# ELEMENTARY DIFFERENTIAL EQUATIONS WITH BOUNDARY VALUES PROBLEMS



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# Elementary Differential Equations with Boundary Values Problems

William F. Trench

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Detailed Licensing

# Licensing

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# CHAPTER OVERVIEW

## 1: Introduction

In this chapter, we begin our study of differential equations.


1.1: Applications Leading to Differential Equations
1.2: Basic Concepts
1.2E: Basic Concepts (Exercises)
1.3: Direction Fields for First Order Equations
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# 1.1: Applications Leading to Differential Equations

In order to apply mathematical methods to a physical or “real life” problem, we must formulate the problem in mathematical terms; that is, we must construct a mathematical model for the problem. Many physical problems concern relationships between changing quantities. Since rates of change are represented mathematically by derivatives, mathematical models often involve equations relating an unknown function and one or more of its derivatives. Such equations are differential equations. They are the subject of this book.

Much of calculus is devoted to learning mathematical techniques that are applied in later courses in mathematics and the sciences; you wouldn’t have time to learn much calculus if you insisted on seeing a specific application of every topic covered in the course. Similarly, much of this book is devoted to methods that can be applied in later courses. Only a relatively small part of the book is devoted to the derivation of specific differential equations from mathematical models, or relating the differential equations that we study to specific applications. In this section we mention a few such applications. The mathematical model for an applied problem is almost always simpler than the actual situation being studied, since simplifying assumptions are usually required to obtain a mathematical problem that can be solved. For example, in modeling the motion of a falling object, we might neglect air resistance and the gravitational pull of celestial bodies other than Earth, or in modeling population growth we might assume that the population grows continuously rather than in discrete steps.

A good mathematical model has two important properties:

- It’s sufficiently simple so that the mathematical problem can be solved.
- It represents the actual situation sufficiently well so that the solution to the mathematical problem predicts the outcome of the real problem to within a useful degree of accuracy. If results predicted by the model don’t agree with physical observations,the underlying assumptions of the model must be revised until satisfactory agreement is obtained.

We will now give examples of mathematical models involving differential equations. We willreturn to these problems at the appropriate times, as we learn how to solve the various types of differential equations that occur in the models. All the examples in this section deal with functions of time, which we denote by $t$. If $y$ is a function of $t$, $y'$ denotes the derivative of $y$ with respect to $t$; thus,

$$y' = \frac{dy}{dt}.$$

## Population Growth and Decay

Although the number of members of a population (people in a given country, bacteria in a laboratory culture, wildflowers in a forest, etc.) at any given time t is necessarily an integer, models that use differential equations to describe the growth and decay of populations usually rest on the simplifying assumption that the number of members of the population can be regarded as a differentiable function $P = P(t)$. In most models it is assumed that the differential equation takes the form

$$P' = a(P)P \tag{1.1.1}$$

where $a$ is a continuous function of $P$ that represents the rate of change of population per unit time per individual. In the **Malthusian model**, it is assumed that $a(P)$ is a constant, so Equation 1.1.1 becomes

$$P' = aP. \tag{1.1.2}$$

(When you see a name in blue italics, just click on it for information about the person.) This model assumes that the numbers of births and deaths per unit time are both proportional to the population. The constants of proportionality are the birth rate (births per unit time per individual) and the death rate (deaths per unit time per individual); a is the birth rate minus the death rate. You learned in calculus that if $c$ is any constant then

$$P = ce^{at} \tag{1.1.3}$$

satisfies Equation 1.1.2, so Equation 1.1.2 has infinitely many solutions. To select the solution of the specific problem that we are considering, we must know the population $P_0$ at an initial time, say $t = 0$. Setting $t = 0$ in Equation 1.1.3 yields $c = P(0) = P_0$, so the applicable solution is

$$P(t) = P_0 e^{at}.$$

This implies that

$$\lim_{t\to\infty} P(t) = \begin{cases} \infty & \text{if } a > 0, \\ 0 & \text{if } a < 0; \end{cases}$$

that is, the population approaches infinity if the birth rate exceeds the death rate, or zero if the death rate exceeds the birth rate.

To see the limitations of the Malthusian model, suppose we are modeling the population of a country, starting from a time $t = 0$ when the birth rate exceeds the death rate (so $a > 0$), and the country's resources in terms of space, food supply, and other necessities of life can support the existing population. Then the prediction $P = P_0 e^{at}$ may be reasonably accurate as long as it remains within limits that the country's resources can support. However, the model must inevitably lose validity when the prediction exceeds these limits. (If nothing else, eventually there will not be enough space for the predicted population!) This flaw in the Malthusian model suggests the need for a model that accounts for limitations of space and resources that tend to oppose the rate of population growth as the population increases.

Perhaps the most famous model of this kind is the **Verhulst model**, where Equation 1.1.2 is replaced by

$$P' = aP(1 - \alpha P), \tag{1.1.4}$$

where $\alpha$ is a positive constant. As long as $P$ is small compared to $1/\alpha$, the ratio $P'/P$ is approximately equal to $a$. Therefore the growth is approximately exponential; however, as $P$ increases, the ratio $P'/P$ decreases as opposing factors become significant.

Equation 1.1.4 is the **logistic equation**. You will learn how to solve it in Section 1.2. (See **Exercise 2.2.28**.) The solution is

$$P = \frac{P_0}{\alpha P_0 + (1 - \alpha P_0)e^{-at}},$$

where $P_0 = P(0) > 0$. Therefore $\lim_{t\to\infty} P(t) = 1/\alpha$, independent of $P_0$.

Figure 1.1.1 shows typical graphs of $P$ versus $t$ for various values of $P_0$.

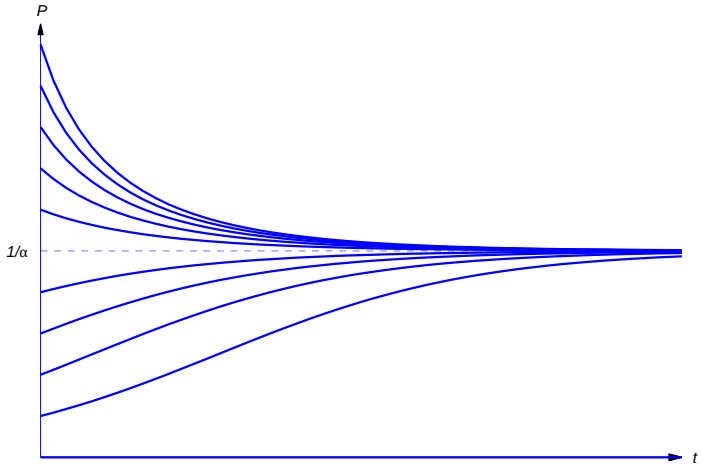



Figure 1.1.1 : Solutions of the logistic equation

## Newton's Law of Cooling

According to **Newton's law of cooling**, the temperature of a body changes at a rate proportional to the difference between the temperature of the body and the temperature of the surrounding medium. Thus, if $T_m$ is the temperature of the medium and $T = T(t)$ is the temperature of the body at time $t$, then

$$T' = -k(T - T_m) \tag{1.1.5}$$

where $k$ is a positive constant and the minus sign indicates; that the temperature of the body increases with time if it is less than the temperature of the medium, or decreases if it is greater. We will see in Section 4.2 that if $T_m$ is constant then the solution of Equation 1.1.5 is

$$T = T_m + (T_0 - T_m)e^{-kt} \tag{1.1.6}$$

where $T_0$ is the temperature of the body when $t = 0$. Therefore

$$\lim_{t\to\infty} T(t) = T_m$$

independent of $T_0$ (Common sense suggests this. Why?). Figure 1.1.2 shows typical graphs of $T$ versus $t$ for various values of $T_0$.

