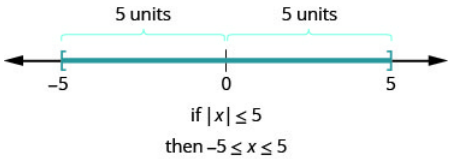


Figure 19.3.2.

In a more general way, we can see that if $|u|\leq a$, then $-a\leq u\leq a$ (Figure 19.3.3).

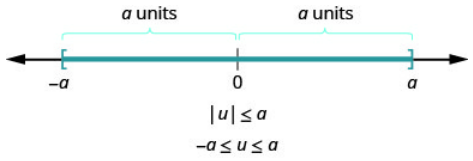


Figure 19.3.3.

This result is summarized here.

ABSOLUTE VALUE INEQUALITIES WITH $<$ OR $\leq$

For any algebraic expression, u, and any positive real number, a,

$$\text{if} \quad |u| < a, \quad \text{then} \quad -a < u < a$$
$$\text{if} \quad |u| \leq a, \quad \text{then} \quad -a \leq u \leq a$$

After solving an inequality, it is often helpful to check some points to see if the solution makes sense. The graph of the solution divides the number line into three sections. Choose a value in each section and substitute it in the original inequality to see if it makes the inequality true or not. While this is not a complete check, it often helps verify the solution.

Example 19.3.16

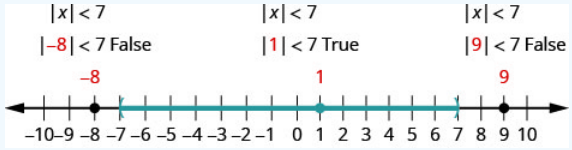
Solve $|x| < 7$. Graph the solution and write the solution in interval notation.

Solution

	$\|x\| < 7$
Write the equivalent inequality.	$-7 < x < 7$
Graph the solution.	
Write the solution using interval notation.	$(-7, 7)$

Check:

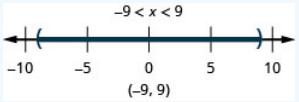
To verify, check a value in each section of the number line showing the solution. Choose numbers such as −8,−8, 1, and 9.



EXERCISE 19.3.17

Graph the solution and write the solution in interval notation: $|x| < 9$.

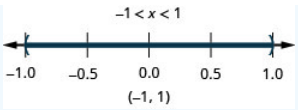
Answer



EXERCISE 19.3.18

Graph the solution and write the solution in interval notation: $|x| < 1$.

Answer



Example 19.3.19

Solve $|5x-6| \leq 4$. Graph the solution and write the solution in interval notation.

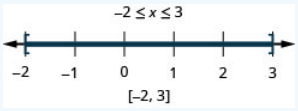
Solution

Step 1. Isolate the absolute value expression. It is isolated.	$\|5x-6\| \leq 4$
Step 2. Write the equivalent compound inequality.	$-4 \leq 5x-6 \leq 4$
Step 3. Solve the compound inequality.	$2 \leq 5x \leq 10$ $\frac{2}{5} \leq x \leq 2$
Step 4. Graph the solution.	
Step 5. Write the solution using interval notation.	$[\frac{2}{5}, 2]$
Check: The check is left to you.	

EXERCISE 19.3.20

Solve $|2x-1| \leq 5$. Graph the solution and write the solution in interval notation:

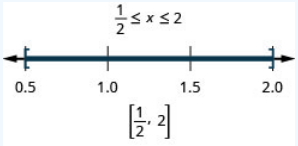
Answer



EXERCISE 19.3.21

Solve $|4x-5| \leq 3$. Graph the solution and write the solution in interval notation:

Answer



SOLVE ABSOLUTE VALUE INEQUALITIES WITH $<$ OR $\leq$

1. Isolate the absolute value expression.
2. Write the equivalent compound inequality.

$$|u| < a \quad \text{is equivalent to} \quad -a < u < a$$
$$|u| \leq a \quad \text{is equivalent to} \quad -a \leq u \leq a$$

3. Solve the compound inequality.
4. Graph the solution

5. Write the solution using interval notation.

Solve Absolute Value Inequalities with “Greater Than”

What happens for absolute value inequalities that have “greater than”? Again we will look at our definition of absolute value. The absolute value of a number is its distance from zero on the number line.

We started with the inequality $|x| \leq 5$. We saw that the numbers whose distance is less than or equal to five from zero on the number line were -5 and 5 and all the numbers between -5 and 5 (Figure 19.3.4).

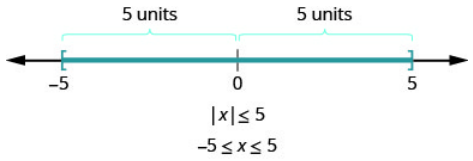


Figure 19.3.4.

Now we want to look at the inequality $|x| \geq 5$. Where are the numbers whose distance from zero is greater than or equal to five?

Again both -5 and 5 are five units from zero and so are included in the solution. Numbers whose distance from zero is greater than five units would be less than -5 and greater than 5 on the number line (Figure 19.3.5).

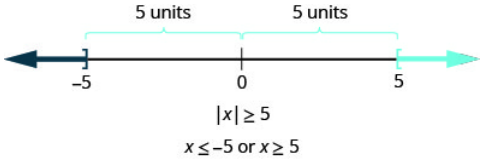


Figure 19.3.5.

In a more general way, we can see that if $|u| \geq a$, then $u \leq -a$ or $u \leq a$. See Figure.

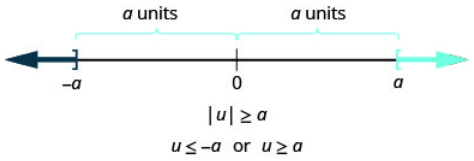


Figure 19.3.6.

This result is summarized here.

ABSOLUTE VALUE INEQUALITIES WITH $>$ OR $\geq$

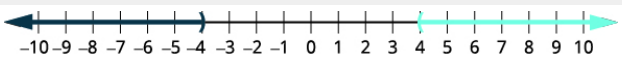
For any algebraic expression, u, and any positive real number, a,

$$\text{if} \quad |u| > a, \quad \text{then } u < -a \text{ or } u > a$$
$$\text{if} \quad |u| \geq a, \quad \text{then } u \leq -a \text{ or } u \geq a$$

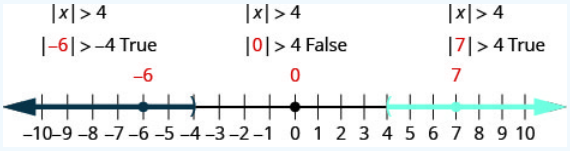
Example 19.3.22

Solve $|x| > 4$. Graph the solution and write the solution in interval notation.

Solution

	$\lvert x\rvert > 4$
Write the equivalent inequality.	$x < -4$ or $x > 4$
Graph the solution.	
Write the solution using interval notation.	$(-\inf, -4) \cup (4, \inf)$
Check:	

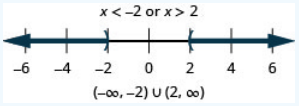
To verify, check a value in each section of the number line showing the solution. Choose numbers such as −6,−6, 0, and 7.



EXERCISE 19.3.23

Solve $|x| > 2$. Graph the solution and write the solution in interval notation.

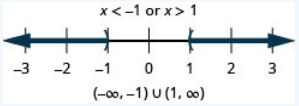
Answer



EXERCISE 19.3.24

Solve $|x| > 1$. Graph the solution and write the solution in interval notation.

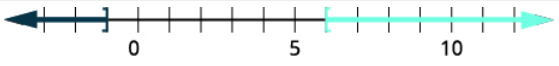
Answer



Example 19.3.25

Solve $|2x-3| \geq 5$. Graph the solution and write the solution in interval notation.

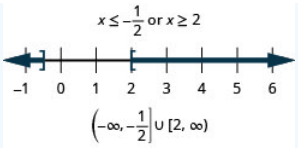
Solution

	$\|2x-3\| \geq 5$
Step 1. Isolate the absolute value expression. It is isolated.	
Step 2. Write the equivalent compound inequality.	$2x-3 \leq -5$ or $2x-3 \geq 5$
Step 3. Solve the compound inequality.	$2x \leq -2$ or $2x \geq 8$ $x \leq -1$ or $x \geq 4$
Step 4. Graph the solution.	
Step 5. Write the solution using interval notation.	$(-\inf, -1] \cup [4, \inf)$
Check: The check is left to you.	

EXERCISE 19.3.26

Solve $|4x-3| \geq 5$. Graph the solution and write the solution in interval notation.

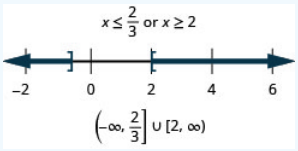
Answer



EXERCISE 19.3.27

Solve $|3x - 4| \ge 2$. Graph the solution and write the solution in interval notation.

Answer



SOLVE ABSOLUTE VALUE INEQUALITIES WITH $>$ OR $\ge$.

1. Isolate the absolute value expression.
2. Write the equivalent compound inequality.

$$\begin{array}{lllll} |u| > a & \text{is equivalent to} & u < -a & \text{or} & u > a \\ |u| \ge a & \text{is equivalent to} & u \le -a & \text{or} & u \ge a \\ |u| > a & \text{is equivalent to} & u < -a & \text{or} & u > a \\ |u| \ge a & \text{is equivalent to} & u \le -a & \text{or} & u \ge a \end{array}$$

3. Solve the compound inequality.
4. Graph the solution
5. Write the solution using interval notation.

Solve Applications with Absolute Value

Absolute value inequalities are often used in the manufacturing process. An item must be made with near perfect specifications. Usually there is a certain *tolerance* of the difference from the specifications that is allowed. If the difference from the specifications exceeds the tolerance, the item is rejected.

$$|\text{actual-ideal}| \le \text{tolerance}$$

Example 19.3.28

The ideal diameter of a rod needed for a machine is 60 mm. The actual diameter can vary from the ideal diameter by 0.075 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Solution

	Let $x =$ the actual measurement
Use an absolute value inequality to express this situation.	$\|\text{actual-ideal}\| \le \text{tolerance}$
	$\|x - 60\| \le 0.075$
Rewrite as a compound inequality.	$-0.075 \le x - 60 \le 0.075$
Solve the inequality.	$59.925 \le x \le 60.075$
Answer the question.	The diameter of the rod can be between $59.925mm$ and $60.075mm$.

EXERCISE 19.3.29

The ideal diameter of a rod needed for a machine is 80 mm. The actual diameter can vary from the ideal diameter by 0.009 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Answer

The diameter of the rod can be between 79.991 and 80.009 mm.

EXERCISE 19.3.30

The ideal diameter of a rod needed for a machine is 75 mm. The actual diameter can vary from the ideal diameter by 0.05 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Answer

The diameter of the rod can be between 74.95 and 75.05 mm.

Access this online resource for additional instruction and practice with solving linear absolute value equations and inequalities.

- Solving Linear Absolute Value Equations and Inequalities

Key Concepts

- **Absolute Value**
  The absolute value of a number is its distance from 0 on the number line.
  The absolute value of a number *n* is written as $|n|$ and $|n| \geq 0$ for all numbers.
  Absolute values are always greater than or equal to zero.
- **Absolute Value Equations**
  For any algebraic expression, *u*, and any positive real number, *a*,
  if $|u| = a$
  then $u = -a$ or $u = a$
  Remember that an absolute value cannot be a negative number.
- **How to Solve Absolute Value Equations**
  1. Isolate the absolute value expression.
  2. Write the equivalent equations.
  3. Solve each equation.
  4. Check each solution.
- **Equations with Two Absolute Values**
  For any algebraic expressions, *u* and *v*,
  if $|u| = |v|$
  then $u = -v$ or $u = v$
- **Absolute Value Inequalities with** $<$ or $\leq$
  For any algebraic expression, *u*, and any positive real number, *a*,
  if $|u| = a$ then $-a < u < a$
  if $|u| \leq a$ then $-a \leq u \leq a$
- **How To Solve Absolute Value Inequalities with** $<$ or $\leq$
  1. Isolate the absolute value expression.
  2. Write the equivalent compound inequality.
     $|u| < a$ is equivalent to $-a < u < a$
     $|u| \leq a$ is equivalent to $-a \leq u \leq a$
  3. Solve the compound inequality.
  4. Graph the solution
  5. Write the solution using interval notation

- **Absolute Value Inequalities with $>$ or $\geq$**
  For any algebraic expression, u, and any positive real number, a,
  if $|u| > a$, then $u < -a$ or $u > a$
  if $|u| \geq a$, then $u \leq -a$ or $u \geq a$
- **How To Solve Absolute Value Inequalities with $>$ or $\geq$**
  1. Isolate the absolute value expression.
  2. Write the equivalent compound inequality.
     $|u| > a$ is equivalent to $u < -a$ or $u > a$
     $|u| \geq a$ is equivalent to $u \leq -a$ or $u \geq a$
  3. Solve the compound inequality.
  4. Graph the solution
  5. Write the solution using interval notation

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19.3E: Exercises

Practice Makes Perfect

Solve Absolute Value Equations

In the following exercises, solve.

1. a. $|x| = 6$ b. $|y| = -3$ c. $|z| = 0$

2. a. $|x| = 4$ b. $|y| = -5$ c. $|z| = 0$

Answer

a. $x = 4, x = -4$ b. no solution c. $z = 0$

3. a. $|x| = 7$ b. $|y| = -11$ c. $|z| = 0$

4. a. $|x| = 3$ b. $|y| = -1$ c. $|z| = 0$

Answer

a. $x = 3, x = -3$ b. no solution c. $z = 0$

5. $|2x - 3| - 4 = 1$

6. $|4x - 1| - 3 = 0$

Answer

$x = 1,\ x = -\frac{1}{2}$

7. $|3x - 4| + 5 = 7$

8. $|4x + 7| + 2 = 5$

Answer

$x = -1,\ x = -\frac{5}{2}$

9. $4|x - 1| + 2 = 10$

10. $3|x - 4| + 2 = 11$

Answer

$x = 7,\ x = 1$

11. $3|4x - 5| - 4 = 11$

12. $3|x + 2| - 5 = 4$

Answer

$x = 1,\ x = -5$

13. $-2|x-3|+8=-4$

14. $-3|x-4|+4=-5$

Answer

$x=7,\ x=1$

15. $|34x-3|+7=2$

16. $|35x-2|+5=2$

Answer

no solution

17. $|12x+5|+4=1$

18. $|14x+3|+3=1$

Answer

no solution

19. $|3x-2|=|2x-3|$

20. $|4x+3|=|2x+1|$

Answer

$x=-1,\ x=-\frac{2}{3}$

21. $|6x-5|=|2x+3|$

22. $|6-x|=|3-2x|$

Answer

$x=-3,\ x=3$

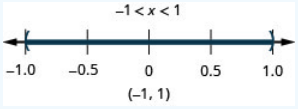
Solve Absolute Value Inequalities with "less than"

In the following exercises, solve each inequality. Graph the solution and write the solution in interval notation.

23. $|x|<5$

24. $|x|<1$

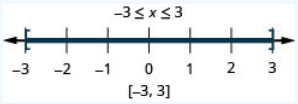
Answer



25. $|x|\leq 8$

26. $|x| \leq 3$

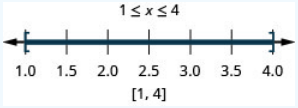
Answer



27. $|3x - 3| \leq 6$

28. $|2x - 5| \leq 3$

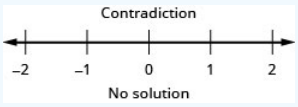
Answer



29. $|2x + 3| + 5 < 4$

30. $|3x - 7| + 3 < 1$

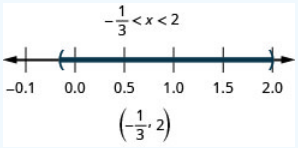
Answer



31. $|4x - 3| < 1$

32. $|6x - 5| < 7$

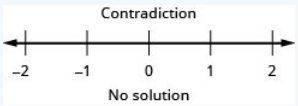
Answer



33. $|x - 4| \leq -1$

34. $|5x + 1| \leq -2$

Answer



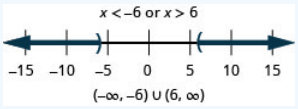
Solve Absolute Value Inequalities with "greater than"

In the following exercises, solve each inequality. Graph the solution and write the solution in interval notation.

35. $|x| > 3$

36. $|x| > 6$

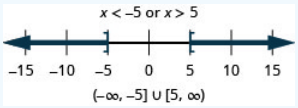
Answer



37. $|x| \geq 2$

38. $|x| \geq 5$

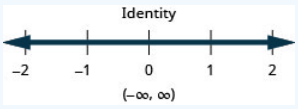
Answer



39. $|3x - 8| > -1$

40. $|x - 5| > -2$

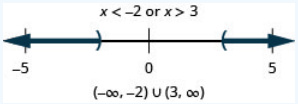
Answer



41. $|3x - 2| > 4$

42. $|2x - 1| > 5$

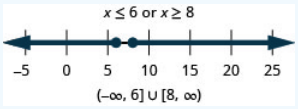
Answer



43. $|x + 3| \geq 5$

44. $|x - 7| \geq 1$

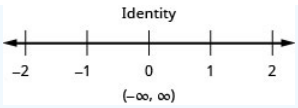
Answer



45. $3|x| + 4 \geq 1$

46. $5|x| + 6 \geq 1$

Answer



In the following exercises, solve. For each inequality, also graph the solution and write the solution in interval notation.

47. $2|x+6|+4=8$

48. $|3x-4| \geq 2$

Answer

$x=4, x=27$

49. $|6x-5|=|2x+3|$

50. $|4x-3|<5$

Answer

$x=3, x=2$

51. $|2x-5|+2=3$

52. $|3x+1|-3=7$

Answer

$x=3, x=-\frac{11}{3}$

53. $|7x+2|+8<4$

54. $5|2x-1|-3=7$

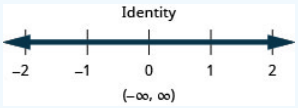
Answer

$x=\frac{3}{2}, x=-\frac{1}{2}$

55. $|x-7|>-3$

56. $|8-x|=|4-3x|$

Answer



Solve Applications with Absolute Value

In the following exercises, solve.

57. A chicken farm ideally produces 200,000 eggs per day. But this total can vary by as much as 25,000 eggs. What is the maximum and minimum expected production at the farm?

58. An organic juice bottler ideally produces 215,000 bottle per day. But this total can vary by as much as 7,500 bottles. What is the maximum and minimum expected production at the bottling company?

Answer

The minimum to maximum expected production is 207,500 to 2,225,000 bottles

59. In order to insure compliance with the law, Miguel routinely overshoots the weight of his tortillas by 0.5 gram. He just received a report that told him that he could be losing as much as $100,000 per year using this practice. He now plans to buy new equipment that guarantees the thickness of the tortilla within 0.005 inches. If the ideal thickness of the tortilla is 0.04 inches, what thickness of tortillas will be guaranteed?

60. At Lilly's Bakery, the ideal weight of a loaf of bread is 24 ounces. By law, the actual weight can vary from the ideal by 1.5 ounces. What range of weight will be acceptable to the inspector without causing the bakery being fined?

Answer

The acceptable weight is 22.5 to 25.5 ounces.

Writing Exercises

61. Write a graphical description of the absolute value of a number

62. In your own words, explain how to solve the absolute value inequality, $|3x-2| \geq 4$.

Answer

Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve absolute value equations.			
solve absolute value inequalities with "less than".			
solve absolute value inequalities with "greater than".			
solve applications with absolute value.			

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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19.4: Solve Rational Inequalities

Learning Objectives

- Solve rational inequalities
- Solve an inequality with rational functions

Be Prepared

Before you get started, take this readiness quiz.

1. Find the value of $x-5$ when ⓐ $x=6$ ⓑ $x=-3$ ⓒ $x=5$
   If you missed this problem, review Example 1.2.16.
2. Solve: $8-2x<12$
   If you missed this problem, review Example 2.6.13.
3. Write in interval notation: $-3\le x<5$
   If you missed this problem, review Example 2.6.4.

Solve Rational Inequalities

We learned to solve linear inequalities after learning to solve linear equations. The techniques were very much the same with one major exception. When we multiplied or divided by a negative number, the inequality sign reversed.

Having just learned to solve rational equations we are now ready to solve rational inequalities. A rational inequality is an inequality that contains a rational expression.

Rational Inequality

A rational inequality is an inequality that contains a rational expression.

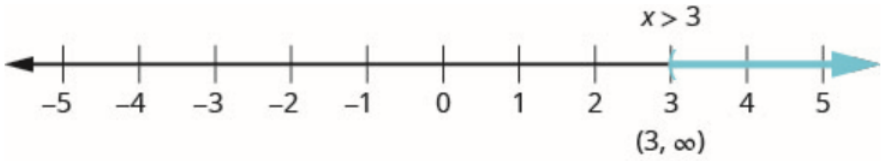
Inequalities such as $\frac{3}{2x}>1$, $\frac{2x}{x-3}<4$, $\frac{2x-3}{x-6}\ge x$, and $\frac{1}{4}-\frac{2}{x^2}\le\frac{3}{x}$ are rational inequalities as they each contain a rational expression.

When we solve a rational inequality, we will use many of the techniques we used solving linear inequalities. We especially must remember that when we multiply or divide by a negative number, the inequality sign must reverse.

Another difference is that we must carefully consider what value might make the rational expression undefined and so must be excluded.

When we solve an equation and the result is $x=3$, we know there is one solution, which is 3.

When we solve an inequality and the result is $x>3$, we know there are many solutions. We graph the result to better help show all the solutions, and we start with 3. Three becomes a critical point and then we decide whether to shade to the left or right of it. The numbers to the right of 3 are larger than 3, so we shade to the right.



To solve a rational inequality, we first must write the inequality with only one quotient on the left and 0 on the right.

Next we determine the critical points to use to divide the number line into intervals. A **critical point** is a number which make the rational expression zero or undefined.

We then will evaluate the factors of the numerator and denominator, and find the quotient in each interval. This will identify the interval, or intervals, that contains all the solutions of the rational inequality.

We write the solution in interval notation being careful to determine whether the endpoints are included.

Example 19.4.1

Solve and write the solution in interval notation: $\dfrac{x-1}{x+3} \geq 0$

Solution

Step 1. Write the inequality as one quotient on the left and zero on the right.

Our inequality is in this form.

$$\frac{x-1}{x+3} \geq 0$$

Step 2. Determine the critical points—the points where the rational expression will be zero or undefined.

The rational expression will be zero when the numerator is zero. Since $x-1=0$ when $x=1$, then 1 is a critical point.

The rational expression will be undefined when the denominator is zero. Since $x+3=0$ when $x=-3$, then -3 is a critical point.

The critical points are 1 and -3.

Step 3. Use the critical points to divide the number line into intervals.

The number line is divided into three intervals:

$$(-\infty,-3) \quad (-3,1) \quad (1,\infty)$$

Step 4. Test a value in each interval. Above the number line show the sign of each factor of the rational expression in each interval. Below the number line show the sign of the quotient.

To find the sign of each factor in an interval, we choose any point in that interval and use it as a test point. Any point in the interval will give the expression the same sign, so we can choose any point in the interval.

$$\text{Interval } (-\infty,-3)$$

The number -4 is in the interval $(-\infty,-3)$. Test $x=-4$ in the expression in the numerator and the denominator.

The numerator:

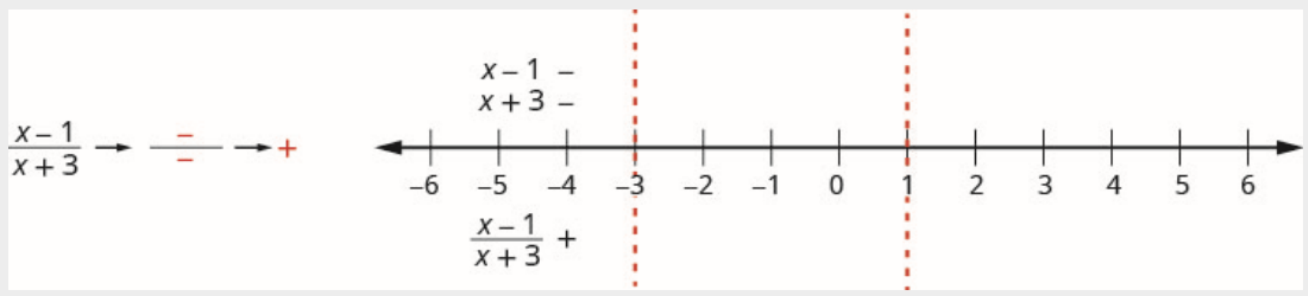
$$\begin{array}{l} x-1 \\ -4-1 \\ -5 \\ \text{Negative} \end{array}$$

The denominator:

$$\begin{array}{l} x+3 \\ -4+3 \\ -1 \\ \text{Negative} \end{array}$$

Above the number line, mark the factor $x-1$ negative and mark the factor $x+3$ negative.

Since a negative divided by a negative is positive, mark the quotient positive in the interval $(-\infty, -3)$.



Interval $(-3, 1)$

The number 0 is in the interval $(-3, 1)$. Test $x = 0$.

The numerator:

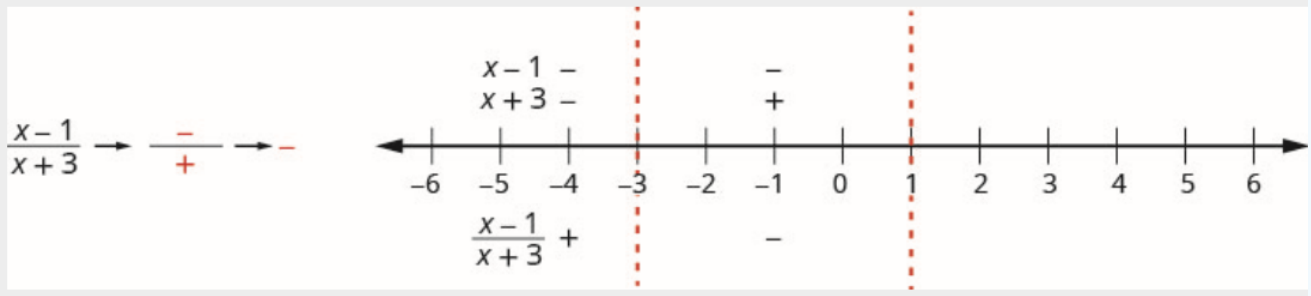
$$x-1$$
$$0-1$$
$$-1$$
Negative

The denominator:

$$x+3$$
$$0+3$$
$$3$$
Positive

Above the number line, mark the factor $x-1$ negative and mark $x+3$ positive.

Since a negative divided by a positive is negative, the quotient is marked negative in the interval $(-3, 1)$.



Interval $(1, \infty)$

The number 2 is in the interval $(1, \infty)$. Test $x = 2$.

The numerator:

$$x-1$$
$$2-1$$
$$1$$
Positive

The denominator:

$x+3$

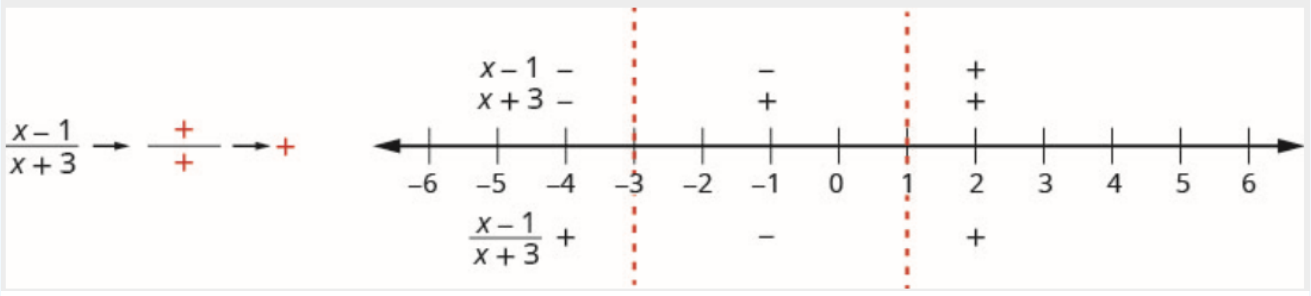
$2+3$

5

Positive

Above the number line, mark the factor $x-1$ positive and mark $x+3$ positive.

Since a positive divided by a positive is positive, mark the quotient positive in the interval $(1,\infty)$.



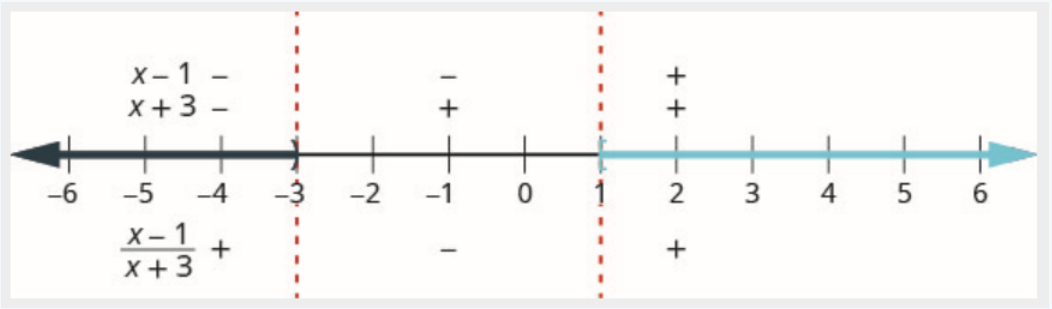
Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

We want the quotient to be greater than or equal to zero, so the numbers in the intervals $(-\infty,-3)$ and $(1,\infty)$are solutions.

But what about the critical points?

The critical point $x=-3$ makes the denominator 0, so it must be excluded from the solution and we mark it with a parenthesis.

The critical point $x=1$ makes the whole rational expression 0. The inequality requires that the rational expression be greater than or equal to. So, 1 is part of the solution and we will mark it with a bracket.



Recall that when we have a solution made up of more than one interval we use the union symbol, $\cup$, to connect the two intervals. The solution in interval notation is $(-\infty,-3)\cup[1,\infty)$.

Exercise 19.4.1

Solve and write the solution in interval notation: $\dfrac{x-2}{x+4}\geq 0$

Answer

$(-\infty,-4)\cup[2,\infty)$

Exercise 19.4.2

Solve and write the solution in interval notation: $\dfrac{x+2}{x-4}\geq 0$

Answer

$(-\infty, -2] \cup (4, \infty)$

We summarize the steps for easy reference.

How to Solve a rational inequality

Step 1. Write the inequality as one quotient on the left and zero on the right.

Step 2. Determine the critical points–the points where the rational expression will be zero or undefined.

Step 3. Use the critical points to divide the number line into intervals.

Step 4. Test a value in each interval. Above the number line show the sign of each factor of the numerator and denominator in each interval. Below the number line show the sign of the quotient.

Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

The next example requires that we first get the rational inequality into the correct form.

Example 19.4.2

Solve and write the solution in interval notation: $\frac{4x}{x-6} < 1$

Solution

$$\frac{4x}{x-6} < 1$$

Subtract 1 to get zero on the right.

$$\frac{4x}{x-6} - 1 < 0$$

Rewrite 1 as a fraction using the LCD.

$$\frac{4x}{x-6} - \frac{x-6}{x-6} < 0$$

Subtract the numerators and place the difference over the common denominator.

$$\frac{4x-(x-6)}{x-6} < 0$$

Simplify.

$$\frac{3x+6}{x-6} < 0$$

Factor the numerator to show all factors.

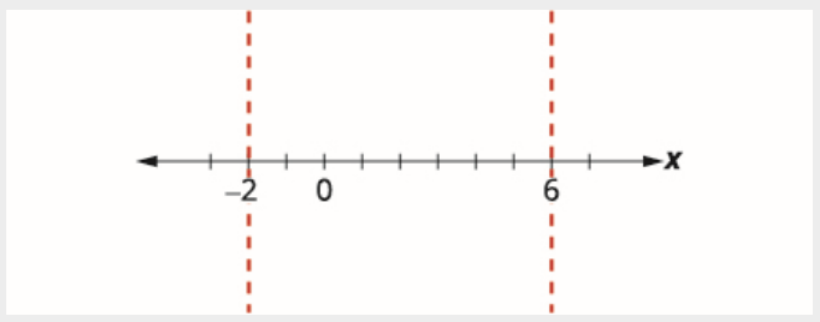
$$\frac{3(x+2)}{x-6} < 0$$

Find the critical points.

The quotient will be zero when the numerator is zero. The quotient is undefined when the denominator is zero.

$$\begin{aligned} x+2 &= 0 \qquad x-6 = 0 \\ x &= -2 \qquad\quad x = 6 \end{aligned}$$

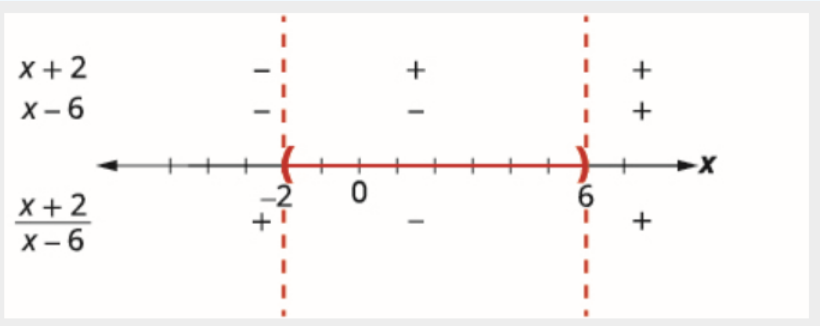
Use the critical points to divide the number line into intervals.



Test a value in each interval.

	$(-\infty, -2)$	$(-2, 6)$	$(6, \infty)$
$x + 2)$	x+2 -3+2 -1 -	x+2 0+2 2 +	x+2 7+2 9 +
$x - 6$	x-6 -3-6 -9 -	x-6 0-6 -6 -	x-6 7-6 1 +

Above the number line show the sign of each factor of the rational expression in each interval. Below the number line show the sign of the quotient.



Determine the intervals where the inequality is correct. We want the quotient to be negative, so the solution includes the points between −2 and 6. Since the inequality is strictly less than, the endpoints are not included.

We write the solution in interval notation as (−2, 6).

? Exercise 19.4.3

Solve and write the solution in interval notation: $\frac{3x}{x-3} < 1$.

Answer

$\left(-\frac{3}{2}, 3\right)$

Exercise 19.4.4

Solve and write the solution in interval notation: $\frac{3x}{x-4} < 2$.

Answer

$(-8, 4)$

In the next example, the numerator is always positive, so the sign of the rational expression depends on the sign of the denominator.

Example 19.4.3

Solve and write the solution in interval notation: $\frac{5}{x^2-2x-15} > 0$.

Solution

The inequality is in the correct form.

$$\frac{5}{x^2-2x-15} > 0$$

Factor the denominator.

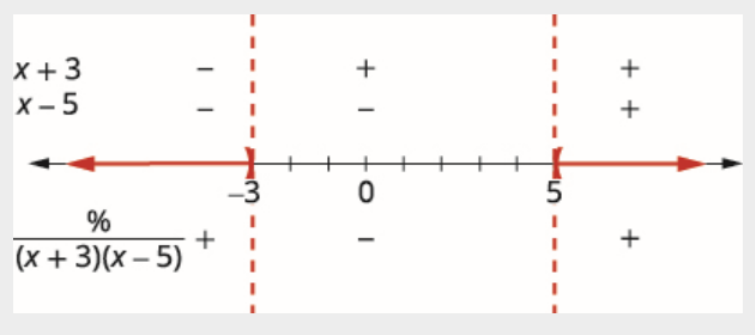
$$\frac{5}{(x+3)(x-5)} > 0$$

Find the critical points. The quotient is 0 when the numerator is 0. Since the numerator is always 5, the quotient cannot be 0.

The quotient will be undefined when the denominator is zero.

$$(x+3)(x-5) = 0$$
$$x = -3, x = 5$$

Use the critical points to divide the number line into intervals.



Test values in each interval. Above the number line show the sign of each factor of the denominator in each interval. Below the number line, show the sign of the quotient.

Write the solution in interval notation.

$$(-\infty, -3) \cup (5, \infty)$$

Exercise 19.4.5

Solve and write the solution in interval notation: $\frac{1}{x^2+2x-8} > 0$.

Answer

$(-\infty, -4) \cup (2, \infty)$

Exercise 19.4.6

Solve and write the solution in interval notation: $\dfrac{3}{x^2+x-12} > 0$.

Answer

$(-\infty, -4) \cup (3, \infty)$

The next example requires some work to get it into the needed form.

Example 19.4.4

Solve and write the solution in interval notation: $\dfrac{1}{3} - \dfrac{2}{x^2} < \dfrac{5}{3x}$.

Solution

$$\frac{1}{3} - \frac{2}{x^2} < \frac{5}{3x}$$

Subtract $\dfrac{5}{3x}$ to get zero on the right.

$$\frac{1}{3} - \frac{2}{x^2} - \frac{5}{3x} < 0$$

Rewrite to get each fraction with the LCD $3x^2$.

$$\frac{1 \cdot x^2}{3 \cdot x^2} - \frac{2 \cdot 3}{x^2 \cdot 3} - \frac{5 \cdot x}{3x - x} < 0$$

Simplify.

$$\frac{x^2}{3x^2} - \frac{6}{3x^2} - \frac{5x}{3x^2} < 0$$

Subtract the numerators and place the difference over the common denominator.

$$\frac{x^2 - 5x - 6}{3x^2} < 0$$

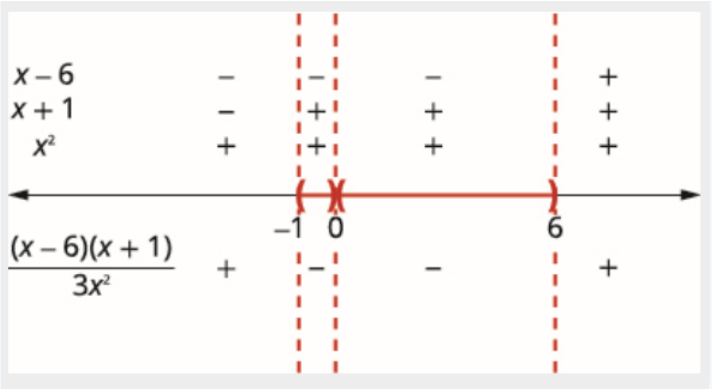
Factor the numerator.

$$\frac{(x-6)(x+1)}{3x^2} < 0$$

Find the critical points.

$$\begin{array}{lll} 3x^2 = 0 & x - 6 = 0 & x + 1 = 0 \\ x = 0 & x = 6 & x = -1 \end{array}$$

Use the critical points to divide the number line into intervals.



Above the number line show the sign of each factor in each interval. Below the number line, show the sign of the quotient.

Since, 0 is excluded, the solution is the two $(-1,0)\cup(0,6)$ intervals, $(-1,0)$ and $(0,6)$.

? Exercise 19.4.7

Solve and write the solution in interval notation: $\frac{1}{2}+\frac{4}{x^2}<\frac{3}{x}$.

Answer

$(2,4)$

? Exercise 19.4.8

Solve and write the solution in interval notation: $\frac{1}{3}+\frac{6}{x^2}<\frac{3}{x}$.

Answer

$(3,6)$

Solve an Inequality with Rational Functions

When working with rational functions, it is sometimes useful to know when the function is greater than or less than a particular value. This leads to a rational inequality.

✓ Example 19.4.5

Given the function $R(x)=\frac{x+3}{x-5}$, find the values of x that make the function less than or equal to 0.

Solution

We want the function to be less than or equal to 0.

$$R(x)\le 0$$

Substitute the rational expression for $R(x)$.

$$\frac{x+3}{x-5}\le 0 \quad x\neq 5$$

Find the critical points.

$$\begin{aligned} x+3&=0 & x-5&=0\\ x&=-3 & x&=5 \end{aligned}$$

Use the critical points to divide the number line into intervals.

Test values in each interval. Above the number line, show the sign of each factor in each interval. Below the number line, show the sign of the quotient. Write the solution in interval notation. Since 5 is excluded we, do not include it in the interval.

$$[-3, 5)$$

Exercise 19.4.9

Given the function $R(x) = \dfrac{x-2}{x+4}$, find the values of x that make the function less than or equal to 0.

Answer

$(-4, 2]$

Exercise 19.4.10

Given the function $R(x) = \dfrac{x+1}{x-4}$, find the values of x that make the function less than or equal to 0.

Answer

$[-1, 4)$

In economics, the function $C(x)$ is used to represent the cost of producing x units of a commodity. The average cost per unit can be found by dividing $C(x)$ by the number of items x. Then, the average cost per unit is \(c(x)=\dfrac{C(x)}{x}).

Example 19.4.6

The function$C(x) = 10x + 3000$ represents the cost to produce x, number of items. Find:

a. The average cost function, $c(x)$
b. How many items should be produced so that the average cost is less than $40.

Solution

a.
$$C(x) = 10x + 3000$$

The average cost function is $c(x) = \dfrac{C(x)}{x}$). To find the average cost function, divide the cost function by x.

$$c(x) = \frac{C(x)}{x}$$
$$c(x) = \frac{10x + 3000}{x}$$

The average cost function is $c(x) = \dfrac{10x + 3000}{x}$

b. We want the function $c(x)$ to be less than 40.

$$c(x) < 40$$

Substitute the rational expression forc(x).

$$\frac{10x+3000}{x} < 40, \quad x \neq 0$$

Subtract 40 to get 0 on the right.

$$\frac{10x+3000}{x} - 40 < 0$$

Rewrite the left side as one quotient by finding the LCD and performing the subtraction.

$$\begin{aligned} \frac{10x+3000}{x} - 40\left(\frac{x}{x}\right) &< 0 \\ \frac{10x+3000}{x} - \frac{40x}{x} &< 0 \\ \frac{10x+3000-40x}{x} &< 0 \\ \frac{-30x+3000}{x} &< 0 \end{aligned}$$

Factor the numerator to show all factors.

$$\begin{aligned} &\frac{-30(x-100)}{x} < 0 \\ &-30(x-100) = 0 \qquad x = 0 \end{aligned}$$

Find the critical points.

$$\begin{aligned} -30 \neq 0 \quad x - 100 &= 0 \\ x &= 100 \end{aligned}$$

More than 100 items must be produced to keep the average cost below $40 per item.

? Exercise 19.4.11

The function$C(x) = 20x + 6000$ represents the cost to produce x, number of items. Find:

a. How many items should be produced so that the average cost is less than $60.

Answer

a. $c(x) = \dfrac{20x+6000}{x}$

b. More than 150 items must be produced to keep the average cost below $60 per item.

? Exercise 19.4.12

The function$C(x) = 5x + 900$ represents the cost to produce x, number of items. Find:

a. How many items should be produced so that the average cost is less than $20.

Answer

a. $c(x) = \dfrac{5x+900}{x}$

b. More than 60 items must be produced to keep the average cost below $20 per item.

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19.4E: Exercises

Solve Rational Inequalities

In the following exercises, solve each rational inequality and write the solution in interval notation.

1. $\frac{x-3}{x+4} \geq 0$

Answer

$(-\infty, -4) \cup [3, \infty)$

2. $\frac{x+6}{x-5} \geq 0$

3. $\frac{x+1}{x-3} \leq 0$

Answer

$[-1, 3)$

4. $\frac{x-4}{x+2} \leq 0$

5. $\frac{x-7}{x-1} > 0$

Answer

$(-\infty, 1) \cup (7, \infty)$

6. $\frac{x+8}{x+3} > 0$

7. $\frac{x-6}{x+5} < 0$

Answer

$(-5, 6)$

8. $\frac{x+5}{x-2} < 0$

9. $\frac{3x}{x-5} < 1$

Answer

$\left(-\frac{5}{2}, 5\right)$

10. $\frac{5x}{x-2} < 1$

11. $\frac{6x}{x-6} > 2$

Answer

$(-\infty, -3) \cup (6, \infty)$

12. $\frac{3x}{x-4} > 2$

13. $\frac{2x+3}{x-6} \leq 1$

Answer

$[-9, 6)$

14. $\frac{4x-1}{x-4} \leq 1$

15. $\frac{3x-2}{x-4} \geq 2$

Answer

$(-\infty, -6] \cup (4, \infty)$

16. $\frac{4x-3}{x-3} \geq 2$

17. $\frac{1}{a} + \frac{2}{5} = \frac{1}{2}$

Answer

$a = 10$

18. $\frac{1}{x^2-4x-12} > 0$

19. $\frac{3}{x^2-5x+4} < 0$

Answer

$(1, 4)$

20. $\frac{4}{x^2+7x+12} < 0$

21. $\frac{2}{2x^2+x-15} \geq 0$

Answer

$(-\infty, -3) \cup \left(\frac{5}{2}, \infty\right)$

22. $\dfrac{6}{3x^2-2x-5} \geq 0$

23. $\dfrac{-2}{6x^2-13x+6} \leq 0$

Answer

$\left(-\infty, \dfrac{2}{3}\right) \cup \left(\dfrac{3}{2}, \infty\right)$

24. $\dfrac{-1}{10x^2+11x-6} \leq 0$

17. $\dfrac{1}{a} + \dfrac{2}{5} = \dfrac{1}{2}$

Answer

$a = 10$

18. $\dfrac{1}{x^2-4x-12} > 0$

19. $\dfrac{3}{x^2-5x+4} < 0$

Answer

$(1, 4)$

20. $\dfrac{4}{x^2+7x+12} < 0$

25. $\dfrac{1}{2} + \dfrac{12}{x^2} > \dfrac{5}{x}$

Answer

$(-\infty, 0) \cup (0, 4) \cup (6, \infty)$

26. $\dfrac{1}{3} + \dfrac{1}{x^2} > \dfrac{4}{3x}$

27. $\dfrac{1}{2} - \dfrac{4}{x^2} \leq \dfrac{1}{x}$

Answer

$[-2, 0) \cup (0, 4]$

28. $\dfrac{1}{2} - \dfrac{3}{2x^2} \geq \dfrac{1}{x}$

29. $\dfrac{1}{x^2-16} < 0$

Answer

$(-4, 4)$

30. $\frac{4}{x^2 - 25} > 0$

31. $\frac{4}{x-2} \geq \frac{3}{x+1}$

Answer

$[-10, -1) \cup (2, \infty)$

32. $\frac{5}{x-1} \leq \frac{4}{x+2}$

Solve an Inequality with Rational Functions

In the following exercises, solve each rational function inequality and write the solution in interval notation.

33. Given the function $R(x) = \frac{x-5}{x-2}$, find the values of x that make the function less than or equal to 0.

Answer

$(2, 5]$

34. Given the function $R(x) = \frac{x+1}{x+3}$, find the values of x that make the function less than or equal to 0.

35. Given the function $R(x) = \frac{x-6}{x+2}$, find the values of x that make the function less than or equal to 0.

Answer

$(-\infty, -2) \cup [6, \infty)$

36. Given the function $R(x) = \frac{x+1}{x-4}$, find the values of x that make the function less than or equal to 0.

Writing Exercises

37. Write the steps you would use to explain solving rational inequalities to your little brother.

Answer

Answers will vary

38. Create a rational inequality whose solution is $(-\infty, -2] \cup [4, \infty)$.

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19.5: Solve Quadratic Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic inequalities graphically
- Solve quadratic inequalities algebraically

Before you get started, take this readiness quiz.

1. Solve: $2x-3=0$.
   If you missed this problem, review Example 2.2.
2. Solve: $2y^2+y=15$.
   If you missed this problem, review Example 6.45.
3. Solve $\frac{1}{x^2+2x-8}>0$
   If you missed this problem, review Example 7.56.

We have learned how to solve linear inequalities and rational inequalities previously. Some of the techniques we used to solve them were the same and some were different. We will now learn to solve inequalities that have a quadratic expression. We will use some of the techniques from solving linear and rational inequalities as well as quadratic equations. We will solve quadratic inequalities two ways —both graphically and algebraically.

Solve Quadratic Inequalities Graphically

A **quadratic equation** is in standard form when written as $ax^2+bx+c=0$. If we replace the equal sign with an inequality sign, we have a **quadratic inequality** in standard form.

Definition 19.5.1: Quadratic Inequality

A **quadratic inequality** is an inequality that contains a quadratic expression. The standard form of a quadratic inequality is written:

$$ax^2+bx+c<0 \quad ax^2+bx+c\le 0$$
$$ax^2+bx+c>0 \quad ax^2+bx+c\ge 0$$

The graph of a quadratic function $f(x)=ax^2+bx+c=0$ is a parabola. When we ask when is $ax^2+bx+c<0$, we are asking when is $f(x)<0$. We want to know when the parabola is below the x-axis.

When we ask when is $ax^2+bx+c>0$, we are asking when is $f(x)>0$. We want to know when the parabola is above the y-axis.

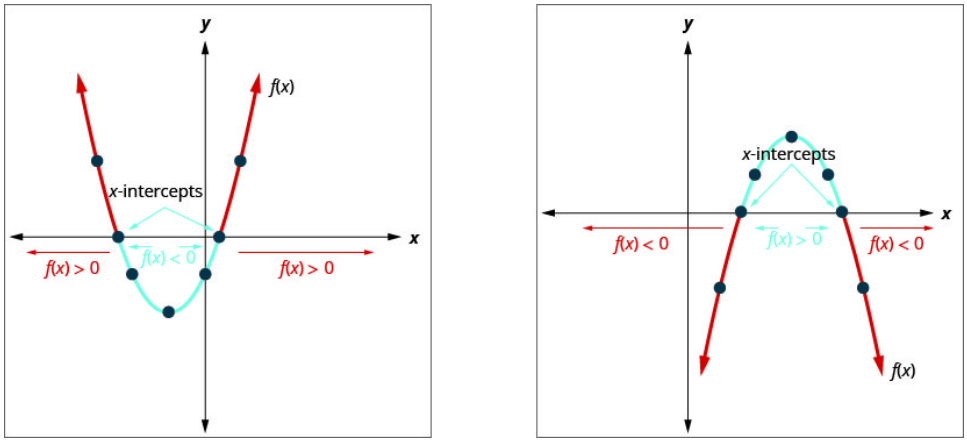


Figure 9.8.1

Example 19.5.1: How to Solve a Quadratic Inequality Graphically

Solve $x^2-6x+8<0$ graphically. Write the solution in interval notation.

Solution:

Step 1: Write the quadratic inequality in standard form.

The inequality is in standard form.

$$x^2-6x+8<0$$

Step 2: Graph the function $f(x)=ax^2+bx+c$ using properties or transformations.

We will graph using the properties.

$$f(x)=x^2-6x+8$$

Look at a in the equation.

$$a=1, b=-6, c=8$$

$f(x)=x^2-6x+8$

Since a is positive, the parabola opens upward.

The parabola opens upward.

Figure 9.8.2

$f(x)=x^2-6x+8$

The axis of symmetry is the line $x=-\frac{b}{2a}$.

Axis of Symmetry

$$x=-\frac{b}{2a}$$
$$x=-\frac{(-6)}{2\cdot 1}$$
$$x=3$$

The axis of symmetry is the line $x=3$.

The vertex is on the axis of symmetry. Substitute $x=3$ into the function.

Vertex

$$f(x)=x^2-6x+8$$
$$f(3)=(3)^2-6(3)+8$$
$$f(3)=-1$$

The vertex is $(3,-1)$.

We find $f(0)$

y-intercept

$$f(x)=x^2-6x+8$$
$$f(0)=(0)^2-6(0)+8$$
$$f(0)=8$$

The y-intercept is $(0,8)$.

We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x=3$. A point 3 units to the right of the axis of symmetry has $x=6$.

Point symmetric to y-intercept

The point is $(6,8)$.

We solve $f(x) = 0$.

x-intercepts

We can solve this quadratic equation by factoring.

$$\begin{aligned} f(x) &= x^2 - 6x + 8 \\ 0 &= x^2 - 6x + 8 \\ 0 &= (x-2)(x-4) \\ x &= 2 \text{ or } x = 4 \end{aligned}$$

The x-intercepts are $(2,0)$ and $(4,0)$.

We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.

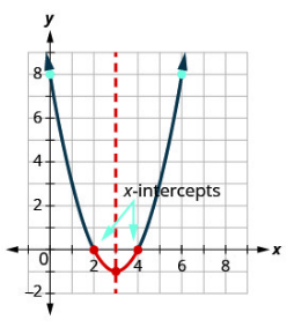


Figure 9.8.3

Step 3: Determine the solution from the graph.

$x^2 - 6x + 8 < 0$

The inequality asks for the values of x which make the function less than 0. Which values of x make the parabola below the x-axis.

We do not include the values 2, 4 as the inequality is less than only.

The solution, in interval notation, is $(2, 4)$.

Exercise 19.5.1

a. Solve $x^2 + 2x - 8 < 0$ graphically
b. Write the solution in interval notation

Answer

a.

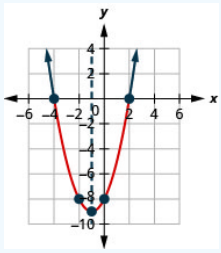


Figure 9.8.4

b. $(-4, 2)$

Exercise 19.5.2

a. Solve $x^2 - 8x + 12 \geq 0$ graphically
b. Write the solution in interval notation

Answer

a.

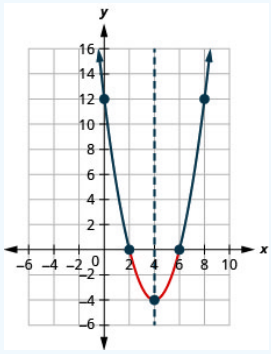


Figure 9.8.5

b. $(-\infty, 2] \cup [6, \infty)$

We list the steps to take to solve a quadratic inequality graphically.

Solve a Quadratic Inequality Graphically

1. Write the quadratic inequality in standard form.
2. Graph the function $f(x) = ax^2 + bx + c$.
3. Determine the solution from the graph.

In the last example, the parabola opened upward and in the next example, it opens downward. In both cases, we are looking for the part of the parabola that is below the x-axis but note how the position of the parabola affects the solution.

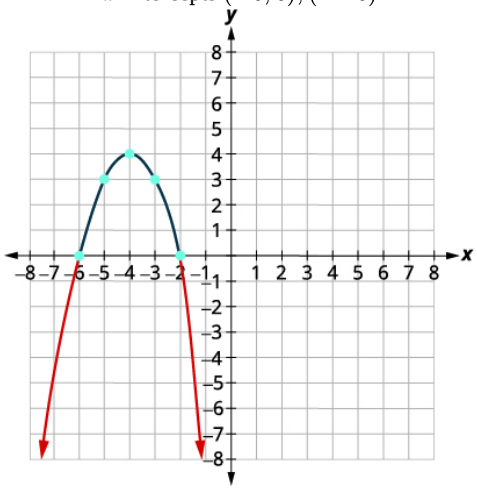
Example 19.5.2

Solve $-x^2 - 8x - 12 \leq 0$ graphically. Write the solution in interval notation.

Solution:

Table 9.8.1

The quadratic inequality in standard form.	$-x^2 - 8x - 12 \leq 0$
Graph the function $f(x) = -x^2 - 8x - 12$	The parabola opens downward. Figure 9.8.6
Find the line of symmetry.	$x = -\frac{b}{2a}$ $x = -\frac{-8}{2(-1)}$ $x = -4$
Find the vertex.	$f(x) = -x^2 - 8x - 12$ $f(-4) = -(-4)^2 - 8(-4) - 12$ $f(-4) = -16 + 32 - 12$ $f(-4) = 4$ Vertex $(-4, 4)$
Find the x-intercepts. Let $f(x) = 0$.	$f(x) = -x^2 - 8x - 12$ $0 = -x^2 - 8x - 12$
Factor: Use the Zero Product Property.	$0 = -1(x + 6)(x + 2)$ $x = -6 \quad x = -2$

Graph the parabola.	x-intercepts $(-6, 0), (-2.0)$  Figure 9.8.7
Determine the solution from the graph. We include the x-intercepts as the inequality is "less than or equal to."	$(-\infty, -6] \cup [-2, \infty)$

? Exercise 19.5.3

a. Solve $-x^2 - 6x - 5 > 0$ graphically
b. Write the solution in interval notation

Answer

a.

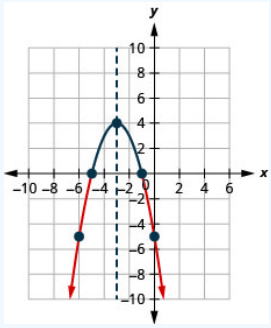


Figure 9.8.8

b. $(-5, -1)$

? Exercise 19.5.4

a. Solve $-x^2 + 10x - 16 \leq 0$ graphically
b. Write the solution in interval notation

Answer

a.

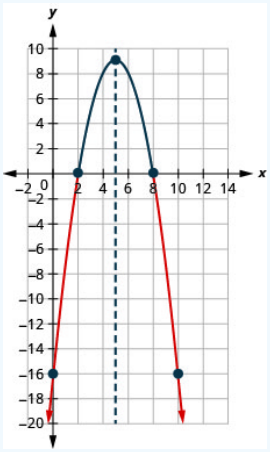


Figure 9.8.9

b. $(-\infty, 2] \cup [8, \infty)$

Solve Quadratic Inequalities Algebraically

The algebraic method we will use is very similar to the method we used to solve rational inequalities. We will find the critical points for the inequality, which will be the solutions to the related quadratic equation. Remember a polynomial expression can change signs only where the expression is zero.

We will use the **critical points** to divide the number line into intervals and then determine whether the quadratic expression will be positive or negative in the interval. We then determine the solution for the inequality.

Example 19.5.3: How to Solve Quadratic Inequalities Algebraically

Solve $x^2 - x - 12 \geq 0$ algebraically. Write the solution in interval notation.

Solution:

Table 9.8.2

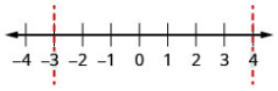
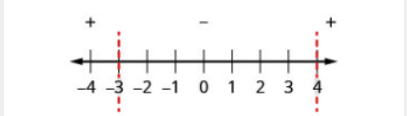
Step 1: Write the quadratic inequality in standard form.	The inequality is in standard form.	$x^2 - x - 12 \geq 0$
Step 2: Determine the critical points--the solutions to the related quadratic equation.	Change the inequality sign to an equal sign and then solve the equation.	$x^2 - x - 12 = 0$ $(x+3)(x-4) = 0$ $x+3=0 \quad x-4=0$ $x=-3 \quad x=4$
Step 3: Use the critical points to divide the number line into intervals.	Use -3 and 4 to divide the number line into intervals.	
Step 4: Above the number line show the sign of each quadratic expression using test points from each interval substituted from the original inequality.	Test: $x=-5$ $x=0$ $x=5$	$x^2 - x - 12$ $\quad$ $x^2 - x - 12$ $\quad$ $x^2 - x -$ $(-5)^2 - (-5) - 12$ $\quad$ $0^2 - 0 - 12$ $\quad$ $5^2 - 5 -$ $18 \quad -12 \quad 8$ 

Figure 9.8.11

Step 5: Determine the intervals where the inequality is correct. Write the solution in interval notation.	$x^2 - x - 12 \geq 0$ The inequality is positive in the first and last intervals and equals 0 at the points $-4, 3$.	The solution, in interval notation, is $(-\infty, -3] \cup [4, \infty)$.

? Exercise 19.5.5

Solve $x^2 + 2x - 8 \geq 0$ algebraically. Write the solution in interval notation.

Answer

$(-\infty, -4] \cup [2, \infty)$

? Exercise 19.5.6

Solve $x^2 - 2x - 15 \leq 0$ algebraically. Write the solution in interval notation.

Answer

$[-3, 5]$

In this example, since the expression $x^2 - x - 12$ factors nicely, we can also find the sign in each interval much like we did when we solved rational inequalities. We find the sign of each of the factors, and then the sign of the product. Our number line would like this:

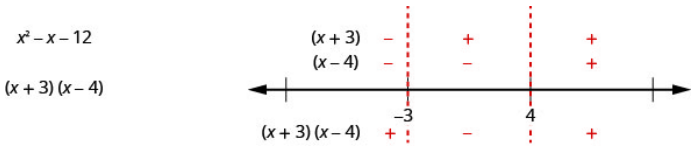


Figure 9.8.12

The result is the same as we found using the other method.

We summarize the steps here.

Solve a Quadratic Inequality Algebraically

1. Write the quadratic inequality in standard form.
2. Determine the critical points—the solutions to the related quadratic equation.
3. Use the critical points to divide the number line into intervals.
4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.
5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

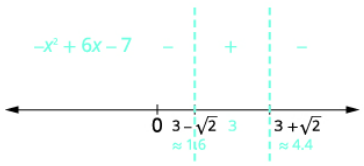
✓ Example 19.5.4

Solve $x^2 + 6x - 7 \geq 0$ algebraically. Write the solution in interval notation.

Solution:

Table 9.8.3

Write the quadratic inequality in standard form.	$-x^2 + 6x - 7 \geq 0$
Multiply both sides of the inequality by -1. Remember to reverse the inequality sign.	$x^2 - 6x + 7 \leq 0$
Determine the critical points by solving the related quadratic equation.	$x^2 - 6x + 7 = 0$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Then substitute in the values of a, b, c.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (7)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{8}}{2}$
Simplify the radical.	$x = \frac{6 \pm 2\sqrt{2}}{2}$
Remove the common factor, 2.	$x = \frac{2(3 \pm \sqrt{2})}{2}$ $x = 3 \pm \sqrt{2}$ $x = 3 + \sqrt{2} \quad x = 3 - \sqrt{2}$ $x \approx 1.6 \quad x \approx 4.4$
Use the critical points to divide the number line into intervals. Test numbers from each interval in the original inequality.	
Determine the intervals where the inequality is correct. Write the solution in interval notation.	$-x^2 + 6x - 7 \geq 0$ in the middle interval $[3 - \sqrt{2}, 3 + \sqrt{2}]$

? Exercise 19.5.7

Solve $-x^2 + 2x + 1 \geq 0$ algebraically. Write the solution in interval notation.

Answer

$[-1 - \sqrt{2}, -1 + \sqrt{2}]$

? Exercise 19.5.8

Solve $-x^2 + 8x - 14 < 0$ algebraically. Write the solution in interval notation.

Answer

$(-\infty, 4 - \sqrt{2}) \cup (4 + \sqrt{2}, \infty)$

The solutions of the quadratic inequalities in each of the previous examples, were either an interval or the union of two intervals. This resulted from the fact that, in each case we found two solutions to the corresponding quadratic equation $ax^2 + bx + c = 0$. These two solutions then gave us either the two x-intercepts for the graph or the two critical points to divide the number line into intervals.

This correlates to our previous discussion of the number and type of solutions to a quadratic equation using the discriminant.

For a quadratic equation of the form $ax^2 + bc + c = 0, a \neq 0$.

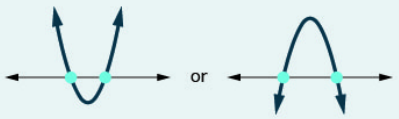
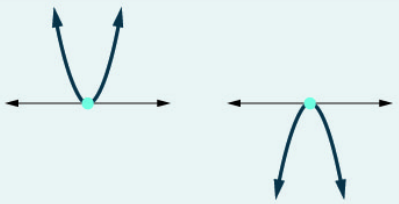
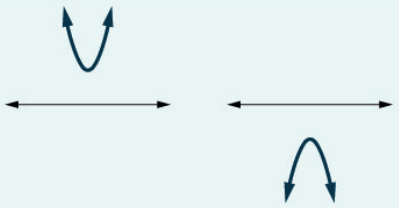
Discriminant	Number/Type of solution	Typical Graph
$b^2 - 4ac > 0$	2 real solutions 2 x-intercepts on graph	
$b^2 - 4ac = 0$	1 real solution 1 x-intercept on graph	
$b^2 - 4ac < 0$	2 complex solutions No x-intercept	

Figure 9.8.14

The last row of the table shows us when the parabolas never intersect the x-axis. Using the Quadratic Formula to solve the quadratic equation, the radicand is a negative. We get two complex solutions.

In the next example, the quadratic inequality solutions will result from the solution of the quadratic equation being complex.

Example 19.5.5

Solve, writing any solution in interval notation:

a. $x^2 - 3x + 4 > 0$
b. $x^2 - 3x + 4 \leq 0$

Solution:

a.

Table 9.8.4

Write the quadratic inequality in standard form.	$-x^2 - 3x + 4 > 0$
Determine the critical points by solving the related quadratic equation.	$x^2 - 3x + 4 = 0$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (4)}}{2 \cdot 1}$
Simplify.	$x = \frac{3 \pm \sqrt{-7}}{2}$
Simplify the radicand.	$x = \frac{3 \pm \sqrt{7}i}{2}$
The complex solutions tell us the parabola does not intercept the x-axis. Also, the parabola opens upward. This tells us that the parabola is completely above the x-axis.	Complex solutions

Figure 9.8.15

We are to find the solution to $x^2-3x+4>0$. Since for all values of x the graph is above the x-axis, all values of x make the inequality true. In interval notation we write $(-\infty,\infty)$.

b. Write the quadratic inequality in standard form.

$$x^2-3x+4\leq 0$$

Determine the critical points by solving the related quadratic equation.

$$x^2-3x+4=0$$

Since the corresponding quadratic equation is the same as in part (a), the parabola will be the same. The parabola opens upward and is completely above the x-axis—no part of it is below the x-axis.

We are to find the solution to $x^2-3x+4\leq 0$. Since for all values of x the graph is never below the x-axis, no values of x make the inequality true. There is no solution to the inequality.

Exercise 19.5.9

Solve and write any solution in interval notation:

a. $-x^2+2x-4\leq 0$
b. $-x^2+2x-4\geq 0$

Answer

a. $(-\infty,\infty)$
b. no solution

Exercise 19.5.10

Solve and write any solution in interval notation:

a. $x^2+3x+3<0$
b. $x^2+3x+3>0$

Answer

a. no solution
b. $(-\infty,\infty)$

Key Concepts

- Solve a Quadratic Inequality Graphically
  1. Write the quadratic inequality in standard form.
  2. Graph the function $f(x)=ax^2+bx+c$ using properties or transformations.
  3. Determine the solution from the graph.
- How to Solve a Quadratic Inequality Algebraically
  1. Write the quadratic inequality in standard form.
  2. Determine the critical points -- the solutions to the related quadratic equation.
  3. Use the critical points to divide the number line into intervals.
  4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.
  5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

Glossary

quadratic inequality

A quadratic inequality is an inequality that contains a quadratic expression.

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19.5E: Exercises

Practice Makes Perfect

Exercise $19.5E.11$ Solve Quadratic Inequalities Graphically

In the following exercises,

a. Solve graphically
b. Write the solution in interval notation

1. $x^2+6x+5>0$
2. $x^2+4x-12<0$
3. $x^2+4x+3\leq 0$
4. $x^2-6x+8\geq 0$
5. $-x^2-3x+18\leq 0$
6. $-x^2+2x+24<0$
7. $-x^2+x+12\geq 0$
8. $-x^2+2x+15>0$

Answer

1.

a.

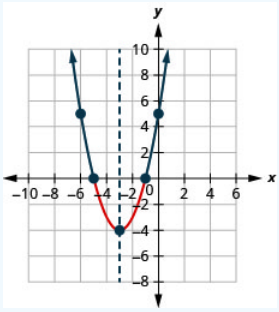

Figure 9.8.16

b. $(-\infty,-5)\cup(-1,\infty)$

3.

a.

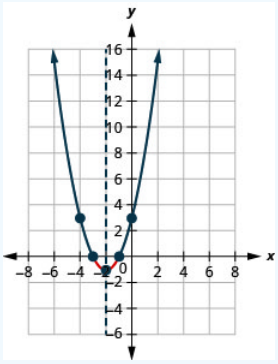

Figure 9.8.17

b. $[-3,-1]$

5.

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a.

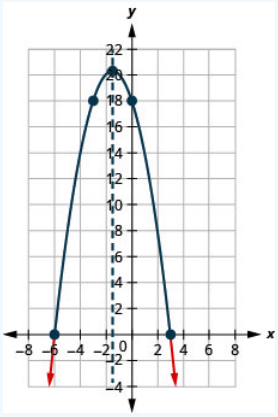

Figure 9.8.18

b. $(-\infty, -6] \cup [3, \infty)$

7.

a.

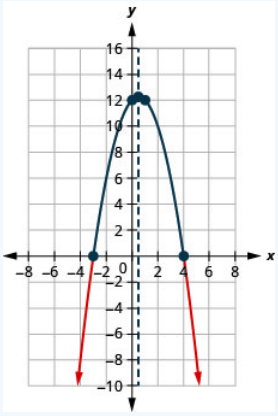

Figure 9.8.19

b. $[-3, 4]$

Exercise $19.5E.12$ Solve Quadratic Inequalities Graphically

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

1. $x^2 + 3x - 4 \geq 0$
2. $x^2 + x - 6 \leq 0$
3. $x^2 - 7x + 10 < 0$
4. $x^2 - 4x + 3 > 0$
5. $x^2 + 8x > -15$
6. $x^2 + 8x < -12$
7. $x^2 - 4x + 2 \leq 0$
8. $-x^2 + 8x - 11 < 0$
9. $x^2 - 10x > -19$
10. $x^2 + 6x < -3$
11. $-6x^2 + 19x - 10 \geq 0$
12. $-3x^2 - 4x + 4 \leq 0$
13. $-2x^2 + 7x + 4 \geq 0$
14. $2x^2 + 5x - 12 > 0$
15. $x^2 + 3x + 5 > 0$

16. $x^2-3x+6\leq 0$
17. $-x^2+x-7>0$
18. $-x^2-4x-5<0$
19. $-2x^2+8x-10<0$
20. $-x^2+2x-7\geq 0$

Answer

1. $(-\infty,-4]\cup[1,\infty)$
3. $(2,5)$
5. $(-\infty,-5)\cup(-3,\infty)$
7. $[2-\sqrt{2},2+\sqrt{2}]$
9. $(-\infty,5-\sqrt{6})\cup(5+\sqrt{6},\infty)$
11. $\left(-\infty,-\frac{5}{2}\right]\cup\left[-\frac{2}{3},\infty\right)$
13. $\left[-\frac{1}{2},4\right]$
15. $(-\infty,\infty)$
17. no solution
19. $(-\infty,\infty)$

? Exercise $19.5E.13$ Writing Exercises

1. Explain critical points and how they are used to solve quadratic inequalities algebraically.
2. Solve $x^2+2x\geq 8$ both graphically and algebraically. Which method do you prefer, and why?
3. Describe the steps needed to solve a quadratic inequality graphically.
4. Describe the steps needed to solve a quadratic inequality algebraically.

Answer

1. Answers may vary.
3. Answers may vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic inequalities graphically.			
solve quadratic inequalities algebraically.			

Figure 9.8.20

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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19.E: Review Exercises 1

Chapter Review Exercises

Use a General Strategy to Solve Linear Equations

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, determine whether each number is a solution to the equation.

1. $10x-1=5x, \quad x=\frac{1}{5}$

2. $-12n+5=8n, \quad n=-\frac{5}{4}$

Answer

no

In the following exercises, solve each linear equation.

3. $6(x+6)=24$

4. $-(s+4)=18$

Answer

$s=-22$ so the solution set is: $\{-22\}$.

5. $23-3(y-7)=8$

6. $\frac{1}{3}(6m+21)=m-7$

Answer

$m=-14$

7. $4(3.5y+0.25)=365$

8. $0.25(q-8)=0.1(q+7)$

Answer

$q=18$

9. $8(r-2)=6(r+10)$

10. $5+7(2-5x)=2(9x+1)-(13x-57)$

Answer

$x=-1$

11. $(9n+5)-(3n-7)=20-(4n-2)$

12. $2[-16+5(8k-6)]=8(3-4k)-32$

Answer

$k = \frac{3}{4}$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

13. $17y - 3(4 - 2y) = 11(y - 1) + 12y - 1$

14. $9u + 32 = 15(u - 4) - 3(2u + 21)$

Answer

contradiction; no solution

15. $-8(7m + 4) = -6(8m + 9)$

Solve Equations with Fraction or Decimal Coefficients

In the following exercises, solve each equation.

16. $\frac{2}{5}n - \frac{1}{10} = \frac{7}{10}$

Answer

$n = 2$

17. $\frac{3}{4}a - \frac{1}{3} = \frac{1}{2}a + \frac{5}{6}$

18. $\frac{1}{2}(k + 3) = \frac{1}{3}(k + 16)$

Answer

$k = 23$

19. $\frac{5y-1}{3} + 4 = \frac{-8y+4}{6}$

20. $0.8x - 0.3 = 0.7x + 0.2$

Answer

$x = 5$

21. $0.10d + 0.05(d - 4) = 2.05$

Use a Problem-Solving Strategy

Use a Problem Solving Strategy for Word Problems

In the following exercises, solve using the problem solving strategy for word problems.

22. Three-fourths of the people at a concert are children. If there are 87 children, what is the total number of people at the concert?

Answer

There are 116 people.

23. There are nine saxophone players in the band. The number of saxophone players is one less than twice the number of tuba players. Find the number of tuba players.

Solve Number Word Problems

In the following exercises, solve each number word problem.

24. The sum of a number and three is forty-one. Find the number.

Answer

38

25. One number is nine less than another. Their sum is negative twenty-seven. Find the numbers.

26. One number is two more than four times another. Their sum is negative thirteen. Find the numbers.

Answer

$-3, -10$

27. The sum of two consecutive integers is -135. Find the numbers.

28. Find three consecutive even integers whose sum is 234.

Answer

76, 78, 80

29. Find three consecutive odd integers whose sum is 51.

30. Koji has $5,502 in his savings account. This is $30 less than six times the amount in his checking account. How much money does Koji have in his checking account?

Answer

$922

Solve Percent Applications

In the following exercises, translate and solve.

31. What number is 67% of 250?

32. 12.5% of what number is 20?

Answer

160

33. What percent of 125 is 150?

In the following exercises, solve.

34. The bill for Dino's lunch was $19.45. He wanted to leave 20% of the total bill as a tip. How much should the tip be?

Answer

$3.89

35. Dolores bought a crib on sale for $350. The sale price was 40% of the original price. What was the original price of the crib?

36. Jaden earns $2,680 per month. He pays $938 a month for rent. What percent of his monthly pay goes to rent?

Answer

35

37. Angel received a raise in his annual salary from $55,400 to $56,785. Find the percent change.

38. Rowena's monthly gasoline bill dropped from $83.75 last month to $56.95 this month. Find the percent change.

Answer

32

39. Emmett bought a pair of shoes on sale at 40% off from an original price of $138. Find ⓐ the amount of discount and ⓑthe sale price.

40. Lacey bought a pair of boots on sale for $95. The original price of the boots was $200. Find ⓐ the amount of discount and ⓑ the discount rate. (Round to the nearest tenth of a percent, if needed.)

Answer

ⓐ $105ⓑ 52.5

41. Nga and Lauren bought a chest at a flea market for $50. They re-finished it and then added a 350% mark-up. Find ⓐthe amount of the mark-up and ⓑ the list price.

Solve Simple Interest Applications

In the following exercises, solve.

42. Winston deposited $3,294 in a bank account with interest rate 2.6% How much interest was earned in five years?

Answer

$428.22

43. Moira borrowed $4,500 from her grandfather to pay for her first year of college. Three years later, she repaid the $4,500 plus $243 interest. What was the rate of interest?

44. Jaime's refrigerator loan statement said he would pay $1,026 in interest for a four-year loan at 13.5%. How much did Jaime borrow to buy the refrigerator?

Answer

$1,900

Solve a formula for a Specific Variable

Solve a Formula for a Specific Variable

In the following exercises, solve the formula for the specified variable.

45. Solve the formula
$V = LWH$ for L.

46. Solve the formula
$A = \frac{1}{2}d_1 d_2$ for d_2.

Answer

$d_2 = \frac{2A}{d_1}$

47. Solve the formula
$h = 48t + \frac{1}{2}at^2$ for t.

48. Solve the formula
4x−3y=12 for y.

Answer

$y = \frac{4x}{3} - 4$

Use Formulas to Solve Geometry Applications

In the following exercises, solve using a geometry formula.

49. What is the height of a triangle with area 67.567.5 square meters and base 9 meters?

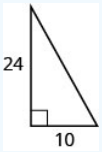
50. The measure of the smallest angle in a right triangle is 45°45° less than the measure of the next larger angle. Find the measures of all three angles.

Answer

$22.5^\circ,\ 67.5^\circ,\ 90^\circ$

51. The perimeter of a triangle is 97 feet. One side of the triangle is eleven feet more than the smallest side. The third side is six feet more than twice the smallest side. Find the lengths of all sides.

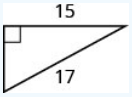
52. Find the length of the hypotenuse.



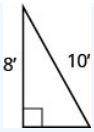
Answer

26

53. Find the length of the missing side. Round to the nearest tenth, if necessary.



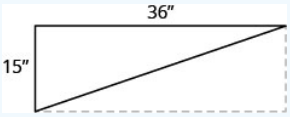
54. Sergio needs to attach a wire to hold the antenna to the roof of his house, as shown in the figure. The antenna is eight feet tall and Sergio has 10 feet of wire. How far from the base of the antenna can he attach the wire? Approximate to the nearest tenth, if necessary.



Answer

6 feet

55. Seong is building shelving in his garage. The shelves are 36 inches wide and 15 inches tall. He wants to put a diagonal brace across the back to stabilize the shelves, as shown. How long should the brace be?



56. The length of a rectangle is 12 cm more than the width. The perimeter is 74 cm. Find the length and the width.

Answer

24.5cm, 12.5cm

57. The width of a rectangle is three more than twice the length. The perimeter is 96 inches. Find the length and the width.

58. The perimeter of a triangle is 35 feet. One side of the triangle is five feet longer than the second side. The third side is three feet longer than the second side. Find the length of each side.

Answer

9 ft, 14 ft, 12 ft

Solve Mixture and Uniform Motion Applications

Solve Coin Word Problems

In the following exercises, solve.

59. Paulette has $140 in $5 and $10 bills. The number of $10 bills is one less than twice the number of $5 bills. How many of each does she have?

60. Lenny has $3.69 in pennies, dimes, and quarters. The number of pennies is three more than the number of dimes. The number of quarters is twice the number of dimes. How many of each coin does he have?

Answer

nine pennies, six dimes, 12 quarters

Solve Ticket and Stamp Word Problems

In the following exercises, solve each ticket or stamp word problem.

61. Tickets for a basketball game cost $2 for students and $5 for adults. The number of students was three less than 10 times the number of adults. The total amount of money from ticket sales was $619. How many of each ticket were sold?

62. 125 tickets were sold for the jazz band concert for a total of $1,022. Student tickets cost $6 each and general admission tickets cost $10 each. How many of each kind of ticket were sold?

Answer

57 students, 68 adults

63. Yumi spent $34.15 buying stamps. The number of $0.56 stamps she bought was 10 less than four times the number of $0.41 stamps. How many of each did she buy?

Solve Mixture Word Problems

In the following exercises, solve.

64. Marquese is making 10 pounds of trail mix from raisins and nuts. Raisins cost $3.45 per pound and nuts cost $7.95 per pound. How many pounds of raisins and how many pounds of nuts should Marquese use for the trail mix to cost him $6.96 per pound?

Answer

2.2 lbs of raisins, 7.8 lbs of nuts

65. Amber wants to put tiles on the backsplash of her kitchen counters. She will need 36 square feet of tile. She will use basic tiles that cost $8 per square foot and decorator tiles that cost $20 per square foot. How many square feet of each tile should she use so that the overall cost of the backsplash will be $10 per square foot?

66. Enrique borrowed $23,500 to buy a car. He pays his uncle 2% interest on the $4,500 he borrowed from him, and he pays the bank 11.5% interest on the rest. What average interest rate does he pay on the total $23,500? (Round your answer to the nearest tenth of a percent.)

Answer

9.7

Solve Uniform Motion Applications

In the following exercises, solve.

67. When Gabe drives from Sacramento to Redding it takes him 2.2 hours. It takes Elsa two hours to drive the same distance. Elsa's speed is seven miles per hour faster than Gabe's speed. Find Gabe's speed and Elsa's speed.

68. Louellen and Tracy met at a restaurant on the road between Chicago and Nashville. Louellen had left Chicago and drove 3.2 hours towards Nashville. Tracy had left Nashville and drove 4 hours towards Chicago, at a speed one mile per hour faster than Louellen's speed. The distance between Chicago and Nashville is 472 miles. Find Louellen's speed and Tracy's speed.

Answer

Louellen 65 mph, Tracy 66 mph

69. Two busses leave Amarillo at the same time. The Albuquerque bus heads west on the I-40 at a speed of 72 miles per hour, and the Oklahoma City bus heads east on the I-40 at a speed of 78 miles per hour. How many hours will it take them to be 375 miles apart?

70. Kyle rowed his boat upstream for 50 minutes. It took him 30 minutes to row back downstream. His speed going upstream is two miles per hour slower than his speed going downstream. Find Kyle's upstream and downstream speeds.

Answer

upstream 3 mph, downstream 5 mph

71. At 6:30, Devon left her house and rode her bike on the flat road until 7:30. Then she started riding uphill and rode until 8:00. She rode a total of 15 miles. Her speed on the flat road was three miles per hour faster than her speed going uphill. Find Devon's speed on the flat road and riding uphill.

72. Anthony drove from New York City to Baltimore, which is a distance of 192 miles. He left at 3:45 and had heavy traffic until 5:30. Traffic was light for the rest of the drive, and he arrived at 7:30. His speed in light traffic was four miles per hour more than twice his speed in heavy traffic. Find Anthony's driving speed in heavy traffic and light traffic.

Answer

heavy traffic 32 mph, light traffic 66 mph

Solve Linear Inequalities

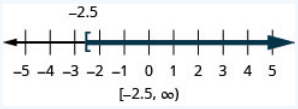
Graph Inequalities on the Number Line

In the following exercises, graph the inequality on the number line and write in interval notation.

73. $x < -1$

74. $x \geq -2.5$

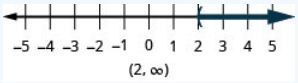
Answer



75. $x \leq \frac{5}{4}$

76. $x > 2$

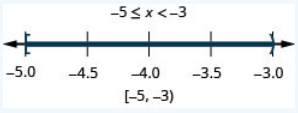
Answer



77. $-2 < x < 0$

78. $-5 \leq x < -3$

Answer



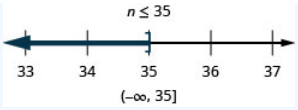
79. $0 \leq x \leq 3.5$

Solve Linear Inequalities

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

80. $n - 12 \leq 23$

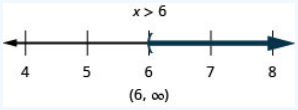
Answer



81. $a+\frac{2}{3} \geq \frac{7}{12}$

82. $9x > 54$

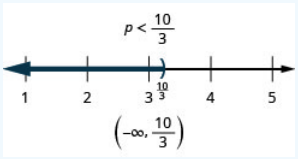
Answer



83. $\frac{q}{-2} \geq -24$

84. $6p > 15p - 30$

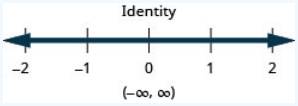
Answer



85. $9h - 7(h-1) \leq 4h - 23$

86. $5n - 15(4-n) < 10(n-6) + 10n$

Answer



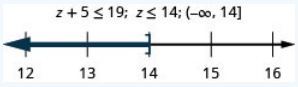
87. $\frac{3}{8}a - \frac{1}{12}a > \frac{5}{12}a + \frac{3}{4}$

Translate Words to an Inequality and Solve

In the following exercises, translate and solve. Then write the solution in interval notation and graph on the number line.

88. Five more than z is at most 19.

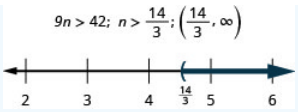
Answer



89. Three less than c is at least 360.

90. Nine times n exceeds 42.

Answer



91. Negative two times a is no more than eight.

Solve Applications with Linear Inequalities

In the following exercises, solve.

92. Julianne has a weekly food budget of $231 for her family. If she plans to budget the same amount for each of the seven days of the week, what is the maximum amount she can spend on food each day?

Answer

$33 per day

93. Rogelio paints watercolors. He got a $100 gift card to the art supply store and wants to use it to buy 12″ × 16″ canvases. Each canvas costs $10.99. What is the maximum number of canvases he can buy with his gift card?

94. Briana has been offered a sales job in another city. The offer was for $42,500 plus 8% of her total sales. In order to make it worth the move, Briana needs to have an annual salary of at least $66,500. What would her total sales need to be for her to move?

Answer

at least $300,000

95. Renee's car costs her $195 per month plus $0.09 per mile. How many miles can Renee drive so that her monthly car expenses are no more than $250?

96. Costa is an accountant. During tax season, he charges $125 to do a simple tax return. His expenses for buying software, renting an office, and advertising are $6,000. How many tax returns must he do if he wants to make a profit of at least $8,000?

Answer

at least 112 jobs

97. Jenna is planning a five-day resort vacation with three of her friends. It will cost her $279 for airfare, $300 for food and entertainment, and $65 per day for her share of the hotel. She has $550 saved towards her vacation and can earn $25 per hour as an assistant in her uncle's photography studio. How many hours must she work in order to have enough money for her vacation?

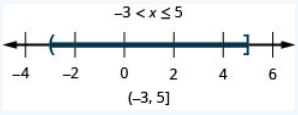
Solve Compound Inequalities

Solve Compound Inequalities with "and"

In each of the following exercises, solve each inequality, graph the solution, and write the solution in interval notation.

98. $x \leq 5$ and $x > -3$

Answer



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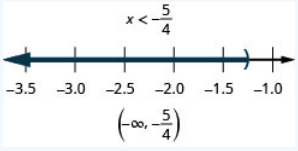
99. $4x-2\leq 4$ and $7x-1>-8$

100. $5(3x-2)\leq 5$ and $4(x+2)<3$

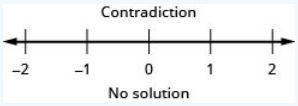
Answer



101. $34(x-8)\leq 3$ and $15(x-5)\leq 3$

102. $34x-5\geq -2$ and $-3(x+1)\geq 6$

Answer



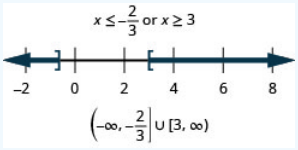
103. $-5\leq 4x-1<7$

Solve Compound Inequalities with "or"

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

104. $5-2x\leq -1$ or $6+3x\leq 4$

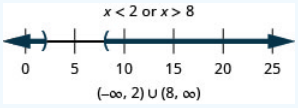
Answer



105. $3(2x-3)<-5$ or $4x-1>3$

106. $34x-2>4$ or $4(2-x)>0$

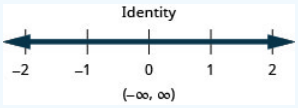
Answer



107. $2(x+3)\geq 0$ or $3(x+4)\leq 6$

108. $12x-3\leq 4$ or $13(x-6)\geq -2$

Answer



Solve Applications with Compound Inequalities

In the following exercises, solve.

109. Liam is playing a number game with his sister Audry. Liam is thinking of a number and wants Audry to guess it. Five more than three times her number is between 2 and 32. Write a compound inequality that shows the range of numbers that Liam might be thinking of.

110. Elouise is creating a rectangular garden in her back yard. The length of the garden is 12 feet. The perimeter of the garden must be at least 36 feet and no more than 48 feet. Use a compound inequality to find the range of values for the width of the garden.

Answer

$6 \leq w \leq 12$

Solve Absolute Value Inequalities

Solve Absolute Value Equations

In the following exercises, solve.

111. $|x| = 8$

112. $|y| = -14$

Answer

no solution

113. $|z| = 0$

114. $|3x - 4| + 5 = 7$

Answer

$x = 2, x = \frac{2}{3}$

115. $4|x - 1| + 2 = 10$

116. $-2|x - 3| + 8 = -4$

Answer

$x = 9, x = -3$

117. $|12x + 5| + 4 = 1$

118. $|6x - 5| = |2x + 3|$

Answer

$x = 2, x = 14$

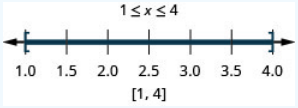
Solve Absolute Value Inequalities with “less than”

In the following exercises, solve each inequality. Graph the solution and write the solution in interval notation.

119. $|x| \leq 8$

120. $|2x-5| \leq 3$

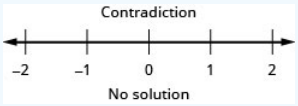
Answer



121. $|6x-5| < 7$

122. $|5x+1| \leq -2$

Answer



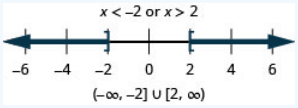
Solve Absolute Value Inequalities with "greater than"

In the following exercises, solve. Graph the solution and write the solution in interval notation.

123. $|x| > 6$

124. $|x| \geq 2$

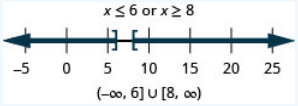
Answer



125. $|x-5| > -2$

126. $|x-7| \geq 1$

Answer



127. $3|x|+4 \geq 1$

Solve Applications with Absolute Value

In the following exercises, solve.

128. A craft beer brewer needs 215,000 bottle per day. But this total can vary by as much as 5,000 bottles. What is the maximum and minimum expected usage at the bottling company?

Answer

The minimum to maximum expected usage is 210,000 to 220,000 bottles

129. At Fancy Grocery, the ideal weight of a loaf of bread is 16 ounces. By law, the actual weight can vary from the ideal by 1.5 ounces. What range of weight will be acceptable to the inspector without causing the bakery being fined?

Practice Test

In the following exercises, solve each equation.

1. $-5(2x+1)=45$

Answer

$x=-5$

2. $\frac{1}{4}(12m+28)=6+2(3m+1)$

3. $8(3a+5)-7(4a-3)=20-3a$

Answer

$a=41$

4. $0.1d+0.25(d+8)=4.1$

5. $14n-3(4n+5)=-9+2(n-8)$

Answer

contradiction; no solution

6. $3(3u+2)+4[6-8(u-1)]=3(u-2)$

7. $\frac{3}{4}x-\frac{2}{3}=\frac{1}{2}x+\frac{5}{6}$

Answer

$x=6$

8. $|3x-4|=8$

9. $|2x-1|=|4x+3|$

Answer

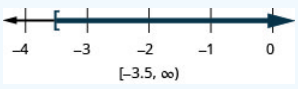
$x=-2, x=-13$

10. Solve the formula $x+2y=5$ for y.

In the following exercises, graph the inequality on the number line and write in interval notation.

11. $x\geq -3.5$

Answer



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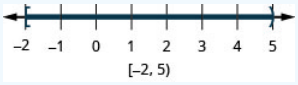
12. $x < \frac{11}{4}$

13. $-2 \leq x < 5$

Answer

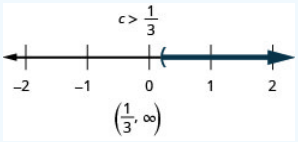


In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

14. $8k \geq 5k - 120$

15. $3c - 10(c-2) < 5c + 16$

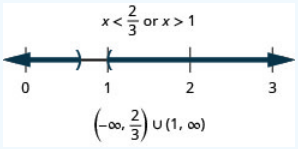
Answer



16. $\frac{3}{4}x - 5 \geq -2$ and $-3(x+1) \geq 6$

17. $3(2x-3) < -5$ or $4x - 1 > 3$

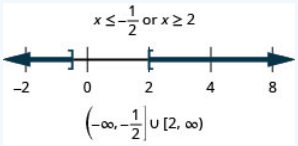
Answer



18. $\frac{1}{2}x - 3 \leq 4$ or $\frac{1}{3}(x-6) \geq -2$

19. $|4x - 3| \geq 5$

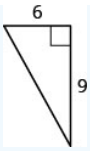
Answer



In the following exercises, translate to an equation or inequality and solve.

20. Four less than twice x is 16.

21. Find the length of the missing side.



Answer

10.8

22. One number is four more than twice another. Their sum is -47. Find the numbers.

23. The sum of two consecutive odd integers is -112. Find the numbers.

Answer

$-57, -55$

24. Marcus bought a television on sale for \$626.50 The original price of the television was \$895. Find ⓐ the amount of discount and ⓑ the discount rate.

25. Bonita has \$2.95 in dimes and quarters in her pocket. If she has five more dimes than quarters, how many of each coin does she have?

Answer

12 dimes, seven quarters

26. Kim is making eight gallons of punch from fruit juice and soda. The fruit juice costs \$6.04 per gallon and the soda costs \$4.28 per gallon. How much fruit juice and how much soda should she use so that the punch costs \$5.71 per gallon?

27. The measure of one angle of a triangle is twice the measure of the smallest angle. The measure of the third angle is three times the measure of the smallest angle. Find the measures of all three angles.

Answer

$30^\circ, 60^\circ, 90^\circ$

28. The length of a rectangle is five feet more than four times the width. The perimeter is 60 feet. Find the dimensions of the rectangle.

29. Two planes leave Dallas at the same time. One heads east at a speed of 428 miles per hour. The other plane heads west at a speed of 382 miles per hour. How many hours will it take them to be 2,025 miles apart?

Answer

2.5 hours

30. Leon drove from his house in Cincinnati to his sister's house in Cleveland, a distance of 252 miles. It took him $4\frac{1}{2}$ hours. For the first half hour, he had heavy traffic, and the rest of the time his speed was five miles per hour less than twice his speed in heavy traffic. What was his speed in heavy traffic?

31. Sara has a budget of \$1,000 for costumes for the 18 members of her musical theater group. What is the maximum she can spend for each costume?

Answer

> At most $55.56 per costume.

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19.E: Review Exercises 2

Chapter Review Exercises

Solve Quadratic Equations Using the Square Root Property

Exercise 19.E.1 Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

1. $y^2 = 144$
2. $n^2 - 80 = 0$
3. $4a^2 = 100$
4. $2b^2 = 72$
5. $r^2 + 32 = 0$
6. $t^2 + 18 = 0$
7. $\frac{2}{3}w^2 - 20 = 30$
8. $5c^2 + 3 = 19$

Answer

1. $y = \pm 12$
3. $a = \pm 5$
5. $r = \pm 4\sqrt{2}i$
7. $w = \pm 5\sqrt{3}$

Exercise 19.E.2 Solve Quadratic Equations of the Form $a(x-h)^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

1. $(p-5)^2 + 3 = 19$
2. $(u+1)^2 = 45$
3. $\left(x - \frac{1}{4}\right)^2 = \frac{3}{16}$
4. $\left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$
5. $(n-4)^2 - 50 = 150$
6. $(4c-1)^2 = -18$
7. $n^2 + 10n + 25 = 12$
8. $64a^2 + 48a + 9 = 81$

Answer

1. $p = -1, 9$
3. $x = \frac{1}{4} \pm \frac{\sqrt{3}}{4}$
5. $n = 4 \pm 10\sqrt{2}$
7. $n = -5 \pm 2\sqrt{3}$

Solve Quadratic Equations by Completing the Square

Exercise 19.E.3 Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

1. $x^2 + 22x$

2. $m^2 - 8m$
3. $a^2 - 3a$
4. $b^2 + 13b$

Answer

1. $(x+11)^2$
3. $\left(a - \frac{3}{2}\right)^2$

Exercise $19.E.4$ Solve Quadratic Equations Using Completing the Square

In the following exercises, solve by completing the square.