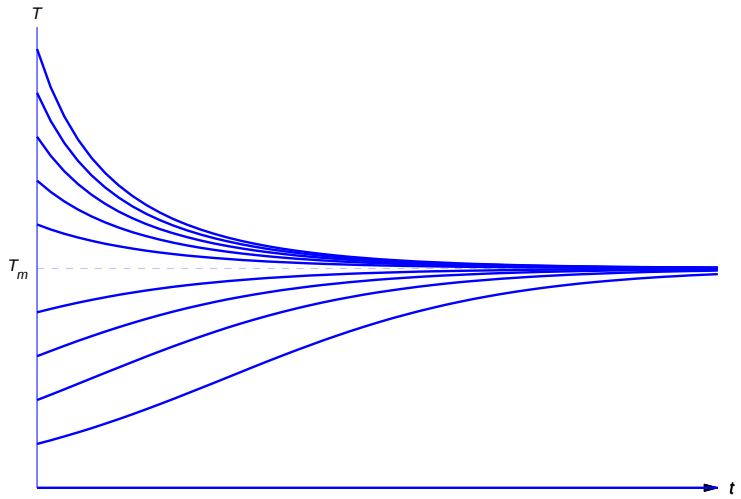



Figure 1.1.2 : Temperature according to Newton's Law of Cooling

Assuming that the medium remains at constant temperature seems reasonable if we are considering a cup of coffee cooling in a room, but not if we are cooling a huge cauldron of molten metal in the same room. The difference between the two situations is that the heat lost by the coffee isn't likely to raise the temperature of the room appreciably, but the heat lost by the cooling metal is. In this second situation we must use a model that accounts for the heat exchanged between the object and the medium. Let $T = T(t)$ and $T_m = T_m(t)$ be the temperatures of the object and the medium respectively, and let $T_0$ and $T_m0$ be their initial values. Again, we assume that T and Tm are related by Equation 1.1.5. We also assume that the change in heat of the object as its temperature changes from $T_0$ to $T$ is $a(T - T_0)$ and the change in heat of the medium as its temperature changes from $T_{m0}$ to $T_m$ is $a_m(T_m - T_{m0})$, where a and am are positive constants depending upon the masses and thermal properties of the object and medium respectively. If we assume that the total heat of the in the object and the medium remains constant (that is, energy is conserved), then

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0.$$

Solving this for $T_m$ and substituting the result into Equation 1.1.6 yields the differential equation

$$T' = -k\left(1 + \frac{a}{a_m}\right)T + k\left(T_{m0} + \frac{a}{a_m}T_0\right)$$

for the temperature of the object. After learning to solve linear first order equations, you'll be able to show (**Exercise 4.2.17**) that

$$T = \frac{aT_0 + a_mT_{m0}}{a + a_m} + \frac{a_m\,(T_0 - T_{m0})}{a + a_m}e^{-k(1+a/a_m)t}$$

## Glucose Absorption by the Body

Glucose is absorbed by the body at a rate proportional to the amount of glucose present in the blood stream. Let $\lambda$ denote the (positive) constant of proportionality. Suppose there are $G_0$ units of glucose in the bloodstream when $t = 0$, and let $G = G(t)$ be the number of units in the bloodstream at time $t > 0$. Then, since the glucose being absorbed by the body is leaving the bloodstream, $G$ satisfies the equation

$$G' = -\lambda G. \tag{1.1.7}$$

From calculus you know that if $c$ is any constant then

$$G = ce^{-\lambda t} \tag{1.1.8}$$

satisfies Equation (1.1.7), so Equation 1.1.7 has infinitely many solutions. Setting $t=0$ in Equation 1.1.8 and requiring that $G(0)=G_0$ yields $c=G_0$, so

$$G(t)=G_0e^{-\lambda t}.$$

Now let's complicate matters by injecting glucose intravenously at a constant rate of $r$ units of glucose per unit of time. Then the rate of change of the amount of glucose in the bloodstream per unit time is

$$G'=-\lambda G+r \tag{1.1.9}$$

where the first term on the right is due to the absorption of the glucose by the body and the second term is due to the injection. After you've studied Section 2.1, you'll be able to show that the solution of Equation 1.1.9 that satisfies $G(0)=G_0$ is

$$G=\frac{r}{\lambda}+\left(G_0-\frac{r}{\lambda}\right)e^{-\lambda t}$$

Graphs of this function are similar to those in Figure 1.1.2 . (Why?)

## Spread of Epidemics

One model for the spread of epidemics assumes that the number of people infected changes at a rate proportional to the product of the number of people already infected and the number of people who are susceptible, but not yet infected. Therefore, if $S$ denotes the total population of susceptible people and $I=I(t)$ denotes the number of infected people at time $t$, then $S-I$ is the number of people who are susceptible, but not yet infected. Thus,

$$I'=rI(S-I)$$

, where $r$ is a positive constant. Assuming that $I(0)=I_0$, the solution of this equation is

$$I=\frac{SI_0}{I_0+(S-I_0)e^{-rSt}}$$

(**Exercise 2.2.29**). Graphs of this function are similar to those in Figure 1.1.1. (Why?) Since $\lim_{t\to\infty} I(t)=S$, this model predicts that all the susceptible people eventually become infected.

## Newton's Second Law of Motion

According to Newton's second law of motion, the instantaneous acceleration a of an object with constant mass $m$ is related to the force $F$ acting on the object by the equation $F=ma$. For simplicity, let's assume that $m=1$ and the motion of the object is along a vertical line. Let $y$ be the displacement of the object from some reference point on Earth's surface, measured positive upward. In many applications, there are three kinds of forces that may act on the object:

a. A force such as gravity that depends only on the position $y$, which we write as $-p(y)$, where $p(y)>0$ if $y\ge 0$.
b. A force such as atmospheric resistance that depends on the position and velocity of the object, which we write as $-q(y,y')y'$, where $q$ is a nonnegative function and we've put $y'$ "outside" to indicate that the resistive force is always in the direction opposite to the velocity.
c. A force $f=f(t)$, exerted from an external source (such as a towline from a helicopter) that depends only on $t$.

In this case, Newton's second law implies that

$$y''=-q(y,y')y'-p(y)+f(t),$$

which is usually rewritten as

$$y''+q(y,y')y'+p(y)=f(t).$$

Since the second (and no higher) order derivative of $y$ occurs in this equation, we say that it is a second order differential equation.

## Interacting Species: Competition

Let $P=P(t)$ and $Q=Q(t)$ be the populations of two species at time $t$, and assume that each population would grow exponentially if the other did not exist; that is, in the absence of competition we would have

$$P'=aP \quad \text{and} \quad Q'=bQ, \tag{1.1.10}$$

where $a$ and $b$ are positive constants. One way to model the effect of competition is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so Equation 1.1.10 is replaced by

$$P' = aP - \alpha Q$$
$$Q' = -\beta P + bQ,$$

where $\alpha$ and $\beta$ are positive constants. (Since negative population doesn't make sense, this system works only while $P$ and $Q$ are both positive.) Now suppose $P(0) = P_0 > 0$ and $Q(0) = Q_0 > 0$. It can be shown (**Exercise 10.4.42**) that there's a positive constant $\rho$ such that if $(P_0, Q_0)$ is above the line $L$ through the origin with slope $\rho$, then the species with population $P$ becomes extinct in finite time, but if $(P_0, Q_0)$ is below $L$, the species with population $Q$ becomes extinct in finite time. Figure 1.1.3 illustrates this. The curves shown there are given parametrically by $P = P(t), Q = Q(t)$, $t > 0$. The arrows indicate direction along the curves with increasing $t$.

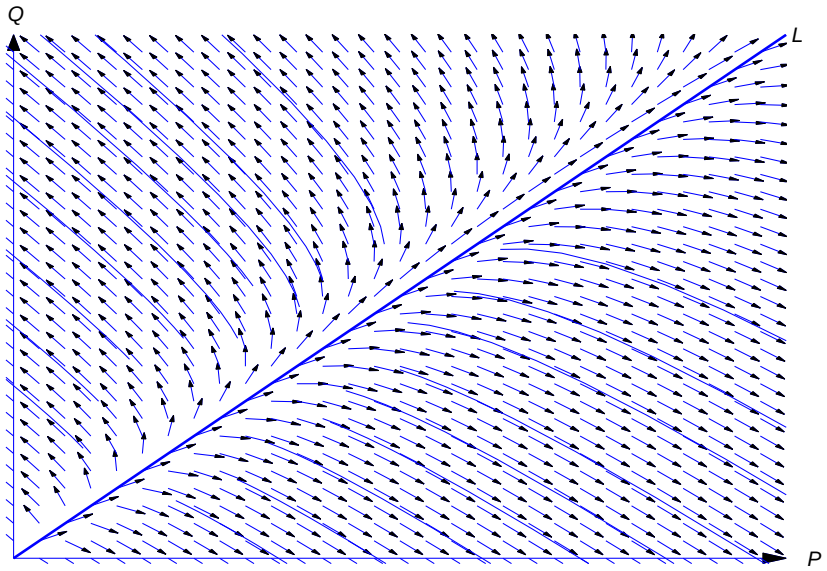



Figure 1.1.3 : Populations of competing species

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## 1.2: Basic Concepts

A **differential equation** is an equation that contains one or more derivatives of an unknown function. The **order** of a differential equation is the order of the highest derivative that it contains. A differential equation is an **ordinary differential equation** if it involves an unknown function of only one variable, or a **partial differential equation** if it involves partial derivatives of a function of more than one variable. For now we'll consider only ordinary differential equations, and we'll just call them **differential equations.**

> Throughout this text, all variables and constants are real unless it is stated otherwise. We'll usually use $x$ for the independent variable unless the independent variable is time; then we'll use $t$.

The simplest differential equations are first order equations of the form

$$\frac{dy}{dx} = f(x)$$

or equivalently

$$y' = f(x),$$

where $f$ is a known function of $x$. We already know from calculus how to find functions that satisfy this kind of equation. For example, if

$$y' = x^3,$$

then

$$y = \int x^3\,dx = \frac{x^4}{4} + c,$$

where $c$ is an arbitrary constant. If $n > 1$ we can find functions $y$ that satisfy equations of the form

$$y^{(n)} = f(x) \tag{1.2.1}$$

by repeated integration. Again, this is a calculus problem.

Except for illustrative purposes in this section, there's no need to consider differential equations like Equation 1.2.1. We'll usually consider differential equations that can be written as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \tag{1.2.2}$$

where at least one of the functions $y$, $y'$, …, $y^{(n-1)}$ actually appears on the right. Here are some examples:

$$\begin{aligned}
\tfrac{dy}{dx} - x^2 &= 0 && \text{(first order)},\\
\tfrac{dy}{dx} + 2xy^2 &= -2 && \text{(first order)},\\
\tfrac{d^2y}{dx^2} + 2\tfrac{dy}{dx} + y &= 2x && \text{(second order)},\\
xy''' + y^2 &= \sin x && \text{(third order)},\\
y^{(n)} + xy' + 3y &= x && \text{(n-th order)}.
\end{aligned}$$

Although none of these equations is written as in Equation 1.2.2, all of them **can** be written in this form:

$$
\begin{aligned}
y' &= x^2, \\
y' &= -2 - 2xy^2, \\
y'' &= 2x - 2y' - y, \\
y''' &= \frac{\sin x - y^2}{x}, \\
y^{(n)} &= x - xy' - 3y.
\end{aligned}
$$

## Solutions of Differential Equations

A **solution** of a differential equation is a function that satisfies the differential equation on some open interval; thus, $y$ is a solution of Equation 1.2.2 if $y$ is $n$ times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$$

for all $x$ in some open interval $(a, b)$. In this case, we also say that $y$ **is a solution of Equation 1.2.2 on $(a, b)$**. Functions that satisfy a differential equation at isolated points are not interesting. For example, $y = x^2$ satisfies

$$xy' + x^2 = 3x$$

if and only if $x = 0$ or $x = 1$, but it is not a solution of this differential equation because it does not satisfy the equation on an open interval.

The graph of a solution of a differential equation is a **solution curve**. More generally, a curve $C$ is said to be an **integral curve** of a differential equation if every function $y = y(x)$ whose graph is a segment of $C$ is a solution of the differential equation. Thus, any solution curve of a differential equation is an integral curve, but an integral curve need not be a solution curve.

### Example 1.2.1

If $a$ is any positive constant, the circle

$$x^2 + y^2 = a^2 \tag{1.2.3}$$

is an integral curve of

$$y' = -\frac{x}{y}. \tag{1.2.4}$$

**Solution**

To see this, note that the only functions whose graphs are segments of Equation 1.2.3 are

$$y_1 = \sqrt{a^2 - x^2} \quad \text{and} \quad y_2 = -\sqrt{a^2 - x^2}.$$

We leave it to you to verify that these functions both satisfy Equation 1.2.4 on the open interval $(-a, a)$. However, Equation 1.2.3 is not a solution curve of Equation 1.2.4, since it is not the graph of a function.

### Example 1.2.2

Verify that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{1.2.5}$$

is a solution of

$$xy' + y = x^2 \tag{1.2.6}$$

on $(0, \infty)$ and on $(-\infty, 0)$.

**Solution**

Substituting Equation 1.2.5 and

$$y' = \frac{2x}{3} - \frac{1}{x^2}$$

into Equation 1.2.6 yields

$$xy'(x) + y(x) = x\left(\frac{2x}{3} - \frac{1}{x^2}\right) + \left(\frac{x^2}{3} + \frac{1}{x}\right) = x^2$$

for all $x \neq 0$. Therefore $y$ is a solution of Equation 1.2.6 on $(-\infty, 0)$ and $(0, \infty)$. However, $y$ isn't a solution of the differential equation on any open interval that contains $x = 0$, since $y$ is not defined at $x = 0$.

Figure 1.2.2 shows the graph of Equation 1.2.5. The part of the graph of Equation 1.2.5 on $(0, \infty)$ is a solution curve of Equation 1.2.6, as is the part of the graph on $(-\infty, 0)$.

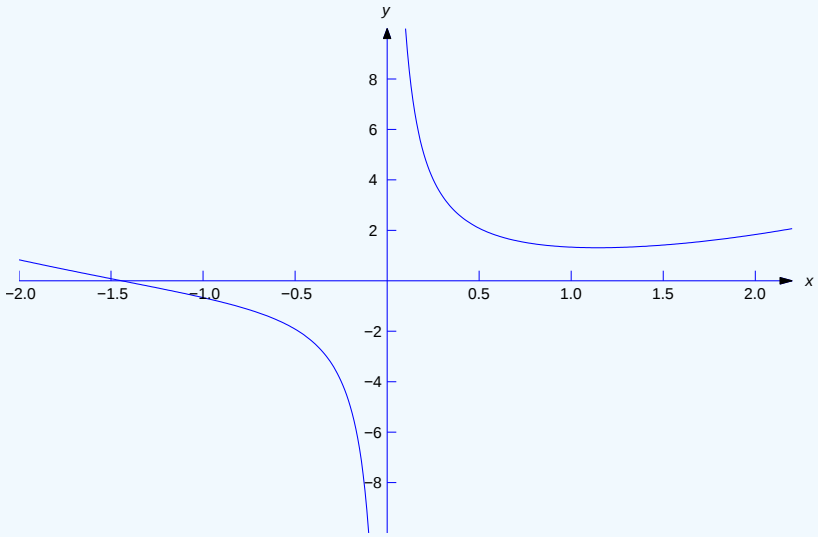



Figure 1.2.2 : $y = \frac{x^2}{3} + \frac{1}{x}$

## Example 1.2.3

Show that if $c_1$ and $c_2$ are constants then

$$y = (c_1 + c_2 x)e^{-x} + 2x - 4 \tag{1.2.7}$$

is a solution of

$$y'' + 2y' + y = 2x \tag{1.2.8}$$

on $(-\infty, \infty)$.

### Solution

Differentiating Equation 1.2.7 twice yields

$$y' = -(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2$$

and

$$y'' = (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x},$$

so

$$\begin{aligned} y'' + 2y' + y &= (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x} + 2\left[-(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2\right] + (c_1 + c_2 x)e^{-x} + 2x - 4 \\ &= (1 - 2 + 1)(c_1 + c_2 x)e^{-x} + (-2 + 2)c_2 e^{-x} + 4 + 2x - 4 \\ &= 2x \end{aligned}$$

for all values of $x$. Therefore $y$ is a solution of Equation 1.2.8 on $(-\infty, \infty)$.

**Example 1.2.4**

Find all solutions of

$$y^{(n)} = e^{2x}. \tag{1.2.9}$$

**Solution**

Integrating Equation 1.2.9 yields

$$y^{(n-1)} = \frac{e^{2x}}{2} + k_1,$$

where $k_1$ is a constant. If $n \geq 2$, integrating again yields

$$y^{(n-2)} = \frac{e^{2x}}{4} + k_1 x + k_2.$$

If $n \geq 3$, repeatedly integrating yields

$$y = \frac{e^{2x}}{2^n} + k_1 \frac{x^{n-1}}{(n-1)!} + k_2 \frac{x^{n-2}}{(n-2)!} + \cdots + k_n, \tag{1.2.10}$$

where $k_1$, $k_2$, …, $k_n$ are constants. This shows that every solution of Equation 1.2.9 has the form Equation 1.2.10 for some choice of the constants $k_1$, $k_2$, …, $k_n$. On the other hand, differentiating Equation 1.2.10 $n$ times shows that if $k_1$, $k_2$, …, $k_n$ are arbitrary constants, then the function $y$ in Equation 1.2.10 satisfies Equation 1.2.9.

Since the constants $k_1$, $k_2$, …, $k_n$ in Equation 1.2.10 are arbitrary, so are the constants

$$\frac{k_1}{(n-1)!}, \frac{k_2}{(n-2)!}, \cdots, k_n.$$

Therefore Example 1.2.4 actually shows that all solutions of Equation 1.2.9 can be written as

$$y = \frac{e^{2x}}{2^n} + c_1 + c_2 x + \cdots + c_n x^{n-1},$$

where we renamed the arbitrary constants in Equation 1.2.10 to obtain a simpler formula. As a general rule, arbitrary constants appearing in solutions of differential equations should be simplified if possible. You'll see examples of this throughout the text.