1. $d^2 + 14d = -13$
2. $y^2 - 6y = 36$
3. $m^2 + 6m = -109$
4. $t^2 - 12t = -40$
5. $v^2 - 14v = -31$
6. $w^2 - 20w = 100$
7. $m^2 + 10m - 4 = -13$
8. $n^2 - 6n + 11 = 34$
9. $a^2 = 3a + 8$
10. $b^2 = 11b - 5$
11. $(u+8)(u+4) = 14$
12. $(z-10)(z+2) = 28$

Answer

1. $d = -13, -1$
3. $m = -3 \pm 10i$
5. $v = 7 \pm 3\sqrt{2}$
7. $m = -9, -1$
9. $a = \frac{3}{2} \pm \frac{\sqrt{41}}{2}$
11. $u = -6 \pm 2\sqrt{2}$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

Exercise $19.E.5$ Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

1. $3p^2 - 18p + 15 = 15$
2. $5q^2 + 70q + 20 = 0$
3. $4y^2 - 6y = 4$
4. $2x^2 + 2x = 4$
5. $3c^2 + 2c = 9$
6. $4d^2 - 2d = 8$
7. $2x^2 + 6x = -5$
8. $2x^2 + 4x = -5$

Answer

1. $p = 0, 6$

3. $y = -\frac{1}{2}, 2$

5. $c = -\frac{1}{3} \pm \frac{2\sqrt{7}}{3}$

7. $x = \frac{3}{2} \pm \frac{1}{2}i$

Exercise 19.E.6 Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

1. $4x^2 - 5x + 1 = 0$
2. $7y^2 + 4y - 3 = 0$
3. $r^2 - r - 42 = 0$
4. $t^2 + 13t + 22 = 0$
5. $4v^2 + v - 5 = 0$
6. $2w^2 + 9w + 2 = 0$
7. $3m^2 + 8m + 2 = 0$
8. $5n^2 + 2n - 1 = 0$
9. $6a^2 - 5a + 2 = 0$
10. $4b^2 - b + 8 = 0$
11. $u(u - 10) + 3 = 0$
12. $5z(z - 2) = 3$
13. $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$
14. $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$
15. $4c^2 + 4c + 1 = 0$
16. $9d^2 - 12d = -4$

Answer

1. $x = \frac{1}{4}, 1$

3. $r = -6, 7$

5. $v = \frac{-1 \pm \sqrt{21}}{8}$

7. $m = \frac{-4 \pm \sqrt{10}}{3}$

9. $a = \frac{5}{12} \pm \frac{\sqrt{23}}{12}i$

11. $u = 5 \pm \sqrt{21}$

13. $p = \frac{4 \pm \sqrt{5}}{5}$

15. $c = -\frac{1}{2}$

Exercise 19.E.7 Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

1. a. $9x^2 - 6x + 1 = 0$
   b. $3y^2 - 8y + 1 = 0$
   c. $7m^2 + 12m + 4 = 0$
   d. $5n^2 - n + 1 = 0$
2. a. $5x^2 - 7x - 8 = 0$
   b. $7x^2 - 10x + 5 = 0$
   c. $25x^2 - 90x + 81 = 0$
   d. $15x^2 - 8x + 4 = 0$

Answer

1.

a. 1
b. 2
c. 2
d. 2

Exercise 19.E.8 Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

1. a. $16r^2-8r+1=0$
   b. $5t^2-8t+3=9$
   c. $3(c+2)^2=15$
2. a. $4d^2+10d-5=21$
   b. $25x^2-60x+36=0$
   c. $6(5v-7)^2=150$

Answer

1.

a. Factor
b. Quadratic Formula
c. Square Root

Solve Equations in Quadratic Form

Exercise 19.E.9 Solve Equations in Quadratic Form

In the following exercises, solve.

1. $x^4-14x^2+24=0$
2. $x^4+4x^2-32=0$
3. $4x^4-5x^2+1=0$
4. $(2y+3)^2+3(2y+3)-28=0$
5. $x+3\sqrt{x}-28=0$
6. $6x+5\sqrt{x}-6=0$
7. $x^{\frac{2}{3}}-10x^{\frac{1}{3}}+24=0$
8. $x+7x^{\frac{1}{2}}+6=0$
9. $8x^{-2}-2x^{-1}-3=0$

Answer

1. $x=\pm\sqrt{2}, x=\pm 2\sqrt{3}$
3. $x=\pm 1, x=\pm\frac{1}{2}$
5. $x=16$
7. $x=64, x=216$
9. $x=-2, x=\frac{4}{3}$

Exercise 19.E.10 Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using the method of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth, if needed.

1. Find two consecutive odd numbers whose product is 323.
2. Find two consecutive even numbers whose product is 624.
3. A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.
4. Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.
5. A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.

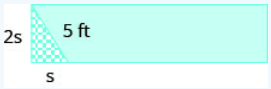

Figure 9.E.1

6. A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

7. The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

8. For Sophia's graduation party, several tables of the same width will be arranged end to end to give serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth . Round answer to the nearest tenth.

9. A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

10. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 360 miles. If the plane was flying at 150 mph, what was the speed of the wind that affected the plane?

11. Ezra kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

12. Two handymen can do a home repair in 2 hours if they work together. One of the men takes 3 hours more than the other man to finish the job by himself. How long does it take for each handyman to do the home repair individually?

Answer

2. Two consecutive even numbers whose product is 624 are 24 and 26, and -24 and -26.

4. The height is 14 inches and the width is 10 inches.

6. The length of the diagonal is 3.6 feet.

8. The width of the serving table is 4.7 feet and the length is 16.1 feet.

Figure 9.E.2

10. The speed of the wind was 30 mph.

12. One man takes 3 hours and the other man 6 hours to finish the repair alone.

Graph Quadratic Functions Using Properties

Exercise 19.E.11 Recognize the Graph of a Quadratic Function

In the following exercises, graph by plotting point.

1. Graph $y = x^2 - 2$
2. Graph $y = -x^2 + 3$

Answer

2.

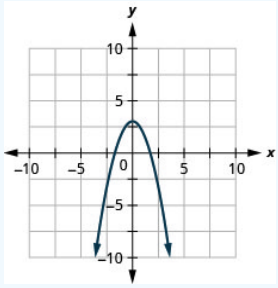

Figure 9.E.3

Exercise 19.E.12 Recognize the Graph of a Quadratic Function

In the following exercises, determine if the following parabolas open up or down.

1. a. $y = -3x^2 + 3x - 1$
   b. $y = 5x^2 + 6x + 3$
2. a. $y = x^2 + 8x - 1$
   b. $y = -4x^2 - 7x + 1$

Answer

2.

a. Up
b. Down

Exercise 19.E.13 Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find

a. The equation of the axis of symmetry
b. The vertex

1. $y = -x^2 + 6x + 8$
2. $y = 2x^2 - 8x + 1$

Answer

2. $x = 2$; $(2, -7)$

Exercise 19.E.14 Find the Intercepts of a Parabola

In the following exercises, find the x- and y-intercepts.

1. $y = x^2 - 4x + 5$

2. $y = x^2 - 8x + 15$
3. $y = x^2 - 4x + 10$
4. $y = -5x^2 - 30x - 46$
5. $y = 16x^2 - 8x + 1$
6. $y = x^2 + 16x + 64$

Answer

2. $y : (0, 15)$
   $x : (3, 0), (5, 0)$
4. $y : (0, -46)$
   $x :$ none
6. $y : (0, -64)$
   $x : (-8, 0)$

Graph Quadratic Functions Using Properties

Exercise $19.E.15$ Graph Quadratic Functions Using Properties

In the following exercises, graph by using its properties.

1. $y = x^2 + 8x + 15$
2. $y = x^2 - 2x - 3$
3. $y = -x^2 + 8x - 16$
4. $y = 4x^2 - 4x + 1$
5. $y = x^2 + 6x + 13$
6. $y = -2x^2 - 8x - 12$

Answer

2.

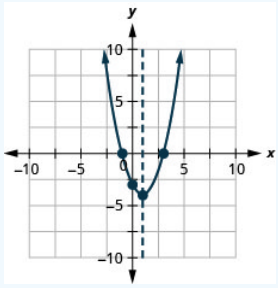

Figure 9.E.4

4.

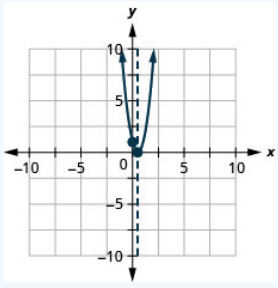

Figure 9.E.5

6.

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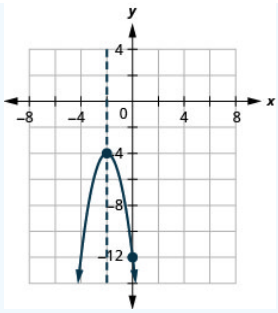

Figure 9.E.6

Exercise 19.E.16 Solve Maximum and Minimum Applications

In the following exercises, find the minimum or maximum value.

1. $y = 7x^2 + 14x + 6$
2. $y = -3x^2 + 12x - 10$

Answer

2. The maximum value is 2 when $x = 2$.

Exercise 19.E.17 Solve Maximum and Minimum Applications

In the following exercises, solve. Rounding answers to the nearest tenth.

1. A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.
2. A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.

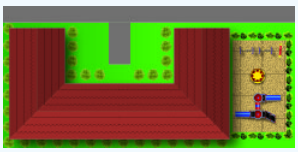
Figure 9.E.7

Answer

2. The length adjacent to the building is 90 feet giving a maximum area of $4,050$ square feet.

Graph Quadratic Functions Using Transformations

Exercise 19.E.18 Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using a vertical shift.

1. $g(x) = x^2 + 4$
2. $h(x) = x^2 - 3$

Answer

2.

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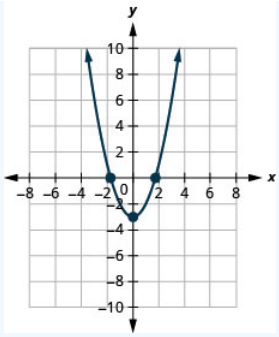

Figure 9.E.8

Exercise $19.E.19$ Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using a horizontal shift.

1. $f(x) = (x+1)^2$
2. $g(x) = (x-3)^2$

Answer

2.

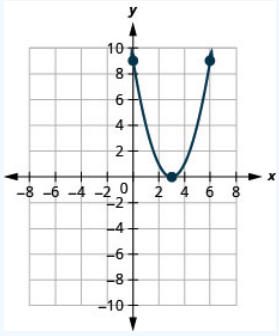

Figure 9.E.9

Exercise $19.E.20$ Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using transformations.

1. $f(x) = (x+2)^2 + 3$
2. $f(x) = (x+3)^2 - 2$
3. $f(x) = (x-1)^2 + 4$
4. $f(x) = (x-4)^2 - 3$

Answer

2.

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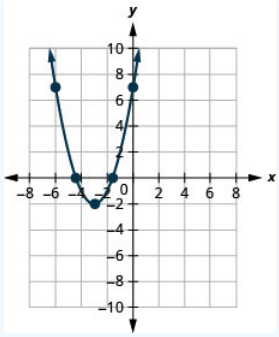

Figure 9.E.10

4.

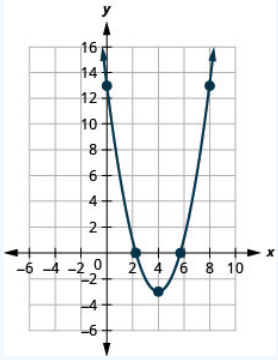

Figure 9.E.11

Exercise $19.E.21$ Graph Quadratic Functions of the Form $f(x) = ax^2$

In the following exercises, graph each function.

1. $f(x) = 2x^2$
2. $f(x) = -x^2$
3. $f(x) = \frac{1}{2}x^2$

Answer

2.

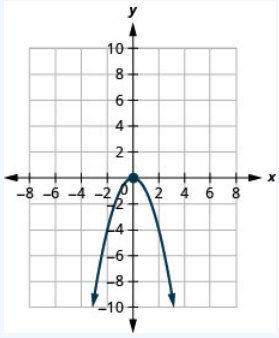

Figure 9.E.12

Exercise $19.E.22$ Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x) = a(x-h)^2 + k$ form by completing the square.

1. $f(x) = 2x^2 - 4x - 4$

2. $f(x) = 3x^2 + 12x + 8$

Answer

1. $f(x) = 2(x-1)^2 - 6$

Exercise $19.E.23$ Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x) = a(x-h)^2 + k$ form
b. Graph it by using transformations

1. $f(x) = 3x^2 - 6x - 1$
2. $f(x) = -2x^2 - 12x - 5$
3. $f(x) = 2x^2 + 4x + 6$
4. $f(x) = 3x^2 - 12x + 7$

Answer

1.

a. $f(x) = 3(x-1)^2 - 4$
b.

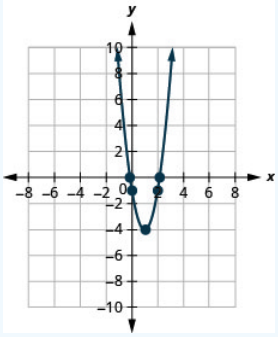

Figure 9.E.13

3.

a. $f(x) = 2(x+1)^2 + 4$
b.

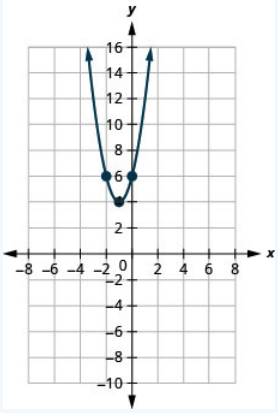

Figure 9.E.14

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Exercise $19.E.24$ Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x)=a(x-h)^2+k$ form
b. Graph it using properties

1. $f(x)=-3x^2-12x-5$
2. $f(x)=2x^2-12x+7$

Answer

1.

a. $f(x)=-3(x+2)^2+7$
b.

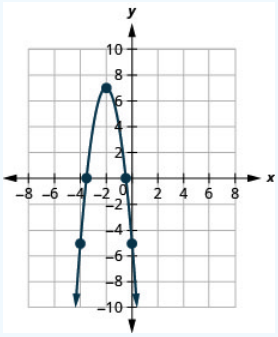

Figure 9.E.15

Exercise $19.E.25$ Find a Quadratic Function From its Graph

In the following exercises, write the quadratic function in $f(x)=a(x-h)^2+k$ form.

1.

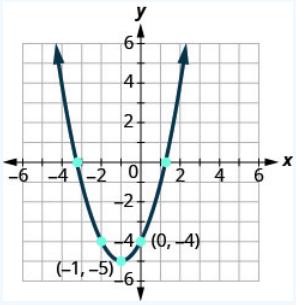

Figure 9.E.16

2.

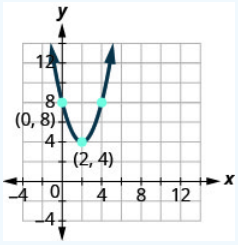

Figure 9.E.17

Answer

1. $f(x)=(x+1)^2-5$

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Exercise $19.E.26$ Solve Quadratic Inequalities Graphically

In the following exercises, solve graphically and write the solution in interval notation.

1. $x^2 - x - 6 > 0$
2. $x^2 + 4x + 3 \leq 0$
3. $-x^2 - x + 2 \geq 0$
4. $-x^2 + 2x + 3 < 0$

Answer

1.

a.

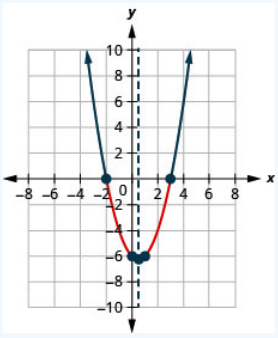

Figure 9.E.18

b. $(-\infty, -2) \cup (3, \infty)$

3.

a.

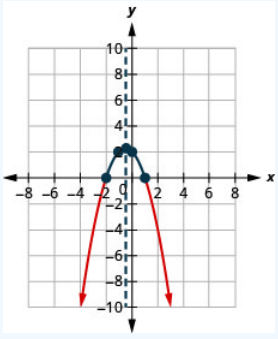

Figure 9.E.19

b. $[-2, 1]$

Exercise $19.E.27$ Solve Quadratic Inequalities Graphically

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

1. $x^2 - 6x + 8 < 0$
2. $x^2 + x > 12$
3. $x^2 - 6x + 4 \leq 0$
4. $2x^2 + 7x - 4 > 0$
5. $-x^2 + x - 6 > 0$
6. $x^2 - 2x + 4 \geq 0$

Answer

1. $(2, 4)$
3. $[3-\sqrt{5}, 3+\sqrt{5}]$
5. no solution

Practice Test

Exercise 19.E.28

1. Use the Square Root Property to solve the quadratic equation $3(w+5)^2 = 27$.
2. Use Completing the Square to solve the quadratic equation $a^2 - 8a + 7 = 23$.
3. Use the Quadratic Formula to solve the quadratic equation $2m^2 - 5m + 3 = 0$.

Answer

1. $w = -2, w = -8$
3. $m = 1, m = \frac{3}{2}$

Exercise 19.E.29

Solve the following quadratic equations. Use any method.

1. $2x(3x-2) - 1 = 0$
2. $\frac{9}{4}y^2 - 3y + 1 = 0$

Answer

2. $y = \frac{2}{3}$

Exercise 19.E.30

Use the discriminant to determine the number and type of solutions of each quadratic equation.

1. $6p^2 - 13p + 7 = 0$
2. $3q^2 - 10q + 12 = 0$

Answer

2. 2 complex

Exercise 19.E.31

Solve each equation.

1. $4x^4 - 17x^2 + 4 = 0$
2. $y^{\frac{2}{3}} + 2y^{\frac{1}{3}} - 3 = 0$

Answer

2. $y = 1, y = -27$

Exercise 19.E.32

For each parabola, find

a. Which direction it opens
b. The equation of the axis of symmetry

c. The vertex
d. The x-and y-intercepts
e. The maximum or minimum value

1. $y = 3x^2 + 6x + 8$
2. $y = -x^2 - 8x + 16$

Answer

2.

a. down
b. $x = -4$
c. $(-4, 0)$
d. $y : (0, 16); x : (-4, 0)$
e. minimum value of -4 when $x = 0$

Exercise $19.E.33$

Graph each quadratic function using intercepts, the vertex, and the equation of the axis of symmetry.

1. $f(x) = x^2 + 6x + 9$
2. $f(x) = -2x^2 + 8x + 4$

Answer

2.

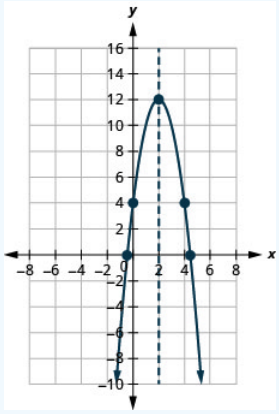

Figure 9.E.20

Exercise $19.E.34$

In the following exercises, graph each function using transformations.

1. $f(x) = (x + 3)^2 + 2$
2. $f(x) = x^2 - 4x - 1$

Answer

2.

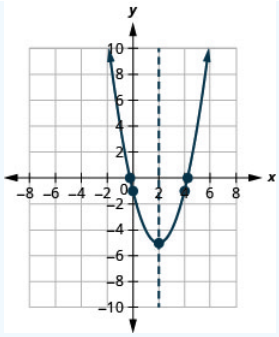

Figure 9.E.21

Exercise $19.E.35$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

1. $x^2 - 6x - 8 \leq 0$
2. $2x^2 + x - 10 > 0$

Answer

2. $\left(-\infty, -\frac{5}{2}\right) \cup (2, \infty)$

Exercise $19.E.36$

Model the situation with a quadratic equation and solve by any method.

1. Find two consecutive even numbers whose product is 360.
2. The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

Answer

2. A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$ find how long it will take the balloon to reach the maximum height, and then find the maximum height. Round to the nearest tenth.

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CHAPTER OVERVIEW

20: Exponential and Logarithmic Functions

20.1: Prelude to Exponential and Logarithmic Functions
20.2: Finding Composite and Inverse Functions
20.2E: Exercises
20.3: Evaluate and Graph Exponential Functions
20.3E: Exercises
20.4: Evaluate and Graph Logarithmic Functions
20.4E: Exercises
20.5: Use the Properties of Logarithms
20.5E: Exercises
20.6: Solve Exponential and Logarithmic Equations
20.6E: Exercises
20.7: Review Exercises

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20.1: Prelude to Exponential and Logarithmic Functions

As the world population continues to grow, food supplies are becoming less able to meet the increasing demand. At the same time, available resources of fertile soil for growing plants is dwindling. One possible solution—grow plants without soil. Botanists around the world are expanding the potential of hydroponics, which is the process of growing plants without soil. To provide the plants with the nutrients they need, the botanists keep careful growth records. Some growth is described by the types of functions you will explore in this chapter—exponential and logarithmic. You will evaluate and graph these functions, and solve equations using them.

Figure 10.0.1: Hydroponic systems allow botanists to grow crops without land. (credit: “Izhamwong”/Wikimedia Commons)

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20.2: Finding Composite and Inverse Functions

Learning Objectives

By the end of this section, you will be able to:

- Find and evaluate composite functions
- Determine whether a function is one-to-one
- Find the inverse of a function

Before you get started, take this readiness quiz.

1. If $f(x) = 2x - 3$ and $g(x) = x^2 + 2x - 3$, find $f(4)$.
   If you missed this problem, review Example 3.48.
2. Solve for x, $3x + 2y = 12$.
   If you missed this problem, review Example 2.31.
3. Simplify: $5\frac{(x+4)}{5} - 4$.
   If you missed this problem, review Example 1.25.

In this chapter, we will introduce two new types of functions, exponential functions and logarithmic functions. These functions are used extensively in business and the sciences as we will see.

Find and Evaluate Composite Functions

Before we introduce the functions, we need to look at another operation on functions called composition. In composition, the output of one function is the input of a second function. For functions f and g, the composition is written $f \circ g$ and is defined by $(f \circ g)(x) = f(g(x))$.

We read $f(g(x))$ as "f of g of x."

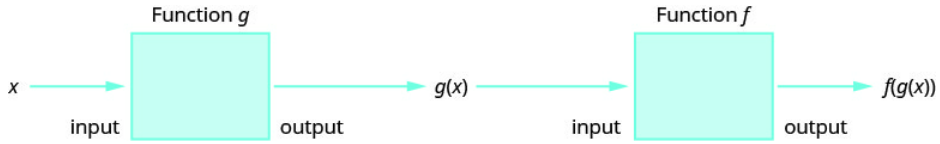

Figure 10.1.1

To do a composition, the output of the first function, $g(x)$, becomes the input of the second function, f, and so we must be sure that it is part of the domain of f.

Definition 20.2.1

The composition of functions f and g is written $f \cdot g$ and is defined by

$$(f \circ g)(x) = f(g(x))$$

We read $f(g(x))$ as f of g of x.

We have actually used composition without using the notation many times before. When we graphed quadratic functions using translations, we were composing functions. For example, if we first graphed $g(x) = x^2$ as a parabola and then shifted it down vertically four units, we were using the composition defined by $(f \circ g)(x) = f(g(x))$ where $f(x) = x - 4$.

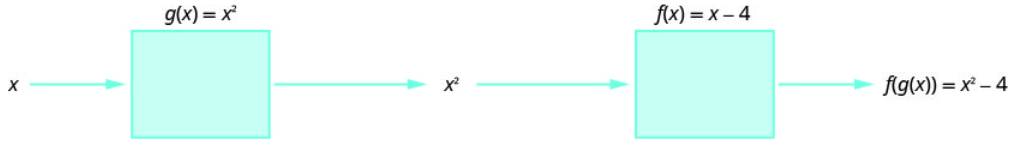

Figure 10.1.2

Example 20.2.1

For functions $f(x) = 4x - 5$ and $g(x) = 2x + 3$, find

a. $(f \circ g)(x)$
b. $(g \circ f)(x)$
c. $(f \cdot g)(x)$

Solution:

Table 10.1.1

a.

Use the definition of $(f \circ g)(x)$.	$(f \circ g)(x) = f(g(x))$
Substitute $2x + 3$ for $g(x)$.	$(f \circ g)(x) = f(2x + 3)$
Find $f(2x + 3)$ where $f(x) = 4x - 5$.	$(f \circ g)(x) = 4(2x + 3) - 5$
Distribute.	$(f \circ g)(x) = 8x + 12 - 5$
Simplify.	$(f \circ g)(x) = 8x + 7$

Table 10.1.2

b.

Use the definition of $(f \circ g)(x)$.	$(g \circ f)(x) = g(f(x))$
Substitute $4x - 5$ for $f(x)$.	$(g \circ f)(x) = g(4x - 5)$
Find $g(4x - 5)$ where $g(x) = 2x + 3$.	$(g \circ f)(x) = 2(4x - 5) + 3$
Distribute.	$(g \circ f)(x) = 8x - 10 + 3$
Simplify.	$(g \circ f)(x) = 8x - 7$

Notice the difference in the result in part a. and part b.

c. Notice that $(f \cdot g)(x)$ is different than $(f \circ g)(x)$. In part a. we did the composition of the functions. Now in part c. we are not composing them, we are multiplying them.

Use the definition of $(f \cdot g)(x)$.

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Substitute $f(x) = 4x - 5$ and $g(x) = 2x + 3$.

$$(f \cdot g)(x) = (4x - 5) \cdot (2x + 3)$$

Multiply.

$$(f \cdot g)(x) = 8x^2 + 2x - 15$$

Exercise 20.2.1

For functions $f(x) = 3x - 2$ and $g(x) = 5x + 1$, find

a. $(f \circ g)(x)$
b. $(g \circ f)(x)$
c. $(f \cdot g)(x)$

Answer

a. $15x + 1$
b. $15x - 9$
c. $15x^2 - 7x - 2$

Exercise 20.2.2

For functions $f(x) = 4x - 3$, and $g(x) = 6x - 5$, find

a. $(f \circ g)(x)$
b. $(g \circ f)(x)$
c. $(f \cdot g)(x)$

Answer

a. $24x - 23$
b. $24x - 23$
c. $24x^2 - 38x + 15$

In the next example we will evaluate a composition for a specific value.

Example 20.2.2

For functions $f(x) = x^2 - 4$, and $g(x) = 3x + 2$, find:

a. $(f \circ g)(-3)$
b. $(g \circ f)(-1)$
c. $(f \circ f)(2)$

Solution:

Table 10.1.3

a.

Use the definition of $(f \circ g)(-3)$.	$(f \circ g)(-3) = f(g(-3))$
Find $g(-3)$ where $g(x) = 3x + 2$.	$(f \circ g)(-3) = f(3 \cdot (-3) + 2)$
Simplify.	$(f \circ g)(-3) = f(-7)$
Find $f(-7)$ where $f(x) = x^2 - 4$.	$(f \circ g)(-3) = (-7)^2 - 4$
Simplify.	$(f \circ g)(-3) = 45$

Table 10.1.4

b.

Use the definition of $(g \circ f)(-1)$.	$(g \circ f)(-1) = g(f(-1))$
Find $f(-1)$ where $f(x) = x^2 - 4$.	$(g \circ f)(-1) = g((-1)^2 - 4)$
Simplify.	$(g \circ f)(-1) = g(-3)$
Find $g(-3)$ where $g(x) = 3x + 2$.	$(g \circ f)(-1) = 3(-3) + 2$
Simplify.	$(g \circ f)(-1) = -7$

Table 10.1.5

c.

Use the definition of $(f \circ f)(2)$.	$(f \circ f)(2) = f(f(2))$
Find $f(2)$ where $f(x) = x^2 - 4$.	$(f \circ f)(2) = f(2^2 - 4)$
Simplify.	$(f \circ f)(2) = f(0)$
Find $f(0)$ where $f(x) = x^2 - 4$.	$(f \circ f)(2) = 0^2 - 4$
Simplify.	$(f \circ f)(2) = -4$

Exercise 20.2.3

For functions $f(x) = x^2 - 9$, and $g(x) = 2x + 5$, find

a. $(f \circ g)(-2)$
b. $(g \circ f)(-3)$
c. $(f \circ f)(4)$

Answer

a. -8
b. 5
c. 40

Exercise 20.2.4

For functions $f(x) = x^2 + 1$, and $g(x) = 3x - 5$, find

a. $(f \circ g)(-1)$
b. $(g \circ f)(2)$
c. $(f \circ f)(-1)$

Answer

a. 65
b. 10
c. 5

Determine Whether a Function is One-to-One

When we first introduced functions, we said a **function** is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each x-value is matched with only one y-value.

We used the birthday example to help us understand the definition. Every person has a birthday, but no one has two birthdays and it is okay for two people to share a birthday. Since each person has exactly one birthday, that relation is a function.

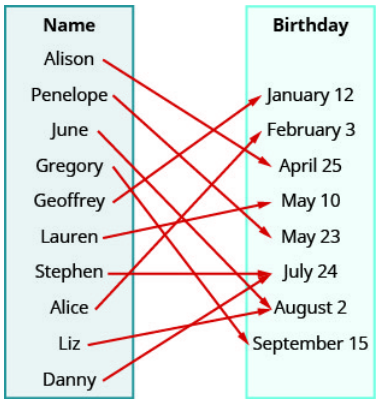

Figure 10.1.38

A function is **one-to-one** if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one x-value.

Our example of the birthday relation is not a one-to-one function. Two people can share the same birthday. The range value August 2 is the birthday of Liz and June, and so one range value has two domain values. Therefore, the function is not one-to-one.

Definition 20.2.2

A function is **one-to-one** if each value in the range corresponds to one element in the domain. For each ordered pair in the function, each y-value is matched with only one x-value. There are no repeated y-values.

Example 20.2.3

For each set of ordered pairs, determine if it represents a function and, if so, if the function is one-to-one.

a. $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$
b. $\{(0,0),(1,1),(4,2),(9,3),(16,4)\}$

Solution:

a. $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

Each x-value is matched with only one y-value. So this relation is a function.

But each y-value is not paired with only one x-value, $(-3,27)$ and $(3,27)$, for example. So this function is not one-to-one.

b. $\{(0,0),(1,1),(4,2),(9,3),(16,4)\}$

Each x-value is matched with only one y-value. So this relation is a function.

Since each y-value is paired with only one x-value, this function is one-to-one.

Exercise 20.2.5

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

a. $\{(-3,-6),(-2,-4),(-1,-2),(0,0),(1,2),(2,4),(3,6)\}$
b. $\{(-4,8),(-2,4),(-1,2),(0,0),(1,2),(2,4),(4,8)\}$

Answer

a. One-to-one function
b. Function; not one-to-one

Exercise 20.2.6

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

a. $\{(27,-3),(8,-2),(1,-1),(0,0),(1,1),(8,2),(27,3)\}$
b. $\{(7,-3),(-5,-4),(8,0),(0,0),(-6,4),(-2,2),(-1,3)\}$

Answer

a. Not a function
b. Function; not one-to-one

To help us determine whether a relation is a function, we use the **vertical line test**. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. Also, if any vertical line intersects the graph in more than one point, the graph does not represent a function.

The vertical line is representing an x-value and we check that it intersects the graph in only one y-value. Then it is a function.

To check if a function is one-to-one, we use a similar process. We use a horizontal line and check that each horizontal line intersects the graph in only one point. The horizontal line is representing a y-value and we check that it intersects the graph in only one x-value. If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function. This is the **horizontal line test**.

Definition 20.2.3

Horizontal Line Test

If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function.

We can test whether a graph of a relation is a function by using the vertical line test. We can then tell if the function is one-to-one by applying the horizontal line test.

Example 20.2.4

Determine

a. whether each graph is the graph of a function and, if so,
b. whether it is one-to-one

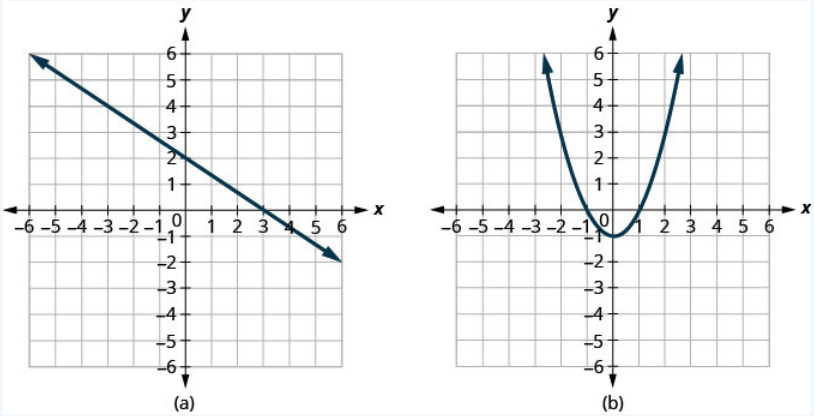

Figure 10.1.39

Solution:

a.

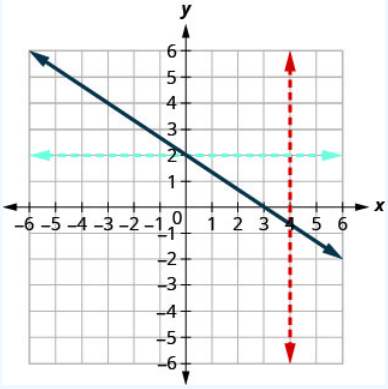

Figure 10.1.40

Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. Since any horizontal line intersects the graph in at most one point, the graph is the graph of a one-to-one function.

b.

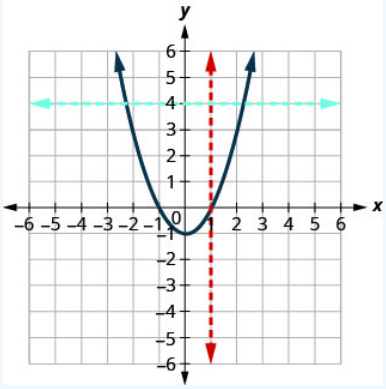

Figure 10.1.41

Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. The horizontal line shown on the graph intersects it in two points. This graph does not represent a one-to-one function.

Exercise 20.2.7

Determine

a. whether each graph is the graph of a function and, if so,
b. whether it is one-to-one

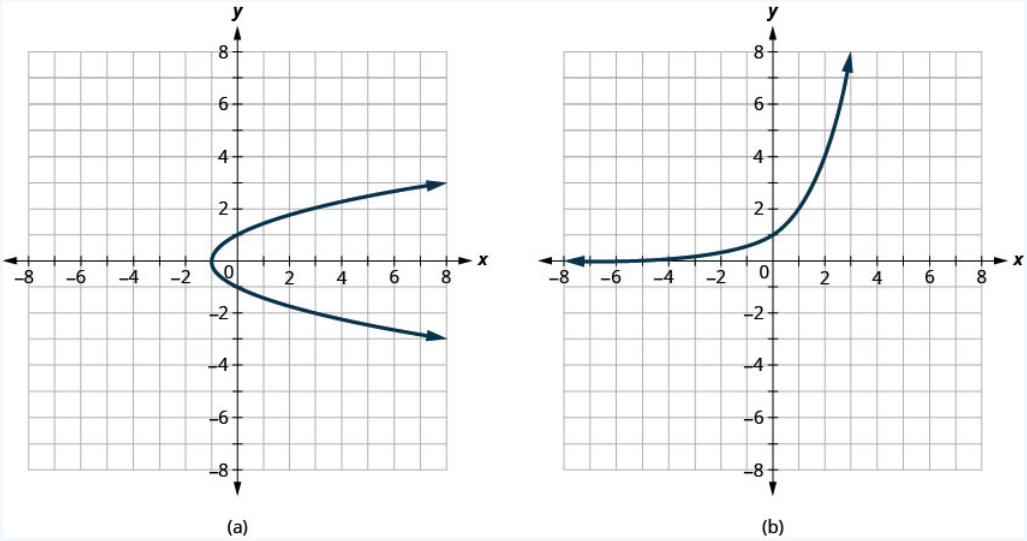

Figure 10.1.42

Answer

a. Not a function
b. One-to-one function

Exercise 20.2.8

Determine

a. whether each graph is the graph of a function and, if so,
b. whether it is one-to-one

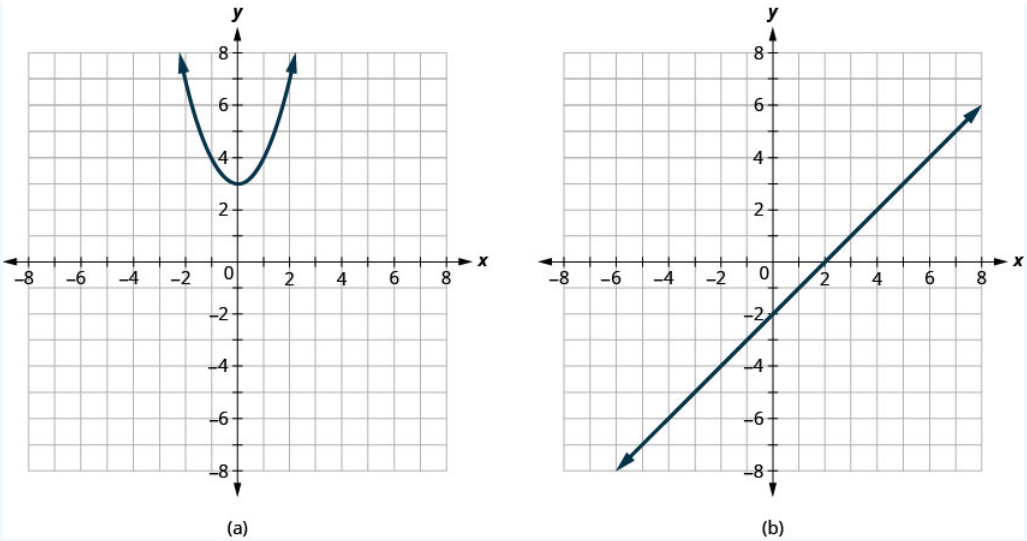

Figure 10.1.43

Answer

a. Function; not one-to-one
b. One-to-one function

Find the Inverse of a Function

Let's look at a one-to one function, f, represented by the ordered pairs $\{(0,5),(1,6),(2,7),(3,8)\}$ For each x-value, f adds 5 to get the y-value. To 'undo' the addition of 5, we subtract 5 from each y-value and get back to the original x-value. We can call this "taking the inverse of f" and name the function f^{-1}.

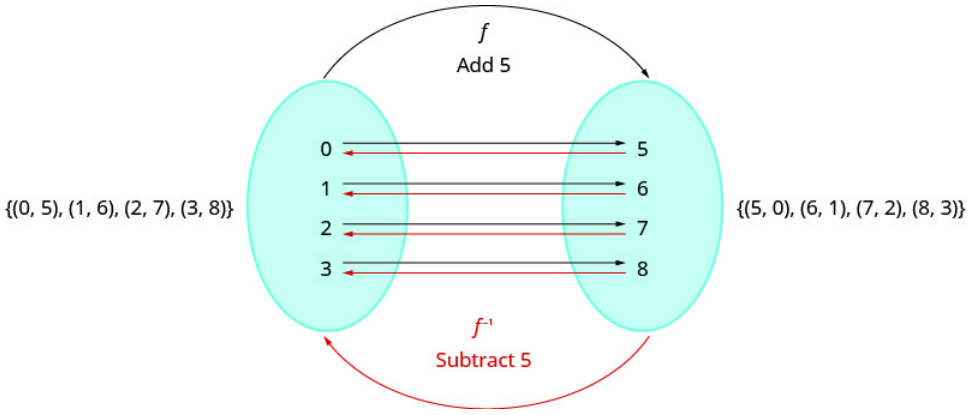

Figure 10.1.44

Notice that that the ordered pairs of f and f^{-1} have their x-values and y-values reversed. The domain of f is the range of f^{-1} and the domain of f^{-1} is the range of f.

Definition 20.2.4

Inverse of a Function Defined by Ordered Pairs

If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x,y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y,x).

In the next example we will find the inverse of a function defined by ordered pairs.

Example 20.2.5

Find the inverse of the function $\{(0,3),(1,5),(2,7),(3,9)\}$ Determine the domain and range of the inverse function.

Solution:

This function is one-to-one since every x-value is paired with exactly one y-value.

To find the inverse we reverse the x-values and y-values in the ordered pairs of the function.

Function	$\{(0,3),(1,5),(2,7),(3,9)\}$
Inverse Function	$\{(3,0),(5,1),(7,2),(9,3)\}$
Domain of Inverse Function	$\{3,5,7,9\}$
Range of Inverse Function	$\{0,1,2,3\}$

Exercise 20.2.9

Find the inverse of $\{(0,4),(1,7),(2,10),(3,13)\}$Determine the domain and range of the inverse function.

Answer

Inverse function: $\{(4,0),(7,1),(10,2),(13,3)\}$Domain: $\{4,7,10,13\}$Range: $\{0,1,2,3\}$

Exercise 20.2.10

Find the inverse of $\{(-1,4),(-2,1),(-3,0),(-4,2)\}$ Determine the domain and range of the inverse function.

Answer

Inverse function: $\{(4,-1),(1,-2),(0,-3),(2,-4)\}$. Domain: $\{0,1,2,4\}$ Range: $\{-4,-3,-2,-1\}$.

We just noted that if $f(x)$ is a one-to-one function whose ordered pairs are of the form (x,y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y,x).

So if a point (a,b) is on the graph of a function $f(x)$, then the ordered pair (b,a) is on the graph of $f^{-1}(x)$. See Figure 10.1.43.

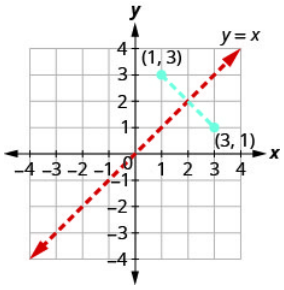

Figure 10.1.45

The distance between any two pairs (a,b) and (b,a) is cut in half by the line $y=x$. So we say the points are mirror images of each other through the line $y=x$.

Since every point on the graph of a function $f(x)$ is a mirror image of a point on the graph of $f^{-1}(x)$, we say the graphs are mirror images of each other through the line $y=x$. We will use this concept to graph the inverse of a function in the next example.

Example 20.2.6

Graph, on the same coordinate system, the inverse of the one-to one function shown.

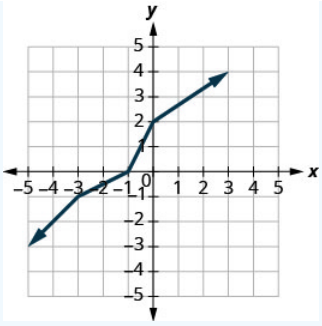

Figure 10.1.46

Solution:

We can use points on the graph to find points on the inverse graph. Some points on the graph are: $(-5,-3),(-3,-1),(-1,0),(0,2),(3,4)$

So, the inverse function will contain the points: $(-3,-5),(-1,-3),(0,-1),(2,0),(4,3)$

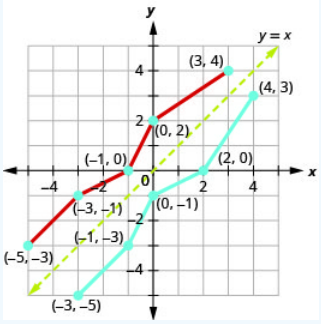

Figure 10.1.47

Notice how the graph of the original function and the graph of the inverse functions are mirror images through the line $y = x$.

Exercise 20.2.11

Graph, on the same coordinate system, the inverse of the one-to one function.

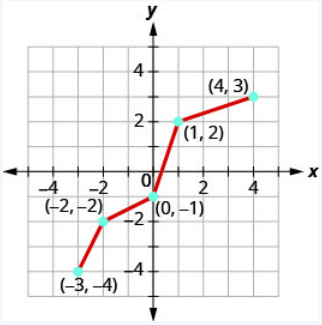

Figure 10.1.48

Answer

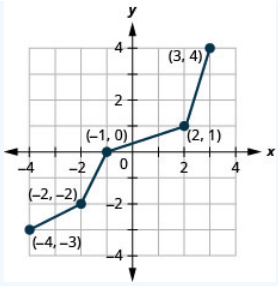

Figure 10.1.49

Exercise 20.2.12

Graph, on the same coordinate system, the inverse of the one-to one function.

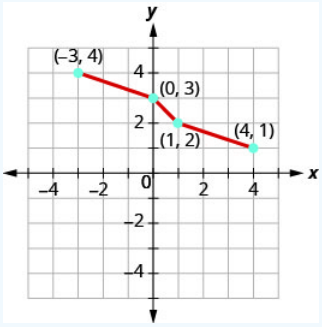

Figure 10.1.50

Answer

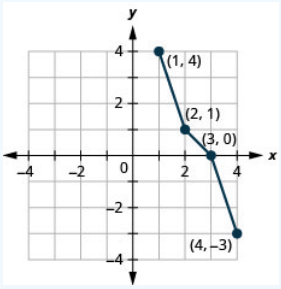

Figure 10.1.51

When we began our discussion of an inverse function, we talked about how the inverse function 'undoes' what the original function did to a value in its domain in order to get back to the original x-value.

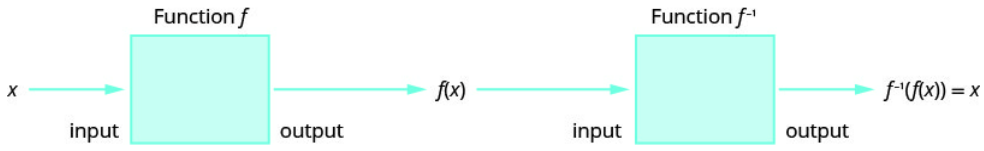

Figure 10.1.52

Definition 20.2.5

Inverse Functions

$f^{-1}(f(x)) = x$, for all x in the domain of f

$f\left(f^{-1}(x)\right) = x$, for all x in the domain of f^{-1}

We can use this property to verify that two functions are inverses of each other.

Example 20.2.7

Verify that $f(x) = 5x - 1$ and $g(x) = \frac{x+1}{5}$ are inverse functions.

Solution:

The functions are inverses of each other if $g(f(x)) = x$ and $f(g(x)) = x$.

Table 10.1.6

	$g(f(x)) \stackrel{?}{=} x$
Substitue $5x - 1$ for $f(x)$.	$g(5x-1) \stackrel{?}{=} x$
Find $g(5x-1)$ where $g(x) = \frac{x+1}{5}$.	$\frac{(5x-1)+1}{5} \stackrel{?}{=} x$
Simplify.	$\frac{5x}{5} \stackrel{?}{=} x$
Simplify.	$x = x$ ✓

$f(g(x)) \stackrel{?}{=} x$

Figure 10.1.59

Substitute $\frac{x+1}{5}$ for $g(x)$.	$f\left(\frac{x+1}{5}\right) \stackrel{?}{=} x$
Find $f\left(\frac{x+1}{5}\right)$ where $f(x) = 5x - 1$.	$5\left(\frac{x+1}{5}\right) - 1 \stackrel{?}{=} x$
Simplify.	$x + 1 - 1 \stackrel{?}{=} x$
Simplify.	$x = x$ ✓

Since both $g(f(x)) = x$ and $f(g(x)) = x$ are true, the functions $f(x) = 5x - 1$ and $g(x) = \frac{x+1}{5}$ are inverse functions. That is, they are inverses of each other.

Exercise 20.2.13

Verify that the functions are inverse functions. $f(x) = 4x - 3$ and $g(x) = \frac{x+3}{4}$.

Answer

$g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

Exercise 20.2.14

Verify that the functions are inverse functions. $f(x) = 2x + 6$ and $g(x) = \frac{x-6}{2}$

Answer

$g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

We have found inverses of function defined by ordered pairs and from a graph. We will now look at how to find an inverse using an algebraic equation. The method uses the idea that if $f(x)$ is a one-to-one function with ordered pairs (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).

If we reverse the x and y in the function and then solve for y, we get our **inverse function**.

Example 20.2.8 How to Find the Inverse of a One-to-One Function

Find the inverse of $f(x) = 4x + 7$.

Solution:

Table 10.1.7

Step 1. Substitute y for $f(x)$.	Replace $f(x)$ with y.	$f(x) = 4x + 7$ $y = 4x + 7$
Step 2: Interchange the variables x and y.	Replace x with y and then y with x.	$x = 4y + 7$
Step 3: Solve for y.	Subtract 7 from each side. Divide by 4.	$x - 7 = 4y$ $\frac{x-7}{4} = y$
Step 4: Substitute $f^{-1}(x)$ for y.	Replace y with $f^{-1}(x)$.	$\frac{x-7}{4} = f^{-1}(x)$
Step 5: Verify that the functions are inverses.	Show $f^{-1}(f(x)) = x$ and $f\left(f^{-1}(x)\right) = x$	$f^{-1}(f(x)) \stackrel{?}{=} x$ $f^{-1}(4x+7) \stackrel{?}{=} x$ $\frac{(4x+7)-7}{4} \stackrel{?}{=} x$ $\frac{4x}{4} \stackrel{?}{=} x$ $x = x$ $f(f^{-1}(x)) \stackrel{?}{=} x$ $f\left(\frac{x-7}{4}\right) \stackrel{?}{=} x$ $4\left(\frac{x-7}{4}\right) + 7 \stackrel{?}{=} x$ $x - 7 + 7 \stackrel{?}{=} x$ $x = x$

Exercise 20.2.15

Find the inverse of the function $f(x) = 5x - 3$.

Answer

$f^{-1}(x) = \frac{x+3}{5}$

Exercise 20.2.16

Find the inverse of the function $f(x) = 8x + 5$.

Answer

$f^{-1}(x) = \frac{x-5}{8}$

We summarize the steps below.

How to Find the Inverse of a One-to-One Function

1. Substitute y for $f(x)$.
2. Interchange the variables x and y.
3. Solve for y.
4. Substitute $f^{-1}(x)$ for y.

5. Verify that the functions are inverses.

Example 20.2.9 How to Find the Inverse of a One-to-One Function

Find the inverse of $f(x)=\sqrt[5]{2x-3}$.

Solution:

$$f(x)=\sqrt[5]{2x-3}$$

Substitute y for $f(x)$.

$$y=\sqrt[5]{2x-3}$$

Interchange the variables x and y.

$$x=\sqrt[5]{2y-3}$$

Solve for y.

$$\begin{aligned}(x)^5&=(\sqrt[5]{2y-3})^5\\ x^5&=2y-3\\ x^5+3&=2y\\ \frac{x^5+3}{2}&=y\end{aligned}$$

Substitute $f^{-1}(x)$ for y.

$$f^{-1}(x)=\tfrac{x^5+3}{2}$$

Verify that the functions are inverses.

$$\begin{aligned} f^{-1}(f(x))&\stackrel{?}{=}x & f\left(f^{-1}(x)\right)&\stackrel{?}{=}x\\ f^{-1}(\sqrt[5]{2x-3})&\stackrel{?}{=}x & f\left(\tfrac{x^5+3}{2}\right)&\stackrel{?}{=}x\\ \tfrac{(\sqrt[5]{2x-3})^5+3}{2}&\stackrel{?}{=}x & \sqrt[5]{2\left(\tfrac{x^5+3}{2}\right)-3}&\stackrel{?}{=}x\\ \tfrac{2x-3+3}{2}&\stackrel{?}{=}x & \sqrt[5]{x^5+3-3}&\stackrel{?}{=}x\\ \tfrac{2x}{2}&\stackrel{?}{=}x & \sqrt[5]{x^5}&\stackrel{?}{=}x\\ x&=x & x&=x\end{aligned}$$

Exercise 20.2.17

Find the inverse of the function $f(x)=\sqrt[5]{3x-2}$.

Answer

$f^{-1}(x)=\tfrac{x^5+2}{3}$

Exercise 20.2.18

Find the inverse of the function $f(x)=\sqrt[4]{6x-7}$.

Answer

$f^{-1}(x)=\tfrac{x^4+7}{6}$

Key Concepts

- **Composition of Functions:** The composition of functions f and g, is written $f \circ g$ and is defined by

$$(f \circ g)(x) = f(g(x))$$

  We read $f(g(x))$ as f of g of x.
- **Horizontal Line Test:** If every horizontal line, intersects the graph of a function in at most one point, it is a one-to-one function.
- **Inverse of a Function Defined by Ordered Pairs:** If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).
- **Inverse Functions:** For every x in the domain of one-to-one function f and f^{-1},

$$f^{-1}(f(x)) = x$$
$$f\left(f^{-1}(x)\right) = x$$

- **How to Find the Inverse of a One-to-One Function:**
  1. Substitute y for $f(x)$.
  2. Interchange the variables x and y.
  3. Solve for y.
  4. Substitute $f^{-1}(x)$ for y.
  5. Verify that the functions are inverses.

Glossary

one-to-one function

A function is one-to-one if each value in the range has exactly one element in the domain. For each ordered pair in the function, each y-value is matched with only one x-value.

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20.2E: Exercises

Practice Makes Perfect

Exercise $20.2E.19$ Find and Evaluate Composite Functions

In the following exercises, find

a. $(f \circ g)(x)$
b. $(g \circ f)(x)$
c. $(f \cdot g)(x)$

1. $f(x) = 4x + 3$ and $g(x) = 2x + 5$
2. $f(x) = 3x - 1$ and $g(x) = 5x - 3$
3. $f(x) = 6x - 5$ and $g(x) = 4x + 1$
4. $f(x) = 2x + 7$ and $g(x) = 3x - 4$
5. $f(x) = 3x$ and $g(x) = 2x^2 - 3x$
6. $f(x) = 2x$ and $g(x) = 3x^2 - 1$
7. $f(x) = 2x - 1$ and $g(x) = x^2 + 2$
8. $f(x) = 4x + 3$ and $g(x) = x^2 - 4$

Answer

1.

a. $8x + 23$
b. $8x + 11$
c. $8x^2 + 26x + 15$

3.

a. $24x + 1$
b. $24x - 19$
c. $24x^2 + 19x - 5$

5.

a. $6x^2 - 9x$
b. $18x^2 - 9x$
c. $6x^3 - 9x^2$

7.

a. $2x^2 + 3$
b. $4x^2 - 4x + 3$
c. $2x^3 - x^2 + 4x - 2$

Exercise $20.2E.20$ Find and Evaluate Composite Functions

In the following exercises, find the values described.

1. For functions $f(x) = 2x^2 + 3$ and $g(x) = 5x - 1$, find
   a. $(f \circ g)(-2)$
   b. $(g \circ f)(-3)$
   c. $(f \circ f)(-1)$
2. For functions $f(x) = 5x^2 - 1$ and $g(x) = 4x - 1$, find
   a. $(f \circ g)(1)$
   b. $(g \circ f)(-1)$
   c. $(f \circ f)(2)$

3. For functions $f(x) = 2x^3$ and $g(x) = 3x^2 + 2$, find
   a. $(f \circ g)(-1)$
   b. $(g \circ f)(1)$
   c. $(g \circ g)(1)$
4. For functions $f(x) = 3x^3 + 1$ and $g(x) = 2x^2 = 3$, find
   a. $(f \circ g)(-2)$
   b. $(g \circ f)(-1)$
   c. $(g \circ g)(1)$

Answer

1.
   a. 245
   b. 104
   c. 53

3.
   a. 250
   b. 14
   c. 77

Exercise $20.2E.21$ Determine Whether a Function is One-to-One

In the following exercises, determine if the set of ordered pairs represents a function and if so, is the function one-to-one.

1. $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$
2. $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$
3. $\{(-3, -5), (-2, -3), (-1, -1), (0, 1), (1, 3), (2, 5), (3, 7)\}$
4. $\{(5, 3), (4, 2), (3, 1), (2, 0), (1, -1), (0, -2), (-1, -3)\}$

Answer

1. Function; not one-to-one

3. One-to-one function

Exercise $20.2E.22$ Determine Whether a Function is One-to-One

In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.

1.

   a.

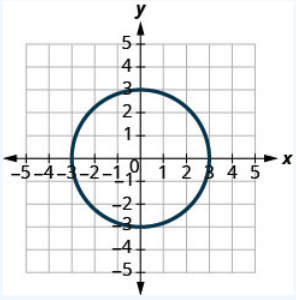

Figure 10.1.65

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b.

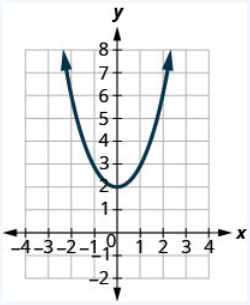

Figure 10.1.66

2.

a.

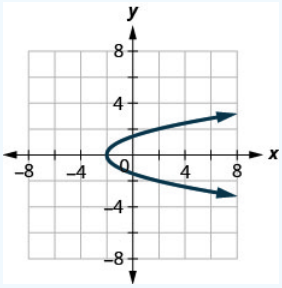

Figure 10.1.67

b.

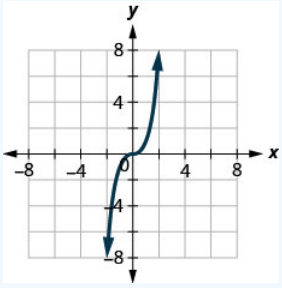

Figure 10.1.68

3.

a.

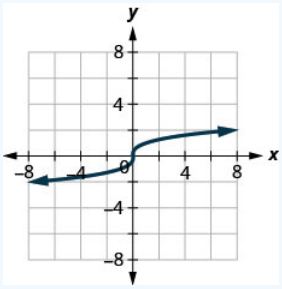

Figure 10.1.69

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b.

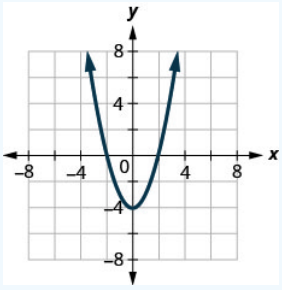

Figure 10.1.70

4.

a.

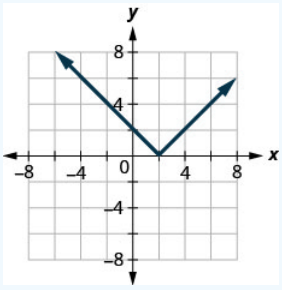

Figure 10.1.71

b.

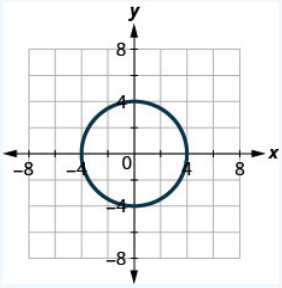

Figure 10.1.72

Answer

1.

a. Not a function
b. Function; not one-to-one

3.

a. One-to-one function
b. Function; not one-to-one

? Exercise $20.2E.23$ Determine Whether a Function is One-to-One

In the following exercises, find the inverse of each function. Determine the domain and range of the inverse function.

1. $\{(2,1),(4,2),(6,3),(8,4)\}$
2. $\{(6,2),(9,5),(12,8),(15,11)\}$
3. $\{(0,-2),(1,3),(2,7),(3,12)\}$
4. $\{(0,0),(1,1),(2,4),(3,9)\}$
5. $\{(-2,-3),(-1,-1),(0,1),(1,3)\}$

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6. $\{(5,3),(4,2),(3,1),(2,0)\}$

Answer

1. Inverse function: $\{(1,2),(2,4),(3,6),(4,8)\}$. Domain: $\{1,2,3,4\}$. Range: $\{2,4,6,8\}$.
3. Inverse function: $\{(-2,0),(3,1),(7,2),(12,3)\}$. Domain: $\{-2,3,7,12\}$. Range: $\{0,1,2,3\}$
5. Inverse function: $\{(-3,-2),(-1,-1),(1,0),(3,1)\}$. Domain: $\{-3,-1,1,3\}$. Range: $\{-2,-1,0,1\}$

Exercise $20.2E.24$ Determine Whether a Function is One-to-One

In the following exercises, graph, on the same coordinate system, the inverse of the one-to-one function shown.

1.

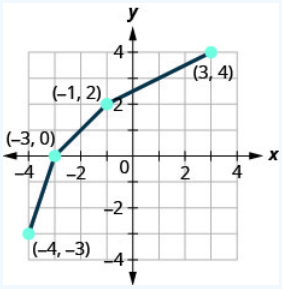

Figure 10.1.73

2.

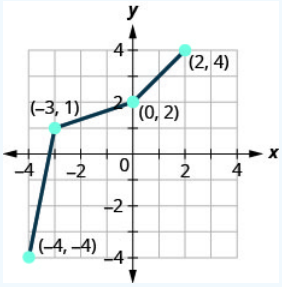

Figure 10.1.74

3.

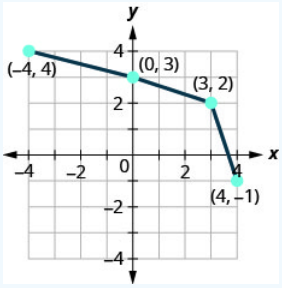

Figure 10.1.75

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4.

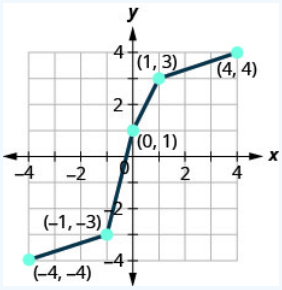

Figure 10.1.76

Answer

1.

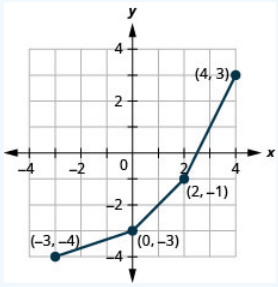

Figure 10.1.77

3.

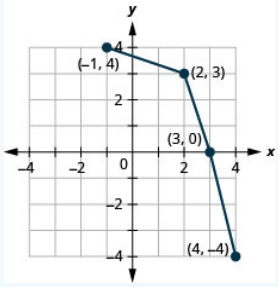

Figure 10.1.78

? Exercise $20.2E.25$ Determine Whether the given functions are inverses

In the following exercises, determine whether or not the given functions are inverses.

1. $f(x) = x + 8$ and $g(x) = x - 8$
2. $f(x) = x - 9$ and $g(x) = x + 9$
3. $f(x) = 7x$ and $g(x) = \frac{x}{7}$
4. $f(x) = \frac{x}{11}$ and $g(x) = 11x$
5. $f(x) = 7x + 3$ and $g(x) = \frac{x-3}{7}$
6. $f(x) = 5x - 4$ and $g(x) = \frac{x-4}{5}$
7. $f(x) = \sqrt{x+2}$ and $g(x) = x^2 - 2$
8. $f(x) = \sqrt[3]{x-4}$ and $g(x) = x^3 + 4$

Answer

1. $g(f(x)) = x,$ and $f(g(x)) = x,$ so they are inverses.
3. $g(f(x)) = x,$ and $f(g(x)) = x,$ so they are inverses.

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5. $g(f(x))=x,$ and $f(g(x))=x,$ so they are inverses.
7. $g(f(x))=x,$ and $f(g(x))=x,$ so they are inverses (for nonnegative x)

Exercise $20.2E.26$ Determine the inverse of a function

In the following exercises, find the inverse of each function.

1. $f(x)=x-12$
2. $f(x)=x+17$
3. $f(x)=9x$
4. $f(x)=8x$
5. $f(x)=\frac{x}{6}$
6. $f(x)=\frac{x}{4}$
7. $f(x)=6x-7$
8. $f(x)=7x-1$
9. $f(x)=-2x+5$
10. $f(x)=-5x-4$
11. $f(x)=x^2+6, x\geq 0$
12. $f(x)=x^2-9, x\geq 0$
13. $f(x)=x^3-4$
14. $f(x)=x^3+6$
15. $f(x)=\frac{1}{x+2}$
16. $f(x)=\frac{1}{x-6}$
17. $f(x)=\sqrt{x-2}, x\geq 2$
18. $f(x)=\sqrt{x+8}, x\geq -8$
19. $f(x)=\sqrt[3]{x-3}$
20. $f(x)=\sqrt[3]{x+5}$
21. $f(x)=\sqrt[4]{9x-5}, x\geq \frac{5}{9}$
22. $f(x)=\sqrt[4]{8x-3}, x\geq \frac{3}{8}$
23. $f(x)=\sqrt[5]{-3x+5}$
24. $f(x)=\sqrt[5]{-4x-3}$

Answer

1. $f^{-1}(x)=x+12$
3. $f^{-1}(x)=\frac{x}{9}$
5. $f^{-1}(x)=6x$
7. $f^{-1}(x)=\frac{x+7}{6}$
9. $f^{-1}(x)=\frac{x-5}{-2}$
11. $f^{-1}(x)=\sqrt{x-6}$
13. $f^{-1}(x)=\sqrt[3]{x+4}$
15. $f^{-1}(x)=\frac{1}{x}-2$
17. $f^{-1}(x)=x^2+2, x\geq 0$
19. $f^{-1}(x)=x^3+3$
21. $f^{-1}(x)=\frac{x^4+5}{9}, x\geq 0$
23. $f^{-1}(x)=\frac{x^5-5}{-3}$

Exercise $20.2E.27$ Writing Exercises

1. Explain how the graph of the inverse of a function is related to the graph of the function.
2. Explain how to find the inverse of a function from its equation. Use an example to demonstrate the steps.

Answer

1. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find and evaluate composite functions.			
determine whether a function is one-to-one.			
find the inverse of a function.			

Figure 10.1.79

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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20.3: Evaluate and Graph Exponential Functions

Learning Objectives

By the end of this section, you will be able to:

- Graph exponential functions
- Solve Exponential equations
- Use exponential models in applications

Before you get started, take this readiness quiz.

1. Simplify: $\left(\frac{x^3}{x^2}\right)$.
   If you missed this problem, review Example 5.13.
2. Evaluate: a. 2^0 b. $\left(\frac{1}{3}\right)^0$.
   If you missed this problem, review Example 5.14.
3. Evaluate: a. 2^{-1} b. $\left(\frac{1}{3}\right)^{-1}$.
   If you missed this problem, review Example 5.15.

Graph Exponential Functions

The functions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential functions.

An **exponential function** is a function of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$.

Definition 20.3.1

An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form

$$f(x) = a^x$$

Notice that in this function, the variable is the exponent. In our functions so far, the variables were the base.

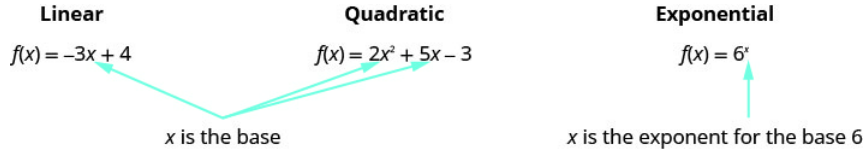

Figure 10.2.1

Our definition says $a \neq 1$. If we let $a = 1$, then $f(x) = a^x$ becomes $f(x) = 1^x$. Since $1^x = 1$ for all real numbers, $f(x) = 1$. This is the constant function.

Our definition also says $a > 0$. If we let a base be negative, say -4, then $f(x) = (-4)^x$ is not a real number when $x = \frac{1}{2}$.

$$f(x) = (-4)^x$$
$$f\left(\frac{1}{2}\right) = (-4)^{\frac{1}{2}}$$
$$f\left(\frac{1}{2}\right) = \sqrt{-4} \text{ not a real number}$$

In fact, $f(x) = (-4)^x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $a > 0$.

By graphing a few exponential functions, we will be able to see their unique properties.

Example 20.3.1

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 3^x$.

Solution:

We will use point plotting to graph the functions.

x	$f(x) = 2^x$	$(x, f(x))$	$g(x) = 3^x$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0 = 1$	(0, 1)	$3^0 = 1$	(0, 1)
1	$2^1 = 2$	(1, 2)	$3^1 = 3$	(1, 3)
2	$2^2 = 4$	(2, 4)	$3^2 = 9$	(2, 9)
3	$2^3 = 8$	(3, 8)	$3^3 = 27$	(3, 27)

Figure 10.2.2

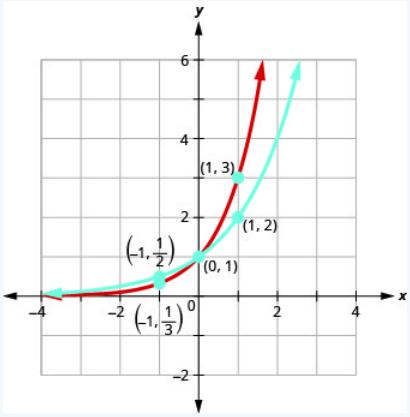

Figure 10.2.3

Exercise 20.3.1

Graph: $f(x) = 4^x$.

Answer

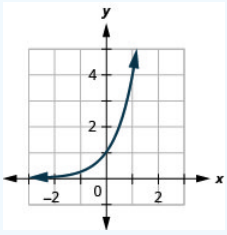

Figure 10.2.4

Exercise 20.3.2

Graph: $g(x) = 5^x$

Answer

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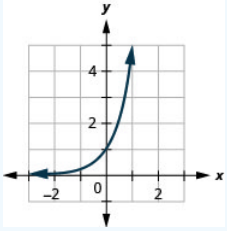

Figure 10.2.5

If we look at the graphs from the previous Example 10.2.1 and Exercises 10.2.1 and 10.2.2, we can identify some of the properties of exponential functions.

The graphs of $f(x) = 2^x$ and $g(x) = 3^x$, as well as the graphs of $f(x) = 4^x$ and $g(x) = 5^x$, all have the same basic shape. This is the shape we expect from an exponential function where $a > 1$.

We notice, that for each function, the graph contains the point $(0, 1)$. This make sense because $a^0 = 1$ for any a.

The graph of each function, $f(x) = a^x$ also contains the point $(1, a)$. The graph of $f(x) = 2^x$ contained $(1, 2)$ and the graph of $g(x) = 3^x$ contained $(1, 3)$. This makes sense as $a^1 = a$.

Notice too, the graph of each function $f(x) = a^x$ also contains the point $(-1, \frac{1}{a})$. The graph of $f(x) = 2^x$ contained $(-1, \frac{1}{2})$ and the graph of $g(x) = 3^x$ contained $(-1, \frac{1}{3})$.This makes sense as $a^{-1} = \frac{1}{a}$.

What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as $(-\infty, \infty)$.

Look at each graph. What is the range of the function? The graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the x-axis very closely but will never cross it, we call the line $y = 0$, the x-axis, a horizontal asymptote.

Properties of the Graph of $f(x) = a^x$ when $a > 1$

Table 10.2.1

Domain	$(-\infty, \infty)$
Range	$(0, \infty)$
x-intercept	None
y-intercept	$(0, 1)$
Contains	$(1, a)$, $(-1, \frac{1}{a})$
Asymptote	x-axis, the line $y = 0$

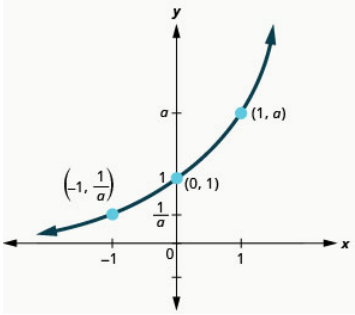

Figure 10.2.6

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Our definition of an exponential function $f(x) = a^x$ says $a > 0$, but the examples and discussion so far has been about functions where $a > 1$. What happens when $0 < a < 1$ The next example will explore this possibility.

Example 20.3.2

On the same coordinate system, graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

Solution:

We will use point plotting to graph the functions.

x	$f(x) = \left(\frac{1}{2}\right)^x$	$(x, f(x))$	$g(x) = \left(\frac{1}{3}\right)^x$	$(x, g(x))$
–2	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$	(–2, 4)	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	(–2, 9)
–1	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$	(–1, 2)	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	(–1, 3)
0	$\left(\frac{1}{2}\right)^0 = 1$	(0, 1)	$\left(\frac{1}{3}\right)^0 = 1$	(0, 1)
1	$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(2, \frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$	$\left(3, \frac{1}{8}\right)$	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(3, \frac{1}{27}\right)$

Figure 10.2.7

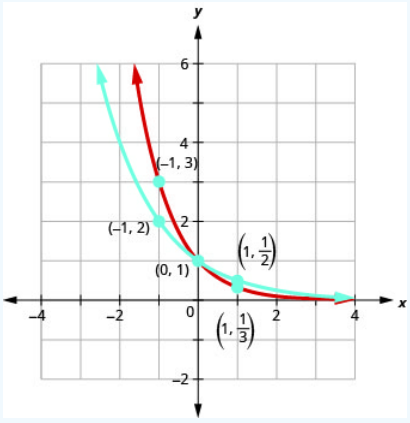

Figure 10.2.8

Exercise 20.3.3

Graph: $f(x) = \left(\frac{1}{4}\right)^x$.

Answer

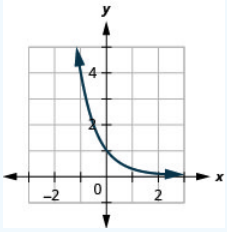

Figure 10.2.9

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Exercise 20.3.4

Graph: $g(x) = \left(\frac{1}{5}\right)^x$.

Answer

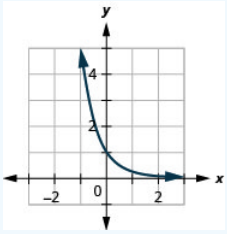

Figure 10.2.10

Now let's look at the graphs from the previous Example 10.2.2 and Exercises 10.2.3 and 10.2.4 so we can now identify some of the properties of exponential functions where $0 < a < 1$.

The graphs of $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$ as well as the graphs of $f(x) = \left(\frac{1}{4}\right)^x$ and $g(x) = \left(\frac{1}{5}\right)^x$ all have the same basic shape. While this is the shape we expect from an exponential function where $0 < a < 1$, the graphs go down from left to right while the previous graphs, when $a > 1$, went from up from left to right.

We notice that for each function, the graph still contains the point $(0, 1)$. This make sense because $a^0 = 1$ for any a.

As before, the graph of each function, $f(x) = a^x$, also contains the point $(1, a)$. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ contained $\left(1, \frac{1}{2}\right)$ and the graph of $g(x) = \left(\frac{1}{3}\right)^x$ contained $\left(1, \frac{1}{3}\right)$. This makes sense as $a^1 = a$.

Notice too that the graph of each function, $f(x) = a^x$, also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ contained $(-1, 2)$ and the graph of $g(x) = \left(\frac{1}{3}\right)^x$ contained $(-1, 3)$. This makes sense as $a^{-1} = \frac{1}{a}$.

What is the domain and range for each function? From the graphs we can see that the domain is the set of all real numbers and we write the domain in interval notation as $(-\infty, \infty)$. Again, the graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Properties of the Graph of $f(x) = a^x$

Table 10.2.2

When $a > 1$		When $0 < a < 1$	
1\)">Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
1\)">Range	$(0, \infty)$	Range	$(0, \infty)$
1\)">x-intercept	none	x-intercept	none
1\)">y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
1\)">Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$	Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
1\)">Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
1\)">Basic Shape	increasing	Basic Shape	decreasing

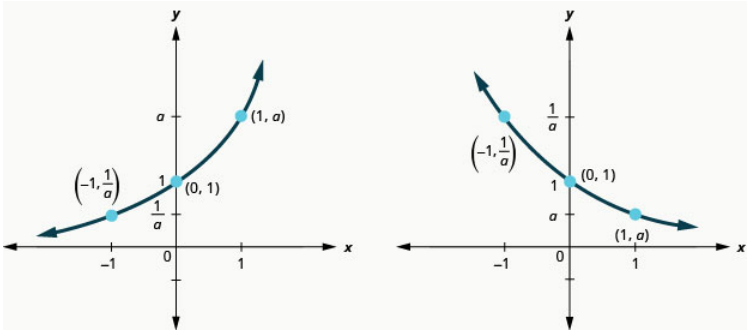

Figure 10.2.11

It is important for us to notice that both of these graphs are one-to-one, as they both pass the horizontal line test. This means the exponential function will have an inverse. We will look at this later.

When we graphed quadratic functions, we were able to graph using translation rather than just plotting points. Will that work in graphing exponential functions?

Example 20.3.3

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 2^{x+1}$.

Solution:

We will use point plotting to graph the functions.

x	$f(x) = 2^x$	$(x, f(x))$	$g(x) = 2^{x+1}$	$(x, g(x))$
–2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$2^{-2+1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-2, \frac{1}{2}\right)$
–1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$2^{-1+1} = 2^0 = 1$	(–1, 1)
0	$2^0 = 1$	(0, 1)	$2^{0+1} = 2^1 = 2$	(0, 2)
1	$2^1 = 2$	(1, 2)	$2^{1+1} = 2^2 = 4$	(1, 4)
2	$2^2 = 4$	(2, 4)	$2^{2+1} = 2^3 = 8$	(2, 8)
3	$2^3 = 8$	(3, 8)	$2^{3+1} = 2^4 = 16$	(3, 16)

Figure 10.2.12

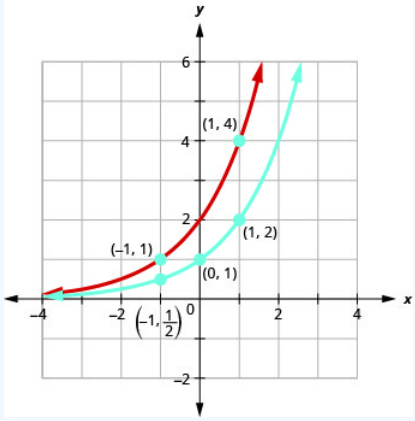

Figure 10.2.13

Exercise 20.3.5

On the same coordinate system, graph: $f(x) = 2^x$ and $g(x) = 2^{x-1}$.

Answer

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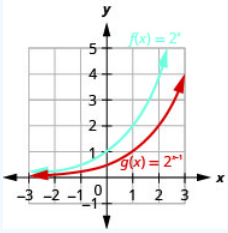

Figure 10.2.14

Exercise 20.3.6

On the same coordinate system, graph $f(x) = 3^x$ and $g(x) = 3^{x+1}$.

Answer

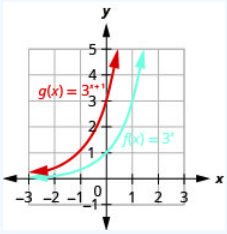

Figure 10.2.15

Looking at the graphs of the functions $f(x) = 2^x$ and $g(x) = 2^{x+1}$ in the last example, we see that adding one in the exponent caused a horizontal shift of one unit to the left. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

Let's now consider another situation that might be graphed more easily by translation, once we recognize the pattern.

Example 20.3.4

On the same coordinate system graph $f(x) = 3^x$ and $g(x) = 3^x - 2$.

Solution:

We will use point plotting to graph the functions.

x	$f(x) = 3^x$	$(x, g(x))$		$g(x) = 3^x - 2$	$(x, g(x))$
−2	$3^{-2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$		$3^{-2} - 2 = \frac{1}{9} - 2 = -\frac{17}{9}$	$\left(-2, -\frac{17}{9}\right)$
−1	$3^{-1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$		$3^{-1} - 2 = \frac{1}{3} - 2 = -\frac{5}{3}$	$\left(-1, -\frac{5}{3}\right)$
0	$3^0 = 1$	(0, 1)		$3^0 - 2 = 1 - 2 = -1$	(0, −1)
1	$3^1 = 3$	(1, 3)		$3^1 - 2 = 3 - 2 = 1$	(1, 1)
2	$3^2 = 9$	(2, 9)		$3^2 - 2 = 9 - 2 = 7$	(2, 8)

Figure 10.2.16

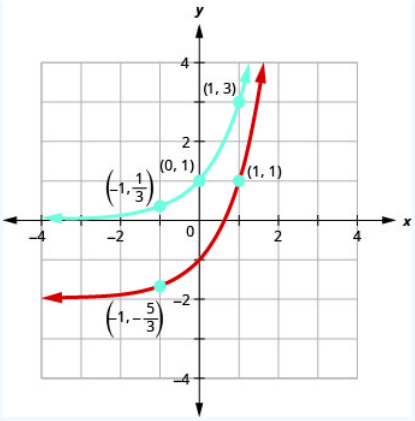

Figure 10.2.17

Exercise 20.3.7

On the same coordinate system, graph $f(x)=3^x$ and $g(x)=3^x+2$.

Answer

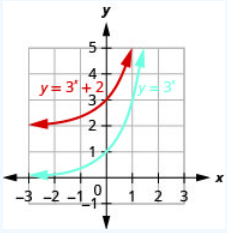

Figure 10.2.18

Exercise 20.3.8

On the same coordinate system, graph $f(x)=4^x$ and $g(x)=4^x-2$.

Answer

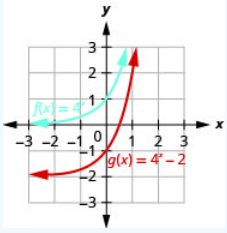

Figure 10.2.19

Looking at the graphs of the functions $f(x)=3^x$ and $g(x)=3^x-2$ in the last example, we see that subtracting 2 caused a vertical shift of down two units. Notice that the horizontal asymptote also shifted down 2 units. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

All of our exponential functions have had either an integer or a rational number as the base. We will now look at an exponential function with an irrational number as the base.

Before we can look at this exponential function, we need to define the irrational number, e. This number is used as a base in many applications in the sciences and business that are modeled by exponential functions. The number is defined as the value of $\left(1+\frac{1}{n}\right)^n$ as n gets larger and larger. We say, as n approaches infinity, or increases without bound. The table shows the value of $\left(1+\frac{1}{n}\right)^n$ for several values of n.

Table 10.2.3

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	$2.704813829\ldots$
$1,000$	$2.716923932\ldots$
$10,000$	$2.718145927\ldots$
$100,000$	$2.718268237\ldots$
$1,000,000$	$2.718280469\ldots$
$1,000,000,000$	$2.718281827\ldots$

$$e \approx 2.718281827$$

The number e is like the number π in that we use a symbol to represent it because its decimal representation never stops or repeats. The irrational number e is called the **natural base**.

Definition 20.3.2

Natural Base e

The number e is defined as the value of $\left(1+\frac{1}{n}\right)^n$, as n increases without bound. We say, as n approaches infinity,

$$e \approx 2.718281827$$

The exponential function whose base is e, $f(x) = e^x$ is called the **natural exponential function**.

Definition 20.3.3

Natural Exponential Function

The natural exponential function is an exponential function whose base is e

$$f(x) = e^x$$

The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Let's graph the function $f(x) = e^x$ on the same coordinate system as $g(x) = 2^x$ and $h(x) = 3^x$.

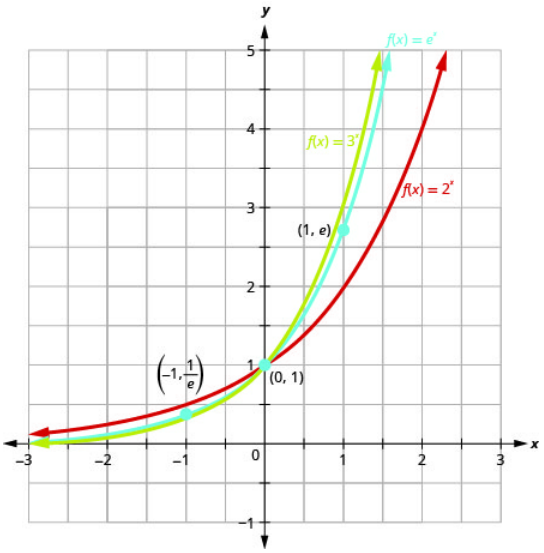

Figure 10.2.20

Notice that the graph of $f(x) = e^x$ is "between" the graphs of $g(x) = 2^x$ and $h(x) = 3^x$.Does this make sense as $2 < e < 3$?

Solve Exponential Equations

Equations that include an exponential expression a^x are called exponential equations. To solve them we use a property that says as long as $a > 0$ and $a \neq 1$, if $a^x = a^y$ then it is true that $x = y$. In other words, in an exponential equation, if the bases are equal then the exponents are equal.

Definition 20.3.4

One-to-One Property of Exponential Equations

For $a > 0$ and $a \neq 1$,

$$\text{If } a^x = a^y \text{,then } x = y.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

Example 20.3.5 How to Solve an Exponential Equation

Solve: $3^{2x-5} = 27$.

Solution:

Table 10.2.4

Step 1: Write both sides of the equation with the same base.	Since the left side has base 3, we write the right side with base 3. $27 = 3^3$	$3^{2x-5} = 27$ $3^{2x-5} = 3^3$
Step 2: Write a new equation by setting the exponents equal.	Since the bases are the same, the exponents must be equal.	$2x - 5 = 3$
Step 3: Solve the equation.	Add 5 to each side. Divide by 2.	$2x = 8$ $x = 4$
Step 4: Check the solution.	Substitute $x = 4$ into the original equation.	$3^{2x-5} = 27$ $3^{2\cdot 4-5} \stackrel{?}{=} 27$ $3^3 \stackrel{?}{=} 27$ $27 = 27$

Exercise 20.3.9

Solve: $3^{3x-2} = 81$.

Answer

$x = 2$

Exercise 20.3.10

Solve: $7^{x-3} = 7$.

Answer

$x = 4$

The steps are summarized below.

How to Solve an Exponential Function

1. Write both sides of the equation with the same base, if possible.
2. Write a new equation by setting the exponents equal.
3. Solve the equation.
4. Check the solution.

In the next example, we will use our properties on exponents.

Example 20.3.6

Solve $\frac{e^{x^2}}{e^3} = e^{2x}$.

Solution:

Table 10.2.5

	$\frac{e^{x^2}}{e^3} = e^{2x}$
Use the Property of Exponents: $\frac{a^m}{a^n} = a^{m-n}$.	$e^{x^2-3} = e^{2x}$
Write a new equation by setting the exponents equal.	$x^2 - 3 = 2x$
Solve the equation.	$x^2 - 2x - 3 = 0$
	$(x-3)(x+1) = 0$
	$x = 3, x = -1$
Check the solutions.	
$x = 3$ $\frac{e^{x^2}}{e^3} \stackrel{?}{=} e^{2x}$ $\frac{e^{3^2}}{e^3} \stackrel{?}{=} e^{2\cdot 3}$ $\frac{e^9}{e^3} \stackrel{?}{=} e^6$ $e^6 = e^6$ ✓	$x = -1$ $\frac{e^{x^2}}{e^3} \stackrel{?}{=} e^{2x}$ $\frac{e^{(-1)^2}}{e^3} \stackrel{?}{=} e^{2\cdot(-1)}$ $\frac{e^1}{e^3} \stackrel{?}{=} e^{-2}$ $e^{-2} = e^{-2}$ ✓

Exercise 20.3.11

Solve: $\frac{e^{x^2}}{e^x} = e^2$.

Answer

$x = -1, x = 2$

Exercise 20.3.12

Solve: $\frac{e^{x^2}}{e^x} = e^6$.

Answer

$x = -2, x = 3$

Use Exponential Models in Applications

Exponential functions model many situations. If you own a bank account, you have experienced the use of an exponential function. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, P, is invested at an interest rate, r, for t years, the new balance, A, will depend on how often the interest is compounded. If the interest is compounded n times a year we use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, we use the formula $A = Pe^{rt}$. These are the formulas for **compound interest**.

Definition 20.3.5

Compound Interest

For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{when compounded } n \text{ times a year.}$$
$$A = Pe^{rt} \quad \text{when compounded continuously.}$$

As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

Example 20.3.7

A total of $\$10,000$ was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Solution:

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.

$$A = ?$$
$$P = \$10,000$$
$$r = 0.05$$
$$t = 18 \text{ years}$$

a. For quarterly compounding, $n = 4$. There are 4 quarters in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10,000\left(1+\frac{0.05}{4}\right)^{4\cdot 18}$$

Compute the amount. Be careful to consider the order of operations as you enter the expression into your calculator.

$$A = \$24,459.20$$

b. For monthly compounding, $n = 12$.There are 12 months in a year.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10,000\left(1+\frac{0.05}{12}\right)^{12\cdot 18}$$

Compute the amount.

$$A = \$24,550.08$$

c. For compounding continuously,

$$A = Pe^{rt}$$

Substitute the values in the formula.

$$A = 10,000e^{0.05\cdot 18}$$

Compute the amount.

$$A = \$24,596.03$$

? Exercise 20.3.13

Angela invested $\$15,000$ in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Answer

a. $\$22,332.96$
b. $\$22,362.49$
c. $\$22,377.37$

? Exercise 20.3.14

Allan invested $\$10,000$ in a mutual fund. If the interest rate is 5%, how much will be in the account in 15 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Answer

a. $\$21,071.81$
b. $\$21,137.04$
c. $\$21,170.00$

Other topics that are modeled by exponential functions involve growth and decay. Both also use the formula $A = Pe^{rt}$ we used for the growth of money. For growth and decay, generally we use A_0, as the original amount instead of calling it P, the principal. We see that **exponential growth** has a positive rate of growth and **exponential decay** has a negative rate of growth.

Definition 20.3.6

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, r, for a certain time, t, the final amount, A, is:

$$A = A_0 e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

Example 20.3.8

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

Solution:

Identify the values of each variable in the formulas. Be sure to put the percent in decimal form. Be sure the units match--the rate is per hour and the time is in hours.

$$\begin{aligned} A &= ? \\ A_0 &= 100 \\ r &= 0.25/\text{ hour} \\ t &= 24 \text{ hours} \end{aligned}$$

Substitute the values in the formula: $A = A_0 e^{rt}$.

$$A = 100e^{0.25 \cdot 24}$$

Compute the amount.

$$A = 40,342.88$$

Round to the nearest whole virus.

$$A = 40,343$$

The researcher will find $40,343$ viruses.

Exercise 20.3.15

Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with 50 of the bacteria that grows at a rate of 15% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

Answer

She will find 166 bacteria.

Exercise 20.3.16

Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

Answer

She will find $1,102$ viruses.

Access these online resources for additional instruction and practice with evaluating and graphing exponential functions.

- Graphing Exponential Functions

- Solving Exponential Equations
- Applications of Exponential Functions
- Continuously Compound Interest
- Radioactive Decay and Exponential Growth

Key Concepts

- **Properties of the Graph of $f(x) = a^x$:**

Table 10.2.2

When $a > 1$		When $0 < a < 1$	
1\)">Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
1\)">Range	$(0, \infty)$	Range	$(0, \infty)$
1\)">x-intercept	none	x-intercept	none
1\)">y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
1\)">Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$	Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
1\)">Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
1\)">Basic Shape	increasing	Basic Shape	decreasing

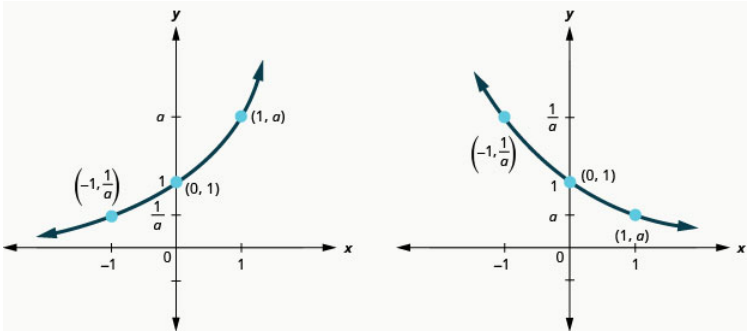

Figure 10.2.11

- **One-to-One Property of Exponential Equations:**
  For $a > 0$ and $a \neq 1$,

$$A = A_0 e^{rt}$$

- **How to Solve an Exponential Equation**
  1. Write both sides of the equation with the same base, if possible.
  2. Write a new equation by setting the exponents equal.
  3. Solve the equation.
  4. Check the solution.
- **Compound Interest:** For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is
  $A = P\left(1 + \frac{r}{n}\right)^{nt}$ when compounded n times a year.
  $A = Pe^{rt}$ when compounded continuously.
- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount, A, is $A = A_0 e^{rt}$.

Glossary

asymptote

A line which a graph of a function approaches closely but never touches.

exponential function

An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form $f(x) = a^x$.

natural base

The number e is defined as the value of $(1 + \frac{1}{n})^n$, as n gets larger and larger. We say, as n increases without bound, $e \approx 2.718281827\ldots$

natural exponential function

The natural exponential function is an exponential function whose base is e: $f(x) = e^x$. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

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20.3E: Exercises

Practice Makes Perfect

Exercise $20.3E.17$ Graph Exponential Functions

In the following exercises, graph each exponential function.

1. $f(x) = 2^x$
2. $g(x) = 3^x$
3. $f(x) = 6^x$
4. $g(x) = 7^x$
5. $f(x) = (1.5)^x$
6. $g(x) = (2.5)^x$
7. $f(x) = \left(\frac{1}{2}\right)^x$
8. $g(x) = \left(\frac{1}{3}\right)^x$
9. $f(x) = \left(\frac{1}{6}\right)^x$
10. $g(x) = \left(\frac{1}{7}\right)^x$
11. $f(x) = (0.4)^x$
12. $g(x) = (0.6)^x$

Answer

1.

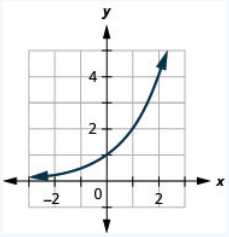

Figure 10.2.22

3.

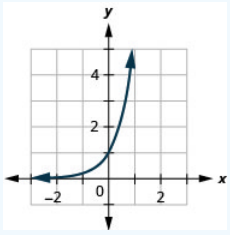

Figure 10.2.23

5.

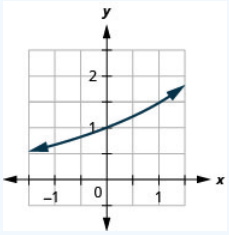

Figure 10.2.24

7.

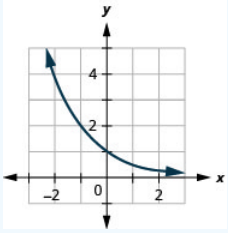

Figure 10.2.25

9.

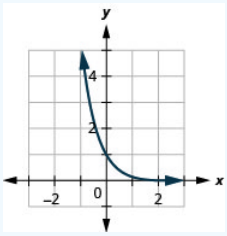

Figure 10.2.26

11.

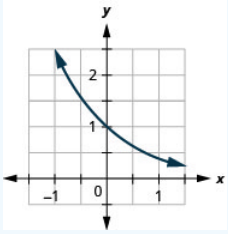

Figure 10.2.27

? Exercise $20.3E.18$ Graph Exponential Functions

In the following exercises, graph each function in the same coordinate system.

1. $f(x) = 4^x, g(x) = 4^{x-1}$
2. $f(x) = 3^x, g(x) = 3^{x-1}$
3. $f(x) = 2^x, g(x) = 2^{x-2}$
4. $f(x) = 2^x, g(x) = 2^{x+2}$
5. $f(x) = 3^x, g(x) = 3^x + 2$
6. $f(x) = 4^x, g(x) = 4^x + 2$
7. $f(x) = 2^x, g(x) = 2^x + 1$
8. $f(x) = 2^x, g(x) = 2^x - 1$

Answer

1.

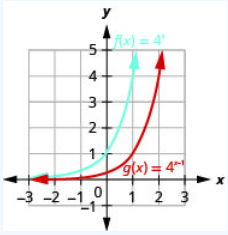

Figure 10.2.28

3.

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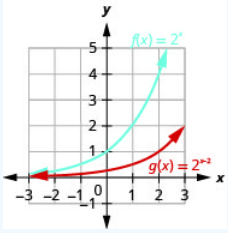

Figure 10.2.29

5.

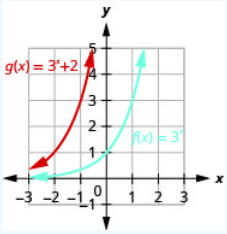

Figure 10.2.30

7.

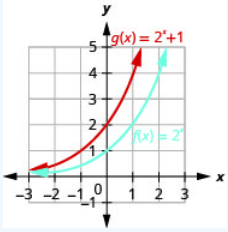

Figure 10.2.31

? Exercise $20.3E.19$ Graph Exponential Functions

In the following exercises, graph each exponential function.

1. $f(x) = 3^{x+2}$
2. $f(x) = 3^{x-2}$
3. $f(x) = 2^x + 3$
4. $f(x) = 2^x - 3$
5. $f(x) = \left(\frac{1}{2}\right)^{x-4}$
6. $f(x) = \left(\frac{1}{2}\right)^x - 3$
7. $f(x) = e^x + 1$
8. $f(x) = e^{x-2}$
9. $f(x) = -2^x$
10. $f(x) = 2^{-x-1} - 1$

Answer

1.

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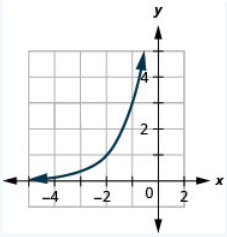

Figure 10.2.32

3.

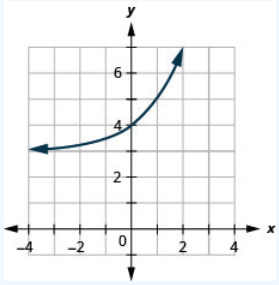

Figure 10.2.33

5.

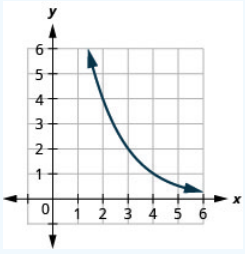

Figure 10.2.34

7.

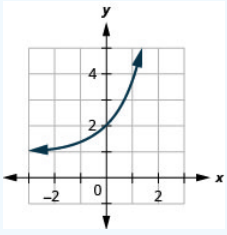

Figure 10.2.35

9.

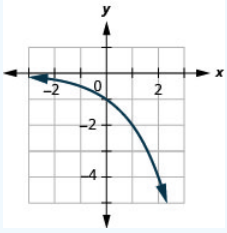

Figure 10.2.36

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Exercise $20.3E.20$ Solve Exponential Equations

In the following exercises, solve each equation.

1. $2^{3x-8} = 16$
2. $2^{2x-3} = 32$
3. $3^{x+3} = 9$
4. $3^{x^2} = 81$
5. $4^{x^2} = 4$
6. $4^{x} = 32$
7. $4^{x+2} = 64$
8. $4^{x+3} = 16$
9. $2^{x^2+2x} = \frac{1}{2}$
10. $3^{x^2-2x} = \frac{1}{3}$
11. $e^{3x} \cdot e^{4} = e^{10}$
12. $e^{2x} \cdot e^{3} = e^{9}$
13. $\frac{e^{x^2}}{e^2} = e^{x}$
14. $\frac{e^{x^2}}{e^3} = e^{2x}$

Answer

1. $x = 4$
3. $x = -1$
5. $x = -1, x = 1$
7. $x = 1$
9. $x = -1$
11. $x = 2$
13. $x = -1, x = 2$

Exercise $20.3E.21$ Solve Exponential Equations

In the following exercises, match the graphs to one of the following functions:

a. 2^{x}
b. 2^{x+1}
c. 2^{x-1}
d. $2^{x} + 2$
e. $2^{x} - 2$
f. 3^{x}

1.

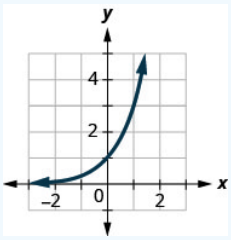

Figure 10.2.37

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2.

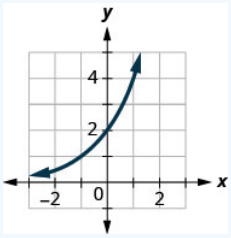

Figure 10.2.38

3.

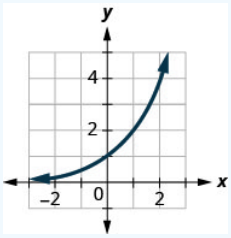

Figure 10.2.39

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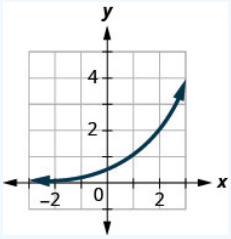

Figure 10.2.40

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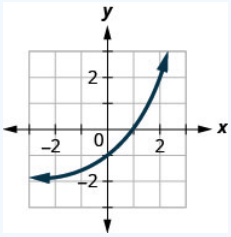

Figure 10.2.41

6.

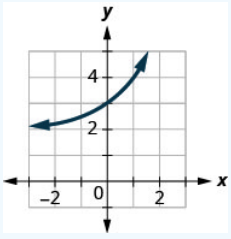

Figure 10.2.42

Answer

1. f
3. a
5. e

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Exercise $20.3E.22$ Use Exponential Models in Applications

In the following exercises, use an exponential model to solve.

1. Edgar accumulated $\$5,000$ in credit card debt. If the interest rate is 20% per year, and he does not make any payments for 2 years, how much will he owe on this debt in 2 years by each method of compounding?
   a. compound quarterly
   b. compound monthly
   c. compound continuously
2. Cynthia invested $\$12,000$ in a savings account. If the interest rate is 6%, how much will be in the account in 10 years by each method of compounding?
   a. compound quarterly
   b. compound monthly
   c. compound continuously
3. Rochelle deposits $\$5,000$ in an IRA. What will be the value of her investment in 25 years if the investment is earning 8% per year and is compounded continuously?
4. Nazerhy deposits $\$8,000$ in a certificate of deposit. The annual interest rate is 6% and the interest will be compounded quarterly. How much will the certificate be worth in 10 years?
5. A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. He starts his experiment with 100 of the bacteria that grows at a rate of 6% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?
6. A biologist is observing the growth pattern of a virus. She starts with 50 of the virus that grows at a rate of 20% per hour. She will check on the virus in 24 hours. How many viruses will she find?
7. In the last ten years the population of Indonesia has grown at a rate of 1.12% per year to $258,316,051$ If this rate continues, what will be the population in 10 more years?
8. In the last ten years the population of Brazil has grown at a rate of 0.9% per year to $205,823,665$ If this rate continues, what will be the population in 10 more years?

Answer

1.
   a. $\$7,387.28$
   b. $\$7,434.57$
   c. $\$7,459.12$
3. $\$36,945.28$
5. 223 bacteria
7. $288,929,825$

Exercise $20.3E.23$ Writing Exercises

1. Explain how you can distinguish between exponential functions and polynomial functions.
2. Compare and contrast the graphs of $y = x^2$ and $y = 2^x$.
3. What happens to an exponential function as the values of x decreases? Will the graph ever cross the x-axis? Explain.

Answer

1. Answers will vary
3. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph exponential eunctions.			
solve exponential equations.			
use exponential models in applications.			

Figure 10.2.43

b. After reviewing this checklist, what will you do to become confident for all objectives?

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20.4: Evaluate and Graph Logarithmic Functions

Learning Objectives

By the end of this section, you will be able to:

- Convert between exponential and logarithmic form
- Evaluate logarithmic functions
- Graph Logarithmic functions
- Solve logarithmic equations
- Use logarithmic models in applications

Before you get started, take this readiness quiz.

1. Solve: $x^2 = 81$.
   If you missed this problem, review Example 6.46.
2. Evaluate: 3^{-2}.
   If you missed this problem, review Example 5.15.
3. Solve: $2^4 = 3x - 5$.
   If you missed this problem, review Example 2.2.

We have spent some time finding the inverse of many functions. It works well to 'undo' an operation with another operation. Subtracting 'undoes' addition, multiplication 'undoes' division, taking the square root 'undoes' squaring.

As we studied the exponential function, we saw that it is one-to-one as its graphs pass the horizontal line test. This means an exponential function does have an inverse. If we try our algebraic method for finding an inverse, we run into a problem.

$$f(x) = a^x$$

Rewrite with $y = f(x)$.

$$y = a^x$$

Interchange the variables x and y.

$$x = a^y$$

Solve for y.

Oops! We have no way to solve for y!

To deal with this we define the logarithm function with base a to be the inverse of the exponential function $f(x) = a^x$. We use the notation $f^{-1}(x) = log_a x$ and say the inverse function of the exponential function is the logarithmic function.

Definition 20.4.1: Logarithmic Function

The function $f(x) = \log_a x$ is the **logarithmic function** with base a, where $a > 0, x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

Convert Between Exponential and Logarithmic Form

Since the equations $y = \log_a x$ and $x = a^y$ are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let's take a close look at the equations. See *Figure 10.3.1*. Notice the positions of the exponent and base.

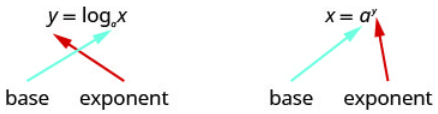


Figure 10.3.1

If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, “base to the exponent give us the number.”

Example 20.4.1

Convert to logarithmic form:

a. $2^3 = 8$
b. $5^{\frac{1}{2}} = \sqrt{5}$
c. $\left(\frac{1}{2}\right)^x = \frac{1}{16}$

Solution:

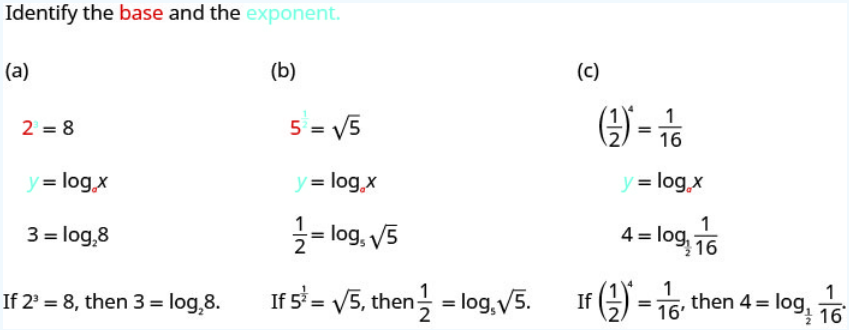


Figure 10.3.2

Exercise 20.4.1

Convert to logarithmic form:

a. $3^2 = 9$
b. $7^{\frac{1}{2}} = \sqrt{7}$
c. $\left(\frac{1}{3}\right)^x = \frac{1}{27}$

Answer

a. $\log_3 9 = 2$
b. $\log_7 \sqrt{7} = \frac{1}{2}$
c. $\log_{\frac{1}{3}} \frac{1}{27} = x$

Exercise 20.4.2

Convert to logarithmic form:

a. $4^3 = 64$
b. $4^{\frac{1}{3}} = \sqrt[3]{4}$
c. $\left(\frac{1}{2}\right)^x = \frac{1}{32}$

Answer

a. $\log_4 64 = 3$
b. $\log_4 \sqrt[3]{4} = \frac{1}{3}$
c. $\log_{\frac{1}{2}} \frac{1}{32} = x$

In the next example we do the reverse—convert logarithmic form to exponential form.

Example 20.4.2

Convert to exponential form:

a. $2 = \log_8 64$
b. $0 = \log_4 1$
c. $-3 = \log_{10} \frac{1}{1000}$

Solution:

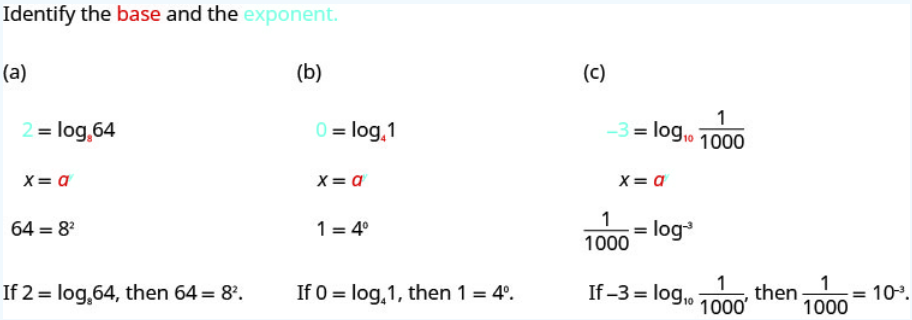


Figure 10.3.3

Exercise 20.4.3

Convert to exponential form:

a. $3 = \log_4 64$
b. $0 = \log_x 1$
c. $-2 = \log_{10} \frac{1}{100}$

Answer

a. $64 = 4^3$
b. $1 = x^0$
c. $\frac{1}{100} = 10^{-2}$

Exercise 20.4.4

Convert to exponential form:

a. $3 = \log_3 27$
b. $0 = \log_x 1$
c. $-1 = \log_{10} \frac{1}{10}$

Answer

a. $27 = 3^3$
b. $1 = x^0$
c. $\frac{1}{10} = 10^{-1}$

Evaluate Logarithmic Functions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

Example 20.4.3

Find the value of x:

a. $\log_x 36 = 2$
b. $\log_4 x = 3$
c. $\log_{\frac{1}{2}} \frac{1}{8} = x$

Solution:

a.

$$\log_x 36 = 2$$

Convert to exponential form.

$$x^2 = 36$$

Solve the quadratic.

$$x = 6, \quad x = \cancel{-6}$$

The base of a logarithmic function must be positive, so we eliminate $x = -6$.

$$x = 6 \quad \text{Therefore, } \log_6 36 = 2$$

b.

$$\log_4 x = 3$$

Convert to exponential form.

$$4^3 = x$$

Simplify.

$$x = 64 \quad \text{Therefore , } \log_4 64 = 3$$

c.

$$\log_{\frac{1}{2}} \frac{1}{8} = x$$

Convert to exponential form.

$$\left(\frac{1}{2}\right)^x = \frac{1}{8}$$

Rewrite $\frac{1}{8}$ as $\left(\frac{1}{2}\right)^3$.

$$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^3$$

With the same base, the exponents must be equal.

$$x = 3 \quad \text{Therefore } \log_{\frac{1}{2}} \frac{1}{8} = 3$$

Exercise 20.4.5

Find the value of x:

a. $\log_x 64 = 2$
b. $\log_5 x = 3$
c. $\log_{\frac{1}{2}} \frac{1}{4} = x$

Answer

a. $x = 8$
b. $x = 125$
c. $x = 2$

Exercise 20.4.6

Find the value of x:

a. $\log_x 81 = 2$
b. $\log_3 x = 5$
c. $\log_{\frac{1}{3}} \frac{1}{27} = x$

Answer

a. $x = 9$
b. $x = 243$
c. $x = 3$

When see an expression such as $log_3 27$, we can find its exact value two ways. By inspection we realize it means "3 to what power will be 27"? Since $3^3 = 27$, we know $log_3 27 = 3$. An alternate way is to set the expression equal to x and then convert it into an exponential equation.

Example 20.4.4

Find the exact value of each logarithm without using a calculator:

a. $\log_5 25$
b. $\log_9 3$
c. $\log_2 \frac{1}{16}$

Solution:

a.

$$\log_5 25$$

5 to what power will be 25?

$$\log_5 25 = 2$$

Or

Set the expression equal to x.

$$\log_5 25 = x$$

Change to exponential form.

$$5^x = 25$$

Rewrite 25 as 5^2.

$$5^x = 5^2$$

With the same base the exponents must be equal.

$$x = 2 \quad \text{Therefore , } \log_5 25 = 2.$$

b.

$$\log_9 3$$

Set the expression equal to x.

$$\log_9 3 = x$$

Change to exponential form.

$$9^x = 3$$

Rewrite 9 as 3^2.

$$\left(3^2\right)^x = 3^1$$

Simplify the exponents.

$$3^{2x} = 3^1$$

With the same base the exponents must be equal.

$$2x = 1$$

Solve the equation.

$$x = \tfrac{1}{2} \quad \text{Therefore , } \log_9 3 = \tfrac{1}{2}\,.$$

c.

$$\log_2 \tfrac{1}{16}$$

Set the expression equal to x.

$$\log_2 \tfrac{1}{16} = x$$

Change to exponential form.

$$2^x = \tfrac{1}{16}$$

Rewrite 16 as 2^4.

$$2^x = \tfrac{1}{2^4}$$

$$2^x = 2^{-4}$$

With the same base the exponents must be equal.

$$x = -4 \quad \text{Therefore , } \log_2 \tfrac{1}{16} = -4\,.$$

Exercise 20.4.7

Find the exact value of each logarithm without using a calculator:

a. $\log_{12} 144$
b. $\log_4 2$
c. $\log_2 \frac{1}{32}$

Answer

a. 2
b. $\frac{1}{2}$
c. -5

Exercise 20.4.8

Find the exact value of each logarithm without using a calculator:

a. $\log_9 81$
b. $\log_8 2$
c. $\log_3 \frac{1}{9}$

Answer

a. 2
b. $\frac{1}{3}$
c. -2

Graph Logarithmic Functions

To graph a logarithmic function $y = log_a x$, it is easiest to convert the equation to its exponential form, $x = a^y$. Generally, when we look for ordered pairs for the graph of a function, we usually choose an x-value and then determine its corresponding y-value. In this case you may find it easier to choose y-values and then determine its corresponding x-value.

Example 20.4.5

Graph $y = \log_2 x$.

Solution:

To graph the function, we will first rewrite the logarithmic equation, $y = \log_2 x$, in exponential form, $2^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of y and then get x.

Table 10.3.1

y	$2^y = x$	(x, y)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$(\frac{1}{4}, 2)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$(\frac{1}{2}, -1)$
0	$2^0 = 1$	$(1, 0)$
1	$2^1 = 2$	$(2, 1)$
2	$2^2 = 4$	$(4, 2)$
3	$2^3 = 8$	$(8, 3)$

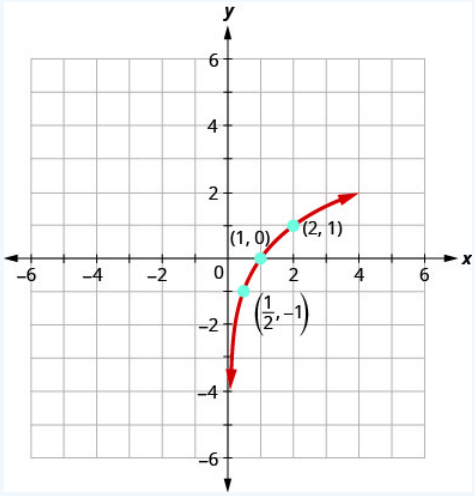


Figure 10.3.4

Exercise 20.4.9

Graph: $y = \log_3 x$.

Answer

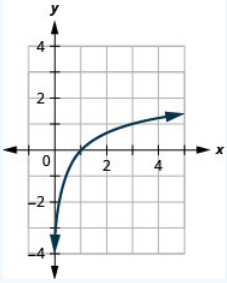


Figure 10.3.5

? Exercise 20.4.10

Graph: $y = \log_5 x$.

Answer

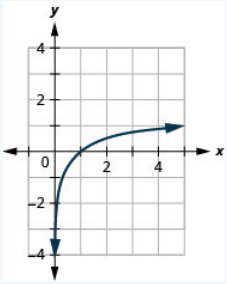


Figure 10.3.6

The graphs of $y = \log_2 x, y = \log_3 x$, and $y = \log_5 x$ are the shape we expect from a logarithmic function where $a > 1$.

We notice that for each function the graph contains the point $(1, 0)$. This make sense because $0 = log_a 1$ means $a^0 = 1$ which is true for any a.

The graph of each function, also contains the point $(a, 1)$. This makes sense as $1 = \log_a a$ means $a^1 = a$. which is true for any a.

Notice too, the graph of each function $y = \log_a x$ also contains the point $\left(\frac{1}{a}, -1\right)$. This makes sense as $-1 = \log_a \frac{1}{a}$ means $a^{-1} = \frac{1}{a}$, which is true for any a.

Look at each graph again. Now we will see that many characteristics of the logarithm function are simply 'mirror images' of the characteristics of the corresponding exponential function.

What is the domain of the function? The graph never hits the y-axis. The domain is all positive numbers. We write the domain in interval notation as $(0, \infty)$.

What is the range for each function? From the graphs we can see that the range is the set of all real numbers. There is no restriction on the range. We write the range in interval notation as $(-\infty, \infty)$.

When the graph approaches the y-axis so very closely but will never cross it, we call the line $x = 0$, the y-axis, a vertical asymptote.

Table 10.3.2: Properties of the Graph of $y = \log_a x$ When $a > 1$

Domain	$(0, \infty)$
Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$
y-intercept	None
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$

Asymptote	y-axis

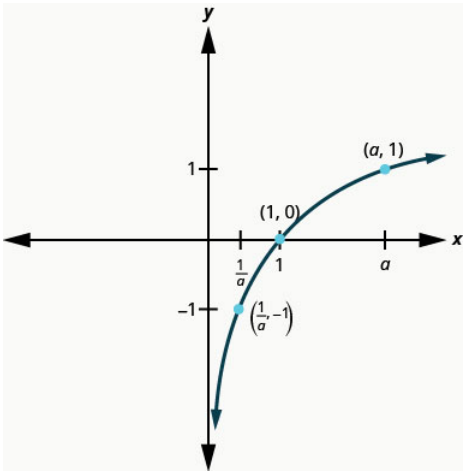


Figure 10.3.7

Our next example looks at the graph of $y = log_a x$ when $0 < a < 1$.

Example 20.4.6

Graph $y = \log_{\frac{1}{3}} x$.

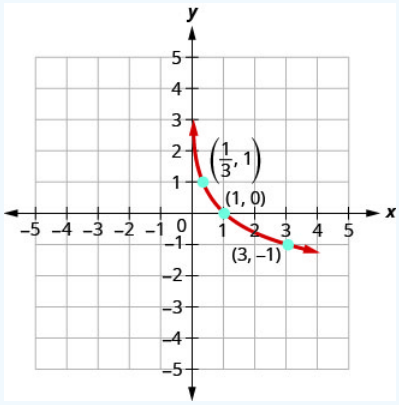
Solution:

To graph the function, we will first rewrite the logarithmic equation, $y = \log_{\frac{1}{3}} x$, in exponential form, $\left(\frac{1}{3}\right)^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of y and then get x.

Table 10.3.3

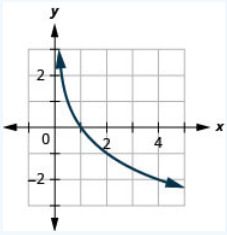
y	$\left(\frac{1}{3}\right)^y = x$	(x, y)
-2	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(9, -2)$
-1	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(3, -1)$
0	$\left(\frac{1}{3}\right)^0 = 1$	$(1, 0)$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(\frac{1}{3}, 1\right)$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(\frac{1}{9}, 2\right)$
3	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(\frac{1}{27}, 3\right)$



Exercise 20.4.11

Graph: $y = \log_{\frac{1}{2}} x$.

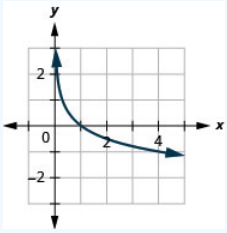
Answer



Exercise 20.4.12

Graph: $y = \log_{\frac{1}{4}} x$.

Answer



Now, let's look at the graphs $y = \log_{\frac{1}{2}} x, y = \log_{\frac{1}{3}} x$ and $y = \log_{\frac{1}{4}} x$, so we can identify some of the properties of logarithmic functions where $0 < a < 1$.

The graphs of all have the same basic shape. While this is the shape we expect from a logarithmic function where $0 < a < 1$.

We notice, that for each function again, the graph contains the points,$(1, 0), (a, 1), \left(\frac{1}{a}, -1\right)$. This make sense for the same reasons we argued above.

We notice the domain and range are also the same—the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$. The y-axis is again the vertical asymptote.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Table 10.3.4: Properties of the Graph of $y = \log_a x$

When $a > 1$		**When $0 < a < 1$**	
1\)">Domain	$(0, \infty)$	Domain	$(0, \infty)$
1\)">Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
1\)">x-intercept	$(1, 0)$	x-intercept	$(1, 0)$
1\)">y-intercept	None	y-intercept	None
1\)">Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
1\)">Asymptote	y -axis	Asymptote	y -axis
1\)">Basic shape	Increasing	Basic shape	Decreasing

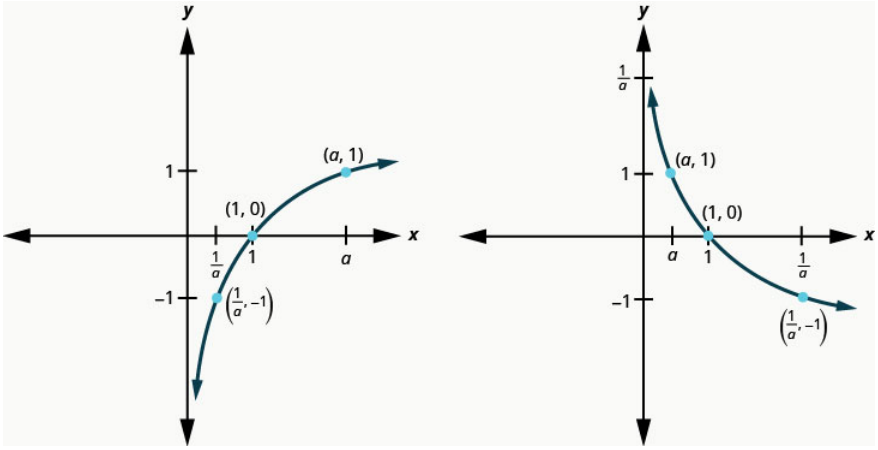


Figure 10.3.11

We talked earlier about how the logarithmic function $f^{-1}(x) = \log_a x$ is the inverse of the exponential function $f(x) = a^x$. The graphs in *Figure 10.3.12* show both the exponential (blue) and logarithmic (red) functions on the same graph for both $a > 1$ and $0 < a < 1$.

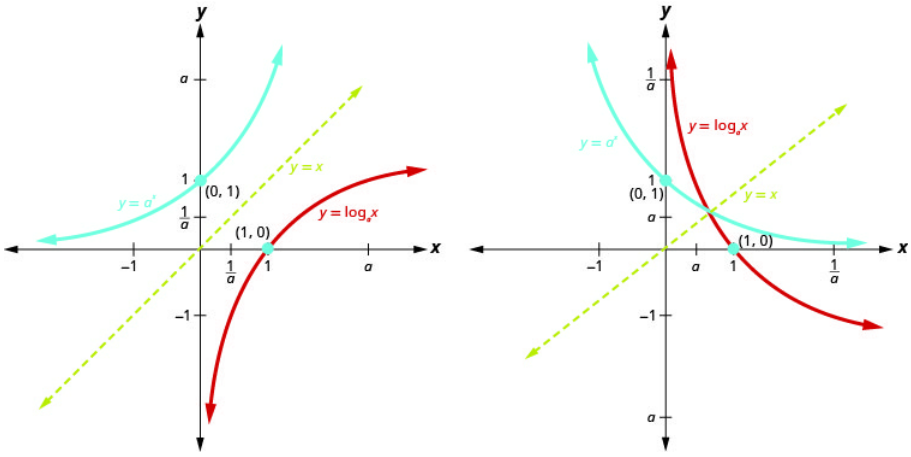


Figure 10.3.12

Notice how the graphs are reflections of each other through the line $y = x$. We know this is true of inverse functions. Keeping a visual in your mind of these graphs will help you remember the domain and range of each function. Notice the x-axis is the horizontal asymptote for the exponential functions and the y-axis is the vertical asymptote for the logarithmic functions.

Solve Logarithmic Equations

When we talked about exponential functions, we introduced the number e. Just as e was a base for an exponential function, it can be used a base for logarithmic functions too. The logarithmic function with base e is called the **natural logarithmic function**. The function $f(x) = \log_e x$ is generally written $f(x) = \ln x$ and we read it as "el en of x."

Definition 20.4.2: Natural Logarithmic Function

The function $f(x) = \ln x$ is the **natural logarithmic function** with base e, where $x > 0$.

$$y = \ln x \text{ is equivalent to } x = e^y$$

When the base of the logarithm function is 10, we call it the **common logarithmic function** and the base is not shown. If the base a of a logarithm is not shown, we assume it is 10.

Definition 20.4.3: Common Logarithmic Function

The function $f(x) = \log x$ is the **common logarithmic function** with base 10, where $x > 0$.

$$y = \log x \text{ is equivalent to } x = 10^y$$

It will be important for you to use your calculator to evaluate both common and natural logarithms.

Look for the log and ln keys on your calculator.

To solve logarithmic equations, one strategy is to change the equation to exponential form and then solve the exponential equation as we did before. As we solve logarithmic equations, $y = log_a x$, we need to remember that for the base a, $a > 0$ and $a \neq 1$. Also, the domain is $x > 0$. Just as with radical equations, we must check our solutions to eliminate any extraneous solutions.

Example 20.4.7

Solve:

a. $\log_a 49 = 2$
b. $\ln x = 3$

Solution:

a.

$$\log_a 49 = 2$$

Rewrite in exponential form.

$$a^2 = 49$$

Solve the equation using the square root property.

$$a = \pm 7$$

The base cannot be negative, so we eliminate $a = -7$.

$$a = 7, \quad \cancel{a = -7}$$

Check. $a = 7$

$$\log_a 49 = 2$$
$$\log_7 49 \stackrel{?}{=} 2$$
$$7^2 \stackrel{?}{=} 49$$
$$49 = 49$$

b.

$$\ln x = 3$$

Rewrite in exponential form.

$$e^3 = x$$

Check. $x = e^3$

$$\ln x = 3$$
$$\ln e^3 \stackrel{?}{=} 3$$
$$e^3 = e^3$$

Exercise 20.4.13

Solve:

a. $\log_a 121 = 2$
b. $\ln x = 7$

Answer

a. $a = 11$
b. $x = e^7$

? Exercise 20.4.14

Solve:

a. $\log_a 64 = 3$
b. $\ln x = 9$

Answer

a. $a = 4$
b. $x = e^9$

✓ Example 20.4.8

Solve:

a. $\log_2(3x - 5) = 4$
b. $\ln e^{2x} = 4$

Solution:

a.

$$\log_2(3x - 5) = 4$$

Rewrite in exponential form.

$$2^4 = 3x - 5$$

Simplify.

$$16 = 3x - 5$$

Solve the equation.

$$21 = 3x$$

$$7 = x$$

Check. $x = 7$

$$\begin{aligned} \log_2(3x - 5) &= 4 \\ \log_2(3 \cdot 7 - 5) &\stackrel{?}{=} 4 \\ \log_2(16) &\stackrel{?}{=} 4 \\ 2^4 &\stackrel{?}{=} 16 \\ 16 &= 16 \end{aligned}$$

b.

$$\ln e^{2x} = 4$$

Rewrite in exponential form.

$$e^4 = e^{2x}$$

Since the bases are the same the exponents are equal.

$$4 = 2x$$

Solve the equation.

$$2 = x$$

Check. $x = 2$

$$\ln e^{2x} = 4$$
$$\ln e^{2 \cdot 2} \stackrel{?}{=} 4$$
$$\ln e^{4} = 4$$
$$e^{4} = e^{4}$$

Exercise 20.4.15

Solve:

a. $\log_2(5x - 1) = 6$
b. $\ln e^{3x} = 6$

Answer

a. $x = 13$
b. $x = 2$

Exercise 20.4.16

Solve:

a. $\log_3(4x + 3) = 3$
b. $\ln e^{4x} = 4$

Answer

a. $x = 6$
b. $x = 1$

Use Logarithmic Models in Applications

There are many applications that are modeled by logarithmic equations. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The 10^{-12} in the formula represents the intensity of sound that is barely audible.

Definition 20.4.4

Decibel Level of Sound

The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is

$$D = 10 \log\left(\frac{I}{10^{-12}}\right)$$

Example 20.4.9

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity 10^{-2} watts per square inch?

Solution:

	$D = 10 \log\left(\frac{I}{10^{-12}}\right)$
Substitute in the intensity level, I.	$D = 10 \log\left(\frac{10^{-2}}{10^{-12}}\right)$
Simplify.	$D = 10 \log(10^{10})$
Since $\log 10^{10} = 10$.	$D = 10 \cdot 10$

Multiply.	$D = 100$
	The decibel level of music coming through earphones is 100 dB.

Exercise 20.4.17

What is the decibel level of one of the new quiet dishwashers with intensity 10^{-7} watts per square inch?

Answer

The quiet dishwashers have a decibel level of 50 dB.

Exercise 20.4.18

What is the decibel level heavy city traffic with intensity 10^{-3} watts per square inch?

Answer

The decibel level of heavy traffic is 90 dB.

The magnitude R of an earthquake is measured by a logarithmic scale called the Richter scale. The model is $R = \log I$, where I is the intensity of the shock wave. This model provides a way to measure **earthquake intensity**.

Definition 20.4.5: Earthquake Intensity

The magnitude R of an earthquake is measured by$R = \log I$, where I is the intensity of its shock wave.

Example 20.4.10

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. Over 80% of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured 5.1 on the Richter scale and caused $108 million dollars of damage. Compare the intensities of the two earthquakes.

Solution:

To compare the intensities, we first need to convert the magnitudes to intensities using the log formula. Then we will set up a ratio to compare the intensities.

Convert the magnitudes to intensities.

$$R = \log I$$

1906 earthquake

$$7.8 = \log I$$

Convert to exponential form.

$$I = 10^{7.8}$$

2014 earthquake

$$5.1 = \log I$$

Convert to exponential form.

$$I = 10^{5.1}$$

Form a ratio of the intensities.

$$\frac{\text{Intensity for 1906}}{\text{Intensity for 2014}}$$

Substitute in the values.

$$\frac{10^{7.8}}{10^{5.1}}$$

Divide by subtracting the exponents.

$$10^{2.7}$$

Evaluate.

$$501$$

The intensity of the 1906 earthquake was about 501 times the intensity of the 2014 earthquake.

? Exercise 20.4.19

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured 6.9 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

The intensity of the 1906 earthquake was about 8 times the intensity of the 1989 earthquake.

? Exercise 20.4.20

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

The intensity of the earthquake in Chile was about $1,259$ times the intensity of the earthquake in Los Angeles.

Access these online resources for additional instruction and practice with evaluating and graphing logarithmic functions.

- Re-writing logarithmic equations in exponential form
- Simplifying Logarithmic Expressions
- Graphing logarithmic functions
- Using logarithms to calculate decibel levels

Key Concepts

- **Properties of the Graph of $y = \log_a x$:**

When $a > 1$		**When $0 < a < 1$**	
1\)">Domain	$(0, \infty)$	Domain	$(0, \infty)$
1\)">Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
1\)">x-intercept	$(1, 0)$	x-intercept	$(1, 0)$
1\)">y-intercept	None	y-intercept	None
1\)">Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
1\)">Asymptote	y -axis	Asymptote	y -axis
1\)">Basic shape	Increasing	Basic shape	Decreasing

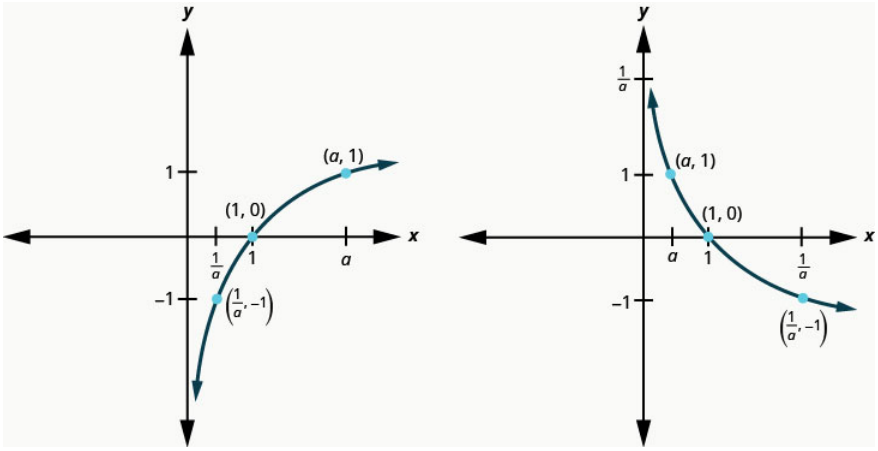


Figure 10.3.11

- **Decibel Level of Sound:** The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is $D = 10 \log\left(\frac{I}{10^{-12}}\right)$.
- **Earthquake Intensity:** The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

Glossary

common logarithmic function

The function $f(x) = \log x$ is the common logarithmic function with base 10, where $x > 0$.

$$y = \log x \text{ is equivalent to } x = 10^y$$

logarithmic function

The function $f(x) = \log_a x$ is the logarithmic function with base a, where $a > 0, x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

natural logarithmic function

The function $f(x) = \ln x$ is the natural logarithmic function with base e, where $x > 0$.

$$y = \ln x \text{ is equivalent to } x = e^y$$

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20.4E: Exercises

Section 10.3 Exercises

Practice Makes Perfect

Exercise $20.4E.21$ Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.

1. $4^2 = 16$
2. $2^5 = 32$
3. $3^3 = 27$
4. $5^3 = 125$
5. $10^3 = 1000$
6. $10^{-2} = \frac{1}{100}$
7. $x^{\frac{1}{2}} = \sqrt{3}$
8. $x^{\frac{1}{3}} = \sqrt[3]{6}$
9. $32^x = \sqrt[4]{32}$
10. $17^x = \sqrt[5]{17}$
11. $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$
12. $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$
13. $3^{-2} = \frac{1}{9}$
14. $4^{-3} = \frac{1}{64}$
15. $e^x = 6$
16. $e^3 = x$

Answer

2. $\log_2 32 = 5$
4. $\log_5 125 = 3$
6. $\log \frac{1}{100} = -2$
8. $\log_x \sqrt[3]{6} = \frac{1}{3}$
10. $\log_{17} \sqrt[5]{17} = x$
12. $\log_{\frac{1}{3}} \frac{1}{81} = 4$
14. $\log_4 \frac{1}{64} = -3$
16. $\ln x = 3$

Exercise $20.4E.22$ Convert Between Exponential and Logarithmic Form

In the following exercises, convert each logarithmic equation to exponential form.

1. $3 = \log_4 64$
2. $6 = \log_2 64$
3. $4 = \log_x 81$
4. $5 = \log_x 32$
5. $0 = \log_{12} 1$
6. $0 = \log_7 1$
7. $1 = \log_3 3$
8. $1 = \log_9 9$

9. $-4 = \log_{10} \frac{1}{10,000}$
10. $3 = \log_{10} 1,000$
11. $5 = \log_e x$
12. $x = \log_e 43$

Answer

2. $64 = 2^6$
4. $32 = x^5$
6. $1 = 7^0$
8. $9 = 9^1$
10. $1,000 = 10^3$
12. $43 = e^x$

Exercise $20.4E.23$ Evaluate Logarithmic Functions

In the following exercises, find the value of x in each logarithmic equation.

1. $\log_x 49 = 2$
2. $\log_x 121 = 2$
3. $\log_x 27 = 3$
4. $\log_x 64 = 3$
5. $\log_3 x = 4$
6. $\log_5 x = 3$
7. $\log_2 x = -6$
8. $\log_3 x = -5$
9. $\log_{\frac{1}{4}} \frac{1}{16} = x$
10. $\log_{\frac{1}{3}} \frac{1}{9} = x$
11. $\log_{\frac{1}{4}} 64 = x$
12. $\log_{\frac{1}{9}} 81 = x$

Answer

2. $x = 11$
4. $x = 4$
6. $x = 125$
8. $x = \frac{1}{243}$
10. $x = 2$
12. $x = -2$

Exercise $20.4E.24$ Evaluate Logarithmic Functions

In the following exercises, find the exact value of each logarithm without using a calculator.

1. $\log_7 49$
2. $\log_6 36$
3. $\log_4 1$
4. $\log_5 1$
5. $\log_{16} 4$
6. $\log_{27} 3$

7. $\log_{\frac{1}{2}} 2$
8. $\log_{\frac{1}{2}} 4$
9. $\log_2 \frac{1}{16}$
10. $\log_3 \frac{1}{27}$
11. $\log_4 \frac{1}{16}$
12. $\log_9 \frac{1}{81}$

Answer

2. 2

4. 0

6. $\frac{1}{3}$

8. -2

10. -3

12. -2

Exercise $20.4E.25$ Graph Logarithmic Functions

In the following exercises, graph each logarithmic function.

1. $y = \log_2 x$
2. $y = \log_4 x$
3. $y = \log_6 x$
4. $y = \log_7 x$
5. $y = \log_{1.5} x$
6. $y = \log_{2.5} x$
7. $y = \log_{\frac{1}{3}} x$
8. $y = \log_{\frac{1}{5}} x$
9. $y = \log_{0.4} x$
10. $y = \log_{0.6} x$

Answer

2.

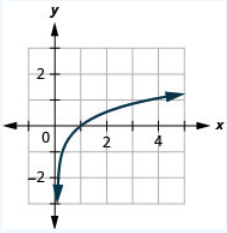

Figure 10.3.19

4.

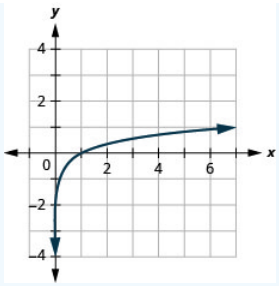

Figure 10.3.20

6.

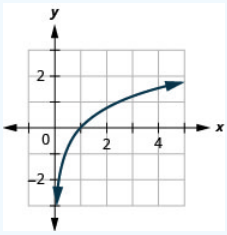

Figure 10.3.21

8.

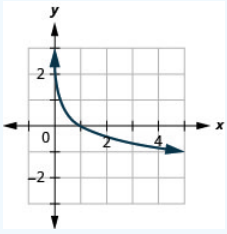

Figure 10.3.22

10.

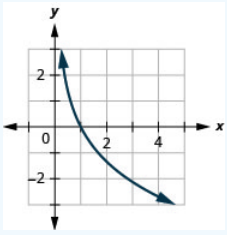

Figure 10.3.23

Exercise $20.4E.26$ Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

1. $\log_a 16 = 2$
2. $\log_a 81 = 2$
3. $\log_a 8 = 3$
4. $\log_a 27 = 3$
5. $\log_a 32 = 2$
6. $\log_a 24 = 3$
7. $\ln x = 5$
8. $\ln x = 4$
9. $\log_2(5x+1) = 4$

https://math.libretexts.org/@go/page/46473

10. $\log_2(6x+2)=5$
11. $\log_3(4x-3)=2$
12. $\log_3(5x-4)=4$
13. $\log_4(5x+6)=3$
14. $\log_4(3x-2)=2$
15. $\ln e^{4x}=8$
16. $\ln e^{2x}=6$
17. $\log x^2=2$
18. $\log\left(x^2-25\right)=2$
19. $\log_2\left(x^2-4\right)=5$
20. $\log_3\left(x^2+2\right)=3$

Answer

2. $a=9$

4. $a=3$

6. $a=\sqrt[3]{24}$

8. $x=e^4$

10. $x=5$

12. $x=17$

14. $x=6$

16. $x=3$

18. $x=-5\sqrt{5}, x=5\sqrt{5}$

20. $x=-5, x=5$

? Exercise $20.4E.27$ Use Logarithmic Models in Applications

In the following exercises, use a logarithmic model to solve.

1. What is the decibel level of normal conversation with intensity 10^{-6} watts per square inch?
2. What is the decibel level of a whisper with intensity 10^{-10} watts per square inch?
3. What is the decibel level of the noise from a motorcycle with intensity 10^{-2} watts per square inch?
4. What is the decibel level of the sound of a garbage disposal with intensity 10^{-2} watts per square inch?
5. In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2010, Haiti also experienced an intense earthquake which measured 7.0 on the Richter scale. Compare the intensities of the two earthquakes.
6. The Los Angeles area experiences many earthquakes. In 1994, the Northridge earthquake measured magnitude of 6.7 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

2. A whisper has a decibel level of 20 dB.

4. The sound of a garbage disposal has a decibel level of 100 dB.

6. The intensity of the 1994 Northridge earthquake in the Los Angeles area was about 40 times the intensity of the 2014 earthquake.

Exercise $20.4E.28$ Writing Exercises

1. Explain how to change an equation from logarithmic form to exponential form.
2. Explain the difference between common logarithms and natural logarithms.
3. Explain why $\log_a a^x = x$.
4. Explain how to find the $\log_7 32$ on your calculator.

Answer

2. Answers may vary

4. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
convert between exponential and logarithmic form.			
evaluate logarithmic functions.			
graph logarithmic functions.			
solve logarithmic functions.			
use logarithmic models in applications.			

Figure 10.3.24

b. After reviewing this checklist, what will you do to become confident for all objectives?

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20.5: Use the Properties of Logarithms

Learning Objectives

By the end of this section, you will be able to:

- Use the properties of logarithms
- Use the Change of Base Formula

Before you get started, take this readiness quiz.

1. Evaluate: a. a^0 b. a^1.
   If you missed this problem, review Example 5.14.
2. Write with a rational exponent: $\sqrt[3]{x^2y}$.
   If you missed this problem, review Example 8.27.
3. Round to three decimal places: 2.5646415
   If you missed this problem, review Example 1.34.

Use the Properties of Logarithms

Now that we have learned about exponential and logarithmic functions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since $a^0 = 1$, we can convert this to logarithmic form and get $\log_a 1 = 0$. Also, since $a^1 = a$, we get $\log_a a = 1$.

Definition 20.5.1

Properties of Logarithms

$$\log_a 1 = 0 \quad \log_a a = 1$$

In the next example we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

Example 20.5.1

Evaluate using the properties of logarithms:

a. $\log_8 1$
b. $\log_6 6$

Solution:

a.

$$\log_8 1$$

Use the property, $\log_a 1 = 0$.

$$0 \quad \log_8 1 = 0$$

b.

$$\log_6 6$$

Use the property, $\log_a a = 1$.

$$1 \quad \log_6 6 = 1$$

Exercise 20.5.1

Evaluate using the properties of logarithms:

a. $\log_{13} 1$
b. $\log_9 9$

Answer

a. 0
b. 1

Exercise 20.5.2

Evaluate using the properties of logarithms:

a. $\log_5 1$
b. $\log_7 7$

Answer

a. 0
b. 1

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation $a^{\log_a x} = x$ converts to the logarithmic equation $\log_a x = \log_a x$, which is a true statement for positive values for x only.

The logarithmic equation $\log_a a^x = x$ converts to the exponential equation $a^x = a^x$, which is also a true statement.

These two properties are called inverse properties because, when we have the same base, raising to a power "undoes" the log and taking the log "undoes" raising to a power. These two properties show the **composition** of functions. Both ended up with the identity function which shows again that the exponential and logarithmic functions are inverse functions.

Definition 20.5.2

Inverse Properties of Logarithms

For $a > 0, x > 0$ and $a \neq 1$,

$$a^{\log_a x} = x \quad \log_a a^x = x$$

In the next example, apply the inverse properties of logarithms.

Example 20.5.2

Evaluate using the properties of logarithms:

a. $4^{\log_4 9}$
b. $\log_3 3^5$

Solution:

a.

$$4^{\log_4 9}$$

Use the property, $a^{\log_a x} = x$.

$$9 \quad 4^{\log_4 9} = 9$$

b.

$$\log_3 3^5$$

Use the property, $a^{\log_a x} = x$.

$$5 \quad \log_3 3^5 = 5$$

Exercise 20.5.3

Evaluate using the properties of logarithms:

a. $5^{\log_5 15}$
b. $\log_7 7^4$

Answer

a. 15
b. 4

Exercise 20.5.4

Evaluate using the properties of logarithms:

a. $2^{\log_2 8}$
b. $\log_2 2^{15}$

Answer

a. 8
b. 15

There are three more properties of logarithms that will be useful in our work. We know exponential functions and logarithmic function are very interrelated. Our definition of logarithm shows us that a logarithm is the exponent of the equivalent exponential. The properties of exponents have related properties for exponents.

In the Product Property of Exponents, $a^m \cdot a^n = a^{m+n}$, we see that to multiply the same base, we add the exponents. The **Product Property of Logarithms**, $\log_a M \cdot N = \log_a M + \log_a N$ tells us to take the log of a product, we add the log of the factors.

Definition 20.5.3

Product Property of Logarithms

If $M > 0, N > 0$, a > 0 and a $\neq 1$, then

$$\log_a (M \cdot N) = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

Example 20.5.3

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_3 7x$
b. $\log_4 64xy$

Solution:

a.

$$\log_3 7x$$

Use the Product Property, $\log_a (M \cdot N) = \log_a M + \log_a N$.

$$\log_3 7 + \log_3 x$$
$$\log_3 7x = \log_3 7 + \log_3 x$$

b.

$$\log_4 64xy$$

Use the Product Property, $\log_a (M \cdot N) = \log_a M + \log_a N$.

$$\log_4 64 + \log_4 x + \log_4 y$$

Simplify be evaluating, $\log_4 64$.

$$3 + \log_4 x + \log_4 y$$
$$\log_4 64xy = 3 + \log_4 x + \log_4 y$$

? Exercise 20.5.5

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_3 3x$
b. $\log_2 8xy$

Answer

a. $1 + \log_3 x$
b. $3 + \log_2 x + \log_2 y$

? Exercise 20.5.6

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_9 9x$
b. $\log_3 27xy$

Answer

a. $1 + \log_9 x$
b. $3 + \log_3 x + \log_3 y$

Similarly, in the Quotient Property of Exponents, $\frac{a^m}{a^n} = a^{m-n}$, we see that to divide the same base, we subtract the exponents. The **Quotient Property of Logarithms**, $\log_a \frac{M}{N} = \log_a M - \log_a N$ tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

Definition 20.5.4

Quotient Property of Logarithms

If $M > 0, N > 0,$ a > 0 and a $\neq 1$, then

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

Note that $\log_a M = \log_a N \neq \log_a (M - N)$.

We use this property to write the log of a quotient as a difference of the logs of each factor.

Example 20.5.4

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_5 \frac{5}{7}$
b. $\log \frac{x}{100}$

Solution:

a.

$$\log_5 \tfrac{5}{7}$$

Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.

$$\log_5 5 - \log_5 7$$

Simplify.

$$1 - \log_5 7$$

$$\log_5 \tfrac{5}{7} = 1 - \log_5 7$$

b.

$$\log \tfrac{x}{100}$$

Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.

$$\log x - \log 100$$

Simplify.

$$\log x - 2$$

$$\log \tfrac{x}{100} = \log x - 2$$

Exercise 20.5.7

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_4 \frac{3}{4}$
b. $\log \frac{x}{1000}$

Answer

a. $\log_4 3 - 1$
b. $\log x - 3$

Exercise 20.5.8

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_2 \frac{5}{4}$
b. $\log \frac{10}{y}$

Answer

a. $\log_2 5 - 2$
b. $1 - \log y$

The third property of logarithms is related to the Power Property of Exponents, $(a^m)^n = a^{m \cdot n}$, we see that to raise a power to a power, we multiply the exponents. The **Power Property of Logarithms**, $\log_a M^p = p \log_a M$ tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

Definition 20.5.5

Power Property of Logarithms

If $M > 0, \mathrm{a} > 0, \mathrm{a} \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power as the product product of the power times the log of the number.

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. We essentially take the exponent and throw it in front of the logarithm.

Example 20.5.5

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_5 4^3$
b. $\log x^{10}$

Solution:

a.

$$\log_5 4^3$$

Use the Power Property, $\log_a M^p = p \log_a M$.

$$3 \log_5 4$$

$$\log_5 4^3 = 3 \log_5 4$$

b.

$$\log x^{10}$$

Use the Power Property, $\log_a M^p = p \log_a M$.

$$10 \log x$$

$$\log x^{10} = 10 \log x$$

Exercise 20.5.9

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_7 5^4$
b. $\log x^{100}$

Answer

a. $4 \log_7 5$
b. $100 \cdot \log x$

Exercise 20.5.10

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_2 3^7$
b. $\log x^{20}$

Answer

a. $7 \log_2 3$
b. $20 \cdot \log x$

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

Properties of Logarithms

If $M > 0, \text{a} > 0, \text{a} \neq 1$ and p is any real number then,

Table 10.4.1

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

Now that we have the properties we can use them to "expand" a logarithmic expression. This means to write the logarithm as a sum or difference and without any powers.

We generally apply the Product and Quotient Properties before we apply the Power Property.

Example 20.5.6

Use the Properties of Logarithms to expand the logarithm $\log_4 \left(2x^3y^2\right)$. Simplify, if possible.

Solution:

Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$.

Use the Power Property, $\log_a M^p = p \log_a M$, on the last two terms. Simplify.

Exercise 20.5.11

Use the Properties of Logarithms to expand the logarithm $\log_2 \left(5x^4y^2\right)$. Simplify, if possible.

Answer

$\log_2 5 + 4 \log_2 x + 2 \log_2 y$

Exercise 20.5.12

Use the Properties of Logarithms to expand the logarithm $\log_3 \left(7x^5y^3\right)$. Simplify, if possible.

Answer

$\log_3 7 + 5 \log_3 x + 3 \log_3 y$

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

Example 20.5.7

Use the Properties of Logarithms to expand the logarithm $\log_2 \sqrt[4]{\frac{x^3}{3y^2z}}$. Simplify, if possible.

Solution

$$\log_2 \sqrt[4]{\frac{x^3}{3y^2z}}$$

Rewrite the radical with a rational exponent.

$$\log_2 \left(\frac{x^3}{3y^2z}\right)^{\frac{1}{4}}$$

Use the Power Property, $\log_a M^p = p \log_a M$.

$$\frac{1}{4}\log_2 \left(\frac{x^3}{3y^2z}\right)$$

Use the Quotient Property, $\log_a M \cdot N = \log_a M - \log_a N$.

$$\frac{1}{4}\left(\log_2\left(x^3\right) - \log_2\left(3y^2z\right)\right)$$

Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$, in the second term.

$$\frac{1}{4}\left(\log_2\left(x^3\right) - \left(\log_2 3 + \log_2 y^2 + \log_2 z\right)\right)$$

Use the Power Property, $\log_a M^p = p \log_a M$, inside the parentheses.

$$\frac{1}{4}(3\log_2 x - (\log_2 3 + 2\log_2 y + \log_2 z))$$

Simplify by distributing.

$$\frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z)$$

$$\log_2 \sqrt[4]{\frac{x^3}{3y^2z}} = \frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z)$$

? Exercise 20.5.13

Use the Properties of Logarithms to expand the logarithm $\log_4 \sqrt[5]{\frac{x^4}{2y^3z^2}}$. Simplify, if possible.

Answer

$\frac{1}{5}\left(4\log_4 x - \frac{1}{2} - 3\log_4 y - 2\log_4 z\right)$

? Exercise 20.5.14

Use the Properties of Logarithms to expand the logarithm $\log_3 \sqrt[3]{\frac{x^2}{5yz}}$. Simplify, if possible.

Answer

$\frac{1}{3}(2\log_3 x - \log_3 5 - \log_3 y - \log_3 z)$

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

✓ Example 20.5.8

Use the Properties of Logarithms to condense the logarithm $\log_4 3 + \log_4 x - \log_4 y$. Simplify, if possible.

Solution:

The log expressions all have the same base, 4.

The first two terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M : N$.

Since the logs are subtracted, we use the Quotient Property, $\log_a M - \log_a N = \log_a \frac{M}{N}$.

Exercise 20.5.15

Use the Properties of Logarithms to condense the logarithm $\log_2 5 + \log_2 x - \log_2 y$. Simplify, if possible.

Answer

$\log_2 \frac{5x}{y}$

Exercise 20.5.16

Use the Properties of Logarithms to condense the logarithm $\log_3 6 - \log_3 x - \log_3 y$. Simplify, if possible.

Answer

$\log_3 \frac{6}{xy}$

Example 20.5.9

Use the Properties of Logarithms to condense the logarithm $2\log_3 x + 4\log_3(x+1)$. Simplify, if possible.

Solution:

The log expressions have the same base, 3.

$$2\log_3 x + 4\log_3(x+1)$$

Use the Power Property, $\log_a M + \log_a N = \log_a M \cdot N$.

$$\log_3 x^2 + \log_3(x+1)^4$$

The terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.

$$\log_3 x^2(x+1)^4$$

$$2\log_3 x + 4\log_3(x+1) = \log_3 x^2(x+1)^4$$

Exercise 20.5.17

Use the Properties of Logarithms to condense the logarithm $3\log_2 x + 2\log_2(x-1)$. Simplify, if possible.

Answer

$\log_2 x^3(x-1)^2$

Exercise 20.5.18

Use the Properties of Logarithms to condense the logarithm $2\log x + 2\log(x+1)$. Simplify, if possible.

Answer

$\log x^2(x+1)^2$

Use the Change-of-Base Formula

To evaluate a logarithm with any other base, we can use the **Change-of-Base Formula**. We will show how this is derived.

Suppose we want to evaluate $\log_a M$	$\log_a M$
Let $y = \log_a M$.	$y = \log_a M$
Rewrite the epression in exponential form.	$a^y = M$
Take the $\log_b$ of each side.	$\log_b a^y = \log_b M$
Use the Power Property.	$y \log_b a = \log_b M$
Solve for y.	$y = \frac{\log_b M}{\log_b a}$
Substiture $y = \log_a M$.	$\log_a M = \frac{\log_b M}{\log_b a}$

The Change-of-Base Formula introduces a new base b. This can be any base b we want where $b > 0, b \neq 1$. Because our calculators have keys for logarithms base 10 and base e, we will rewrite the Change-of-Base Formula with the new base as 10 or e.

Definition 20.5.6

Change-of-Base Formula

For any logarithmic bases a, b and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \quad \log_a M = \frac{\log M}{\log a} \quad \log_a M = \frac{\ln M}{\ln a}$$
$$\text{new base } b \qquad \text{new base } 10 \qquad \text{new base } e$$

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol ($\approx$).

Example 20.5.10

Rounding to three decimal places, approximate $\log_4 35$.

Solution:

Table 10.4.2

	$\log_4 35$
Use the Change-of-Base Formula.	$\log_a M = \frac{\log_b M}{\log_b a}$
Identify a and M. Choose 10 for b.	$\log_4 35 = \frac{\log 35}{\log 4}$
Enter the expression $\frac{\log 35}{\log 4}$ in the calculator using the log button for base 10. Round to three decimal places.	$\log_4 35 \approx 2.565$

Exercise 20.5.19

Rounding to three decimal places, approximate $\log_3 42$.

Answer

3.402

Exercise 20.5.20

Rounding to three decimal places, approximate $\log_5 46$.

Answer

2.379

Access these online resources for additional instruction and practice with using the properties of logarithms.

- Using Properties of Logarithms to Expand Logs
- Using Properties of Logarithms to Condense Logs
- Change of Base

Key Concepts

- $\log_a 1 = 0 \quad \log_a a = 1$
- **Inverse Properties of Logarithms**
  - For $a > 0, x > 0$ and $a \neq 1$

$$a^{\log_a x} = x \quad \log_a a^x = x$$

- **Product Property of Logarithms**
  - If $M > 0, N > 0, a > 0$ and $a \neq 1$, then,

$$\log_a M \cdot N = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

- **Quotient Property of Logarithms**
  - If $M > 0, N > 0, \mathrm{a} > 0$ and $a \neq 1$, then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

- **Power Property of Logarithms**
  - If $M > 0, a > 0, a \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power is the product of the power times the log of the number.

- **Properties of Logarithms Summary**
  If $M > 0, a > 0, a \neq 1$ and p is any real number then,

Table 10.4.1

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

- **Change-of-Base Formula**
  For any logarithmic bases a and b, and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \quad \log_a M = \frac{\log M}{\log a} \quad \log_a M = \frac{\ln M}{\ln a}$$

new base b new base 10 new base e

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20.5E: Exercises

Practice Makes Perfect

Exercise $20.5E.21$ Use the Properties of Logarithms

In the following exercises, use the properties of logarithms to evaluate.

1. a. $\log_4 1$
   b. $\log_8 8$
2. a. $\log_{12} 1$
   b. $\ln e$
3. a. $3^{\log_3 6}$
   b. $\log_2 2^7$
4. a. $5^{\log_5 10}$
   b. $\log_4 4^{10}$
5. a. $8^{\log_8 7}$
   b. $\log_6 6^{-2}$
6. a. $6^{\log_6 15}$
   b. $\log_8 8^{-4}$
7. a. $10^{\log \sqrt{5}}$
   b. $\log 10^{-2}$
8. a. $10^{\log \sqrt{3}}$
   b. $\log 10^{-1}$
9. a. $e^{\ln 4}$
   b. $\ln e^2$
10. a. $e^{\ln 3}$
    b. $\ln e^7$

Answer

2.
   a. 0
   b. 1

4.
   a. 10
   b. 10

6.
   a. 15
   b. -4

8.
   a. $\sqrt{3}$
   b. -1

10.
   a. 3
   b. 7

https://math.libretexts.org/@go/page/46475

Exercise $20.5E.22$ Use the Properties of Logarithms

In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

1. $\log_4 6x$
2. $\log_5 8y$
3. $\log_2 32xy$
4. $\log_3 81xy$
5. $\log 100x$
6. $\log 1000y$

Answer

2. $\log_5 8 + \log_5 y$
4. $4 + \log_3 x + \log_3 y$
6. $3 + \log y$

Exercise $20.5E.23$ Use the Properties of Logarithms

In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

1. $\log_3 \frac{3}{8}$
2. $\log_6 \frac{5}{6}$
3. $\log_4 \frac{16}{y}$
4. $\log_5 \frac{125}{x}$
5. $\log \frac{x}{10}$
6. $\log \frac{10{,}000}{y}$
7. $\ln \frac{e^3}{3}$
8. $\ln \frac{e^4}{16}$

Answer

2. $\log_6 5 - 1$
4. $3 - \log_5 x$
6. $4 - \log y$
8. $4 - \ln 16$

Exercise $20.5E.24$ Use the Properties of Logarithms

In the following exercises, use the Power Property of Logarithms to expand each. Simplify if possible.

1. $\log_3 x^2$
2. $\log_2 x^5$
3. $\log x^{-2}$
4. $\log x^{-3}$
5. $\log_4 \sqrt{x}$
6. $\log_5 \sqrt[3]{x}$
7. $\ln x^{\sqrt{3}}$
8. $\ln x^{\sqrt[3]{4}}$

Answer

2. $5\log_2 x$

4. $-3\log x$

6. $\frac{1}{3}\log_5 x$

8. $\sqrt[3]{4}\ln x$

Exercise $20.5E.25$ Use the Properties of Logarithms

In the following exercises, use the Properties of Logarithms to expand the logarithm. Simplify if possible.

1. $\log_5\left(4x^6y^4\right)$
2. $\log_2\left(3x^5y^3\right)$
3. $\log_3\left(\sqrt{2}x^2\right)$
4. $\log_5\left(\sqrt[4]{21}y^3\right)$
5. $\log_3 \frac{xy^2}{z^2}$
6. $\log_5 \frac{4ab^3c^4}{d^2}$
7. $\log_4 \frac{\sqrt{x}}{16y^4}$
8. $\log_3 \frac{\sqrt[3]{x^2}}{27y^4}$
9. $\log_2 \frac{\sqrt{2x+y^2}}{z^2}$
10. $\log_3 \frac{\sqrt{3x+2y^2}}{5z^2}$
11. $\log_2 \sqrt[4]{\frac{5x^3}{2y^2z^4}}$
12. $\log_5 \sqrt[3]{\frac{3x^2}{4y^3z}}$

Answer

2. $\log_2 3+5\log_2 x+3\log_2 y$

4. $\frac{1}{4}\log_5 21+3\log_5 y$

6. $\log_5 4+\log_5 a+3\log_5 b+4\log_5 c-2\log_5 d$

8. $\frac{2}{3}\log_3 x-3-4\log_3 y$

10. $\frac{1}{2}\log_3\left(3x+2y^2\right)-\log_3 5-2\log_3 z$

12. $\frac{1}{3}(\log_5 3+2\log_5 x-\log_5 4-3\log_5 y-\log_5 z)$

Exercise $20.5E.26$ Use the Properties of Logarithms

In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.

1. $\log_6 4+\log_6 9$
2. $\log 4+\log 25$
3. $\log_2 80-\log_2 5$
4. $\log_3 36-\log_3 4$
5. $\log_3 4+\log_3(x+1)$
6. $\log_2 5-\log_2(x-1)$
7. $\log_7 3+\log_7 x-\log_7 y$
8. $\log_5 2-\log_5 x-\log_5 y$
9. $4\log_2 x+6\log_2 y$
10. $6\log_3 x+9\log_3 y$
11. $\log_3\left(x^2-1\right)-2\log_3(x-1)$
12. $\log\left(x^2+2x+1\right)-2\log(x+1)$

13. $4\log x - 2\log y - 3\log z$
14. $3\ln x + 4\ln y - 2\ln z$
15. $\frac{1}{3}\log x - 3\log(x+1)$
16. $2\log(2x+3) + \frac{1}{2}\log(x+1)$

Answer

2. 2

4. 2

6. $\log_2 \frac{5}{x-1}$

8. $\log_5 \frac{2}{xy}$

10. $\log_3 x^6 y^9$

12. 0

14. $\ln \frac{x^3 y^4}{z^2}$

16. $\log(2x+3)^2 \cdot \sqrt{x+1}$

Exercise $20.5E.27$ Use the Change-of-Base Formula

In the following exercises, use the Change-of-Base Formula, rounding to three decimal places, to approximate each logarithm.

1. $\log_3 42$
2. $\log_5 46$
3. $\log_{12} 87$
4. $\log_{15} 93$
5. $\log_{\sqrt{2}} 17$
6. $\log_{\sqrt{3}} 21$

Answer

2. 2.379

4. 1.674

6. 5.542

Exercise $20.5E.28$ Writing Exercises

1. Write the Product Property in your own words. Does it apply to each of the following? $\log_a 5x, \log_a(5+x)$. Why or why not?
2. Write the Power Property in your own words. Does it apply to each of the following? $\log_a x^p, (\log_a x)^r$. Why or why not?
3. Use an example to show that $\log(a+b) \neq \log a + \log b$?
4. Explain how to find the value of $\log_7 15$ using your calculator.

Answer

2. Answers may vary

4. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the properties of logarithms.			
use the Change of Base Formula.			

Figure 10.4.5

b. On a scale of $1-10$, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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20.6: Solve Exponential and Logarithmic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve logarithmic equations using the properties of logarithms
- Solve exponential equations using logarithms
- Use exponential models in applications

Before you get started, take this readiness quiz.

1. Solve: $x^2 = 16$.
   If you missed this problem, review Example 6.46.
2. Solve: $x^2 - 5x + 6 = 0$.
   If you missed this problem, review Example 6.45.
3. Solve: $x(x+6) = 2x + 5$.
   If you missed this problem, review Example 6.47.

Solve Logarithmic Equations Using the Properties of Logarithms

In the section on logarithmic functions, we solved some equations by rewriting the equation in exponential form. Now that we have the properties of logarithms, we have additional methods we can use to solve logarithmic equations.

If our equation has two logarithms we can use a property that says that if $\log_a M = \log_a N$ then it is true that $M = N$. This is the **One-to-One Property of Logarithmic Equations**.

Definition 20.6.1

One-to-One Property of Logarithmic Equations

For $M > 0, N > 0, a > 0$, and $a \neq 1$ is any real number:

$$\text{If } \log_a M = \log_a N, \text{ then } M = N.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

Remember that logarithms are defined only for positive real numbers. Check your results in the original equation. You may have obtained a result that gives a logarithm of zero or a negative number.

Example 20.6.1

Solve: $2\log_5 x = \log_5 81$.

Solution:

$$2\log_5 x = \log_5 81$$

Use the Power Property.

$$\log_5 x^2 = \log_5 81$$

Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.

$$x^2 = 81$$

Solve using the Square Root Property.

$$x = \pm 9$$

We eliminate $x = -9$ as we cannot take the logarithm of a negative number.

$$x = 9, \ \cancel{x = -9}$$

Check. $x = 9$

$$2\log_5 x = \log_5 81$$
$$2\log_5 9 \stackrel{?}{=} \log_5 81$$
$$\log_5 9^2 \stackrel{?}{=} \log_5 81$$
$$\log_5 81 = \log_5 81$$

Exercise 20.6.1

Solve: $2\log_3 x = \log_3 36$

Answer

$x = 6$

Exercise 20.6.2

Solve: $3\log x = \log 64$

Answer

$x = 4$

Another strategy to use to solve logarithmic equations is to condense sums or differences into a single logarithm.

Example 20.6.2

Solve: $\log_3 x + \log_3(x-8) = 2$.

Solution:

$$\log_3 x + \log_3(x-8) = 2$$

Use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.

$$\log_3 x(x-8) = 2$$

Rewrite in exponential form.

$$3^2 = x(x-8)$$

Simplify.

$$9 = x^2 - 8x$$

Subtract 9 from each side.

$$0 = x^2 - 8x - 9$$

Factor.

$$0 = (x-9)(x+1)$$

Use the Zero-Product-Property

$$x - 9 = 0, \quad x + 1 = 0$$

Solve each equation.

$$x = 9, \quad x = \cancel{-1}$$

Check. $x = -1$

$$\log_3 x + \log_3(x-8) = 2$$
$$\log_3(-1) + \log_3(-1-8) \stackrel{?}{=} 2$$

We cannot take the log of a negative number.

Check. $x = 9$

$$\begin{aligned} \log_3 x + \log_3 (x-8) &= 2 \\ \log_3 9 + \log_3 (9-8) &\stackrel{?}{=} 2 \\ 2 + 0 &\stackrel{?}{=} 2 \\ 2 &= 2 \end{aligned}$$

Exercise 20.6.3

Solve: $\log_2 x + \log_2 (x-2) = 3$

Answer

$x = 4$

Exercise 20.6.4

Solve: $\log_2 x + \log_2 (x-6) = 4$

Answer

$x = 8$

When there are logarithms on both sides, we condense each side into a single logarithm. Remember to use the Power Property as needed.

Example 20.6.3

Solve: $\log_4 (x+6) - \log_4 (2x+5) = -\log_4 x$.

Solution:

$$\log_4 (x+6) - \log_4 (2x+5) = -\log_4 x$$

Use the Quotient Property on the left side and the PowerProperty on the right.

$$\log_4 \left(\tfrac{x+6}{2x+5} \right) = \log_4 x^{-1}$$

Rewrite $x^{-1} = \frac{1}{x}$.

$$\log_4 \left(\tfrac{x+6}{2x+5} \right) = \log_4 \tfrac{1}{x}$$

Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.

$$\tfrac{x+6}{2x+5} = \tfrac{1}{x}$$

Solve the rational equation.

$$x(x+6) = 2x+5$$

Distribute.

$$x^2 + 6x = 2x + 5$$

Write in standard form.

$$x^2 + 4x - 5 = 0$$

Factor.

$$(x+5)(x-1) = 0$$

Use the Zero-Product-Property.

$$x+5=0, \quad x-1=0$$

Solve each equation.

$$x=\cancel{-5}, \quad x=1$$

Check.

We leave the check for you.

Exercise 20.6.5

Solve: $\log(x+2)-\log(4x+3)=-\log x$.

Answer

$x=3$

Exercise 20.6.6

Solve: $\log(x-2)-\log(4x+16)=\log\frac{1}{x}$.

Answer

$x=8$

Example 20.6.4 Solve Exponential Equations Using Logarithms

Solve $5^x=11$. Find the exact answer and then approximate it to three decimal places.

Solution:

$$5^x=11$$

Since the exponential is isolated, take the logarithm of both sides.

$$\log 5^x=\log 11$$

Use the Power Property to get the x as a factor, not an exponent.

$$x\log 5=\log 11$$

Solve for x. Find the exact answer.

$$x=\frac{\log 11}{\log 5}$$

Approximate the answer.

$$x\approx 1.490$$

Since $5^1=5$ and $5^2=25$, does it makes sense that $5^{1.490}\approx 11$?

Exercise 20.6.7

Solve $7^x=43$. Find the exact answer and then approximate it to three decimal places.

Answer

$x=\frac{\log 43}{\log 7}\approx 1.933$

Exercise 20.6.8

Solve $8^x = 98$. Find the exact answer and then approximate it to three decimal places.

Answer

$x = \frac{\log 98}{\log 8} \approx 2.205$

When we take the logarithm of both sides we will get the same result whether we use the common or the natural logarithm (try using the natural log in the last example. Did you get the same result?) When the exponential has base e, we use the natural logarithm.

Example 20.6.5

Solve $3e^{x+2} = 24$. Find the exact answer and then approximate it to three decimal places.

Solution:

$$3e^{x+2} = 24$$

Isolate the exponential by dividing both sides by 3.

$$e^{x+2} = 8$$

Take the natural logarithm of both sides.

$$\ln e^{x+2} = \ln 8$$

Use the Power Property to get the x as a factor, not an exponent.

$$(x+2)\ln e = \ln 8$$

Use the property $\ln e = 1$ to simplify.

$$x + 2 = \ln 8$$

Solve the equation. Find the exact answer.

$$x = \ln 8 - 2$$

Approximate the answer.

$$x \approx 0.079$$

Exercise 20.6.9

Solve $2e^{x-2} = 18$. Find the exact answer and then approximate it to three decimal places.

Answer

$x = \ln 9 + 2 \approx 4.197$

Exercise 20.6.10

Solve $5e^{2x} = 25$. Find the exact answer and then approximate it to three decimal places.

Answer

$x = \frac{\ln 5}{2} \approx 0.805$

Use Exponential Models in Applications

In previous sections we were able to solve some applications that were modeled with exponential equations. Now that we have so many more options to solve these equations, we are able to solve more applications.

We will again use the Compound Interest Formulas and so we list them here for reference.

Definition 20.6.2

Compound Interest

For a principal, P, invested at an interest rate, r, for t years, the new balance, A is:

$$A = P\left(1+\frac{r}{n}\right)^{nt} \quad \text{when compounded } n \text{ times a year.}$$
$$A = Pe^{rt} \quad \text{when compounded continuously.}$$

Example 20.6.6

Jermael's parents put $\$10,000$ in investments for his college expenses on his first birthday. They hope the investments will be worth $\$50,000$ when he turns 18. If the interest compounds continuously, approximately what rate of growth will they need to achieve their goal?

Solution:

Identify the variables in the formula.

$$A = \$50,000$$
$$P = \$10,000$$
$$r = ?$$
$$t = 17 \text{ years}$$
$$A = Pe^{rt}$$

Substitute the values into the formula.

$$50,000 = 10,000e^{r\cdot 17}$$

Solve for r. Divide each side by $10,000$.

$$5 = e^{17r}$$

Take the natural log of each side.

$$\ln 5 = \ln e^{17r}$$

Use the Power Property.

$$\ln 5 = 17r \ln e$$

Simplify.

$$\ln 5 = 17r$$

Divide each side by 17.

$$\frac{\ln 5}{17} = r$$

Approximate the answer.

$$r \approx 0.095$$

Convert to a percentage.

$$r \approx 9.5\%$$

They need the rate of growth to be approximately 9.5%.

Exercise 20.6.11

Hector invests $\$10,000$ at age 21. He hopes the investments will be worth $\$150,000$ when he turns 50. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal?

Answer

$r \approx 9.3\%$

Exercise 20.6.12

Rachel invests $\$15,000$ at age 25. She hopes the investments will be worth $\$90,000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

Answer

$r \approx 11.9\%$

We have seen that growth and decay are modeled by exponential functions. For growth and decay we use the formula $A = A_0 e^{kt}$. Exponential growth has a positive rate of growth or growth constant, k, and **exponential decay** has a negative rate of growth or decay constant, k.

Definition 20.6.3

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, k, for a certain time, t, the final amount, A, is:

$$A = A_0 e^{kt}$$

We can now solve applications that give us enough information to determine the rate of growth. We can then use that rate of growth to predict other situations.

Example 20.6.7

Researchers recorded that a certain bacteria population grew from 100 to 300 in 3 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

Solution:

This problem requires two main steps. First we must find the unknown rate, k. Then we use that value of k to help us find the unknown number of bacteria.

Identify the variables in the formula.

$$\begin{aligned} A &= 300 \\ A_0 &= 100 \\ k &= ? \\ t &= 3 \text{ hours} \\ A &= A_0 e^{kt} \end{aligned}$$

Substitute the values in the formula.

$$300 = 100 e^{k \cdot 3}$$

Solve for k. Divide each side by 100.

$$3 = e^{3k}$$

Take the natural log of each side.

$$\ln 3 = \ln e^{3k}$$

Use the Power Property.

$$\ln 3 = 3k \ln e$$

Simplify.

$$\ln 3 = 3k$$

Divide each side by 3.

$$\frac{\ln 3}{3} = k$$

Approximate the answer.

$$k \approx 0.366$$

We use this rate of growth to predict the number of bacteria there will be in 24 hours.

$$\begin{aligned} A &= ? \\ A_0 &= 100 \\ k &= \frac{\ln 3}{3} \\ t &= 24 \text{ hours} \\ A &= A_0 e^{kt} \end{aligned}$$

Substitute in the values.

$$A = 100e^{\frac{\ln 3}{3} \cdot 24}$$

Evaluate.

$$A \approx 656,100$$

At this rate of growth, they can expect $656,100$ bacteria.

? Exercise 20.6.13

Researchers recorded that a certain bacteria population grew from 100 to 500 in 6 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

Answer

There will be $62,500$ bacteria.

? Exercise 20.6.14

Researchers recorded that a certain bacteria population declined from $700,000$ to $400,000$ in 5 hours after the administration of medication. At this rate of decay, how many bacteria will there be 24 hours from the start of the experiment?

Answer

There will be $5,870,061$ bacteria.

Radioactive substances decay or decompose according to the exponential decay formula. The amount of time it takes for the substance to decay to half of its original amount is called the **half-life** of the substance.

Similar to the previous example, we can use the given information to determine the constant of decay, and then use that constant to answer other questions.

✓ Example 20.6.8

The half-life of radium-226 is $1,590$ years. How much of a 100 mg sample will be left in 500 years?

Solution:

This problem requires two main steps. First we must find the decay constant k. If we start with 100-mg, at the half-life there will be 50-mg remaining. We will use this information to find k. Then we use that value of k to help us find the amount of sample that will be left in 500 years.

Identify the variables in the formula.

$$A = 50$$
$$A_0 = 100$$
$$k = ?$$
$$t = 1590 \text{ years}$$
$$A = A_0 e^{kt}$$

Substitute the values in the formula.

$$50 = 100e^{k \cdot 1590}$$

Solve for k. Divide each side by 100.

$$0.5 = e^{1590k}$$

Take the natural log of each side.

$$\ln 0.5 = \ln e^{1590k}$$

Use the Power Property.

$$\ln 0.5 = 1590k \ln e$$

Simplify.

$$\ln 0.5 = 1590k$$

Divide each side by 1590.

$$\frac{\ln 0.5}{1590} = k \text{ exact answer}$$

We use this rate of growth to predict the amount that will be left in 500 years.

$$A = ?$$
$$A_0 = 100$$
$$k = \frac{\ln 0.5}{1590}$$
$$t = 500 \text{ years}$$
$$A = A_0 e^{kt}$$

Substitute in the values.

$$A = 100e^{\frac{\ln 0.5}{1500} \cdot 500}$$

Evaluate.

$$A \approx 80.4 \text{mg}$$

In 500 years there would be approximately 80.4 mg remaining.

? Exercise 20.6.15

The half-life of magnesium-27 is 9.45 minutes. How much of a 10-mg sample will be left in 6 minutes?

Answer

There will be 6.43mg left.

? Exercise 20.6.16

The half-life of radioactive iodine is 60 days. How much of a 50-mg sample will be left in 40 days?

Answer

There will be 31.5mg left.

Access these online resources for additional instruction and practice with solving exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Logarithm Equations
- Finding the rate or time in a word problem on exponential growth or decay
- Finding the rate or time in a word problem on exponential growth or decay

Key Concepts

- **One-to-One Property of Logarithmic Equations:** For $M > 0, N > 0, a > 0$, and $a \neq 1$ is any real number:

$$\text{If } \log_a M = \log_a N, \text{ then } M = N$$

- **Compound Interest:**
  For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is:

$$\begin{aligned} A &= P\left(1+\tfrac{r}{n}\right)^{nt} && \text{when compounded } n \text{ times a year.} \\ A &= Pe^{rt} && \text{when compounded continuously.} \end{aligned}$$

- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount, A, is $A = A_0 e^{rt}$.

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20.6E: Exercises

Practice Makes Perfect

Exercise $20.6E.17$ Solve Logarithmic Equations Using the Properties of Logarithms

In the following exercises, solve for x.

1. $\log_4 64 = 2\log_4 x$
2. $\log 49 = 2\log x$
3. $3\log_3 x = \log_3 27$
4. $3\log_6 x = \log_6 64$
5. $\log_5(4x-2) = \log_5 10$
6. $\log_3\left(x^2+3\right) = \log_3 4x$
7. $\log_3 x + \log_3 x = 2$
8. $\log_4 x + \log_4 x = 3$
9. $\log_2 x + \log_2(x-3) = 2$
10. $\log_3 x + \log_3(x+6) = 3$
11. $\log x + \log(x+3) = 1$
12. $\log x + \log(x-15) = 2$
13. $\log(x+4) - \log(5x+12) = -\log x$
14. $\log(x-1) - \log(x+3) = \log \frac{1}{x}$
15. $\log_5(x+3) + \log_5(x-6) = \log_5 10$
16. $\log_5(x+1) + \log_5(x-5) = \log_5 7$
17. $\log_3(2x-1) = \log_3(x+3) + \log_3 3$
18. $\log(5x+1) = \log(x+3) + \log 2$

Answer

2. $x = 7$
4. $x = 4$
6. $x = 1, x = 3$
8. $x = 8$
10. $x = 3$
12. $x = 20$
14. $x = 3$
16. $x = 6$
18. $x = \frac{5}{3}$

Exercise $20.6E.18$ Solve Exponential Equations Using Logarithms

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

1. $3^x = 89$
2. $2^x = 74$
3. $5^x = 110$
4. $4^x = 112$
5. $e^x = 16$
6. $e^x = 8$
7. $\left(\frac{1}{2}\right)^x = 6$

8. $\left(\frac{1}{3}\right)^x = 8$
9. $4e^{x+1} = 16$
10. $3e^{x+2} = 9$
11. $6e^{2x} = 24$
12. $2e^{3x} = 32$
13. $\frac{1}{4}e^x = 3$
14. $\frac{1}{3}e^x = 2$
15. $e^{x+1} + 2 = 16$
16. $e^{x-1} + 4 = 12$

Answer

2. $x = \frac{\log 74}{\log 2} \approx 6.209$
4. $x = \frac{\log 112}{\log 4} \approx 3.404$
6. $x = \ln 8 \approx 2.079$
8. $x = \frac{\log 8}{\log \frac{1}{3}} \approx -1.893$
10. $x = \ln 3 - 2 \approx -0.901$
12. $x = \frac{\ln 16}{3} \approx 0.924$
14. $x = \ln 6 \approx 1.792$
16. $x = \ln 8 + 1 \approx 3.079$

Exercise $20.6E.19$ Solve Exponential Equations Using Logarithms

In the following exercises, solve each equation.

1. $3^{3x+1} = 81$
2. $6^{4x-17} = 216$
3. $\frac{e^{x^2}}{e^{14}} = e^{5x}$
4. $\frac{e^{x^2}}{e^x} = e^{20}$
5. $\log_a 64 = 2$
6. $\log_a 81 = 4$
7. $\ln x = -8$
8. $\ln x = 9$
9. $\log_5(3x - 8) = 2$
10. $\log_4(7x + 15) = 3$
11. $\ln e^{5x} = 30$
12. $\ln e^{6x} = 18$
13. $3 \log x = \log 125$
14. $7 \log_3 x = \log_3 128$
15. $\log_6 x + \log_6(x - 5) = \log_6 24$
16. $\log_9 x + \log_9(x - 4) = \log_9 12$
17. $\log_2(x + 2) - \log_2(2x + 9) = -\log_2 x$
18. $\log_6(x + 1) - \log_6(4x + 10) = \log_6 \frac{1}{x}$

Answer

2. $x = 5$
4. $x = -4, x = 5$
6. $a = 3$

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8. $x = e^9$

10. $x = 7$

12. $x = 3$

14. $x = 2$

16. $x = 6$

18. $x = 5$

Exercise $20.6E.20$ Solve Exponential Equations Using Logarithms

In the following exercises, solve for x, giving an exact answer as well as an approximation to three decimal places.

1. $6^x = 91$
2. $\left(\frac{1}{2}\right)^x = 10$
3. $7e^{x-3} = 35$
4. $8e^{x+5} = 56$

Answer

2. $x = \frac{\log 10}{\log \frac{1}{2}} \approx -3.322$

4. $x = \ln 7 - 5 \approx -3.054$

Exercise $20.6E.21$ Use Exponential Models in Applications

In the following exercises, solve.

1. Sung Lee invests $\$5,000$ at age 18. He hopes the investments will be worth $\$10,000$ when he turns 25. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?
2. Alice invests $\$15,000$ at age 30 from the signing bonus of her new job. She hopes the investments will be worth $\$30,000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?
3. Coralee invests $\$5,000$ in an account that compounds interest monthly and earns 7%. How long will it take for her money to double?
4. Simone invests $\$8,000$ in an account that compounds interest quarterly and earns 5%. How long will it take for his money to double?
5. Researchers recorded that a certain bacteria population declined from $100,000$ to 100 in 24 hours. At this rate of decay, how many bacteria will there be in 16 hours?
6. Researchers recorded that a certain bacteria population declined from $800,000$ to $500,000$ in 6 hours after the administration of medication. At this rate of decay, how many bacteria will there be in 24 hours?
7. A virus takes 6 days to double its original population $(A = 2A_0)$. How long will it take to triple its population?
8. A bacteria doubles its original population in 24 hours $(A = 2A_0)$. How big will its population be in 72 hours?
9. Carbon-14 is used for archeological carbon dating. Its half-life is $5,730$ years. How much of a 100-gram sample of Carbon-14 will be left in 1000 years?
10. Radioactive technetium-99m is often used in diagnostic medicine as it has a relatively short half-life but lasts long enough to get the needed testing done on the patient. If its half-life is 6 hours, how much of the radioactive material form a 0.5 ml injection will be in the body in 24 hours?

Answer

2. 6.9%

4. 13.9 years

6. $122,070$bacteria

8. 8 times as large as the original population

10. 0.03mL

Exercise $20.6E.22$ Writing Exercises

1. Explain the method you would use to solve these equations: $3^{x+1}=81$, $3^{x+1}=75$. Does your method require logarithms for both equations? Why or why not?
2. What is the difference between the equation for exponential growth versus the equation for exponential decay?

Answer

2. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve logarithmic equations using the properties of logarithms.			
solve exponential equations using logarithms.			
use exponential models in applications.			

Figure 10.5.1

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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20.7: Review Exercises

Chapter Review Exercises

Finding Composite and Inverse Functions

Exercise 20.7.1 Find and Evaluate Composite Functions

In the following exercises, for each pair of functions, find

a. $(f \circ g)(x)$
b. $(g \circ f)(x)$
c. $(f \cdot g)(x)$

1. $f(x) = 7x - 2$ and $g(x) = 5x + 1$
2. $f(x) = 4x$ and $g(x) = x^2 + 3x$

Answer

2.

a. $4x^2 + 12x$
b. $16x^2 + 12x$
c. $4x^3 + 12x^2$

Exercise 20.7.2 Find and Evaluate Composite Functions

In the following exercises, evaluate the composition.

1. For functions $f(x) = 3x^2 + 2$ and $g(x) = 4x - 3$, find
   a. $(f \circ g)(-3)$
   b. $(g \circ f)(-2)$
   c. $(f \circ f)(-1)$
2. For functions $f(x) = 2x^3 + 5$ and $g(x) = 3x^2 - 7$, find
   a. $(f \circ g)(-1)$
   b. $(g \circ f)(-2)$
   c. $(g \circ g)(1)$

Answer

2.

a. -123
b. 356
c. 41

Exercise 20.7.3 Determine Whether a Function is One-to-One

In the following exercises, for each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

1. $\{(-3,-5),(-2,-4),(-1,-3),(0,-2),(-1,-1),(-2,0),(-3,1)\}$
2. $\{(-3,0),(-2,-2),(-1,0),(0,1),(1,2),(2,1),(3,-1)\}$
3. $\{(-3,3),(-2,1),(-1,-1),(0,-3),(1,-5),(2,-4),(3,-2)\}$

Answer

2. Function; not one-to-one

Exercise 20.7.4 Determine Whether a Function is One-to-One

In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.

1. a.

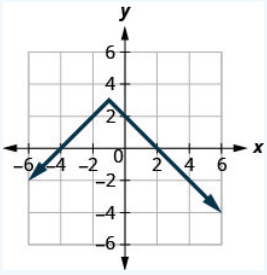

Figure 10.E.1

b.

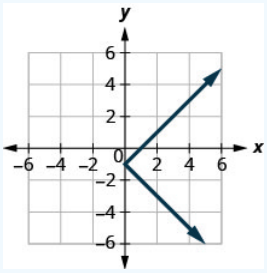

Figure 10.E.2

2. a.

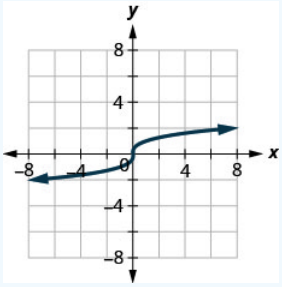

Figure 10.E.3

b.

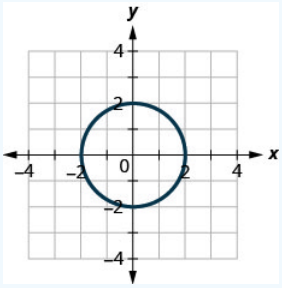

Figure 10.E.4

Answer

1.

a. Function; not one-to-one
b. Not a function

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Exercise 20.7.5 Find the Inverse of a Function

In the following exercise, find the inverse of the function. Determine the domain and range of the inverse function.

1. $\{(-3,10),(-2,5),(-1,2),(0,1)\}$

Answer

1. Inverse function: $\{(10,-3),(5,-2),(2,-1),(1,0)\}$ Domain: $\{1,2,5,10\}$ Range: $\{-3,-2,-1,0\}$.

Exercise 20.7.6 Find the Inverse of a Function

In the following exercise, graph the inverse of the one-to-one function shown.

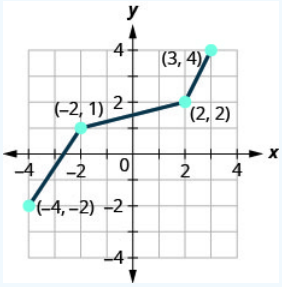

Figure 10.E.5

Answer

Solve on your own

Exercise 20.7.7 Find the Inverse of a Function

In the following exercises, verify that the functions are inverse functions.

1. $f(x)=3x+7$ and $g(x)=\frac{x-7}{3}$
2. $f(x)=2x+9$ and $g(x)=\frac{x+9}{2}$

Answer

1. $g(f(x))=x,$ and $f(g(x))=x,$ so they are inverses.

Exercise 20.7.8 Find the Inverse of a Function

1. $f(x)=6x-11$
2. $f(x)=x^3+13$
3. $f(x)=\frac{1}{x+5}$
4. $f(x)=\sqrt[5]{x-1}$

Answer

1. $f^{-1}(x)=\frac{x+11}{6}$
3. $f^{-1}(x)=\frac{1}{x}-5$

Evaluate and Graph Exponential Functions

Exercise 20.7.9 Graph Exponential Functions

In the following exercises, graph each of the following functions.

1. $f(x) = 4^x$
2. $f(x) = \left(\frac{1}{5}\right)^x$
3. $g(x) = (0.75)^x$
4. $g(x) = 3^{x+2}$
5. $f(x) = (2.3)^x - 3$
6. $f(x) = e^x + 5$
7. $f(x) = -e^x$

Answer

1.

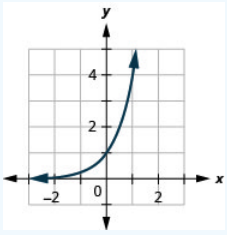

Figure 10.E.6

3.

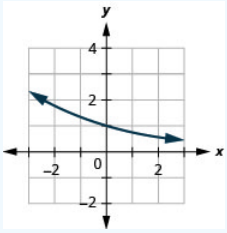

Figure 10.E.7

5.

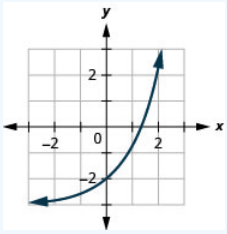

Figure 10.E.8

7.

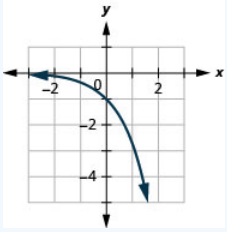

Figure 10.E.9

Exercise 20.7.10 Solve Exponential Equations

In the following exercises, solve each equation.

1. $3^{5x-6} = 81$
2. $2^{x^2} = 16$
3. $9^x = 27$
4. $5^{x^2+2x} = \frac{1}{5}$
5. $e^{4x} \cdot e^7 = e^{19}$
6. $\frac{e^{x^2}}{e^{15}} = e^{2x}$

Answer

2. $x = -2, x = 2$
4. $x = -1$
6. $x = -3, x = 5$

Exercise 20.7.11 Use Exponential Models in Applications

In the following exercises, solve.