## Initial Value Problems

In Example 1.2.4 we saw that the differential equation $y^{(n)} = e^{2x}$ has an infinite family of solutions that depend upon the $n$ arbitrary constants $c_1$, $c_2$, …, $c_n$. In the absence of additional conditions, there's no reason to prefer one solution of a differential equation over another. However, we'll often be interested in finding a solution of a differential equation that satisfies one or more specific conditions. The next example illustrates this.

**Example 1.2.5**

Find a solution of

$$y' = x^3$$

such that $y(1) = 2$.

**Solution**

At the beginning of this section we saw that the solutions of $y' = x^3$ are

$$y = \frac{x^4}{4} + c.$$

To determine a value of $c$ such that $y(1) = 2$, we set $x = 1$ and $y = 2$ here to obtain

$$2 = y(1) = \frac{1}{4} + c$$

so

$$c = \frac{7}{4}.$$

Therefore the required solution is

$$y = \frac{x^4 + 7}{4}.$$

Figure 1.2.2 shows the graph of this solution. Note that imposing the condition $y(1) = 2$ is equivalent to requiring the graph of $y$ to pass through the point $(1, 2)$.

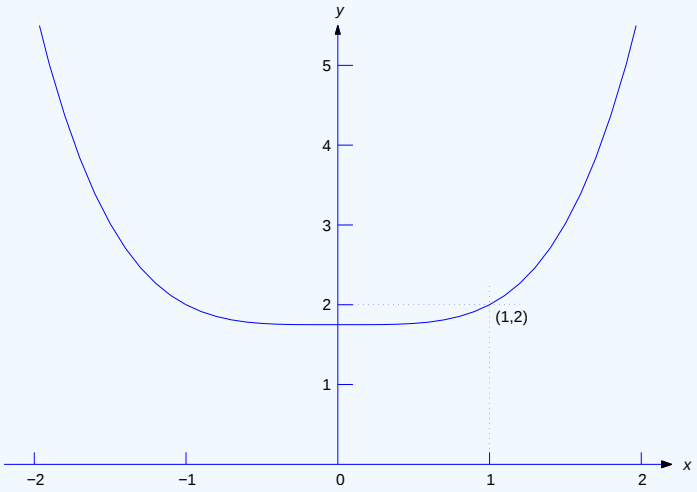



Figure 1.2.2 : $y = \frac{x^2+7}{4}$

We can rewrite the problem considered in Example 1.2.5 more briefly as

$$y' = x^3, \quad y(1) = 2.$$

We call this an **initial value problem**. The requirement $y(1) = 2$ is an **initial condition**. Initial value problems can also be posed for higher order differential equations. For example,

$$y'' - 2y' + 3y = e^x, \quad y(0) = 1, \quad y'(0) = 2 \tag{1.2.11}$$

is an initial value problem for a second order differential equation where $y$ and $y'$ are required to have specified values at $x = 0$. In general, an initial value problem for an $n$-th order differential equation requires $y$ and its first $n - 1$ derivatives to have specified values at some point $x_0$. These requirements are the **initial conditions**.We'll denote an initial value problem for a differential equation by writing the initial conditions after the equation, as in Equation 1.2.11. For example, we would write an initial value problem for Equation 1.2.2 as

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),\ y(x_0) = k_0,\ y'(x_0) = k_1,\ \ldots,\ y^{(n-1)} = k_{n-1}. \tag{1.2.12}$$

Consistent with our earlier definition of a solution of the differential equation in Equation 1.2.12, we say that $y$ is a solution of the initial value problem Equation 1.2.12 if $y$ is $n$ times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$$

for all $x$ in some open interval $(a, b)$ that contains $x_0$, and $y$ satisfies the initial conditions in Equation 1.2.12. The largest open interval that contains $x_0$ on which $y$ is defined and satisfies the differential equation is the **interval of validity** of $y$.

### Example 1.2.6

In Example 1.2.5 we saw that

$$y = \frac{x^4 + 7}{4} \tag{1.2.13}$$

is a solution of the initial value problem

$$y' = x^3, \quad y(1) = 2.$$

Since the function in Equation 1.2.13 is defined for all $x$, the interval of validity of this solution is $(-\infty, \infty)$.

### Example 1.2.7

In Example 1.2.2 we verified that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{1.2.14}$$

is a solution of

$$xy' + y = x^2$$

on $(0, \infty)$ and on $(-\infty, 0)$. By evaluating Equation 1.2.14 at $x = \pm 1$, you can see that Equation 1.2.14 is a solution of the initial value problems

$$xy' + y = x^2, \quad y(1) = \frac{4}{3} \tag{1.2.15}$$

and

$$xy' + y = x^2, \quad y(-1) = -\frac{2}{3}. \tag{1.2.16}$$

The interval of validity of Equation 1.2.14 as a solution of Equation 1.2.15 is $(0, \infty)$, since this is the largest interval that contains $x_0 = 1$ on which Equation 1.2.14 is defined. Similarly, the interval of validity of Equation 1.2.14 as a solution of Equation 1.2.16 is $(-\infty, 0)$, since this is the largest interval that contains $x_0 = -1$ on which Equation 1.2.14 is defined.

## Free Fall Under Constant Gravity

The term **initial value problem** originated in problems of motion where the independent variable is $t$ (representing elapsed time), and the initial conditions are the position and velocity of an object at the initial (starting) time of an experiment.

### Example 1.2.8

An object falls under the influence of gravity near Earth's surface, where it can be assumed that the magnitude of the acceleration due to gravity is a constant $g$.

a. Construct a mathematical model for the motion of the object in the form of an initial value problem for a second order differential equation, assuming that the altitude and velocity of the object at time $t = 0$ are known. Assume that gravity is the only force acting on the object.
b. Solve the initial value problem derived above to obtain the altitude as a function of time.

**Solution a**

Let $y(t)$ be the altitude of the object at time $t$. Since the acceleration of the object has constant magnitude $g$ and is in the downward (negative) direction, $y$ satisfies the second order equation

$$y'' = -g,$$

where the prime now indicates differentiation with respect to $t$. If $y_0$ and $v_0$ denote the altitude and velocity when $t = 0$, then $y$ is a solution of the initial value problem

$$y'' = -g, \quad y(0) = y_0, \quad y'(0) = v_0. \tag{1.2.17}$$

**Solution b**

Integrating Equation 1.2.17 twice yields

$$y' = -gt + c_1,$$

$$y = -\frac{gt^2}{2} + c_1 t + c_2.$$

Imposing the initial conditions $y(0) = y_0$ and $y'(0) = v_0$ in these two equations shows that $c_1 = v_0$ and $c_2 = y_0$. Therefore the solution of the initial value problem Equation 1.2.17 is

$$y = -\frac{gt^2}{2} + v_0 t + y_0.$$

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# 1.2E: Basic Concepts (Exercises)

1. Find the order of the equation.

a. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}\ \frac{d^3y}{dx^3}+x=0$

b. $y''-3y'+2y=x^7$

c. $y'-y^7=0$

d. $y''y-(y')^2=2$

2. Verify that the function is a solution of the differential equation on some interval, for any choice of the arbitrary constants appearing in the function.

a. $y=ce^{2x};\quad y'=2y$

b. $y=\frac{x^2}{3}+\frac{c}{x};\quad xy'+y=x^2$

c. $y=\frac{1}{2}+ce^{-x^2};\quad y'+2xy=x$

d. $y=(1+ce^{-x^2/2})(1-ce^{-x^2/2})^{-1};\quad 2y'+x(y^2-1)=0$

e. $y=\tan\left(\frac{x^3}{3}+c\right);\quad y'=x^2(1+y^2)$

f. $y=(c_1+c_2x)e^x+\sin x+x^2;\quad y''-2y'+y=-2\cos x+x^2-4x+2$

g. $y=c_1e^x+c_2x+\frac{2}{x};\quad (1-x)y''+xy'-y=4(1-x-x^2)x^{-3}$

h. $y=x^{-1/2}(c_1\sin x+c_2\cos x)+4x+8\ ;x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=4x^3+8x^2+3x-2$