1. Felix invested $\$12,000$ in a savings account. If the interest rate is 4% how much will be in the account in 12 years by each method of compounding?
   a. compound quarterly
   b. compound monthly
   c. compound continuously
2. Sayed deposits $\$20,000$ in an investment account. What will be the value of his investment in 30 years if the investment is earning 7% per year and is compounded continuously?
3. A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. She starts her experiment with 150 of the bacteria that grows at a rate of 15% per hour. She will check on the bacteria every 24 hours. How many bacteria will he find in 24 hours?
4. In the last five years the population of the United States has grown at a rate of 0.7% per year to about $318,900,000$ If this rate continues, what will be the population in 5 more years?

Answer

2. $\$163,323.40$
4. $330,259,000$

Evaluate and Graph Logarithmic Functions

Exercise 20.7.12 Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.

1. $5^4 = 625$
2. $10^{-3} = \frac{1}{1,000}$
3. $63^{\frac{1}{5}} = \sqrt[5]{63}$
4. $e^y = 16$

Answer

2. $\log \frac{1}{1,000} = -3$
4. $\ln 16 = y$

Exercise 20.7.13 Convert Between Exponential and Logarithmic Form

In the following exercises, convert each logarithmic equation to exponential form.

1. $7 = \log_2 128$
2. $5 = \log 100,000$
3. $4 = \ln x$

Answer

2. $100000 = 10^5$

Exercise 20.7.14 Evaluate Logarithmic Functions

In the following exercises, solve for x.

1. $\log_x 125 = 3$
2. $\log_7 x = -2$
3. $\log_{\frac{1}{2}} \frac{1}{16} = x$

Answer

1. $x = 5$
3. $x = 4$

Exercise 20.7.15 Evaluate Logarithmic Functions

In the following exercises, find the exact value of each logarithm without using a calculator.

1. $\log_2 32$
2. $\log_8 1$
3. $\log_3 \frac{1}{9}$

Answer

2. 0

Exercise 20.7.16 Graph Logarithmic Functions

In the following exercises, graph each logarithmic function.

1. $y = \log_5 x$
2. $y = \log_{\frac{1}{4}} x$
3. $y = \log_{0.8} x$

Answer

1.

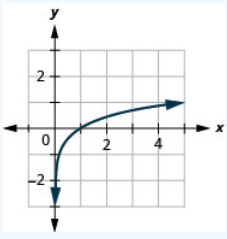

Figure 10.E.10

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3.

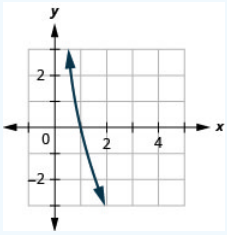

Figure 10.E.11

Exercise 20.7.17 Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

1. $\log_a 36 = 5$
2. $\ln x = -3$
3. $\log_2(5x - 7) = 3$
4. $\ln e^{3x} = 24$
5. $\log\left(x^2 - 21\right) = 2$

Answer

2. $x = e^{-3}$

4. $x = 8$

Exercise 20.7.18 Use Logarithmic Models in Applications

What is the decibel level of a train whistle with intensity 10^{-3} watts per square inch?

Answer

90 dB

Use the Properties of Logarithms

Exercise 20.7.19 Use the Properties of Logarithms

In the following exercises, use the properties of logarithms to evaluate.

1. a. $\log_7 1$
   b. $\log_{12} 12$
2. a. $5^{\log_5 13}$
   b. $\log_3 3^{-9}$
3. a. $10^{\log\sqrt{5}}$
   b. $\log 10^{-3}$
4. a. $e^{\ln 8}$
   b. $\ln e^5$

Answer

2.

a. 13
b. -9

4.

a. 8
b. 5

Exercise 20.7.20 Use the Properties of Logarithms

In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

1. $\log_4(64xy)$
2. $\log 10,000m$

Answer

2. $4+\log m$

Exercise 20.7.21 Use the Properties of Logarithms

In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

1. $\log_7 \frac{49}{y}$
2. $\ln \frac{e^5}{2}$

Answer

2. $5-\ln 2$

Exercise 20.7.22 Use the Properties of Logarithms

In the following exercises, use the Power Property of Logarithms to expand each logarithm. Simplify, if possible.

1. $\log x^{-9}$
2. $\log_4 \sqrt[7]{z}$

Answer

2. $\frac{1}{7}\log_4 z$

Exercise 20.7.23 Use the Properties of Logarithms

In the following exercises, use properties of logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

1. $\log_3\left(\sqrt{4}x^7y^8\right)$
2. $\log_5 \frac{8a^2b^6c}{d^3}$
3. $\ln \frac{\sqrt{3x^2-y^2}}{z^4}$
4. $\log_6 \sqrt[3]{\frac{7x^2}{6y^3z^5}}$

Answer

2. $\log_5 8+2\log_5 a+6\log_5 b+\log_5 c-3\log_5 d$
4. $\frac{1}{3}(\log_6 7+2\log_6 x-1-3\log_6 y-5\log_6 z)$

Exercise 20.7.24 Use the Properties of Logarithms

In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.

1. $\log_2 56 - \log_2 7$
2. $3\log_3 x + 7\log_3 y$
3. $\log_5\left(x^2-16\right) - 2\log_5(x+4)$
4. $\frac{1}{4}\log y - 2\log(y-3)$

Answer

2. $\log_3 x^3y^7$
4. $\log \frac{\sqrt[4]{y}}{(y-3)^2}$

Exercise 20.7.25 Use the Change-of-Base Formula

In the following exercises, rounding to three decimal places, approximate each logarithm.

1. $\log_5 97$
2. $\log_{\sqrt{3}} 16$

Answer

2. 5.047

Solve Exponential and Logarithmic Equations

Exercise 20.7.26 Solve Logarithmic Equations Using the Properties of Logarithms

In the following exercises, solve for x.

1. $3\log_5 x = \log_5 216$
2. $\log_2 x + \log_2(x-2) = 3$
3. $\log(x-1) - \log(3x+5) = -\log x$
4. $\log_4(x-2) + \log_4(x+5) = \log_4 8$
5. $\ln(3x-2) = \ln(x+4) + \ln 2$

Answer

2. $x = 4$
4. $x = 3$

Exercise 20.7.27 Solve Exponential Equations Using Logarithms

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

1. $2^x = 101$
2. $e^x = 23$
3. $\left(\frac{1}{3}\right)^x = 7$
4. $7e^{x+3} = 28$
5. $e^{x-4} + 8 = 23$

Answer

1. $x = \frac{\log 101}{\log 2} \approx 6.658$

3. $x = \frac{\log 7}{\log \frac{1}{3}} \approx -1.771$

5. $x = \ln 15 + 4 \approx 6.708$

Exercise 20.7.28 Use Exponential Models in Applications

1. Jerome invests $\$18,000$ at age 17. He hopes the investments will be worth $\$30,000$ when he turns 26. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?
2. Elise invests $\$4500$ in an account that compounds interest monthly and earns 6%.How long will it take for her money to double?
3. Researchers recorded that a certain bacteria population grew from 100 to 300 in 8 hours. At this rate of growth, how many bacteria will there be in 24 hours?
4. Mouse populations can double in 8 months $(A = 2A_0)$. How long will it take for a mouse population to triple?
5. The half-life of radioactive iodine is 60 days. How much of a 50 mg sample will be left in 40 days?

Answer

2. 11.6 years

4. 12.7 months

Practice Test

Exercise 20.7.29

1. For the functions, $f(x) = 6x + 1$ and $g(x) = 8x - 3$, find
   a. $(f \circ g)(x)$
   b. $(g \circ f)(x)$
   c. $(f \cdot g)(x)$
2. Determine if the following set of ordered pairs represents a function and if so, is the function one-to-one.
   $\{(-2, 2), (-1, -3), (0, 1), (1, -2), (2, -3)\}$
3. Determine whether each graph is the graph of a function and if so, is it one-to-one.
   a.

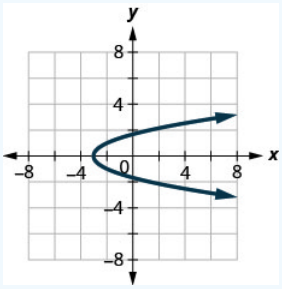

Figure 10.E.12

   b.

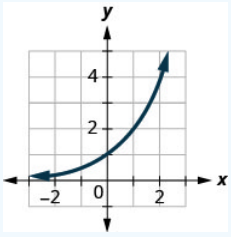

Figure 10.E.13

https://math.libretexts.org/@go/page/46478

4. Graph, on the same coordinate system, the inverse of the one-to-one function shown.

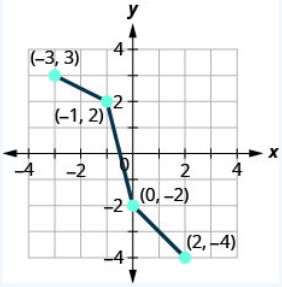

Figure 10.E.14

5. Find the inverse of the function $f(x) = x^5 - 9$.

6. Graph the function $g(x) = 2^{x-3}$.

7. Solve the equation $2^{2x-4} = 64$.

8. Solve the equation $\frac{e^{x^2}}{e^4} = e^{3x}$.

9. Megan invested $\$21,000$ in a savings account. If the interest rate is 5%, how much will be in the account in 8 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

10. Convert the equation from exponential to logarithmic form: $10^{-2} = \frac{1}{100}$.

11. Convert the equation from logarithmic equation to exponential form: $3 = \log_7 343$.

12. Solve for x: $\log_5 x = -3$

13. Evaluate $\log_{11} 1$.

14. Evaluate $\log_4 \frac{1}{64}$.

15. Graph the function $y = \log_3 x$.

16. Solve for x: $\log\left(x^2 - 39\right) = 1$

17. What is the decibel level of a small fan with intensity 10^{-8} watts per square inch?

18. Evaluate each.

a. $6^{\log_6 17}$
b. $\log_9 9^{-3}$

Answer

1.

a. $48x - 17$
b. $48x + 5$
c. $48x^2 - 10x - 3$

3.

a. Not a function
b. One-to-one function

5. $f^{-1}(x) = \sqrt[5]{x+9}$

7. $x = 5$

9.

a. $\$31,250.74$
b. $\$31,302.29$
c. $\$31,328.32$

11. $343 = 7^3$

13. 0

15.

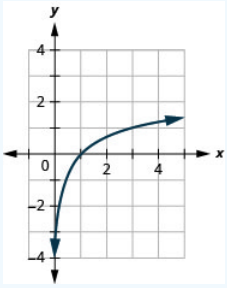

Figure 10.E.15

17. 40 dB

Exercise 20.7.30

In the following exercises, use properties of logarithms to write each expression as a sum of logarithms, simplifying if possible.

1. $\log_5 25ab$
2. $\ln \frac{e^{12}}{8}$
3. $\log_2 \sqrt[4]{\frac{5x^3}{16y^2z^7}}$

Answer

1. $2 + \log_5 a + \log_5 b$
3. $\frac{1}{4}(\log_2 5 + 3\log_2 x - 4 - 2\log_2 y - 7\log_2 z)$

Exercise 20.7.31

In the following exercises, use the Properties of Logarithms to condense the logarithm, simplifying if possible.

1. $5\log_4 x + 3\log_4 y$
2. $\frac{1}{6}\log x - 3\log(x+5)$
3. Rounding to three decimal places, approximate $\log_4 73$.
4. Solve for x: $\log_7(x+2) + \log_7(x-3) = \log_7 24$

Answer

2. $\log \frac{\sqrt[6]{x}}{(x+5)^3}$
4. $x = 6$

Exercise 20.7.32

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

1. $\left(\frac{1}{5}\right)^x = 9$
2. $5e^{x-4} = 40$

3. Jacob invests $\$14,000$ in an account that compounds interest quarterly and earns 4%. How long will it take for his money to double?
4. Researchers recorded that a certain bacteria population grew from 500 to 700 in 5 hours. At this rate of growth, how many bacteria will there be in 20 hours?
5. A certain beetle population can double in 3 months $(A = 2A_0)$. How long will it take for that beetle population to triple?

Answer

2. $x = \ln 8 + 4 \approx 6.079$

4. $1,921$ bacteria

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CHAPTER OVERVIEW

21: Systems of Linear Equations

Topic hierarchy

21.1: Introduction
21.2: Solve Systems of Linear Equations with Two Variables
21.2E: Exercises
21.3: Solve Applications with Systems of Equations
21.3E: Exercises
21.4: Solve Mixture Applications with Systems of Equations
21.4E: Exercises
21.5: Solve Systems of Equations with Three Variables
21.5E: Exercises
21.6: Solve Systems of Equations Using Matrices
21.6E: Exercises
21.7: Solve Systems of Equations Using Determinants
21.7E: Exercises
21.8: Review Exercises

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21.1: Introduction

Figure 21.1.1: In the future, car drivers may become passengers because cars will be able to drive themselves. (credit: jingoba/Pixabay)

Climb into your car. Put on your seatbelt. Choose your destination and then…relax. That's right. You don't have to do anything else because you are in an autonomous car, or one that navigates its way to your destination! No cars are fully autonomous at the moment and so you theoretically still need to have your hands on the wheel. Self-driving cars may help ease traffic congestion, prevent accidents, and lower pollution. The technology is thanks to computer programmers who are developing software to control the navigation of the car. These programmers rely on their understanding of mathematics, including relationships between equations. In this chapter, you will learn how to solve systems of linear equations in different ways and use them to analyze real-world situations.

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21.2: Solve Systems of Linear Equations with Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Solve a system of equations by substitution
- Solve a system of equations by elimination
- Choose the most convenient method to solve a system of linear equations

Before you get started, take this readiness quiz.

1. For the equation $y = \frac{2}{3}x - 4$,
   ⓐ Is $(6, 0)$ a solution? ⓑ Is $(-3, -2)$ a solution?
   If you missed this problem, review [link].
2. Find the slope and y-intercept of the line $3x - y = 12$.
   If you missed this problem, review [link].
3. Find the x- and y-intercepts of the line $2x - 3y = 12$.
   If you missed this problem, review [link].

Determine Whether an Ordered Pair is a Solution of a System of Equations

In Solving Linear Equations, we learned how to solve linear equations with one variable. Now we will work with two or more linear equations grouped together, which is known as a **system of linear equations**.

SYSTEM OF LINEAR EQUATIONS

When two or more linear equations are grouped together, they form a **system of linear equations**.

In this section, we will focus our work on systems of two linear equations in two unknowns. We will solve larger systems of equations later in this chapter.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, such as $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to *both* equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the **solutions of a system of equations**.

SolutionS OF A SYSTEM OF EQUATIONS

The **solutions of a system of equations** are the values of the variables that make *all* the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Example 21.2.1

Determine whether the ordered pair is a solution to the system $\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$.

ⓐ $(-2, -1)$ ⓑ $(-4, -3)$

Answer

ⓐ

$$\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$$

We substitute $x=-2$ and $y=-1$ into both equations.

$x-y=-1$	$2x-y=-5$
$-2-(-1)\stackrel{?}{=}-1$	$2\cdot -2-(-1)\stackrel{?}{=}-5$
$-1=-1$ ✓	$5\neq -5$
$(-2,-1)$ does not make both equations true.	$(-2,-1)$ is not a solution.

ⓑ

We substitute $x=-4$ and $y=-3$ into both equations.

$x-y=-1$	$2x-y=-5$
$-4-(-3)\stackrel{?}{=}-1$	$2\cdot(-4)-(-3)\stackrel{?}{=}-5$
$-1=-1$ ✓	$-5=-5$ ✓
$(-4,-3)$ does make both equations true.	$(-4,-3)$ is a solution.

? Example 21.2.2

Determine whether the ordered pair is a solution to the system $\begin{cases} x+y=0 \\ x+2y=-5 \end{cases}$.

ⓐ $(1,-3)$ ⓑ $(0,0)$

Answer

ⓐ yes ⓑ no

? Example 21.2.3

Determine whether the ordered pair is a solution to the system $\begin{cases} -3y=-8 \\ -3x-y=4 \end{cases}$.

ⓐ $(2,-2)$ ⓑ $(-2,2)$

Answer

ⓐ no ⓑ yes

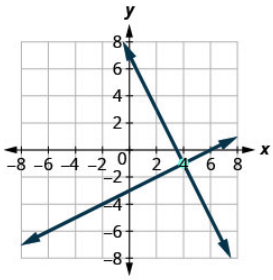
Solve a System of Linear Equations by Graphing

In this section, we will use three methods to solve a system of linear equations. The first method we'll use is graphing.

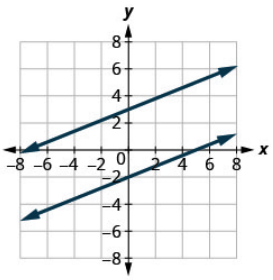
The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions.

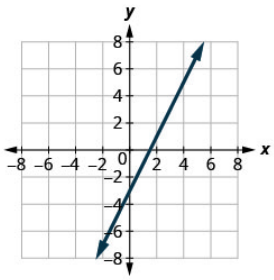
Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.



The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.



The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.



Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

Figure 21.2.1

Each time we demonstrate a new method, we will use it on the same system of linear equations. At the end of the section you'll decide which method was the most convenient way to solve this system.

Example 21.2.4: How to Solve a System of Equations by Graphing

Solve the system by graphing $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$.

Answer

Step 1. Graph the first equation.

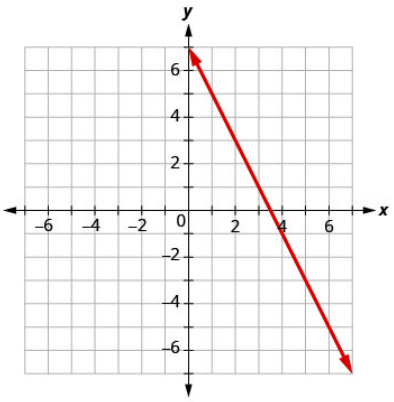
To graph the first line, write the equation in slope–intercept form.

$2x+y=7$

$y=-2x+7$

$m=-2 \qquad b=7$

$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

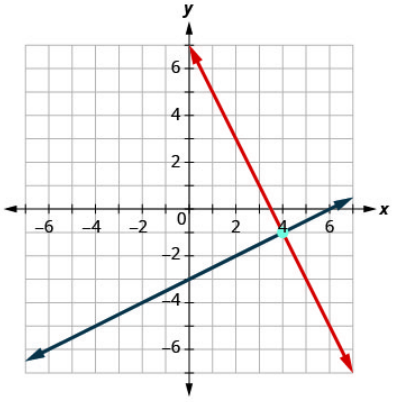


Step 2. Graph the second equation on the same rectangular coordinate system.

To graph the second line, use intercepts.

$x-2y=6$

$(0,-3) \quad (6,0)$



Step 3. Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Step 4. Identify the solution to the system. • If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. • If the lines are parallel, the system has no solution. • If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at (4, –1). $2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $8 - 1 \stackrel{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6$ ✓ The solution is (4, –1).

Example 21.2.5

Solve the system by graphing: $\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$.

Answer

$(3, 2)$

Example 21.2.6

Solve the system by graphing: $\begin{cases} -x + y = 1 \\ 3x + 2y = 12 \end{cases}$

Answer

$(2, 3)$

The steps to use to solve a system of linear equations by graphing are shown here.

SOLVE A SYSTEM OF LINEAR EQUATIONS BY GRAPHING.

1. Graph the first equation.
2. Graph the second equation on the same rectangular coordinate system.
3. Determine whether the lines intersect, are parallel, or are the same line.
4. Identify the solution to the system.
   - If the lines intersect, identify the point of intersection. This is the solution to the system.
   - If the lines are parallel, the system has no solution.
   - If the lines are the same, the system has an infinite number of solutions.
5. Check the solution in both equations.

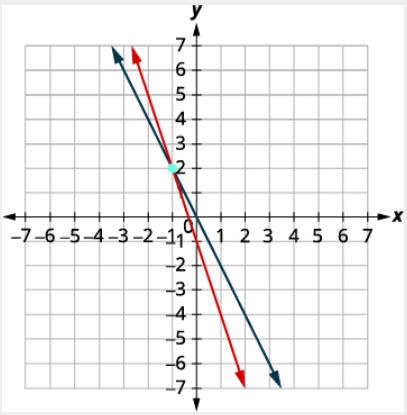
In the next example, we'll first re-write the equations into slope–intercept form as this will make it easy for us to quickly graph the lines.

Example 21.2.7

Solve the system by graphing: $\begin{cases} 3x + y = -1 \\ 2x + y = 0 \end{cases}$

Answer

We'll solve both of these equations for y so that we can easily graph them using their slopes and y-intercepts.

	$\begin{cases} 3x+y=-1 \\ 2x+y=0 \end{cases}$
Solve the first equation for y.	$\begin{aligned} 3x+y&=-1 \\ y&=-3x-1 \end{aligned}$
Find the slope and y-intercept.	$\begin{aligned} m&=-3 \\ b&=-1 \end{aligned}$
Solve the second equation for y.	$\begin{aligned} 2x+y&=0 \\ y&=-2x \end{aligned}$
Find the slope and y-intercept.	$\begin{aligned} m&=-2 \\ b&=0 \end{aligned}$
Graph the lines.	
Determine the point of intersection.	The lines intersect at $(-1, 2)$.
Check the solution in both equations. $\begin{aligned} 3x+y&=-1 \\ 3(-1)+2&\stackrel{?}{=}-1 \\ -1&=-1\ \checkmark \end{aligned}$ $\begin{aligned} 2x+y&=0 \\ 2(-1)+2&\stackrel{?}{=}0 \\ 0&=0\ \checkmark \end{aligned}$	
	The solution is $(-1, 2)$.

Example 21.2.8

Solve the system by graphing: $\begin{cases} -x+y=1 \\ 2x+y=10 \end{cases}$.

Answer

$(3, 4)$

Example 21.2.9

Solve the system by graphing: $\begin{cases} 2x+y=6 \\ x+y=1 \end{cases}$.

Answer

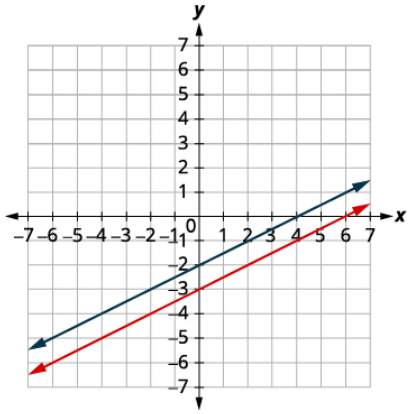
$(5, -4)$

In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

Example 21.2.10

Solve the system by graphing: $\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$.

Answer

	$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$
To graph the first equation, we will use its slope and y-intercept.	$y = \frac{1}{2}x - 3$ $m = \frac{1}{2}$ $b = -3$
To graph the second equation, we will use the intercepts.	$x - 2y = 4$
	x | y 0 | −2 4 | 0
Graph the lines.	
Determine the points of intersection.	The lines are parallel. Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

? Example 21.2.11

Solve the system by graphing: $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = 4 \end{cases}$.

Answer

no solution

? Example 21.2.12

Solve the system by graphing: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 6 \end{cases}$.

Answer

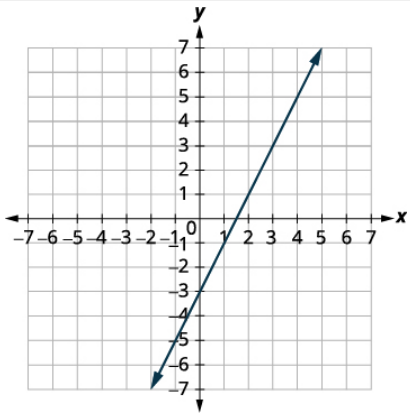
no solution

Sometimes the equations in a system represent the same line. Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to the system.

Example 21.2.13

Solve the system by graphing: $\begin{cases} y = 2x - 3 \\ -6x + 3y = 9 \end{cases}$.

Answer

	$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$
Find the slope and y-intercept of the first equation.	$y = 2x - 3$ $m = 2$ $b = -3$
Find the intercepts of the second equation.	$-6x + 3y = -9$
	x | y 0 | -3 $\frac{3}{2}$ | 0
Graph the lines.	
	The lines are the same! Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to this system.

If you write the second equation in slope-intercept form, you may recognize that the equations have the same slope and same y-intercept.

Example 21.2.14

Solve the system by graphing: $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$.

Answer

infinitely many solutions

Example 21.2.15

Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x - 4 \\ 2x - 4y = 16 \end{cases}$.

Answer

infinitely many solutions

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are **coincident**. Coincident lines have the same slope and same *y*-intercept.

COINCIDENT LINES

Coincident lines have the same slope and same *y*-intercept.

The systems of equations in Example and Example each had two intersecting lines. Each system had one solution.

In Example, the equations gave coincident lines, and so the system had infinitely many solutions.

The systems in those three examples had at least one solution. A system of equations that has at least one solution is called a *consistent* system.

A system with parallel lines, like Example, has no solution. We call a system of equations like this *inconsistent*. It has no solution.

CONSISTENT AND INCONSISTENT SYSTEMS

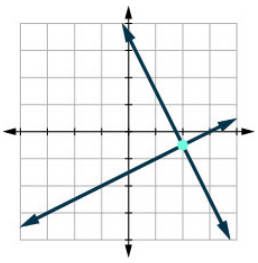
A **consistent system of equations** is a system of equations with at least one solution.

An **inconsistent system of equations** is a system of equations with no solution.

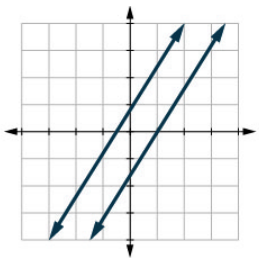
We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are independent, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines.

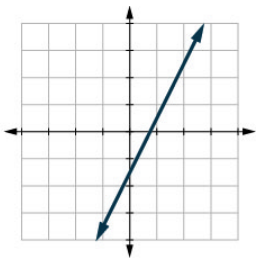
Let's sum this up by looking at the graphs of the three types of systems. See below and Table.



Intersecting



Parallel



Coincident

Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/ independent	Independent	Independent	Dependent

Example 21.2.16

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$ ⓑ $\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$

Answer

ⓐ We will compare the slopes and intercepts of the two lines.

$$\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$$

$y = 3x - 1$

The first equation is already in slope-intercept form.
Write the second equation in slope-intercept form.

$$6x - 2y = 12$$
$$-2y = -6x + 12$$
$$\frac{-2y}{-2} = \frac{-6x+12}{-2}$$
$$y = 3x - 6$$

$y = 3x - 1$
$m = 3$
$b = -1$

$y = 3x - 6$
$m = 3$
$b = -6$

Find the slope and intercept of each line.

Since the slopes are the same andy-intercepts are different, the lines are parallel.

ⓑ We will compare the slope and intercepts of the two lines.

$$\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$$

Write both equations in slope–intercept form.

$2x + y = -3$
$y = -2x - 3$

$x - 5y = 5$
$-5y = -x + 5$
$\frac{-5y}{-5} = \frac{-x+5}{-5}$
$y = \frac{1}{5} - 1$

Find the slope and intercept of each line.

$y = -2x - 3$
$m = -2$
$b = -3$

$y = \frac{1}{5} - 1$
$m = \frac{1}{5}$
$b = -1$

Since the slopes are different, the lines intersect.

A system of equations whose graphs are intersect has 1 solution and is consistent and independent.

Example 21.2.17

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y = -2x - 4 \\ 4x + 2y = 9 \end{cases}$ ⓑ $\begin{cases} 3x + 2y = 2 \\ 2x + y = 1 \end{cases}$

Answer

ⓐ no solution, inconsistent, independent ⓑ one solution, consistent, independent

Example 21.2.18

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y = \frac{1}{3}x - 5 \\ x - 3y = 6 \end{cases}$ ⓑ $\begin{cases} x + 4y = 12 \\ -x + y = 3 \end{cases}$

Answer

ⓐ no solution, inconsistent, independent ⓑ one solution, consistent, independent

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between -10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

Solve a System of Equations by Substitution

We will now solve systems of linear equations by the substitution method.

We will use the same system we used first for graphing.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

Example 21.2.19: How to Solve a System of Equations by Substitution

Solve the system by substitution: $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

Answer

Step 1. Solve one of the equations for either variable.	We'll solve the first equation for y.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $2x + y = 7$ $y = 7 - 2x$
Step 2. Substitute the expression from Step 1 into the other equation.	We replace y in the second equation with the expression $7 - 2x$.	$x - 2y = 6$ $x - 2(7 - 2x) = 6$

Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x-2(7-2x)=6$ $x-14+4x=6$ $5x=20$ $x=4$
Step 4. Substitute the solution from Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace x with 4.	$2x+y=7$ $2(4)+y=7$ $8+y=7$ $y=-1$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $x=4$, $y=-1$ into both equations and make sure they are both true.	$2x+y=7$ $\quad$ $x-2y=6$ $2(4)+(-1)\stackrel{?}{=}7$ $\quad$ $4-2(-1)\stackrel{?}{=}6$ $7=7$ ✓ $\quad$ $6=6$ ✓ Both equations are true. $(4, -1)$ is the solution to the system.

? Example 21.2.20

Solve the system by substitution: $\begin{cases} -2x+y=-11 \\ x+3y=9 \end{cases}$

Answer

$(6, 1)$

? Example 21.2.21

Solve the system by substitution: $\begin{cases} 2x+y=-1 \\ 4x+3y=3 \end{cases}$

Answer

$(-3, 5)$

SOLVE A SYSTEM OF EQUATIONS BY SUBSTITUTION.

1. Solve one of the equations for either variable.
2. Substitute the expression from Step 1 into the other equation.
3. Solve the resulting equation.
4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to **both** original equations.

Be very careful with the signs in the next example.

? Example 21.2.22

Solve the system by substitution: $\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$

Answer

We need to solve one equation for one variable. We will solve the first equation for y.

	$\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$
Solve the first equation for y. Substitute $-2x+2$ for y in the second equation.	$4x+2y=4$ $2y=-4x+4$ $y=-2x+2$ $6x-y=8$
Replace the y with $-2x+2$.	$6x-(-2x+2)=8$
Solve the equation for x.	$6x+2x-2=8$ $8x-2=8$ $8x=10$ $x=\frac{5}{4}$
Substitute $x=54$ into $4x+2y=4$ to find y.	$4x+2y=4$ $4\left(\frac{5}{4}\right)+2y=4$ $5+2y=4$ $2y=-1$ $y=-\frac{1}{2}$
	The ordered pair is $(54,-12)$.
Check the ordered pair in both equations.	$4x+2y=4$ $4\left(\frac{5}{4}\right)+2\left(-\frac{1}{2}\right)\stackrel{?}{=}4$ $5-1\stackrel{?}{=}4$ $4=4\checkmark$ $6x-y=8$ $6\left(\frac{5}{4}\right)-\left(-\frac{1}{2}\right)\stackrel{?}{=}8$ $\frac{15}{4}-\left(-\frac{1}{2}\right)\stackrel{?}{=}8$ $\frac{16}{2}\stackrel{?}{=}8$ $8=8\checkmark$
	The solution is $(54,-12)$.

? Example 21.2.23

Solve the system by substitution: $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$

Answer

$(2,32)$

? Example 21.2.24

Solve the system by substitution: $\begin{cases} 4x-y=0 \\ 2x-3y=5 \end{cases}$

Answer

$(-12,-2)$

Solve a System of Equations by Elimination

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Elimination Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination

method, too, but we'll have a different way to get there.

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions *a*, *b*, *c*, and *d*.

$$\begin{aligned} &\text{if} && a=b \\ &\text{and} && c=d \\ &\text{then} && a+c=b+d. \end{aligned}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:

$$\begin{cases} 3x+y=5 \\ 2x-y=0 \end{cases}$$
$$\underline{\qquad\qquad}$$
$$5x=5$$

The *y*'s add to zero and we have one equation with one variable.

Let's try another one:

$$\begin{cases} +4y=2 \\ 2x+5y=-2 \end{cases}$$

This time we don't see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by -2, we will make the coefficients of *x* opposites. We must multiply every term on both sides of the equation by -2.

$$\begin{aligned} -2(x+4y) &= -2(2) \\ 2x+5y &= -2 \end{aligned}$$

Then rewrite the system of equations.

$$\begin{aligned} -2x-8y &= -4 \\ 2x+5y &= -2 \end{aligned}$$

Now we see that the coefficients of the *x* terms are opposites, so *x* will be eliminated when we add these two equations.

$$\begin{array}{r} -2x-8y=-4 \\ 2x+5y=-2 \\ \hline -3y=-6 \end{array}$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we'll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

? Exercise 21.2.25: How to Solve a System of Equations by Elimination

Solve the system by elimination: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Answer

Step 1. Write both equations in standard form. • If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
Step 2. Make the coefficients of one variable opposites. • Decide which variable you will eliminate. • Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $\begin{cases} 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$
Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$\begin{array}{r} 4x + 2y = 14 \\ x - 2y = 6 \\ \hline 5x \quad\quad = 20 \end{array}$
Step 4. Solve for the remaining variable.	Solve for x.	$x = 4$
Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$
Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $x = 4$, $y = -1$ into $2x + y = 7$ and $x - 2y = 6$. Do they make both equations true? Yes!	$2x + y = 7$ $\quad$ $x - 2y = 6$ $2(4) + (-1) \stackrel{?}{=} 7$ $\quad$ $4 - 2(-1) \stackrel{?}{=} 6$ $7 = 7$ ✓ $\quad$ $6 = 6$ ✓ The solution is $(4, -1)$.

? Exercise 21.2.26

Solve the system by elimination: $\begin{cases} 3x + y = 5 \\ 2x - 3y = 7 \end{cases}$

Answer

$(2, -1)$

? Exercise 21.2.27

Solve the system by elimination: $\begin{cases} 4x + y = -5 \\ -2x - 2y = -2 \end{cases}$

Answer

$(-2, 3)$

The steps are listed here for easy reference.

SOLVE A SYSTEM OF EQUATIONS BY ELIMINATION.

1. Write both equations in standard form. If any coefficients are fractions, clear them.
2. Make the coefficients of one variable opposites.
   - Decide which variable you will eliminate.

- Multiply one or both equations so that the coefficients of that variable are opposites.

3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to **both** original equations.

Now we'll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Exercise 21.2.28

Solve the system by elimination: $\begin{cases} 4x-3y=9 \\ 7x+2y=-6 \end{cases}$

Answer

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by different constants to get the opposites.

	$\begin{aligned} 4x-3y&=9 \\ 7x+2y&=-6 \end{aligned}$
Both equations are in standard form. To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.	$\begin{aligned} 2(4x-3y)&=2(9) \\ 3(7x+2y)&=3(-6) \end{aligned}$
Simplify.	$\begin{aligned} 8x-6y&=18 \\ 21x+6y&=-18 \end{aligned}$
Add the two equations to eliminate y.	$\begin{aligned} 8x-6y&=18 \\ 21x+6y&=-18 \\ \hline 29x&=0 \end{aligned}$
Solve for x.	$\begin{aligned} x&=0 \\ 7x+2y&=-6 \end{aligned}$
Substitute x=0x=0 into one of the original equations.	$7\cdot 0+2y=-6$
Solve for y.	$\begin{aligned} 2y&=-6 \\ y&=-3 \end{aligned}$
Write the solution as an ordered pair.	The ordered pair is $(0,-3)$.
Check that the ordered pair is a solution to **both** original equations. $\begin{aligned} 4x-3y&=9 \\ 4(0)-3(-3)&\stackrel{?}{=}9 \\ 9&=9\checkmark \end{aligned}$ $\begin{aligned} 7x+2y&=-6 \\ 7(0)+2(-3)&\stackrel{?}{=}-6 \\ -6&=-6\checkmark \end{aligned}$	
	The solution is $(0,-3)$.

Exercise 21.2.29

Solve the system by elimination: $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$

Answer

$(1,3)$

Exercise 21.2.30

Solve each system by elimination: $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$

Answer

$(4,-3)$

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by the LCD of all the fractions in the equation.

Exercise 21.2.31

Solve the system by elimination: $\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$

Answer

In this example, both equations have fractions. Our first step will be to multiply each equation by the LCD of all the fractions in the equation to clear the fractions.

	$\begin{aligned} x+\tfrac{1}{2}y&=6 \\ \tfrac{3}{2}x+\tfrac{2}{3}y&=\tfrac{17}{2} \end{aligned}$
To clear the fractions, multiply each equation by its LCD.	$\begin{aligned} 2\left(x+\tfrac{1}{2}y\right)&=2(6) \\ 6\left(\tfrac{3}{2}x+\tfrac{2}{3}y\right)&=6\left(\tfrac{17}{2}\right) \end{aligned}$
Simplify.	$\begin{aligned} 2x+\ y&=12 \\ 9x+4y&=51 \end{aligned}$
Now we are ready to eliminate one of the variables. Notice that both equations are in standard form.	
We can eliminate y by multiplying the top equation by -4.	$\begin{aligned} -4(2x+\ y)&=-4(12) \\ 9x+4y\ &=51 \end{aligned}$
Simplify and add. Substitute $x=3$ into one of the original equations.	$\begin{aligned} -8x-4y&=-48 \\ 9x+4y\ &=\ 51 \\ \hline x\quad &=\ 3 \end{aligned}$ $x+\frac{1}{2}y=6$
Solve for y.	$\begin{aligned} 3+\tfrac{1}{2}y&=6 \\ \tfrac{1}{2}y&=3 \\ y&=6 \end{aligned}$
Write the solution as an ordered pair.	The ordered pair is $(3,6)$.
Check that the ordered pair is a solution to both original equations.	$\begin{aligned} x+\tfrac{1}{2}y&=6 \\ 3+\tfrac{1}{2}(6)&\stackrel{?}{=}6 \\ 3+3&\stackrel{?}{=}6 \\ 6&=6\ \checkmark \end{aligned}$ $\begin{aligned} \tfrac{3}{2}x+\tfrac{2}{3}y&=\tfrac{17}{2} \\ \tfrac{3}{2}(3)+\tfrac{2}{3}(6)&\stackrel{?}{=}\tfrac{17}{2} \\ \tfrac{9}{2}+4&\stackrel{?}{=}\tfrac{17}{2} \\ \tfrac{9}{2}+\tfrac{8}{2}&\stackrel{?}{=}\tfrac{17}{2} \\ \tfrac{17}{2}&=\tfrac{17}{2}\ \checkmark \end{aligned}$
	The solution is $(3,6)$.

Exercise 21.2.32

Solve each system by elimination: $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 1 \\ \frac{3}{4}x - y = \frac{5}{2} \end{cases}$

Answer

$(6, 2)$

Exercise 21.2.33

Solve each system by elimination: $\begin{cases} x + \frac{3}{5}y = -\frac{1}{5} \\ -\frac{1}{2}x - \frac{2}{3}y = \frac{5}{6} \end{cases}$

Answer

$(1, -2)$

When we solved the system by graphing, we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

The same is true using substitution or elimination. If the equation at the end of substitution or elimination is a true statement, we have a consistent but dependent system and the system of equations has infinitely many solutions. If the equation at the end of substitution or elimination is a false statement, we have an inconsistent system and the system of equations has no solution.

Exercise 21.2.34

Solve the system by elimination: $\begin{cases} 3x + 4y = 12 \\ y = 3 - \frac{3}{4}x \end{cases}$

Answer

$$\begin{cases} 3x + 4y = 12 \\ y = 3 - \frac{3}{4}x \end{cases}$$

Write the second equation in standard form.

$$\begin{cases} 3x + 4y = 12 \\ \frac{3}{4}x + y = 3 \end{cases}$$

Clear the fractions by multiplying the second equation by 4.

$$\begin{cases} 3x + 4y = 12 \\ 4(\frac{3}{4}x + y) = 4(3) \end{cases}$$

Simplify.

$$\begin{cases} 3x + 4y = 12 \\ 3x + 4y = 12 \end{cases}$$

To eliminate a variable, we multiply the second equation by -1. Simplify and add.

$$\begin{cases} 3x + 4y = 12 \\ -3x - 4y = -12 \end{cases}$$
$$0 = 0$$

This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

Exercise 21.2.35

Solve the system by elimination: $\begin{cases} 5x-3y=15 \\ 5y=-5+\frac{5}{3}x \end{cases}$

Answer

infinitely many solutions

Exercise 21.2.36

Solve the system by elimination: $\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$

Answer

infinitely many solutions

Choose the Most Convenient Method to Solve a System of Linear Equations

When you solve a system of linear equations in in an application, you will not be told which method to use. You will need to make that decision yourself. So you'll want to choose the method that is easiest to do and minimizes your chance of making mistakes.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations a rein standard form.

Example 21.2.37

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$ ⓑ $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Answer

ⓐ

$$\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$$

Since both equations are in standard form, using elimination will be most convenient.

ⓑ

$$\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$$

Since one equation is already solved for y, using substitution will be most convenient.

Example 21.2.38

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 4x-5y=-32 \\ 3x+2y=-1 \end{cases}$ ⓑ $\begin{cases} x=2y-1 \\ 3x-5y=-7 \end{cases}$

Answer

ⓐ Since both equations are in standard form, using elimination will be most convenient. ⓑ Since one equation is already solved for x, using substitution will be most convenient.

? Example 21.2.39

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} y = 2x - 1 \\ 3x - 4y = -6 \end{cases}$ ⓑ $\begin{cases} 6x - 2y = 12 \\ 3x + 7y = -13 \end{cases}$

Answer

ⓐ Since one equation is already solved for y, using substitution will be most convenient. ⓑ Since both equations are in standard form, using elimination will be most convenient.

Key Concepts

- **How to solve a system of linear equations by graphing.**
  1. Graph the first equation.
  2. Graph the second equation on the same rectangular coordinate system.
  3. Determine whether the lines intersect, are parallel, or are the same line.
  4. Identify the solution to the system.
     If the lines intersect, identify the point of intersection. This is the solution to the system.
     If the lines are parallel, the system has no solution.
     If the lines are the same, the system has an infinite number of solutions.
  5. Check the solution in both equations.
- **How to solve a system of equations by substitution.**
  1. Solve one of the equations for either variable.
  2. Substitute the expression from Step 1 into the other equation.
  3. Solve the resulting equation.
  4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
  5. Write the solution as an ordered pair.
  6. Check that the ordered pair is a solution to **both** original equations.
- **How to solve a system of equations by elimination.**
  1. Write both equations in standard form. If any coefficients are fractions, clear them.
  2. Make the coefficients of one variable opposites.
     Decide which variable you will eliminate.
     Multiply one or both equations so that the coefficients of that variable are opposites.
  3. Add the equations resulting from Step 2 to eliminate one variable.
  4. Solve for the remaining variable.
  5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
  6. Write the solution as an ordered pair.
  7. Check that the ordered pair is a solution to **both** original equations.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations a rein standard form.

Glossary

coincident lines

Coincident lines have the same slope and same y-intercept.

consistent and inconsistent systems

Consistent system of equations is a system of equations with at least one solution; inconsistent system of equations is a system of equations with no solution.

solutions of a system of equations

Solutions of a system of equations are the values of the variables that make *all* the equations true; solution is represented by an ordered pair (x,y).(x,y).

system of linear equations

When two or more linear equations are grouped together, they form a system of linear equations.

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21.2E: Exercises

Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Equations

In the following exercises, determine if the following points are solutions to the given system of equations.

1. $\begin{cases} 2x-6y=0 \\ 3x-4y=5 \end{cases}$

ⓐ $(3,1)$
ⓑ $(-3,4)$

Answer

ⓐ yes ⓑ no

2. $\begin{cases} -3x+y=8 \\ -x+2y=-9 \end{cases}$

ⓐ $(-5,-7)$
ⓑ $(-5,7)$

3. $\begin{cases} x+y=2 \\ y=\frac{3}{4}x \end{cases}$

ⓐ $(87,67)$
ⓑ $(1,34)$

Answer

ⓐ yes ⓑ no

4. $\begin{cases} 2x+3y=6 \\ y=\frac{2}{3}x+2 \end{cases}$

ⓐ $(-6,2)$
ⓑ $(-3,4)$

Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.

5. $\begin{cases} 3x+y=-3 \\ 2x+3y=5 \end{cases}$

Answer

$(-3,2)$

6. $\begin{cases} -x+y=2 \\ 2x+y=-4 \end{cases}$

7. $\begin{cases} y=x+2 \\ y=-2x+2 \end{cases}$

Answer

$(0, 2)$

8. $\begin{cases} y = x - 2 \\ y = -3x + 2 \end{cases}$

9. $\begin{cases} y = \frac{3}{2}x + 1 \\ y = -\frac{1}{2}x + 5 \end{cases}$

Answer

$(2, 4)$

10. $\begin{cases} y = \frac{2}{3}x - 2 \\ y = -\frac{1}{3}x - 5 \end{cases}$

11. $\begin{cases} x + y = -4 \\ -x + 2y = -2 \end{cases}$

Answer

$(-2, 2)$

12. $\begin{cases} -x + 3y = 3 \\ x + 3y = 3 \end{cases}$

13. $\begin{cases} -2x + 3y = 3 \\ x + 3y = 12 \end{cases}$

Answer

$(3, 3)$

14. $\begin{cases} 2x - y = 4 \\ 2x + 3y = 12 \end{cases}$

15. $\begin{cases} x + 3y = -6 \\ y = -\frac{4}{3}x + 4 \end{cases}$

Answer

$(6, -4)$

16. $\begin{cases} -x + 2y = -6 \\ y = -\frac{1}{2}x - 1 \end{cases}$

17. $\begin{cases} -2x + 4y = 4 \\ y = \frac{1}{2}x \end{cases}$

Answer

no solution

18. $\begin{cases} 3x+5y=10 \\ y=-\frac{3}{5}x+1 \end{cases}$

19. $\begin{cases} 4x-3y=8 \\ 8x-6y=14 \end{cases}$

Answer

no solution

20. $\begin{cases} x+3y=4 \\ -2x-6y=3 \end{cases}$

21. $\begin{cases} x=-3y+4 \\ 2x+6y=8 \end{cases}$

Answer

infinite solutions with solution set: $\{(x,y)|2x+6y=8\}$

22. $\begin{cases} 4x=3y+7 \\ 8x-6y=14 \end{cases}$

23. $\begin{cases} 2x+y=6 \\ -8x-4y=-24 \end{cases}$

Answer

infinite solutions with solution set: $\{(x,y)|2x+y=6\}$

24. $\begin{cases} 5x+2y=7 \\ -10x-4y=-14 \end{cases}$

Without graphing, determine the number of solutions and then classify the system of equations.

25. $\begin{cases} y=\frac{2}{3}x+1 \\ -2x+3y=5 \end{cases}$

Answer

1 point, consistent and independent

26. $\begin{cases} y=\frac{3}{2}x+1 \\ 2x-3y=7 \end{cases}$

27. $\begin{cases} 5x+3y=4 \\ 2x-3y=5 \end{cases}$

Answer

1 point, consistent and independent

28. $\begin{cases} y=-12x+5 \\ x+2y=10 \end{cases}$

29. $\begin{cases} 5x-2y=10 \\ y=52x-5 \end{cases}$

Answer

infinite solutions, consistent, dependent

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

30. $\begin{cases} 2x+y=-4 \\ 3x-2y=-6 \end{cases}$

31. $\begin{cases} 2x+y=-2 \\ 3x-y=7 \end{cases}$

Answer

$(1,-4)$

32. $\begin{cases} x-2y=-5 \\ 2x-3y=-4 \end{cases}$

33. $\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$

Answer

$(-3,2)$

34. $\begin{cases} 5x-2y=-6 \\ y=3x+3 \end{cases}$

35. $\begin{cases} -2x+2y=6 \\ y=-3x+1 \end{cases}$

Answer

$(-1/2,5/2)$

36. $\begin{cases} 2x+5y=1 \\ y=\frac{1}{3}x-2 \end{cases}$

37. $\begin{cases} 3x+4y=1 \\ y=-\frac{2}{5}x+2 \end{cases}$

Answer

$(-5,4)$

38. $\begin{cases} 2x+y=5 \\ x-2y=-15 \end{cases}$

39. $\begin{cases} 4x+y=10 \\ x-2y=-20 \end{cases}$

Answer

$(0, 10)$

40. $\begin{cases} y=-2x-1 \\ y=-\frac{1}{3}x+4 \end{cases}$

41. $\begin{cases} y=x-6 \\ y=-\frac{3}{2}x+4 \end{cases}$

Answer

$(4, -2)$

42. $\begin{cases} x=2y \\ 4x-8y=0 \end{cases}$

43. $\begin{cases} 2x-16y=8 \\ -x-8y=-4 \end{cases}$

Answer

$(4, 0)$

44. $\begin{cases} y=\frac{7}{8}x+4 \\ -7x+8y=6 \end{cases}$

45. $\begin{cases} y=-\frac{2}{3}x+5 \\ 2x+3y=11 \end{cases}$

Answer

no solution

Solve a System of Equations by Elimination

In the following exercises, solve the systems of equations by elimination.

46. $\begin{cases} 5x+2y=2 \\ -3x-y=0 \end{cases}$

47. $\begin{cases} 6x-5y=-1 \\ 2x+y=13 \end{cases}$

Answer

$(4, 5)$

48. $\begin{cases} 2x-5y=7 \\ 3x-y=17 \end{cases}$

49. $\begin{cases} 5x-3y=-1 \\ 2x-y=2 \end{cases}$

Answer

$(7,12)$

50. $\begin{cases} 3x-5y=-9 \\ 5x+2y=16 \end{cases}$

51. $\begin{cases} 4x-3y=3 \\ 2x+5y=-31 \end{cases}$

Answer

$(-3,-5)$

52. $\begin{cases} 3x+8y=-3 \\ 2x+5y=-3 \end{cases}$

53. $\begin{cases} 11x+9y=-5 \\ 7x+5y=-1 \end{cases}$

Answer

$(2,-3)$

54. $\begin{cases} 3x+8y=67 \\ 5x+3y=60 \end{cases}$

55. $\begin{cases} 2x+9y=-4 \\ 3x+13y=-7 \end{cases}$

Answer

$(-11,2)$

56. $\begin{cases} \frac{1}{3}x-y=-3 \\ x+\frac{5}{2}y=2 \end{cases}$

57. $\begin{cases} x+\frac{1}{2}y=\frac{3}{2} \\ \frac{1}{5}x-\frac{1}{5}y=3 \end{cases}$

Answer

$(6/-9,24/7)$

58. $\begin{cases} x+\frac{1}{3}y=-1 \\ \frac{1}{3}x+\frac{1}{2}y=1 \end{cases}$

59. $\begin{cases} \frac{1}{3}x-y=-3 \\ \frac{2}{3}x+\frac{5}{2}y=3 \end{cases}$

Answer

$(-3, 2)$

60. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$

61. $\begin{cases} x-4y=-1 \\ -3x+12y=3 \end{cases}$

Answer

infinitely many solutions with solution set: $\{(x, y)|x-4y=-1\}$

62. $\begin{cases} -3x-y=8 \\ 6x+2y=-16 \end{cases}$

63. $\begin{cases} 4x+3y=2 \\ 20x+15y=10 \end{cases}$

Answer

infinitely many solutions with solution set: $\{(x, y)|4x+3y=2\}$

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

64.

ⓐ $\begin{cases} 8x-15y=-32 \\ 6x+3y=-5 \end{cases}$

ⓑ $\begin{cases} x=4y-3 \\ 4x-2y=-6 \end{cases}$

65.

ⓐ $\begin{cases} y=7x-5 \\ 3x-2y=16 \end{cases}$

ⓑ $\begin{cases} 12x-5y=-42 \\ 3x+7y=-15 \end{cases}$

Answer

ⓐ substitution ⓑ elimination

66.

ⓐ $\begin{cases} y=4x+95 \\ x-2y=-21 \end{cases}$

ⓑ $\begin{cases} 9x-4y=24 \\ 3x+5y=-14 \end{cases}$

67.

ⓐ $\begin{cases} 14x-15y=-30 \\ 7x+2y=10 \end{cases}$

ⓑ $\begin{cases} x = 9y - 11 \\ 2x - 7y = -27 \end{cases}$

Answer

ⓐ elimination ⓑ substitution

Writing Exercises

68. In a system of linear equations, the two equations have the same intercepts. Describe the possible solutions to the system.

69. Solve the system of equations by substitution and explain all your steps in words: $\begin{cases} 3x + y = 1 \\ 2x = y - 8 \end{cases}$

Answer

Answers will vary.

70. Solve the system of equations by elimination and explain all your steps in words: $\begin{cases} 5x + 4y = 10 \\ 2x = 3y + 27 \end{cases}$

71. Solve the system of equations $\begin{cases} x + y = 10 \\ x - y = 6 \end{cases}$

ⓐ by graphing ⓑ by substitution
ⓒ Which method do you prefer? Why?

Answer

Answers will vary.

Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether an ordered pair is a solution of a system of equations.			
solve a system of linear equations by graphing.			
solve a system of equations by substitution.			
solve a system of equations by elimination.			
choose the most convenient method to solve a system of linear equations.			

If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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21.3: Solve Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve direct translation applications
- Solve geometry applications
- Solve uniform motion applications

Before you get started, take this readiness quiz.

1. The sum of twice a number and nine is 31. Find the number.
   If you missed this problem, review [link].
2. Twins Jon and Ron together earned $96,000 last year. Ron earned $8000 more than three times what Jon earned. How much did each of the twins earn?
   If you missed this problem, review [link].
3. An express train and a local train leave Pittsburgh to travel to Washington, D.C. The express train can make the trip in four hours and the local train takes five hours for the trip. The speed of the express train is 12 miles per hour faster than the speed of the local train. Find the speed of both trains.
   If you missed this problem, review [link].

Solve Direct Translation Applications

Systems of linear equations are very useful for solving applications. Some people find setting up word problems with two variables easier than setting them up with just one variable. To solve an application, we'll first translate the words into a system of linear equations. Then we will decide the most convenient method to use, and then solve the system.

SOLVE APPLICATIONS WITH SYSTEMS OF EQUATIONS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose variables to represent those quantities.
4. **Translate** into a system of equations.
5. **Solve** the system of equations using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We solved number problems with one variable earlier. Let's see how differently it works using two variables.

?

The sum of two numbers is zero. One number is nine less than the other. Find the numbers.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let $n =$ the first number. $m =$ the second number
Step 4. Translate into a system of equations.	The sum of two numbers is zero.
	$n + m = 0$
	One number is nine less than the other.

	$n = m - 9$
The system is:	$n + m = 0$ $n = m - 9$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for n.	
Substitute $m - 9$ for n in the first equation.	$n = (m - 9)$ $n + m = 0$
Solve for m.	$m - 9 + m = 0$
	$2m - 9 = 0$
	$2m = 9$
Substitute $m = \frac{9}{2}$ into the second equation and then solve for n.	$m = \frac{9}{2}$ $n = m - 9$
	$m = \frac{9}{2} - 9$ $m = \frac{9}{2} - \frac{18}{2}$
	$n = -\frac{9}{2}$
Step 6. Check the answer in the problem.	Do these numbers make sense in the problem? We will leave this to you!
Step 7. Answer the question.	The numbers are $\frac{9}{2}$ and $-\frac{9}{2}$.

? Example 21.3.2

The sum of two numbers is 10. One number is 4 less than the other. Find the numbers.

Answer

$3, 7$

? Example 21.3.3

The sum of two numbers is -6. One number is 10 less than the other. Find the numbers.

Answer

$2, -8$

?

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her \$25,000 plus \$15 for each training session. Option B would pay her $\$10,000 + \$40$ for each training session. How many training sessions would make the salary options equal?

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of training sessions that would make the pay equal.
Step 3. Name what we are looking for.	Let s=s= Heather's salary. n=n= the number of training sessions
Step 4. Translate into a system of equations.	Option A would pay her $25,000 plus $15 for each training session.
	$s = 25{,}000 + 15n$
	Option B would pay her $10,000 + $40 for each training session.
	$s = 10{,}000 + 40n$
The system is shown.	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Step 5. Solve the system of equations. We will use substitution.	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Substitute 25,000 +15n for s in the second equation.	$25{,}000 + 15n = 10{,}000 + 40n$
Solve for n.	$25{,}000 = 10{,}000 + 25n$ $15{,}000 = 25n$ $600 = n$
Step 6. Check the answer.	Are 600 training sessions a year reasonable? Are the two options equal when n = 600?
Step 7. Answer the question.	The salary options would be equal for 600 training sessions.

Example 21.3.5

Geraldine has been offered positions by two insurance companies. The first company pays a salary of $12,000 plus a commission of $100 for each policy sold. The second pays a salary of $20,000 plus a commission of $50 for each policy sold. How many policies would need to be sold to make the total pay the same?

Answer

160 policies

Example 21.3.6

Kenneth currently sells suits for company A at a salary of $22,000 plus a $10 commission for each suit sold. Company B offers him a position with a salary of $28,000 plus a $4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Answer

1000 suits

As you solve each application, remember to analyze which method of solving the system of equations would be most convenient.

Example 21.3.7

Translate to a system of equations and then solve:

When Jenna spent 10 minutes on the elliptical trainer and then did circuit training for 20 minutes, her fitness app says she burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes circuit training she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories for each minute of circuit training?

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of calories burned each minute on the elliptical trainer and each minute of circuit training.
Step 3. Name what we are looking for.	Let e=e= number of calories burned per minute on the elliptical trainer. c=c= number of calories burned per minute while circuit training
Step 4. Translate into a system of equations.	10 minutes on the elliptical and circuit training for 20 minutes, burned 278 calories
	$10e + 20c = 278$
	20 minutes on the elliptical and 30 minutes of circuit training burned 473 calories
	$20e + 30c = 473$
The system is:	$\begin{cases} 10e + 20c = 278 \\ 20e + 30c = 473 \end{cases}$
Step 5. Solve the system of equations.	
Multiply the first equation by −2 to get opposite coefficients of e.	$\begin{cases} -2(10e + 20c) = -2(278) \\ 20e + 30c = 473 \end{cases}$
Simplify and add the equations. Solve for c.	$\begin{cases} -20e - 40c = -556 \\ 20e + 30c = 473 \end{cases}$ $-10c = -83$ $c = 8.3$
Substitute $c = 8.3$ into one of the original equations to solve for e.	$10e + 20c = 278$ $10e + 20(8.3) = 278$ $10e + 166 = 278$ $10e = 112$ $e = 11.2$
Step 6. Check the answer in the problem.	Check the math on your own.
	$\begin{cases} 10(11.2) + 20(8.3) \stackrel{?}{=} 278 \\ 20(11.2) + 30(8.3) \stackrel{?}{=} 473 \end{cases}$
Step 7. Answer the question.	Jenna burns 8.3 calories per minute circuit training and 11.2 calories per minute while on the elliptical trainer.

Example 21.3.8

Translate to a system of equations and then solve:

Mark went to the gym and did 40 minutes of Bikram hot yoga and 10 minutes of jumping jacks. He burned 510 calories. The next time he went to the gym, he did 30 minutes of Bikram hot yoga and 20 minutes of jumping jacks burning 470 calories. How many calories were burned for each minute of yoga? How many calories were burned for each minute of jumping jacks?

Answer

Mark burned 11 calories for each minute of yoga and 7 calories for each minute of jumping jacks.

Example 21.3.9

Translate to a system of equations and then solve:

Erin spent 30 minutes on the rowing machine and 20 minutes lifting weights at the gym and burned 430 calories. During her next visit to the gym she spent 50 minutes on the rowing machine and 10 minutes lifting weights and burned 600 calories. How many calories did she burn for each minutes on the rowing machine? How many calories did she burn for each minute of weight lifting?

Answer

Erin burned 11 calories for each minute on the rowing machine and 5 calories for each minute of weight lifting.

Solve Geometry Applications

We will now solve geometry applications using systems of linear equations. We will need to add complementary angles and supplementary angles to our list some properties of angles.

The measures of two **complementary angles** add to 90 degrees. The measures of two **supplementary angles** add to 180 degrees.

COMPLEMENTARY AND SUPPLEMENTARY ANGLES

Two angles are complementary if the sum of the measures of their angles is 90 degrees.

Two angles are supplementary if the sum of the measures of their angles is 180 degrees

If two angles are complementary, we say that *one angle is the complement of the other.*

If two angles are supplementary, we say that *one angle is the supplement of the other.*

Example 21.3.10

Translate to a system of equations and then solve.

The difference of two complementary angles is 26 degrees. Find the measures of the angles.

Answer

Step 1. Read the problem.

Step 2. Identify what we are looking for. We are looking for the measure of each angle.

Step 3. Name what we are looking for. Let $x =$ the measure of the first angle.
$y =$ the measure of the second angle

Step 4. Translate into a system of equations. The angles are complementary.

$$x + y = 90$$

The difference of the two angles is 26 degrees.

$$x - y = 26$$

The system is shown.

$$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$$

Step 5. Solve the system of equations by elimination.

$$\begin{cases} x + y = 90 \\ \underline{x - y = 26} \end{cases}$$
$$2x = 116$$
$$x = 58$$

Substitute $x = 58$ into the first equation.

$$x + y = 90$$
$$58 + y = 90$$
$$y = 32$$

Step 6. Check the answer in the problem.

$58 + 32 = 90$✓
$58 - 32 = 26$✓

Step 7. Answer the question. The angle measures are 58 and 32 degrees.

Example 21.3.11

Translate to a system of equations and then solve:

The difference of two complementary angles is 20 degrees. Find the measures of the angles.

Answer

The angle measures are 55 and 35.

Example 21.3.12

Translate to a system of equations and then solve:

The difference of two complementary angles is 80 degrees. Find the measures of the angles.

Answer

The angle measures are 5 and 85.

In the next example, we remember that the measures of supplementary angles add to 180.

Example 21.3.13

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is twelve degrees less than five times the measure of the smaller angle. Find the measures of both angles.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for measure of each angle.
Step 3. Name what we are looking for.	Let x=x= the measure of the first angle. y=y= the measure of the second angle
Step 4. Translate into a system of equations.	The angles are supplementary.
	$x+y=180$
	The larger angle is twelve less than five times the smaller angle.
	$y=5x-12$
The system is shown: **Step 5. Solve** the system of equations substitution.	$\begin{cases} x+y=180 \\ y=5x-12 \end{cases}$ $x+y=180$
Substitute $5x-12$ for y in the first equation. Solve for x.	$x+5x-12=180$ $6x-12=180$
Substitute 32 for x in the second equation, then solve for y.	$6x=192$ $x=32$ $y=5x-12$ $y=5\cdot 32-12$ $y=160-12$ $y=148$
Step 6. Check the answer in the problem.	$32+148=180$ ✓ $5\cdot 32-12=148$ ✓
Step 7. Answer the question.	The angle measures are 148 and 32 degrees.

Example 21.3.14

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 12 degrees more than three times the smaller angle. Find the measures of the angles.

Answer

The angle measures are 42 and 138.

Example 21.3.15

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 18 less than twice the measure of the smaller angle. Find the measures of the angles.

Answer

The angle measures are 66 and 114.

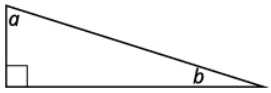
Recall that the angles of a triangle add up to 180 degrees. A right triangle has one angle that is 90 degrees. What does that tell us about the other two angles? In the next example we will be finding the measures of the other two angles.

Example 21.3.16

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle. Find the measures of both angles.

Answer

We will draw and label a figure.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the measures of the angles.
Step 3. Name what we are looking for.	Let a=a= the measure of the first angle. b=b= the measure of the second angle
Step 4. Translate into a system of equations.	The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle.
	$a = 3b + 10$
	The sum of the measures of the angles of a triangle is 180.
	$a + b + 90 = 180$
The system is shown.	$a = 3b + 10$ $a + b + 90 = 180$
Step 5. Solve the system of equations. We will use substitution since the first equation is solved for a.	$a = 3b + 10$ $a + b + 90 = 180$
Substitute 3b+103b+10 for a in the second equation.	$(3b + 10) + b + 90 = 180$
Solve for b.	$4b + 100 = 180$ $4b = 80$ $b = 20$ $a = 3b + 10$
Substitute b=20b=20 into the first equation and then solve for a.	$a = 3 \cdot 20 + 10$ $a = 70$
Step 6. Check the answer in the problem.	We will leave this to you!
Step 7. Answer the question.	The measures of the small angles are 20 and 70 degrees.

Example 21.3.17

The measure of one of the small angles of a right triangle is 2 more than 3 times the measure of the other small angle. Find the measure of both angles.

Answer

$22, 68$

Example 21.3.18

The measure of one of the small angles of a right triangle is 18 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

$36, 54$

Often it is helpful when solving geometry applications to draw a picture to visualize the situation.

Example 21.3.19

Translate to a system of equations and then solve:

Randall has 125 feet of fencing to enclose the part of his backyard adjacent to his house. He will only need to fence around three sides, because the fourth side will be the wall of the house. He wants the length of the fenced yard (parallel to the house wall) to be 5 feet more than four times as long as the width. Find the length and the width.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let L=L= the length of the fenced yard. W=W= the width of the fenced yard
Step 4. Translate into a system of equations.	One lenth and two widths equal 125.
	$L + 2W = 125$
	The length will be 5 feet more than four times the width.
	$L = 4W + 5$
The system is shown. **Step 5. Solve** The system of equations by substitution.	$\begin{cases} L + 2W = 125 \\ L = 4W + 5 \end{cases}$ $L + 2W = 125$
Substitute $L = 4W + 5$ into the first equation, then solve for W.	$4W + 5 + 2W = 125$

	$6W + 5 = 125$ $6W = 120$ $W = 20$
Substitute 20 for W in the second equation, then solve for L.	$L = 4W + 5$ $L = 4 \cdot 20 + 5$ $L = 80 + 5$ $L = 85$
Step 6. Check the answer in the problem.	$20 + 85 + 20 = 125$ $85 = 4 \cdot 20 + 5$ $85 = 85$ ✓
Step 7. Answer the equation.	The length is 85 feet and the width is 20 feet.

? Example 21.3.20

Translate to a system of equations and then solve:

Mario wants to put a fence around the pool in his backyard. Since one side is adjacent to the house, he will only need to fence three sides. There are two long sides and the one shorter side is parallel to the house. He needs 155 feet of fencing to enclose the pool. The length of the long side is 10 feet less than twice the width. Find the length and width of the pool area to be enclosed.

Answer

The length is 60 feet and the width is 35 feet.

? Example 21.3.21

Translate to a system of equations and then solve:

Alexis wants to build a rectangular dog run in her yard adjacent to her neighbor's fence. She will use 136 feet of fencing to completely enclose the rectangular dog run. The length of the dog run along the neighbor's fence will be 16 feet less than twice the width. Find the length and width of the dog run.

Answer

The length is 60 feet and the width is 38 feet.

Solve uniform motion applications

We used a table to organize the information in uniform motion problems when we introduced them earlier. We'll continue using the table here. The basic equation was $D = rt$ where D is the distance traveled, r is the rate, and t is the time.

Our first example of a uniform motion application will be for a situation similar to some we have already seen, but now we can use two variables and two equations.

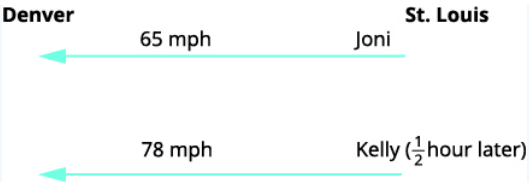
? Example 21.3.22

Translate to a system of equations and then solve:

Joni left St. Louis on the interstate, driving west towards Denver at a speed of 65 miles per hour. Half an hour later, Kelly left St. Louis on the same route as Joni, driving 78 miles per hour. How long will it take Kelly to catch up to Joni?

Answer

A diagram is useful in helping us visualize the situation.



Identify and name what we are looking for. A chart will help us organize the data. We know the rates of both Joni and Kelly, and so we enter them in the chart. We are looking for the length of time Kelly, *k*, and Joni, *j*, will each drive.

	Rate · Time = Distance		
Joni	65	j	$65j$
Kelly	78	k	$78k$

Since $D = r \cdot t$ we can fill in the Distance column.

Translate into a system of equations.

To make the system of equations, we must recognize that Kelly and Joni will drive the same distance. So,

$$65j = 78k$$

Also, since Kelly left later, her time will be $\frac{1}{2}$ hour less than Joni's time. So,

$$k = j - \tfrac{1}{2}$$

Now we have the system.

$$\begin{cases} k = j - \frac{1}{2} \\ 65j = 78k \end{cases}$$

Solve the system of equations by substitution.

Substitute $k = j - 12$ into the second equation, then solve for j.

$$\begin{aligned} 65j &= 78k \\ 65j &= 78(j - \tfrac{1}{2}) \\ 65j &= 78j - 39 \\ -13j &= -39 \\ j &= 3 \end{aligned}$$

To find Kelly's time, substitute $j = 3$ into the first equation, then solve for k.

$$\begin{aligned} k &= j - \tfrac{1}{2} \\ k &= 3 - \tfrac{1}{2} \\ k &= \tfrac{5}{2} \text{ or } k = 2\tfrac{1}{2} \end{aligned}$$

Check the answer in the problem.

Joni 3 hours (65 mph) = 195 miles

Kelly $2\frac{1}{2}$ hours (78 mph) = 195 miles

Yes, they will have traveled the same distance when they meet.

Answer the question.

Kelly will catch up to Joni in $2\frac{1}{2}$ hours. By then, Joni will have traveled 3 hours.

Example 21.3.23

Translate to a system of equations and then solve:

Mitchell left Detroit on the interstate driving south towards Orlando at a speed of 60 miles per hour. Clark left Detroit 1 hour later traveling at a speed of 75 miles per hour, following the same route as Mitchell. How long will it take Clark to catch Mitchell?

Answer

It will take Clark 4 hours to catch Mitchell.

Example 21.3.24

Translate to a system of equations and then solve:

Charlie left his mother's house traveling at an average speed of 36 miles per hour. His sister Sally left 15 minutes $(\frac{1}{4}\text{ hour})$ later traveling the same route at an average speed of 42 miles per hour. How long before Sally catches up to Charlie?

Answer

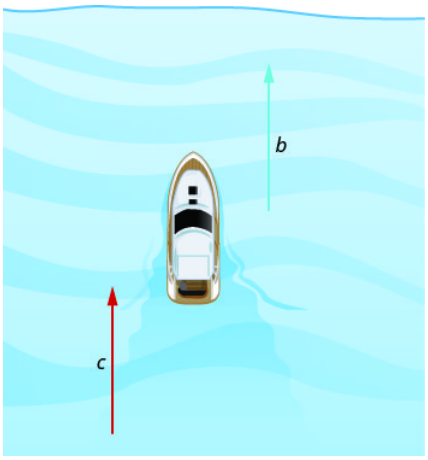
It will take Sally 112 hours to catch up to Charlie.

Many real-world applications of uniform motion arise because of the effects of currents—of water or air—on the actual speed of a vehicle. Cross-country airplane flights in the United States generally take longer going west than going east because of the prevailing wind currents.

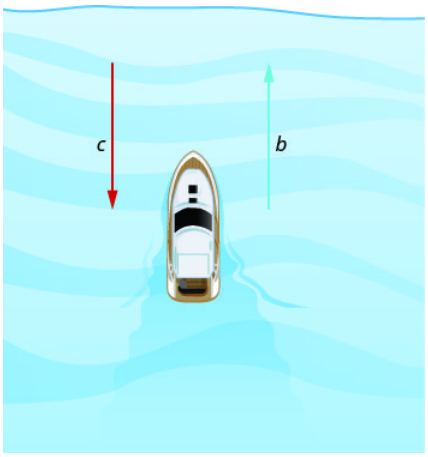
Let's take a look at a boat travelling on a river. Depending on which way the boat is going, the current of the water is either slowing it down or speeding it up.

The images below show how a river current affects the speed at which a boat is actually travelling. We'll call the speed of the boat in still water b and the speed of the river current c.

The boat is going downstream, in the same direction as the river current. The current helps push the boat, so the boat's actual speed is faster than its speed in still water. The actual speed at which the boat is moving is $b+c$.



Now, the boat is going upstream, opposite to the river current. The current is going against the boat, so the boat's actual speed is slower than its speed in still water. The actual speed of the boat is $b-c$.



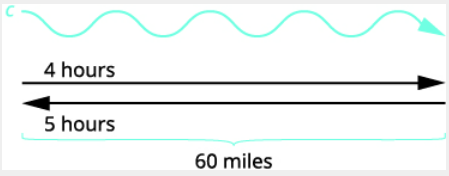
We'll put some numbers to this situation in the next example.

? Example 21.3.25

Translate to a system of equations and then solve.

A river cruise ship sailed 60 miles downstream for 4 hours and then took 5 hours sailing upstream to return to the dock. Find the speed of the ship in still water and the speed of the river current.

Answer

Read the problem.	This is a uniform motion problem and a picture will help us visualize the situation.
	
Identify what we are looking for.	We are looking for the speed of the ship in still water and the speed of the current.
Name what we are looking for.	Let $s =$ the rate of the ship in still water. $c =$ the rate of the current
A chart will help us organize the information. The ship goes downstream and then upstream. Going downstream, the current helps the ship and so the ship's actual rate is $s + c$. Going upstream, the current slows the ship and so the actual rate is $s - c$.	<table><tr><th></th><th colspan="3">Rate • Time = Distance</th></tr><tr><td>downstream</td><td>$s + c$</td><td>4</td><td>60</td></tr><tr><td>upstream</td><td>$s - c$</td><td>5</td><td>60</td></tr></table>
Downstream it takes 4 hours. Upstream it takes 5 hours. Each way the distance is 60 miles.	
Translate into a system of equations. Since rate times time is distance, we can write the system of equations.	$\begin{cases} 4(s + c) = 60 \\ 5(s - c) = 60 \end{cases}$
Solve the system of equations. Distribute to put both equations in standard form, then solve by elimination.	$\begin{cases} 4s + 4c = 60 \\ 5s - 5c = 60 \end{cases}$

Multiply the top equation by 5 and the bottom equation by 4. Add the equations, then solve for s.	$20s + 20c = 300$ $20s - 20c = 240$ $40s = 540$ $s = 13.5$ $4(s + c) = 60$
Substitute $s = 13.5$ into of the original equations.	$4(13.5 + c) = 60$ $54 + 4c = 60$ $4c = 6$ $c = 1.5$
Check the answer in the problem. The downstream rate would be $13.5 + 1.5 = 15$ mph. In 4 hours the ship would travel $15 \cdot 4 = 60$ miles. The upstream rate would be $13.5 - 1.5 = 12$ mph. In 5 hours the ship would travel $12 \cdot 5 = 60$ miles.	
Answer the question.	The rate of the ship is 13.5 mph and the rate of the current is 1.5 mph.

Example 21.3.26

Translate to a system of equations and then solve:

A Mississippi river boat cruise sailed 120 miles upstream for 12 hours and then took 10 hours to return to the dock. Find the speed of the river boat in still water and the speed of the river current.

Answer

The rate of the boat is 11 mph and the rate of the current is 1 mph.

Example 21.3.27

Translate to a system of equations and then solve:

Jason paddled his canoe 24 miles upstream for 4 hours. It took him 3 hours to paddle back. Find the speed of the canoe in still water and the speed of the river current.

Answer

The speed of the canoe is 7 mph and the speed of the current is 1 mph.

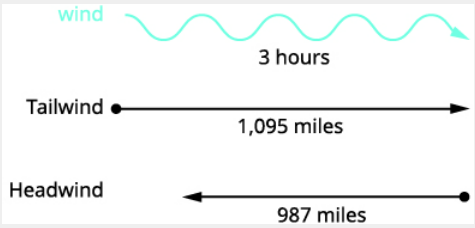
Wind currents affect airplane speeds in the same way as water currents affect boat speeds. We'll see this in the next example. A wind current in the same direction as the plane is flying is called a *tailwind*. A wind current blowing against the direction of the plane is called a *headwind*.

Example 21.3.28

Translate to a system of equations and then solve:

A private jet can fly 1,095 miles in three hours with a tailwind but only 987 miles in three hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

Read the problem.	This is a uniform motion problem and a picture will help us visualize.
	
Identify what we are looking for.	We are looking for the speed of the jet in still air and the speed of the wind.
Name what we are looking for.	Let j=j= the speed of the jet in still air. w=w= the speed of the wind.
A chart will help us organize the information. The jet makes two trips—one in a tailwind and one in a headwind. In a tailwind, the wind helps the jet and so the rate is $j + w$. In a headwind, the wind slows the jet and so the rate is $j - w$.	

	Rate •	Time =	Distance
tailwind	$j+w$	3	1095
headwind	$j-w$	3	987

Each trip takes 3 hours. In a tailwind the jet flies 1,095 miles. In a headwind the jet flies 987 miles.	
Translate into a system of equations. Since rate times time is distance, we get the system of equations.	$3(j+w)=1095$ $3(j-w)=\ 987$
Solve the system of equations. Distribute, then solve by elimination. Add, and solve for j.	$3j+3w=1095$ $3j-3w=\ 987$ $6j\ \ =2082$ $j\ \ =347$ $4(j+w)=1095$
Substitute $j = 347$ into one of the original equations, then solve for w.	$3(347+w)=1095$ $1041+3w=1095$ $3w=54$ $w=18$
Check the answer in the problem. With the tailwind, the actual rate of the jet would be $347+18=365$ mph. In 3 hours the jet would travel $365 \cdot 3 = 1,095$ miles Going into the headwind, the jet's actual rate would be $347-18=329$ mph. In 3 hours the jet would travel $329 \cdot 3 = 987$ miles.	
Answer the question.	The rate of the jet is 347 mph and the rate of the wind is 18 mph.

Example 21.3.29

Translate to a system of equations and then solve:

A small jet can fly 1,325 miles in 5 hours with a tailwind but only 1,035 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

The speed of the jet is 235 mph and the speed of the wind is 30 mph.

Example 21.3.30

Translate to a system of equations and then solve:

A commercial jet can fly 1,728 miles in 4 hours with a tailwind but only 1,536 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

The speed of the jet is 408 mph and the speed of the wind is 24 mph.

Access this online resource for additional instruction and practice with systems of equations.

- Systems of Equations

Key Concepts

- **How To Solve Applications with Systems of Equations**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose variables to represent those quantities.
  4. **Translate** into a system of equations.
  5. **Solve** the system of equations using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Glossary

complementary angles

Two angles are complementary if the sum of the measures of their angles is 90 degrees.

supplementary angles

Two angles are supplementary if the sum of the measures of their angles is 180 degrees.

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21.3E: Exercises

Practice Makes Perfect

Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

1. The sum of two number is 15. One number is 3 less than the other. Find the numbers.

2. The sum of two number is 30. One number is 4 less than the other. Find the numbers.

Answer

13 and 17

3. The sum of two number is −16. One number is 20 less than the other. Find the numbers.

4. The sum of two number is -26. One number is 12 less than the other. Find the numbers.

Answer

-7 and -19

5. The sum of two numbers is 65. Their difference is 25. Find the numbers.

6. The sum of two numbers is 37. Their difference is 9. Find the numbers.

Answer

14 and 23

7. The sum of two numbers is -27. Their difference is -59. Find the numbers.

8. The sum of two numbers is -45. Their difference is -89. Find the numbers.

Answer

22 and -67

9. Maxim has been offered positions by two car companies. The first company pays a salary of $10,000 plus a commission of $1000 for each car sold. The second pays a salary of $20,000 plus a commission of $500 for each car sold. How many cars would need to be sold to make the total pay the same?

10. Jackie has been offered positions by two cable companies. The first company pays a salary of $14,000 plus a commission of $100 for each cable package sold. The second pays a salary of $20,000 plus a commission of $25 for each cable package sold. How many cable packages would need to be sold to make the total pay the same?

Answer

Eighty cable packages would need to be sold to make the total pay the same.

11. Amara currently sells televisions for company A at a salary of $17,000 plus a $100 commission for each television she sells. Company B offers her a position with a salary of $29,000 plus a $20 commission for each television she sells. How televisions would Amara need to sell for the options to be equal?

12. Mitchell currently sells stoves for company A at a salary of $12,000 plus a $150 commission for each stove he sells. Company B offers him a position with a salary of $24,000 plus a $50 commission for each stove he sells. How many stoves would Mitchell need to sell for the options to be equal?

Answer

Mitchell would need to sell 120 stoves for the companies to be equal.

13. Two containers of gasoline hold a total of fifty gallons. The big container can hold ten gallons less than twice the small container. How many gallons does each container hold?

14. June needs 48 gallons of punch for a party and has two different coolers to carry it in. The bigger cooler is five times as large as the smaller cooler. How many gallons can each cooler hold?

Answer

8 and 40 gallons

15. Shelly spent 10 minutes jogging and 20 minutes cycling and burned 300 calories. The next day, Shelly swapped times, doing 20 minutes of jogging and 10 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?

16. Drew burned 1800 calories Friday playing one hour of basketball and canoeing for two hours. Saturday he spent two hours playing basketball and three hours canoeing and burned 3200 calories. How many calories did he burn per hour when playing basketball? How many calories did he burn per hour when canoeing?

Answer

1000 calories playing basketball and 400 calories canoeing

17. Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and thumb drive. Troy bought four notebooks and five thumb drives for $116. Lisa bought two notebooks and three thumb dives for $68. Find the cost of each notebook and each thumb drive.

18. Nancy bought seven pounds of oranges and three pounds of bananas for $17. Her husband later bought three pounds of oranges and six pounds of bananas for $12. What was the cost per pound of the oranges and the bananas?

Answer

Oranges cost $2 per pound and bananas cost $1 per pound

19. Andrea is buying some new shirts and sweaters. She is able to buy 3 shirts and 2 sweaters for $114 or she is able to buy 2 shirts and 4 sweaters for $164. How much does a shirt cost? How much does a sweater cost?

20. Peter is buying office supplies. He is able to buy 3 packages of paper and 4 staplers for $40 or he is able to buy 5 packages of paper and 6 staplers for $62. How much does a package of paper cost? How much does a stapler cost?

Answer

Package of paper $4, stapler $7

21. The total amount of sodium in 2 hot dogs and 3 cups of cottage cheese is 4720 mg. The total amount of sodium in 5 hot dogs and 2 cups of cottage cheese is 6300 mg. How much sodium is in a hot dog? How much sodium is in a cup of cottage cheese?

22. The total number of calories in 2 hot dogs and 3 cups of cottage cheese is 960 calories. The total number of calories in 5 hot dogs and 2 cups of cottage cheese is 1190 calories. How many calories are in a hot dog? How many calories are in a cup of cottage cheese?

Answer

Hot dog 150 calories, cup of cottage cheese 220 calories

23. Molly is making strawberry infused water. For each ounce of strawberry juice, she uses three times as many ounces of water as juice. How many ounces of strawberry juice and how many ounces of water does she need to make 64 ounces of strawberry infused water?

24. Owen is making lemonade from concentrate. The number of quarts of water he needs is 4 times the number of quarts of concentrate. How many quarts of water and how many quarts of concentrate does Owen need to make 100 quarts of lemonade?

Answer

Owen will need 80 quarts of water and 20 quarts of concentrate to make 100 quarts of lemonade.

Solve Geometry Applications

In the following exercises, translate to a system of equations and solve.

25. The difference of two complementary angles is 55 degrees. Find the measures of the angles.

26. The difference of two complementary angles is 17 degrees. Find the measures of the angles.

Answer

53.5degrees and 36.5degree.

27. Two angles are complementary. The measure of the larger angle is twelve less than twice the measure of the smaller angle. Find the measures of both angles.

28. Two angles are complementary. The measure of the larger angle is ten more than four times the measure of the smaller angle. Find the measures of both angles.

Answer

16 degrees and 74 degrees

29. The difference of two supplementary angles is 8 degrees. Find the measures of the angles.

30. The difference of two supplementary angles is 88 degrees. Find the measures of the angles.

Answer

134 degrees and 46 degrees

31. Two angles are supplementary. The measure of the larger angle is four more than three times the measure of the smaller angle. Find the measures of both angles.

32. Two angles are supplementary. The measure of the larger angle is five less than four times the measure of the smaller angle. Find the measures of both angles.

Answer

37 degrees and 143 degrees

33. The measure of one of the small angles of a right triangle is 14 more than 3 times the measure of the other small angle. Find the measure of both angles.

34. The measure of one of the small angles of a right triangle is 26 more than 3 times the measure of the other small angle. Find the measure of both angles.

Answer

16° and 74°

35. The measure of one of the small angles of a right triangle is 15 less than twice the measure of the other small angle. Find the measure of both angles.

36. The measure of one of the small angles of a right triangle is 45 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

45° and 45°

37. Wayne is hanging a string of lights 45 feet long around the three sides of his patio, which is adjacent to his house. The length of his patio, the side along the house, is five feet longer than twice its width. Find the length and width of the patio.

38. Darrin is hanging 200 feet of Christmas garland on the three sides of fencing that enclose his front yard. The length is five feet less than three times the width. Find the length and width of the fencing.

Answer

Width is 41 feet and length is 118 feet.

39. A frame around a family portrait has a perimeter of 90 inches. The length is fifteen less than twice the width. Find the length and width of the frame.

40. The perimeter of a toddler play area is 100 feet. The length is ten more than three times the width. Find the length and width of the play area.

Answer

Width is 10 feet and length is 40 feet.

Solve Uniform Motion Applications

In the following exercises, translate to a system of equations and solve.

41. Sarah left Minneapolis heading east on the interstate at a speed of 60 mph. Her sister followed her on the same route, leaving two hours later and driving at a rate of 70 mph. How long will it take for Sarah's sister to catch up to Sarah?

42. College roommates John and David were driving home to the same town for the holidays. John drove 55 mph, and David, who left an hour later, drove 60 mph. How long will it take for David to catch up to John?

Answer

11 hours

43. At the end of spring break, Lucy left the beach and drove back towards home, driving at a rate of 40 mph. Lucy's friend left the beach for home 30 minutes (half an hour) later, and drove 50 mph. How long did it take Lucy's friend to catch up to Lucy?

44. Felecia left her home to visit her daughter driving 45 mph. Her husband waited for the dog sitter to arrive and left home twenty minutes (1/3 hour) later. He drove 55 mph to catch up to Felecia. How long before he reaches her?

Answer

1.5 hour

45. The Jones family took a 12-mile canoe ride down the Indian River in two hours. After lunch, the return trip back up the river took three hours. Find the rate of the canoe in still water and the rate of the current.

46. A motor boat travels 60 miles down a river in three hours but takes five hours to return upstream. Find the rate of the boat in still water and the rate of the current.

Answer

Boat rate is 16 mph and current rate is 4 mph.

47. A motor boat traveled 18 miles down a river in two hours but going back upstream, it took 4.54.5 hours due to the current. Find the rate of the motor boat in still water and the rate of the current. (Round to the nearest hundredth.)

48. A river cruise boat sailed 80 miles down the Mississippi River for four hours. It took five hours to return. Find the rate of the cruise boat in still water and the rate of the current.

Answer

Boat rate is 18 mph and current rate is 2 mph.

49. A small jet can fly 1072 miles in 4 hours with a tailwind but only 848 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

50. A small jet can fly 1435 miles in 5 hours with a tailwind but only 1,215 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

Jet rate is 265 mph and wind speed is 22 mph.

51. A commercial jet can fly 868 miles in 2 hours with a tailwind but only 792 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

52. A commercial jet can fly 1,320 miles in 3 hours with a tailwind but only 1170 miles in 3 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

Jet rate is 415 mph and wind speed is 25 mph.

Writing Exercises

53. Write an application problem similar to Example. Then translate to a system of equations and solve it.\

54. Write a uniform motion problem similar to Example that relates to where you live with your friends or family members. Then translate to a system of equations and solve it.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve direct translation applications.			
solve geometry applications.			
solve uniform motion applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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21.4: Solve Mixture Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve mixture applications
- Solve interest applications
- Solve applications of cost and revenue functions

Before you get started, take this readiness quiz.

1. Multiply: $4.025(1,562)$
   If you missed this problem, review [link].
2. Write 8.2% as a decimal.
   If you missed this problem, review [link].
3. Earl's dinner bill came to $32.50 and he wanted to leave an 18% tip. How much should the tip be?
   If you missed this problem, review [link].

Solve Mixture Applications

Mixture application involve combining two or more quantities. When we solved mixture applications with coins and tickets earlier, we started by creating a table so we could organize the information. For a coin example with nickels and dimes, the table looked like this:

	Number •	Value =	Total Value
nickels			
dimes			

Using one variable meant that we had to relate the number of nickels and the number of dimes. We had to decide if we were going to let n be the number of nickels and then write the number of dimes in terms of n, or if we would let d be the number of dimes and write the number of nickels in terms of d.

Now that we know how to solve systems of equations with two variables, we'll just let n be the number of nickels and d be the number of dimes. We'll write one equation based on the total value column, like we did before, and the other equation will come from the number column.

For the first example, we'll do a ticket problem where the ticket prices are in whole dollars, so we won't need to use decimals just yet.

Example 21.4.1

Translate to a system of equations and solve:

A science center sold 1,363 tickets on a busy weekend. The receipts totaled $12,146. How many $12 adult tickets and how many $7 child tickets were sold?

Answer

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of adult tickets and the number of child tickets sold.
Step 3. Name what we are looking for.	Let $a =$ the number of adult tickets. $c =$ the number of child tickets
A table will help us organize the data. We have two types of tickets, adult and child.	Write in a and c for the number of tickets.

Write the total number of tickets sold at the bottom of the Number column.	Altogether 1,363 were sold.
Write the value of each type of ticket in the Value column.	The value of each adult ticket is \$12. The value of each child tickets is \$7.
The number times the value gives the total value, so the total value of adult tickets is $a \cdot 12 = 12a$, and the total value of child tickets is $c \cdot 7 = 7c$.	Fill in the Total Value column.
Altogether the total value of the tickets was \$12,146.	<table><tr><th>Type</th><th>Number</th><th>•</th><th>Value</th><th>=</th><th>Total Value</th></tr><tr><td>adult</td><td>a</td><td></td><td>12</td><td></td><td>$11a$</td></tr><tr><td>child</td><td>c</td><td></td><td>7</td><td></td><td>$8c$</td></tr><tr><td></td><td>1,363</td><td></td><td></td><td></td><td>12,146</td></tr></table>
Step 4. Translate into a system of equations.	
The Number column and the Total value column give us the system of equations.	$\begin{cases} a + c = 1{,}363 \\ 12a + 7c = 12{,}146 \end{cases}$
We will use the elimination method to solve this system. Multiply the first equation by -7.	$\begin{cases} -7(a + c) = -7(1{,}363) \\ 12a + 7c = 12{,}146 \end{cases}$
Simplify and add, then solve for a.	$\begin{cases} -7a - 7c = -9{,}541 \\ 12a + 7c = 12{,}146 \end{cases}$ $5a = 2{,}605$ $a = 521$
Substitute $a = 521$ into the first equation, then solve for c.	$a + c = 1{,}363$ $521 + c = 1{,}363$ $c = 842$
Step 6. Check the answer in the problem. 521 adult at \$12 per ticket makes \$ 6,252 842 child at \$7 per ticket makes \$58,994 The total receipts are \$12,146✓	
Step 7. Answer the question.	The science center sold 521 adult tickets and 842 child tickets.

Example 21.4.2

Translate to a system of equations and solve:

The ticket office at the zoo sold 553 tickets one day. The receipts totaled \$3,936. How many \$9 adult tickets and how many \$6 child tickets were sold?

Answer

206 adults, 347 children

Example 21.4.3

Translate to a system of equations and solve:

The box office at a movie theater sold 147 tickets for the evening show, and receipts totaled \$1,302. How many \$11 adult and how many \$8 child tickets were sold?

Answer

42 adults, 105 children

In the next example, we'll solve a coin problem. Now that we know how to work with systems of two variables, naming the variables in the 'number' column will be easy.

Example 21.4.4

Translate to a system of equations and solve:

Juan has a pocketful of nickels and dimes. The total value of the coins is \$8.10. The number of dimes is 9 less than twice the number of nickels. How many nickels and how many dimes does Juan have?

Answer

Step 1. Read the problem. We will create a table to organize the information.	
Step 2. Identify what we are looking for.	We are looking for the number of nickels and the number of dimes.
Step 3. Name what we are looking for.	Let $n =$ the number of nickels. $d =$ the number of dimes
A table will help us organize the data. We have two types of coins, nickels and dimes.	Write *n* and *d* for the number of each type of coin.
Fill in the Value column with the value of each type of coin.	The value of each nickel is \$0.05. The value of each dime is \$0.10.
The number times the value gives the total value, so, the total value of the nickels is $n(0.05) = 0.05n$ and the total value of dimes is $d(0.10) = 0.10d$. Altogether the total value of the coins is \$8.10.	**Type** | **Number** • **Value** = **Total Value** nickels | n | 0.05 | $0.05n$ dimes | d | 0.10 | $0.10d$ | | | 8.10
Step 4. Translate into a system of equations.	
The Total Value column gives one equation.	$0.05n + 0.10d = 8.10$
We also know the number of dimes is 9 less than twice the number of nickels.	
Translate to get the second equation.	$d = 2n - 9$
Now we have the system to solve.	$\begin{cases} 0.05n + 0.10d = 8.10 \\ d = 2n - 9 \end{cases}$
Step 5. Solve the system of equations We will use the substitution method.	
Substitute $d = 2n - 9$ into the first equation.	$0.05n + 0.10d = 8.10$ $0.05n + 0.10(2n - 9) = 8.10$
Simplify and solve for *n*.	$0.05n + 0.2n - 0.90 = 8.10$ $0.25n - 0.90 = 8.10$ $0.25n = 9.00$ $n = 36$
To find the number of dimes, substitute $n = 36$ into the second equation.	$d = 2n - 9$ $d = 2 \cdot 36 - 9$ $d = 63$

Step 6. Check the answer in the problem 63 dimes at $\$0.10 = \$6.30$ 36 nickels at $\$0.05 = \$1.80$ Total $= \$8.10$✓	
Step 7. Answer the question.	Juan has 36 nickels and 63 dimes.

? Example 21.4.5

Translate to a system of equations and solve:

Matilda has a handful of quarters and dimes, with a total value of $8.55. The number of quarters is 3 more than twice the number of dimes. How many dimes and how many quarters does she have?

Answer

13 dimes and 29 quarters

? Example 21.4.6

Translate to a system of equations and solve:

Priam has a collection of nickels and quarters, with a total value of $7.30. The number of nickels is six less than three times the number of quarters. How many nickels and how many quarters does he have?

Answer

19 quarters and 51 nickels

Some mixture applications involve combining foods or drinks. Example situations might include combining raisins and nuts to make a trail mix or using two types of coffee beans to make a blend.

? Example 21.4.7

Translate to a system of equations and solve:

Carson wants to make 20 pounds of trail mix using nuts and chocolate chips. His budget requires that the trail mix costs him $7.60. per pound. Nuts cost $9.00 per pound and chocolate chips cost $2.00 per pound. How many pounds of nuts and how many pounds of chocolate chips should he use?

Answer

Step 1. Read the problem. We will create a table to organize the information.	
Step 2. Identify what we are looking for.	We are looking for the number of pounds of nuts and the number of pounds of chocolate chips.
Step 3. Name what we are looking for.	Let $n =$ the number of pound of nuts. $c =$ the number of pounds of chips
Carson will mix nuts and chocolate chips to get trail mix. Write in n and c for the number of pounds of nuts and chocolate chips.	

Type	Number of pounds	• Value	= Total Value
Nuts	n	9.00	$9n$
Chocolate chips	c	2.00	$2c$
Trail mix	20	7.60	$7.60(20) = 152$

There will be 20 pounds of trail mix. Put the price per pound of each item in the Value column. Fill in the last column using $\text{Number} \bullet \text{Value} = \text{Total Value}$	
Step 4. Translate into a system of equations. We get the equations from the Number and Total Value columns.	$\begin{cases} n + c = 20 \\ 9n + 2c = 152 \end{cases}$
Step 5. Solve the system of equations We will use elimination to solve the system. Multiply the first equation by -2 to eliminate c.	$\begin{cases} -2(n + c) = -2(20) \\ 9n + 2c = 152 \end{cases}$
Simplify and add. Solve for n.	$\begin{cases} -2n - 2c = -40 \\ 9n + 2c = 152 \end{cases}$ $7n = 112$ $n = 16$
To find the number of pounds of chocolate chips, substitute $n = 16$ into the first equation, then solve for c.	$n + c = 20$ $16 + c = 20$ $c = 4$
Step 6. Check the answer in the problem. $16 + 4 = 20\checkmark$ $9 \cdot 16 + 2 \cdot 4 = 152\checkmark$	
Step 7. Answer the question.	Carson should mix 16 pounds of nuts with 4 pounds of chocolate chips to create the trail mix.

? Example 21.4.8

Translate to a system of equations and solve:

Greta wants to make 5 pounds of a nut mix using peanuts and cashews. Her budget requires the mixture to cost her \$6 per pound. Peanuts are \$4 per pound and cashews are \$9 per pound. How many pounds of peanuts and how many pounds of cashews should she use?

Answer

3 pounds peanuts and 2 pounds cashews

? Example 21.4.9

Translate to a system of equations and solve:

Sammy has most of the ingredients he needs to make a large batch of chili. The only items he lacks are beans and ground beef. He needs a total of 20 pounds combined of beans and ground beef and has a budget of \$3 per pound. The price of beans is \$1 per pound and the price of ground beef is \$5 per pound. How many pounds of beans and how many pounds of ground beef should he purchase?

Answer

10 pounds of beans, 10 pounds of ground beef

Another application of mixture problems relates to concentrated cleaning supplies, other chemicals, and mixed drinks. The concentration is given as a percent. For example, a 20% concentrated household cleanser means that 20% of the total amount is

cleanser, and the rest is water. To make 35 ounces of a 20% concentration, you mix 7 ounces (20% of 35) of the cleanser with 28 ounces of water.

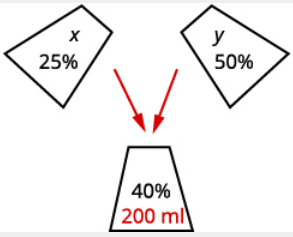
For these kinds of mixture problems, we'll use "percent" instead of "value" for one of the columns in our table.

? Example 21.4.10

Translate to a system of equations and solve:

Sasheena is lab assistant at her community college. She needs to make 200 milliliters of a 40% solution of sulfuric acid for a lab experiment. The lab has only 25% and 50% solutions in the storeroom. How much should she mix of the 25% and the 50% solutions to make the 40% solution?

Answer

Step 1. Read the problem. A figure may help us visualize the situation, then we will create a table to organize the information.	Sasheena must mix some of the 25 solution and some of the 50 solution together to get 200 ml of the 40 solution.
	
Step 2. Identify what we are looking for.	We are looking for how much of each solution she needs.
Step 3. Name what we are looking for.	Let $x =$ number of ml of 25 \(y= \text{number of }ml\text{ of }50%\text{ solution}\)
A table will help us organize the data. She will mix x ml of 25 with y ml of 50 to get 200 ml of 40 solution. We write the percents as decimals in the chart. We multiply the number of units times the concentration to get the total amount of sulfuric acid in each solution.	

Type	Number of units	• Concentration %	= Amount
25%	x	0.25	$0.25x$
50%	y	0.50	$0.50y$
40%	200	0.40	$0.40(200)$

Step 4. Translate into a system of equations. We get the equations from the Number column and the Amount column. Now we have the system.	$\begin{aligned} x + y &= 200 \\ 0.25x + 0.50y &= 0.40(200) \end{aligned}$
Step 5. Solve the system of equations We will solve the system by elimination. Multiply the first equation by -0.5 to eliminate y.	$\begin{aligned} -0.5(x + y) &= -0.5(200) \\ 0.25x + 0.50y &= 80 \end{aligned}$
Simplify and add to solve for x.	$\begin{array}{rcl} -0.5x - 0.5y &=& -100 \\ 0.25x + 0.5y &=& 80 \\ \hline -0.25x &=& -20 \\ x &=& 80 \end{array}$
To solve for y, substitute $x = 80$ into the first equation.	$\begin{aligned} x + y &= 200 \\ 80 + y &= 200 \\ y &= 120 \end{aligned}$

Step 6. Check the answer in the problem. $80+120 = 200$✓ $0.25(80)+0.50(120) = 200$✓ Yes!	
Step 7. Answer the question.	Sasheena should mix 80 ml of the 25 solution with 120 ml of the 50 solution to get the 200 ml of the 40 solution.

? Example 21.4.11

Translate to a system of equations and solve:

LeBron needs 150 milliliters of a 30% solution of sulfuric acid for a lab experiment but only has access to a 25% and a 50% solution. How much of the 25% and how much of the 50% solution should he mix to make the 30% solution?

Answer

120 ml of 25% solution and 30 ml of 50% solution

? Example 21.4.12

Translate to a system of equations and solve:

Anatole needs to make 250 milliliters of a 25% solution of hydrochloric acid for a lab experiment. The lab only has a 10% solution and a 40% solution in the storeroom. How much of the 10% and how much of the 40% solutions should he mix to make the 25% solution?

Answer

125 ml of 10% solution and 125 ml of 40% solution

Solve Interest Applications

The formula to model simple interest applications is $I = Prt$. Interest, I, is the product of the principal, P, the rate, r, and the time, t. In our work here, we will calculate the interest earned in one year, so t will be 1.

We modify the column titles in the mixture table to show the formula for interest, as you'll see in the next example.

Translate to a system of equations and solve:

Adnan has $40,000 to invest and hopes to earn 7.1 interest per year. He will put some of the money into a stock fund that earns 8% per year and the rest into bonds that earns 3% per year. How much money should he put into each fund?

Answer

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the amount to invest in each fund.
Step 3. Name what we are looking for.	Let $s =$ the amount invested in stocks. $b =$ the amount invested in stocks
Write the interest rate as a decimal for each fund. Multiply: Principal · Rate · Time	

Account	Principal •	Rate	• Time =	Interest
Stock fund	s	0.08	1	$0.08s$
Bonds	b	0.03	1	$0.03b$
Total	40,000	0.071		0.071(40,000)

Step 4. Translate into a system of equations. We get our system of equations from the Principal column and the Interest column.	$\begin{cases} s+b=40{,}000 \\ 0.08s+0.03b=0.071(40{,}000) \end{cases}$
Step 5. Solve the system of equations by elimination. Multiply the top equation by -0.03.	$\begin{cases} -0.03(s+b)=-0.03(40{,}000) \\ 0.08s+0.03b=2{,}840 \end{cases}$
Simplify and add to solve for s.	$\begin{cases} -0.03s-0.03b=-1{,}200 \\ 0.08s+0.03b=2{,}840 \end{cases}$ $0.05s=1{,}640$ $s=32{,}800$
To find b, substitute $s = 32{,}800$ into the first equation.	$s+b=40{,}000$ $32{,}800+b=40{,}000$ $b=7{,}200$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	Adnan should invest \$32,800 in stock and \$7,200 in bonds.

Did you notice that the Principal column represents the total amount of money invested while the Interest column represents only the interest earned? Likewise, the first equation in our system, $s+b=40,000$, represents the total amount of money invested and the second equation, $0.08s+0.03b=0.071(40,000)$represents the interest earned.

Example 21.4.14

Translate to a system of equations and solve:

Leon had \$50,000 to invest and hopes to earn 6.2 interest per year. He will put some of the money into a stock fund that earns 7% per year and the rest in to a savings account that earns 2% per year. How much money should he put into each fund?

Answer

\$42,000 in the stock fund and \$8000 in the savings account

Example 21.4.15

Translate to a system of equations and solve:

Julius invested \$7000 into two stock investments. One stock paid 11% interest and the other stock paid 13% interest. He earned 12.5 interest on the total investment. How much money did he put in each stock?

Answer

\$1750 at 11% and \$5250 at 13%

The next example requires that we find the principal given the amount of interest earned.

Example 21.4.16

Translate to a system of equations and solve:

Rosie owes \$21,540 on her two student loans. The interest rate on her bank loan is 10.5 and the interest rate on the federal loan is 5.9. The total amount of interest she paid last year was $\$1,669.68$ What was the principal for each loan?

Answer

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the principal of each loan.
Step 3. Name what we are looking for.	Let $b =$ the principal for the bank loan. $f =$ the principal on the federal loan
The total loans are $21,540.	
Record the interest rates as decimals in the chart. Multiply using the formula $I = Prt$ to get the Interest.	

Account	Principal	• Rate	• Time	= Interest
Bank	b	0.105	1	$0.105b$
Federal	f	0.059	1	$0.059f$
Total	21,540			1669.68

Step 4. Translate into a system of equations. The system of equations comes from the Principal column and the Interest column.	$b + f = 21{,}540$ $0.105b + 0.059f = 1669.68$
Step 5. Solve the system of equations We will use substitution to solve. Solve the first equation for b.	$b + f = 21{,}540$ $b = -f + 21{,}540$
	$0.105b + 0.059f = 1669.68$
Substitute $b = -f + 21.540$ into the second equation.	$0.105(-f + 21{,}540) + 0.059f = 1669.68$ $-0.105f + 2261.70 + 0.059f = 1669.68$
Simplify and solve for f.	$-0.046f + 2261.70 = 1669.68$ $-0.046f = -592.02$ $f = 12{,}870$
To find b, substitute $f = 12{,}870$ into the first equation.	$b + f = 21{,}540$ $b + 12{,}870 = 21{,}540$ $b = 8{,}670$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	The principal of the federal loan was $12,870 and the principal for the bank loan was $8,670.

Example 21.4.17

Translate to a system of equations and solve:

Laura owes $18,000 on her student loans. The interest rate on the bank loan is 2.5% and the interest rate on the federal loan is 6.9%. The total amount of interest she paid last year was $1,066. What was the principal for each loan?

Answer

Bank $4,000; Federal $14,000

Example 21.4.18

Translate to a system of equations and solve:

Jill's Sandwich Shoppe owes $65,200 on two business loans, one at 4.5% interest and the other at 7.2% interest. The total amount of interest owed last year was $3,582. What was the principal for each loan?

Answer

$41,200 at 4.5%, $24,000 at 7.2%

Solve applications of cost and revenue functions

Suppose a company makes and sells *x* units of a product. The cost to the company is the total costs to produce *x* units. This is the cost to manufacture for each unit times *x*, the number of units manufactured, plus the fixed costs.

The **revenue** is the money the company brings in as a result of selling *x* units. This is the selling price of each unit times the number of units sold.

When the costs equal the revenue we say the business has reached the **break-even point**.

COST AND REVENUE FUNCTIONS

The **cost function** is the cost to manufacture each unit times *x*, the number of units manufactured, plus the fixed costs.

$$C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$$

The **revenue** function is the selling price of each unit times *x*, the number of units sold.

$$R(x) = (\text{selling price per unit}) \cdot x$$

The **break-even point** is when the revenue equals the costs.

$$C(x) = R(x)$$

Example 21.4.19

The manufacturer of a weight training bench spends $105 to build each bench and sells them for $245. The manufacturer also has fixed costs each month of $7,000.

ⓐ Find the cost function *C* when *x* benches are manufactured.

ⓑ Find the revenue function *R* when *x* benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ The manufacturer has $7,000 of fixed costs no matter how many weight training benches it produces. In addition to the fixed costs, the manufacturer also spends $105 to produce each bench. Suppose *x* benches are sold.

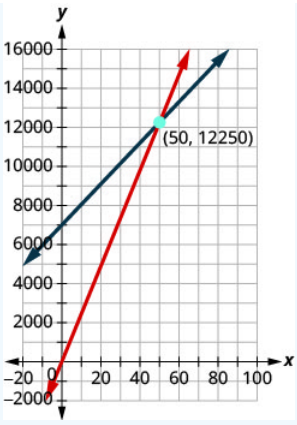
Write the general Cost function formula.	$C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$
Substitute in the cost values.	$C(x) = 105x + 7000$

ⓑ The manufacturer sells each weight training bench for $245. We get the total revenue by multiplying the revenue per unit times the number of units sold.

Write the general Revenue function.	$C(x) = (\text{selling price per unit}) \cdot x$
Substitute in the revenue per unit.	$R(x) = 245x$

ⓒ Essentially we have a system of linear equations. We will show the graph of the system as this helps make the idea of a break-even point more visual.

$$\begin{cases} C(x) = 105x + 7000 \\ R(x) = 245x \end{cases} \quad \text{or} \quad \begin{cases} y = 105x + 7000 \\ y = 245x \end{cases}$$



ⓓ To find the actual value, we remember the break-even point occurs when costs equal revenue.

Write the break-even formula. $C(x) = R(x)$

Solve.
$$105x + 7000 = 245x$$
$$7000 = 140x$$
$$50 = x$$

When 50 benches are sold, the costs equal the revenue.

$C(x) = 105x + 7000$	$R(x) = 245x$
$C(50) = 105(50) + 7000$	$R(50) = 245 \cdot 50$
$C(50) = 12{,}250$	$R(50) = 12{,}250$

When 50 benches are sold, the revenue and costs are both $12,250. Notice this corresponds to the ordered pair $(50, 12250)$.

Example 21.4.20

The manufacturer of a weight training bench spends $15 to build each bench and sells them for $32. The manufacturer also has fixed costs each month of $25,500.

ⓐ Find the cost function C when x benches are manufactured.

ⓑ Find the revenue function R when x benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

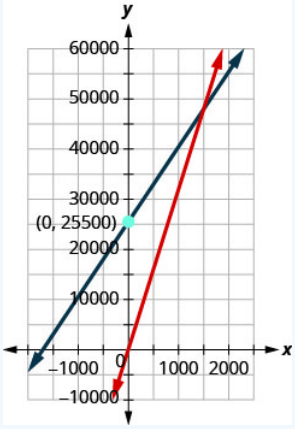
ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ $C(x) = 15x + 25,500$

ⓑ $R(x) = 32x$

ⓒ



ⓓ 1,5001,500; when 1,500 benches are sold, the cost and revenue will be both 48,000

Example 21.4.21

The manufacturer of a weight training bench spends $120 to build each bench and sells them for $170. The manufacturer also has fixed costs each month of $150,000.

ⓐ Find the cost function C when x benches are manufactured.

ⓑ Find the revenue function R when x benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

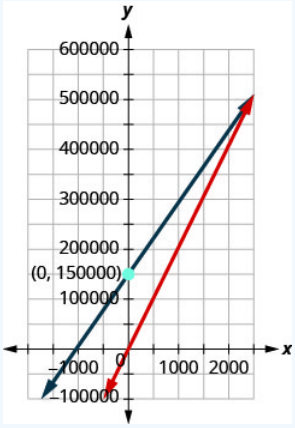
ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ $C(x) = 120x + 150,000$

ⓑ $R(x) = 170x$

ⓒ



ⓓ $3,000$; when 3,000 benches are sold, the revenue and costs are both $510,000

Access this online resource for additional instruction and practice with interest and mixtures.

- Interest and Mixtures

Key Concepts

- **Cost function:** The cost function is the cost to manufacture each unit times *x*, the number of units manufactured, plus the fixed costs.

  $C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$

- **Revenue:** The revenue function is the selling price of each unit times *x*, the number of units sold.

  $R(x) = (\text{selling price per unit}) \cdot x$

- **Break-even point:** The break-even point is when the revenue equals the costs.

  $C(x) = R(x)$

Glossary

cost function

The cost function is the cost to manufacture each unit times xx, the number of units manufactured, plus the fixed costs; $C(x)$ = (cost per unit)x + fixed costs.

revenue

The revenue is the selling price of each unit times *x*, the number of units sold; $R(x)$ = (selling price per unit)x.

break-even point

The point at which the revenue equals the costs is the break-even point; C(x)=R(x).

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21.4E: Exercises

Practice Makes Perfect

Solve Mixture Applications

In the following exercises, translate to a system of equations and solve.

1. Tickets to a Broadway show cost $35 for adults and $15 for children. The total receipts for 1650 tickets at one performance were $47,150. How many adult and how many child tickets were sold?

2. Tickets for the Cirque du Soleil show are $70 for adults and $50 for children. One evening performance had a total of 300 tickets sold and the receipts totaled $17,200. How many adult and how many child tickets were sold?

Answer

110 adult tickets, 190 child tickets

3. Tickets for an Amtrak train cost $10 for children and $22 for adults. Josie paid $1200 for a total of 72 tickets. How many children tickets and how many adult tickets did Josie buy?

4. Tickets for a Minnesota Twins baseball game are $69 for Main Level seats and $39 for Terrace Level seats. A group of sixteen friends went to the game and spent a total of $804 for the tickets. How many of Main Level and how many Terrace Level tickets did they buy?

Answer

6 good seats, 10 cheap seats

5. Tickets for a dance recital cost $15 for adults and $7 dollars for children. The dance company sold 253 tickets and the total receipts were $2771. How many adult tickets and how many child tickets were sold?

6. Tickets for the community fair cost $12 for adults and $5 dollars for children. On the first day of the fair, 312 tickets were sold for a total of $2204. How many adult tickets and how many child tickets were sold?

Answer

92 adult tickets, 220 children tickets

7. Brandon has a cup of quarters and dimes with a total value of $\$3.80$. The number of quarters is four less than twice the number of quarters. How many quarters and how many dimes does Brandon have?

8. Sherri saves nickels and dimes in a coin purse for her daughter. The total value of the coins in the purse is $\$0.95$. The number of nickels is two less than five times the number of dimes. How many nickels and how many dimes are in the coin purse?

Answer

13 nickels, 3 dimes

9. Peter has been saving his loose change for several days. When he counted his quarters and nickels, he found they had a total value $\$13.10$. The number of quarters was fifteen more than three times the number of dimes. How many quarters and how many dimes did Peter have?

10. Lucinda had a pocketful of dimes and quarters with a value of $6.20. The number of dimes is eighteen more than three times the number of quarters. How many dimes and how many quarters does Lucinda have?

Answer

42 dimes, 8 quarters

11. A cashier has 30 bills, all of which are $10 or $20 bills. The total value of the money is $460. How many of each type of bill does the cashier have?

12. A cashier has 54 bills, all of which are $10 or $20 bills. The total value of the money is $910. How many of each type of bill does the cashier have?

Answer

17 $10 bills, 37 $20 bills

13. Marissa wants to blend candy selling for $1.80 per pound with candy costing $1.20 per pound to get a mixture that costs her $1.40 per pound to make. She wants to make 90 pounds of the candy blend. How many pounds of each type of candy should she use?

14. How many pounds of nuts selling for $6 per pound and raisins selling for $3 per pound should Kurt combine to obtain 120 pounds of trail mix that cost him $5 per pound?

Answer

80 pounds nuts and 40 pounds raisins

15. Hannah has to make twenty-five gallons of punch for a potluck. The punch is made of soda and fruit drink. The cost of the soda is $1.79 per gallon and the cost of the fruit drink is $2.49 per gallon. Hannah's budget requires that the punch cost $2.21 per gallon. How many gallons of soda and how many gallons of fruit drink does she need?

16. Joseph would like to make twelve pounds of a coffee blend at a cost of $6 per pound. He blends Ground Chicory at $5 a pound with Jamaican Blue Mountain at $9 per pound. How much of each type of coffee should he use?

Answer

9 pounds of Chicory coffee, 3 pounds of Jamaican Blue Mountain coffee

17. Julia and her husband own a coffee shop. They experimented with mixing a City Roast Columbian coffee that cost $7.80 per pound with French Roast Columbian coffee that cost $8.10 per pound to make a twenty-pound blend. Their blend should cost them $7.92 per pound. How much of each type of coffee should they buy?

18. Twelve-year old Melody wants to sell bags of mixed candy at her lemonade stand. She will mix M&M's that cost $4.89 per bag and Reese's Pieces that cost $3.79 per bag to get a total of twenty-five bags of mixed candy. Melody wants the bags of mixed candy to cost her $4.23 a bag to make. How many bags of M&M's and how many bags of Reese's Pieces should she use?

Answer

10 bags of M&M's, 15 bags of Reese's Pieces

19. Jotham needs 70 liters of a 50% solution of an alcohol solution. He has a 30% and an 80% solution available. How many liters of the 30% and how many liters of the 80% solutions should he mix to make the 50% solution?

20. Joy is preparing 15 liters of a 25% saline solution. She only has 40% and 10% solution in her lab. How many liters of the 40% and how many liters of the 10% should she mix to make the 25% solution?

Answer

7.5 liters of each solution

21. A scientist needs 65 liters of a 15% alcohol solution. She has available a 25% and a 12% solution. How many liters of the 25% and how many liters of the 12% solutions should she mix to make the 15% solution?

22. A scientist needs 120 milliliters of a 20% acid solution for an experiment. The lab has available a 25% and a 10% solution. How many liters of the 25% and how many liters of the 10% solutions should the scientist mix to make the 20% solution?

Answer

80 liters of the 25% solution and 40 liters of the 10% solution

23. A 40% antifreeze solution is to be mixed with a 70% antifreeze solution to get 240 liters of a 50% solution. How many liters of the 40% and how many liters of the 70% solutions will be used?

24. A 90% antifreeze solution is to be mixed with a 75% antifreeze solution to get 360 liters of an 85% solution. How many liters of the 90% and how many liters of the 75% solutions will be used?

Answer

240 liters of the 90% solution and 120 liters of the 75% solution

Solve Interest Applications

In the following exercises, translate to a system of equations and solve.

25. Hattie had $3000 to invest and wants to earn 10.6% interest per year. She will put some of the money into an account that earns 12% per year and the rest into an account that earns 10% per year. How much money should she put into each account?

26. Carol invested $2560 into two accounts. One account paid 8% interest and the other paid 6% interest. She earned 7.25% interest on the total investment. How much money did she put in each account?

Answer

$1600 at 8%, 960 at 6%

27. Sam invested $48,000, some at 6% interest and the rest at 10%. How much did he invest at each rate if he received $4000 in interest in one year?

28. Arnold invested $64,000, some at 5.5% interest and the rest at 9%. How much did he invest at each rate if he received $4500 in interest in one year?

Answer

$28,000 at 9%, $36,000 at 5.5%

29. After four years in college, Josie owes $65, 800 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owes for one year was $2878.50 What is the amount of each loan?

30. Mark wants to invest $10,000 to pay for his daughter's wedding next year. He will invest some of the money in a short term CD that pays 12% interest and the rest in a money market savings account that pays 5% interest. How much should he

invest at each rate if he wants to earn $1095 in interest in one year?

Answer

$8500 CD, $1500 savings account

31. A trust fund worth $25,000 is invested in two different portfolios. This year, one portfolio is expected to earn 5.25% interest and the other is expected to earn 4%. Plans are for the total interest on the fund to be $1150 in one year. How much money should be invested at each rate?

32. A business has two loans totaling $85,000. One loan has a rate of 6% and the other has a rate of 4.5% This year, the business expects to pay $4,650 in interest on the two loans. How much is each loan?

Answer

$55,000 on loan at 6% and $30,000 on loan at 4.5%

Solve Applications of Cost and Revenue Functions

33. The manufacturer of an energy drink spends $1.20 to make each drink and sells them for $2. The manufacturer also has fixed costs each month of $8,000.

ⓐ Find the cost function C when x energy drinks are manufactured.

ⓑ Find the revenue function R when x drinks are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

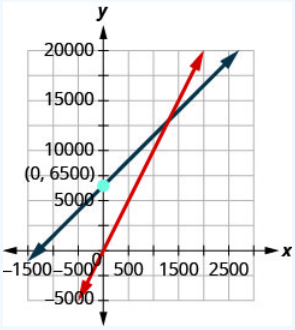
34. The manufacturer of a water bottle spends $5 to build each bottle and sells them for $10. The manufacturer also has fixed costs each month of $6500. ⓐ Find the cost function C when x bottles are manufactured. ⓑ Find the revenue function R when x bottles are sold. ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid. ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ $C(x) = 5x + 6500$

ⓑ $R(x) = 10x$

ⓒ



ⓓ 1,500; when 1,500 water bottles are sold, the cost and the revenue equal $15,000

Writing Exercises

35. Take a handful of two types of coins, and write a problem similar to Example relating the total number of coins and their total value. Set up a system of equations to describe your situation and then solve it.

36. In Example, we used elimination to solve the system of equations
$\begin{cases} s+b=40,000 \\ 0.08s+0.03b=0.071(40,000). \end{cases}$
Could you have used substitution or elimination to solve this system? Why?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve mixture applications.			
solve interest applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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21.5: Solve Systems of Equations with Three Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered triple is a solution of a system of three linear equations with three variables
- Solve a system of linear equations with three variables
- Solve applications using systems of linear equations with three variables

Before you get started, take this readiness quiz.

1. Evaluate $5x-2y+3z$ when $x=-2, y=-4$, and $z=3$.
   If you missed this problem, review [link].
2. Classify the equations as a conditional equation, an identity, or a contradiction and then state the solution.
   $\begin{cases} -2x+y=-11 \\ x+3y=9 \end{cases}$
   If you missed this problem, review [link].
3. Classify the equations as a conditional equation, an identity, or a contradiction and then state the solution. $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$
   If you missed this problem, review [link].

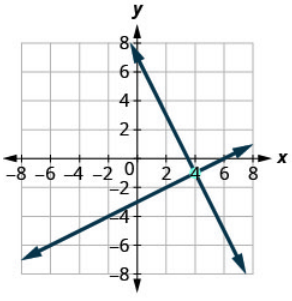
Determine Whether an Ordered Triple is a Solution of a System of Three Linear Equations with Three Variables

In this section, we will extend our work of solving a system of linear equations. So far we have worked with **systems of equations** with two equations and two variables. Now we will work with systems of three equations with three variables. But first let's review what we already know about solving equations and systems involving up to two variables.

We learned earlier that the graph of a **linear equation**, $ax+by=c$, is a line. Each point on the line, an ordered pair (x,y), is a solution to the equation. For a system of two equations with two variables, we graph two lines. Then we can see that all the points that are solutions to each equation form a line. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions

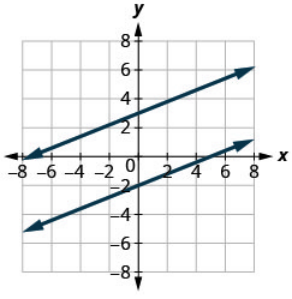
We know when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.



One solution

The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.

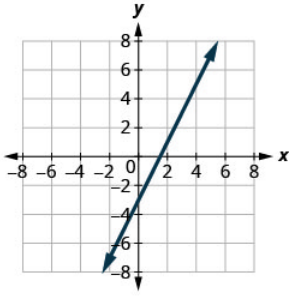
Consistent
Independent



No solution

The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.

Inconsistent



Infinitely many solutions

Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

Consistent
Dependent

Similarly, for a linear equation with three variables ax+by+cz=d,ax+by+cz=d, every solution to the equation is an ordered triple, (x,y,z)(x,y,z)that makes the equation true.

LINEAR EQUATION IN THREE VARIABLES

A linear equation with three variables, where a, b, c, and d are real numbers and a, b, and c are not all 0, is of the form

$$ax + by + cz = d$$

Every solution to the equation is an ordered triple, (x, y, z) that makes the equation true.

All the points that are solutions to one equation form a plane in three-dimensional space. And, by finding what the planes have in common, we'll find the solution to the system.

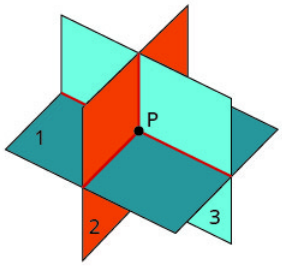
When we solve a system of three linear equations represented by a graph of three planes in space, there are three possible cases.

One solution

Consistent system and Independent equations

The 3 planes intersect.

The three intersecting planes have one point in common.

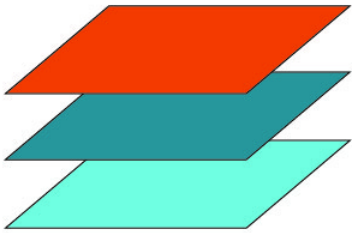


No solution

Inconsistent system

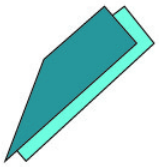
The planes are parallel.

Parallel planes have no points in common.



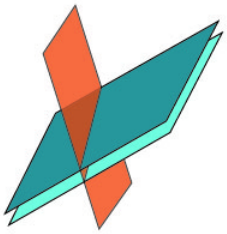
Two planes are coincident and parallel to the third plane.

The planes have no points in common.



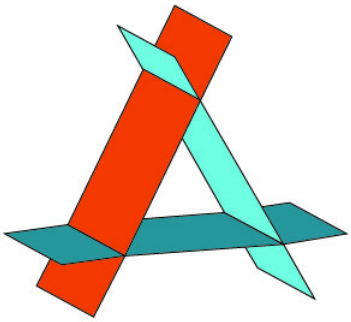
Two planes are parallel and each intersect the third plane.

The planes have no points in common.



Each plane intersects the other two, but all three share no points.

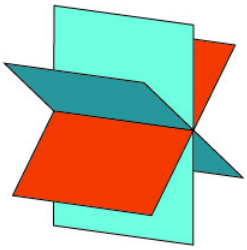
The planes have no points in common.



Infinitely many solutions
Consistent system and dependent equations

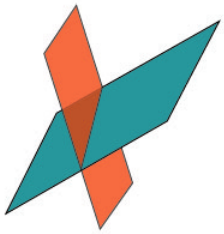
Three planes intersect in one line.

There is just one line, so there are infinitely many solutions.



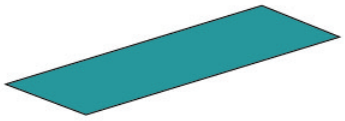
Two planes are coincident and intersect the third plane in a line.

There is just one line, so there are infinitely many solutions.



Three planes are coincident.

There is just one plane, so there are infinitely many solutions.



To solve a system of three linear equations, we want to find the values of the variables that are solutions to all three equations. In other words, we are looking for the ordered triple (x, y, z) that makes all three equations true. These are called the **solutions of the system of three linear equations with three variables**.

SolutionS OF A SYSTEM OF LINEAR EQUATIONS WITH THREE VARIABLES

Solutions of a system of equations are the values of the variables that make all the equations true. A solution is represented by an **ordered triple** (x, y, z).

To determine if an ordered triple is a solution to a system of three equations, we substitute the values of the variables into each equation. If the ordered triple makes all three equations true, it is a solution to the system.

Example 21.5.1

Determine whether the ordered triple is a solution to the system: $\begin{cases} x-y+z=2 \\ 2x-y-z=-6 \\ 2x+2y+z=-3 \end{cases}$

ⓐ $(-2,-1,3)$ ⓑ $(-4,-3,4)$

Answer

ⓐ

$$\begin{cases} x-y+z=2 \\ 2x-y-z=-6 \\ 2x+2y+z=-3 \end{cases}$$

We substitute $x=-2$ and $y=-1$ and $z=3$ into all three equations.

$x-y+z=2$	$2x-y-z=-6$	$2x+2y+z=-3$
$-2-(-1)+3 \stackrel{?}{=} 2$	$2\cdot(-2)-(-1)-3 \stackrel{?}{=} -6$	$2\cdot(-2)+2(-1)+3 \stackrel{?}{=} -3$
$2=2$ ✓	$-6=-6$ ✓	$-3=-3$ ✓
(–2, –1, 3) does make all three equations true.		(–2, –1, 3) is a solution.

ⓑ

$$\begin{cases} x - y + z = 2 \\ 2x - y - z = -6 \\ 2x + 2y + z = -3 \end{cases}$$

We substitute $x = -4$ and $y = -3$ and $z = 4$ into all three equations.

$x - y + z = 2$	$2x - y - z = -6$	$2x + 2y + z = -3$
$-4 - (-3) + 4 \stackrel{?}{=} 2$	$2 \cdot (-4) - (-3) - 4 \stackrel{?}{=} -6$	$2 \cdot (-4) + 2(-3) + 4 \stackrel{?}{=} -3$
$3 \neq 2$	$-9 \neq -6$	$-10 \neq -3$
$(-4, -3, 4)$ does not make all three equations true.		$(-4, -3, 4)$ is not a solution.

Example 21.5.2

Determine whether the ordered triple is a solution to the system: $\begin{cases} 3x + y + z = 2 \\ x + 2y + z = -3 \\ 3x + y + 2z = 4 \end{cases}$

ⓐ $(1, -3, 2)$ ⓑ $(4, -1, -5)$

Answer

ⓐ yes ⓑ no

Example 21.5.3

Determine whether the ordered triple is a solution to the system: $\begin{cases} x - 3y + z = -5 \\ -3x - y - z = 1 \\ 2x - 2y + 3z = 1 \end{cases}$

ⓐ $(2, -2, 3)$ ⓑ $(-2, 2, 3)$

Answer

ⓐ no ⓑ yes

Solve a System of Linear Equations with Three Variables

To solve a system of linear equations with three variables, we basically use the same techniques we used with systems that had two variables. We start with two pairs of equations and in each pair we eliminate the same variable. This will then give us a system of equations with only two variables and then we know how to solve that system!

Next, we use the values of the two variables we just found to go back to the original equation and find the third variable. We write our answer as an ordered triple and then check our results.

Example 21.5.4: How to Solve a System of Equations With Three Variables by Elimination

Solve the system by elimination: $\begin{cases} x - 2y + z = 3 \\ 2x + y + z = 4 \\ 3x + 4y + 3z = -1 \end{cases}$

Answer

Step 1. Write the equations in standard form. • If any coefficients are fractions, clear them.	The equations are in standard form. There are no fractions.	$\begin{cases} x-2y+z=3 & (1)\\ 2x+y+z=4 & (2)\\ 3x+4y+3z=-1 & (3)\end{cases}$
Step 2. Eliminate the same variable from two equations. • Decide which variable you will eliminate. • Work with a pair of equations to eliminate the chosen variable. • Multiply one or both equations so that the coefficients of that variable are opposites. • Add the equations to eliminate one variable.	We can eliminate the y's from equations (1) and (2) by multiplying equation (2) by 2. Multiply. We add the x's, y's, and constants. The new equation, (4), has only x and z.	$x-2y+z=3 \quad (1)$ $2(2x+y+z)=(4)2 \quad (2)$ $x-2y+z=3 \quad (1)$ $4x+2y+2z=8 \quad (2)$ $x-2y+z=3 \quad (1)$ $4x+2y+2z=8 \quad (2)$ $5x+3z=11 \quad (4)$
Step 3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.	We can again eliminate the y's using the equations (1) and (3) by multiplying equation (1) by 2. Add the new equations and the result will be equation (5).	$2(x-2y+z)=(3)2 \quad (1)$ $3x+4y+3z=-1 \quad (3)$ $2x-4y+2z=6 \quad (1)$ $3x+4y+3z=-1 \quad (3)$ $5x+5z=5 \quad (5)$
Step 4. The two new equations form a system of two equations with two variables. Solve this system.	Use equations (4) and (5) to eliminate either x or z. Multiply (5) by -1. Use either (4) or (5) to solve for x using substitution.	$\begin{cases} 5x+3z=11 & (4)\\ 5x+5z=5 & (5)\end{cases}$ $5x+3z=11 \quad (4)$ $-1(5x+5z)=5(-1) \quad (5)$ $5x+3z=11 \quad (4)$ $-5x-5z=-5 \quad (5)$ $-2z=6$ $z=-3$ $5x+3z=11 \quad (4)$ $5x+3(-3)=11$ $5x-9=11$ $5x=20$ $x=4$
Step 5. Use the values of the two variables found in Step 4 to find the third variable.	Use an original equation to find y.	$2x+y+z=4 \quad (2)$ $2\cdot 4+y+z=4$ $8+y-3=4$ $y=-1$
Step 6. Write the solution as an ordered triple.	Write it as (x, y, z).	$(4, -1, -3)$
Step 7. Check that the ordered triple is a solution to **all three** original equations.	Substitute $x=4$, $y=-1$ and $z=-3$ into all three equations. Do they make the equations true?	We'll leave the checks to you. The solution is $(4, -1, -3)$.

Example 21.5.5

Solve the system by elimination: $\begin{cases} 3x+y-z=2 \\ 2x-3y-2z=1 \\ 4x-y-3z=0 \end{cases}$

Answer

$(2,-1,3)$

Example 21.5.6

Solve the system by elimination: $\begin{cases} 4x+y+z=-1 \\ -2x-2y+z=2 \\ 2x+3y-z=1 \end{cases}$

Answer

$(-2,3,4)$

The steps are summarized here.

SOLVE A SYSTEM OF LINEAR EQUATIONS WITH THREE VARIABLES.

1. Write the equations in standard form
   - If any coefficients are fractions, clear them.
2. Eliminate the same variable from two equations.
   - Decide which variable you will eliminate.
   - Work with a pair of equations to eliminate the chosen variable.
   - Multiply one or both equations so that the coefficients of that variable are opposites.
   - Add the equations resulting from Step 2 to eliminate one variable
3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.
4. The two new equations form a system of two equations with two variables. Solve this system.
5. Use the values of the two variables found in Step 4 to find the third variable.
6. Write the solution as an ordered triple.
7. Check that the ordered triple is a solution to **all three** original equations.

Example 21.5.7

Solve: $\begin{cases} 3x-4z=0 \\ 3y+2z=-3 \\ 2x+3y=-5 \end{cases}$

Answer

$$\begin{cases} 3x-4z=0 & (1) \\ 3y+2z=-3 & (2) \\ 2x+3y=-5 & (3) \end{cases}$$

We can eliminate z from equations (1) and (2) by multiplying equation (2) by 2 and then adding the resulting equations.

$$\begin{array}{rl} 3x & -4z=0 \quad (1) \\ 3y+2z= & -3 \quad (2) \end{array} \rightarrow \begin{array}{rl} 3x & -4z=0 \\ 2(3y+2z)= & (-3)2 \end{array} \rightarrow \begin{array}{rl} 3x & -4z=0 \\ 6y+4z= & -6 \\ \hline 3x+6y & =-6 \quad (4) \end{array}$$

Notice that equations (3) and (4) both have the variables x and y. We will solve this new system for x and y.

$$\begin{array}{l} 2x+3y=-5 \quad (3) \\ 3x+6y=-6 \quad (4) \end{array} \longrightarrow \begin{array}{l} -2(2x+3y)=-2(-5) \\ 3x+6y=-6 \end{array} \longrightarrow \begin{array}{r} -4x-6y=10 \\ 3x+6y=-6 \\ \hline -x \quad = \quad 4 \\ x=-4 \end{array}$$

To solve for y, we substitute $x=-4$ into equation (3).

$$\begin{array}{r} 2x+3y=-5 \quad (3) \\ 2(-4)+3y=-5 \\ -8+3y=-5 \\ 3y=3 \\ y=1 \end{array}$$

We now have $x=-4$ and $y=1$. We need to solve for z. We can substitute $x=-4$ into equation (1) to find z.

$$\begin{array}{r} 3x-4z=0 \quad (1) \\ 3(-4)-4z=0 \\ -12-4z=0 \\ -4z=12 \\ z=-3 \end{array}$$

We write the solution as an ordered triple. $(-4,1,-3)$

We check that the solution makes all three equations true.

$$\begin{array}{lll} 3x-4z=0\ (1) & 3y+2z=-3\ (2) & 2x+3y=-5\ (3) \\ 3(-4)-4(-3)\stackrel{?}{=}0 & 3(1)+2(-3)\stackrel{?}{=}-3 & 2(-4)+3(1)\stackrel{?}{=}-5 \\ 0=0\checkmark & -3=-3\checkmark & -5=-5\checkmark \\ & & \text{The solution is } (-4,1,-3) \end{array}$$

Example 21.5.8

Solve: $\begin{cases} 3x-4z=-1 \\ 2y+3z=2 \\ 2x+3y=6 \end{cases}$

Answer

$(-3,4,-2)$

Example 21.5.9

Solve: $\begin{cases} 4x-3z=-5 \\ 3y+2z=7 \\ 3x+4y=6 \end{cases}$

Answer

$(-2,3,-1)$

When we solve a system and end up with no variables and a false statement, we know there are no solutions and that the system is inconsistent. The next example shows a system of equations that is inconsistent.

Example 21.5.10

Solve the system of equations: $\begin{cases} x+2y-3z=-1 \\ x-3y+z=1 \\ 2x-y-2z=2 \end{cases}$

Answer

$$\begin{cases} x+2y-3z=-1 & (1) \\ x-3y+z=1 & (2) \\ 2x-y-2z=2 & (3) \end{cases}$$

Use equation (1) and (2) to eliminate z.

$$\begin{array}{l} x+2y-3z=-1 \quad (1) \\ x-3y+\ z=\ 1 \quad (2) \end{array} \xrightarrow{\text{multiply by 3}} \begin{array}{r} x+2y-3z=-1 \\ 3x-9y+3z=\ 3 \\ \hline 4x-7y\ \ =\ 2 \quad (4) \end{array}$$

Use (2) and (3) to eliminate z again.

$$\begin{array}{l} x-3y+\ z=1 \quad (2) \\ 2x-\ y-2z=2 \quad (3) \end{array} \xrightarrow{\text{multiply by 2}} \begin{array}{r} 2x-6y+2z=2 \\ 2x-\ y-2z=2 \\ \hline 4x-7y\ \ =4 \quad (5) \end{array}$$

Use (4) and (5) to eliminate a variable.

$$\begin{array}{l} 4x-7y=2 \quad (4) \\ 4x-7y=4 \quad (5) \end{array} \xrightarrow{\text{multiply by } -1} \begin{array}{r} 4x-7y=2 \\ -4x+7y=-4 \\ \hline 0=-2\ \textit{false} \end{array}$$

There is no solution.

We are left with a false statement and this tells us the system is inconsistent and has no solution.

Example 21.5.11

Solve the system of equations: $\begin{cases} x+2y+6z=5 \\ -x+y-2z=3 \\ x-4y-2z=1 \end{cases}$

Answer

no solution

Example 21.5.12

Solve the system of equations: $\begin{cases} 2x-2y+3z=6 \\ 4x-3y+2z=0 \\ -2x+3y-7z=1 \end{cases}$

Answer

no solution

When we solve a system and end up with no variables but a true statement, we know there are infinitely many solutions. The system is consistent with dependent equations. Our solution will show how two of the variables depend on the third.

Example 21.5.13

Solve the system of equations: $\begin{cases} x+2y-z=1 \\ 2x+7y+4z=11 \\ x+3y+z=4 \end{cases}$

Answer

$$\begin{cases} x+2y-z=1 & (1) \\ 2x+7y+4z=11 & (2) \\ x+3y+z=4 & (3) \end{cases}$$

Use equation (1) and (3) to eliminate x.

$$\begin{array}{rl} x+2y-z=1 & (1) \\ x+3y+z=4 & (3) \end{array} \xrightarrow{\text{multiply by } -1} \begin{array}{rl} -x-2y+z=-1 & (1) \\ x+3y+z=4 & (3) \\ \hline y+2z=3 & (4) \end{array}$$

Use equation (1) and (2) to eliminate x again.

$$\begin{array}{rl} x+2y-z=1 & (1) \\ 2x+7y+4z=11 & (2) \end{array} \xrightarrow{\text{multiply by } -2} \begin{array}{rl} -2x-4y+2z=-2 & (1) \\ 2x+7y+4z=11 & (2) \\ \hline 3y+6z=9 & (5) \end{array}$$

Use equation (4) and (5) to eliminate y.

$$\begin{array}{rl} y+2z=3 & (4) \\ 3y+6z=9 & (5) \end{array} \xrightarrow{\text{multiply by } -3} \begin{array}{rl} -3y-6z=-9 & (4) \\ 3y+6z=9 & (5) \\ \hline 0=0 & \end{array}$$

	There are infinitely many solutions.
Solve equation (4) for y.	Represent the solution showing how x and y are dependent on z. $y+2z=3$ $y=-2z+3$
Use equation (1) to solve for x.	$x+2y-z=1$
Substitute $y=-2z+3$.	$x+2(-2z+3)-z=1$ $x-4z+6-z=1$ $x-5z+6=1$ $x=5z-5$

The true statement $0=0$ tells us that this is a dependent system that has infinitely many solutions. The solutions are of the form (x,y,z)(x,y,z) where $x=5z-5;\ y=-2z+3$ and z is any real number.

Example 21.5.14

Solve the system by equations: $\begin{cases} x+y-z=0 \\ 2x+4y-2z=6 \\ 3x+6y-3z=9 \end{cases}$

Answer

infinitely many solutions $(x,3,z)$ where $x=z-3;\ y=3;\ z$ is any real number

Example 21.5.15

Solve the system by equations: $\begin{cases} x-y-z=1 \\ -x+2y-3z=-4 \\ 3x-2y-7z=0 \end{cases}$

Answer

infinitely many solutions (x, y, z) where $x=5z-2;\ y=4z-3;\ z$ is any real number

Solve Applications using Systems of Linear Equations with Three Variables

Applications that are modeled by a systems of equations can be solved using the same techniques we used to solve the systems. Many of the application are just extensions to three variables of the types we have solved earlier.

Example 21.5.16

The community college theater department sold three kinds of tickets to its latest play production. The adult tickets sold for \$15, the student tickets for \$10 and the child tickets for \$8. The theater department was thrilled to have sold 250 tickets and brought in \$2,825 in one night. The number of student tickets sold is twice the number of adult tickets sold. How many of each type did the department sell?

Answer

We will use a chart to organize the information.

Type	Number	· Value	= Total Value
adult	x	15	$15x$
student	y	10	$10y$
child	z	8	$8z$
	250		2825

Number of students is twice number of adults.

Rewrite the equation in standard form.

$$y=2x$$
$$2x-y=0$$

Write the system of equations.

$$\begin{cases} x+y+z=250 & (1) \\ 15x+10y+8z=2825 & (2) \\ -2x+y=0 & (3) \end{cases}$$

Use equations (1) and (2) to eliminate z.

$$\begin{array}{rl} x+y+z=250 & (1) \\ 15x+10y+8z=2825 & (2) \end{array} \xrightarrow{\text{Multiply by } -8} \begin{array}{rl} -8x-8y-8z=250 & (1) \\ 15x+10y+8z=2825 & (2) \\ \hline 7x+2y=825 & (4) \end{array}$$

Use (3) and (4) to eliminate y.

$$\begin{array}{rl} -2x+y=0 & (3) \\ 7x+2y=825 & (4) \end{array} \xrightarrow{\text{Multiply by } -2} \begin{array}{rl} 4x-2y=0 & (3) \\ 7x+2y=825 & (4) \\ \hline 11x=825 & \end{array}$$

Solve for x. $x=75$ adult tickets

Use equation (3) to find y. $-2x+y=0$

Substitute $x=75$.

$$-2(75)+y=0$$
$$-150+y=0$$
$$y=150 \text{ student tickets}$$

Use equation (1) to find z. $x+y+z=250$

Substitute in the values $x = 75,\ y = 150$.	$\begin{aligned} 75 + 150 + z &= 250 \\ 225 + z &= 250 \\ z &= 25 \text{ child tickets} \end{aligned}$
Write the solution.	The theater department sold 75 adult tickets, 150 student tickets, and 25 child tickets.

Example 21.5.17

The community college fine arts department sold three kinds of tickets to its latest dance presentation. The adult tickets sold for $20, the student tickets for $12 and the child tickets for $10.The fine arts department was thrilled to have sold 350 tickets and brought in $4,650 in one night. The number of child tickets sold is the same as the number of adult tickets sold. How many of each type did the department sell?

Answer

The fine arts department sold 75 adult tickets, 200 student tickets, and 75 child tickets.

Example 21.5.18

The community college soccer team sold three kinds of tickets to its latest game. The adult tickets sold for $10, the student tickets for $8 and the child tickets for $5. The soccer team was thrilled to have sold 600 tickets and brought in $4,900 for one game. The number of adult tickets is twice the number of child tickets. How many of each type did the soccer team sell?

Answer

The soccer team sold 200 adult tickets, 300 student tickets, and 100 child tickets.

Access this online resource for additional instruction and practice with solving a linear system in three variables with no or infinite solutions.

- Solving a Linear System in Three Variables with No or Infinite Solutions
- 3 Variable Application

Key Concepts

- **Linear Equation in Three Variables:** A linear equation with three variables, where *a*, *b*, *c*, and *d* are real numbers and *a*, *b*,and *c* are not all 0, is of the form

$$ax + by + cz = d$$

  Every solution to the equation is an ordered triple, (x, y, z) that makes the equation true.
- **How to solve a system of linear equations with three variables.**
  1. Write the equations in standard form
     If any coefficients are fractions, clear them.
  2. Eliminate the same variable from two equations.
     Decide which variable you will eliminate.
     Work with a pair of equations to eliminate the chosen variable.
     Multiply one or both equations so that the coefficients of that variable are opposites.
     Add the equations resulting from Step 2 to eliminate one variable
  3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.
  4. The two new equations form a system of two equations with two variables. Solve this system.
  5. Use the values of the two variables found in Step 4 to find the third variable.
  6. Write the solution as an ordered triple.

7. Check that the ordered triple is a solution to **all three** original equations.

Glossary

solutions of a system of linear equations with three variables

The solutions of a system of equations are the values of the variables that make all the equations true; a solution is represented by an ordered triple (x,y,z).

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21.5E: Exercises

Practice Makes Perfect

Determine Whether an Ordered Triple is a Solution of a System of Three Linear Equations with Three Variables

In the following exercises, determine whether the ordered triple is a solution to the system.

1. $\begin{cases} 2x-6y+z=3 \\ 3x-4y-3z=2 \\ 2x+3y-2z=3 \end{cases}$

ⓐ $(3,1,3)$
ⓑ $(4,3,7)$

2. $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$

ⓐ $(-5,-7,4)$
ⓑ $(5,7,4)$

Answer

ⓐ no ⓑ yes

3. $\begin{cases} y-10z=-8 \\ 2x-y=2 \\ x-5z=3 \end{cases}$

ⓐ $(7,12,2)$
ⓑ $(2,2,1)$

4. $\begin{cases} x+3y-z=1 \\ 5y=\frac{2}{3}x \\ -2x-3y+z=-2 \end{cases}$

ⓐ $(-6,5,12)$
ⓑ $(5,\frac{4}{3},-3)$

Answer

ⓐ no ⓑ yes

Solve a System of Linear Equations with Three Variables

In the following exercises, solve the system of equations.

5. $\begin{cases} 5x+2y+z=5 \\ -3x-y+2z=6 \\ 2x+3y-3z=5 \end{cases}$

6. $\begin{cases} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{cases}$

Answer

$(4, 5, 2)$

7. $\begin{cases} 2x - 5y + 3z = 8 \\ 3x - y + 4z = 7 \\ x + 3y + 2z = -3 \end{cases}$

8. $\begin{cases} 5x - 3y + 2z = -5 \\ 2x - y - z = 4 \\ 3x - 2y + 2z = -7 \end{cases}$

Answer

$(7, 12, -2)$

9. $\begin{cases} 3x - 5y + 4z = 5 \\ 5x + 2y + z = 0 \\ 2x + 3y - 2z = 3 \end{cases}$

10. $\begin{cases} 4x - 3y + z = 7 \\ 2x - 5y - 4z = 3 \\ 3x - 2y - 2z = -7 \end{cases}$

Answer

$(-3, -5, 4)$

11. $\begin{cases} 3x + 8y + 2z = -5 \\ 2x + 5y - 3z = 0 \\ x + 2y - 2z = -1 \end{cases}$

12. $\begin{cases} 11x + 9y + 2z = -9 \\ 7x + 5y + 3z = -7 \\ 4x + 3y + z = -3 \end{cases}$

Answer

$(2, -3, -2)$

13. $\begin{cases} \frac{1}{3}x - y - z = 1 \\ x + \frac{5}{2}y + z = -2 \\ 2x + 2y + \frac{1}{2}z = -4 \end{cases}$

14. $\begin{cases} x + \frac{1}{2}y + \frac{1}{2}z = 0 \\ \frac{1}{5}x - \frac{1}{5}y + z = 0 \\ \frac{1}{3}x - \frac{1}{3}y + 2z = -1 \end{cases}$

Answer

$(6, -9, -3)$

15. $\begin{cases} x+\frac{1}{3}y-2z=-1 \\ \frac{1}{3}x+y+\frac{1}{2}z=0 \\ \frac{1}{2}x+\frac{1}{3}y-\frac{1}{2}z=-1 \end{cases}$

16. $\begin{cases} \frac{1}{3}x-y+\frac{1}{2}z=4 \\ \frac{2}{3}x+\frac{5}{2}y-4z=0 \\ x-\frac{1}{2}y+\frac{3}{2}z=2 \end{cases}$

Answer

$(3,-4,-2)$

17. $\begin{cases} x+2z=0 \\ 4y+3z=-2 \\ 2x-5y=3 \end{cases}$

18. $\begin{cases} 2x+5y=4 \\ 3y-z=\frac{3}{4} \\ x+3z=-3 \end{cases}$

Answer

$(-3,2,3)$

19. $\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$

20. $\begin{cases} 3x-z=-3 \\ 5y+2z=-6 \\ 4x+3y=-8 \end{cases}$

Answer

$(-2,0,-3)$

21. $\begin{cases} 4x-3y+2z=0 \\ -2x+3y-7z=1 \\ 2x-2y+3z=6 \end{cases}$

22. $\begin{cases} x-2y+2z=1 \\ -2x+y-z=2 \\ x-y+z=5 \end{cases}$

Answer

no solution

23. $\begin{cases} 2x+3y+z=1 \\ 2x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$

24. $\begin{cases} x+4y+z=-8 \\ 4x-y+3z=9 \\ 2x+7y+z=0 \end{cases}$

Answer

$x=\frac{203}{16};\ y=-\frac{25}{16};\ z=-\frac{231}{16};$

25. $\begin{cases} x+2y+z=4 \\ x+y-2z=3 \\ -2x-3y+z=-7 \end{cases}$

26. $\begin{cases} x+y-2z=3 \\ -2x-3y+z=-7 \\ x+2y+z=4 \end{cases}$

Answer

(x,y,z) where $x=5z+2;\ y=-3z+1;\ z$ is any real number

27. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

28. $\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$

Answer

(x,y,z) where $x=5z-2;\ y=4z-3;\ z$ is any real number

Solve Applications using Systems of Linear Equations with Three Variables

In the following exercises, solve the given problem.

29. The sum of the measures of the angles of a triangle is 180. The sum of the measures of the second and third angles is twice the measure of the first angle. The third angle is twelve more than the second. Find the measures of the three angles.

30. The sum of the measures of the angles of a triangle is 180. The sum of the measures of the second and third angles is three times the measure of the first angle. The third angle is fifteen more than the second. Find the measures of the three angles.

Answer

42, 50, 58

31. After watching a major musical production at the theater, the patrons can purchase souvenirs. If a family purchases 4 t-shirts, the video, and 1 stuffed animal, their total is $135.

A couple buys 2 t-shirts, the video, and 3 stuffed animals for their nieces and spends $115. Another couple buys 2 t-shirts, the video, and 1 stuffed animal and their total is $85. What is the cost of each item?

32. The church youth group is selling snacks to raise money to attend their convention. Amy sold 2 pounds of candy, 3 boxes of cookies and 1 can of popcorn for a total sales of $65. Brian sold 4 pounds of candy, 6 boxes of cookies and 3 cans of popcorn for a total sales of $140. Paulina sold 8 pounds of candy, 8 boxes of cookies and 5 cans of popcorn for a total sales of $250. What is the cost of each item?

Answer

$20, $5, $10

Writing Exercises

33. In your own words explain the steps to solve a system of linear equations with three variables by elimination.

34. How can you tell when a system of three linear equations with three variables has no solution? Infinitely many solutions?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether an ordered triple is a solution of a system of three linear equations with three variables.			
solve a system of linear equations with three variables.			
solve applications using systems of linear equations with three variables.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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21.6: Solve Systems of Equations Using Matrices

Learning Objectives

By the end of this section, you will be able to:

- Write the augmented matrix for a system of equations
- Use row operations on a matrix
- Solve systems of equations using matrices

Before you get started, take this readiness quiz.

1. Solve: $3(x+2)+4=4(2x-1)+9$.
   If you missed this problem, review [link].
2. Solve: $0.25p+0.25(x+4)=5.20$.
   If you missed this problem, review [link].
3. Evaluate when $x=-2$ and $y=3$: $2x^2-xy+3y^2$.
   If you missed this problem, review [link].

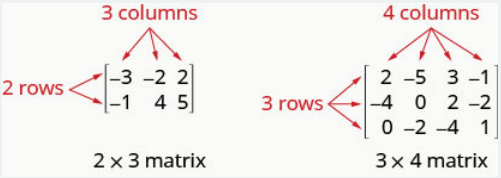
Write the Augmented Matrix for a System of Equations

Solving a system of equations can be a tedious operation where a simple mistake can wreak havoc on finding the solution. An alternative method which uses the basic procedures of elimination but with notation that is simpler is available. The method involves using a **matrix**. A matrix is a rectangular array of numbers arranged in rows and columns.

MATRIX

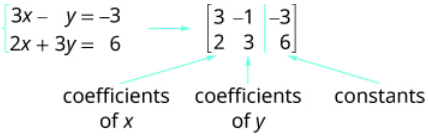
A **matrix** is a rectangular array of numbers arranged in rows and columns.

A matrix with *m* rows and *n* columns has order $m \times n$. The matrix on the left below has 2 rows and 3 columns and so it has order 2×3. We say it is a 2 by 3 matrix.



Each number in the matrix is called an element or entry in the matrix.

We will use a matrix to represent a system of linear equations. We write each equation in standard form and the coefficients of the variables and the constant of each equation becomes a row in the matrix. Each column then would be the coefficients of one of the variables in the system or the constants. A vertical line replaces the equal signs. We call the resulting matrix the augmented matrix for the system of equations.



Notice the first column is made up of all the coefficients of *x*, the second column is the all the coefficients of *y*, and the third column is all the constants.

Example 21.6.1

ⓐ $\begin{cases} 5x-3y=-1 \\ y=2x-2 \end{cases}$ ⓑ $\begin{cases} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{cases}$

Answer

ⓐ The second equation is not in standard form. We rewrite the second equation in standard form.

$$y=2x-2$$
$$-2x+y=-2$$

We replace the second equation with its standard form. In the augmented matrix, the first equation gives us the first row and the second equation gives us the second row. The vertical line replaces the equal signs.

$$\begin{array}{l} 5x-3y=-1 \\ 2x-\ \ y=\ \ 2 \end{array} \qquad \left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \quad \begin{array}{cc} x & y \end{array}$$

ⓑ All three equations are in standard form. In the augmented matrix the first equation gives us the first row, the second equation gives us the second row, and the third equation gives us the third row. The vertical line replaces the equal signs.

$$\begin{array}{l} 6x-5y+2z=\ \ 3 \\ 2x+\ \ y-4z=\ \ 5 \\ 3x-3y+\ \ z=-1 \end{array} \qquad \left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \quad \begin{array}{ccc} x & y & z \end{array}$$

Example 21.6.2

Write each system of linear equations as an augmented matrix:

ⓐ $\begin{cases} 3x+8y=-3 \\ 2x=-5y-3 \end{cases}$ ⓑ $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

Answer

ⓐ $\left[\begin{array}{cc|c} 3 & 8 & -3 \\ 2 & 5 & -3 \end{array}\right]$

ⓑ $\left[\begin{array}{ccc|c} 2 & 5 & -3 & 8 \\ 3 & -1 & 4 & 7 \\ 1 & 3 & 2 & -3 \end{array}\right]$

Example 21.6.3

Write each system of linear equations as an augmented matrix:

ⓐ $\begin{cases} 11x=-9y-5 \\ 7x+5y=-1 \end{cases}$ ⓑ $\begin{cases} 5x-3y+2z=-5 \\ 2x-y-z=4 \\ 3x-2y+2z=-7 \end{cases}$

Answer

ⓐ $\left[\begin{array}{cc|c} 11 & 9 & -5 \\ 7 & 5 & -1 \end{array}\right]$

ⓑ $\left[\begin{array}{ccc|c} 5 & -3 & 2 & -5 \\ 2 & -1 & -1 & 4 \\ 3 & -2 & 2 & -7 \end{array}\right]$

It is important as we solve systems of equations using matrices to be able to go back and forth between the system and the matrix. The next example asks us to take the information in the matrix and write the system of equations.

? Example 21.6.4

Write the system of equations that corresponds to the augmented matrix:

$$\left[\begin{array}{ccc|c} 4 & -3 & 3 & -1 \\ 1 & 2 & -1 & 2 \\ -2 & -1 & 3 & -4 \end{array}\right].$$

Answer

We remember that each row corresponds to an equation and that each entry is a coefficient of a variable or the constant. The vertical line replaces the equal sign. Since this matrix is a 4×3, we know it will translate into a system of three equations with three variables.

$$\begin{array}{ccc} x & y & z \\ \left[\begin{array}{ccc|c} 4 & -3 & 3 & -1 \\ 1 & 2 & -1 & 2 \\ -2 & -1 & 3 & -4 \end{array}\right] & \longrightarrow & \begin{array}{r} 4x-3y+3z=-1 \\ x+2y-z=2 \\ -2x-y+3z=-4 \end{array} \end{array}$$

? Example 21.6.5

Write the system of equations that corresponds to the augmented matrix: $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & 1 & -2 & 1 \\ 4 & -1 & 2 & 0 \end{array}\right]$.

Answer

$$\begin{cases} x-y+2z=3 \\ 2x+y-2z=1 \\ 4x-y+2z=0 \end{cases}$$

? Example 21.6.6

Write the system of equations that corresponds to the augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 3 & -1 & 8 \\ 1 & 1 & -1 & 3 \end{array}\right]$.

Answer

$$\begin{cases} x+y+z=4 \\ 2x+3y-z=8 \\ x+y-z=3 \end{cases}$$

Use Row Operations on a Matrix

Once a system of equations is in its augmented matrix form, we will perform operations on the rows that will lead us to the solution.

To solve by elimination, it doesn't matter which order we place the equations in the system. Similarly, in the matrix we can interchange the rows.

When we solve by elimination, we often multiply one of the equations by a constant. Since each row represents an equation, and we can multiply each side of an equation by a constant, similarly we can multiply each entry in a row by any real number except 0.

In elimination, we often add a multiple of one row to another row. In the matrix we can replace a row with its sum with a multiple of another row.

These actions are called row operations and will help us use the matrix to solve a system of equations.

ROW OPERATIONS

In a matrix, the following operations can be performed on any row and the resulting matrix will be equivalent to the original matrix.

1. Interchange any two rows.
2. Multiply a row by any real number except 0.
3. Add a nonzero multiple of one row to another row.

Performing these operations is easy to do but all the arithmetic can result in a mistake. If we use a system to record the row operation in each step, it is much easier to go back and check our work.

We use capital letters with subscripts to represent each row. We then show the operation to the left of the new matrix. To show interchanging a row:

$$\begin{bmatrix} 5 & -3 & -1 \\ 2 & -1 & 2 \end{bmatrix} \xrightarrow[R_1]{R_2} \begin{bmatrix} 2 & -1 & 2 \\ 5 & -3 & -1 \end{bmatrix}$$

To multiply row 2 by -3:

$$\begin{bmatrix} 5 & -3 & -1 \\ 2 & -1 & 2 \end{bmatrix} \xrightarrow{-3R_2} \begin{bmatrix} 5 & -3 & -1 \\ -6 & 3 & -6 \end{bmatrix}$$

To multiply row 2 by -3 and add it to row 1:

$$\begin{bmatrix} 5 & -3 & -1 \\ 2 & -1 & 2 \end{bmatrix} \xrightarrow{-3R_2 + R_1} \begin{bmatrix} -1 & 0 & -7 \\ 2 & -1 & 2 \end{bmatrix}$$

Example 21.6.7

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 2 and 3.

ⓑ Multiply row 2 by 5.

ⓒ Multiply row 3 by $-2-2$ and add to row 1.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

Answer

ⓐ We interchange rows 2 and 3.

$$\begin{bmatrix} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{bmatrix} \xrightarrow[R_2]{R_3} \begin{bmatrix} 6 & -5 & 2 & 3 \\ 3 & -3 & 1 & -1 \\ 2 & 1 & -4 & 5 \end{bmatrix}$$

ⓑ We multiply row 2 by 5.

$$\left[\begin{array}{rrr|r} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{5R_2} \left[\begin{array}{rrr|r} 6 & -5 & 2 & 3 \\ 10 & 5 & -20 & 25 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

ⓒ We multiply row 3 by -2 and add to row 1.

$$\left[\begin{array}{rrr|r} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{-2R_3 + R_1} \left[\begin{array}{rrr|r} (-2 \cdot 3) + 6 & (-2 \cdot (-3)) + (-5) & (-2 \cdot 1) + 2 & (-2 \cdot (-1)) + 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-2R_3 + R_1} \left[\begin{array}{rrr|r} 0 & 1 & 0 & 5 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

Example 21.6.8

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 1 and 3.

ⓑ Multiply row 3 by 3.

ⓒ Multiply row 1 by 2 and add to row 2.

$$\left[\begin{array}{rrr|r} 5 & -2 & -2 & -2 \\ 4 & -1 & -4 & 4 \\ -2 & 3 & 0 & -1 \end{array}\right]$$

Answer

ⓐ $\left[\begin{array}{rrr|r} -2 & 3 & 0 & -2 \\ 4 & -1 & -4 & 4 \\ 5 & -2 & -2 & -2 \end{array}\right]$

ⓑ $\left[\begin{array}{rrr|r} -2 & 3 & 0 & -2 \\ 4 & -1 & -4 & 4 \\ 15 & -6 & -6 & -6 \end{array}\right]$

ⓒ $\left[\begin{array}{rrr|r} -2 & 3 & 0 & 2 \\ 0 & 5 & -4 & 0 \\ 15 & -6 & -6 & -6 \end{array}\right]$

Example 21.6.9

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 1 and 2,

ⓑ Multiply row 1 by 2,

ⓒ Multiply row 2 by 3 and add to row 1.

$$\left[\begin{array}{rrr|r} 2 & -3 & -2 & -4 \\ 4 & 1 & -3 & 2 \\ 5 & 0 & 4 & -1 \end{array}\right]$$

Answer

ⓐ $\left[\begin{array}{ccc|c} 4 & 1 & -3 & 2 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{array}\right]$

ⓑ $\left[\begin{array}{ccc|c} 8 & 2 & -6 & 4 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{array}\right]$

ⓒ $\left[\begin{array}{ccc|c} 14 & -7 & -12 & -8 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{array}\right]$

Now that we have practiced the row operations, we will look at an augmented matrix and figure out what operation we will use to reach a goal. This is exactly what we did when we did elimination. We decided what number to multiply a row by in order that a variable would be eliminated when we added the rows together.

Given this system, what would you do to eliminate x?

$$\begin{cases} x - y = 2 \\ 4x - 8y = 0 \end{cases} \xrightarrow{\text{multiply the first equation by } -4} \begin{cases} -4x + 4y = -8 \\ 4x - 8y = 0 \end{cases} \xrightarrow{\text{then add}} \begin{array}{r} -4x + 4y = -8 \\ 4x - 8y = 0 \\ \hline -4y = -8 \end{array}$$

This next example essentially does the same thing, but to the matrix.

? Example 21.6.10

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 4 & -8 & 0 \end{array}\right]$

Answer

To make the 4 a 0, we could multiply row 1 by -4 and then add it to row 2.

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 4 & -8 & 0 \end{array}\right] \xrightarrow{-4R_1 + R_2} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & -4 & -8 \end{array}\right]$$

? Example 21.6.11

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 3 & -6 & 2 \end{array}\right]$

Answer

$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & -3 & -4 \end{array}\right]$

? Example 21.6.12

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 3 \\ -2 & -3 & 2 \end{array}\right]$

Answer

$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & -5 & 8 \end{array}\right]$

Solve Systems of Equations Using Matrices

To solve a system of equations using matrices, we transform the augmented matrix into a matrix in **row-echelon form** using row operations. For a consistent and independent system of equations, its **augmented matrix** is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

ROW-ECHELON FORM

For a consistent and independent system of equations, its augmented matrix is in **row-echelon form** when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \end{bmatrix}$ $\begin{bmatrix} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{bmatrix}$ a, b, c, d, e, f are real numbers

Once we get the augmented matrix into row-echelon form, we can write the equivalent system of equations and read the value of at least one variable. We then substitute this value in another equation to continue to solve for the other variables. This process is illustrated in the next example.

How to Solve a System of Equations Using a Matrix

Solve the system of equations using a matrix: $\begin{cases} 3x+4y=5 \\ x+2y=1 \end{cases}$

Answer

Step 1. Write the augmented matrix for the system of equations.		$\begin{matrix} 3x+4y=5 \\ x+2y=1 \end{matrix}$ $\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 1 \end{bmatrix}$
Step 2. Using row operations get the entry in row 1, column 1 to be 1.	Interchange the rows, so 1 will be in row 1, column 1.	$\begin{matrix} R_2 \\ R_1 \end{matrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 5 \end{bmatrix}$
Step 3. Using row operations, get zeros in column 1 below the 1.	Multiply row 1 by –3 and add it to row 2.	$\begin{matrix} \\ -3R_1+R_2 \end{matrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \end{bmatrix}$
Step 4. Using row operations, get the entry in row 2, column 2 to be 1.	Multiply row 2 by $-\frac{1}{2}$.	$\begin{matrix} \\ -\frac{1}{2}R_2 \end{matrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
Step 5. Continue the process until the matrix is in row-echelon form.	The matrix is now in row-echelon form.	$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
Step 6. Write the corresponding system of equations.	$\begin{matrix} x & y & \\ \end{matrix}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$	$\begin{matrix} x+2y=1 \\ y=-1 \end{matrix}$
Step 7. Use substitution to find the remaining variables.	Substitute $y=-1$ into $x+2y=1$.	$\begin{matrix} y=-1 \\ x+2y=1 \\ x+2(-1)=1 \\ x-2=1 \\ x=3 \end{matrix}$

Step 8. Write the solution as an ordered pair or triple.		(3, –1)
Step 9. Check that the solution makes the original equations true.		We leave the check to you.

? Example 21.6.14

Solve the system of equations using a matrix: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Answer

The solution is $(4,-1)$.

? Example 21.6.15

Solve the system of equations using a matrix: $\begin{cases} 2x+y=-4 \\ x-y=-2 \end{cases}$

Answer

The solution is $(-2,0)$.

The steps are summarized here.

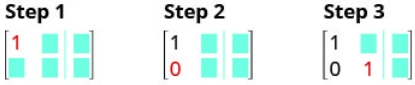
SOLVE A SYSTEM OF EQUATIONS USING MATRICES.

1. Write the augmented matrix for the system of equations.
2. Using row operations get the entry in row 1, column 1 to be 1.
3. Using row operations, get zeros in column 1 below the 1.
4. Using row operations, get the entry in row 2, column 2 to be 1.
5. Continue the process until the matrix is in row-echelon form.
6. Write the corresponding system of equations.
7. Use substitution to find the remaining variables.
8. Write the solution as an ordered pair or triple.
9. Check that the solution makes the original equations true.

Here is a visual to show the order for getting the 1's and 0's in the proper position for row-echelon form.

2 × 3 matrix

Step 1 **Step 2** **Step 3**



3 × 4 matrix

Step 1 **Step 2** **Step 3** **Step 4**

Step 5 **Step 6**

We use the same procedure when the system of equations has three equations.

Example 21.6.16

Solve the system of equations using a matrix: $\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$

Answer

	$\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{rrr\|r} 3 & 8 & 2 & -5 \\ 2 & 5 & -3 & 0 \\ 1 & 2 & -2 & -1 \end{array}\right]$
Interchange row 1 and 3 to get the entry in row 1, column 1 to be 1.	$\begin{matrix} R_3 \\ \\ R_1 \end{matrix}\left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 2 & 5 & -3 & 0 \\ 3 & 8 & 2 & -5 \end{array}\right]$
Using row operations, get zeros in column 1 below the 1.	$\begin{matrix} \\ -2R_1+R_2 \\ \\ \end{matrix}\left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 3 & 8 & 2 & -5 \end{array}\right]$
	$\begin{matrix} \\ \\ -3R_1+R_3 \end{matrix}\left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 8 & -2 \end{array}\right]$
The entry in row 2, column 2 is now 1.	
Continue the process until the matrix is in row-echelon form.	$\begin{matrix} \\ \\ -2R_2+R_3 \end{matrix}\left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & -6 \end{array}\right]$
	$\begin{matrix} \\ \\ \frac{1}{6}R_3 \end{matrix}\left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$
The matrix is now in row-echelon form.	$\left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$
Write the corresponding system of equations.	$\begin{cases} x+2y-2z=-1 \\ y+z=2 \\ z=-1 \end{cases}$
Use substitution to find the remaining variables.	$\begin{aligned} y+z&=2 \\ y+(-1)&=2 \\ y&=3 \end{aligned}$
	$\begin{aligned} x+2y-2z&=-1 \\ x+2(3)-2(-1)&=-1 \\ x+6+2&=-1 \\ x&=-9 \end{aligned}$
Write the solution as an ordered pair or triple.	$(-9, 3, -1)$
Check that the solution makes the original equations true.	We leave the check for you.

Example 21.6.17

Solve the system of equations using a matrix: $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

Answer

$(6,-1,-3)$

Example 21.6.18

Solve the system of equations using a matrix: $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$

Answer

$(5,7,4)$

So far our work with matrices has only been with systems that are consistent and independent, which means they have exactly one solution. Let's now look at what happens when we use a matrix for a dependent or inconsistent system.

Example 21.6.19

Solve the system of equations using a matrix: $\begin{cases} x+y+3z=0 \\ x+3y+5z=0 \\ 2x+4z=1 \end{cases}$

Answer

	$\begin{cases} x+y+3z=0 \\ x+3y+5z=0 \\ 2x+4z=1 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 1 & 3 & 5 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$
The entry in row 1, column 1 is 1.	
Using row operations, get zeros in column 1 below the 1.	$\begin{matrix} \\ -1R_1+R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$
	$\begin{matrix} \\ \\ -2R_1+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$
Continue the process until the matrix is in row-echelon form.	$\begin{matrix} \\ \frac{1}{2}R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$
Multiply row 2 by 2 and add it to row 3.	$\begin{matrix} \\ \\ 2R_2+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$
At this point, we have all zeros on the left of row 3.	

Write the corresponding system of equations.	$\begin{cases} x+y+3z=0 \\ y+z=0 \\ 0\neq 1 \end{cases}$

Since $0 \neq 1$ we have a false statement. Just as when we solved a system using other methods, this tells us we have an inconsistent system. There is no solution.

Example 21.6.20

Solve the system of equations using a matrix: $\begin{cases} x-2y+2z=1 \\ -2x+y-z=2 \\ x-y+z=5 \end{cases}$

Answer

no solution

Example 21.6.21

Solve the system of equations using a matrix: $\begin{cases} 3x+4y-3z=-2 \\ -2x+3y-z=-1 \\ 2x+y-2z=6 \end{cases}$

Answer

no solution

The last system was inconsistent and so had no solutions. The next example is dependent and has infinitely many solutions.

Example 21.6.22

Solve the system of equations using a matrix: $\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$

Answer

	$\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 1 & 1 & -3 & 7 \\ 3 & -4 & 5 & 7 \end{array}\right]$
The entry in row 1, column 1 is 1.	
Using row operations, get zeros in column 1 below the 1.	$\begin{matrix} \\ -1R_1+R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 0 & 3 & -6 & 6 \\ 3 & -4 & 5 & 7 \end{array}\right]$
	$\begin{matrix} \\ \\ -3R_1+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 0 & 3 & -6 & 6 \\ 0 & 2 & -4 & 4 \end{array}\right]$

Continue the process until the matrix is in row-echelon form.	$\frac{1}{3}R_2 \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -4 & 4 \end{bmatrix}$
Multiply row 2 by -2 and add it to row 3.	$-2R_2 + R_3 \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
At this point, we have all zeros in the bottom row.	
Write the corresponding system of equations.	$\begin{cases} x-2y+3z=1 \\ y-2z=2 \\ 0=0 \end{cases}$
Since $0=0$ we have a true statement. Just as when we solved by substitution, this tells us we have a dependent system. There are infinitely many solutions.	
Solve for y in terms of z in the second equation.	$\begin{aligned} y-2z&=2 \\ y&=2z+2 \end{aligned}$
Solve the first equation for x in terms of z.	$x-2y+3z=1$
Substitute $y=2z+2$.	$x-2(2z+2)+3z=1$
Simplify.	$x-4z-4+3z=1$
Simplify.	$x-z-4=1$
Simplify.	$x=z+5$
The system has infinitely many solutions (x,y,z), where $x=z+5;\ y=2z+2;\ z$ is any real number.	

Example 21.6.23

Solve the system of equations using a matrix: $\begin{cases} x+y-z=0 \\ 2x+4y-2z=6 \\ 3x+6y-3z=9 \end{cases}$

Answer

infinitely many solutions (x,y,z), where $x=z-3;\ y=3;\ z$ is any real number.

Example 21.6.24

Solve the system of equations using a matrix: $\begin{cases} x-y-z=1 \\ -x+2y-3z=-4 \\ 3x-2y-7z=0 \end{cases}$

Answer

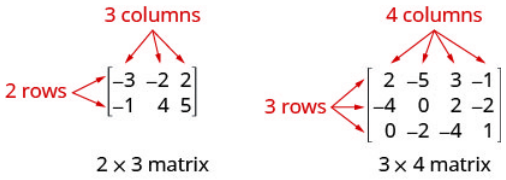
infinitely many solutions (x,y,z), where $x=5z-2;\ y=4z-3;\ z$ is any real number.

Access this online resource for additional instruction and practice with Gaussian Elimination.

- Gaussian Elimination

Key Concepts

- **Matrix:** A matrix is a rectangular array of numbers arranged in rows and columns. A matrix with *m* rows and *n* columns has *order* $m \times n$. The matrix on the left below has 2 rows and 3 columns and so it has order 2×3. We say it is a 2 by 3 matrix.



Each number in the matrix is called an *element* or *entry* in the matrix.

- **Row Operations:** In a matrix, the following operations can be performed on any row and the resulting matrix will be equivalent to the original matrix.
  - Interchange any two rows
  - Multiply a row by any real number except 0
  - Add a nonzero multiple of one row to another row
- **Row-Echelon Form:** For a consistent and independent system of equations, its augmented matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \end{bmatrix}$ $\begin{bmatrix} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{bmatrix}$ a, b, c, d, e, f are real numbers

- **How to solve a system of equations using matrices.**
  1. Write the augmented matrix for the system of equations.
  2. Using row operations get the entry in row 1, column 1 to be 1.
  3. Using row operations, get zeros in column 1 below the 1.
  4. Using row operations, get the entry in row 2, column 2 to be 1.
  5. Continue the process until the matrix is in row-echelon form.
  6. Write the corresponding system of equations.
  7. Use substitution to find the remaining variables.
  8. Write the solution as an ordered pair or triple.
  9. Check that the solution makes the original equations true.

Glossary

matrix

A matrix is a rectangular array of numbers arranged in rows and columns.

row-echelon form

A matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

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21.6E: Exercises

Practice Makes Perfect

Write the Augmented Matrix for a System of Equations

In the following exercises, write each system of linear equations as an augmented matrix.

ⓐ $\begin{cases} 3x-y=-1 \\ 2y=2x+5 \end{cases}$

ⓑ $\begin{cases} 4x+3y=-2 \\ x-2y-3z=7 \\ 2x-y+2z=-6 \end{cases}$

ⓐ $\begin{cases} 2x+4y=-5 \\ 3x-2y=2 \end{cases}$

ⓑ $\begin{cases} 3x-2y-z=-2 \\ -2x+y=5 \\ 5x+4y+z=-1 \end{cases}$

Answer

ⓐ $\begin{bmatrix} 2 & 4 & -5 \\ 3 & -2 & 2 \end{bmatrix}$

ⓑ $\begin{bmatrix} 3 & -2 & -1 & -2 \\ -2 & 1 & 0 & 5 \\ 5 & 4 & 1 & -1 \end{bmatrix}$

ⓐ $\begin{cases} 3x-y=-4 \\ 2x=y+2 \end{cases}$

ⓑ $\begin{cases} x-3y-4z=-2 \\ 4x+2y+2z=5 \\ 2x-5y+7z=-8 \end{cases}$

ⓐ $\begin{cases} 2x-5y=-3 \\ 4x=3y-1 \end{cases}$

ⓑ $\begin{cases} 4x+3y-2z=-3 \\ -2x+y-3z=4 \\ -x-4y+5z=-2 \end{cases}$

Answer

ⓐ $\begin{bmatrix} 2 & -5 & -3 \\ 4 & -3 & -1 \end{bmatrix}$

ⓑ $\begin{bmatrix} 4 & 3 & -2 & -3 \\ -2 & 1 & -3 & 4 \\ -1 & -4 & 5 & -2 \end{bmatrix}$

Write the system of equations that corresponds to the augmented matrix.

$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ 1 & -3 & 2 \end{array}\right]$

$$\left[\begin{array}{cc|c} 2 & -4 & -2 \\ 3 & -3 & -1 \end{array}\right]$$

Answer

$$\begin{cases} 2x-4y=-2 \\ 3x-3y=-1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 1 & -2 & 0 & -2 \\ 0 & -1 & 2 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & -1 \\ 0 & 2 & -1 & 2 \\ 3 & 0 & -1 & -2 \end{array}\right]$$

Answer

$$\begin{cases} 2x-2y=-1 \\ 2y-z=2 \\ 3x-z=-2 \end{cases}$$

Use Row Operations on a Matrix

In the following exercises, perform the indicated operations on the augmented matrices.

$$\left[\begin{array}{cc|c} 6 & -4 & 3 \\ 3 & -2 & 1 \end{array}\right]$$

ⓐ Interchange rows 1 and 2

ⓑ Multiply row 2 by 3

ⓒ Multiply row 2 by -2 and add row 1 to it.

$$\left[\begin{array}{cc|c} 4 & -6 & -3 \\ 3 & 2 & 1 \end{array}\right]$$

ⓐ Interchange rows 1 and 2

ⓑ Multiply row 1 by 4

ⓒ Multiply row 2 by 3 and add row 1 to it.

Answer

ⓐ $\left[\begin{array}{ccc} 3 & 2 & 1 \\ 4 & -6 & -3 \end{array}\right]$

ⓑ $\left[\begin{array}{ccc} 12 & 8 & 4 \\ 4 & -6 & -3 \end{array}\right]$

ⓒ $\left[\begin{array}{ccc} 12 & 8 & 4 \\ 24 & -10 & -5 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 4 & -12 & -8 & 16 \\ 4 & -2 & -3 & -1 \\ -6 & 2 & -1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

Answer

ⓐ $\begin{bmatrix} 2 & 1 & -4 & 5 \\ 6 & -5 & 2 & 3 \\ 3 & -3 & 1 & -1 \end{bmatrix}$

ⓑ $\begin{bmatrix} 2 & 1 & -4 & 5 \\ 6 & -5 & 2 & 3 \\ 3 & -3 & 1 & -1 \end{bmatrix}$

ⓒ $\begin{bmatrix} 2 & 1 & -4 & 5 \\ 6 & -5 & 2 & 3 \\ -4 & 7 & -6 & 7 \end{bmatrix}$

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ -3 & -4 & -1 \end{array}\right]$

Perform the needed row operations that will get the first entry in both row 2 and row 3 to be zero in the augmented matrix: $\left[\begin{array}{ccc|c} 1 & -2 & 3 & -4 \\ 3 & -1 & -2 & 5 \\ 2 & -3 & -4 & 1 \end{array}\right]$

Answer

$\begin{bmatrix} 1 & -2 & 3 & -4 \\ 0 & 5 & -11 & 17 \\ 0 & 1 & -10 & 7 \end{bmatrix}$

Solve Systems of Equations Using Matrices

In the following exercises, solve each system of equations using a matrix.

$\begin{cases} 2x+y=2 \\ x-y=-2 \end{cases}$

$\begin{cases} 3x+y=2 \\ x-y=2 \end{cases}$

Answer

$(1,-1)$

$\begin{cases} -x+2y=-2 \\ x+y=-4 \end{cases}$

$\begin{cases} -2x+3y=3 \\ x+3y=12 \end{cases}$

Answer

$(3,3)$

In the following exercises, solve each system of equations using a matrix.

$$\begin{cases} 2x-3y+z=19 \\ -3x+y-2z=-1 \\ 5x+y+z=0 \end{cases}$$

$$\begin{cases} 2x-y+3z=-3 \\ -x+2y-z=10 \\ x+y+z=5 \end{cases}$$

Answer

$(-2,5,2)$

$$\begin{cases} 2x-6y+z=3 \\ 3x+2y-3z=2 \\ 2x+3y-2z=3 \end{cases}$$

$$\begin{cases} 4x-3y+z=7 \\ 2x-5y-4z=3 \\ 3x-2y-2z=-7 \end{cases}$$

Answer

$(-3,-5,4)$

$$\begin{cases} x+2z=0 \\ 4y+3z=-2 \\ 2x-5y=3 \end{cases}$$

$$\begin{cases} 2x+5y=4 \\ 3y-z=3 \\ 4x+3z=-3 \end{cases}$$

Answer

$(-3,2,3)$

$$\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$$

$$\begin{cases} 3x-z=-3 \\ 5y+2z=-6 \\ 4x+3y=-8 \end{cases}$$

Answer

$(-2,0,-3)$

$$\begin{cases} 2x+3y+z=1 \\ 2x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$$

$$\begin{cases} x+2y+6z=5 \\ -x+y-2z=3 \\ x-4y-2z=1 \end{cases}$$

Answer

no solution

$$\begin{cases} x+2y-3z=-1 \\ x-3y+z=1 \\ 2x-y-2z=2 \end{cases}$$

$$\begin{cases} 4x-3y+2z=0 \\ -2x+3y-7z=1 \\ 2x-2y+3z=6 \end{cases}$$

Answer

no solution

$$\begin{cases} x-y+2z=-4 \\ 2x+y+3z=2 \\ -3x+3y-6z=12 \end{cases}$$

$$\begin{cases} -x-3y+2z=14 \\ -x+2y-3z=-4 \\ 3x+y-2z=6 \end{cases}$$

Answer

infinitely many solutions (x, y, z) where $x=12z+4;\ y=12z-6;\ z$ is any real number

$$\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$$

$$\begin{cases} x+2y+z=4 \\ x+y-2z=3 \\ -2x-3y+z=-7 \end{cases}$$

Answer

infinitely many solutions (x, y, z) where $x=5z+2;\ y=-3z+1;\ z$ is any real number

Writing Exercises

Solve the system of equations $\begin{cases} x+y=10 \\ x-y=6 \end{cases}$ ⓐ by graphing and ⓑ by substitution. ⓒ Which method do you prefer? Why?

Solve the system of equations $\begin{cases} 3x+y=1 \\ 2x=y-8 \end{cases}$ by substitution and explain all your steps in words.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
write the augmented matrix for a system of equations.			
use row operations on a matrix.			
solve systems of equations using matrices.			
write the augmented matrix for a system of equations.			
use row operations on a matrix.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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21.7: Solve Systems of Equations Using Determinants

Learning Objectives

By the end of this section, you will be able to:

- Evaluate the determinant of a 2×2 matrix
- Evaluate the determinant of a 3×3 matrix
- Use Cramer's Rule to solve systems of equations
- Solve applications using determinants

Before you get started, take this readiness quiz.

1. Simplify: $5(-2)-(-4)(1)$.
   If you missed this problem, review [link].
2. Simplify: $-3(8-10)+(-2)(6-3)-4(-3-(-4))$.
   If you missed this problem, review [link].
3. Simplify: $\frac{-12}{-8}$.
   If you missed this problem, review [link].

In this section we will learn of another method to solve systems of linear equations called Cramer's rule. Before we can begin to use the rule, we need to learn some new definitions and notation.

Evaluate the Determinant of a 2×2 Matrix

If a matrix has the same number of rows and columns, we call it a **square matrix**. Each square matrix has a real number associated with it called its **determinant**. To find the determinant of the square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we first write it as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. To get the real number value of the determinate we subtract the products of the diagonals, as shown.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

DETERMINANT

The determinant of any square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, and d are real numbers, is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 21.7.1

Evaluate the determinate of ⓐ $\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$ ⓑ $\begin{bmatrix} -3 & -4 \\ -2 & 0 \end{bmatrix}$.

Answer

ⓐ

	$\begin{vmatrix} 4 & -2 \\ 3 & -1 \end{vmatrix}$
Write the determinant.	$\begin{vmatrix} 4 & -2 \\ 3 & -1 \end{vmatrix}$
Subtract the products of the diagonals.	$4(-1) - 3(-2)$
Simplify.	$-4 + 6$

Simplify.	2

ⓑ

	$\begin{bmatrix} -3 & -4 \\ -2 & 0 \end{bmatrix}$
Write the determinant.	$\begin{vmatrix} -3 & -4 \\ -2 & 0 \end{vmatrix}$
Subtract the products of the diagonals.	$-3(0) - (-2)(-4)$
Simplify.	$0 - 8$
Simplify.	-8

Example 21.7.2

Evaluate the determinate of ⓐ $\begin{bmatrix} 5 & -3 \\ 2 & -4 \end{bmatrix}$ ⓑ $\begin{bmatrix} -4 & -6 \\ 0 & 7 \end{bmatrix}$.

Answer

ⓐ -14; ⓑ -28

Example 21.7.3

Evaluate the determinate of ⓐ $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ ⓑ $\begin{bmatrix} -7 & -3 \\ -5 & 0 \end{bmatrix}$.

Answer

ⓐ 2 ⓑ -15

Evaluate the Determinant of a 3×3 Matrix

To evaluate the determinant of a 3×3 matrix, we have to be able to evaluate the **minor of an entry** in the determinant. The minor of an entry is the 2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.

MINOR OF AN ENTRY IN 3×3 A DETERMINANT

The **minor of an entry** in a 3×3 determinant is the 2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.

To find the minor of entry a_1, we eliminate the row and column which contain it. So we eliminate the first row and first column. Then we write the 2×2 determinant that remains.

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ minor of a_1 $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

To find the minor of entry b_2, we eliminate the row and column that contain it. So we eliminate the 2nd row and 2nd column. Then we write the 2×2 determinant that remains.

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ minor of b_2 $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

Example 21.7.4

For the determinant $\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$, find and then evaluate the minor of ⓐ a_1 ⓑ b_3 ⓒ c_2.

Answer

ⓐ

	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$
Eliminate the row and column that contains a_1.	$\begin{vmatrix} \cancel{4} & \cancel{-2} & \cancel{3} \\ \cancel{1} & 0 & -3 \\ \cancel{-2} & -4 & 2 \end{vmatrix}$
Write the 2×2 determinant that remains.	minor of $a_1 \begin{vmatrix} 0 & -3 \\ -4 & 2 \end{vmatrix}$
Evaluate.	$0(2) - (-3)(-4)$
Simplify.	-12

ⓑ

Eliminate the row and column that contains b_3.	$\begin{vmatrix} 4 & \cancel{-2} & 3 \\ 1 & \cancel{0} & -3 \\ \cancel{-2} & \cancel{-4} & \cancel{2} \end{vmatrix}$
Write the 2×2 determinant that remains.	minor of $b_3 \begin{vmatrix} 4 & 3 \\ 1 & -3 \end{vmatrix}$
Evaluate.	$4(-3) - (1)(3)$
Simplify.	-15

ⓒ

	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$
Eliminate the row and column that contains c_2.	$\begin{vmatrix} 4 & -2 & \cancel{3} \\ \cancel{1} & \cancel{0} & \cancel{-3} \\ -2 & -4 & \cancel{2} \end{vmatrix}$
Write the 2×2 determinant that remains.	minor of $c_2 \begin{vmatrix} 4 & -2 \\ -2 & 4 \end{vmatrix}$
Evaluate.	$4(4) - (-2)(-2)$
Simplify.	12

? Example 21.7.5

For the determinant $\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & -1 \\ -2 & -3 & 3 \end{vmatrix}$, find and then evaluate the minor of ⓐ a_1 ⓑ b_2 ⓒ c_3.

Answer

ⓐ 3 ⓑ 11 ⓒ 2

? Example 21.7.6

For the determinant $\begin{vmatrix} -2 & -1 & 0 \\ 3 & 0 & -1 \\ -1 & -2 & 3 \end{vmatrix}$, find and then evaluate the minor of ⓐ a_2 ⓑ b_3 ⓒ c_2.

Answer

ⓐ -3 ⓑ 2 ⓒ 3

We are now ready to evaluate a 3×3 determinant. To do this we expand by minors, which allows us to evaluate the 3×3 determinant using 2×2 determinants—which we already know how to evaluate!

To evaluate a 3×3 determinant by expanding by minors along the first row, we use the following pattern:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

Remember, to find the minor of an entry we eliminate the row and column that contains the entry.

EXPANDING BY MINORS ALONG THE FIRST ROW TO EVALUATE A 3×3 DETERMINANT

To evaluate a 3×3 determinant by **expanding by minors along the first row**, the following pattern:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

? Example 21.7.7

Evaluate the determinant $\begin{vmatrix} 2 & -3 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & -2 \end{vmatrix}$ by expanding by minors along the first row.