3. Find all solutions of the equation.

a. $y'=-x$

b. $y'=-x\sin x$

c. $y'=x\ln x$

d. $y''=x\cos x$

e. $y''=2xe^x$

f. $y''=2x+\sin x+e^x$

g. $y'''=-\cos x$

h. $y'''=-x^2+e^x$

i. $y'''=7e^{4x}$

4. Solve the initial value problem.

a. $y'=-xe^x,\quad y(0)=1$

b. $y'=x\sin x^2,\quad y\left(\sqrt{\frac{\pi}{2}}\right)=1$

c. $y'=\tan x,\quad y(\pi/4)=3$

d. $y''=x^4,\quad y(2)=-1,\quad y'(2)=-1$

e. $y''=xe^{2x},\quad y(0)=7,\quad y'(0)=1$

f. $y''=-x\sin x,\quad y(0)=1,\quad y'(0)=-3$

g. $y'''=x^2e^x,\quad y(0)=1,\quad y'(0)=-2,\quad y''(0)=3$

h. $y'''=2+\sin 2x,\quad y(0)=1,\quad y'(0)=-6,\quad y''(0)=3$

i. $y'''=2x+1,\quad y(2)=1,\quad y'(2)=-4,\quad y''(2)=7$

5. Verify that the function is a solution of the initial value problem.

a. $y = x\cos x; \quad y' = \cos x - y\tan x, \quad y(\pi/4) = \frac{\pi}{4\sqrt{2}}$

b. $y = \frac{1+2\ln x}{x^2} + \frac{1}{2}; \quad y' = \frac{x^2-2x^2y+2}{x^3}, \quad y(1) = \frac{3}{2}$

c. $y = \tan\left(\frac{x^2}{2}\right); \quad y' = x(1+y^2), \quad y(0) = 0$

d. $y = \frac{2}{x-2}; \quad y' = \frac{-y(y+1)}{x}, \quad y(1) = -2$

6. Verify that the function is a solution of the initial value problem.

a. $y = x^2(1+\ln x); \quad y'' = \frac{3xy'-4y}{x^2}, \quad y(e) = 2e^2, \quad y'(e) = 5e$

b. $y = \frac{x^2}{3} + x - 1; \quad y'' = \frac{x^2-xy'+y+1}{x^2}, \quad y(1) = \frac{1}{3}, \quad y'(1) = \frac{5}{3}$

c. $y = (1+x^2)^{-1/2}; \quad y'' = \frac{(x^2-1)y-x(x^2+1)y'}{(x^2+1)^2}, \quad y(0) = 1, \; y'(0) = 0$

d. $y = \frac{x^2}{1-x}; \quad y'' = \frac{2(x+y)(xy'-y)}{x^3}, \quad y(1/2) = 1/2, \quad y'(1/2) = 3$

7. Suppose an object is launched from a point 320 feet above the earth with an initial velocity of 128 ft/sec upward, and the only force acting on it thereafter is gravity. Take $g = 32$ ft/sec$^2$.

a. Find the highest altitude attained by the object.
b. Determine how long it takes for the object to fall to the ground.

8. Let $a$ be a nonzero real number.

a. Verify that if $c$ is an arbitrary constant then

$$y = (x-c)^a \tag{A}$$

is a solution of

$$y' = ay^{(a-1)/a} \tag{B}$$

on $(c, \infty)$.

b. Suppose $a < 0$ or $a > 1$. Can you think of a solution of (B) that isn't of the form (A)?

9. Verify that

$$y = \begin{cases} e^x - 1, & x \geq 0, \\ 1 - e^{-x}, & x < 0, \end{cases}$$

is a solution of

$$y' = |y| + 1$$

on $(-\infty, \infty)$.

10.

(a) Verify that if $c$ is any real number then

$$y = c^2 + cx + 2c + 1 \tag{A}$$

satisfies

$$y' = \frac{-(x+2) + \sqrt{x^2+4x+4y}}{2} \tag{B}$$

on some open interval. Identify the open interval.

(b) Verify that

$$y_1 = \frac{-x(x+4)}{4}$$

also satisfies (B) on some open interval, and identify the open interval. (Note that $y_1$ can't be obtained by selecting a value of $c$ in (A).)

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# 1.3: Direction Fields for First Order Equations

It's impossible to find explicit formulas for solutions of some differential equations. Even if there are such formulas, they may be so complicated that they're useless. In this case we may resort to graphical or numerical methods to get some idea of how the solutions of the given equation behave.

In Section 2.3 we'll take up the question of existence of solutions of a first order equation

$$y' = f(x, y). \tag{1.3.1}$$

In this section we'll simply assume that Equation 1.3.1 has solutions and discuss a graphical method for approximating them. In Chapter 3 we discuss numerical methods for obtaining approximate solutions of Equation 1.3.1. Recall that a solution of Equation 1.3.1 is a function $y = y(x)$ such that

$$y'(x) = f(x, y(x))$$

for all values of $x$ in some interval, and an integral curve is either the graph of a solution or is made up of segments that are graphs of solutions. Therefore, not being able to solve Equation 1.3.1 is equivalent to not knowing the equations of integral curves of Equation 1.3.1. However, it is easy to calculate the slopes of these curves. To be specific, the slope of an integral curve of Equation 1.3.1 through a given point $(x_0, y_0)$ is given by the number $f(x_0, y_0)$. This is the basis of **the method of direction fields**.

If $f$ is defined on a set $R$, we can construct a **direction field** for Equation 1.3.1 in $R$ by drawing a short line segment through each point $(x, y)$ in $R$ with slope $f(x, y)$. Of course, as a practical matter, we can't actually draw line segments through **every** point in $R$; rather, we must select a finite set of points in $R$. For example, suppose $f$ is defined on the closed rectangular region

$$R : \{a \le x \le b, c \le y \le d\}.$$

Let

$$a = x_0 < x_1 < \cdots < x_m = b$$

be equally spaced points in $[a, b]$ and

$$c = y_0 < y_1 < \cdots < y_n = d$$

form a **rectangular grid** (Figure 1.3.1 ). Through each point in the grid we draw a short line segment with slope $f(x_i, y_j)$. The result is an approximation to a direction field for Equation 1.3.1 in $R$. If the grid points are sufficiently numerous and close together, we can draw approximate integral curves of Equation 1.3.1 by drawing curves through points in the grid tangent to the line segments associated with the points in the grid.

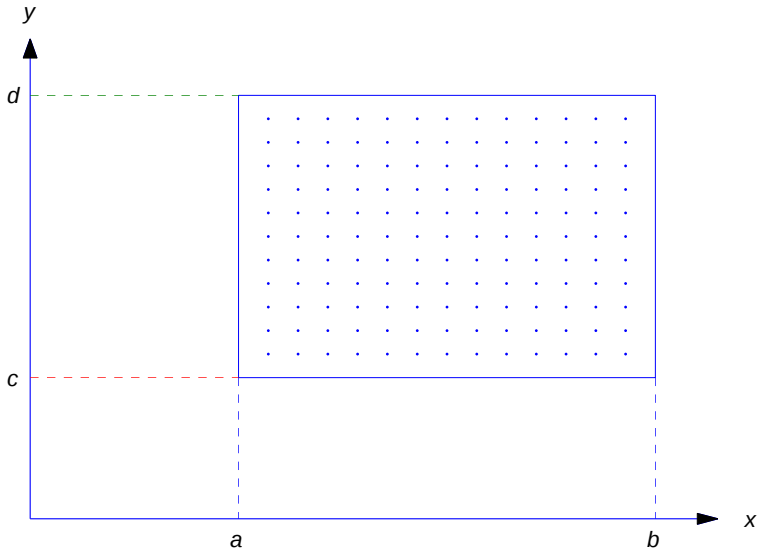


Figure 1.3.1 : A rectangular grid

Unfortunately, approximating a direction field and graphing integral curves in this way is too tedious to be done effectively by hand. However, there is software for doing this. As you'll see, the combination of direction fields and integral curves gives useful

insights into the behavior of the solutions of the differential equation even if we can't obtain exact solutions.

We'll study numerical methods for solving a single first order equation Equation 1.3.1 in Chapter 3. These methods can be used to plot solution curves of Equation 1.3.1 in a rectangular region $R$ **if $f$ is continuous on** $R$. Figures 1.3.2 , 1.3.3 , and 1.3.4 show direction fields and solution curves for the differential equations:

- $y' = \dfrac{x^2 - y^2}{1 + x^2 + y^2}$ ,
- $y' = 1 + xy^2$ , and
- $y' = \dfrac{x - y}{1 + x^2}$ .

which are all of the form Equation 1.3.1 with $f$ continuous for all $(x, y)$.