Answer

	$\begin{vmatrix} 2 & -3 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & -2 \end{vmatrix}$
Expand by minors along the first row	$2\underbrace{\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix}}_{\text{minor of } 2} - (-3)\underbrace{\begin{vmatrix} 3 & 0 \\ -1 & -2 \end{vmatrix}}_{\text{minor of } -3} + (-1)\underbrace{\begin{vmatrix} 3 & 2 \\ -1 & -1 \end{vmatrix}}_{\text{minor of } -1}$
Evaluate each determinant.	$2(-4-0) + 3(-6-0) - 1(-3-(-2))$
Simplify.	$2(-4) + 3(-6) - 1(-1)$

Simplify.	$-8-18+1$
Simplify.	-25

Example 21.7.8

Evaluate the determinant $\begin{vmatrix} 3 & -2 & 4 \\ 0 & -1 & -2 \\ 2 & 3 & -1 \end{vmatrix}$, by expanding by minors along the first row.

Answer

37

Example 21.7.9

Evaluate the determinant $\begin{vmatrix} 3 & -2 & -2 \\ 2 & -1 & 4 \\ -1 & 0 & -3 \end{vmatrix}$, by expanding by minors along the first row.

Answer

7

To evaluate a 3×3 determinant we can expand by minors using any row or column. Choosing a row or column other than the first row sometimes makes the work easier.

When we expand by any row or column, we must be careful about the sign of the terms in the expansion. To determine the sign of the terms, we use the following sign pattern chart.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

SIGN PATTERN

When expanding by minors using a row or column, the sign of the terms in the expansion follow the following pattern.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Notice that the sign pattern in the first row matches the signs between the terms in the expansion by the first row.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

minor of a_1 minor of b_1 minor of c_1

Since we can expand by any row or column, how do we decide which row or column to use? Usually we try to pick a row or column that will make our calculation easier. If the determinant contains a 0, using the row or column that contains the 0 will make the calculations easier.

Example 21.7.10

Evaluate the determinant $\begin{vmatrix} 4 & -1 & -3 \\ 3 & 0 & 2 \\ 5 & -4 & -3 \end{vmatrix}$ by expanding by minors.

Answer

To expand by minors, we look for a row or column that will make our calculations easier. Since 0 is in the second row and second column, expanding by either of those is a good choice. Since the second row has fewer negatives than the second column, we will expand by the second row.

	$\begin{vmatrix} 4 & -1 & -3 \\ 3 & 0 & 2 \\ 5 & -4 & -3 \end{vmatrix}$
Expand using the second row.	
Be careful of the signs.	$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ $-3\underbrace{\begin{vmatrix} -1 & -3 \\ -4 & -3 \end{vmatrix}}_{\text{minor of 3}} + (0)\underbrace{\begin{vmatrix} 4 & -3 \\ 5 & -3 \end{vmatrix}}_{\text{minor of 0}} - 2\underbrace{\begin{vmatrix} 4 & -1 \\ 5 & -4 \end{vmatrix}}_{\text{minor of 2}}$
Evaluate each determinant.	$-3(3-12)+0(-12-(-15))-2(-16-(-5))$
Simplify.	$-3(-9)+0-2(-11)$
Simplify.	$27+0+22$
Add.	49

Example 21.7.11

Evaluate the determinant $\begin{vmatrix} 2 & -1 & -3 \\ 0 & 3 & -4 \\ 3 & -4 & -3 \end{vmatrix}$ by expanding by minors.

Answer

-11

Example 21.7.12

Evaluate the determinant $\begin{vmatrix} -2 & -1 & -3 \\ -1 & 2 & 2 \\ 4 & -4 & 0 \end{vmatrix}$ by expanding by minors.

Answer

8

Use Cramer's Rule to Solve Systems of Equations

Cramer's Rule is a method of solving systems of equations using determinants. It can be derived by solving the general form of the systems of equations by elimination. Here we will demonstrate the rule for both systems of two equations with two variables and for systems of three equations with three variables.

Let's start with the systems of two equations with two variables.

CRAMER'S RULE FOR SOLVING A SYSTEM OF TWO EQUATIONS

For the system of equations $\begin{cases} a_1x+b_1y=k_1 \\ a_2x+b_2y=k_2 \end{cases}$, the solution (x, y) can be determined by

$$x=\frac{D_x}{D} \text{ and } y=\frac{D_y}{D}$$

where $D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ use the coefficients of the variables.

$D_x=\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y=\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$ replace the y coefficients with the constants.

Notice that to form the determinant D, we use take the coefficients of the variables.

$$\begin{cases} a_1x+b_1y=k_1 \\ a_2x+b_2y=k_2 \end{cases} \qquad D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Coefficients — Coefficient of x — Coefficient of y

Notice that to form the determinant D_x and D_y, we substitute the constants for the coefficients of the variable we are finding.

$$\begin{cases} a_1x+b_1y=k_1 \\ a_2x+b_2y=k_2 \end{cases} \qquad D_x=\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix} \qquad \begin{cases} a_1x+b_1y=k_1 \\ a_2x+b_2y=k_2 \end{cases} \qquad D_y=\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$$

Constants — Coefficients of y — Coefficients of x — Constants

Example 21.7.13: How to Solve a System of Equations Using Cramer's Rule

Solve using Cramer's Rule: $\begin{cases} 2x+y=-4 \\ 3x-2y=-6 \end{cases}$

Answer

Step 1. Evaluate the determinant D, using the coefficients of the variables.		$\begin{cases} 2x+y=-4 \\ 3x-2y=-6 \end{cases}$ $D=\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}$ $D=-4-3$ $D=-7$
Step 2. Evaluate the determinant D_x. Use the constants in place of the x coefficients.	We replace the coefficients of x, 2 and 3, with the constants, –4 and –6.	$D_x=\begin{vmatrix} -4 & 1 \\ -6 & -2 \end{vmatrix}$ $D_x=8-(-6)$ $D_x=14$
Step 3. Evaluate the determinant D_y. Use the constants in place of the y coefficients.	We replace the coefficients of y, 1 and 2, with the constants, –4 and –6.	$D_y=\begin{vmatrix} 2 & -4 \\ 3 & -6 \end{vmatrix}$ $D_y=-12-(-12)$ $D_y=0$
Step 4. Find x and y.	Substitute in the values of D, D_x and D_y.	$x=\frac{D_x}{D}$ and $y=\frac{D_y}{D}$ $x=\frac{14}{-7}$ and $y=\frac{0}{-7}$ $x=-2$ and $y=0$

Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(-2, 0)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $x = -2$, $y = 0$ into both equations and make sure they are both true.	$(-2, 0)$ is the solution to the system.

? Example 21.7.14

Solve using Cramer's rule: $\begin{cases} 3x+y=-3 \\ 2x+3y=6 \end{cases}$

Answer

$(-\frac{15}{7}, \frac{24}{7})$

? Example 21.7.15

Solve using Cramer's rule: $\begin{cases} -x+y=2 \\ 2x+y=-4 \end{cases}$

Answer

$(-2, 0)$

SOLVE A SYSTEM OF TWO EQUATIONS USING CRAMER'S RULE.

1. Evaluate the determinant D, using the coefficients of the variables.
2. Evaluate the determinant D_x. Use the constants in place of the x coefficients.
3. Evaluate the determinant D_y. Use the constants in place of the y coefficients.
4. Find x and y. $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$
5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to both original equations.

To solve a system of three equations with three variables with Cramer's Rule, we basically do what we did for a system of two equations. However, we now have to solve for three variables to get the solution. The determinants are also going to be 3×3 which will make our work more interesting!

CRAMER'S RULE FOR SOLVING A SYSTEM OF THREE EQUATIONS

For the system of equations $\begin{cases} a_1x+b_1y+c_1z=k_1 \\ a_2x+b_2y+c_2z=k_2 \\ a_3x+b_3y+c_3z=k_3 \end{cases}$, the solution (x, y, z) can be determined by

$$x = \frac{D_x}{D}, y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ use the coefficients of the variables.

$D_x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}$ replace the y coefficients with the constants.

$D_z = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}$ replace the z coefficients with the constants.

Example 21.7.16

Solve the system of equations using Cramer's Rule: $\begin{cases} 3x - 5y + 4z = 5 \\ 5x + 2y + z = 0 \\ 2x + 3y - 2z = 3 \end{cases}$

Answer

Evaluate the determinant D.	$D = \begin{vmatrix} 3 & -5 & 4 \\ 5 & 2 & 1 \\ 2 & 3 & -2 \end{vmatrix}$
Expand by minors using column 1.	
Be careful of the signs. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$	$D = 3\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} - 5\begin{vmatrix} -5 & 4 \\ 3 & -2 \end{vmatrix} + 2\begin{vmatrix} -5 & 4 \\ 2 & 1 \end{vmatrix}$
Evaluate the determinants.	$D = 3(-4-3) - 5(10-12) + 2(-5-8)$
Simplify.	$D = 3(-7) - 5(-2) + 2(-13)$
Simplify.	$D = -21 + 10 - 26$
Simplify.	$D = -37$
Evaluate the determinant D_x. Use the constants to replace the coefficients of x.	$D_x = \begin{vmatrix} 5 & -5 & 4 \\ 0 & 2 & 1 \\ 3 & 3 & -2 \end{vmatrix}$
Expand by minors using column 1.	$D_x = 5\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} - 0\begin{vmatrix} -5 & 4 \\ 3 & -2 \end{vmatrix} + 3\begin{vmatrix} -5 & 4 \\ 2 & 1 \end{vmatrix}$
Evaluate the determinants.	$D_x = 5(-4-3) - 0(10-12) + 3(-5-8)$
Simplify.	$D_x = 5(-7) - 0 + 3(-13)$
Simplify.	$D_x = -74$
Evaluate the determinant Dy.Dy. Use the constants to replace the coefficients of y.	$D_y = \begin{vmatrix} 3 & 5 & 4 \\ 5 & 0 & 1 \\ 2 & 3 & -2 \end{vmatrix}$

Expand by minors using column 2. Be careful of the signs. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$	$D_y = -5\begin{vmatrix} 5 & 1 \\ 2 & -2 \end{vmatrix} + 0\begin{vmatrix} 5 & 4 \\ 3 & -2 \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 5 & 1 \end{vmatrix}$
Evaluate the determinants.	$D_y = -5(-10-2) + 0(-10-12) - 3(3-20)$
Simplify.	$D_y = -5(-12) + 0 - 3(-17)$
Simplify.	$D_y = 60 + 0 + 51$
Simplify.	$D_y = 111$
Evaluate the determinant Dz.Dz. Use the constants to replace the coefficients of z.	$D_z = \begin{vmatrix} 3 & -5 & 5 \\ 5 & 2 & 0 \\ 2 & 3 & 3 \end{vmatrix}$
Expand by minors using column 3. Be careful of the signs. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$	$D_z = 5\begin{vmatrix} 5 & 2 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 3 & -5 \\ 5 & 2 \end{vmatrix}$
Evaluate the determinants.	$D_z = 5(15-4) - 0(9-(-10)) + 3(6-(-25))$
Simplify.	$D_z = 5(11) - 0 + 3(31)$
Simplify.	$D_z = 55 - 0 + 93$
Simplify.	$D_z = 148$
Find x, y, and z.	$x = \frac{D_x}{D}, y = \frac{D_y}{D}$ and $z = \frac{D_z}{D}$
Substitute in the values.	$x = \frac{-74}{-37}, y = \frac{111}{-37}$ and $z = \frac{148}{-37}$
Simplify.	$x = 2, y = -3$ and $z = -4$
Write the solution as an ordered triple.	$(2, -3, -4)$
Check that the ordered triple is a solution to **all three** original equations.	We leave the check to you.
	The solution is $(2, -3, -4)$.

? Example 21.7.17

Solve the system of equations using Cramer's Rule: $\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$

Answer

$(-9, 3, -1)$

? Example 21.7.18

Solve the system of equations using Cramer's Rule: $\begin{cases} 3x+y-6z=-3 \\ 2x+6y+3z=0 \\ 3x+2y-3z=-6 \end{cases}$

Answer

$(-6, 3, -2)$

Cramer's rule does not work when the value of the D determinant is 0, as this would mean we would be dividing by 0. But when $D = 0$, the system is either inconsistent or dependent.

When the value of $D = 0$ and D_x, D_y and D are all zero, the system is consistent and dependent and there are infinitely many solutions.

When the value of $D = 0$ and D_x, D_y and D_z are not all zero, the system is inconsistent and there is no solution.

DEPENDENT AND INCONSISTENT SYSTEMS OF EQUATIONS

For any system of equations, where the **value of the determinant** $D = 0$,

Value of determinants	**Type of system**	**Solution**
$D = 0$ and D_x, D_y and D_z are all zero	consistent and dependent	infinitely many solutions
$D = 0$ and D_x, D_y and D_z are not all zero	inconsistent	no solution

In the next example, we will use the values of the determinants to find the solution of the system.

Example 21.7.19

Solve the system of equations using Cramer's rule : $\begin{cases} x + 3y = 4 \\ -2x - 6y = 3 \end{cases}$

Answer

Evaluate the determinantD,using the coefficients of the variables.

$$\begin{cases} x + 3y = 4 \\ -2x - 6y = 3 \end{cases}$$

$$D = \begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix}$$

$$D = -6 - (-6)$$

$$D = 0$$

We cannot use Cramer's Rule to solve this system. But by looking at the value of the determinants D_x and D_y, we can determine whether the system is dependent or inconsistent.

Evaluate the determinant D_x.

$$D_x = \begin{vmatrix} 4 & 3 \\ 3 & -6 \end{vmatrix}$$

$$D_x = -24 - 9$$

$$D_x = 15$$

Since all the determinants are not zero, the system is inconsistent. There is no solution.

Example 21.7.20

Solve the system of equations using Cramer's rule: $\begin{cases} 4x - 3y = 8 \\ 8x - 6y = 14 \end{cases}$

Answer

no solution

Example 21.7.21

Solve the system of equations using Cramer's rule: $\begin{cases} x = -3y + 4 \\ 2x + 6y = 8 \end{cases}$

Answer

infinite solutions

Solve Applications using Determinants

An interesting application of determinants allows us to test if points are collinear. Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if the determinant below is zero.

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

TEST FOR COLLINEAR POINTS

Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

We will use this property in the next example.

Example 21.7.22

Determine whether the points $(5, -5)$, $(4, -3)$, and $(3, -1)$ are collinear.

Answer

	$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$
Substitute the values into the determinant. $(5, -5)$, $(4, -3)$, and $(3, -1)$	$\begin{vmatrix} 5 & -5 & 1 \\ 4 & -3 & 1 \\ 3 & -1 & 1 \end{vmatrix}$
Evaluate the determinant by expanding by minors using column 3.	$1\begin{vmatrix} 4 & -3 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 5 & -5 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 5 & -5 \\ 4 & -3 \end{vmatrix}$
Evaluate the determinants.	$1(-4-(-9)) - 1(-5-(-15)) + 1(-15-(-20))$
Simplify.	$1(5) - 1(10) + 1(5)$
Simplify.	0
	The value of the determinate is 0, so the points are collinear.

Example 21.7.23

Determine whether the points $(3, -2)$, $(5, -3)$, and $(1, -1)$ are collinear.

Answer

yes

Example 21.7.24

Determine whether the points $(-4,-1)$, $(-6,2)$, and $(-2,-4)$ are collinear.

Answer

yes

Access these online resources for additional instruction and practice with solving systems of linear inequalities by graphing.

- Solving Systems of Linear Inequalities by Graphing
- Systems of Linear Inequalities

Key Concepts

- **Determinant:** The determinant of any square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where *a*, *b*, *c*, and *d* are real numbers, is
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
- **Expanding by Minors along the First Row to Evaluate a 3 × 3 Determinant:** To evaluate a 3×3 determinant by expanding by minors along the first row, the following pattern:
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$
- **Sign Pattern:** When expanding by minors using a row or column, the sign of the terms in the expansion follow the following pattern.
$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$
- **Cramer's Rule:** For the system of equations $\begin{cases} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{cases}$, the solution (x, y) can be determined by
$$x = \frac{D_x}{D} \text{ and } y = \frac{D_y}{D}$$
where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ use the coefficients of the variables.
$D_x = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$ replace the *x* coefficients with the constants.
$D_y = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$ replace the *y* coefficients with the constants.
Notice that to form the determinant *D*, we use take the coefficients of the variables.
- **How to solve a system of two equations using Cramer's rule.**
  1. Evaluate the determinant *D*, using the coefficients of the variables.
  2. Evaluate the determinant D_x. Use the constants in place of the *x* coefficients.
  3. Evaluate the determinant D_y. Use the constants in place of the *y* coefficients.
  4. Find *x* and *y*. $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$.
  5. Write the solution as an ordered pair.
  6. Check that the ordered pair is a solution to **both** original equations.
  7. **Dependent and Inconsistent Systems of Equations:** For any system of equations, where the **value of the determinant** $D = 0$,

Value of determinants	Type of system	Solution
$D=0$ and $D_x,\ D_y$ and D_z are all zero	consistent and dependent	infinitely many solutions
$D=0$ and $D_x,\ D_y$ and D_z are not all zero	inconsistent	no solution

8. **Test for Collinear Points:** Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Glossary

determinant

Each square matrix has a real number associated with it called its determinant.

minor of an entry in a 3×33×3 determinant

The minor of an entry in a 3×33×3 determinant is the 2×22×2 determinant found by eliminating the row and column in the 3×33×3determinant that contains the entry.

square matrix

A square matrix is a matrix with the same number of rows and columns.

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21.7E: Exercises

Practice Makes Perfect

Evaluate the Determinant of a 2 × 2 Matrix

In the following exercises, evaluate the determinate of each square matrix.

$\begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix}$

$\begin{bmatrix} -4 & 8 \\ -3 & 5 \end{bmatrix}$

Answer

4

$\begin{bmatrix} -3 & 5 \\ 0 & -4 \end{bmatrix}$

$\begin{bmatrix} -2 & 0 \\ 7 & -5 \end{bmatrix}$

Answer

10

Evaluate the Determinant of a 3 × 3 Matrix

In the following exercises, find and then evaluate the indicated minors.

$\begin{vmatrix} 3 & -1 & 4 \\ -1 & 0 & -2 \\ -4 & 1 & 5 \end{vmatrix}$

Find the minor ⓐ a_1 ⓑ b_2 ⓒ c_3

$\begin{vmatrix} -1 & -3 & 2 \\ 4 & -2 & -1 \\ -2 & 0 & -3 \end{vmatrix}$

Find the minor ⓐ a_1 ⓑ b_1 ⓒ c_2

Answer

ⓐ 6 ⓑ -14 ⓒ -6

$\begin{vmatrix} 2 & -3 & -4 \\ -1 & 2 & -3 \\ 0 & -1 & -2 \end{vmatrix}$

Find the minor ⓐ a_2 ⓑ b_2 ⓒ c_2

$\begin{vmatrix} -2 & -2 & 3 \\ 1 & -3 & 0 \\ -2 & 3 & -2 \end{vmatrix}$

Find the minor ⓐ a_3 ⓑ b_3 ⓒ c_3

Answer

ⓐ 9 ⓑ -3 ⓒ 8

In the following exercises, evaluate each determinant by expanding by minors along the first row.

$$\begin{vmatrix} -2 & 3 & -1 \\ -1 & 2 & -2 \\ 3 & 1 & -3 \end{vmatrix}$$

$$\begin{vmatrix} 4 & -1 & -2 \\ -3 & -2 & 1 \\ -2 & -5 & 7 \end{vmatrix}$$

Answer

-77

$$\begin{vmatrix} -2 & -3 & -4 \\ 5 & -6 & 7 \\ -1 & 2 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 3 & -2 \\ 5 & -6 & 4 \\ 0 & -2 & -1 \end{vmatrix}$$

Answer

49

In the following exercises, evaluate each determinant by expanding by minors.

$$\begin{vmatrix} -5 & -1 & -4 \\ 4 & 0 & -3 \\ 2 & -2 & 6 \end{vmatrix}$$

$$\begin{vmatrix} 4 & -1 & 3 \\ 3 & -2 & 2 \\ -1 & 0 & 4 \end{vmatrix}$$

Answer

-24

$$\begin{vmatrix} 3 & 5 & 4 \\ -1 & 3 & 0 \\ -2 & 6 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -4 & -3 \\ 5 & -1 & -4 \\ 3 & 2 & 0 \end{vmatrix}$$

Answer

25

Use Cramer's Rule to Solve Systems of Equations

In the following exercises, solve each system of equations using Cramer's Rule.

$$\begin{cases} -2x+3y=3 \\ x+3y=12 \end{cases}$$

$$\begin{cases} x-2y=-5 \\ 2x-3y=-4 \end{cases}$$

Answer

$(7,6)$

$$\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$$

$$\begin{cases} 2x+y=-4 \\ 3x-2y=-6 \end{cases}$$

Answer

$(-2,0)$

$$\begin{cases} x-2y=-5 \\ 2x-3y=-4 \end{cases}$$

$$\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$$

Answer

$(-3,2)$

$$\begin{cases} 5x-3y=-1 \\ 2x-y=2 \end{cases}$$

$$\begin{cases} 3x+8y=-3 \\ 2x+5y=-3 \end{cases}$$

Answer

$(-9,3)$

$$\begin{cases} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{cases}$$

$$\begin{cases} 4x-3y+z=7 \\ 2x-5y-4z=3 \\ 3x-2y-2z=-7 \end{cases}$$

Answer

$(-3,-5,4)$

$$\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$$

$$\begin{cases} 11x+9y+2z=-9 \\ 7x+5y+3z=-7 \\ 4x+3y+z=-3 \end{cases}$$

Answer

$(2,-3,-2)$

$$\begin{cases} x+2z=0 \\ 4y+3z=-2 \\ 2x-5y=3 \end{cases}$$

$$\begin{cases} 2x+5y=4 \\ 3y-z=3 \\ 4x+3z=-3 \end{cases}$$

Answer

$(-3,2,3)$

$$\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$$

$$\begin{cases} 3x-z=-3 \\ 5y+2z=-6 \\ 4x+3y=-8 \end{cases}$$

Answer

$(-2,0,-3)$

$$\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$$

$$\begin{cases} x-4y=-1 \\ -3x+12y=3 \end{cases}$$

Answer

infinitely many solutions

$$\begin{cases} -3x-y=4 \\ 6x+2y=-16 \end{cases}$$

$$\begin{cases} 4x+3y=2 \\ 20x+15y=5 \end{cases}$$

Answer

inconsistent

$$\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$$

$$\begin{cases} 2x+3y+z=1 \\ 2x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$$

Answer

inconsistent

$$\begin{cases} 3x+4y-3z=-2 \\ 2x+3y-z=-1 \\ 2x+y-2z=6 \end{cases}$$

$$\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$$

Answer

infinitely many solutions

Solve Applications Using Determinants

In the following exercises, determine whether the given points are collinear.

$(0,1)$, $(2,0)$, and $(-2,2)$.

$(0,-5)$, $(-2,-2)$, and $(2,-8)$.

Answer

yes

$(4,-3)$, $(6,-4)$, and $(2,-2)$.

$(-2,1)$, $(-4,4)$, and $(0,-2)$.

Answer

no

Writing Exercises

Explain the difference between a square matrix and its determinant. Give an example of each.

Explain what is meant by the minor of an entry in a square matrix.

Answer

Answers will vary.

Explain how to decide which row or column you will use to expand a 3×3 determinant.

Explain the steps for solving a system of equations using Cramer's rule.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate the determinant of a 2×2 matrix.			
evaluate the determinant of a 3×3 matrix.			
use Cramer's rule to solve systems of equations.			
solve applications using determinants.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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21.8: Review Exercises

Chapter Review Exercises

Solve Systems of Linear Equations with Two Variables

Determine Whether an Ordered Pair is a Solution of a System of Equations.

In the following exercises, determine if the following points are solutions to the given system of equations.

1. $\begin{cases} x+3y=-9 \\ 2x-4y=12 \end{cases}$

ⓐ $(-3,-2)$
ⓑ $(0,-3)$

2. $\begin{cases} x+y=8 \\ y=x-4 \end{cases}$

ⓐ $(6,2)$
ⓑ $(9,-1)$

Answer

ⓐ yes ⓑ no

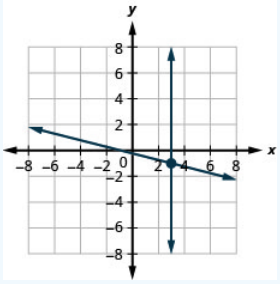
Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.

3. $\begin{cases} 3x+y=6 \\ x+3y=-6 \end{cases}$

4. $\begin{cases} x+4y=-1 \\ x=3 \end{cases}$

Answer

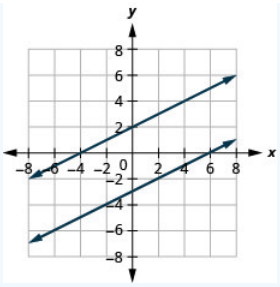


$(3,-1)$

5. $\begin{cases} 2x-y=5 \\ 4x-2y=10 \end{cases}$

6. $\begin{cases} -x+2y=4 \\ y=\frac{1}{2}x-3 \end{cases}$

Answer



no solution

In the following exercises, without graphing determine the number of solutions and then classify the system of equations.

7. $\begin{cases} y = \frac{2}{5}x + 2 \\ -2x + 5y = 10 \end{cases}$

8. $\begin{cases} 3x + 2y = 6 \\ y = -3x + 4 \end{cases}$

Answer

one solution, consistent system, independent equations

9. $\begin{cases} 5x - 4y = 0 \\ y = \frac{5}{4}x - 5 \end{cases}$

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

10. $\begin{cases} 3x - 2y = 2 \\ y = \frac{1}{2}x + 3 \end{cases}$

Answer

$(4, 5)$

11. $\begin{cases} x - y = 0 \\ 2x + 5y = -14 \end{cases}$

12. $\begin{cases} y = -2x + 7 \\ y = \frac{2}{3}x - 1 \end{cases}$

Answer

$(3, 1)$

13. $\begin{cases} y = -5x \\ 5x + y = 6 \end{cases}$

14. $\begin{cases} y = -\frac{1}{3}x + 2 \\ x + 3y = 6 \end{cases}$

Answer

infinitely many solutions

Solve a System of Equations by Elimination

In the following exercises, solve the systems of equations by elimination

15. $\begin{cases} x+y=12 \\ x-y=-10 \end{cases}$

16. $\begin{cases} 3x-8y=20 \\ x+3y=1 \end{cases}$

Answer

$(4,-1)$

17. $\begin{cases} 9x+4y=2 \\ 5x+3y=5 \end{cases}$

18. $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{3}{4}x-y=\frac{5}{2} \end{cases}$

Answer

$(6,2)$

19. $\begin{cases} -x+3y=8 \\ 2x-6y=-20 \end{cases}$

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

20. $\begin{cases} 6x-5y=27 \\ 3x+10y=-24 \end{cases}$

Answer

elimination

21. $\begin{cases} y=3x-9 \\ 4x-5y=23 \end{cases}$

Solve Applications with Systems of Equations

Solve Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

22. Mollie wants to plant 200 bulbs in her garden, all irises and tulips. She wants to plant three times as many tulips as irises. How many irises and how many tulips should she plant?

Answer

50 irises and 150 tulips

23. Ashanti has been offered positions by two phone companies. The first company pays a salary of $22,000 plus a commission of $100 for each contract sold. The second pays a salary of $28,000 plus a commission of $25 for each contract sold. How many contract would need to be sold to make the total pay the same?

24. Leroy spent 20 minutes jogging and 40 minutes cycling and burned 600 calories. The next day, Leroy swapped times, doing 40 minutes of jogging and 20 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?

Answer

10 calories jogging and 10 calories cycling

25. Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and calculator. Troy bought four notebooks and five calculators for $116. Lisa bought two notebooks and three calculators for $68. Find the cost of each notebook and each thumb drive.

Solve Geometry Applications

In the following exercises, translate to a system of equations and solve.

26. The difference of two supplementary angles is 58 degrees. Find the measures of the angles.

Answer

119, 61

27. Two angles are complementary. The measure of the larger angle is five more than four times the measure of the smaller angle. Find the measures of both angles.

28. The measure of one of the small angles of a right triangle is 15 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

35° and 55°

29. Becca is hanging a 28 foot floral garland on the two sides and top of a pergola to prepare for a wedding. The height is four feet less than the width. Find the height and width of the pergola.

30. The perimeter of a city rectangular park is 1428 feet. The length is 78 feet more than twice the width. Find the length and width of the park.

Answer

the length is 450 feet, the width is 264 feet

Solve Uniform Motion Applications

In the following exercises, translate to a system of equations and solve.

31. Sheila and Lenore were driving to their grandmother's house. Lenore left one hour after Sheila. Sheila drove at a rate of 45 mph, and Lenore drove at a rate of 60 mph. How long will it take for Lenore to catch up to Sheila?

32. Bob left home, riding his bike at a rate of 10 miles per hour to go to the lake. Cheryl, his wife, left 45 minutes (34(34 hour) later, driving her car at a rate of 25 miles per hour. How long will it take Cheryl to catch up to Bob?

Answer

12 an hour

33. Marcus can drive his boat 36 miles down the river in three hours but takes four hours to return upstream. Find the rate of the boat in still water and the rate of the current.

34. A passenger jet can fly 804 miles in 2 hours with a tailwind but only 776 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

the rate of the jet is 395 mph, the rate of the wind is 7 mph

Solve Mixture Applications with Systems of Equations

Solve Mixture Applications with Systems of Equations

For the following exercises, translate to a system of equations and solve.

35. Lynn paid a total of $2,780 for 261 tickets to the theater. Student tickets cost $10 and adult tickets cost $15. How many student tickets and how many adult tickets did Lynn buy?

36. Priam has dimes and pennies in a cup holder in his car. The total value of the coins is $4.21. The number of dimes is three less than four times the number of pennies. How many dimes and how many pennies are in the cup?

Answer

41 dimes and 11 pennies

37. Yumi wants to make 12 cups of party mix using candies and nuts. Her budget requires the party mix to cost her $1.29 per cup. The candies are $2.49 per cup and the nuts are $0.69 per cup. How many cups of candies and how many cups of nuts should she use?

38. A scientist needs 70 liters of a 40% solution of alcohol. He has a 30% and a 60% solution available. How many liters of the 30% and how many liters of the 60% solutions should he mix to make the 40% solution?

Answer

$46\frac{2}{3}$ liters of 30% solution, $23\frac{1}{3}$ liters of 60% solution

Solve Interest Applications

For the following exercises, translate to a system of equations and solve.

39. Jack has $12,000 to invest and wants to earn 7.5% interest per year. He will put some of the money into a savings account that earns 4% per year and the rest into CD account that earns 9% per year. How much money should he put into each account?

40. When she graduates college, Linda will owe $43,000 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owes for one year was $1,585. What is the amount of each loan?

Answer

$29,000 for the federal loan, $14,000 for the private loan

Solve Systems of Equations with Three Variables

Solve Systems of Equations with Three Variables

In the following exercises, determine whether the ordered triple is a solution to the system.

41. $\begin{cases} 3x-4y-3z=2 \\ 2x-6y+z=3 \\ 2x+3y-2z=3 \end{cases}$

ⓐ $(2,3,-1)$
ⓑ $(3,1,3)$

42. $\begin{cases} y=\frac{2}{3}x-2 \\ x+3y-z=15 \\ x-3y+z=-2 \end{cases}$

ⓐ $(-6,5,\frac{1}{2})$
ⓑ $(5,\frac{4}{3},-3)$

Answer

ⓐ no ⓑ yes

Solve a System of Linear Equations with Three Variables

In the following exercises, solve the system of equations.

43. $\begin{cases} 3x-5y+4z=5 \\ 5x+2y+z=0 \\ 2x+3y-2z=3 \end{cases}$

44. $\begin{cases} x+\frac{5}{2}y+z=-2 \\ 2x+2y+\frac{1}{2}z=-4 \\ \frac{1}{3}x-y-z=1 \end{cases}$

Answer

$(-3,2,-4)$

45. $\begin{cases} 5x+3y=-6 \\ 2y+3z=-1 \\ 7x+z=1 \end{cases}$

46. $\begin{cases} 2x+3y+z=12 \\ x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$

Answer

no solution

47. $\begin{cases} -x-3y+2z=14 \\ -x+2y-3z=-4 \\ 3x+y-2z=6 \end{cases}$

Solve Applications using Systems of Linear Equations with Three Variables

48. After attending a major league baseball game, the patrons often purchase souvenirs. If a family purchases 4 t-shirts, a cap and 1 stuffed animal their total is $135. A couple buys 2 t-shirts, a cap and 3 stuffed animals for their nieces and spends $115. Another couple buys 2 t-shirts, a cap and 1 stuffed animal and their total is $85. What is the cost of each item?

Answer

$25, 20, 15$

Solve Systems of Equations Using Matrices

Write the Augmented Matrix for a System of Equations.

Write each system of linear equations as an augmented matrix.

49. $\begin{cases} 3x - y = -1 \\ -2x + 2y = 5 \end{cases}$

50. $\begin{cases} 4x + 3y = -2 \\ x - 2y - 3z = 7 \\ 2x - y + 2z = -6 \end{cases}$

Answer

$$\begin{bmatrix} 4 & 3 & 0 & -2 \\ 1 & -2 & -3 & 7 \\ 2 & -1 & 2 & -6 \end{bmatrix}$$

Write the system of equations that that corresponds to the augmented matrix.

51. $\left[\begin{array}{cc|c} 2 & -4 & -2 \\ 3 & -3 & -1 \end{array}\right]$

52. $\left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 1 & -2 & 0 & -2 \\ 0 & -1 & 2 & 3 \end{array}\right]$

Answer

$$\begin{cases} x - 3z = -1 \\ x - 2y = -27 \\ -y + 2z = 3 \end{cases}$$

In the following exercises, perform the indicated operations on the augmented matrices.

53. $\left[\begin{array}{cc|c} 4 & -6 & -3 \\ 3 & 2 & 1 \end{array}\right]$

ⓐ Interchange rows 2 and 1.
ⓑ Multiply row 1 by 4.
ⓒ Multiply row 2 by 3 and add to row 1.

54. $\left[\begin{array}{ccc|c} 1 & -3 & -2 & 4 \\ 2 & 2 & -1 & -3 \\ 4 & -2 & -3 & -1 \end{array}\right]$

ⓐ Interchange rows 2 and 3.
ⓑ Multiply row 1 by 2.
ⓒ Multiply row 3 by $-2-2$ and add to row 2.

Answer

ⓐ $\begin{bmatrix} 1 & -3 & -2 & 4 \\ 4 & -2 & -3 & -1 \\ 2 & 2 & -1 & -3 \end{bmatrix}$

ⓑ $\begin{bmatrix} 2 & -6 & -4 & 8 \\ 4 & -2 & -3 & -1 \\ 2 & 2 & -1 & -3 \end{bmatrix}$

ⓒ $\begin{bmatrix} 2 & -6 & -4 & 8 \\ 4 & -2 & -3 & -1 \\ 0 & -6 & -1 & 5 \end{bmatrix}$

Solve Systems of Equations Using Matrices

In the following exercises, solve each system of equations using a matrix.

55. $\begin{cases} 4x+y=6 \\ x-y=4 \end{cases}$

56. $\begin{cases} 2x-y+3z=-3 \\ -x+2y-z=10 \\ x+y+z=5 \end{cases}$

Answer

$(-2, 5, -2)$

57. $\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$

58. $\begin{cases} x+2y-3z=-1 \\ x-3y+z=1 \\ 2x-y-2z=2 \end{cases}$

Answer

no solution

59. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

Solve Systems of Equations Using Determinants

Evaluate the Determinant of a 2 × 2 Matrix

In the following exercise, evaluate the determinate of the square matrix.

60. $\begin{bmatrix} 8 & -4 \\ 5 & -3 \end{bmatrix}$

Answer

-4

Evaluate the Determinant of a 3 × 3 Matrix

In the following exercise, find and then evaluate the indicated minors.

61. $\begin{vmatrix} -1 & -3 & 2 \\ 4 & -2 & -1 \\ -2 & 0 & -3 \end{vmatrix}$; Find the minor ⓐ a_1 ⓑ b_1 ⓒ c_2

In the following exercise, evaluate each determinant by expanding by minors along the first row.

62. $\begin{vmatrix} -2 & -3 & -4 \\ 5 & -6 & 7 \\ -1 & 2 & 0 \end{vmatrix}$

Answer

21In the following exercise, evaluate each determinant by expanding by minors.

63. $\begin{vmatrix} 3 & 5 & 4 \\ -1 & 3 & 0 \\ -2 & 6 & 1 \end{vmatrix}$

Use Cramer's Rule to Solve Systems of Equations

In the following exercises, solve each system of equations using Cramer's rule

64. $\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$

Answer

$(-3, 2)$

65. $\begin{cases} 4x-3y+z=7 \\ 2x-5y-4z=3 \\ 3x-2y-2z=-7 \end{cases}$

66. $\begin{cases} 2x+5y=4 \\ 3y-z=3 \\ 4x+3z=-3 \end{cases}$

Answer

$(-3, 2, 3)$

67. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

68. $\begin{cases} 3x+4y-3z=-2 \\ 2x+3y-z=-1 \\ 2x+y-2z=6 \end{cases}$

Answer

inconsistent

Solve Applications Using Determinants

In the following exercises, determine whether the given points are collinear.

69. $(0, 2)$, $(-1, -1)$, and $(-2, 4)$

Graphing Systems of Linear Inequalities

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

In the following exercises, determine whether each ordered pair is a solution to the system.

70. $\begin{cases} 4x + y > 6 \\ 3x - y \leq 12 \end{cases}$

ⓐ $(2, -1)$
ⓑ $(3, -2)$

Answer

ⓐ yes ⓑ no

71. $\begin{cases} y > \frac{1}{3}x + 2 \\ x - \frac{1}{4}y \leq 10 \end{cases}$

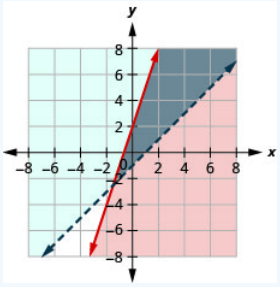
ⓐ $(6, 5)$
ⓑ $(15, 8)$

Solve a System of Linear Inequalities by Graphing

In the following exercises, solve each system by graphing.

72. $\begin{cases} y < 3x + 1 \\ y \geq -x - 2 \end{cases}$

Answer

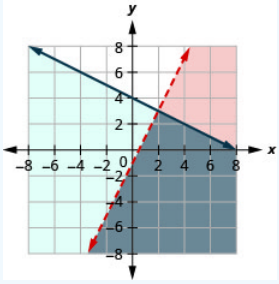


The solution is the grey region.

73. $\begin{cases} x - y > -1 \\ y < \frac{1}{3}x - 2 \end{cases}$

74. $\begin{cases} 2x - 3y < 6 \\ 3x + 4y \geq 12 \end{cases}$

Answer

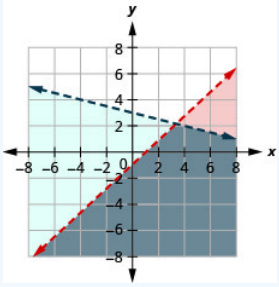


The solution is the grey region.

75. $\begin{cases} y \leq -\frac{3}{4}x + 1 \\ x \geq -5 \end{cases}$

76. $\begin{cases} x + 3y < 5 \\ y \geq -\frac{1}{3}x + 6 \end{cases}$

Answer



No solution.

77. $\begin{cases} y \geq 2x - 5 \\ -6x + 3y > -4 \end{cases}$

Solve Applications of Systems of Inequalities

In the following exercises, translate to a system of inequalities and solve.

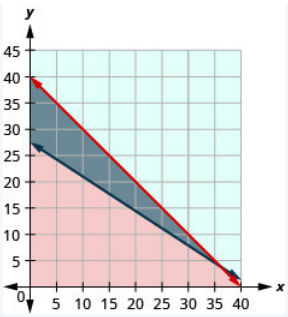
78. Roxana makes bracelets and necklaces and sells them at the farmers' market. She sells the bracelets for $12 each and the necklaces for $18 each. At the market next weekend she will have room to display no more than 40 pieces, and she needs to sell at least $500 worth in order to earn a profit.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Should she display 26 bracelets and 14 necklaces?
ⓓ Should she display 39 bracelets and 1 necklace?

Answer

ⓐ $\begin{cases} b \geq 0 \\ n \geq 0 \\ b + n \leq 40 \\ 12b + 18n \geq 500 \end{cases}$

ⓑ



ⓒ yes
ⓓ no

79. Annie has a budget of $600 to purchase paperback books and hardcover books for her classroom. She wants the number of hardcover to be at least 5 more than three times the number of paperback books. Paperback books cost $4 each and hardcover books cost $15 each.

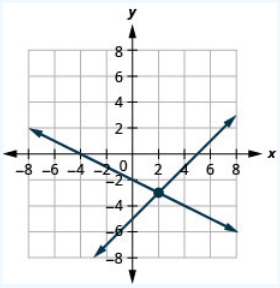
ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can she buy 8 paperback books and 40 hardcover books?
ⓓ Can she buy 10 paperback books and 37 hardcover books?

Chapter Practice Test

In the following exercises, solve the following systems by graphing.

1. $\begin{cases} x-y=5 \\ x+2y=-4 \end{cases}$

Answer



$(2,-3)$

2. $\begin{cases} x-y>-2 \\ y\leq 3x+1 \end{cases}$

In the following exercises, solve each system of equations. Use either substitution or elimination.

3. $\begin{cases} x+4y=6 \\ -2x+y=-3 \end{cases}$

Answer

$(2,1)$

4. $\begin{cases} -3x+4y=2 \\ 5x-5y=-23 \end{cases}$

5. $\begin{cases} x+y-z=-1 \\ 2x-y+2z=8 \\ -3x+2y+z=-9 \end{cases}$

Answer

$(2,-2,1)$

Solve the system of equations using a matrix.

6. $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

7. $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$

Answer

$(5,7,4)$

Solve using Cramer's rule.

8. $\begin{cases} 3x+y=-3 \\ 2x+3y=6 \end{cases}$

9. Evaluate the determinant by expanding by minors:

$$\begin{vmatrix} 3 & -2 & -2 \\ 2 & -1 & 4 \\ -1 & 0 & -3 \end{vmatrix}$$

Answer

99

In the following exercises, translate to a system of equations and solve.

10. Greg is paddling his canoe upstream, against the current, to a fishing spot 10 miles away. If he paddles upstream for 2.5 hours and his return trip takes 1.25 hours, find the speed of the current and his paddling speed in still water.

11. A pharmacist needs 20 liters of a 2% saline solution. He has a 1% and a 5% solution available. How many liters of the 1% and how many liters of the 5% solutions should she mix to make the 2% solution?

Answer

15 liters of 1% solution, 5 liters of 5% solution

12. Arnold invested $64,000, some at 5.5% interest and the rest at 9%. How much did he invest at each rate if he received $4,500 in interest in one year?

13. The church youth group is selling snacks to raise money to attend their convention. Amy sold 2 pounds of candy, 3 boxes of cookies and 1 can of popcorn for a total sales of $65. Brian sold 4 pounds of candy, 6 boxes of cookies and 3 cans of

popcorn for a total sales of $140. Paulina sold 8 pounds of candy, 8 boxes of cookies and 5 can of popcorn for a total sales of $250. What is the cost of each item?

Answer

The candy cost $20; the cookies cost $5; and the popcorn cost $10.

14. The manufacturer of a granola bar spends $1.20 to make each bar and sells them for $2. The manufacturer also has fixed costs each month of $8,000.

ⓐ Find the cost function C when x granola bars are manufactured
ⓑ Find the revenue function R when x granola bars are sold.
ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.
ⓓ Find the break-even point. Interpret what the break-even point means.

15. Translate to a system of inequalities and solve.

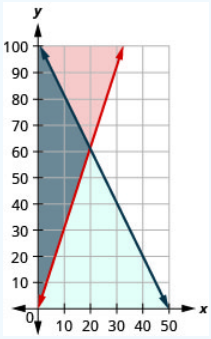
Andi wants to spend no more than $50 on Halloween treats. She wants to buy candy bars that cost $1 each and lollipops that cost $0.50 each, and she wants the number of lollipops to be at least three times the number of candy bars.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can she buy 20 candy bars and 40 lollipops?

Answer

ⓐ $$\begin{cases} C \geq 0 \\ L \geq 0 \\ C + 0.5L \leq 50 \\ L \geq 3C \end{cases}$$

ⓑ



ⓒ no
ⓓ yes

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CHAPTER OVERVIEW

22: Conics

In this chapter, you will learn about conics, including circles, parabolas, ellipses, and hyperbolas. Then you will use what you learn to investigate systems of nonlinear equations.

22.1: Prelude to Conics
22.2: Distance and Midpoint Formulas and Circles
22.2E: Exercises
22.3: Parabolas
22.3E: Exercises
22.4: Ellipses
22.4E: Exercises
22.5: Hyperbolas
22.5E: Exercises
22.6: Solving Systems of Nonlinear Equations
22.6E: Exercises
22.7: Review Exercises

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22.1: Prelude to Conics

Five, Four. Three. Two. One. Lift off. The rocket launches off the ground headed toward space. Unmanned spaceships, and spaceships in general, are designed by aerospace engineers. These engineers are investigating reusable rockets that return safely to Earth to be used again. Someday, rockets may carry passengers to the International Space Station and beyond. One essential math concept for aerospace engineers is that of conics. In this chapter, you will learn about conics, including circles, parabolas, ellipses, and hyperbolas. Then you will use what you learn to investigate systems of nonlinear equations.

Figure 11.0.1: Aerospace engineers use rockets such as this one to launch people and objects into space. (credit: WikiImages/Pixabay)

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22.2: Distance and Midpoint Formulas and Circles

Learning Objectives

By the end of this section, you will be able to:

- Use the Distance Formula
- Use the Midpoint Formula
- Write the equation of a circle in standard form
- Graph a circle

Before you get started, take this readiness quiz.

1. Find the length of the hypotenuse of a right triangle whose legs are 12 and 16 inches.
   If you missed this problem, review Example 2.34.
2. Factor: $x^2-18x+81$.
   If you missed this problem, review Example 6.24.
3. Solve by completing the square: $x^2-12x-12=0$.
   If you missed this problem, review Example 9.22.

In this chapter we will be looking at the conic sections, usually called the conics, and their properties. The conics are curves that result from a plane intersecting a double cone—two cones placed point-to-point. Each half of a double cone is called a nappe.

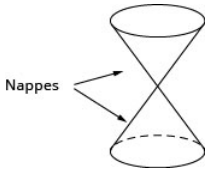


Figure 11.1.1

There are four conics—the **circle**, **parabola**, **ellipse**, and **hyperbola**. The next figure shows how the plane intersecting the double cone results in each curve.

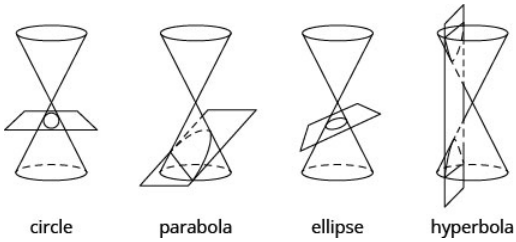


Figure 11.1.2

Each of the curves has many applications that affect your daily life, from your cell phone to acoustics and navigation systems. In this section we will look at the properties of a circle.

Use the Distance Formula

We have used the Pythagorean Theorem to find the lengths of the sides of a right triangle. Here we will use this theorem again to find distances on the rectangular coordinate system. By finding distance on the rectangular coordinate system, we can make a connection between the geometry of a conic and algebra—which opens up a world of opportunities for application.

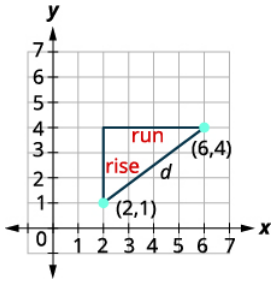
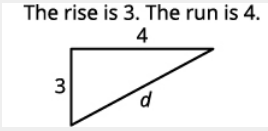
Our first step is to develop a formula to find distances between points on the rectangular coordinate system. We will plot the points and create a right triangle much as we did when we found slope in Graphs and Functions. We then take it one step further and use the Pythagorean Theorem to find the length of the hypotenuse of the triangle—which is the distance between the points.

Example 22.2.1

Use the rectangular coordinate system to find the distance between the points $(6, 4)$ and $(2, 1)$.

Solution

Table 11.1.1

Plot the two points. Connect the two points with a line. Draw a right triangle as if you were going to find slope.	 Figure 11.1.3
Find the length of each leg.	 Figure 11.1.4
Use the Pythagorean Theorem to find d, the distance between the two points.	$a^2 + b^2 = c^2$
Substitute in the values.	$3^2 + 4^2 = d^2$
Simplify.	$9 + 16 = d^2$
	$25 = d^2$
Use the Square Root Property.	$d = 5 \quad d = \cancel{-5}$
Since distance, d is positive, we can eliminate $d = -5$.	The distance between the points $(6, 4)$ and $(2, 1)$ is 5.

Exercise 22.2.1

Use the rectangular coordinate system to find the distance between the points $(6, 1)$ and $(2, -2)$.

Answer

$d = 5$

Exercise 22.2.2

Use the rectangular coordinate system to find the distance between the points $(5, 3)$ and $(-3, -3)$.

Answer

$d = 10$

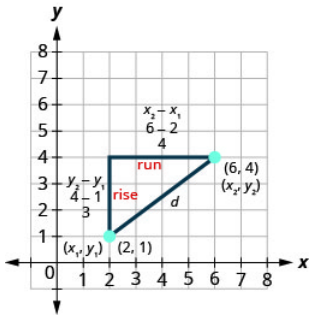


Figure 11.1.5

The method we used in the last example leads us to the formula to find the distance between the two points (x_1, y_1) and (x_2, y_2).

When we found the length of the horizontal leg we subtracted $6 - 2$ which is $x_2 - x_1$.

When we found the length of the vertical leg we subtracted $4 - 1$ which is $y_2 - y_1$.

If the triangle had been in a different position, we may have subtracted $x_1 - x_2$ or $y_1 - y_2$. The expressions $x_2 - x_1$ and $x_1 - x_2$ vary only in the sign of the resulting number. To get the positive value-since distance is positive- we can use absolute value. So to generalize we will say $|x_2 - x_1|$ and $|y_2 - y_1|$.

In the Pythagorean Theorem, we substitute the general expressions $|x_2 - x_1|$ and $|y_2 - y_1|$ rather than the numbers.

	$a^2 + b^2 = c^2$
Substitute in the values.	$(\|x_2 - x_1\|)^2 + (\|y_2 - y_1\|)^2 = d^2$
Squaring the expressions makes them positive, so we eliminate the absolute value bars.	$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$
Use the Square Root Property.	$d = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Distance is positive, so eliminate the negative value.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

This is the Distance Formula we use to find the distance d between the two points (x_1, y_1) and (x_2, y_2).

Definition 22.2.1

Distance Formula

The distance d between the two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 22.2.2

Use the Distance Formula to find the distance between the points $(-5, -3)$ and $(7, 2)$.

Solution:

Write the Distance Formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Label the points, $\begin{pmatrix} x_1, y_1 \\ -5, -3 \end{pmatrix}, \begin{pmatrix} x_2, y_2 \\ 7, 2 \end{pmatrix}$ and substitute.

$$d = \sqrt{(7 - (-5))^2 + (2 - (-3))^2}$$

Simplify.

$$d = \sqrt{12^2 + 5^2}$$
$$d = \sqrt{144 + 25}$$

$$d = \sqrt{169}$$
$$d = 13$$

Answer:

$d = 13$

Exercise 22.2.3

Use the Distance Formula to find the distance between the points $(-4, -5)$ and $(5, 7)$.

Answer

$d = 15$

Exercise 22.2.4

Use the Distance Formula to find the distance between the points $(-2, -5)$ and $(-14, -10)$.

Answer

$d = 13$

Example 22.2.3

Use the Distance Formula to find the distance between the points $(10, -4)$ and $(-1, 5)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Solution:

Write the Distance Formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Label the points, $\begin{pmatrix} x_1, y_1 \\ 10, -4 \end{pmatrix}$, $\begin{pmatrix} x_2, y_2 \\ -1, 5 \end{pmatrix}$ and substitute.

$$d = \sqrt{(-1 - 10)^2 + (5 - (-4))^2}$$

Simplify.

$$d = \sqrt{(-11)^2 + 9^2}$$
$$d = \sqrt{121 + 81}$$
$$d = \sqrt{202}$$

Since 202 is not a perfect square, we can leave the answer in exact form or find a decimal approximation.

$$d = \sqrt{202}$$
or
$$d \approx 14.2$$

Exercise 22.2.5

Use the Distance Formula to find the distance between the points $(-4, -5)$ and $(3, 4)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Answer

$d = \sqrt{130}, d \approx 11.4$

Exercise 22.2.6

Use the Distance Formula to find the distance between the points $(-2,-5)$ and $(-3,-4)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Answer

$d=\sqrt{2}, d \approx 1.4$

Use the Midpoint Formula

It is often useful to be able to find the midpoint of a segment. For example, if you have the endpoints of the diameter of a circle, you may want to find the center of the circle which is the midpoint of the diameter. To find the midpoint of a line segment, we find the average of the x-coordinates and the average of the y-coordinates of the endpoints.

Definition 22.2.2

Midpoint Formula

The midpoint of the line segment whose endpoints are the two points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

To find the midpoint of a line segment, we find the average of the x-coordinates and the average of the y-coordinates of the endpoints.

Example 22.2.4

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-5,-4)$ and $(7,2)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

Solution:

Table 11.1.2

Write the Midpoint Formula.	$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
Label the points, $\begin{pmatrix} x_1, y_1 \\ -5, -4 \end{pmatrix}, \begin{pmatrix} x_2, y_2 \\ 7, 2 \end{pmatrix}$ and substitute.	$\left(\frac{-5+7}{2}, \frac{-4+2}{2}\right)$
Simplify.	$\left(\frac{2}{2}, \frac{-2}{2}\right)$
	$(1,-1)$ The midpoint of the segment is the point $(1,-1)$.

Plot the endpoints and midpoint.

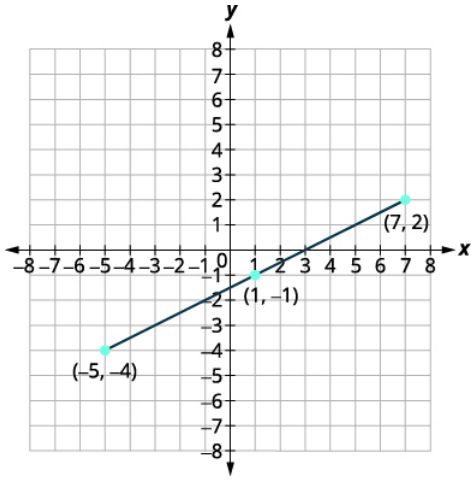


Figure 11.1.6

Exercise 22.2.7

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-3,-5)$ and $(5,7)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

Answer

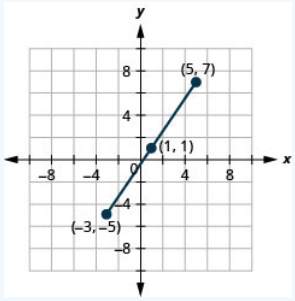


Figure 11.1.7

Exercise 22.2.8

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-2,-5)$ and $(6,-1)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

Answer

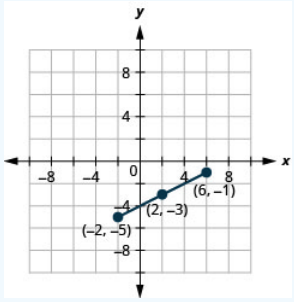


Figure 11.1.8

Both the Distance Formula and the Midpoint Formula depend on two points, (x_1, y_1) and (x_2, y_2). It is easy to confuse which formula requires addition and which subtraction of the coordinates. If we remember where the formulas come from, is may be easier to remember the formulas.

Distance Formula	Midpoint Formula
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Subtract the coordinates.	Add the coordinates.

Figure 11.1.9

Write the Equation of a Circle in Standard Form

As we mentioned, our goal is to connect the geometry of a conic with algebra. By using the coordinate plane, we are able to do this easily.

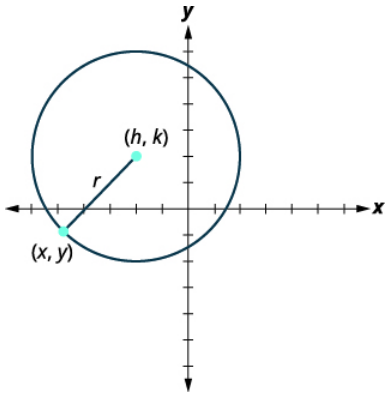
Figure 11.1.10

We define a **circle** as all points in a plane that are a fixed distance from a given point in the plane. The given point is called the *center*, (h, k), and the fixed distance is called the *radius*, r, of the circle.

Definition 22.2.3

A circle is all points in a plane that are a fixed distance from a given point in the plane. The given point is called the **center**, (h, k), and the fixed distance is called the **radius**, r, of the circle.

Table 11.1.3

We look at a circle in the rectangular coordinate system. The radius is the distance from the center, (h, k),to a point on the circle, (x, y).	 Figure 11.1.11
To derive the equation of a circle, we can use the distance formula with the points (h, k), (x, y) and the distance, r.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Substitute the values.	$r = \sqrt{(x - h)^2 + (y - k)^2}$
Square both sides.	$r^2 = (x - h)^2 + (y - k)^2$

This is the standard form of the equation of a circle with center, (h, k), and radius, r.

Definition 22.2.4

The standard form of the equation of a circle with center, (h, k), and radius, r, is

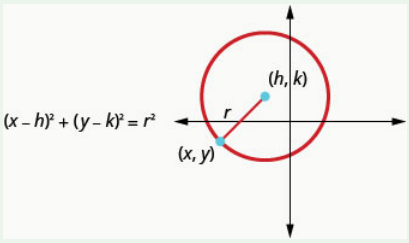


Figure 11.1.12

Example 22.2.5

Write the standard form of the equation of the circle with radius 3 and center $(0, 0)$.

Solution:

Table 11.1.4

Use the standard form of the equation of a circle	$(x-h)^2+(y-k)^2=r^2$
Substitute in the values $r=3$, $h=0$, and $k=0$.	$(x-0)^2+(y-0)^2=3^2$
Center: $\begin{pmatrix} h, & k \\ 0, & 0 \end{pmatrix}$	
Simplify.	$x^2+y^2=9$

Exercise 22.2.9

Write the standard form of the equation of the circle with a radius of 6 and center $(0, 0)$.

Answer

$x^2+y^2=36$

Exercise 22.2.10

Write the standard form of the equation of the circle with a radius of 8 and center $(0, 0)$.

Answer

$x^2+y^2=64$

In the last example, the center was $(0, 0)$. Notice what happened to the equation. Whenever the center is $(0, 0)$, the standard form becomes $x^2+y^2=r^2$.

Example 22.2.6

Write the standard form of the equation of the circle with radius 2 and center $(-1, 3)$.

Solution:

Table 11.1.5

Use the standard form of the equation of a circle.	$(x-h)^2+(y-k)^2=r^2$
Substitute in the values.	$(x-(-1))^2+(y-3)^2=2^2$

Center: $\begin{pmatrix} h, & k \\ -1, & 3 \end{pmatrix}$	
Simplify.	$(x+1)^2+(y-3)^2=4$

? Exercise 22.2.11

Write the standard form of the equation of the circle with a radius of 7 and center $(2,-4)$.

Answer

$(x-2)^2+(y+4)^2=49$

? Exercise 22.2.12

Write the standard form of the equation of the circle with a radius of 9 and center $(-3,-5)$.

Answer

$(x+3)^2+(y+5)^2=81$

In the next example, the radius is not given. To calculate the radius, we use the Distance Formula with the two given points.

✓ Example 22.2.7

Write the standard form of the equation of the circle with center $(2,4)$ that also contains the point $(-2,1)$.

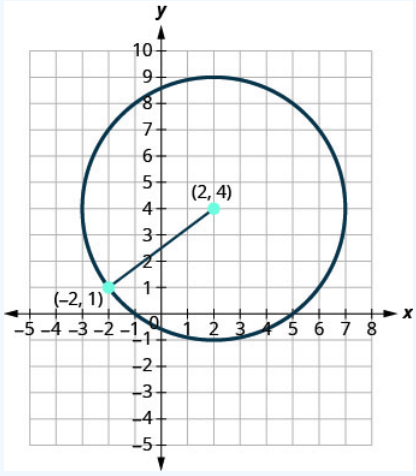


Figure 11.1.15

Solution:

The radius is the distance from the center to any point on the circle so we can use the distance formula to calculate it. We will use the center $(2,4)$ and point $(-2,1)$

Use the Distance Formula to find the radius.

$$r=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$$

Substitute the values. $\begin{pmatrix} x_1, y_1 \\ 2, 4 \end{pmatrix}, \begin{pmatrix} x_2, y_2 \\ -2, 1 \end{pmatrix}$

$$r=\sqrt{(-2-2)^2+(1-4)^2}$$

Simplify.

$$r=\sqrt{(-4)^2+(-3)^2}$$
$$r=\sqrt{16+9}$$
$$r=\sqrt{25}$$
$$r=5$$

Now that we know the radius, $r=5$, and the center, $(2,4)$, we can use the standard form of the equation of a circle to find the equation.

Use the standard form of the equation of a circle.

$$(x-h)^2+(y-k)^2=r^2$$

Substitute in the values.

$$(x-2)^2+(y-4)^2=5^2$$

Simplify.

$$(x-2)^2+(y-4)^2=25$$

Exercise 22.2.13

Write the standard form of the equation of the circle with center $(2,1)$ that also contains the point $(-2,-2)$.

Answer

$(x-2)^2+(y-1)^2=25$

Exercise 22.2.14

Write the standard form of the equation of the circle with center $(7,1)$ that also contains the point $(-1,-5)$.

Answer

$(x-7)^2+(y-1)^2=100$

Graph a Circle

Any equation of the form $(x-h)^2+(y-k)^2=r^2$ is the standard form of the equation of a **circle** with center, (h,k), and radius, r. We can then graph the circle on a rectangular coordinate system.

Note that the standard form calls for subtraction from x and y. In the next example, the equation has $x+2$, so we need to rewrite the addition as subtraction of a negative.

Example 22.2.8

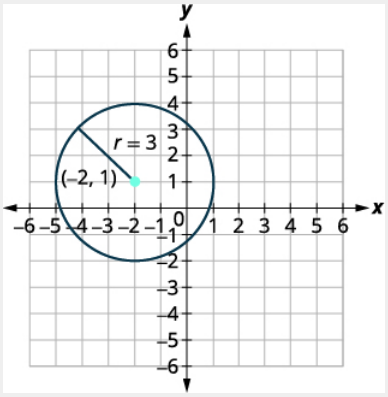
Find the center and radius, then graph the circle: $(x+2)^2+(y-1)^2=9$.

Solution:

Table 11.1.6

	$(x+2)^2+(y-1)^2=9$
Use the standard form of the equation of a circle. Identify the center, (h,k) and radius, r.	$(x-h)^2+(y-k)^2=r^2$ $(x-(-2))^2+(y-1)^2=3^2$
	Center: $(-2,1)$ radius: 3

Graph the circle.



Exercise 22.2.15

a. Find the center and radius, then
b. Graph the circle: $(x-3)^2+(y+4)^2=4$.

Answer

a. The circle is centered at $(3,-4)$ with a radius of 2.
b.

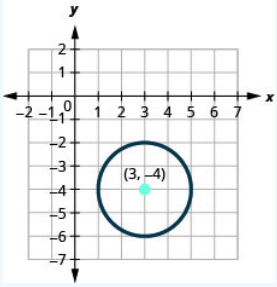


Figure 11.1.19

Exercise 22.2.16

a. Find the center and radius, then
b. Graph the circle: $(x-3)^2+(y-1)^2=16$.

Answer

a. The circle is centered at $(3,1)$ with a radius of 4.
b.

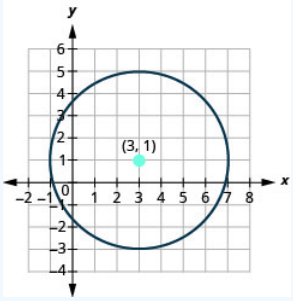


Figure 11.1.20

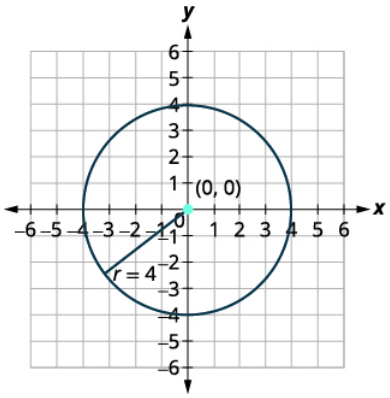
To find the center and radius, we must write the equation in standard form. In the next example, we must first get the coefficient of x^2, y^2 to be one.

Example 22.2.9

Find the center and radius and then graph the circle, $4x^2 + 4y^2 = 64$.

Solution:

Table 11.1.7

	$4x^2 + 4y^2 = 64$
Divide each side by 4.	$x^2 + y^2 = 16$
Use the standard form of the equation of a circle. Identify the center, (h, k) and radius, r.	$(x-h)^2 + (y-k)^2 = r^2$ $(x-0)^2 + (y-0)^2 = 4^2$
	Center: $(0, 0)$ radius: 4
Graph the circle.	

Exercise 22.2.17

a. Find the center and radius, then
b. Graph the circle: $3x^2 + 3y^2 = 27$

Answer

a. The circle is centered at $(0, 0)$ with a radius of 3.
b.

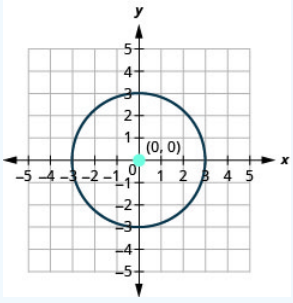


Figure 11.1.25

Exercise 22.2.18

a. Find the center and radius, then
b. Graph the circle: $5x^2+5y^2=125$

Answer

a. The circle is centered at $(0,0)$ with a radius of 5.
b.

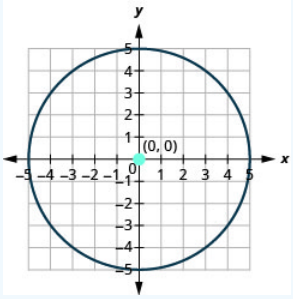


Figure 11.1.26

If we expand the equation from Example 11.1.8, $(x+2)^2+(y-1)^2=9$, the equation of the circle looks very different.

$$(x+2)^2+(y-1)^2=9$$

Square the binomials.

$$x^2+4x+4+y^2-2y+1=9$$

Arrange the terms in descending degree order, and get zero on the right

$$x^2+y^2+4x-2y-4=0$$

This form of the equation is called the general form of the equation of the **circle**.

Definition 22.2.5

The general form of the equation of a circle is

$$x^2+y^2+ax+by+c=0$$

If we are given an equation in general form, we can change it to standard form by completing the squares in both x and y. Then we can graph the circle using its center and radius.

Example 22.2.10

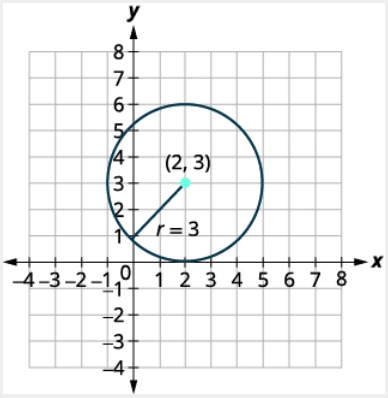
a. Find the center and radius, then
b. Graph the circle: $x^2+y^2-4x-6y+4=0$

Solution:

We need to rewrite this general form into standard form in order to find the center and radius.

Table 11.1.8

	$x^2+y^2-4x-6y+4=0$ Figure 11.1.27
Group the x-terms and y-terms. Collect the constants on the right side.	$x^2-4x+y^2-6y=-4$ Figure 11.1.28
Complete the squares.	$x^2-4x+4+y^2-6y+9=-4+4+9$ Figure 11.1.29

Rewrite as binomial squares.	$(x-2)^2+(y-3)^2=9$ Figure 11.1.30
Identify the center and radius.	Center: $(2,3)$ radius: 3
Graph the circle.	 Figure 11.1.31

Exercise 22.2.19

a. Find the center and radius, then
b. Graph the circle: $x^2+y^2-6x-8y+9=0$.

Answer

a. The circle is centered at $(3,4)$ with a radius of 4.
b.

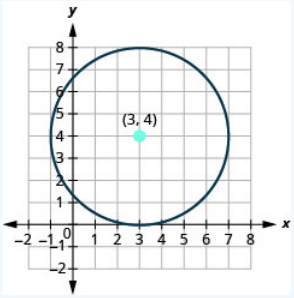


Figure 11.1.32

Exercise 22.2.20

a. Find the center and radius, then
b. Graph the circle: $x^2+y^2+6x-2y+1=0$

Answer

a. The circle is centered at $(-3,1)$ with a radius of 3.
b.

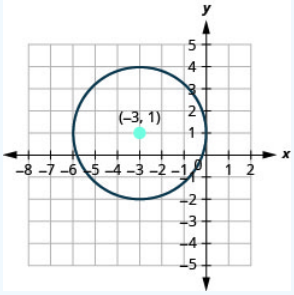


Figure 11.1.33

In the next example, there is a y-term and a y^2-term. But notice that there is no x-term, only an x^2-term. We have seen this before and know that it means h is 0. We will need to complete the square for the y terms, but not for the x terms.

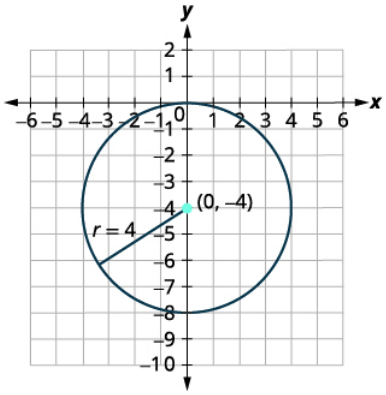
Example 22.2.11

a. Find the center and radius, then
b. Graph the circle: $x^2 + y^2 + 8y = 0$

Solution:

We need to rewrite this general form into standard form in order to find the center and radius.

Table 11.1.9

	$x^2 + y^2 + 8y = 0$
Group the x-terms and y-terms.	$x^2 + y^2 + 8y = 0$
There are no constants to collect on the right side.	
Complete the square for $y^2 + 8y$.	$x^2 + y^2 + 8y + 16 = 0 + 16$
Rewrite as binomial squares.	$(x - 0)^2 + (y + 4)^2 = 16$
Identify the center and radius.	Center: $(0, -4)$ radius: 4
Graph the circle.	

Exercise 22.2.21

a. Find the center and radius, then
b. Graph the circle: $x^2 + y^2 - 2x - 3 = 0$.

Answer

a. The circle is centered at $(-1, 0)$ with a radius of 2.
b.

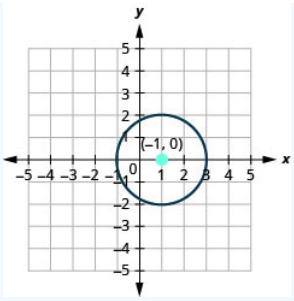


Figure 11.1.39

Exercise 22.2.22

a. Find the center and radius, then
b. Graph the circle: $x^2+y^2-12y+11=0$.

Answer

a. The circle is centered at $(0,6)$ with a radius of 5.
b.

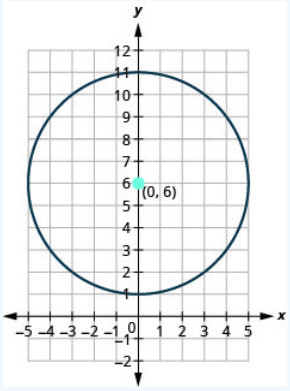


Figure 11.1.40

Access these online resources for additional instructions and practice with using the distance and midpoint formulas, and graphing circles.

- Distance-Midpoint Formulas and Circles
- Finding the Distance and Midpoint Between Two Points
- Completing the Square to Write Equation in Standard Form of a Circle

Key Concepts

- **Distance Formula:** The distance d between the two points (x_1, y_1) and (x_2, y_2) is

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

- **Midpoint Formula:** The midpoint of the line segment whose endpoints are the two points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

  To find the midpoint of a line segment, we find the average of the x-coordinates and the average of the y-coordinates of the endpoints.
- **Circle:** A circle is all points in a plane that are a fixed distance from a fixed point in the plane. The given point is called the *center*, (h, k), and the fixed distance is called the *radius*, r, of the circle.
- **Standard Form of the Equation a Circle:** The standard form of the equation of a circle with center, (h, k), and radius, r, is

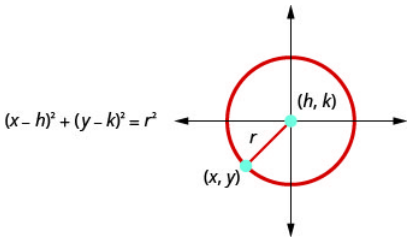


Figure 11.1.41

- **General Form of the Equation of a Circle:** The general form of the equation of a circle is

$$x^2+y^2+ax+by+c=0$$

Glossary

circle

A circle is all points in a plane that are a fixed distance from a fixed point in the plane.

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22.2E: Exercises

Practice Makes Perfect

Exercise $22.2E.23$ Use the Distance Formula

In the following exercises, find the distance between the points. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

1. $(2, 0)$ and $(5, 4)$
2. $(-4, -3)$ and $(2, 5)$
3. $(-4, -3)$ and $(8, 2)$
4. $(-7, -3)$ and $(8, 5)$
5. $(-1, 4)$ and $(2, 0)$
6. $(-1, 3)$ and $(5, -5)$
7. $(1, -4)$ and $(6, 8)$
8. $(-8, -2)$ and $(7, 6)$
9. $(-3, -5)$ and $(0, 1)$
10. $(-1, -2)$ and $(-3, 4)$
11. $(3, -1)$ and $(1, 7)$
12. $(-4, -5)$ and $(7, 4)$

Answer

1. $d = 5$

3. 13

5. 5

7. 13

9. $d = 3\sqrt{5}, d \approx 6.7$

11. $d = \sqrt{68}, d \approx 8.2$

Exercise $22.2E.24$ Use the Midpoint Formula

In the following exercises,

a. find the midpoint of the line segments whose endpoints are given and
b. plot the endpoints and the midpoint on a rectangular coordinate system.

1. $(0, -5)$ and $(4, -3)$
2. $(-2, -6)$ and $(6, -2)$
3. $(3, -1)$ and $(4, -2)$
4. $(-3, -3)$ and $(6, -1)$

Answer

1.

a. Midpoint: $(2, -4)$
b.

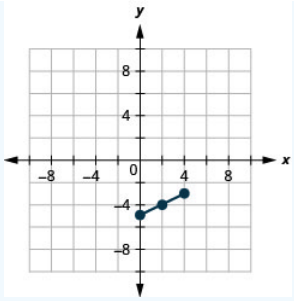

Figure 11.1.42

3.

a. Midpoint: $\left(3\frac{1}{2}, -1\frac{1}{2}\right)$

b.

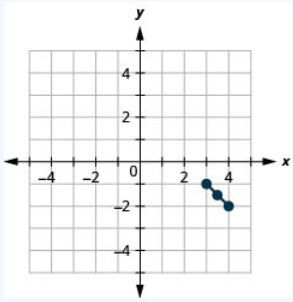

Figure 11.1.43

Exercise $22.2E.25$ Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given radius and center $(0, 0)$.

1. Radius: 7
2. Radius: 9
3. Radius: $\sqrt{2}$
4. Radius: $\sqrt{5}$

Answer

1. $x^2 + y^2 = 49$
3. $x^2 + y^2 = 2$

Exercise $22.2E.26$ Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given radius and center

1. Radius: 1, center: $(3, 5)$
2. Radius: 10, center: $(-2, 6)$
3. Radius: 2.5, center: $(1.5, -3.5)$
4. Radius: 1.5, center: $(-5.5, -6.5)$

Answer

1. $(x-3)^2 + (y-5)^2 = 1$
3. $(x-1.5)^2 + (y+3.5)^2 = 6.25$

Exercise $22.2E.27$ Write the Equation of a Circle in Standard Form

For the following exercises, write the standard form of the equation of the circle with the given center with point on the circle.

1. Center $(3,-2)$ with point $(3,6)$
2. Center $(6,-6)$ with point $(2,-3)$
3. Center $(4,4)$ with point $(2,2)$
4. Center $(-5,6)$ with point $(-2,3)$

Answer

1. $(x-3)^2+(y+2)^2=64$
3. $(x-4)^2+(y-4)^2=8$

Exercise $22.2E.28$ Graph a Circle

In the following exercises,

a. find the center and radius, then
b. graph each circle.

1. $(x+5)^2+(y+3)^2=1$
2. $(x-2)^2+(y-3)^2=9$
3. $(x-4)^2+(y+2)^2=16$
4. $(x+2)^2+(y-5)^2=4$
5. $x^2+(y+2)^2=25$
6. $(x-1)^2+y^2=36$
7. $(x-1.5)^2+(y+2.5)^2=0.25$
8. $(x-1)^2+(y-3)^2=\frac{9}{4}$
9. $x^2+y^2=64$
10. $x^2+y^2=49$
11. $2x^2+2y^2=8$
12. $6x^2+6y^2=216$

Answer

1.

a. The circle is centered at $(-5,-3)$ with a radius of 1.
b.

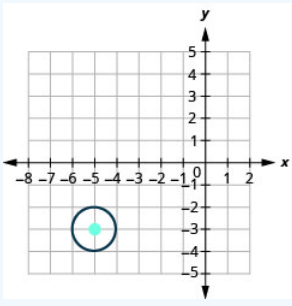

Figure 11.1.44

3.

a. The circle is centered at $(4,-2)$ with a radius of 4.
b.

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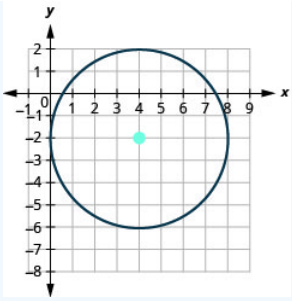

Figure 11.1.45

5.

a. The circle is centered at $(0, -2)$ with a radius of 5.
b.

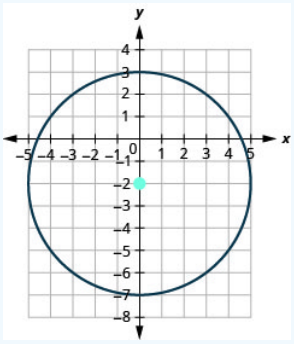

Figure 11.1.46

7.

a. The circle is centered at $(1.5, 2.5)$with a radius of 0.5.
b.

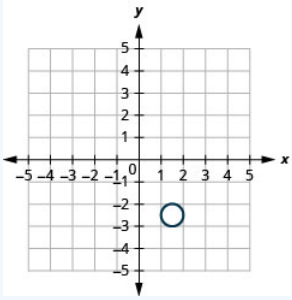

Figure 11.1.47

9.

a. The circle is centered at $(0, 0)$ with a radius of 8.
b.

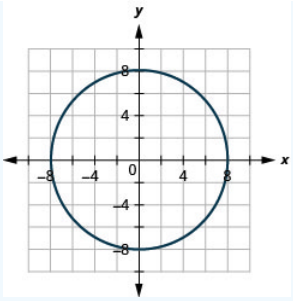

Figure 11.1.48

11.

a. The circle is centered at $(0, 0)$ with a radius of 2.
b.

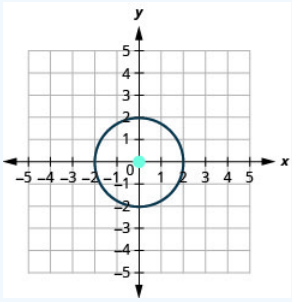

Figure 11.1.49

Exercise $22.2E.29$ Graph a Circle

In the following exercises,

a. identify the center and radius and
b. graph.

1. $x^2 + y^2 + 2x + 6y + 9 = 0$
2. $x^2 + y^2 - 6x - 8y = 0$
3. $x^2 + y^2 - 4x + 10y - 7 = 0$
4. $x^2 + y^2 + 12x - 14y + 21 = 0$
5. $x^2 + y^2 + 6y + 5 = 0$
6. $x^2 + y^2 - 10y = 0$
7. $x^2 + y^2 + 4x = 0$
8. $x^2 + y^2 - 14x + 13 = 0$

Answer

1.

a. Center: $(-1, -3)$, radius: 1
b.

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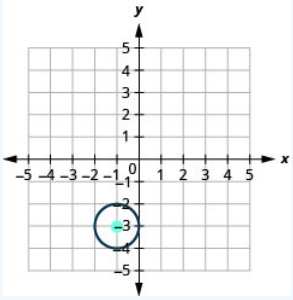

Figure 11.1.49

3.

a. Center: $(2, -5)$, radius: 6

b.

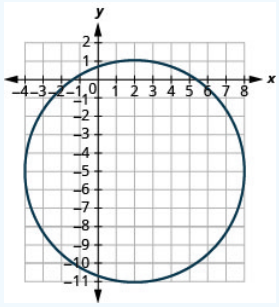

Figure 11.1.50

5.

a. Center: $(0, -3)$, radius: 2

b.

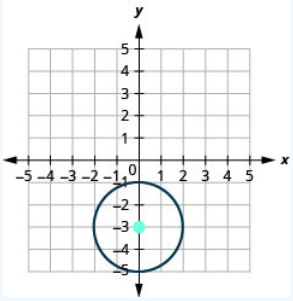

Figure 11.1.51

7.

a. Center: $(-2, 0)$, radius: -2

b.

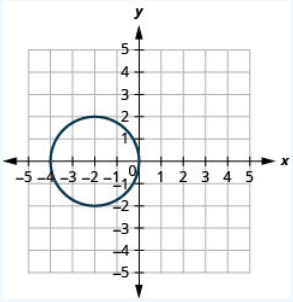

Figure 11.1.52

Exercise $22.2E.30$ Writing Exercises

1. Explain the relationship between the distance formula and the equation of a circle.
2. Is a circle a function? Explain why or why not.
3. In your own words, state the definition of a circle.
4. In your own words, explain the steps you would take to change the general form of the equation of a circle to the standard form.

Answer

1. Answers will vary.

3. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the distance formula.			
use the midpoint formula.			
write the equation of a circle in standard form.			
graph a circle.			

Figure 11.1.53

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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22.3: Parabolas

Learning Objectives

By the end of this section, you will be able to:

- Graph vertical parabolas
- Graph horizontal parabolas
- Solve applications with parabolas

Be Prepared

Before you get started, take this readiness quiz.

1. Graph: $y = -3x^2 + 12x - 12$.
   If you missed this problem, review Example 9.47.
2. Solve by completing the square: $x^2 - 6x + 6 = 0$.
   If you missed this problem, review Example 9.12.
3. Write in standard form: $y = 3x^2 - 6x + 5$.
   If you missed this problem, review Example 9.59.

Graph Vertical Parabolas

The next conic section we will look at is a **parabola**. We define a parabola as all points in a plane that are the same distance from a fixed point and a fixed line. The fixed point is called the **focus,**and the fixed line is called the **directrix** of the parabola.

parabola

Figure 11.2.1

Definition 22.3.1: Parabola, Focus, and Directrix

A **parabola** is all points in a plane that are the same distance from a fixed point and a fixed line. The fixed point is called the **focus,** and the fixed line is called the **directrix** of the parabola.

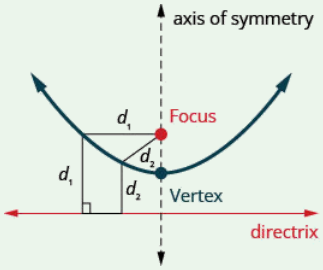


Figure 11.2.2

Previously, we learned to graph vertical parabolas from the general form or the standard form using properties. Those methods will also work here. We will summarize the properties here.

Vertical Parabolas

Table 11.2.1

	General form $y = ax^2 + bx + c$	Standard Form $y = a(x - h)^2 + k$

	General form $y = ax^2 + bx + c$	Standard Form $y = a(x - h)^2 + k$
Orientation	$a > 0$ up; $a < 0$ down	$a > 0$ up; $a < 0$ down
Axis of Symmetry	$x = -\dfrac{b}{2a}$	$x = h$
Vertex	Substitute $x = -\dfrac{b}{2a}$ and solve for y.	(h, k)
y-intercept	Let $x = 0$	Let $x = 0$
x-intercepts	Let $y = 0$	Let $y = 0$

The graphs show what the parabolas look like when they open up or down. Their position in relation to the x- or y-axis is merely an example.

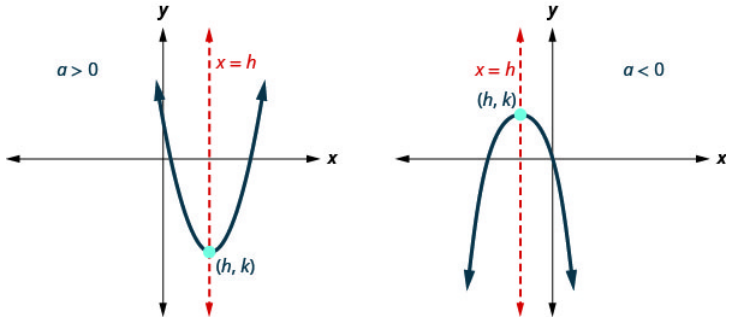


Figure 11.2.3

To graph a parabola from these forms, we used the following steps.

Graphing Vertical Parabolas

How to Graph Vertical Parabolas $y = ax^2 + bx + c$ or $f(x) = a(x - h)^2 + k$ using Properties.

- Step 1: Determine whether the parabola opens upward or downward.
- Step 2. Find the axis of symmetry.
- Step 3. Find the vertex.
- Step 4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
- Step 5. Find the x-intercepts.
- Step 6. Graph the parabola.

The next example reviews the method of graphing a parabola from the general form of its equation.

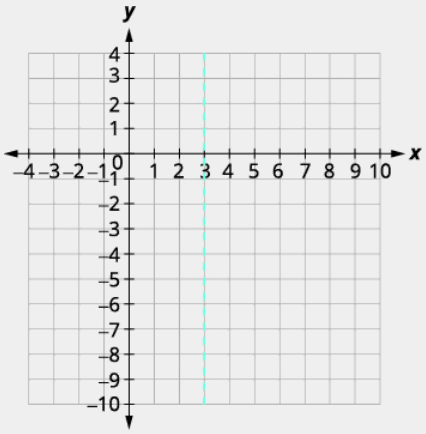
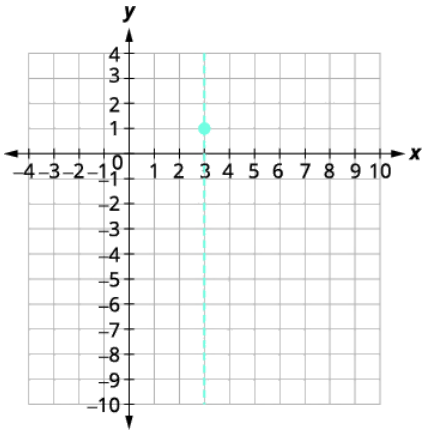
Example 22.3.1

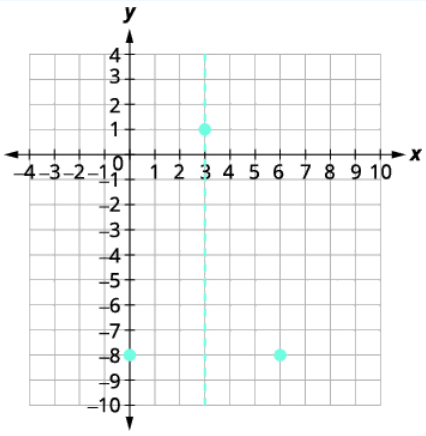
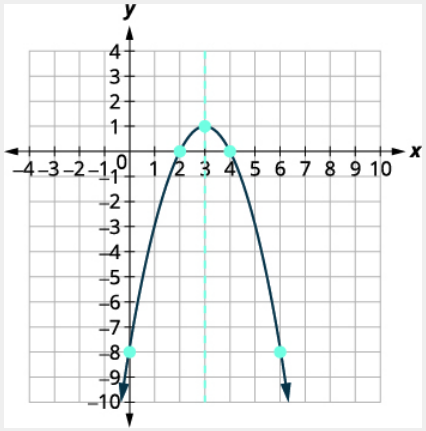
Graph $y = -x^2 + 6x - 8$ by using properties.

Solution:

Table 11.2.2

	$y=ax^2 + bx + c$ $y = -x^2 + 6x - 8$
Since a is -1, the parabola opens downward.	

To find the axis of symmetry, find $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$ $x = -\frac{6}{2(-1)}$ $x = 3$
	The axis of symmetry is $x = 3$.
	
The vertex is on the line $x = 3$.	$y = -x^2 + 6x - 8$
Let $x = 3$.	$y = -3^2 + 6 \cdot 3 - 8$
	$y = -9 + 18 - 8$ $y = 1$
	The vertex is $(3, 1)$.
	
The y-intercept occurs when $x = 0$.	$y = -x^2 + 6x - 8$
Substitute $x = 0$.	$y = -0^2 + 6 \cdot 0 - 8$
Simplify.	$y = -8$
	The y-intercept is $(0, -8)$.
The point $(0, -8)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -8)$.	Point symmetric to the y-intercept is $(6, -8)$.

	
The x-intercept occurs when $y = 0$.	$y = -x^2 + 6x - 8$
Let $y = 0$.	$0 = -x^2 + 6x - 8$
Factor the GCF.	$0 = -\left(x^2 - 6x + 8\right)$
Factor the trinomial.	$0 = -(x - 4)(x - 2)$
Solve for x.	$x = 4, \quad x = 2$
	The x-intercepts are $(4, 0), (2, 0)$.
Graph the parabola.	

? Exercise 22.3.1

Graph $y = -x^2 + 5x - 6$ by using properties.

Answer

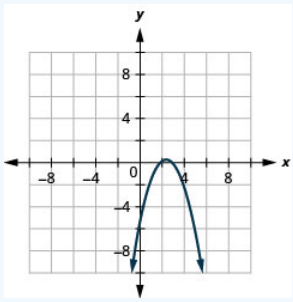


Figure 11.2.24

Exercise 22.3.2

Graph $y = -x^2 + 8x - 12$ by using properties.

Answer

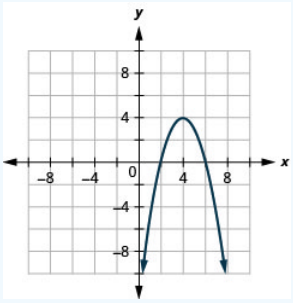


Figure 11.2.25

The next example reviews the method of graphing a **parabola** from the standard form of its equation, $y = a(x-h)^2 + k$.

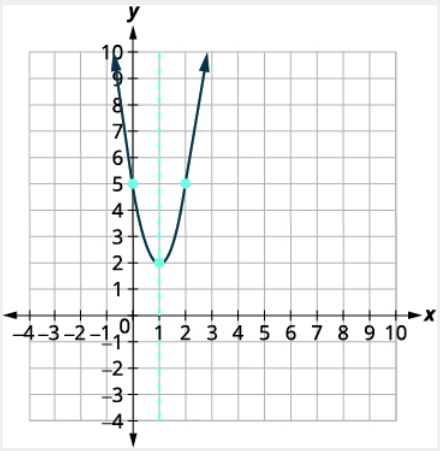
Example 22.3.2

Write $y = 3x^2 - 6x + 5$ in standard form and then use properties of standard form to graph the equation.

Solution:

Table 11.2.3

Rewrite the function in $y = a(x-h)^2 + k$ form by completing the square.	$y = 3x^2 - 6x + 5$ $y = 3\left(x^2 - 2x\right) + 5$ $y = 3\left(x^2 - 2x + 1\right) + 5 - 3$ $y = 3(x-1)^2 + 2$
Identify the constants a, h, k.	$a = 3, h = 1, k = 2$
Since $a = 2$, the parabola opens upward.	
The axis of symmetry is $x = h$.	The axis of symmetry is $x = 1$.
The vertex is (h, k).	The vertex is $(1, 2)$.
Find the y-intercept by substituting $x = 0$,	$y = 3(x-1)^2 + 2$ $y = 3 \cdot 0^2 - 6 \cdot 0 + 5$ $y = 0$
	y-intercept $(0, 5)$
Find the point symmetric to $(0, 5)$ across the axis of symmetry.	$(2, 5)$
Find the x-intercepts.	$y = 3(x-1)^2 + 2$ $0 = 3(x-1)^2 + 2$ $-2 = 3(x-1)^2$ $-\frac{2}{3} = (x-1)^2$ $\pm\sqrt{-\frac{2}{3}} = x - 1$

	The square root of a negative number tells us the solutions are complex numbers. So there are no x-intercepts.
Graph the parabola.	

Exercise 22.3.3

a. Write $y = 2x^2 + 4x + 5$ in standard form and
b. use properties of standard form to graph the equation.

Answer

a. $y = 2(x+1)^2 + 3$
b.

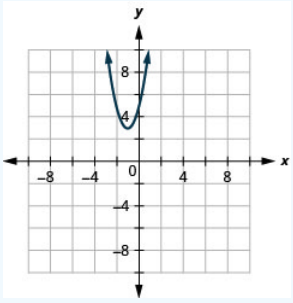


Figure 11.2.28

Exercise 22.3.4

a. Write $y = -2x^2 + 8x - 7$ in standard form and
b. use properties of standard form to graph the equation.

Answer

a. $y = -2(x-2)^2 + 1$
b.

https://math.libretexts.org/@go/page/46498

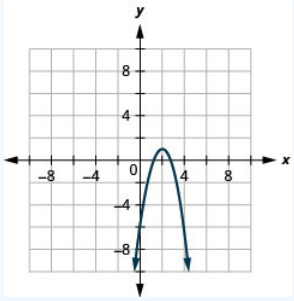


Figure 11.2.29

Graph Horizontal Parabolas

Our work so far has only dealt with parabolas that open up or down. We are now going to look at horizontal parabolas. These parabolas open either to the left or to the right. If we interchange the x and y in our previous equations for parabolas, we get the equations for the parabolas that open to the left or to the right.

Horizontal Parabolas

Table 11.2.4

	General form $x = ay^2 + by + c$	**Standard form** $x = a(y-k)^2 + h$
Orientation	$a > 0$ right; $a < 0$ left	$a > 0$ right; $a < 0$ left
Axis of Symmetry	$y = -\dfrac{b}{2a}$	$y = k$
Vertex	Substitute $y = -\dfrac{b}{2a}$ and solve for x.	(h, k)
x-intercepts	Let $x = 0$	Let $x = 0$
y-intercept	Let $y = 0$	Let $y = 0$

The graphs show what the parabolas look like when they to the left or to the right. Their position in relation to the x- or y-axis is merely an example.

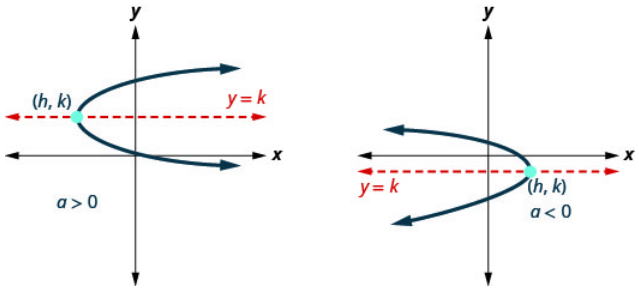


Figure 11.2.30

Looking at these parabolas, do their graphs represent a function? Since both graphs would fail the vertical line test, they do not represent a function.

To graph a **parabola** that opens to the left or to the right is basically the same as what we did for parabolas that open up or down, with the reversal of the x and y variables.

Howto: Graph Horizontal Parabolas $y = ax^2 + bx + c$ or $f(x) = a(x - h)^2 + k$ using Properties

- Step 1: Determine whether the parabola opens to the left or to the right.
- Step 2: Find the axis of symmetry.
- Step 3: Find the vertex.
- Step 4: Find the x-intercept. Find the point symmetric to the x-intercept across the axis of symmetry.
- Step 5: Find the y-intercepts.
- Step 6: Graph the parabola.

Example 22.3.3

Graph $x = 2y^2$ by using properties.

Solution:

Table 11.2.5

	$x = ay^2 + by + c$ $x = 2y^2$
Since $a = 2$, the parabola opens to the right.	
To find the axis of symmetry, find $y = -\frac{b}{2a}$	$y = -\frac{b}{2a}$
	$y = -\frac{0}{2(2)}$
	$y = 0$
	The axis of symmetry is $y = 0$.
The vertex is on the line $y = 0$.	$x = 2y^2$
Let $y = 0$.	$x = 2 \cdot 0^2$
	$x = 0$
	The vertex is $(0, 0)$.

Since the vertex is $(0, 0)$, both the x- and y-intercepts are the point $(0, 0)$. To graph the parabola we need more points. In this case it is easiest to choose values of y.

	$x = 2y^2$		$x = 2y^2$	x	y	
$y = 1$	$x = 2 \cdot 1^2$	$y = 2$	$x = 2 \cdot 2^2$	2	1	(2, 1)
	$x = 2$		$x = 8$	8	2	(8, 2)

Figure 11.2.38

We also plot the points symmetric to $(2, 1)$ and $(8, 2)$ across the y-axis, the points $(2, -1), (8, -2)$.

Graph the parabola.

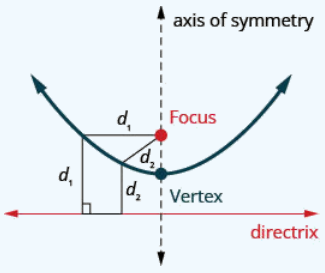


Figure 11.2.39

Exercise 22.3.5

Graph $x = y^2$ by using properties.

Answer

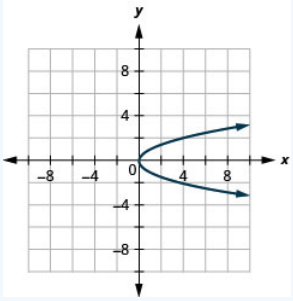


Figure 11.2.40

Exercise 22.3.6

Graph $x = -y^2$ by using properties.

Answer

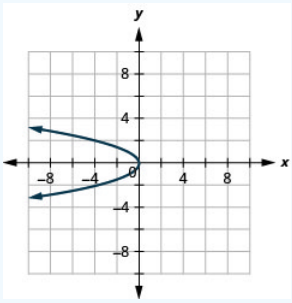


Figure 11.2.41

In the next example, the vertex is not the origin.

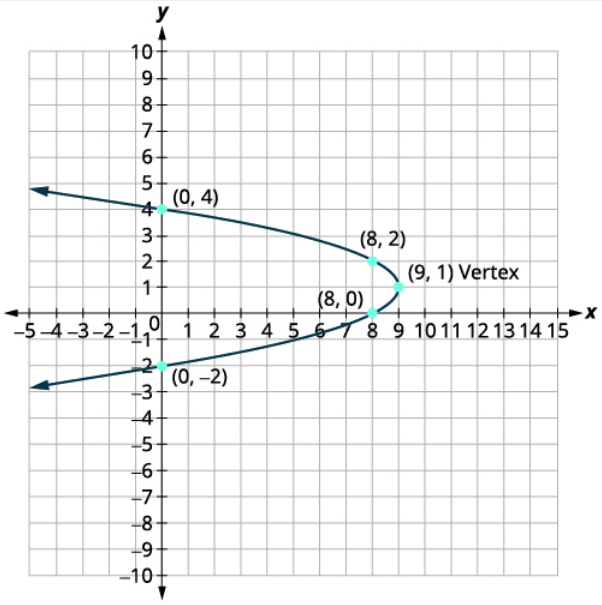
Example 22.3.4

Graph $x = -y^2 + 2y + 8$ by using properties.

Solution:

Table 11.2.6

	$x = ay^2 + by + c$ $x = -y^2 + 2y + 8$
Since $a = -1$, the parabola opens to the left.	
To find the axis of symmetry, find $y = -\frac{b}{2a}$	$y = -\frac{b}{2a}$
	$y = -\frac{2}{2(-1)}$
	$y = 1$

	The axis of symmetry is $y = 1$.
The vertex is on the line $y = 1$.	$x = -y^2 + 2y + 8$
Let $y = 1$.	$x = -1^2 + 2 \cdot 1 + 8$
	$x = 9$
	The vertex is $(9, 1)$.
The x-intercept occurs when $y = 0$.	$x = -y^2 + 2y + 8$
	$x = -0^2 + 2 \cdot 0 + 8$
	$x = 8$
	The x-intercept is $(8, 0)$.
The point $(8, 0)$ is one unit below the line of symmetry. The symmetric point one unit above the line of symmetry is $(8, 2)$	Symmetric point is $(8, 2)$.
The y-intercept occurs when $x = 0$.	$x = -y^2 + 2y + 8$
Substitute $x = 0$.	$0 = -y^2 + 2y + 8$
Solve.	$y^2 - 2y - 8 = 0$
	$(y - 4)(y + 2) = 0$
	$y = 4, \quad y = -2$
	The y -intercepts are $(0, 4)$ and $(0, -2)$.
Connect the points to graph the parabola.	

Exercise 22.3.7

Graph $x = -y^2 - 4y + 12$ by using properties.

Answer

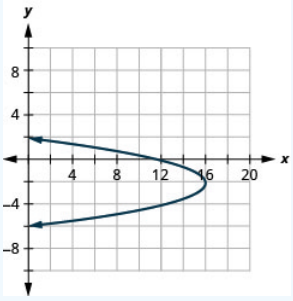


Figure 11.2.58

? Exercise 22.3.8

Graph $x = -y^2 + 2y - 3$ by using properties.

Answer

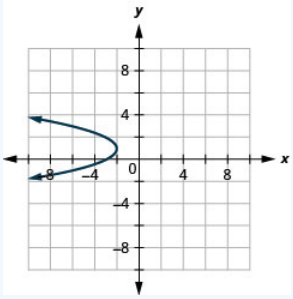


Figure 11.2.59

In Table 11.2.4, we see the relationship between the equation in standard form and the properties of the parabola. The How To box lists the steps for graphing a parabola in the standard form $x = a(y-k)^2 + h$. We will use this procedure in the next example.

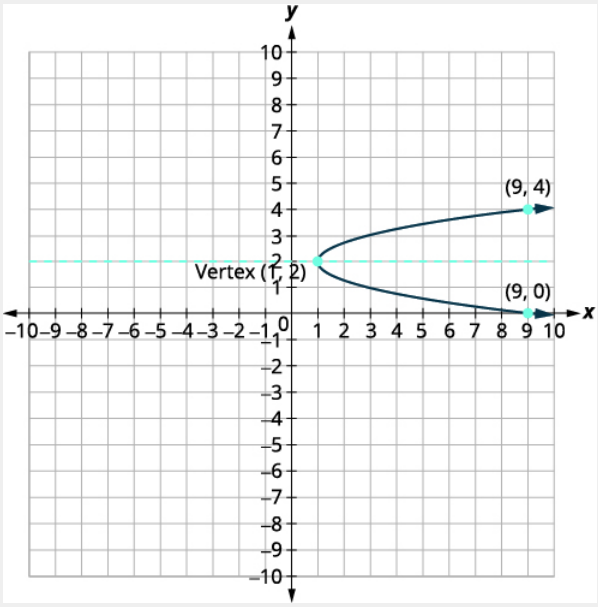
✓ Example 22.3.5

Graph $x = 2(y-2)^2 + 1$ using properties.

Solution:

Table 11.2.7

	$x = a(y-k)^2 + h$ $x = 2(y-2)^2 + 1$
Identify the constants a, h, k.	$a = 2, h = 1, k = 2$
Since $a = 2$, the parabola opens to the right.	
The axis of symmetry is $y = k$.	The axis of symmetry is $y = 2$.
The vertex is (h, k).	The vertex is $(1, 2)$.
Find the x-intercept by substituting $y = 0$.	$x = 2(y-2)^2 + 1$ $x = 2(0-2)^2 + 1$ $x = 9$
	The x-intercept is $(9, 0)$.
Find the point symmetric to $(9, 0)$ across the axis of symmetry.	$(9, 4)$

Find the y-intercepts. Let $x = 0$.	$x = 2(y-2)^2 + 1$ $0 = 2(y-2)^2 + 1$ $-1 = 2(y-2)^2$
	A square cannot be negative, so there is no real solution. So there are no y-intercepts.
Graph the parabola.	

Exercise 22.3.9

Graph $x = 3(y-1)^2 + 2$ using properties.

Answer

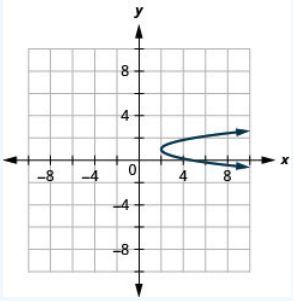


Figure 11.2.63

Exercise 22.3.10

Graph $x = 2(y-3)^2 + 2$ using properties.

Answer

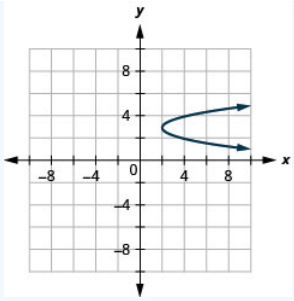


Figure 11.2.64

In the next example, we notice the a is negative and so the parabola opens to the left.

Example 22.3.6

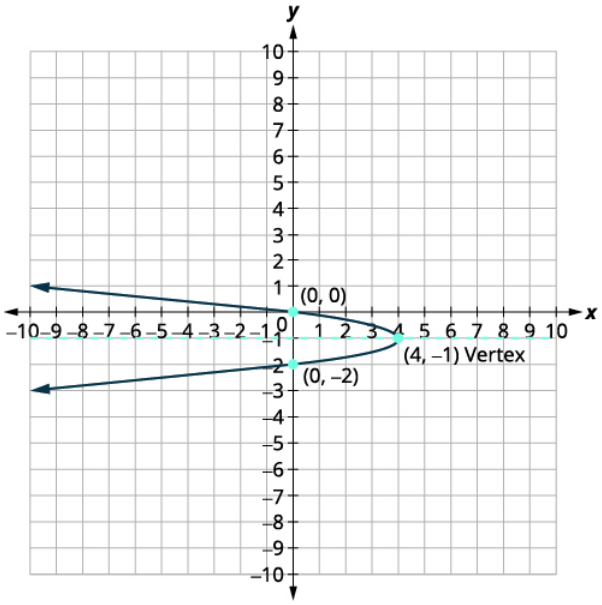
Graph $x = -4(y+1)^2 + 4$ using properties.

Solution:

Table 11.2.8

	$x = a(y-k)^2 + h$ $x = -4(y+1)^2 + 4$
Identify the constants a, h, k.	$a = -4, h = 4, k = -1$
Since $a = -4$, the parabola opens to the left.	
The axis of symmetry is $y = k$.	The axis of symmetry is $y = -1$.
The vertex is (h, k).	The vertex is $(4, -1)$.
Find the x-intercept by substituting $y = 0$.	$x = -4(y+1)^2 + 4$ $x = -4(0+1)^2 + 4$ $x = 0$
	The x-intercept is $(0, 0)$.
Find the point symmetric to $(0, 0)$ across the axis of symmetry.	$(0, -2)$
Find the y-intercepts.	$x = -4(y+1)^2 + 4$
Let $x = 0$.	$0 = -4(y+1)^2 + 4$ $-4 = -4(y+1)^2$ $1 = (y+1)^2$ $y+1 = \pm 1$
	$y = -1+1 \quad y = -1-1$
	$y = 0 \qquad y = -2$
	The y-intercepts are $(0, 0)$ and $(0, -2)$.

Graph the parabola.



Exercise 22.3.11

Graph $x = -4(y+2)^2 + 4$ using properties.

Answer

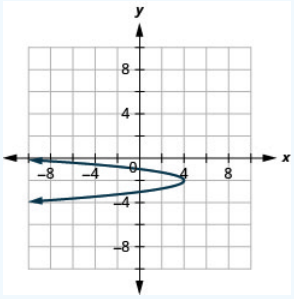


Figure 11.2.68

Exercise 22.3.12

Graph $x = -2(y+3)^2 + 2$ using properties.

Answer

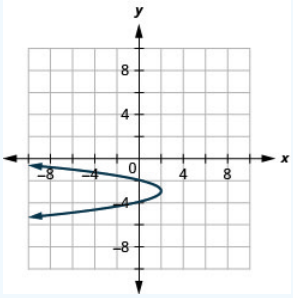


Figure 11.2.69

The next example requires that we first put the equation in standard form and then use the properties.

Example 22.3.7

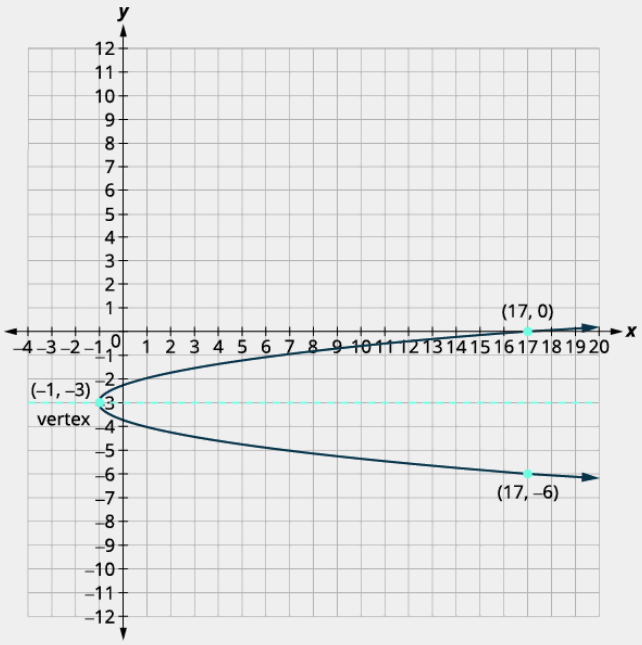
Write $x = 2y^2 + 12y + 17$ in standard form and then use the properties of the standard form to graph the equation.

Solution:

Table 11.2.9

	$x = 2y^2 + 12y + 17$
Rewrite the function in $x = a(y-k)^2 + h$ form by completing the square.	$x = 2\left(y^2 + 6y\right) + 17$
	$x = 2(y^2 + 6y + 9) + 17 - 18$
	$x = 2(y+3)^2 - 1$
	$x = a(y-k)^2 + h$ $x = 2(y+3)^2 - 1$
Identify the constants a, h, k.	$a = 2, h = -1, k = -3$
Since $a = 2$, the parabola opens to the right.	
The axis of symmetry is $y = k$.	The axis of symmetry is $y = -3$.
The vertex is (h, k).	The vertex is $(-1, -3)$.
Find the x-intercept by substituting $y = 0$.	$x = 2(y+3)^2 - 1$ $x = 2(0+3)^2 - 1$ $x = 17$
	The x-intercept is $(17, 0)$.
Find the point symmetric to $(17, 0)$ across the axis of symmetry.	$(17, -6)$
Find the y-intercepts. Let $x = 0$.	$x = 2(y+3)^2 - 1$ $0 = 2(y+3)^2 - 1$ $1 = 2(y+3)^2$ $\frac{1}{2} = (y+3)^2$ $y + 3 = \pm\sqrt{\frac{1}{2}}$ $y = -3 \pm \frac{\sqrt{2}}{2}$
	$y = -3 + \frac{\sqrt{2}}{2} \quad y = -3 - \frac{\sqrt{2}}{2}$
	$y \approx -2.3 \quad y \approx -3.7$
	The y-intercepts are $\left(0, -3 + \frac{\sqrt{2}}{2}\right), \left(0, -3 - \frac{\sqrt{2}}{2}\right)$.

Graph the parabola.



Exercise 22.3.13

a. Write $x = 3y^2 + 6y + 7$ in standard form and
b. Use properties of the standard form to graph the equation.

Answer

a. $x = 3(y+1)^2 + 4$
b.

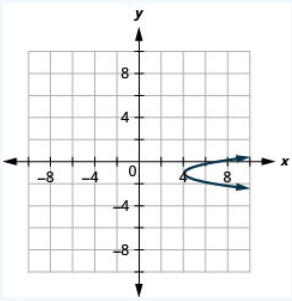


Figure 11.2.77

Exercise 22.3.14

a. Write $x = -4y^2 - 16y - 12$ in standard form and
b. Use properties of the standard form to graph the equation.

Answer

a. $x = -4(y+2)^2 + 4$
b.

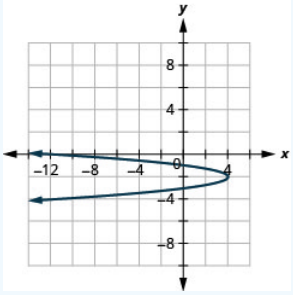


Figure 11.2.78

Solve Applications with Parabolas

Many architectural designs incorporate parabolas. It is not uncommon for bridges to be constructed using parabolas as we will see in the next example.

Example 22.3.8

Find the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.

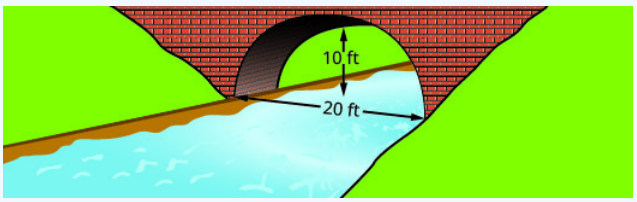
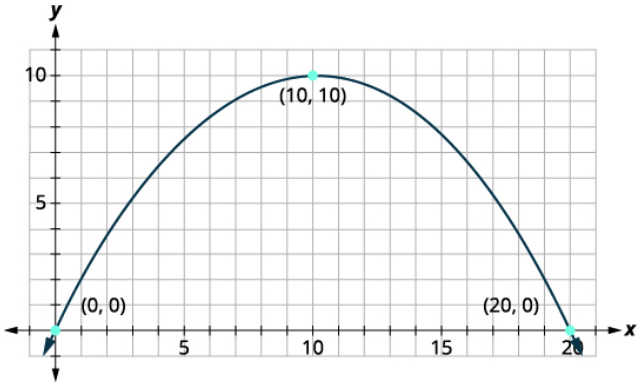


Figure 11.2.79

Solution:

We will first set up a coordinate system and draw the parabola. The graph will give us the information we need to write the equation of the graph in the standard form $y = a(x-h)^2 + k$.

Table 11.2.10

Let the lower left side of the bridge be the origin of the coordinate grid at the point $(0, 0)$. Since the base is 20 feet wide the point $(20, 0)$ represents the lower right side. The bridge is 10 feet high at the highest point. The highest point is the vertex of the parabola so the y-coordinate of the vertex will be 10. Since the bridge is symmetric, the vertex must fall halfway between the left most point, $(0, 0)$, and the rightmost point $(20, 0)$. From this we know that the x-coordinate of the vertex will also be 10.	
Identify the vertex, (h, k).	$(h, k) = (10, 10)$
	$h = 10, \quad k = 10$
Substitute the values into the standard form. The value of a is still unknown. To find the value of a use one of the other points on the parabola.	$y = a(x-h)^2 + k$ $y = a(x-10)^2 + 10$ $(x, y) = (0, 0)$
Substitute the values of the other point into the equation.	$y = a(x-10)^2 + 10$ $0 = a(0-10)^2 + 10$

Solve for a.	$0 = a(0-10)^2 + 10$ $-10 = a(-10)^2$ $-10 = 100a$ $\frac{-10}{100} = a$ $a = -\frac{1}{10}$
	$y = a(x-10)^2 + 10$
Substitute the value for a into the equation.	$y = -\frac{1}{10}(x-10)^2 + 10$

? Exercise 22.3.15

Find the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.

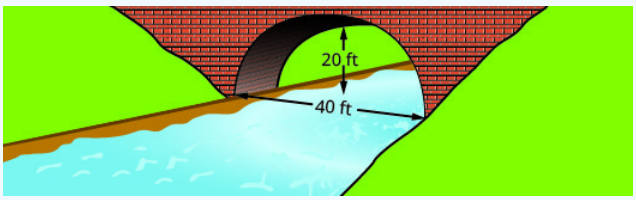


Figure 11.2.81

Answer

$$y = -\frac{1}{20}(x-20)^2 + 20$$

? Exercise 22.3.16

Find the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.

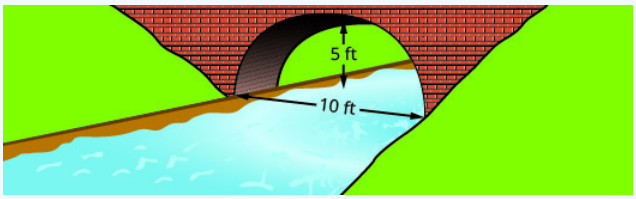


Figure 11.2.82

Answer

$$y = -\frac{1}{5}x^2 + 2xy = -\frac{1}{5}(x-5)^2 + 5$$

Access these online resources for additional instructions and practice with quadratic functions and parabolas.

- Quadratic Functions
- Introduction to Conics and Graphing Horizontal Parabolas

Key Concepts

- **Parabola:** A **parabola** is all points in a plane that are the same distance from a fixed point and a fixed line. The fixed point is called the **focus,** and the fixed line is called the **directrix** of the parabola.

Vertical Parabolas

Table 11.2.1

	General form $y = ax^2 + bx + c$	**Standard Form** $y = a(x-h)^2 + k$
Orientation	$a > 0$ up; $a < 0$ down	$a > 0$ up; $a < 0$ down
Axis of Symmetry	$x = -\dfrac{b}{2a}$	$x = h$
Vertex	Substitute $x = -\dfrac{b}{2a}$ and solve for y.	(h, k)
y-intercept	Let $x = 0$	Let $x = 0$
x-intercepts	Let $y = 0$	Let $y = 0$

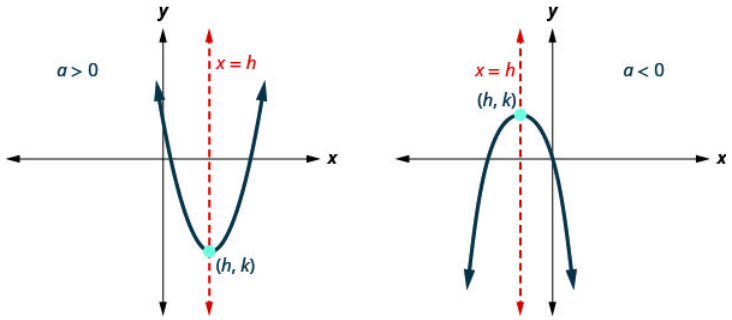


Figure 11.2.3

- **How to graph vertical parabolas $y = ax^2 + bx + c$ or $f(x) = a(x-h)^2 + k$ using properties.**

1. Determine whether the parabola opens upward or downward.
2. Find the axis of symmetry.
3. Find the vertex.
4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
5. Find the x-intercepts.
6. Graph the parabola.

Horizontal Parabolas

Table 11.2.4

	General form $x = ay^2 + by + c$	**Standard form** $x = a(y-k)^2 + h$
Orientation	$a > 0$ right; $a < 0$ left	$a > 0$ right; $a < 0$ left
Axis of Symmetry	$y = -\dfrac{b}{2a}$	$y = k$
Vertex	Substitute $y = -\dfrac{b}{2a}$ and solve for x.	(h, k)
x-intercepts	Let $x = 0$	Let $x = 0$
y-intercept	Let $y = 0$	Let $y = 0$

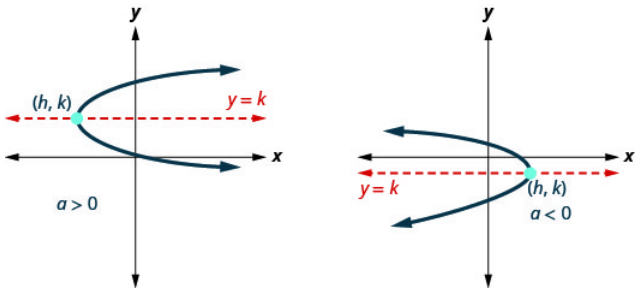


Figure 11.2.30

Graphing Horizontal Parabolas

How to graph horizontal parabolas $x = ay^2 + by + c$ or $x = a(y - k)^2 + h$ using properties.

1. Determine whether the parabola opens to the left or to the right.
2. Find the axis of symmetry.
3. Find the vertex.
4. Find the x-intercept. Find the point symmetric to the x-intercept across the axis of symmetry.
5. Find the y-intercepts.
6. Graph the parabola.

Glossary

parabola

A parabola is all points in a plane that are the same distance from a fixed point and a fixed line.

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22.3E: Exercises

Practice Makes Perfect

Exercise 22.3E. 17 Graph Vertical Parabolas

In the following exercises, graph each equation by using properties.

1. $y = -x^2 + 4x - 3$
2. $y = -x^2 + 8x - 15$
3. $y = 6x^2 + 2x - 1$
4. $y = 8x^2 - 10x + 3$

Answer

1.

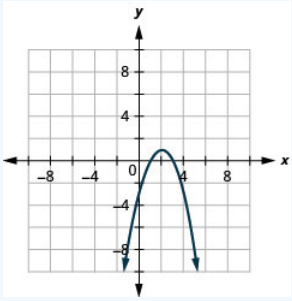

Figure 11.2.83

3.

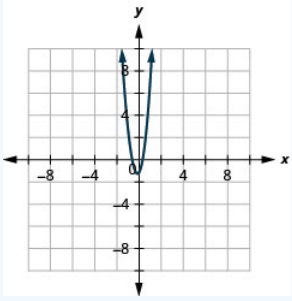

Figure 11.2.84

Exercise 22.3E. 18 Graph Vertical Parabolas

In the following exercises,

a. Write the equation in standard form and
b. Use properties of the standard form to graph the equation.

1. $y = -x^2 + 2x - 4$
2. $y = 2x^2 + 4x + 6$
3. $y = -2x^2 - 4x - 5$
4. $y = 3x^2 - 12x + 7$

Answer

1.

a. $y = -(x-1)^2 - 3$
b.

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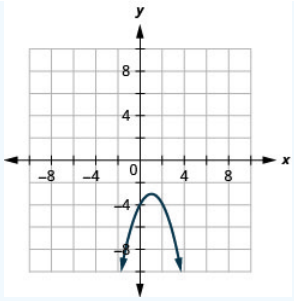

Figure 11.2.85

3.

a. $y = -2(x+1)^2 - 3$

b.

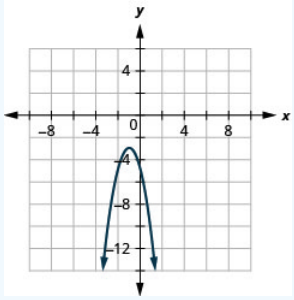

Figure 11.2.86

Exercise $22.3E.19$ Graph Horizontal Parabolas

In the following exercises, graph each equation by using properties.

1. $x = -2y^2$
2. $x = 3y^2$
3. $x = 4y^2$
4. $x = -4y^2$
5. $x = -y^2 - 2y + 3$
6. $x = -y^2 - 4y + 5$
7. $x = y^2 + 6y + 8$
8. $x = y^2 - 4y - 12$
9. $x = (y-2)^2 + 3$
10. $x = (y-1)^2 + 4$
11. $x = -(y-1)^2 + 2$
12. $x = -(y-4)^2 + 3$
13. $x = (y+2)^2 + 1$
14. $x = (y+1)^2 + 2$
15. $x = -(y+3)^2 + 2$
16. $x = -(y+4)^2 + 3$
17. $x = -3(y-2)^2 + 3$
18. $x = -2(y-1)^2 + 2$
19. $x = 4(y+1)^2 - 4$
20. $x = 2(y+4)^2 - 2$

Answer

1.

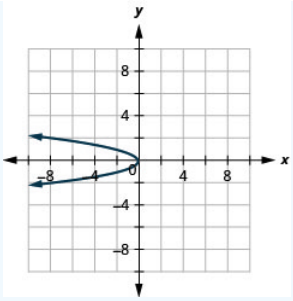

Figure 11.2.87

3.

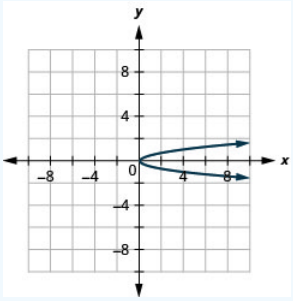

Figure 11.2.88

5.

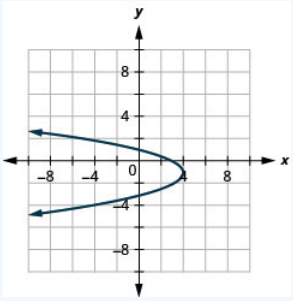

Figure 11.2.89

7.

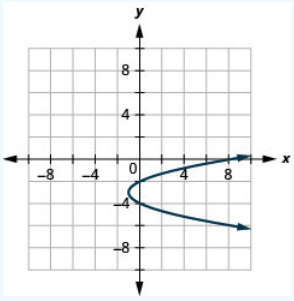

Figure 11.2.90

9.

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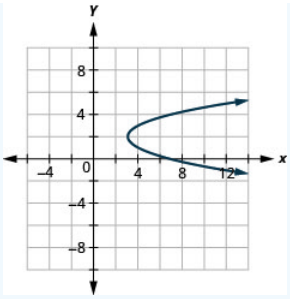

Figure 11.2.91

11.

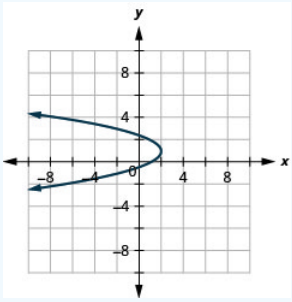

Figure 11.2.92

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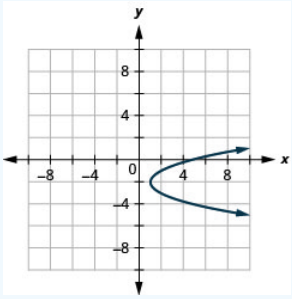

Figure 11.2.93

15.

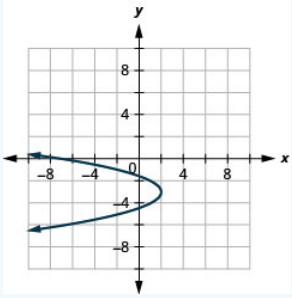

Figure 11.2.94

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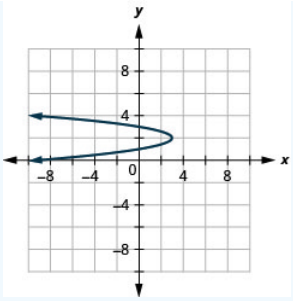

Figure 11.2.95

19.

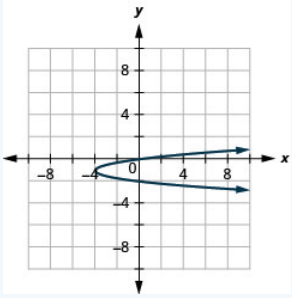

Figure 11.2.96

Exercise $22.3E.20$ Graph Horizontal Parabolas

In the following exercises,

a. Write the equation in standard form and
b. Use properties of the standard form to graph the equation.

1. $x = y^2 + 4y - 5$
2. $x = y^2 + 2y - 3$
3. $x = -2y^2 - 12y - 16$
4. $x = -3y^2 - 6y - 5$

Answer

1.

a. $x = (y+2)^2 - 9$
b.

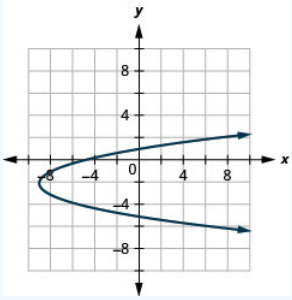

Figure 11.2.97

3.

a. $x = -2(y+3)^2 + 2$
b.

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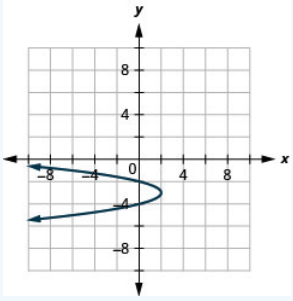

Figure 11.2.98

Exercise $22.3E.21$ Mixed Practice

In the following exercises, match each graph to one of the following equations:

a. $x^2+y^2=64$
b. $x^2+y^2=49$
c. $(x+5)^2+(y+2)^2=4$
d. $(x-2)^2+(y-3)^2=9$
e. $y=-x^2+8x-15$
f. $y=6x^2+2x-1$

1.

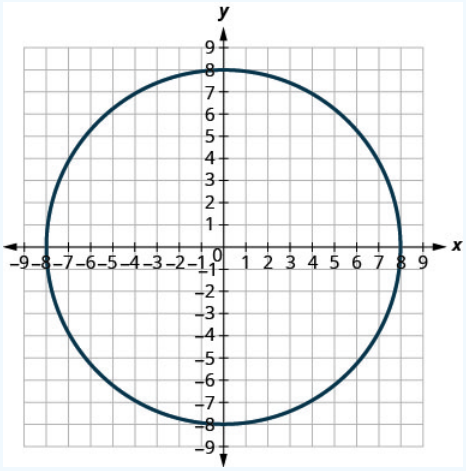

Figure 11.2.99

2.

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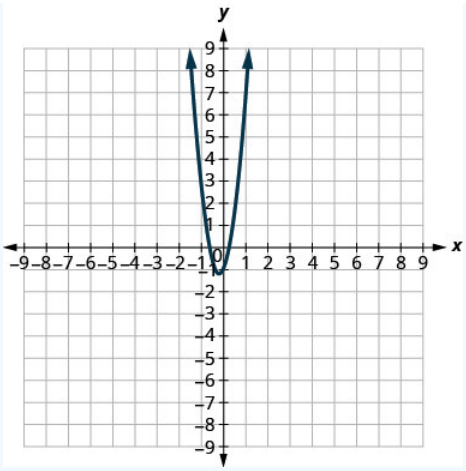

Figure 11.2.100

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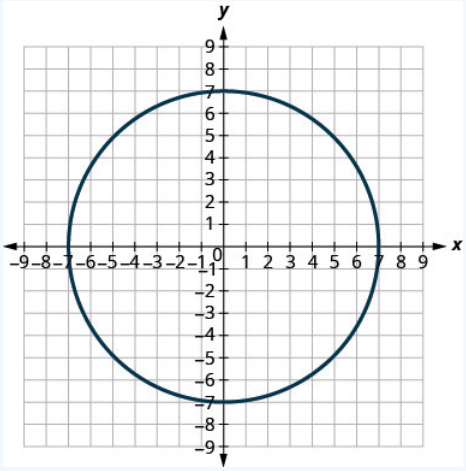

Figure 11.2.101

4.

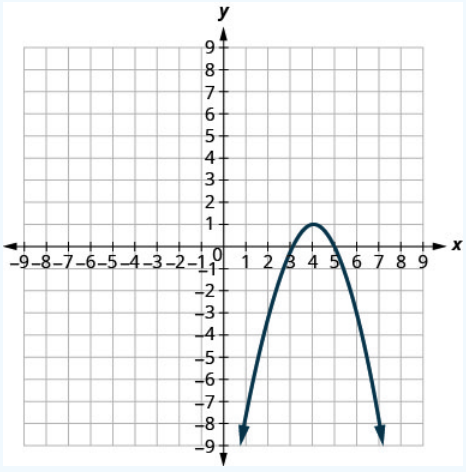

Figure 11.2.102

5.

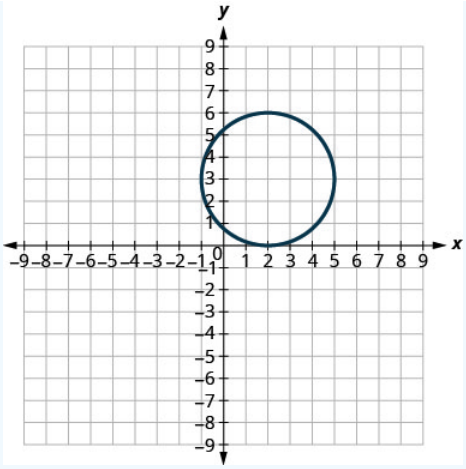

Figure 11.2.103

6.

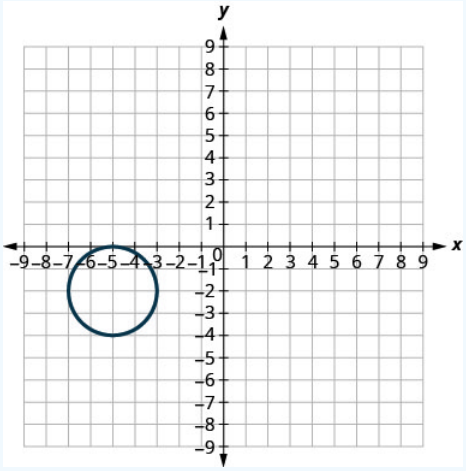

Figure 11.2.104

Answer

1. a
3. b
5. d

Exercise $22.3E.22$ Solve Applications with Parabolas

Write the equation in standard form of the parabolic arch formed in the foundation of the bridge shown. Use the lower left side of the bridge as the origin $(0, 0)$.

1.

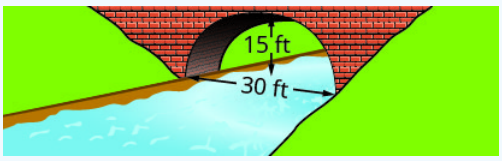

Figure 11.2.105

2.

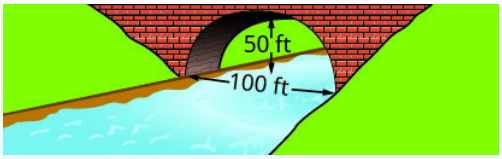

Figure 11.2.106

3.

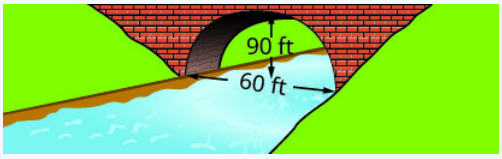

Figure 11.2.107

4.

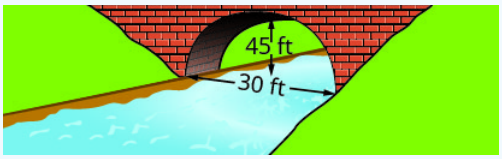

Figure 11.2.108

Answer

1. $y = -\frac{1}{15}(x-15)^2 + 15$
3. $y = -\frac{1}{10}(x-30)^2 + 90$

Exercise $22.3E.23$ Writing Exercises

1. In your own words, define a parabola.
2. Is the parabola $y = x^2$ a function? Is the parabola $x = y^2$ a function? Explain why or why not.
3. Write the equation of a parabola that opens up or down in standard form and the equation of a parabola that opens left or right in standard form. Provide a sketch of the parabola for each one, label the vertex and axis of symmetry.
4. Explain in your own words, how you can tell from its equation whether a parabola opens up, down, left or right.

Answer

1. Answers may vary
3. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph vertical parabolas.			
graph horizontal parabolas.			
solve applications with parabolas.			

Figure 11.2.109

b. After reviewing this checklist, what will you do to become confident for all objectives?

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22.4: Ellipses

Learning Objectives

By the end of this section, you will be able to:

- Graph an ellipse with center at the origin
- Find the equation of an ellipse with center at the origin
- Graph an ellipse with center not at the origin
- Solve application with ellipses

Before you get started, take this readiness quiz.

1. Graph $y=(x-1)^2-2$ using transformations.
   If you missed this problem, review Example 9.57.
2. Complete the square: $x^2-8x=8$.
   If you missed this problem, review Example 9.12.
3. Write in standard form. $y=2x^2-12x+14$
   If you missed this problem, review Example 9.59.

Graph an Ellipse with Center at the Origin

The next conic section we will look at is an **ellipse**. We define an ellipse as all points in a plane where the sum of the distances from two fixed points is constant. Each of the given points is called a **focus** of the ellipse.

Definition 22.4.1

An **ellipse** is all points in a plane where the sum of the distances from two fixed points is constant. Each of the fixed points is called a **focus** of the ellipse.

Figure 11.3.1

We can draw an ellipse by taking some fixed length of flexible string and attaching the ends to two thumbtacks. We use a pen to pull the string taut and rotate it around the two thumbtacks. The figure that results is an ellipse.

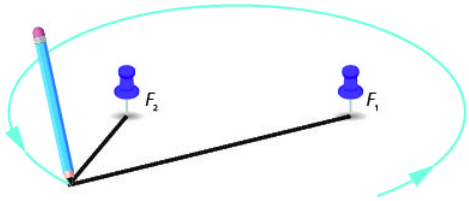


Figure 11.3.2

A line drawn through the foci intersect the ellipse in two points. Each point is called a **vertex** of the ellipse. The segment connecting the vertices is called the **major axis**. The midpoint of the segment is called the **center** of the ellipse. A segment perpendicular to the major axis that passes through the center and intersects the ellipse in two points is called the **minor axis**.

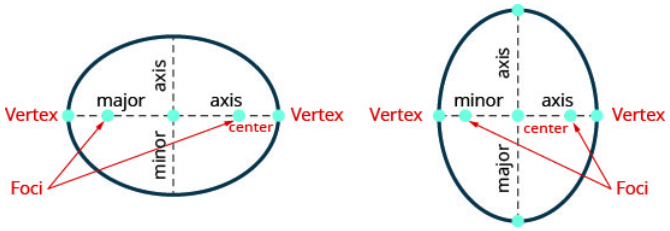


Figure 11.3.3

We mentioned earlier that our goal is to connect the geometry of a conic with algebra. Placing the ellipse on a rectangular coordinate system gives us that opportunity. In the figure, we placed the ellipse so the foci $((-c,0),(c,0))$ are on the x-axis and the center is the origin.

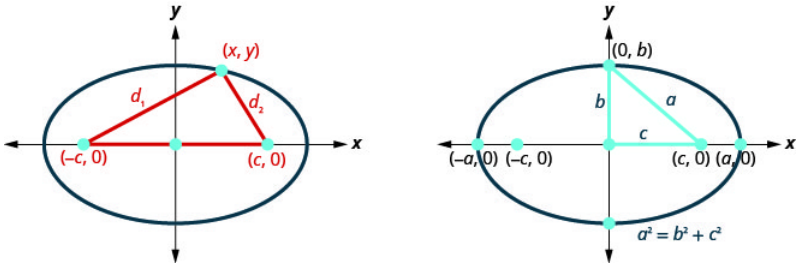


Figure 11.3.4

The definition states the sum of the distance from the foci to a point (x,y) is constant. So d_1+d_2 is a constant that we will call $2a$ so, $d_1+d_2=2a$. We will use the distance formula to lead us to an algebraic formula for an ellipse.

$$d_1 \quad + \quad d_2 \quad = \quad 2a$$

Use the distance formula to find d_1, d_2.

$$\sqrt{(x-(-c))^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$

After eliminating radicals and simplifying,we get:

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

To simplify the equation of the ellipse, we let $a^2-c^2=b^2$ .So, the equation of an ellipse centered at the origin in standard form is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

To graph the ellipse, it will be helpful to know the intercepts. We will find the x-intercepts and y-intercepts using the formula.

y-intercepts

Let $x=0$.

$$\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2}&=1\\ \frac{0^2}{a^2}+\frac{y^2}{a^2}&=1\\ \frac{y^2}{b^2}&=1\\ y^2&=b^2\\ y&=\pm b\end{aligned}$$

The y-intercepts are $(0,b)$ and $(0,-b)$.

x-intercepts

Let $y=0$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
$$\frac{x^2}{a^2}+\frac{0^2}{b^2}=1$$
$$\frac{x^2}{a^2}=1$$
$$x^2=a^2$$
$$x=\pm a$$

The x-intercepts are $(a,0)$ and $(-a,0)$.

Definition 22.4.2

Standard Form of the Equation an Ellipse with Center $(0,0)$

The standard form of the equation of an ellipse with center $(0,\ 0)$, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The x-intercepts are $(a,0)$ and $(-a,0)$.

The y-intercepts are $(0,b)$ and $(0,-b)$.

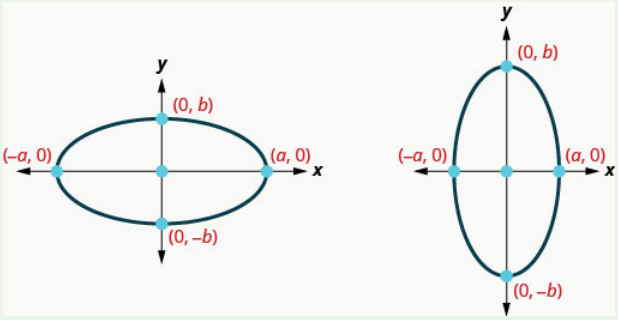


Figure 11.3.5

Notice that when the major axis is horizontal, the value of a will be greater than the value of b and when the major axis is vertical, the value of b will be greater than the value of a. We will use this information to graph an **ellipse** that is centered at the origin.

Ellipse with Center $(0,0)$

Table 11.3.1

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$a>b$	$b>a$
Major axis	on the x-axis.	on the y-axis
x-intercepts	$(-a,0),(a,0)$	
y-intercepts	$(0,-b),(0,b)$	

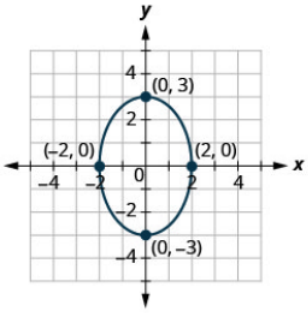
Example 22.4.1

Graph: $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Solution:

Table 11.3.2

Step 1. Write the equation in standard form.	It is in standard form.	$\frac{x^2}{4}+\frac{y^2}{9}=1$

Step 2. Determine whether the major axis is horizontal or vertical.	Since $9 > 4$ and 9 is in the y^2 term, the major axis is vertical.	Major axis is vertical.
Step 3. Find the endpoints of the major axis.	The endpoints will be the y-intercepts. Since $b^2 = 9$, then $b = \pm 3$. The endpoints of the major axis are $(0, 3), (0, -3)$.	The endpoints of the major axis are $(0, 3), (0, -3)$.
Step 4. Find the endpoints of the minor axis.	The endpoints will be the x-intercepts. Since $a^2 = 4$, then $a = \pm 2$. The endpoints of the major axis are $(2, 0), (-2, 0)$.	The endpoints of the major axis are $(2, 0), (-2, 0)$.
Step 5. Sketch the ellipse.		

? Exercise 22.4.1

Graph: $\frac{x^2}{4} + \frac{y^2}{16} = 1$.

Answer

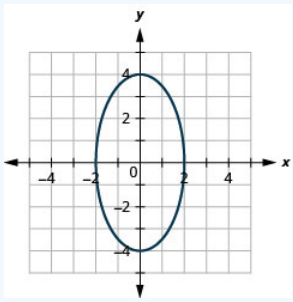


Figure 11.3.7

? Exercise 22.4.2

Graph: $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

Answer

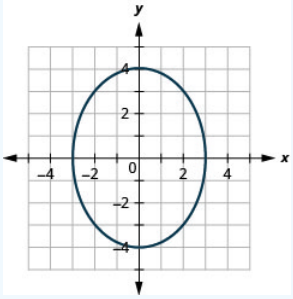


Figure 11.3.8

We summarize the steps for reference.

HOW TO GRAPH AN ELLIPSE WITH CENTER $(0,0)$.

1. Write the equation in standard form.
2. Determine whether the major axis is horizontal or vertical.
3. Find the endpoints of the major axis.
4. Find the endpoints of the minor axis
5. Sketch the ellipse.

Sometimes our equation will first need to be put in standard form.

Example 22.4.2

Graph $x^2+4y^2=16$.

Solution:

Table 11.3.3

We recognize this as the equation of an ellipse since both the x and y terms are squared and have different coefficients.	$x^2+4y^2=16$
To get the equation in standard form, divide both sides by 16 so that the equation is equal to 1.	$\frac{x^2}{16}+\frac{4y^2}{16}=\frac{16}{16}$
Simplify.	$\frac{x^2}{16}+\frac{y^2}{4}=1$
The equation is in standard form. The ellipse is centered at the origin.	The center is $(0,0)$.
Since $16>4$ and 16 is in the x^2 term, the major axis is horizontal.	
$a^2=16, a=\pm 4$ $b^2=4, \quad b=\pm 2$	The vertices are $(4,0), (-4,0)$. The endpoints of the minor axis are $(0,2), (0,-2)$.

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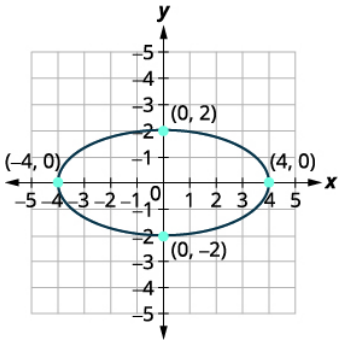
Sketch the parabola.



Exercise 22.4.3

Graph $9x^2 + 16y^2 = 144$.

Answer

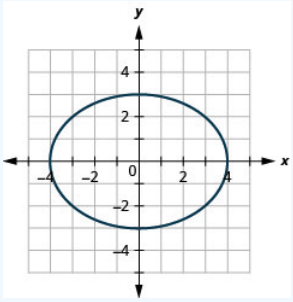


Figure 11.3.10

Exercise 22.4.4

Graph $16x^2 + 25y^2 = 400$.

Answer

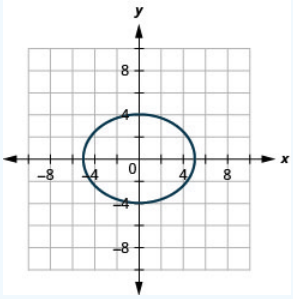


Figure 11.3.11

Find the Equation of an Ellipse with Center at the Origin

If we are given the graph of an **ellipse**, we can find the equation of the ellipse.

Example 22.4.3

Find the equation of the ellipse shown.

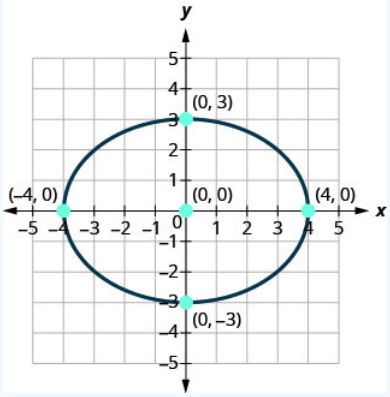


Figure 11.3.12

Solution:

We recognize this as an ellipse that is centered at the origin.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Since the major axis is horizontal and the distance from the center to the vertex is 4, we know $a=4$ and so $a^2=16$.

$$\frac{x^2}{16}+\frac{y^2}{b^2}=1$$

The minor axis is vertical and the distance from the center to the ellipse is 3, we know $b=3$ and so $b^2=9$.

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

Exercise 22.4.5

Find the equation of the ellipse shown.

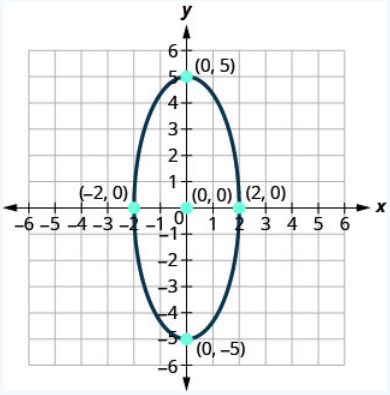


Figure 11.3.13

Answer

$\frac{x^2}{4}+\frac{y^2}{25}=1$

Exercise 22.4.6

Find the equation of the ellipse shown.

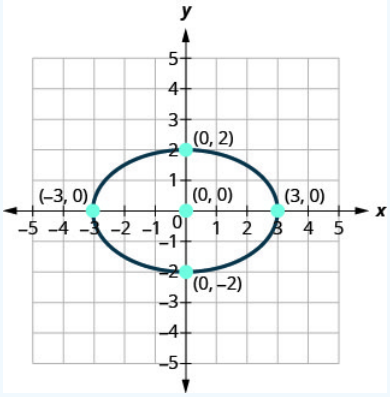


Figure 11.3.14

Answer

$\frac{x^2}{9}+\frac{y^2}{4}=1$

Graph an Ellipse with Center Not at the Origin

The ellipses we have looked at so far have all been centered at the origin. We will now look at ellipses whose center is (h,k).

The equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ and when $a>b$, the major axis is horizontal so the distance from the center to the vertex is a. When $b>a$, the major axis is vertical so the distance from the center to the vertex is b.

Definition 22.4.3

Standard Form of the Equation an Ellipse with Center (h,k)

The standard form of the equation of an ellipse with center (h,k), is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

When $a>b$, the major axis is horizontal so the distance from the center to the vertex is a.

When $b>a$, the major axis is vertical so the distance from the center to the vertex is b.

Example 22.4.4

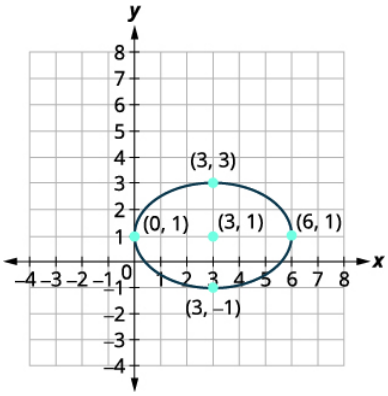
Graph: $\frac{(x-3)^2}{9}+\frac{(y-1)^2}{4}=1$.

Solution:

Table 11.3.4

The equation is in standard form, $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.	$\frac{(x-3)^2}{9}+\frac{(y-1)^2}{4}=1$
The ellipse is centered at (h,k).	The center is $(3,1)$.
Since $9>4$ and 9 is in the x^2 term, the major axis is horizontal.	
$a^2=9, a=\pm 3$ $b^2=4, b=\pm 2$	The distance from the center to the vertices is 3. The distance from the center to the endpoints of the minor axis is 2.

Sketch the ellipse.



Exercise 22.4.7

Graph: $\frac{(x+3)^2}{4}+\frac{(y-5)^2}{16}=1$.

Answer

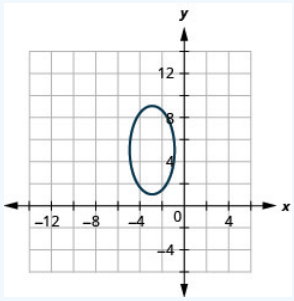


Figure 11.3.16

Exercise 22.4.8

Graph: $\frac{(x-1)^2}{25}+\frac{(y+3)^2}{16}=1$.

Answer

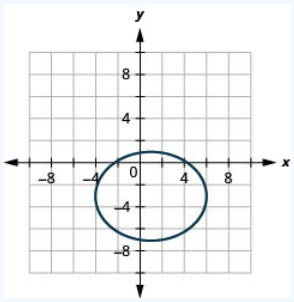


Figure 11.3.17

If we look at the equations of $\frac{x^2}{9}+\frac{y^2}{4}=1$ and $\frac{(x-3)^2}{9}+\frac{(y-1)^2}{4}=1$, we see that they are both ellipses with $a=3$ and $b=2$. So they will have the same size and shape. They are different in that they do not have the same center.

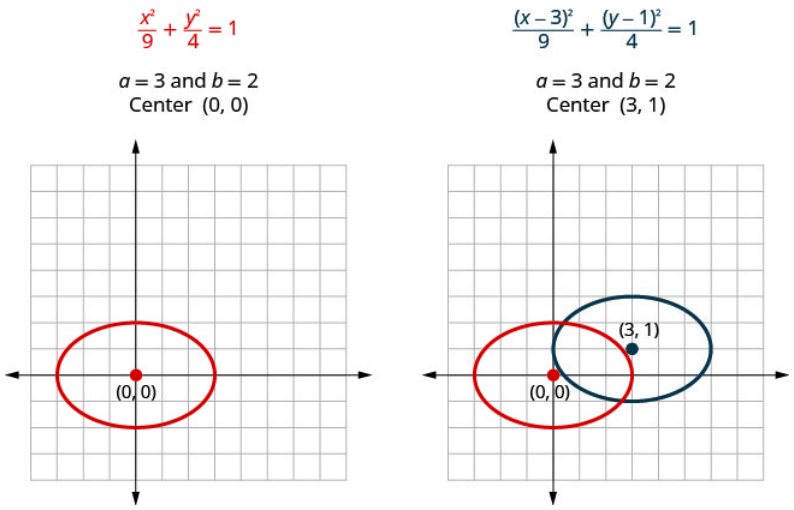


Figure 11.3.18

Notice in the graph above that we could have graphed $\frac{(x-3)^2}{9}+\frac{(y-1)^2}{4}=1$ by translations. We moved the original ellipse to the right 3 units and then up 1 unit.

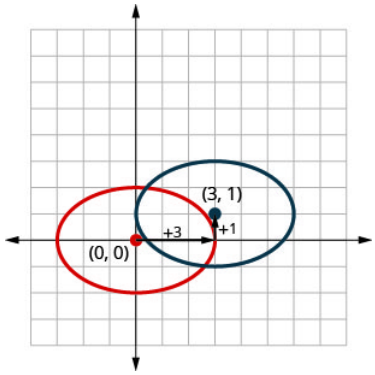


Figure 11.3.19

In the next example we will use the translation method to graph the ellipse.

Example 22.4.5

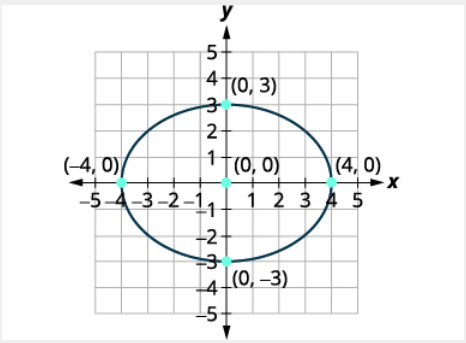
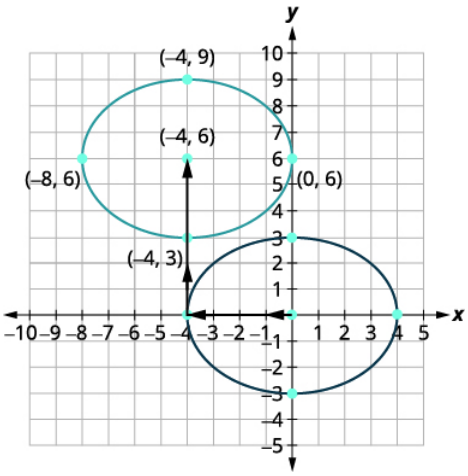
Graph $\frac{(x+4)^2}{16}+\frac{(y-6)^2}{9}=1$ by translation.

Solution:

This ellipse will have the same size and shape as $\frac{x^2}{16}+\frac{y^2}{9}=1$ whose center is $(0,0)$. We graph this ellipse first.

Table 11.3.5

The center is $(0,0)$.	Center $(0,0)$
Since $16>9$, the major axis is horizontal.	
$a^2=16, a=\pm 4$ $b^2=9, \quad b=\pm 3$	The vertices are $(4,0),(-4,0)$. The endpoints of the minor axis are $(0,3),(0,-3)$.

Sketch the ellipse.	
The original equation is in standard form, $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.	$\frac{(x-(-4))^2}{16}+\frac{(y-6)^2}{9}=1$
The ellipse is centered at (h, k).	The center is $(-4, 6)$.
We translate the graph of $\frac{x^2}{16}+\frac{y^2}{9}=1$ four units to the left and then up 6 units. Verify that the center is $(-4, 6)$. The new ellipse is the ellipse whose equation is $\frac{(x+4)^2}{16}+\frac{(y-6)^2}{9}=1$.	

? Exercise 22.4.9

Graph $\frac{(x-5)^2}{9}+\frac{(y+4)^2}{4}=1$ by translation.

Answer

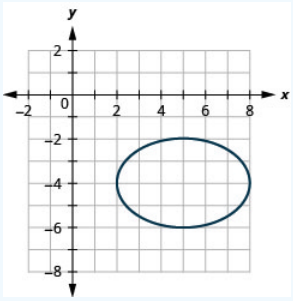


Figure 11.3.22

? Exercise 22.4.10

Graph $\frac{(x+6)^2}{16}+\frac{(y+2)^2}{25}=1$ by translation.

Answer

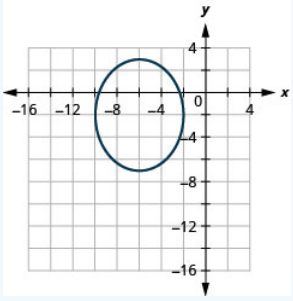


Figure 11.3.23

When an equation has both an x^2 and a y^2 with different coefficients, we verify that it is an ellipsis by putting it in standard form. We will then be able to graph the equation.

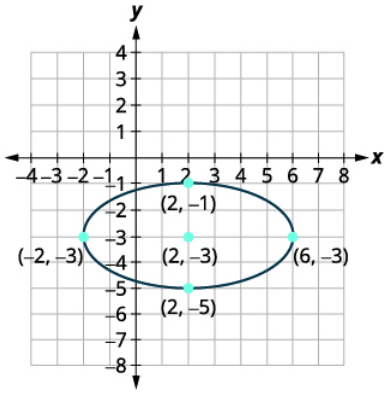
Example 22.4.6

Write the equation $x^2+4y^2-4x+24y+24=0$ in standard form and graph.

Solution:

We put the equation in standard form by completing the squares in both x and y.

Table 11.3.6

	$x^2+4y^2-4x+24y+24=0$
Rewrite grouping the x terms and y terms.	$(x^2-4x+\,)+(4y^2+24y+\,)=-24$
Make the coefficients of x^2 and y^2 equal 1.	$(x^2-4x+\,)+4(y^2+6y+\,)=-24$
Complete the squares.	$(x^2-4x+4)+4(y^2+6y+9)=-24+4+36$
Write as binomial squares.	$(x-2)^2+4(y+3)^2=16$
Divide both sides by 16 to get 1 on the right.	$\frac{(x-2)^2}{16}+\frac{4(y+3)^2}{16}=\frac{16}{16}$
Simplify.	$\frac{(x-2)^2}{16}+\frac{(y+3)^2}{4}=1$
The equation is in standard form, $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	$\frac{(x-2)^2}{16}+\frac{(y+3)^2}{4}=1$
The ellipse is centered at (h,k).	The center is $(2,-3)$.
Since $16>4$ and 16 is in the x^2 term, the major axis is horizontal. $a^2=16, a=\pm 4$ $b^2=4, \quad b=\pm 2$	The distance from the center to the vertices is 4. The distance from the center to the endpoints of the minor axis is 2.
Sketch the ellipse.	

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Exercise 22.4.11

a. Write the equation $6x^2+4y^2+12x-32y+34=0$ in standard form and
b. Graph.

Answer

a. $\frac{(x+1)^2}{6}+\frac{(y-4)^2}{9}=1$
b.

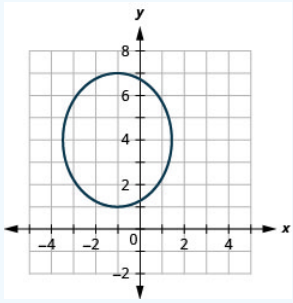

Figure 11.3.32

Exercise 22.4.12

a. Write the equation $4x^2+y^2-16x-6y+9=0$ in standard form and
b. Graph.

Answer

a. $\frac{(x-2)^2}{4}+\frac{(y-3)^2}{16}=1$
b.

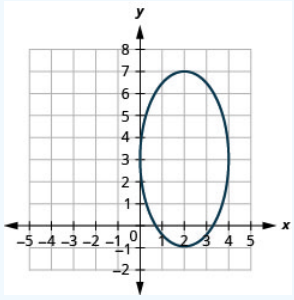

Figure 11.3.33

Solve Application with Ellipses

The orbits of the planets around the sun follow elliptical paths.

Example 22.4.7

Pluto (a dwarf planet) moves in an elliptical orbit around the Sun. The closest Pluto gets to the Sun is approximately 30 astronomical units (AU) and the furthest is approximately 50 AU. The Sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of Pluto.

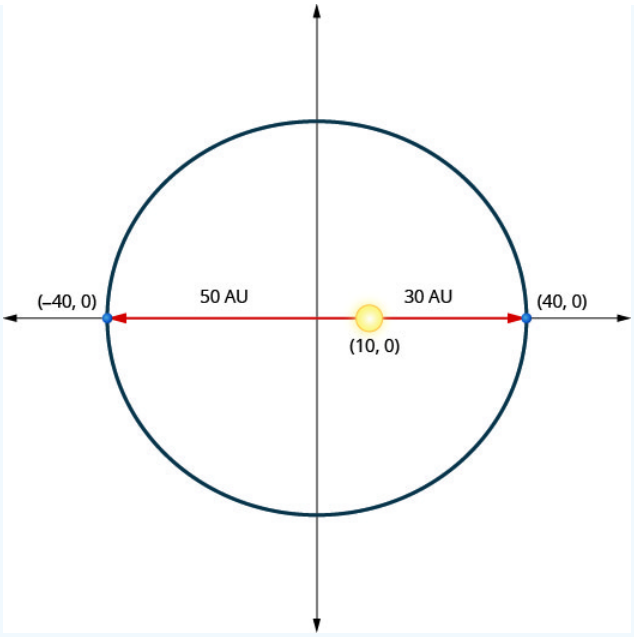

Figure 11.3.34

Solution:

We recognize this as an ellipse that is centered at the origin.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Since the major axis is horizontal and the distance from the center to the vertex is 40, we know $a=40$ and so $a^2=1600$.

$$\frac{x^2}{1600}+\frac{y^2}{b^2}=1$$

The minor axis is vertical but the end points aren't given.To find b we will use the location of the Sun. Since the Sun is a focus of the ellipse at the point $(10,0)$,we know $c=10$. Use this to solve for b^2.

$$\begin{aligned} b^2 &= a^2-c^2 \\ b^2 &= 40^2-10^2 \\ b^2 &= 1600-100 \\ b^2 &= 1500 \end{aligned}$$

Substitute a^2 and b^2 into the standard form of the ellipse.

$$\frac{x^2}{1600}+\frac{y^2}{1500}=1$$

Exercise 22.4.13

A planet moves in an elliptical orbit around its sun. The closest the planet gets to the sun is approximately 20 AU and the furthest is approximately 30 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the planet.

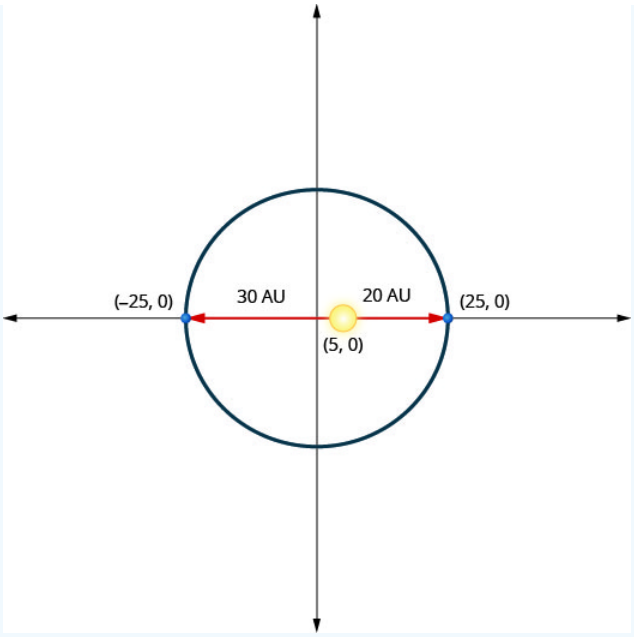

Figure 11.3.35

Answer

$\frac{x^2}{625} + \frac{y^2}{600} = 1$

Exercise 22.4.14

A planet moves in an elliptical orbit around its sun. The closest the planet gets to the sun is approximately 20 AU and the furthest is approximately 50 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the planet.

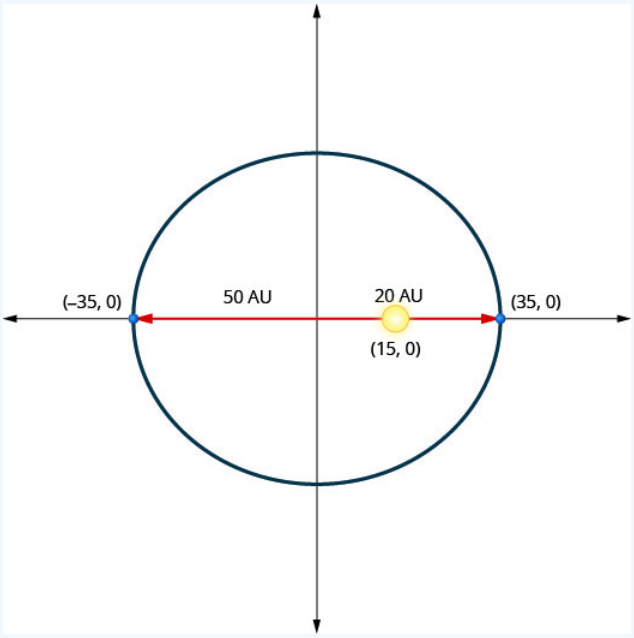

Figure 11.3.36

Answer

$\frac{x^2}{1225}+\frac{y^2}{1000}=1$

Access these online resources for additional instructions and practice with ellipses.

- Conic Sections: Graphing Ellipses Part 1
- Conic Sections: Graphing Ellipses Part 2
- Equation for Ellipse From Graph

Key Concepts

- **Ellipse:** An **ellipse** is all points in a plane where the sum of the distances from two fixed points is constant. Each of the fixed points is called a **focus** of the ellipse.

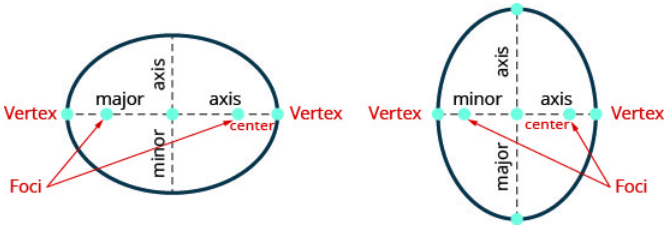

Figure 11.3.37

- If we draw a line through the foci intersects the ellipse in two points—each is called a **vertex** of the ellipse.
  The segment connecting the vertices is called the **major axis**.
  The midpoint of the segment is called the **center** of the ellipse.
  A segment perpendicular to the major axis that passes through the center and intersects the ellipse in two points is called the **minor axis**.
- **Standard Form of the Equation an Ellipse with Center** $(0,0)$: The standard form of the equation of an ellipse with center $(0,0)$, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

  The x-intercepts are $(a,0)$ and $(-a,0)$.
  The y-intercepts are $(0,b)$ and $(0,-b)$.
- **How to an Ellipse with Center** $(0,0)$
  1. Write the equation in standard form.
  2. Determine whether the major axis is horizontal or vertical.
  3. Find the endpoints of the major axis.
  4. Find the endpoints of the minor axis
  5. Sketch the ellipse.
- **Standard Form of the Equation an Ellipse with Center** (h,k): The standard form of the equation of an ellipse with center (h,k), is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

  When $a>b$, the major axis is horizontal so the distance from the center to the vertex is a.
  When $b>a$, the major axis is vertical so the distance from the center to the vertex is b.

Glossary

ellipse

An ellipse is all points in a plane where the sum of the distances from two fixed points is constant.

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22.4E: Exercises

Practice Makes Perfect

Exercise $22.4E.15$ Graph an Ellipse with Center at the Origin

In the following exercises, graph each ellipse.

1. $\frac{x^2}{4}+\frac{y^2}{25}=1$
2. $\frac{x^2}{9}+\frac{y^2}{25}=1$
3. $\frac{x^2}{25}+\frac{y^2}{36}=1$
4. $\frac{x^2}{16}+\frac{y^2}{36}=1$
5. $\frac{x^2}{36}+\frac{y^2}{16}=1$
6. $\frac{x^2}{25}+\frac{y^2}{9}=1$
7. $x^2+\frac{y^2}{4}=1$
8. $\frac{x^2}{9}+y^2=1$
9. $4x^2+25y^2=100$
10. $16x^2+9y^2=144$
11. $16x^2+36y^2=576$
12. $9x^2+25y^2=225$

Answer

1.

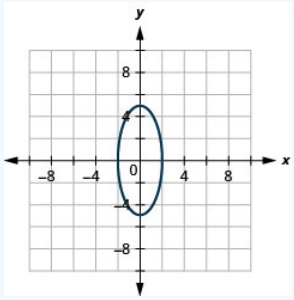

Figure 11.3.38

3.

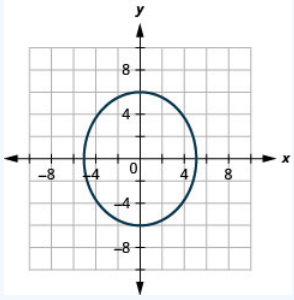

Figure 11.3.39

5.

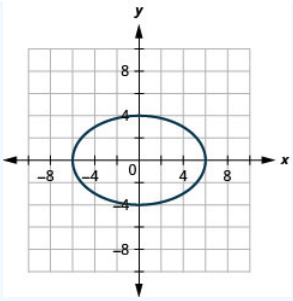

Figure 11.3.40

7.

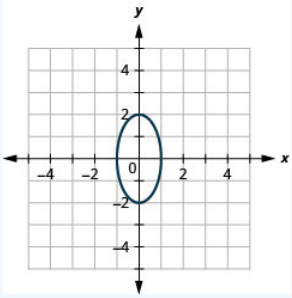

Figure 11.3.41

9.

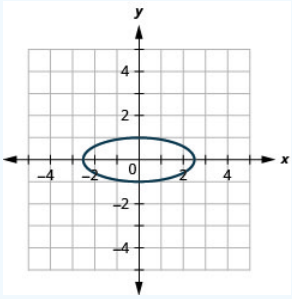

Figure 11.3.42

11.

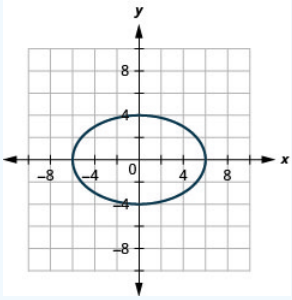

Figure 11.3.43

Exercise $22.4E.16$ Find the Equation of an Ellipse with Center at the Origin

In the following exercises, find the equation of the ellipse shown in the graph.

1.

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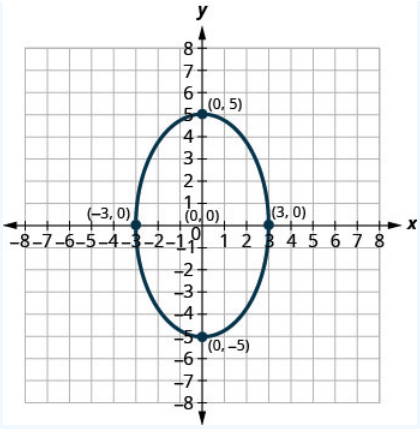

Figure 11.3.44

2.

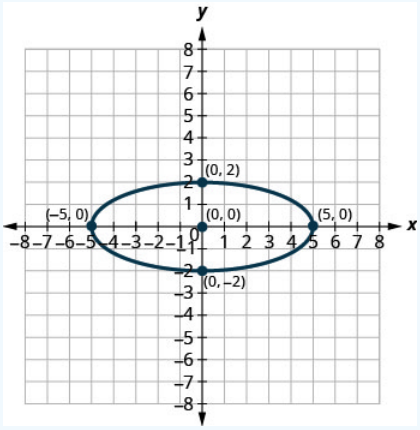

Figure 11.3.45

3.

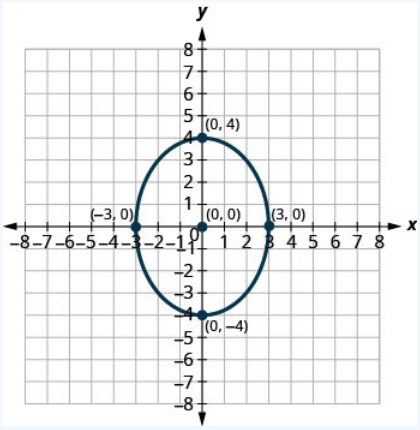

Figure 11.3.46

4.

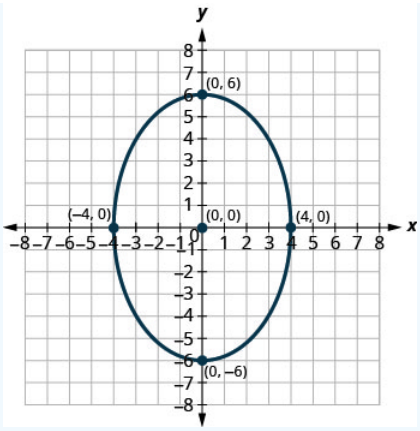

Figure 11.3.47

Answer

1. $\frac{x^2}{9}+\frac{y^2}{25}=1$
3. $\frac{x^2}{9}+\frac{y^2}{16}=1$

Exercise $22.4E.17$ Graph an Ellipse with Center Not at the Origin

In the following exercises, graph each ellipse.

1. $\frac{(x+1)^2}{4}+\frac{(y+6)^2}{25}=1$
2. $\frac{(x-3)^2}{25}+\frac{(y+2)^2}{9}=1$
3. $\frac{(x+4)^2}{4}+\frac{(y-2)^2}{9}=1$
4. $\frac{(x-4)^2}{9}+\frac{(y-1)^2}{16}=1$

Answer

1.

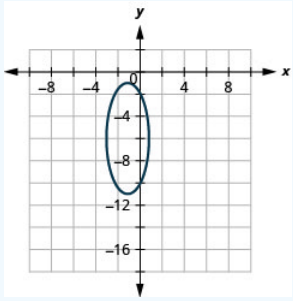

Figure 11.3.48

3.

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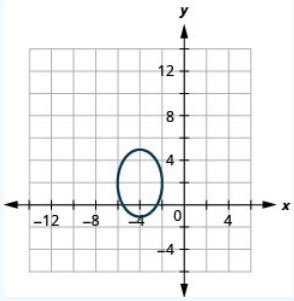

Figure 11.3.49

Exercise $22.4E.18$ Graph an Ellipse with Center Not at the Origin

In the following exercises, graph each equation by translation.

1. $\frac{(x-3)^2}{4}+\frac{(y-7)^2}{25}=1$
2. $\frac{(x+6)^2}{16}+\frac{(y+5)^2}{4}=1$
3. $\frac{(x-5)^2}{9}+\frac{(y+4)^2}{25}=1$
4. $\frac{(x+5)^2}{36}+\frac{(y-3)^2}{16}=1$

Answer

1.

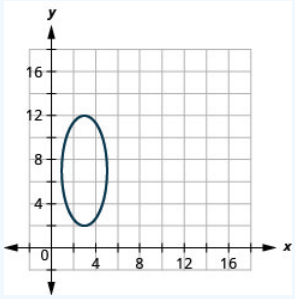

Figure 11.3.50

3.

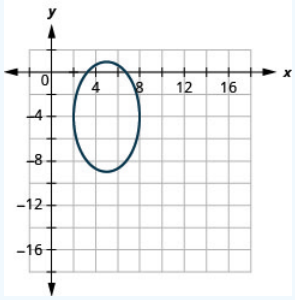

Figure 11.3.51

Exercise $22.4E.19$ Graph an Ellipse with Center Not at the Origin

In the following exercises,

a. Write the equation in standard form and
b. Graph.

1. $25x^2+9y^2-100x-54y-44=0$

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2. $4x^2+25y^2+8x+100y+4=0$
3. $4x^2+25y^2-24x-64=0$
4. $9x^2+4y^2+56y+160=0$

Answer

1.

a. $\frac{(x-2)^2}{9}+\frac{(y-3)^2}{25}=1$

b.

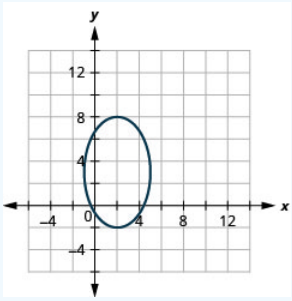

Figure 11.3.52

3.

a. $\frac{y^2}{4}+\frac{(x-3)^2}{25}=1$

b.

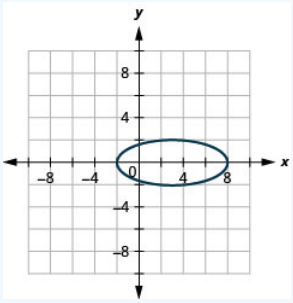

Figure 11.3.53

Exercise $22.4E.20$ Graph an Ellipse with Center Not at the Origin

In the following exercises, graph the equation.

1. $x=-2(y-1)^2+2$
2. $x^2+y^2=49$
3. $(x+5)^2+(y+2)^2=4$
4. $y=-x^2+8x-15$
5. $\frac{(x+3)^2}{16}+\frac{(y+1)^2}{4}=1$
6. $(x-2)^2+(y-3)^2=9$
7. $\frac{x^2}{25}+\frac{y^2}{36}=1$
8. $x=4(y+1)^2-4$
9. $x^2+y^2=64$
10. $\frac{x^2}{9}+\frac{y^2}{25}=1$
11. $y=6x^2+2x-1$
12. $\frac{(x-2)^2}{9}+\frac{(y+3)^2}{25}=1$

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Answer

1.

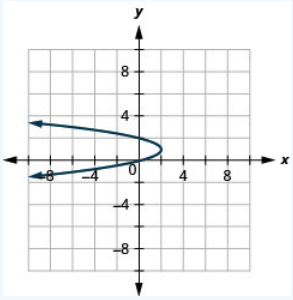

Figure 11.3.54

3.

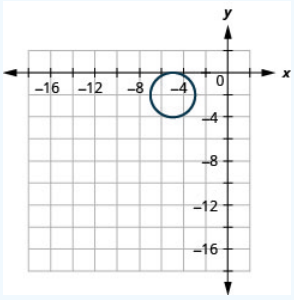

Figure 11.3.55

5.

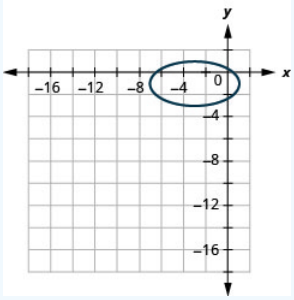

Figure 11.3.56

7.

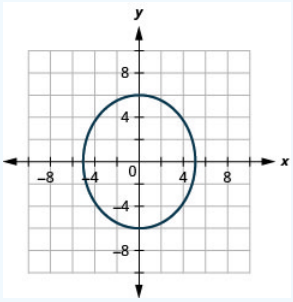

Figure 11.3.57

9.

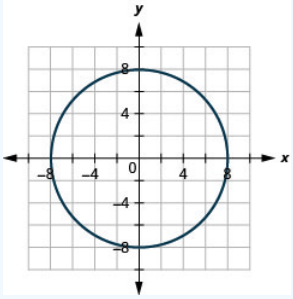

Figure 11.3.58

11.

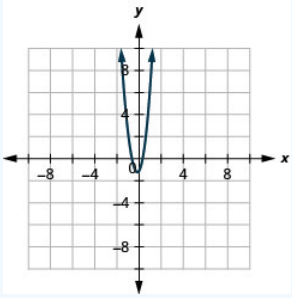

Figure 11.3.59

Exercise $22.4E.21$ Solve Application with Ellipses

1. A planet moves in an elliptical orbit around its sun. The closest the planet gets to the sun is approximately 10 AU and the furthest is approximately 30 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the planet.

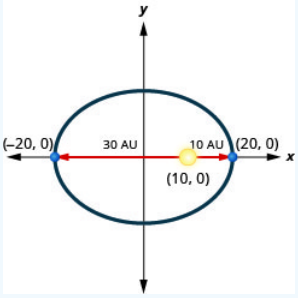

Figure 11.3.60

2. A planet moves in an elliptical orbit around its sun. The closest the planet gets to the sun is approximately 10 AU and the furthest is approximately 70 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the planet.

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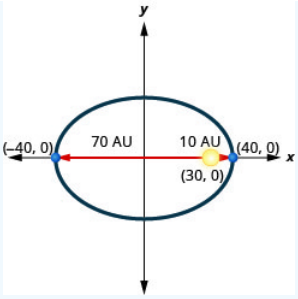

Figure 11.3.61

3. A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 15 AU and the furthest is approximately 85 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.

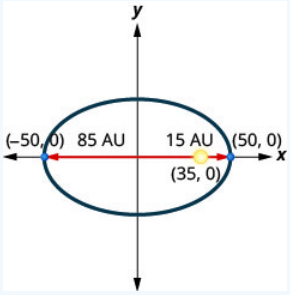

Figure 11.3.62

4. A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 15 AU and the furthest is approximately 95 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.

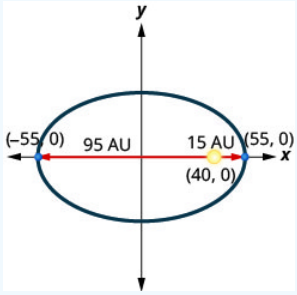

Figure 11.3.63

Answer

1. $\frac{x^2}{400}+\frac{y^2}{300}=1$
3. $\frac{x^2}{2500}+\frac{y^2}{1275}=1$

Exercise $22.4E.22$ Writing Exercises

1. In your own words, define an ellipse and write the equation of an ellipse centered at the origin in standard form. Draw a sketch of the ellipse labeling the center, vertices and major and minor axes.
2. Explain in your own words how to get the axes from the equation in standard form.
3. Compare and contrast the graphs of the equations $\frac{x^2}{4}+\frac{y^2}{9}=1$ and $\frac{x^2}{9}+\frac{y^2}{4}=1$.
4. Explain in your own words, the difference between a vertex and a focus of the ellipse.

Answer

1. Answers may vary
3. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph an ellipse with center at the origin.			
find the equation of an ellipse with center at the origin.			
graph an ellipse with center not at the origin.			
solve applications with ellipses.			

Figure 11.3.64

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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22.5: Hyperbolas

Learning Objectives

By the end of this section, you will be able to:

- Graph a hyperbola with center at $(0, 0)$
- Graph a hyperbola with center at (h, k)
- Identify conic sections by their equations

Before you get started, take this readiness quiz.

1. Solve: $x^2 = 12$.
   If you missed this problem, review Example 9.1.
2. Expand: $(x - 4)^2$.
   If you missed this problem, review Example 5.32.
3. Graph $y = -\frac{2}{3}x$.
   If you missed this problem, review Example 3.4.

Graph a Hyperbola with Center at $(0, 0)$

The last conic section we will look at is called a **hyperbola**. We will see that the equation of a hyperbola looks the same as the equation of an ellipse, except it is a difference rather than a sum. While the equations of an ellipse and a hyperbola are very similar, their graphs are very different.

We define a **hyperbola** as all points in a plane where the difference of their distances from two fixed points is constant. Each of the fixed points is called a **focus** of the hyperbola.

Definition 22.5.1

A **hyperbola** is all points in a plane where the difference of their distances from two fixed points is constant. Each of the fixed points is called a **focus** of the hyperbola.

Figure 11.4.1

The line through the foci, is called the **transverse axis**. The two points where the transverse axis intersects the hyperbola are each a **vertex** of the hyperbola. The midpoint of the segment joining the foci is called the **center** of the hyperbola. The line perpendicular to the transverse axis that passes through the center is called the **conjugate axis**. Each piece of the graph is called a **branch** of the hyperbola.

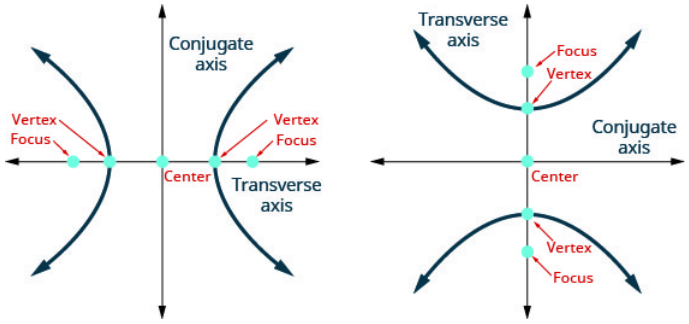


Figure 11.4.2

Again our goal is to connect the geometry of a conic with algebra. Placing the hyperbola on a rectangular coordinate system gives us that opportunity. In the figure, we placed the hyperbola so the foci $((-c,0),(c,0))$ are on the x-axis and the center is the origin.

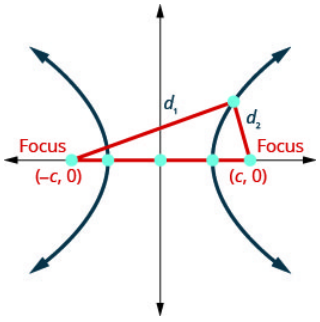


Figure 11.4.3

The definition states the difference of the distance from the foci to a point (x,y) is constant. So $|d_1-d_2|$ is a constant that we will call $2a$ so $|d_1-d_2|=2a$. We will use the distance formula to lead us to an algebraic formula for an ellipse.

$$|d_1-d_2|=2a$$

Use the distance formula to find d_1, d_2

$$\left|\sqrt{(x-(-c))^2+(y-0)^2}\ -\sqrt{(x-c)^2+(y-0)^2}\ \right|=2a$$

Eliminate the radicals. To simplify the equation of the ellipse, we let $c^2-a^2=b^2$.

$$\frac{x^2}{a^2}+\frac{y^2}{c^2-a^2}=1$$

So, the equation of a hyperbola centered at the origin in standard form is:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

To graph the hyperbola, it will be helpful to know about the intercepts. We will find the x-intercepts and y-intercepts using the formula.

x-intercepts

Let $y=0$.

$$\begin{aligned}\frac{x^2}{a^2}-\frac{y^2}{b^2}&=1\\ \frac{x^2}{a^2}-\frac{0^2}{b^2}&=1\\ \frac{x^2}{a^2}&=1\\ x^2&=a^2\\ x&=\pm a\end{aligned}$$

The x-intercepts are $(a,0)$ and $(-a,0)$.

y-intercepts

Let $x = 0$.

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \\ \frac{0^2}{a^2} - \frac{y^2}{b^2} &= 1 \\ -\frac{y^2}{b^2} &= 1 \\ y^2 &= -b^2 \\ y &= \pm\sqrt{-b^2} \end{aligned}$$

There are no y-intercepts.

The \(a, b\) values in the equation also help us find the asymptotes of the hyperbola. The asymptotes are intersecting straight lines that the branches of the graph approach but never intersect as the \(x, y\) values get larger and larger.

To find the asymptotes, we sketch a rectangle whose sides intersect the x-axis at the vertices $(-a, 0), (a, 0)$, and intersect the y-axis at $(0, -b), (0, b)$. The lines containing the diagonals of this rectangle are the asymptotes of the hyperbola. The rectangle and asymptotes are not part of the hyperbola, but they help us graph the hyperbola.

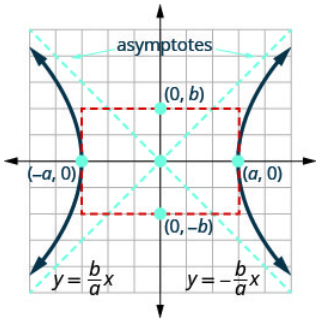


Figure 11.4.4

The asymptotes pass through the origin and we can evaluate their slope using the rectangle we sketched. They have equations $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

There are two equations for hyperbolas, depending whether the transverse axis is vertical or horizontal. We can tell whether the transverse axis is horizontal by looking at the equation. When the equation is in standard form, if the x^2-term is positive, the transverse axis is horizontal. When the equation is in standard form, if the y^2-term is positive, the transverse axis is vertical.

The second equations could be derived similarly to what we have done. We will summarize the results here.

Definition 22.5.2

Standard Form of the Equation a Hyperbola with Center $(0, 0)$

The standard form of the equation of a hyperbola with center $(0, 0)$, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

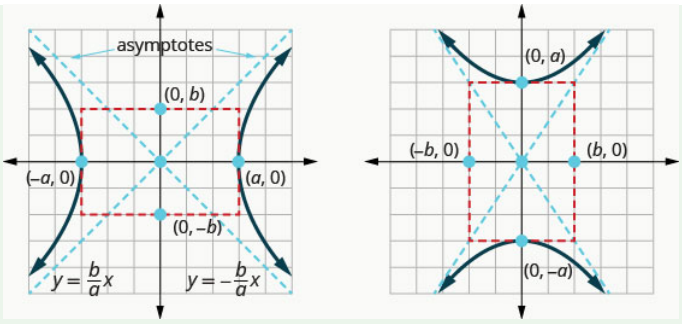


Figure 11.4.5

Notice that, unlike the equation of an ellipse, the denominator of x^2 is not always a^2 and the denominator of y^2 is not always b^2.

Notice that when the x^2-term is positive, the transverse axis is on the x-axis. When the y^2-term is positive, the transverse axis is on the y-axis.

Standard Forms of the Equation a Hyperbola with Center $(0,0)$

Table 11.4.1

	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Orientation	Transverse axis on the x-axis. Opens left and right	Transverse axis on the y-axis. Opens up and down
Vertices	$(-a,0),(a,0)$	$(0,-a),(0,a)$
x-intercepts	$(-a,0),(a,0)$	none
y-intercepts	none	$(0,-a),(0,a)$
Rectangle	Use $(\pm a,0)(0,\pm b)$	Use $(0,\pm a)(\pm b,0)$
Asymptotes	$y=\frac{b}{a}x, y=-\frac{b}{a}x$	$y=\frac{a}{b}x, y=-\frac{a}{b}x$

We will use these properties to graph hyperbolas.