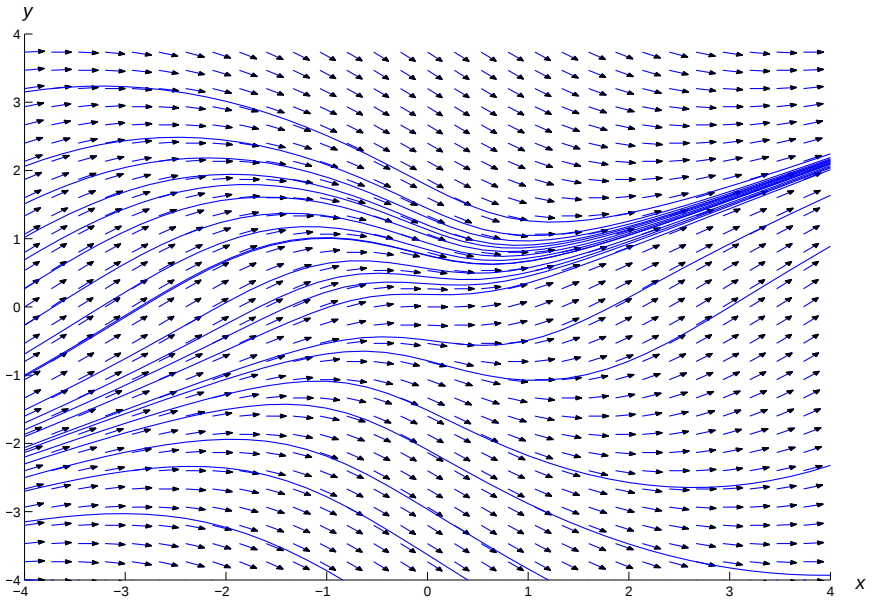



Figure 1.3.2 : A direction and integral curves for $y' = \frac{x^2 - y^2}{1 + x^2 + y^2}$.

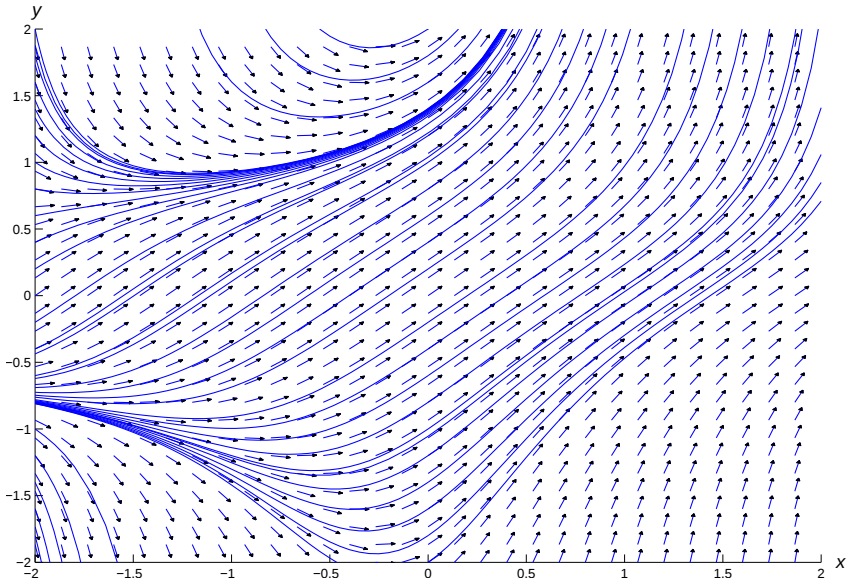



Figure 1.3.3 : A direction and integral curves for $y' = 1 + xy^2$ .

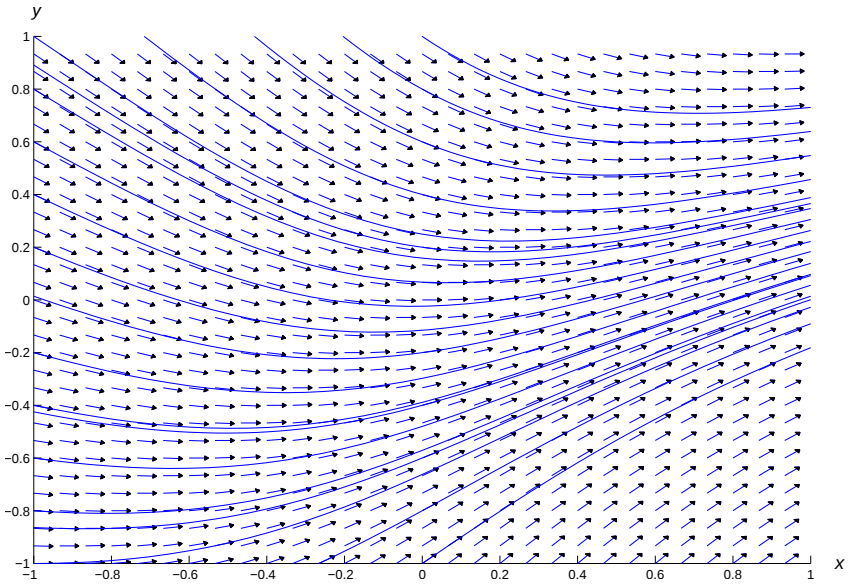


Figure 1.3.4 : A direction and integral curves for $y' = \frac{x-y}{1+x^2}$.

The methods of Chapter 3 will not work for the equation

$$y' = -x/y \tag{1.3.2}$$

if $R$ contains part of the $x$-axis, since $f(x,y) = -x/y$ is undefined when $y = 0$. Similarly, they will not work for the equation

$$y' = \frac{x^2}{1-x^2-y^2} \tag{1.3.3}$$

if $R$ contains any part of the unit circle $x^2+y^2 = 1$, because the right side of Equation 1.3.3 is undefined if $x^2+y^2 = 1$. However, Equation 1.3.2 and Equation 1.3.3 can written as

$$y' = \frac{A(x,y)}{B(x,y)} \tag{1.3.4}$$

where $A$ and $B$ are continuous on any rectangle $R$. Because of this, some differential equation software is based on numerically solving pairs of equations of the form

$$\frac{dx}{dt} = B(x,y), \quad \frac{dy}{dt} = A(x,y) \tag{1.3.5}$$

where $x$ and $y$ are regarded as functions of a parameter $t$. If $x = x(t)$ and $y = y(t)$ satisfy these equations, then

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{A(x,y)}{B(x,y)},$$

so $y = y(x)$ satisfies Equation 1.3.4.

Equations 1.3.2 and 1.3.3 can be reformulated as in Equation 1.3.4 with

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x$$

and

$$\frac{dx}{dt} = 1-x^2-y^2, \quad \frac{dy}{dt} = x^2,$$

respectively. Even if $f$ is continuous and otherwise "nice" throughout $R$, your software may require you to reformulate the equation $y' = f(x,y)$ as

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = f(x,y),$$

which is of the form Equation 1.3.5 with $A(x,y)=f(x,y)$ and $B(x,y)=1$.

Figure 1.3.5 shows a direction field and some integral curves for Equation 1.3.2. As we saw in Example [example:1.2.1} and will verify again in Section 2.2, the integral curves of Equation 1.3.2 are circles centered at the origin.

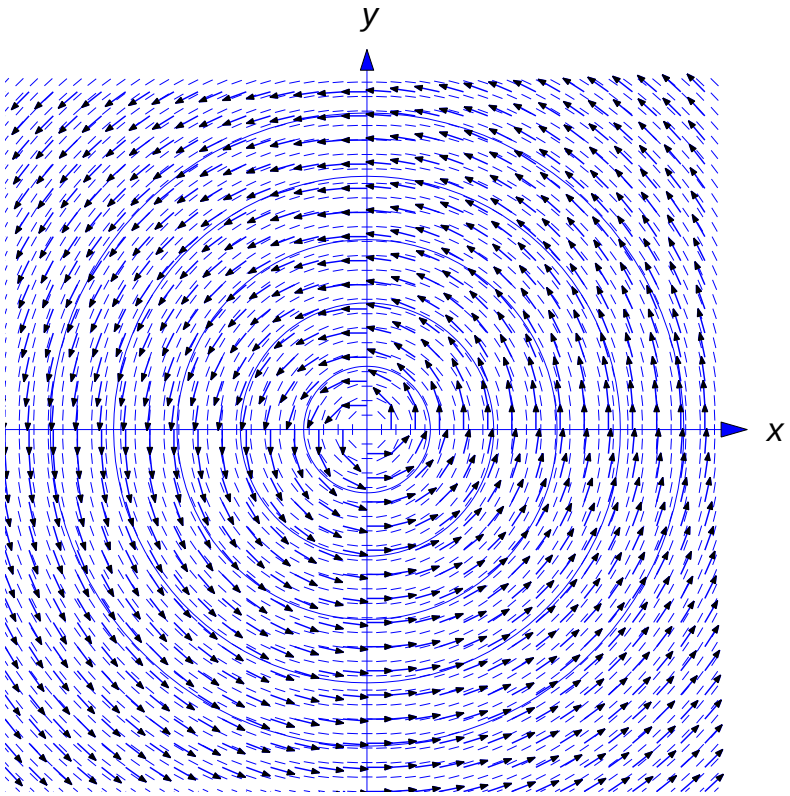



Figure 1.3.5 : A direction field and integral curves for $y'=-\frac{x}{y}$

Figure 1.3.6 shows a direction field and some integral curves for Equation 1.3.3. The integral curves near the top and bottom are solution curves. However, the integral curves near the middle are more complicated.