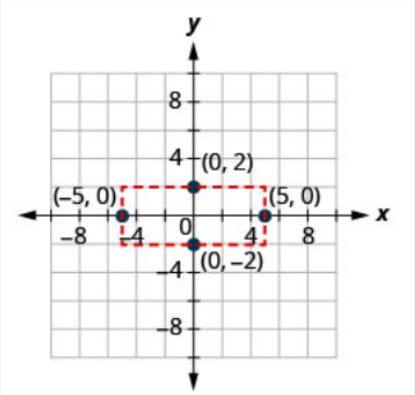
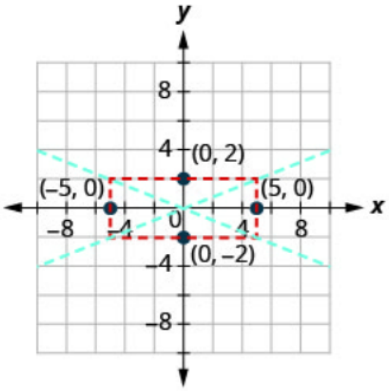
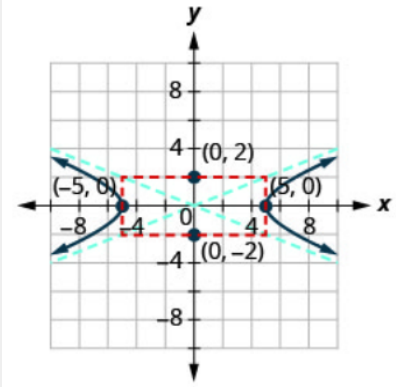
Example 22.5.1 How to Graph a Hyperbola with Center $(0,0)$

Graph $\frac{x^2}{25}-\frac{y^2}{4}=1$.

Solution:

Table 11.4.2

Step 1: Write the equation in standard form.	The equation is in standard form.	$\frac{x^2}{25}-\frac{y^2}{4}=1$
Step 2: Determine whether the transverse axis is horizontal or vertical.	Since the x^2-term is positive, the transverse axis is horizontal.	The transverse axis is horizontal.
Step 3: Find the vertices.	Since $a^2=25$ then $a=\pm 5$. The vertices are on the x-axis.	$(-5,0),(5,0)$

Step 4: Sketch the rectangle centered at the origin intersection one axis at $\pm a$ and the other at $\pm b$.	Since $a = \pm 5$, the rectangle will intersect the x-axis at the vertices. Since $b = \pm 2$, the rectangle will intersect the y-axis at $(0, -2)$ and $(0, 2)$.	
Step 5: Sketch the asymptotes--the lines through the diagonals of the rectangle.	The asymptotes have the equations $y = \frac{5}{2}x, y = -\frac{5}{2}x$.	
Step 6: Draw the two branches of the hyperbola.	Start at each vertex and use the asymptotes as a guide.	

? Exercise 22.5.1

Graph $\frac{x^2}{16} - \frac{y^2}{4} = 1$.

Answer

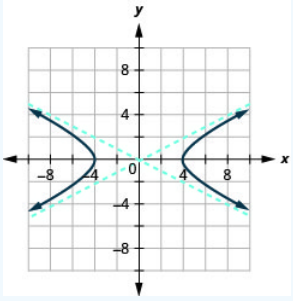


Figure 11.4.9

? Exercise 22.5.2

Graph $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

Answer

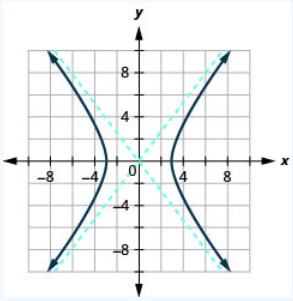


Figure 11.4.10

We summarize the steps for reference.

Graph a Hyperbola Centered at $(0,0)$

1. Write the equation in standard form.
2. Determine whether the transverse axis is horizontal or vertical.
3. Find the vertices.
4. Sketch the rectangle centered at the origin intersecting one axis at $\pm a$ and the other at $\pm b$.
5. Sketch the asymptotes—the lines through the diagonals of the rectangle.
6. Draw the two branches of the hyperbola.

Sometimes the equation for a hyperbola needs to be first placed in standard form before we graph it.

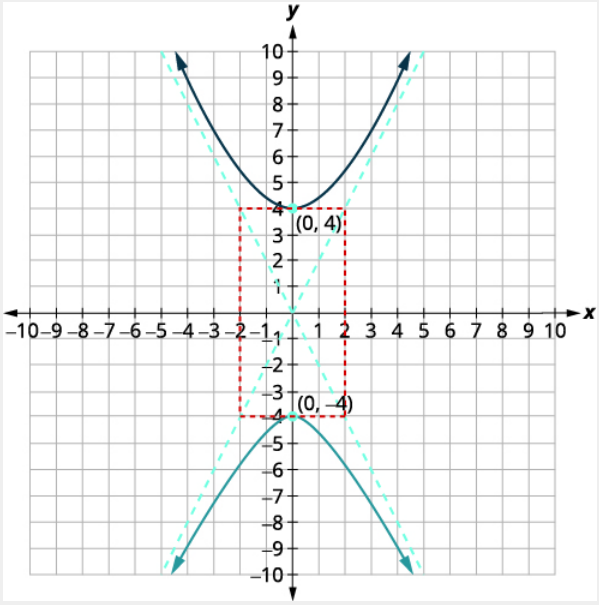
✓ Example 22.5.2

Graph $4y^2 - 16x^2 = 64$.

Solution:

Table 11.4.3

	$4y^2 - 16x^2 = 64$
To write the equation in standard form, divide each term by 64 to make the equation equal to 1.	$\frac{4y^2}{64} - \frac{16x^2}{64} = \frac{64}{64}$
Simplify.	$\frac{y^2}{16} - \frac{x^2}{4} = 1$
Since the y^2-term is positive, the transverse axis is vertical. Since $a^2 = 16$ then $a = \pm 4$.	

The vertices are on the y-axis, $(0, -a), (0, a)$. Since $b^2 = 4$ then $b = \pm 2$.	$(0, -4), (0, 4)$
Sketch the rectangle intersecting the x-axis at $(-2, 0), (2, 0)$ and the y-axis at the vertices. Sketch the asymptotes through the diagonals of the rectangle. Draw the two branches of the hyperbola.	

Exercise 22.5.3

Graph $4y^2 - 25x^2 = 100$.

Answer

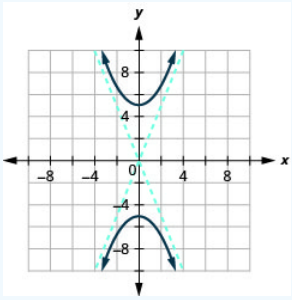


Figure 11.4.12

Exercise 22.5.4

Graph $25y^2 - 9x^2 = 225$.

Answer

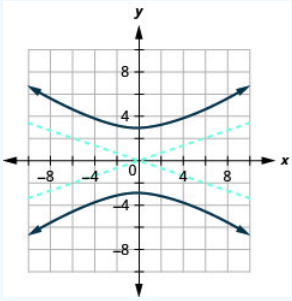


Figure 11.4.13

Graph a Hyperbola with Center at (h, k)

Hyperbolas are not always centered at the origin. When a hyperbola is centered at (h, k) the equations changes a bit as reflected in the table.

Standard Forms of the Equation a Hyperbola with Center (h, k)

Table 11.4.4

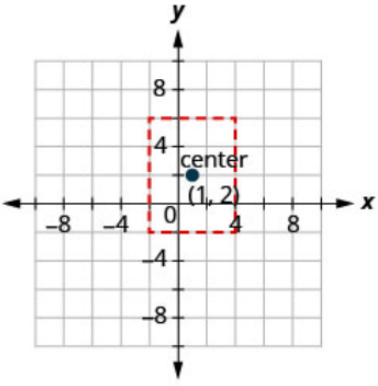
	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Orientation	Transverse axis is horizontal. Opens left and right	Transverse axis is vertical. Opens up and down
Center	(h, k)	(h, k)
Vertices	a units to the left and right of the center	a units above and below the center
Rectangle	Use a units left/right of center b units above/below the center	Use a units above/below the center b units left/right of center

Example 22.5.3 How to Graph a Hyperbola with Center (h, k)

Graph $\frac{(x-1)^2}{9} - \frac{(y-2)^2}{16} = 1$

Solution:

Table 11.4.5

Step 1: Write the equation in standard form.	The equation is in standard form.	$\frac{(x-1)^2}{9} - \frac{(y-2)^2}{16} = 1$
Step 2: Determine whether the transverse axis is horizontal or vertical.	Since the x^2-term is positive, the hyperbola opens left and right.	The transverse axis is horizontal. The hyperbola opens left and right.
Step 3: Find the center and a, b.	$h = 1$ and $k = 2$ $a^2 = 9$ $b^2 = 16$	$\frac{\left(\overset{x-h}{x-1}\right)^2}{9} - \frac{\left(\overset{y-k}{y-2}\right)^2}{16} = 1$ Center: $(1, 2)$ $a = 3$ $b = 4$
Step 4: Sketch the rectangle centered at (h, k) using a, b.	Mark the center, $(1, 2)$. Sketch the rectangle that goes through the points 3 units to the left/right of the center and 4 units above and below the center.	

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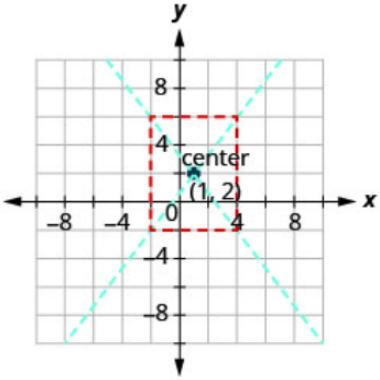
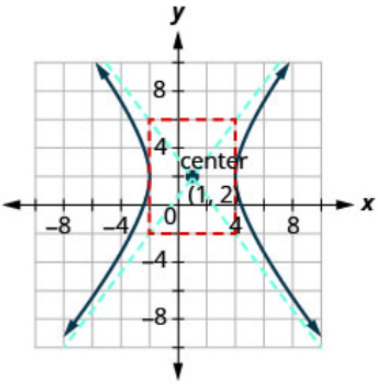
Step 5: Sketch the asymptotes--the lines through the diagonals of the rectangle. Mark the vertices.	Sketch the diagonals. Mark the vertices, which are on the rectangle 3 units to the left and right of the center.	
Step 6: Draw the two branches of the hyperbola.	Start at each vertex and use the asymptotes as a guide.	

? Exercise 22.5.5

Graph $\frac{(x-3)^2}{25}-\frac{(y-1)^2}{9}=1$.

Answer

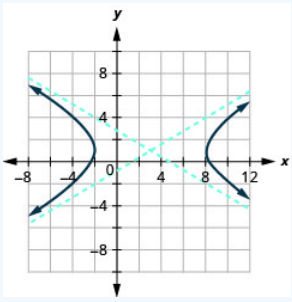


Figure 11.4.17

? Exercise 22.5.6

Graph $\frac{(x-2)^2}{4}-\frac{(y-2)^2}{9}=1$.

Answer

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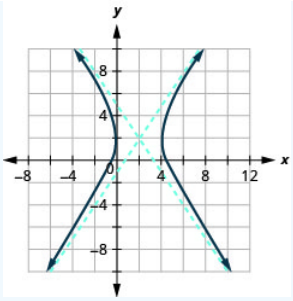


Figure 11.4.18

We summarize the steps for easy reference.

Graph a Hyperbola Centered at (h, k)

1. Write the equation in standard form.
2. Determine whether the transverse axis is horizontal or vertical.
3. Find the center and a, b.
4. Sketch the rectangle centered at (h, k) using a, b.
5. Sketch the asymptotes—the lines through the diagonals of the rectangle. Mark the vertices.
6. Draw the two branches of the hyperbola.

Be careful as you identify the center. The standard equation has $x-h$ and $y-k$ with the center as (h, k).

Example 22.5.4

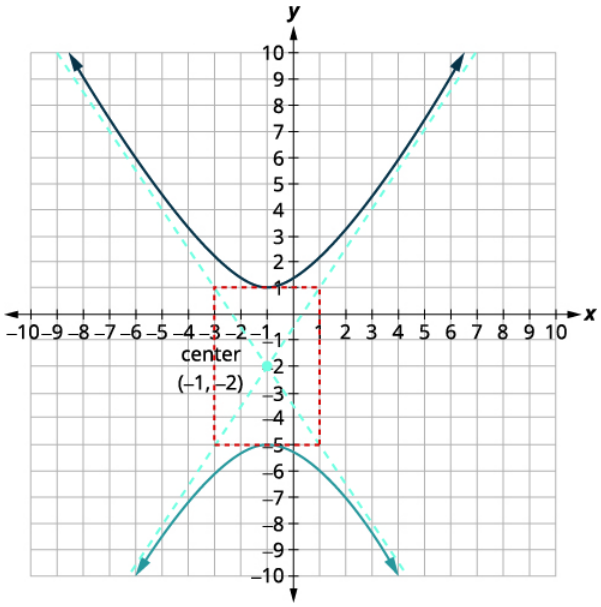
Graph $\frac{(y+2)^2}{9}-\frac{(x+1)^2}{4}=1$.

Solution:

Table 11.4.6

	$\frac{(y+2)^2}{9}-\frac{(x+1)^2}{4}=1$
Since the y^2-term is positive, the hyperbola opens up and down.	$\frac{\left(\frac{y-k}{y-(-2)}\right)^2}{9}-\frac{\left(\frac{x-h}{x-(-1)}\right)^2}{4}=1$
Find the center, (h, k).	Center: $(-1,-2)$
Find a, b.	$a=3 b=2$

Sketch the rectangle that goes through the points 3 units above and below the center and
2 units to the left/right of the center.
Sketch the asymptotes—the lines through the diagonals of the rectangle.
Mark the vertices.
Graph the branches.



Exercise 22.5.7

Graph $\frac{(y+3)^2}{16}-\frac{(x+2)^2}{9}=1$.

Answer

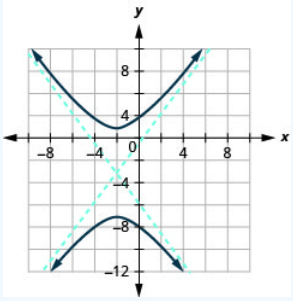


Figure 11.4.22

Exercise 22.5.8

Graph $\frac{(y+2)^2}{9}-\frac{(x+2)^2}{9}=1$.

Answer

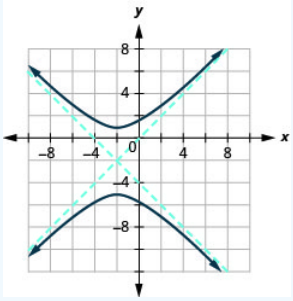


Figure 11.4.23

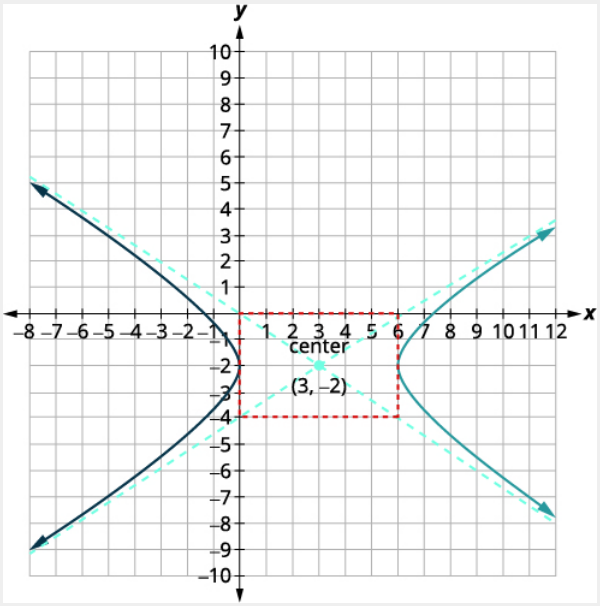
Again, sometimes we have to put the equation in standard form as our first step.

Example 22.5.5

Write the equation in standard form and graph $4x^2 - 9y^2 - 24x - 36y - 36 = 0$.

Solution:

Table 11.4.7

	$4x^2 - 9y^2 - 24x - 36y - 36 = 0$
To get to standard form, complete the squares.	$4(x^2 - 6x) - 9(y^2 + 4y) = 36$
	$4(x^2 - 6x + 9) - 9(y^2 + 4y + 4) = 36 + 36 - 36$
	$4(x-3)^2 - 9(y+2)^2 = 36$
Divide each term by 36 to get the constant to be 1.	$\frac{4(x-3)^2}{36} - \frac{9(y+2)^2}{36} = \frac{36}{36}$
	$\frac{(x-3)^2}{9} - \frac{(y+2)^2}{4} = 1$
Since the x^2-term is positive, the hyperbola opens left and right.	
Find the center, (h, k).	Center: $(3, -2)$
Find a, b.	$a = 3$ $b = 4$
Sketch the rectangle that goes through the points 3 units to the left/right of the center and 2 units above and below the center. Sketch the asymptotes—the lines through the diagonals of the rectangle. Mark the vertices. Graph the branches.	

Exercise 22.5.9

a. Write the equation in standard form and
b. Graph $9x^2 - 16y^2 + 18x + 64y - 199 = 0$.

Answer

a. $\frac{(x+1)^2}{16} - \frac{(y-2)^2}{9} = 1$
b.

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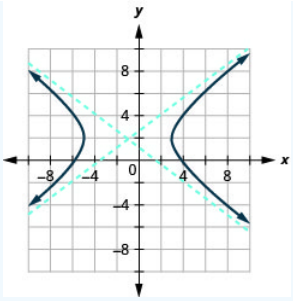

Figure 11.4.31

? Exercise 22.5.10

a. Write the equation in standard form and
b. Graph $16x^2 - 25y^2 + 96x - 50y - 281 = 0$.

Answer

a. $\frac{(x+3)^2}{25} - \frac{(y+1)^2}{16} = 1$
b.

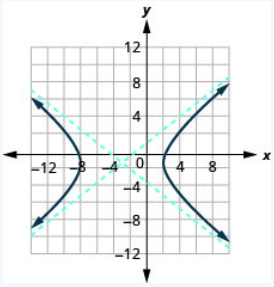

Figure 11.4.32

Identify Conic Sections by Their Equations

Now that we have completed our study of the conic sections, we will take a look at the different equations and recognize some ways to identify a conic by its equation. When we are given an equation to graph, it is helpful to identify the conic so we know what next steps to take.

To identify a conic from its equation, it is easier if we put the variable terms on one side of the equation and the constants on the other.

Table 11.4.8

Conic	Characteristics of x^2- and y^2-terms	Example
Parabola	Either x^2 OR y^2. Only one variable is squared.	$x = 3y^2 - 2y + 1$
Circle	x^2- and y^2- terms have the same coefficients.	$x^2 + y^2 = 49$
Ellipse	x^2- and y^2- terms have the **same** sign, different coefficients.	$4x^2 + 25y^2 = 100$
Hyperbola	x^2- and y^2- terms have the **different** signs, different coefficients.	$25y^2 - 4x^2 = 100$

Example 22.5.6

Identify the graph of each equation as a circle, parabola, ellipse, or hyperbola.

a. $9x^2+4y^2+56y+160=0$
b. $9x^2-16y^2+18x+64y-199=0$
c. $x^2+y^2-6x-8y=0$
d. $y=-2x^2-4x-5$

Solution:

a. The x^2- and y^2-terms have the same sign and different coefficients.

$$9x^2+4y^2+56y+160=0$$

Ellipse

b. The x^2- and y^2-terms have different signs and different coefficients.

$$9x^2-16y^2+18x+64y-199=0$$

Hyperbola

c. The x^2- and y^2-terms have the same coefficients.

$$x^2+y^2-6x-8y=0$$

Circle

d. Only one variable, x, is squared.

$$y=-2x^2-4x-5$$

Parabola

Exercise 22.5.11

Identify the graph of each equation as a circle, parabola, ellipse, or hyperbola.

a. $x^2+y^2-8x-6y=0$
b. $4x^2+25y^2=100$
c. $y=6x^2+2x-1$
d. $16y^2-9x^2=144$

Answer

a. Circle
b. Ellipse
c. Parabola
d. Hyperbola

Exercise 22.5.12

Identify the graph of each equation as a circle, parabola, ellipse, or hyperbola.

a. $16x^2+9y^2=144$
b. $y=2x^2+4x+6$
c. $x^2+y^2+2x+6y+9=0$
d. $4x^2-16y^2=64$

Answer

a. Ellipse
b. Parabola
c. Circle

d. Hyperbola

Access these online resources for additional instructions and practice with hyperbolas.

- Graph a Hyperbola with Center at the Origin
- Graph a Hyperbola with Center not at the Origin
- Graph a Hyperbola in General Form
- Identifying Conic Sections in General Form

Key Concepts

- **Hyperbola:** A **hyperbola** is all points in a plane where the difference of their distances from two fixed points is constant.

hyperbola

Figure 11.4.1

- Each of the fixed points is called a **focus** of the hyperbola.
  The line through the foci, is called the **transverse axis**.
  The two points where the transverse axis intersects the hyperbola are each a **vertex** of the hyperbola.
  The midpoint of the segment joining the foci is called the **center** of the hyperbola.
  The line perpendicular to the transverse axis that passes through the center is called the **conjugate axis**.
  Each piece of the graph is called a **branch** of the hyperbola.

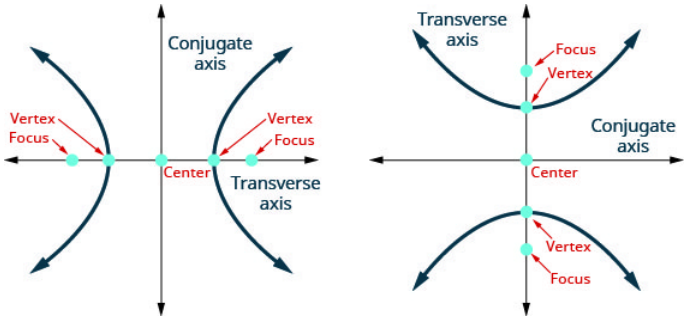

Figure 11.4.2

Standard Forms of the Equation a Hyperbola with Center $(0,0)$

Table 11.4.1

	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Orientation	Transverse axis on the x-axis. Opens left and right	Transverse axis on the y-axis. Opens up and down
Vertices	$(-a,0),(a,0)$	$(0,-a),(0,a)$
x-intercepts	$(-a,0),(a,0)$	none
y-intercepts	none	$(0,-a),(0,a)$
Rectangle	Use $(\pm a,0)(0,\pm b)$	Use $(0,\pm a)(\pm b,0)$
Asymptotes	$y=\frac{b}{a}x, y=-\frac{b}{a}x$	$y=\frac{a}{b}x, y=-\frac{a}{b}x$

- **How to graph a hyperbola centered at $(0,0)$.**

1. Write the equation in standard form.
2. Determine whether the transverse axis is horizontal or vertical.
3. Find the vertices.
4. Sketch the rectangle centered at the origin intersecting one axis at $\pm a$ and the other at $\pm b$.
5. Sketch the asymptotes—the lines through the diagonals of the rectangle.
6. Draw the two branches of the hyperbola.

Standard Forms of the Equation a Hyperbola with Center (h, k)

Table 11.4.4

	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Orientation	Transverse axis is horizontal. Opens left and right	Transverse axis is vertical. Opens up and down
Center	(h, k)	(h, k)
Vertices	a units to the left and right of the center	a units above and below the center
Rectangle	Use a units left/right of center b units above/below the center	Use a units above/below the center b units left/right of center

- **How to graph a hyperbola centered at (h, k).**
  1. Write the equation in standard form.
  2. Determine whether the transverse axis is horizontal or vertical.
  3. Find the center and a, b.
  4. Sketch the rectangle centered at (h, k) using a, b.
  5. Sketch the asymptotes—the lines through the diagonals of the rectangle. Mark the vertices.
  6. Draw the two branches of the hyperbola.

Table 11.4.8

Conic	Characteristics of x^2- and y^2-terms	Example
Parabola	Either x^2 OR y^2. Only one variable is squared.	$x = 3y^2 - 2y + 1$
Circle	x^2- and y^2- terms have the same coefficients.	$x^2 + y^2 = 49$
Ellipse	x^2- and y^2- terms have the **same** sign, different coefficients.	$4x^2 + 25y^2 = 100$
Hyperbola	x^2- and y^2- terms have the **different** signs, different coefficients.	$25y^2 - 4x^2 = 100$

Glossary

hyperbola

A hyperbola is defined as all points in a plane where the difference of their distances from two fixed points is constant.

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22.5E: Exercises

Practice Makes Perfect

Exercise $22.5E.13$ Graph a Hyperbola with Center at $(0,0)$

In the following exercises, graph.

1. $\frac{x^2}{9}-\frac{y^2}{4}=1$
2. $\frac{x^2}{25}-\frac{y^2}{9}=1$
3. $\frac{x^2}{16}-\frac{y^2}{25}=1$
4. $\frac{x^2}{9}-\frac{y^2}{36}=1$
5. $\frac{y^2}{25}-\frac{x^2}{4}=1$
6. $\frac{y^2}{36}-\frac{x^2}{16}=1$
7. $16y^2-9x^2=144$
8. $25y^2-9x^2=225$
9. $4y^2-9x^2=36$
10. $16y^2-25x^2=400$
11. $4x^2-16y^2=64$
12. $9x^2-4y^2=36$

Answer

1.

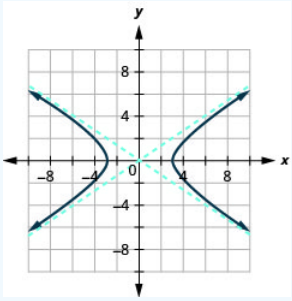

Figure 11.4.33

3.

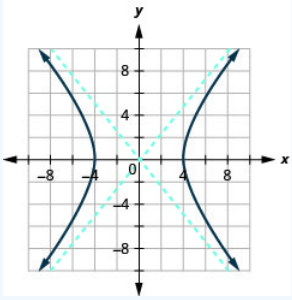

Figure 11.4.34

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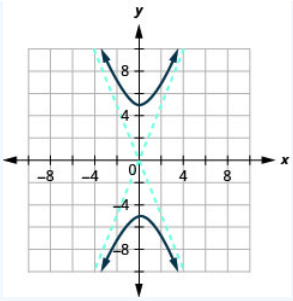

Figure 11.4.35

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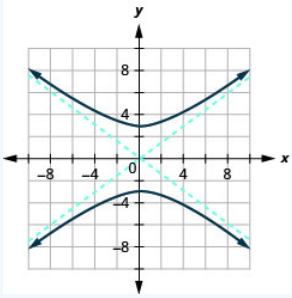

Figure 11.4.36

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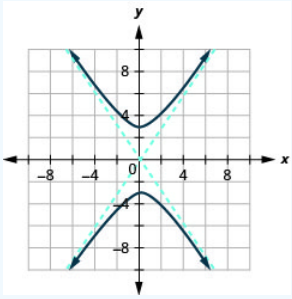

Figure 11.4.37

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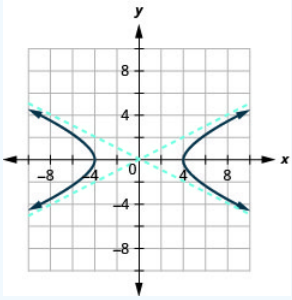

Figure 11.4.38

Exercise $22.5E.14$ Graph a Hyperbola with Center at (h, k)

In the following exercises, graph.

1. $\frac{(x-1)^2}{16} - \frac{(y-3)^2}{4} = 1$
2. $\frac{(x-2)^2}{4} - \frac{(y-3)^2}{16} = 1$

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3. $\frac{(y-4)^2}{9} - \frac{(x-2)^2}{25} = 1$
4. $\frac{(y-1)^2}{25} - \frac{(x-4)^2}{16} = 1$
5. $\frac{(y+4)^2}{25} - \frac{(x+1)^2}{36} = 1$
6. $\frac{(y+1)^2}{16} - \frac{(x+1)^2}{4} = 1$
7. $\frac{(y-4)^2}{16} - \frac{(x+1)^2}{25} = 1$
8. $\frac{(y+3)^2}{16} - \frac{(x-3)^2}{36} = 1$
9. $\frac{(x-3)^2}{25} - \frac{(y+2)^2}{9} = 1$
10. $\frac{(x+2)^2}{4} - \frac{(y-1)^2}{9} = 1$

Answer

1.

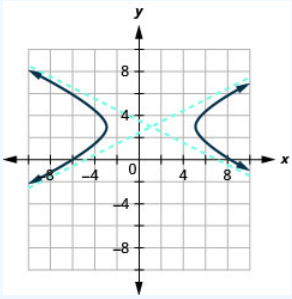

Figure 11.4.39

3.

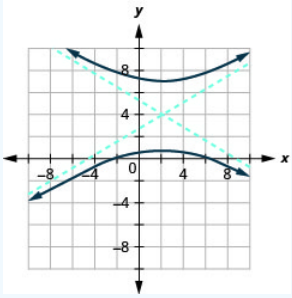

Figure 11.4.40

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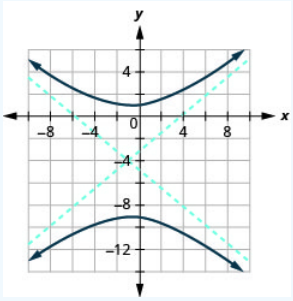

Figure 11.4.41

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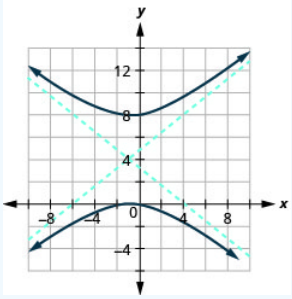

Figure 11.4.42

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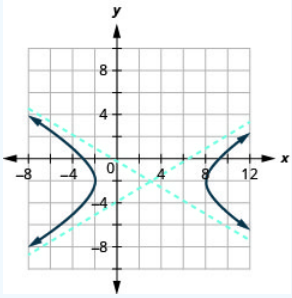

Figure 11.4.43

Exercise $22.5E.15$ Graph a Hyperbola with Center at (h, k)

In the following exercises,

a. Write the equation in standard form and
b. Graph.

1. $9x^2 - 4y^2 - 18x + 8y - 31 = 0$
2. $16x^2 - 4y^2 + 64x - 24y - 36 = 0$
3. $y^2 - x^2 - 4y + 2x - 6 = 0$
4. $4y^2 - 16x^2 - 24y + 96x - 172 = 0$
5. $9y^2 - x^2 + 18y - 4x - 4 = 0$

Answer

1.

a. $\frac{(x-1)^2}{4} - \frac{(y-1)^2}{9} = 1$

b.

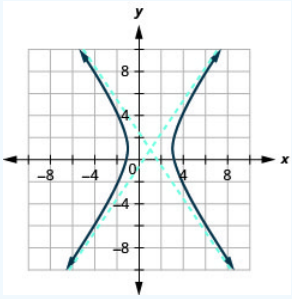

Figure 11.4.44

3.

a. $\frac{(y-2)^2}{9} - \frac{(x-1)^2}{9} = 1$

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b.

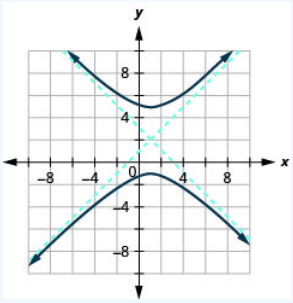

Figure 11.4.45

5.

a. $\frac{(y+1)^2}{1}-\frac{(x+2)^2}{9}=1$

b.

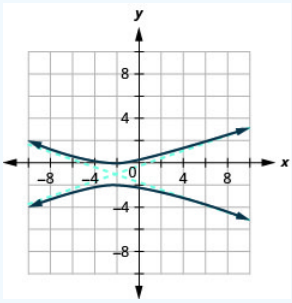

Figure 11.4.46

Exercise $22.5E.16$ Identify the Graph of each Equation as a Circle, Parabola, Ellipse, or Hyperbola

In the following exercises, identify the type of graph.

1. a. $x=-y^2-2y+3$
   b. $9y^2-x^2+18y-4x-4=0$
   c. $9x^2+25y^2=225$
   d. $x^2+y^2-4x+10y-7=0$
2. a. $x=-2y^2-12y-16$
   b. $x^2+y^2=9$
   c. $16x^2-4y^2+64x-24y-36=0$
   d. $16x^2+36y^2=576$

Answer

2.

a. Parabola
b. Circle
c. Hyperbola
d. Ellipse

Exercise $22.5E.17$ Mixed Practice

In the following exercises, graph each equation.

1. $\frac{(y-3)^2}{9}-\frac{(x+2)^2}{16}=1$

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2. $x^2+y^2-4x+10y-7=0$
3. $y=(x-1)^2+2$
4. $\frac{x^2}{9}+\frac{y^2}{25}=1$
5. $(x+2)^2+(y-5)^2=4$
6. $9x^2-4y^2+54x+8y+41=0$
7. $x=-y^2-2y+3$
8. $16x^2+9y^2=144$

Answer

2.

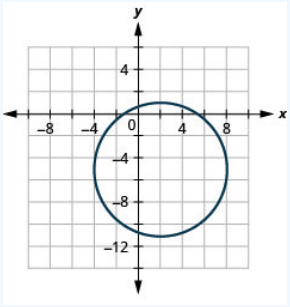

Figure 11.4.47

4.

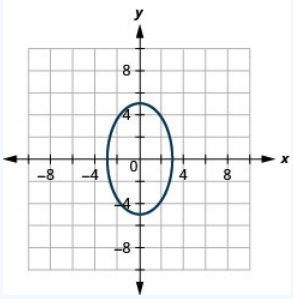

Figure 11.4.48

6.

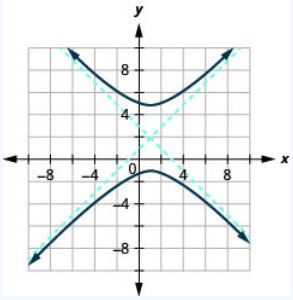

Figure 11.4.49

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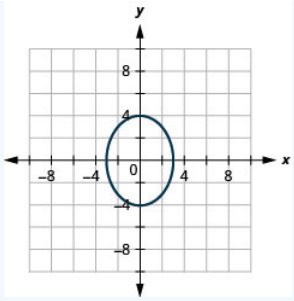

Figure 11.4.50

Exercise $22.5E.18$ Writing Exercises

1. In your own words, define a hyperbola and write the equation of a hyperbola centered at the origin in standard form. Draw a sketch of the hyperbola labeling the center, vertices, and asymptotes.
2. Explain in your own words how to create and use the rectangle that helps graph a hyperbola.
3. Compare and contrast the graphs of the equations $\frac{x^2}{4}-\frac{y^2}{9}=1$ and $\frac{y^2}{9}-\frac{x^2}{4}=1$.
4. Explain in your own words, how to distinguish the equation of an ellipse with the equation of a hyperbola.

Answer

2. Answers may vary

4. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph a hyperbola with center at (0, 0).			
graph a hyperbola with center at (*h*, *k*).			
identify conic sections by their equations.			

Figure 11.4.51

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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22.6: Solving Systems of Nonlinear Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve a system of nonlinear equations using graphing
- Solve a system of nonlinear equations using substitution
- Solve a system of nonlinear equations using elimination
- Use a system of nonlinear equations to solve applications

Before you get started, take this readiness quiz.

1. Solve the system by graphing: $\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$.
   If you missed this problem, review Example 4.2.
2. Solve the system by substitution: $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$
   If you missed this problem, review Example 4.7.
3. Solve the system by elimination: $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$
   If you missed this problem, review Example 4.9.

Solve a System of Nonlinear Equations using Graphing

We learned how to solve systems of linear equations with two variables by graphing, substitution and elimination. We will be using these same methods as we look at nonlinear systems of equations with two equations and two variables. A **system of nonlinear equations** is a system where at least one of the equations is not linear.

For example each of the following systems is a **system of nonlinear equations**.

$$\begin{cases} x^2+y^2=9 \\ x^2-y=9 \end{cases} \quad \begin{cases} 9x^2+y^2=9 \\ y=3x-3 \end{cases} \quad \begin{cases} x+y=4 \\ y=x^2+2 \end{cases}$$

Definition 22.6.1

A **system of nonlinear equations** is a system where at least one of the equations is not linear.

Just as with systems of linear equations, a solution of a nonlinear system is an ordered pair that makes both equations true. In a nonlinear system, there may be more than one solution. We will see this as we solve a system of nonlinear equations by graphing.

When we solved systems of linear equations, the solution of the system was the point of intersection of the two lines. With systems of nonlinear equations, the graphs may be circles, parabolas or hyperbolas and there may be several points of intersection, and so several solutions. Once you identify the graphs, visualize the different ways the graphs could intersect and so how many solutions there might be.

To solve systems of nonlinear equations by graphing, we use basically the same steps as with systems of linear equations modified slightly for nonlinear equations. The steps are listed below for reference.

Solve a System of Nonlinear Equations by Graphing.

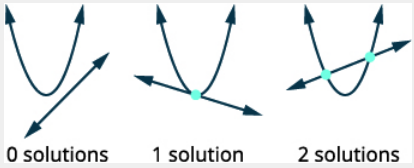
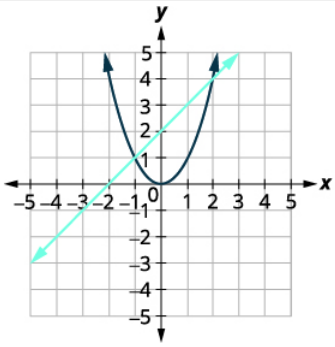
1. Identify the graph of each equation. Sketch the possible options for intersection.
2. Graph the first equation.
3. Graph the second equation on the same rectangular coordinate system.
4. Determine whether the graphs intersect.
5. Identify the points of intersection.
6. Check that each ordered pair is a solution to both original equations.

Example 22.6.1

Solve the system by graphing: $\begin{cases} x-y=-2 \\ y=x^2 \end{cases}$

Solution:

Table 11.5.1

Identify each graph.	$\begin{cases} x-y=-2 & \text{line} \\ y=x^2 & \text{parabola} \end{cases}$
Sketch the possible options for intersection of a parabola and a line.	
Graph the line, $x-y=-2$. Slope-intercept form $y=x+2$. Graph the parabola, $y=x^2$.	
Identify the points of intersection.	The points of intersection appear to be $(2,3)$ and $(-1,1)$.
Check to make sure each solution makes both equations true. $(2,4)$ $x-y=-2 \quad y=x^2$ $2-4\stackrel{?}{=}-2 \quad 4\stackrel{?}{=}2^2$ $-2=-2 \quad 4=4$ $(-1,1)$ $x-y=-2 \quad y=x^2$ $-1-1\stackrel{?}{=}-2 \quad 1\stackrel{?}{=}(-1)^2$ $-2=-2 \quad 1=1$	
	The solutions are $(2,4)$ and $(-1,1)$.

Exercise 22.6.1

Solve the system by graphing: $\begin{cases} x+y=4 \\ y=x^2+2 \end{cases}$.

Answer

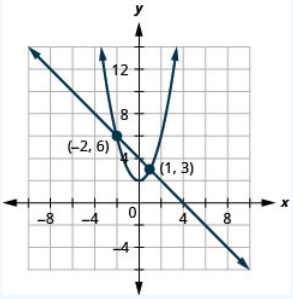


Figure 11.5.3

? Exercise 22.6.2

Solve the system by graphing: $\begin{cases} x-y=-1 \\ y=-x^2+3 \end{cases}$

Answer

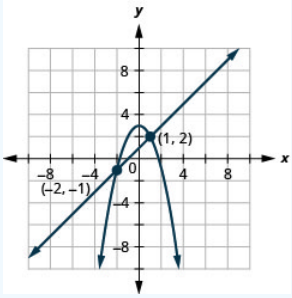


Figure 11.5.4

To identify the graph of each equation, keep in mind the characteristics of the x^2 and y^2 terms of each conic.

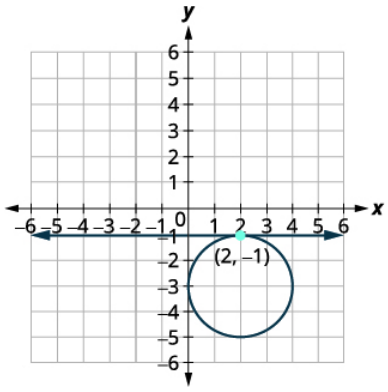
✓ Example 22.6.2

Solve the system by graphing: $\begin{cases} y=-1 \\ (x-2)^2+(y+3)^2=4 \end{cases}$.

Solution:

Table 11.5.2

Identify each graph.	$\begin{cases} y=-1 & \text{line} \\ (x-2)^2+(y+3)^2=4 & \text{circle} \end{cases}$
Sketch the possible options for the intersection of a circle and a line.	

Graph the circle, $(x-2)^2+(y+3)^2=4$ Center: $(2,-3)$ radius: 2 Graph the line, $y=-1$. It is a horizontal line.	
Identify the points of intersection.	The point of intersection appears to be$(2,-1)$.
Check to make sure the solution makes both equations true. $(2,-1)$ $(x-2)^2+(y+3)^2=4 \quad y=-1$ $(2-2)^2+(-1+3)^2 \stackrel{?}{=} 4 \quad -1=-1$ $(0)^2+(2)^2 \stackrel{?}{=} 4$ $4=4$	
	The solution is $(2,-1)$

Exercise 22.6.3

Solve the system by graphing: $\begin{cases} x=-6 \\ (x+3)^2+(y-1)^2=9 \end{cases}$

Answer

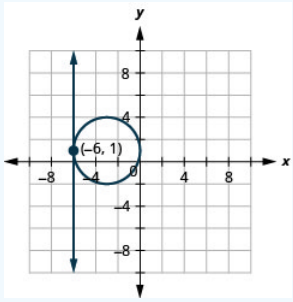


Figure 11.5.7

Exercise 22.6.4

Solve the system by graphing: $\begin{cases} y=4 \\ (x-2)^2+(y+3)^2=4 \end{cases}$

Answer

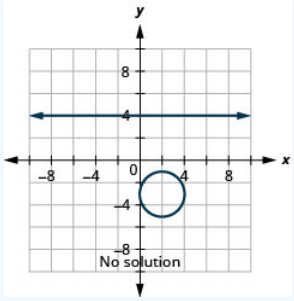


Figure 11.5.8

Solve a System of Nonlinear Equations Using Substitution

The graphing method works well when the points of intersection are integers and so easy to read off the graph. But more often it is difficult to read the coordinates of the points of intersection. The substitution method is an algebraic method that will work well in many situations. It works especially well when it is easy to solve one of the equations for one of the variables.

The substitution method is very similar to the substitution method that we used for systems of linear equations. The steps are listed below for reference.

Solve a System of Nonlinear Equations by Substitution

1. Identify the graph of each equation. Sketch the possible options for intersection.
2. Solve one of the equations for either variable.
3. Substitute the expression from Step 2 into the other equation.
4. Solve the resulting equation.
5. Substitute each solution in Step 4 into one of the original equations to find the other variable.
6. Write each solution as an ordered pair.
7. Check that each ordered pair is a solution to **both** original equations.

Example 22.6.3

Solve the system by using substitution: $\begin{cases} 9x^2+y^2=9 \\ y=3x-3 \end{cases}$

Solution:

Table 11.5.3

Identify each graph.	$\begin{cases} 9x^2+y^2=9 & \text{ellipse} \\ y=3x-3 & \text{line} \end{cases}$
Sketch the possible options for intersection of an ellipse and a line.	
The equation $y=3x-3$ is solved for y.	$y=3x-3$
	$9x^2+y^2=9$
Substitute $3x-3$ for y in the first equation.	$9x^2+(3x-3)^2=9$
Solve the equation for x.	$9x^2+9x^2-18x+9=9$
	$18x^2-18x=0$ $18x(x-1)=0$ $x=0 \quad x=1$
Substitute $x=0$ and $x=1$ into $y=3x-3$ to find y-.	$y=3x-3 \qquad y=3x-3$
	$y=3\cdot 0-3 \qquad y=3\cdot 1-3$ $y=-3 \qquad y=0$

	The ordered pairs are $(0,-3),(1,0)$.
Check **both** ordered pairs in **both** equations. $(0,-3)$ $9x^2+y^2=9 \quad y=3x-3$ $9\cdot 0^2+(-3)^2 \stackrel{?}{=} 9 \quad -3 \stackrel{?}{=} 3\cdot 0-3$ $0+9 \stackrel{?}{=} 9 \quad -3 \stackrel{?}{=} 0-3$ $9=9 \quad -3=-3$ $(1,0)$ $9x^2+y^2=9 \quad y=3x-3$ $9\cdot 1^2+(0)^2 \stackrel{?}{=} 9 \quad 0 \stackrel{?}{=} 3\cdot 1-3$ $9+0 \stackrel{?}{=} 9 \quad 0 \stackrel{?}{=} 3-3$ $9=9 \quad 0=0$	
	The solutions are $(0,-3),(1,0)$.

? Exercise 22.6.5

Solve the system by using substitution: $\begin{cases} x^2+9y^2=9 \\ y=\frac{1}{3}x-3 \end{cases}$

Answer

No solution

? Exercise 22.6.6

Solve the system by using substitution: $\begin{cases} 4x^2+y^2=4 \\ y=x+2 \end{cases}$

Answer

$\left(-\frac{4}{5},\frac{6}{5}\right),(0,2)$

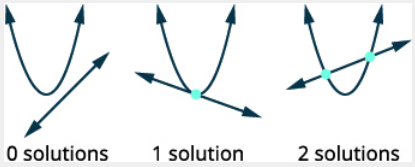
So far, each system of nonlinear equations has had at least one solution. The next example will show another option.

✓ Example 22.6.4

Solve the system by using substitution: $\begin{cases} x^2-y=0 \\ y=x-2 \end{cases}$

Solution:

Table 11.5.4

Identify each graph.	$\begin{cases} x^2-y=0 & \text{parabola} \\ y=x-2 & \text{line} \end{cases}$
Sketch the possible options for intersection of a parabola and a line.	
The equation $y=x-2$ is solved for y.	$y=x-2$

	$x^2 - y = 0$
Substitute $x - 2$ for y in the first equation.	$x^2 - (x - 2) = 0$
Solve the equation for x.	$x^2 - x + 2 = 0$
This doesn't factor easily, so we can check the discriminant.	
$b^2 - 4ac$ $(-1)^2 - 4 \cdot 1 \cdot 2$ -7	The discriminant is negative, so there is no real solution. The system has no solution.

Exercise 22.6.7

Solve the system by using substitution: $\begin{cases} x^2 - y = 0 \\ y = 2x - 3 \end{cases}$

Answer

No solution

Exercise 22.6.8

Solve the system by using substitution: $\begin{cases} y^2 - x = 0 \\ y = 3x - 2 \end{cases}$

Answer

$\left(\frac{4}{9}, -\frac{2}{3}\right), (1, 1)$

Solve a System of Nonlinear Equations Using Elimination

When we studied systems of linear equations, we used the method of elimination to solve the system. We can also use elimination to solve systems of nonlinear equations. It works well when the equations have both variables squared. When using elimination, we try to make the coefficients of one variable to be opposites, so when we add the equations together, that variable is eliminated.

The elimination method is very similar to the elimination method that we used for systems of linear equations. The steps are listed for reference.

Solve a System of Equations by Elimination

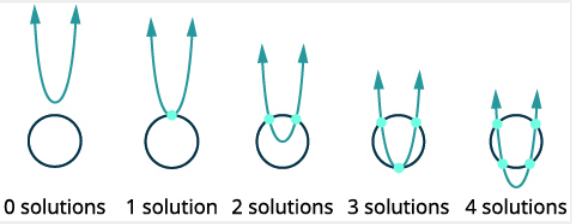
1. Identify the graph of each equation. Sketch the possible options for intersection.
2. Write both equations in standard form.
3. Make the coefficients of one variable opposites.
   Decide which variable you will eliminate.
   Multiply one or both equations so that the coefficients of that variable are opposites.
4. Add the equations resulting from Step 3 to eliminate one variable.
5. Solve for the remaining variable.
6. Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable.
7. Write each solution as an ordered pair.
8. Check that each ordered pair is a solution to **both** original equations.

Example 22.6.5

Solve the system by elimination: $\begin{cases} x^2 + y^2 = 4 \\ x^2 - y = 4 \end{cases}$

Solution:

Table 11.5.5

Identify each graph.	$\begin{cases} x^2+y^2=4 & \text{circle} \\ x^2-y=4 & \text{parabola} \end{cases}$
Sketch the possible options for intersection of a circle and a parabola.	
Both equations are in standard form.	$\begin{cases} x^2+y^2=4 \\ x^2-y=4 \end{cases}$
To get opposite coefficients of x^2, we will multiply the second equation by -1.	$\begin{cases} x^2+y^2=4 \\ -1(x^2-y)=-1(4) \end{cases}$
Simplify.	$\begin{cases} x^2+y^2=4 \\ -x^2+y=4 \end{cases}$
Add the two equation to eliminate x^2/	$\begin{cases} x^2+y^2=4 \\ -x^2+y=4 \\ \hline y^2+y=0 \end{cases}$
Solve for y.	$y(y+1)=0$
	$y=0 \quad y+1=0$ $y=-1$
Substitute $y=0$ and $y=-1$ into one of the original equations. Then solve for x.	$y=0 \quad y=-1$
	$x^2-y=4$ $x^2-0=4$ $x^2=4$ $x=\pm 2$ $x^2-y=4$ $x^2-(-1)=4$ $x^2=3$ $x=\pm\sqrt{3}$
Write each solution as an ordered pair.	The ordered pairs are $(-2,0)(2,0).\ (\sqrt{3},-1)(-\sqrt{3},-1)$
Check that each ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four solutions to you.	The solutions are $(-2,0)$, $(2,0)$, $(\sqrt{3},-1)$, and $(-\sqrt{3},-1)$.

Exercise 22.6.9

Solve the system by elimination: $\begin{cases} x^2+y^2=9 \\ x^2-y=9 \end{cases}$

Answer

$(-3,0),(3,0),(-2\sqrt{2},-1),(2\sqrt{2},-1)$

Exercise 22.6.10

Solve the system by elimination: $\begin{cases} x^2+y^2=1 \\ -x+y^2=1 \end{cases}$

Answer

$(-1,0),(0,1),(0,-1)$

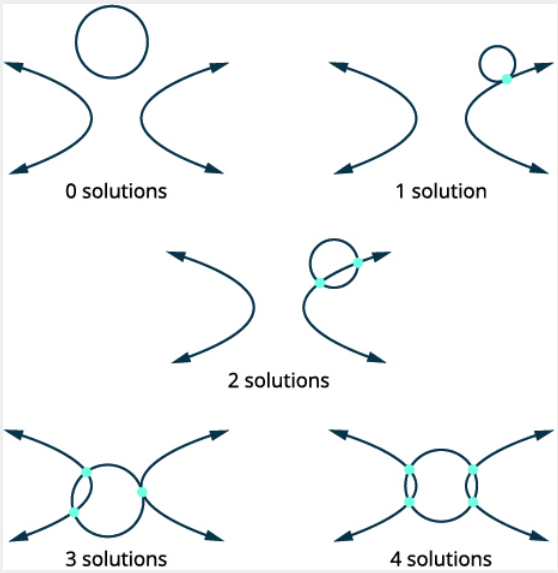
There are also four options when we consider a circle and a hyperbola.

Example 22.6.6

Solve the system by elimination: $\begin{cases} x^2+y^2=7 \\ x^2-y^2=1 \end{cases}$

Solution:

Table 11.5.6

Identify each graph.	$\begin{cases} x^2+y^2=7 & \text{circle} \\ x^2-y^2=1 & \text{hyperbola} \end{cases}$
Sketch the possible options for intersection of a circle and hyperbola.	
Both equations are in standard form.	$\begin{cases} x^2+y^2=7 \\ x^2-y^2=1 \end{cases}$
The coefficients of y^2 are opposite, so we will add the equations.	$\begin{cases} x^2+y^2=7 \\ x^2-y^2=1 \end{cases}$ $2x^2=8$
Simplify.	$x^2=4$ $x=\pm 2$ $x=2 \quad x=-2$
Substitue $x=2$ and $x=-2$ into one of the original equations. Then solve for y.	$x^2+y^2=7$ $2^2+y^2=7$ $4+y^2=7$ $y^2=3$ $y=\pm\sqrt{3}$ $x^2+y^2=7$ $(-2)^2+y^2=7$ $4+y^2=7$ $y^2=3$ $y=\pm\sqrt{3}$
Write each solution as an ordered pair.	The ordered pairs are $(-2,\sqrt{3})$, $(-2,-\sqrt{3})$, $(2,\sqrt{3})$, and $(2,-\sqrt{3})$.
Check that the ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four solutions to you.	The solutions are $(-2,\sqrt{3})$, $(-2,-\sqrt{3})$, $(2,\sqrt{3})$, and $(2,-\sqrt{3})$.

Exercise 22.6.11

Solve the system by elimination: $\begin{cases} x^2+y^2=25 \\ y^2-x^2=7 \end{cases}$

Answer

$(-3,-4),(-3,4),(3,-4),(3,4)$

Exercise 22.6.12

Solve the system by elimination: $\begin{cases} x^2+y^2=4 \\ x^2-y^2=4 \end{cases}$

Answer

$(-2,0),(2,0)$

Use a System of Nonlinear Equations to Solve Applications

Systems of nonlinear equations can be used to model and solve many applications. We will look at an everyday geometric situation as our example.

Example 22.6.7

The difference of the squares of two numbers is 15. The sum of the numbers is 5. Find the numbers.

Solution:

Table 11.5.7

Identify what we are looking for.	Two different numbers.
Define the variables.	x=first number y=second number
Translate the information into a system of equations.	
First sentence.	The difference of the squares of two numbers is 15.
	$x^2-y^2=15$
Second sentence.	The sum of the numbers is 5.
	$x+y=5$
Solve the system by substitution.	$\begin{cases} x^2-y^2=15 \\ x+y=5 \end{cases}$
Solve the second equation for x.	$x=5-y$
Substitute x into the first equation.	$x^2-y^2=15$
	$(5-y)^2-y^2=15$
Expand and simplify.	$(25-10y+y^2)-y^2=15$
	$25-10y+y^2-y^2=15$ $25-10y=15$
Solve for y.	$-10y=-10$
	$y=1$
Substitute back into the second equation.	$x+y=5$

	$x+(1)=5$ $x=4$
	The numbers are 1 and 4.

Exercise 22.6.13

The difference of the squares of two numbers is -20. The sum of the numbers is 10. Find the numbers.

Answer

4 and 6

Exercise 22.6.14

The difference of the squares of two numbers is 35. The sum of the numbers is -1. Find the numbers.

Answer

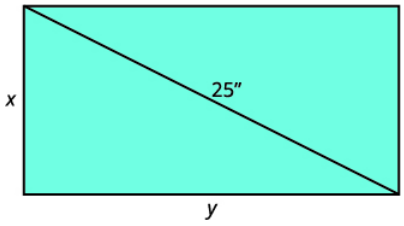
-18 and 17

Example 22.6.8

Myra purchased a small 25” TV for her kitchen. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 300 square inches. What are the length and width of the TV screen?

Solution:

Table 11.5.8

Identify what we are looking for.	The length and width of the rectangle.
Define the variables.	Let x= width of the rectangle y=length of the rectangle
Draw a diagram to help visualize the situation.	
	Area is 300 square inches.
Translate the information into a system of equations.	The diagonal of the right triangle is 25 inches.
	$x^2+y^2=25^2$ $x^2+y^2=625$
	The area of the rectangle is 300 square inches.
	$x \cdot y=300$ $x^2+y^2=625$ $x \cdot y=300$
Solve the system using substitution.	$x \cdot y=300$
Solve the second equation for x.	$x=\frac{300}{y}$
Substitute x into the first equation.	$x^2+y^2=625$

	$\left(\frac{300}{y}\right)^2 + y^2 = 625$
Simplify.	$\frac{90000}{y^2} + y^2 = 625$
Multiply by y^2 to clear the fractions.	$90000 + y^4 = 625y^2$
Put in standard form.	$y^4 - 625y^2 + 90000 = 0$
Solve by factoring.	$(y^2 - 225)(y^2 - 400) = 0$
	$y^2 - 225 = 0 \quad y^2 - 400 = 0$
	$y^2 = 225 \quad y^2 = 400$ $y = \pm 15 \quad y = \pm 20$
Since y is a side of the rectangle, we discard the negative values.	$y = 15 \quad y = 20$
Substitute back into the second equation.	$x \cdot y = 300 \quad x \cdot y = 300$
	$x \cdot 15 = 300 \quad x \cdot 20 = 300$ $x = 20 \quad x = 15$
	If the length is 15 inches, the width is 20 inches.
	If the length is 20 inches, the width is 15 inches.

? Exercise 22.6.15

Edgar purchased a small 20” TV for his garage. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 192 square inches. What are the length and width of the TV screen?

Answer

If the length is 12 inches, the width is 16 inches. If the length is 16 inches, the width is 12 inches.

? Exercise 22.6.16

The Harper family purchased a small microwave for their family room. The diagonal of the door measures 15 inches. The door also has an area of 108 square inches. What are the length and width of the microwave door?

Answer

If the length is 12 inches, the width is 9 inches. If the length is 9 inches, the width is 12 inches.

Access these online resources for additional instructions and practice with solving nonlinear equations.

- Nonlinear Systems of Equations
- Solve a System of Nonlinear Equations
- Solve a System of Nonlinear Equations by Elimination
- System of Nonlinear Equations – Area and Perimeter Application

Key Concepts

- **How to solve a system of nonlinear equations by graphing.**
  1. Identify the graph of each equation. Sketch the possible options for intersection.
  2. Graph the first equation.
  3. Graph the second equation on the same rectangular coordinate system.
  4. Determine whether the graphs intersect.
  5. Identify the points of intersection.

6. Check that each ordered pair is a solution to both original equations.

- **How to solve a system of nonlinear equations by substitution.**
  1. Identify the graph of each equation. Sketch the possible options for intersection.
  2. Solve one of the equations for either variable.
  3. Substitute the expression from Step 2 into the other equation.
  4. Solve the resulting equation.
  5. Substitute each solution in Step 4 into one of the original equations to find the other variable.
  6. Write each solution as an ordered pair.
  7. Check that each ordered pair is a solution to **both** original equations.
- **How to solve a system of equations by elimination.**
  1. Identify the graph of each equation. Sketch the possible options for intersection.
  2. Write both equations in standard form.
  3. Make the coefficients of one variable opposites.
     Decide which variable you will eliminate.
     Multiply one or both equations so that the coefficients of that variable are opposites.
  4. Add the equations resulting from Step 3 to eliminate one variable.
  5. Solve for the remaining variable.
  6. Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable.
  7. Write each solution as an ordered pair.
  8. Check that each ordered pair is a solution to **both** original equations.

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22.6E: Exercises

Practice Makes Perfect

Exercise 22.6E.17 Solve a System of Nonlinear Equations Using Graphing

In the following exercises, solve the system of equations by using graphing.

1. $\begin{cases} y = 2x + 2 \\ y = -x^2 + 2 \end{cases}$
2. $\begin{cases} y = 6x - 4 \\ y = 2x^2 \end{cases}$
3. $\begin{cases} x + y = 2 \\ x = y^2 \end{cases}$
4. $\begin{cases} x - y = -2 \\ x = y^2 \end{cases}$
5. $\begin{cases} y = \frac{3}{2}x + 3 \\ y = -x^2 + 2 \end{cases}$
6. $\begin{cases} y = x - 1 \\ y = x^2 + 1 \end{cases}$
7. $\begin{cases} x = -2 \\ x^2 + y^2 = 4 \end{cases}$
8. $\begin{cases} y = -4 \\ x^2 + y^2 = 16 \end{cases}$
9. $\begin{cases} x = 2 \\ (x+2)^2 + (y+3)^2 = 16 \end{cases}$
10. $\begin{cases} y = -1 \\ (x-2)^2 + (y-4)^2 = 25 \end{cases}$
11. $\begin{cases} y = -2x + 4 \\ y = \sqrt{x} + 1 \end{cases}$
12. $\begin{cases} y = -\frac{1}{2}x + 2 \\ y = \sqrt{x} - 2 \end{cases}$

Answer

2.

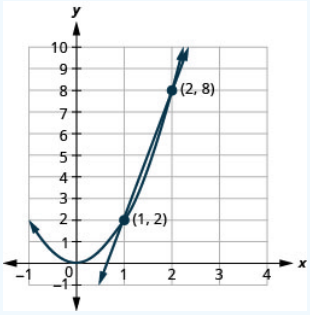

Figure 11.5.61

4.

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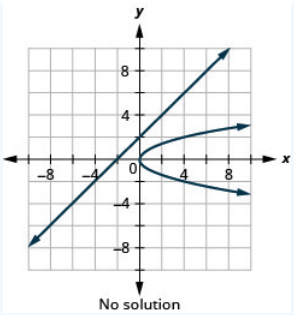

Figure 11.5.62

6.

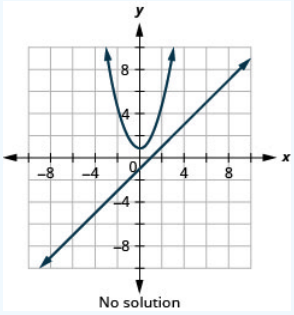

Figure 11.5.63

8.

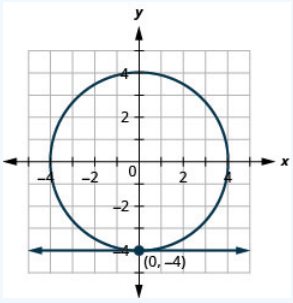

Figure 11.5.64

10.

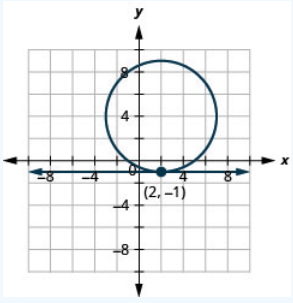

Figure 11.5.65

12.

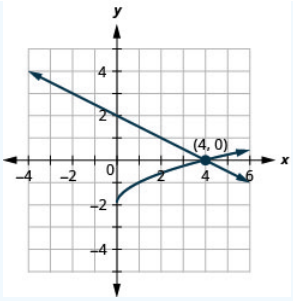

Figure 11.5.66

Exercise $22.6E.18$ Solve a System of Nonlinear Equations Using Substitution

In the following exercises, solve the system of equations by using substitution.

1. $\begin{cases} x^2+4y^2=4 \\ y=\frac{1}{2}x-1 \end{cases}$
2. $\begin{cases} 9x^2+y^2=9 \\ y=3x+3 \end{cases}$
3. $\begin{cases} 9x^2+y^2=9 \\ y=x+3 \end{cases}$
4. $\begin{cases} 9x^2+4y^2=36 \\ x=2 \end{cases}$
5. $\begin{cases} 4x^2+y^2=4 \\ y=4 \end{cases}$
6. $\begin{cases} x^2+y^2=169 \\ x=12 \end{cases}$
7. $\begin{cases} 3x^2-y=0 \\ y=2x-1 \end{cases}$
8. $\begin{cases} 2y^2-x=0 \\ y=x+1 \end{cases}$
9. $\begin{cases} y=x^2+3 \\ y=x+3 \end{cases}$
10. $\begin{cases} y=x^2-4 \\ y=x-4 \end{cases}$
11. $\begin{cases} x^2+y^2=25 \\ x-y=1 \end{cases}$
12. $\begin{cases} x^2+y^2=25 \\ 2x+y=10 \end{cases}$

Answer

2. $(-1,0),(0,3)$
4. $(2,0)$
6. $(12,-5),(12,5)$
8. No solution
10. $(0,-4),(1,-3)$
12. $(3,4),(5,0)$

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Exercise $22.6E.19$ Solve a System of Nonlinear Equations Using Elimination

In the following exercises, solve the system of equations by using elimination.

1. $\begin{cases} x^2+y^2=16 \\ x^2-2y=8 \end{cases}$
2. $\begin{cases} x^2+y^2=16 \\ x^2-y=4 \end{cases}$
3. $\begin{cases} x^2+y^2=4 \\ x^2+2y=1 \end{cases}$
4. $\begin{cases} x^2+y^2=4 \\ x^2-y=2 \end{cases}$
5. $\begin{cases} x^2+y^2=9 \\ x^2-y=3 \end{cases}$
6. $\begin{cases} x^2+y^2=4 \\ y^2-x=2 \end{cases}$
7. $\begin{cases} x^2+y^2=25 \\ 2x^2-3y^2=5 \end{cases}$
8. $\begin{cases} x^2+y^2=20 \\ x^2-y^2=-12 \end{cases}$
9. $\begin{cases} x^2+y^2=13 \\ x^2-y^2=5 \end{cases}$
10. $\begin{cases} x^2+y^2=16 \\ x^2-y^2=16 \end{cases}$
11. $\begin{cases} 4x^2+9y^2=36 \\ 2x^2-9y^2=18 \end{cases}$
12. $\begin{cases} x^2-y^2=3 \\ 2x^2+y^2=6 \end{cases}$
13. $\begin{cases} 4x^2-y^2=4 \\ 4x^2+y^2=4 \end{cases}$
14. $\begin{cases} x^2-y^2=-5 \\ 3x^2+2y^2=30 \end{cases}$
15. $\begin{cases} x^2-y^2=1 \\ x^2-2y=4 \end{cases}$
16. $\begin{cases} 2x^2+y^2=11 \\ x^2+3y^2=28 \end{cases}$

Answer

2. $(0,-4),(-\sqrt{7},3),(\sqrt{7},3)$
4. $(0,-2),(-\sqrt{3},1),(\sqrt{3},1)$
6. $(-2,0),(1,-\sqrt{3}),(1,\sqrt{3})$
8. $(-2,-4),(-2,4),(2,-4),(2,4)$
10. $(-4,0),(4,0)$
12. $(-\sqrt{3},0),(\sqrt{3},0)$
14. $(-2,-3),(-2,3),(2,-3),(2,3)$
16. $(-1,-3),(-1,3),(1,-3),(1,3)$

Exercise $22.6E.20$ Use a System of Nonlinear Equations to Solve Applications

In the following exercises, solve the problem using a system of equations.

1. The sum of two numbers is -6 and the product is 8. Find the numbers.
2. The sum of two numbers is 11 and the product is -42. Find the numbers.
3. The sum of the squares of two numbers is 65. The difference of the number is 3. Find the numbers.
4. The sum of the squares of two numbers is 113. The difference of the number is 1. Find the numbers.
5. The difference of the squares of two numbers is 15. The difference of twice the square of the first number and the square of the second number is 30. Find the numbers.
6. The difference of the squares of two numbers is 20. The difference of the square of the first number and twice the square of the second number is 4. Find the numbers.
7. The perimeter of a rectangle is 32 inches and its area is 63 square inches. Find the length and width of the rectangle.
8. The perimeter of a rectangle is 52 cm and its area is $165\,\text{cm}^2$. Find the length and width of the rectangle.
9. Dion purchased a new microwave. The diagonal of the door measures 17 inches. The door also has an area of 120 square inches. What are the length and width of the microwave door?
10. Jules purchased a microwave for his kitchen. The diagonal of the front of the microwave measures 26 inches. The front also has an area of 240 square inches. What are the length and width of the microwave?
11. Roman found a widescreen TV on sale, but isn't sure if it will fit his entertainment center. The TV is 60". The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1728 square inches. His entertainment center has an insert for the TV with a length of 50 inches and width of 40 inches. What are the length and width of the TV screen and will it fit Roman's entertainment center?
12. Donnette found a widescreen TV at a garage sale, but isn't sure if it will fit her entertainment center. The TV is 50". The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1200 square inches. Her entertainment center has an insert for the TV with a length of 38 inches and width of 27 inches. What are the length and width of the TV screen and will it fit Donnette's entertainment center?

Answer

2. -3 and 14

4. -7 and -8 or 8 and 7

6. -6 and -4 or -6 and 4 or 6 and -4 or 6 and 4

8. If the length is 11 cm, the width is 15 cm. If the length is 15 cm, the width is 11 cm.

10. If the length is 10 inches, the width is 24 inches. If the length is 24 inches, the width is 10 inches.

12. The length is 40 inches and the width is 30 inches. The TV will not fit Donnette's entertainment center.

Exercise $22.6E.21$ Writing Exercises

1. In your own words, explain the advantages and disadvantages of solving a system of equations by graphing.
2. Explain in your own words how to solve a system of equations using substitution.
3. Explain in your own words how to solve a system of equations using elimination.
4. A circle and a parabola can intersect in ways that would result in $0, 1, 2, 3,$ or 4 solutions. Draw a sketch of each of the possibilities.

Answer

2. Answers may vary

4. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve a system of nonlinear equations using graphing.			
solve a system of nonlinear equations using substitution.			
solve a system of nonlinear equations using elimination.			
use a system of nonlinear equations to solve applications.			

Figure 11.5.67

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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22.7: Review Exercises

Chapter Review Exercises

Distance and Midpoint Formulas; Circles

Exercise 22.7.1 Use the Distance Formula

In the following exercises, find the distance between the points. Round to the nearest tenth if needed.

1. $(-5, 1)$ and $(-1, 4)$
2. $(-2, 5)$ and $(1, 5)$
3. $(8, 2)$ and $(-7, -3)$
4. $(1, -4)$ and $(5, -5)$

Answer

2. $d = 3$
4. $d = \sqrt{17}, d \approx 4.1$

Exercise 22.7.2 Use the Midpoint Formula

In the following exercises, find the midpoint of the line segments whose endpoints are given.

1. $(-2, -6)$ and $(-4, -2)$
2. $(3, 7)$ and $(5, 1)$
3. $(-8, -10)$ and $(9, 5)$
4. $(-3, 2)$ and $(6, -9)$

Answer

2. $(4, 4)$
4. $\left(\frac{3}{2}, -\frac{7}{2}\right)$

Exercise 22.7.3 Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given information.

1. radius is 15 and center is $(0, 0)$
2. radius is $\sqrt{7}$ and center is $(0, 0)$
3. radius is 9 and center is $(-3, 5)$
4. radius is 7 and center is $(-2, -5)$
5. center is $(3, 6)$ and a point on the circle is $(3, -2)$
6. center is $(2, 2)$ and a point on the circle is $(4, 4)$

Answer

2. $x^2 + y^2 = 7$
4. $(x + 2)^2 + (y + 5)^2 = 49$
6. $(x - 2)^2 + (y - 2)^2 = 8$

Exercise 22.7.4 Graph a Circle

In the following exercises,

a. Find the center and radius, then
b. Graph each circle.

1. $2x^2+2y^2=450$
2. $3x^2+3y^2=432$
3. $(x+3)^2+(y-5)^2=81$
4. $(x+2)^2+(y+5)^2=49$
5. $x^2+y^2-6x-12y-19=0$
6. $x^2+y^2-4y-60=0$

Answer

2.

a. radius: 12, center: $(0,0)$
b.

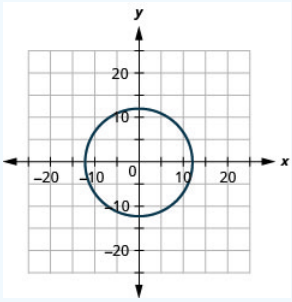

Figure 11.E.1

4.

a. radius: 7, center: $(-2,-5)$
b.

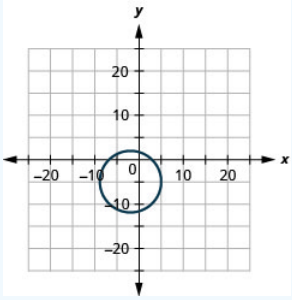

Figure 11.E.2

6.

a. radius: 8, center: $(0,2)$
b.

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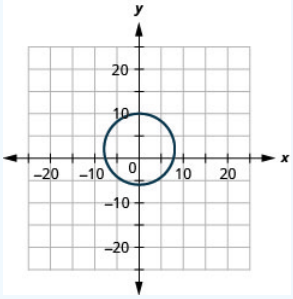

Figure 11.E.3

Parabolas

Exercise 22.7.5 Graph Vertical Parabolas

In the following exercises, graph each equation by using its properties.

1. $y = x^2 + 4x - 3$
2. $y = 2x^2 + 10x + 7$
3. $y = -6x^2 + 12x - 1$
4. $y = -x^2 + 10x$

Answer

2.

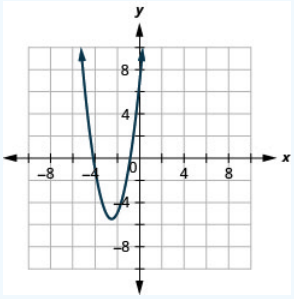

Figure 11.E.4

4.

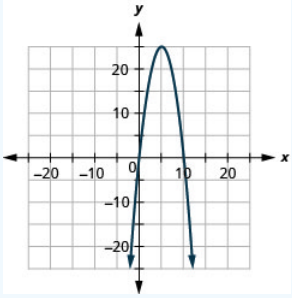

Figure 11.E.5

Exercise 22.7.6 Graph Vertical Parabolas

In the following exercises,

a. Write the equation in standard form, then
b. Use properties of the standard form to graph the equation.

1. $y = x^2 + 4x + 7$

2. $y = 2x^2 - 4x - 2$
3. $y = -3x^2 - 18x - 29$
4. $y = -x^2 + 12x - 35$

Answer

2.

a. $y = 2(x-1)^2 - 4$

b.

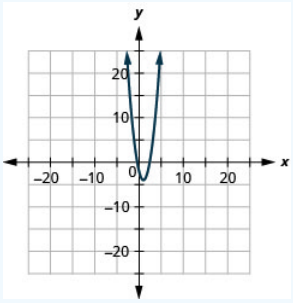

Figure 11.E.6

4.

a. $y = -(x-6)^2 + 1$

b.

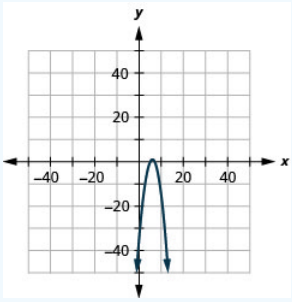

Figure 11.E.7

Exercise 22.7.7 Graph Horizontal Parabolas

In the following exercises, graph each equation by using its properties.

1. $x = 2y^2$
2. $x = 2y^2 + 4y + 6$
3. $x = -y^2 + 2y - 4$
4. $x = -3y^2$

Answer

2.

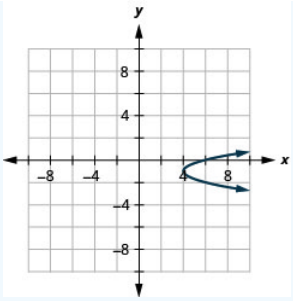

Figure 11.E.8

4.

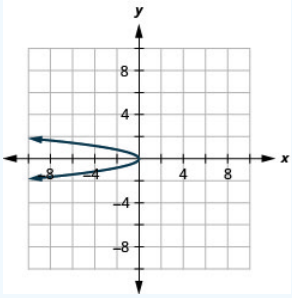

Figure 11.E.9

Exercise 22.7.8 Graph Horizontal Parabolas

In the following exercises,

a. Write the equation in standard form, then
b. Use properties of the standard form to graph the equation.

1. $x = 4y^2 + 8y$
2. $x = y^2 + 4y + 5$
3. $x = -y^2 - 6y - 7$
4. $x = -2y^2 + 4y$

Answer

2.

a. $x = (y+2)^2 + 1$
b.

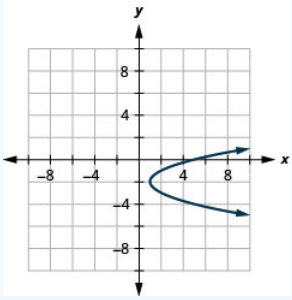

Figure 11.E.10

4.

a. $x = -2(y-1)^2 + 2$
b.

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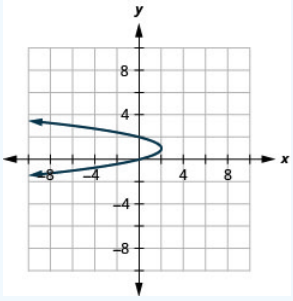

Figure 11.E.11

Exercise 22.7.9 Solve Applications with Parabolas

In the following exercises, create the equation of the parabolic arch formed in the foundation of the bridge shown. Give the answer in standard form.

1.

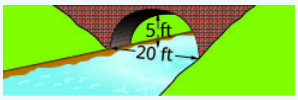

Figure 11.E.12

2.

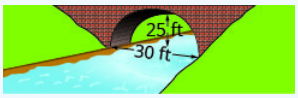

Figure 11.E.13

Answer

2. $y=-\frac{1}{9}x^2+\frac{10}{3}x$

Ellipses

Exercise 22.7.10 Graph an Ellipse with Center at the Origin

In the following exercises, graph each ellipse.

1. $\frac{x^2}{36}+\frac{y^2}{25}=1$
2. $\frac{x^2}{4}+\frac{y^2}{81}=1$
3. $49x^2+64y^2=3136$
4. $9x^2+y^2=9$

Answer

2.

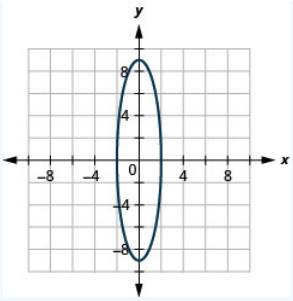

Figure 11.E.14

4.

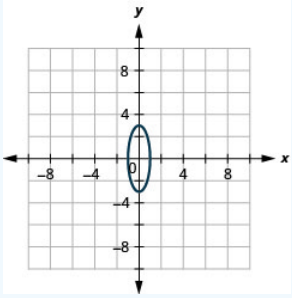

Figure 11.E.15

Exercise 22.7.11 Find the Equation of an Ellipse with Center at the Origin

In the following exercises, find the equation of the ellipse shown in the graph.

1.

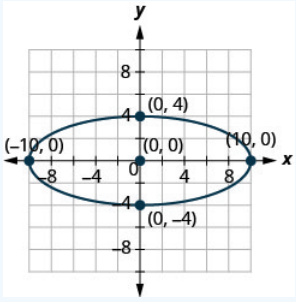

Figure 11.E.16

2.

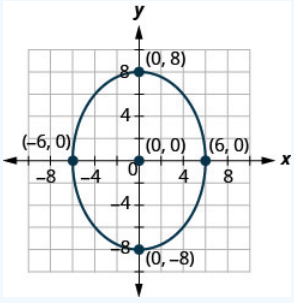

Figure 11.E.17

Answer

2. $\frac{x^2}{36}+\frac{y^2}{64}=1$

Exercise 22.7.12 Graph an Ellipse with Center Not at the Origin

In the following exercises, graph each ellipse.

1. $\frac{(x-1)^2}{25}+\frac{(y-6)^2}{4}=1$
2. $\frac{(x+4)^2}{16}+\frac{(y+1)^2}{9}=1$
3. $\frac{(x-5)^2}{16}+\frac{(y+3)^2}{36}=1$
4. $\frac{(x+3)^2}{9}+\frac{(y-2)^2}{25}=1$

Answer

2.

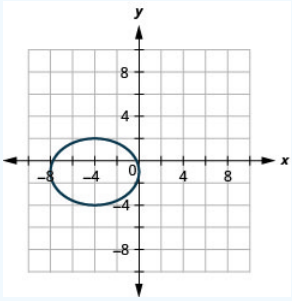

Figure 11.E.18

4.

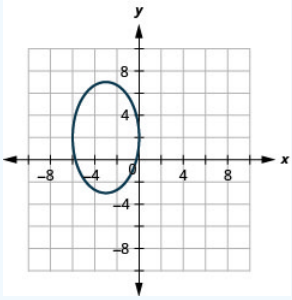

Figure 11.E.19

Exercise 22.7.13 Graph an Ellipse with Center Not at the Origin

In the following exercises,

a. Write the equation in standard form and
b. Graph.

1. $x^2+y^2+12x+40y+120=0$
2. $25x^2+4y^2-150x-56y+321=0$
3. $25x^2+4y^2+150x+125=0$
4. $4x^2+9y^2-126x+405=0$

Answer

2.

a. $\frac{(x-3)^2}{4}+\frac{(y-7)^2}{25}=1$

b.

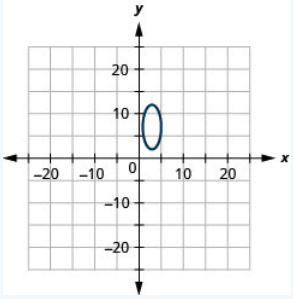

Figure 11.E.20

4.

a. $\frac{x^2}{9}+\frac{(y-7)^2}{4}=1$

b.

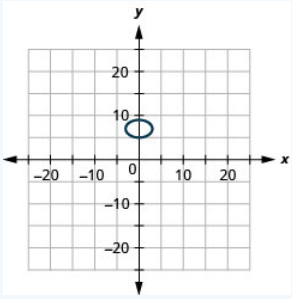

Figure 11.E.21

Exercise 22.7.14 Solve Applications with Ellipses

In the following exercises, write the equation of the ellipse described.

1. A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 10 AU and the furthest is approximately 90 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.

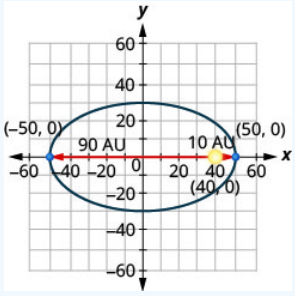

Figure 11.E.22

Answer

1. Solve

Hyperbolas

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Exercise 22.7.15 Graph a Hyperbola with Center at $(0,0)$

In the following exercises, graph.

1. $\frac{x^2}{25}-\frac{y^2}{9}=1$
2. $\frac{y^2}{49}-\frac{x^2}{16}=1$
3. $9y^2-16x^2=144$
4. $16x^2-4y^2=64$

Answer

1.

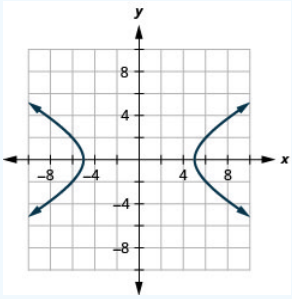

Figure 11.E.23

3.

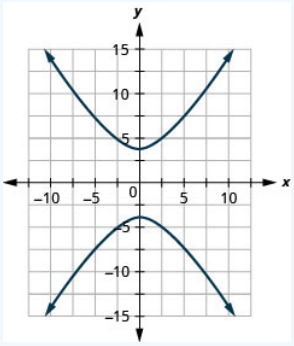

Figure 11.E.24

Exercise 22.7.16 Graph a Hyperbola with Center at (h,k)

In the following exercises, graph.

1. $\frac{(x+1)^2}{4}-\frac{(y+1)^2}{9}=1$
2. $\frac{(x-2)^2}{4}-\frac{(y-3)^2}{16}=1$
3. $\frac{(y+2)^2}{9}-\frac{(x+1)^2}{9}=1$
4. $\frac{(y-1)^2}{25}-\frac{(x-2)^2}{9}=1$

Answer

1.

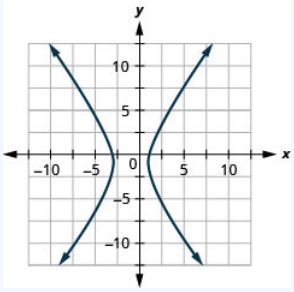

Figure 11.E.25

3.

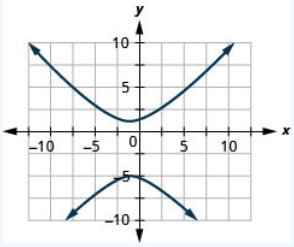

Figure 11.E.26

Exercise 22.7.17 Graph a Hyperbola with Center at (h, k)

In the following exercises,

a. Write the equation in standard form and
b. Graph.

1. $4x^2 - 16y^2 + 8x + 96y - 204 = 0$
2. $16x^2 - 4y^2 - 64x - 24y - 36 = 0$
3. $4y^2 - 16x^2 + 32x - 8y - 76 = 0$
4. $36y^2 - 16x^2 - 96x + 216y - 396 = 0$

Answer

1.

a. $\frac{(x+1)^2}{16} - \frac{(y-3)^2}{4} = 1$

b.

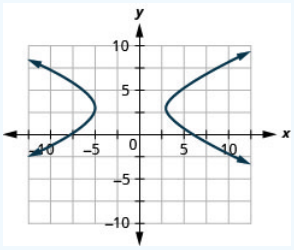

Figure 11.E.27

3.

a. $\frac{(y-1)^2}{16} - \frac{(x-1)^2}{4} = 1$

b.

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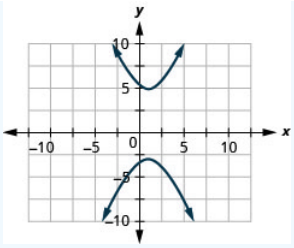

Figure 11.E.28

Exercise 22.7.18 Identify the Graph of Each Equation as a Circle, Parabola, Ellipse, or Hyperbola

In the following exercises, identify the type of graph.

1. a. $16y^2-9x^2-36x-96y-36=0$
   b. $x^2+y^2-4x+10y-7=0$
   c. $y=x^2-2x+3$
   d. $25x^2+9y^2=225$
2. a. $x^2+y^2+4x-10y+25=0$
   b. $y^2-x^2-4y+2x-6=0$
   c. $x=-y^2-2y+3$
   d. $16x^2+9y^2=144$

Answer

1.

a. Hyperbola
b. Circle
c. Parabola
d. Ellipse

Solve Systems of Nonlinear Equations

Exercise 22.7.19 Solve a System of Nonlinear Equations Using Graphing

In the following exercises, solve the system of equations by using graphing.

1. $\begin{cases} 3x^2-y=0 \\ y=2x-1 \end{cases}$
2. $\begin{cases} y=x^2-4 \\ y=x-4 \end{cases}$
3. $\begin{cases} x^2+y^2=169 \\ x=12 \end{cases}$
4. $\begin{cases} x^2+y^2=25 \\ y=-5 \end{cases}$

Answer

1.

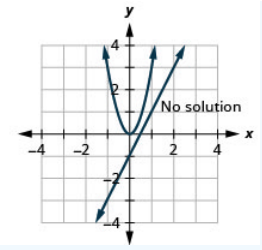

Figure 11.E.29

3.

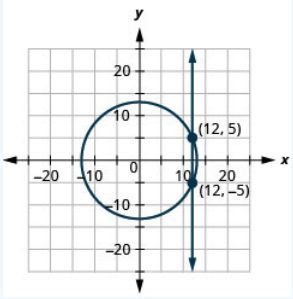

Figure 11.E.30

Exercise 22.7.20 Solve a System of Nonlinear Equations Using Substitution

In the following exercises, solve the system of equations by using substitution.

1. $\begin{cases} y = x^2 + 3 \\ y = -2x + 2 \end{cases}$
2. $\begin{cases} x^2 + y^2 = 4 \\ x - y = 4 \end{cases}$
3. $\begin{cases} 9x^2 + 4y^2 = 36 \\ y - x = 5 \end{cases}$
4. $\begin{cases} x^2 + 4y^2 = 4 \\ 2x - y = 1 \end{cases}$

Answer

1. $(-1, 4)$
3. No solution

Exercise 22.7.21 Solve a System of Nonlinear Equations Using Elimination

In the following exercises, solve the system of equations by using elimination.

1. $\begin{cases} x^2 + y^2 = 16 \\ x^2 - 2y - 1 = 0 \end{cases}$
2. $\begin{cases} x^2 - y^2 = 5 \\ -2x^2 - 3y^2 = -30 \end{cases}$
3. $\begin{cases} 4x^2 + 9y^2 = 36 \\ 3y^2 - 4x = 12 \end{cases}$
4. $\begin{cases} x^2 + y^2 = 14 \\ x^2 - y^2 = 16 \end{cases}$

Answer

1. $(-\sqrt{7}, 3), (\sqrt{7}, 3)$

3. $(-3, 0), (0, -2), (0, 2)$

Exercise 22.7.22 Use a System of Nonlinear Equations to Solve Applications

In the following exercises, solve the problem using a system of equations.

1. The sum of the squares of two numbers is 25. The difference of the numbers is 1. Find the numbers.
2. The difference of the squares of two numbers is 45. The difference of the square of the first number and twice the square of the second number is 9. Find the numbers.
3. The perimeter of a rectangle is 58 meters and its area is 210 square meters. Find the length and width of the rectangle.
4. Colton purchased a larger microwave for his kitchen. The diagonal of the front of the microwave measures 34 inches. The front also has an area of 480 square inches. What are the length and width of the microwave?

Answer

1. -3 and -4 or 4 and 3

3. If the length is 14 inches, the width is 15 inches. If the length is 15 inches, the width is 14 inches.

Practice Test

Exercise 22.7.23

In the following exercises, find the distance between the points and the midpoint of the line segment with the given endpoints. Round to the nearest tenth as needed.

1. $(-4, -3)$ and $(-10, -11)$
2. $(6, 8)$ and $(-5, -3)$

Answer

1. distance: 10, midpoint: $(-7, -7)$

Exercise 22.7.24

In the following exercises, write the standard form of the equation of the circle with the given information.

1. radius is 11 and center is $(0, 0)$
2. radius is 12 and center is $(10, -2)$
3. center is $(-2, 3)$ and a point on the circle is $(2, -3)$
4. Find the equation of the ellipse shown in the graph.

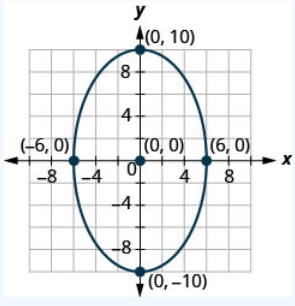

Figure 11.E.31

Answer

1. $x^2 + y^2 = 121$

3. $(x + 2)^2 + (y - 3)^2 = 52$

Exercise 22.7.25

In the following exercises,

a. Identify the type of graph of each equation as a circle, parabola, ellipse, or hyperbola, and
b. Graph the equation.

1. $4x^2+49y^2=196$
2. $y=3(x-2)^2-2$
3. $3x^2+3y^2=27$
4. $\frac{y^2}{100}-\frac{x^2}{36}=1$
5. $\frac{x^2}{16}+\frac{y^2}{81}=1$
6. $x=2y^2+10y+7$
7. $64x^2-9y^2=576$

Answer

1.

a. Ellipse
b.

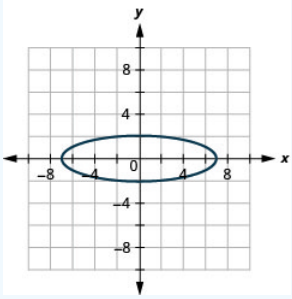

Figure 11.E.32

3.

a. Circle
b.

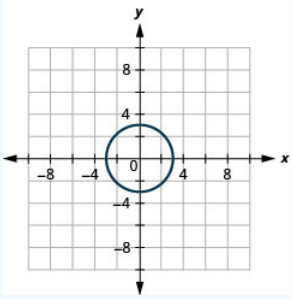

Figure 11.E.33

5.

a. Ellipse
b.

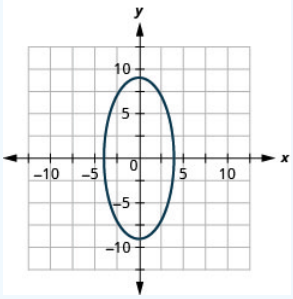

Figure 11.E.34

7.

a. Hyperbola
b.

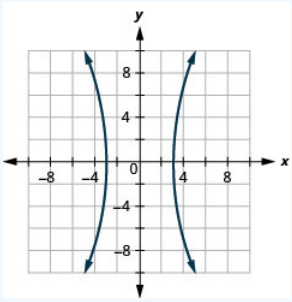

Figure 11.E.35

Exercise 22.7.26

In the following exercises,

a. Identify the type of graph of each equation as a circle, parabola, ellipse, or hyperbola,
b. Write the equation in standard form, and
c. Graph the equation.

1. $25x^2+64y^2+200x-256y-944=0$
2. $x^2+y^2+10x+6y+30=0$
3. $x=-y^2+2y-4$
4. $9x^2-25y^2-36x-50y-214=0$
5. $y=x^2+6x+8$
6. Solve the nonlinear system of equations by graphing: $\begin{cases} 3y^2-x=0 \\ y=-2x-1 \end{cases}$.
7. Solve the nonlinear system of equations using substitution: $\begin{cases} x^2+y^2=8 \\ y=-x-4 \end{cases}$.
8. Solve the nonlinear system of equations using elimination: $\begin{cases} x^2+9y^2=9 \\ 2x^2-9y^2=18 \end{cases}$
9. Create the equation of the parabolic arch formed in the foundation of the bridge shown. Give the answer in $y=ax^2+bx+c$ form.

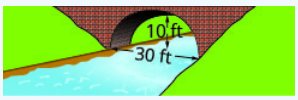

Figure 11.E.36

10. A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 20 AU and the furthest is approximately 70 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and

labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.

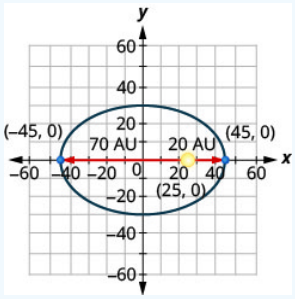

Figure 11.E.37

11. The sum of two numbers is 22 and the product is -240. Find the numbers.

12. For her birthday, Olive's grandparents bought her a new widescreen TV. Before opening it she wants to make sure it will fit her entertainment center. The TV is 55". The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1452 square inches. Her entertainment center has an insert for the TV with a length of 50 inches and width of 40 inches. What are the length and width of the TV screen and will it fit Olive's entertainment center?

Answer

2.

a. Circle
b. $(x+5)^2+(y+3)^2=4$
c.

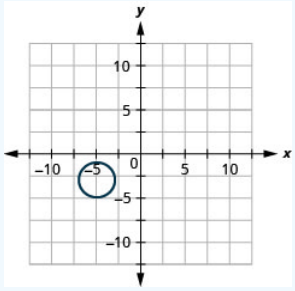

Figure 11.E.38

4.

a. Hyperbola
b. $\frac{(x-2)^2}{25}-\frac{(y+1)^2}{9}=1$
c.

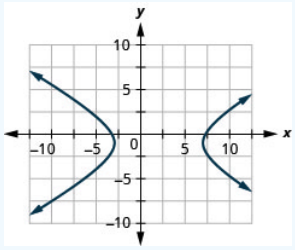

Figure 11.E.39

6. No solution

8. $(0,-3),(0,3)$

10. $\frac{x^2}{2025} + \frac{y^2}{1400} = 1$

12. The length is 44 inches and the width is 33 inches. The TV will fit Olive's entertainment center.

Glossary

system of nonlinear equations

A system of nonlinear equations is a system where at least one of the equations is not linear.

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