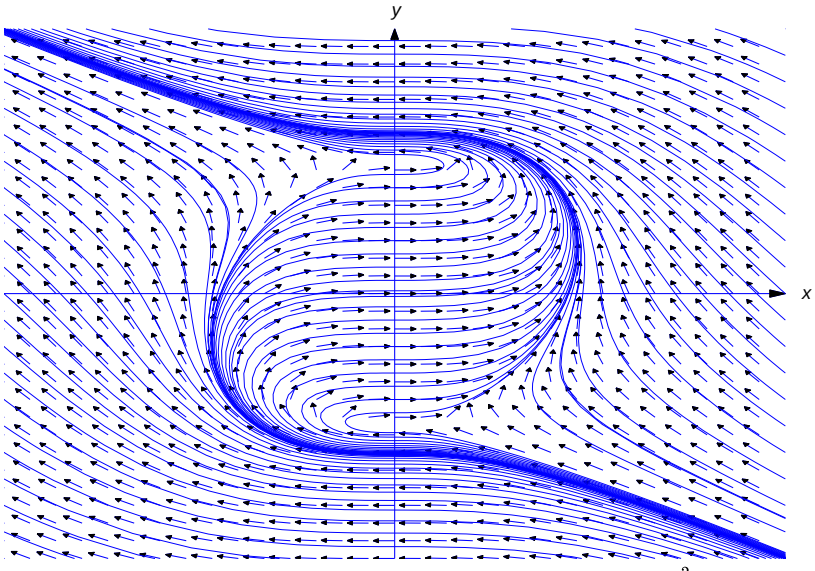



Figure 1.3.6 : A direction field and integral curves for $y'=\dfrac{x^2}{1-x^2-y^2}$.

For example, Figure 1.3.7 shows the integral curve through the origin. The vertices of the dashed rectangle are on the circle $x^2+y^2=1$ ( $a\approx .846$, $b\approx .533$), where all integral curves of Equation 1.3.3 have infinite slope. There are three solution curves of Equation 1.3.3 on the integral curve in the figure: the segment above the level $y=b$ is the graph of a solution on $(-\infty,a)$, the

segment below the level $y = -b$ is the graph of a solution on $(-a, \infty)$, and the segment between these two levels is the graph of a solution on $(-a, a)$.

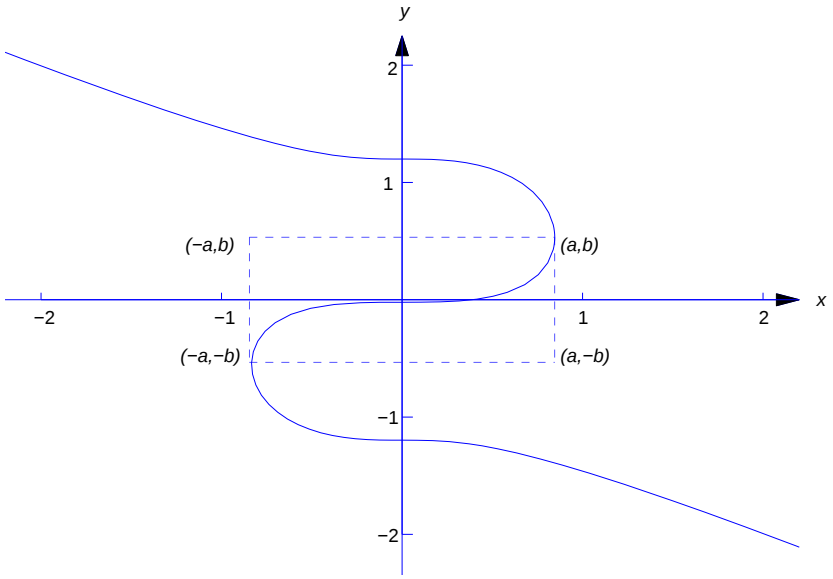



Figure 1.3.7 : An integral curve for $y' = \dfrac{x^2}{1 - x^2 - y^2}$.

**Using Technology**

As you study from this book, you'll often be asked to use computer software and graphics. Exercises with this intent are marked as (computer or calculator required), (computer and/or graphics required), or (laboratory work requiring software and/or graphics). Often you may not completely understand how the software does what it does. This is similar to the situation most people are in when they drive automobiles or watch television, and it doesn't decrease the value of using modern technology as an aid to learning. Just be careful that you use the technology as a supplement to thought rather than a substitute for it.

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# 1.3E: Direction Fields for First Order Equations (Exercises)

## Exercises for Section 1.3

In Exercises 1–11 a direction field is drawn for the given equation. Sketch some integral curves.

1.

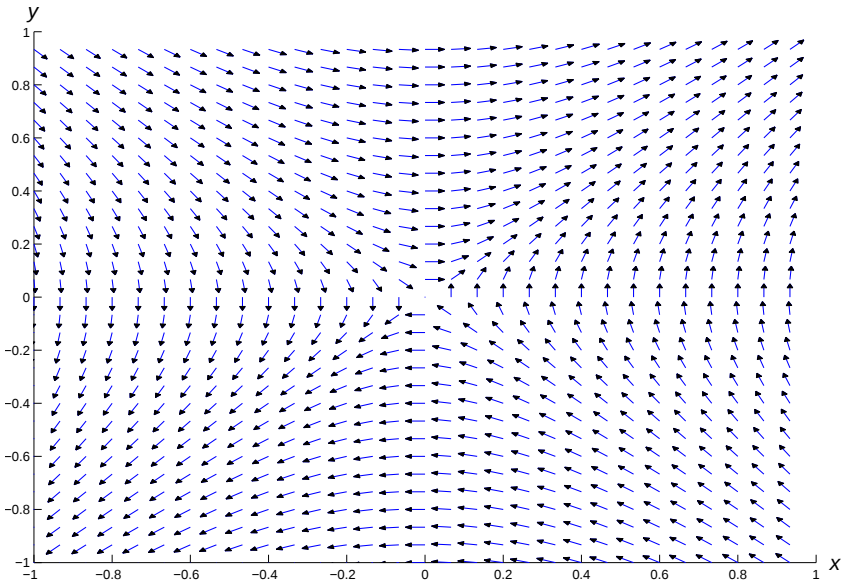



Figure 1.3E.1 : A direction field for $y' = \frac{x}{y}$

2.

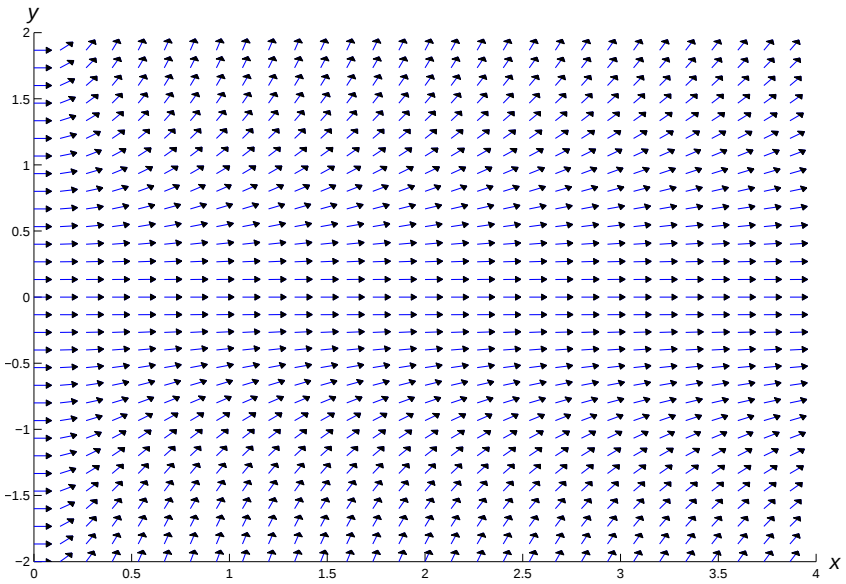



Figure 1.3E.2 : A direction field for $y' = \frac{2xy^2}{1+x^2}$

3.

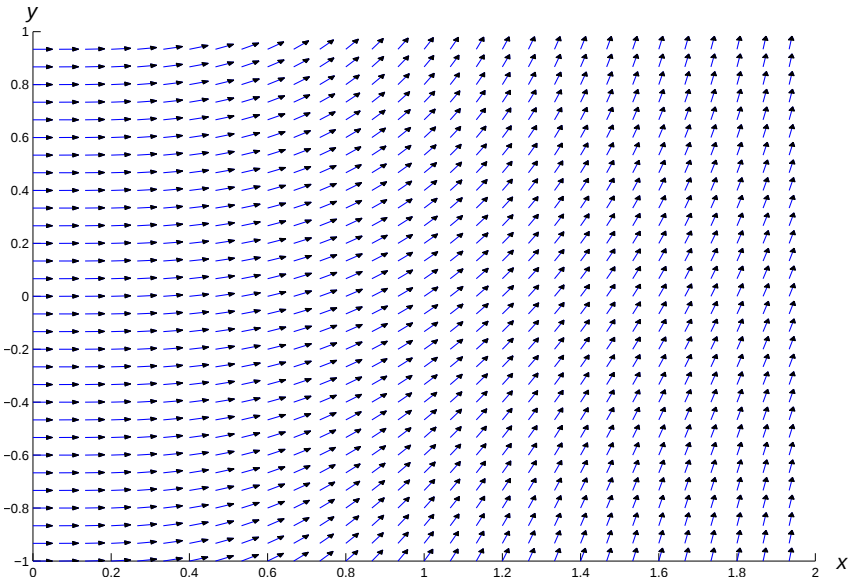



Figure 1.3E.3 : A direction field for $y' = x^2(1 + y^2)$

4.

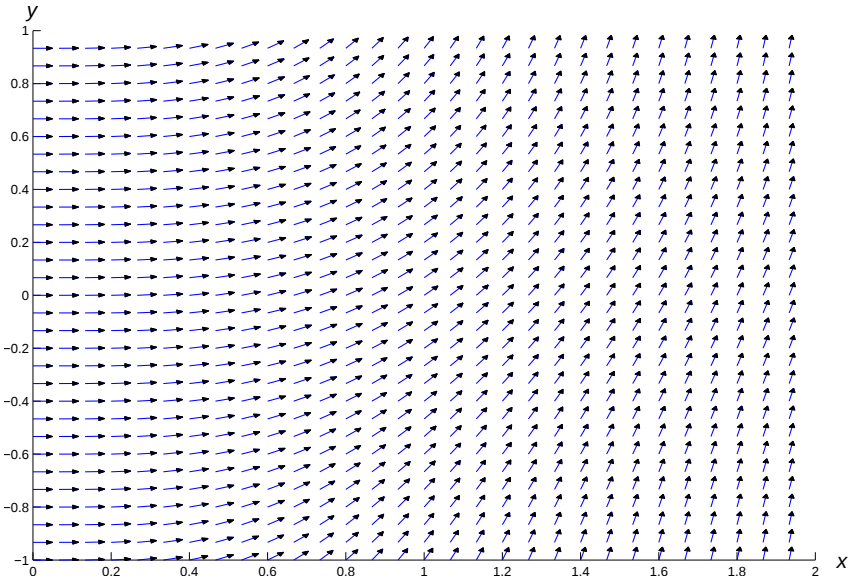



Figure 1.3E.4 : A direction field for $y' = \frac{1}{1+x^2+y^2}$

5.


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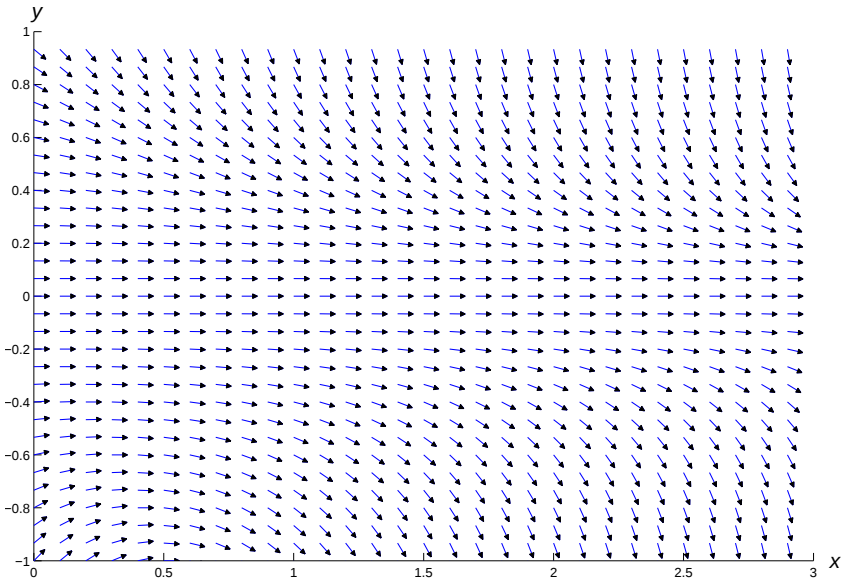



Figure 1.3E.5 : A direction field for $y' = -(2xy^2 + y^3)$

6.

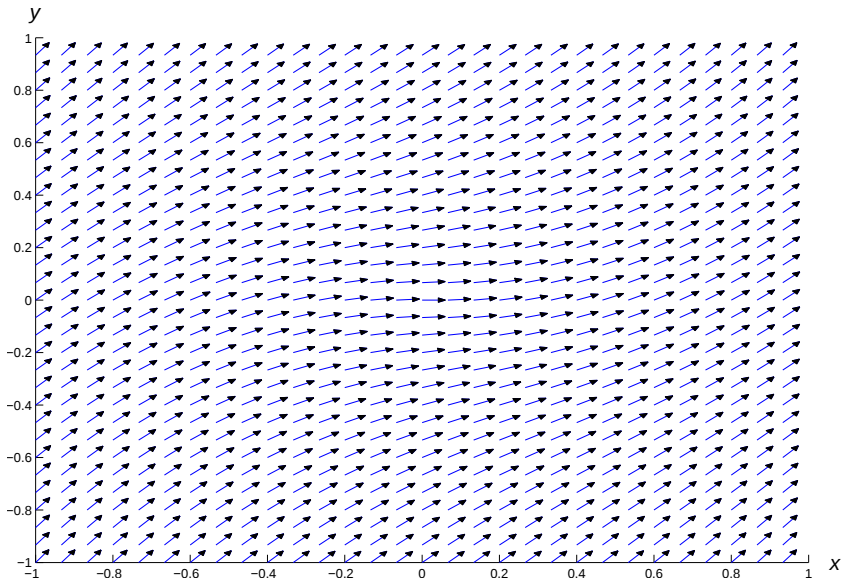



Figure 1.3E.6 : A direction field for $y' = (x^2 + y^2)^{1/2}$

7.


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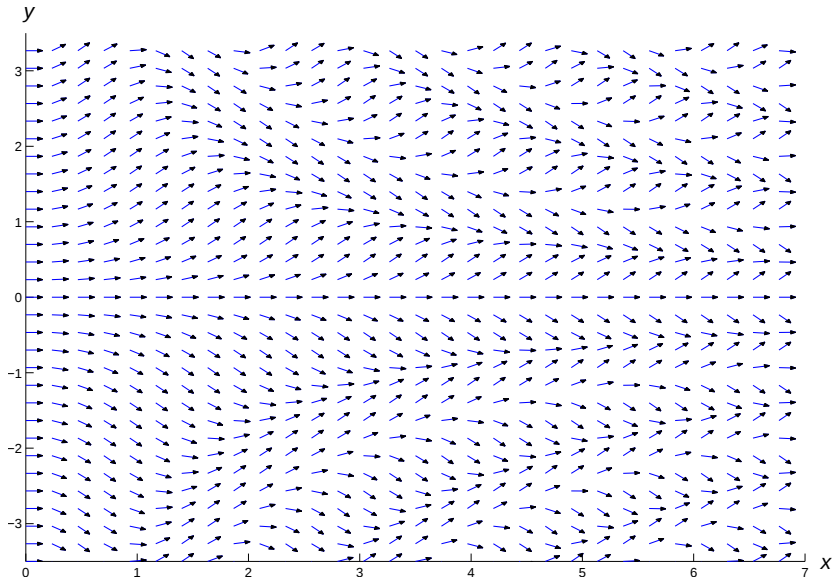


Figure 1.3E.7 : A direction field for $y' = \sin xy$

8.

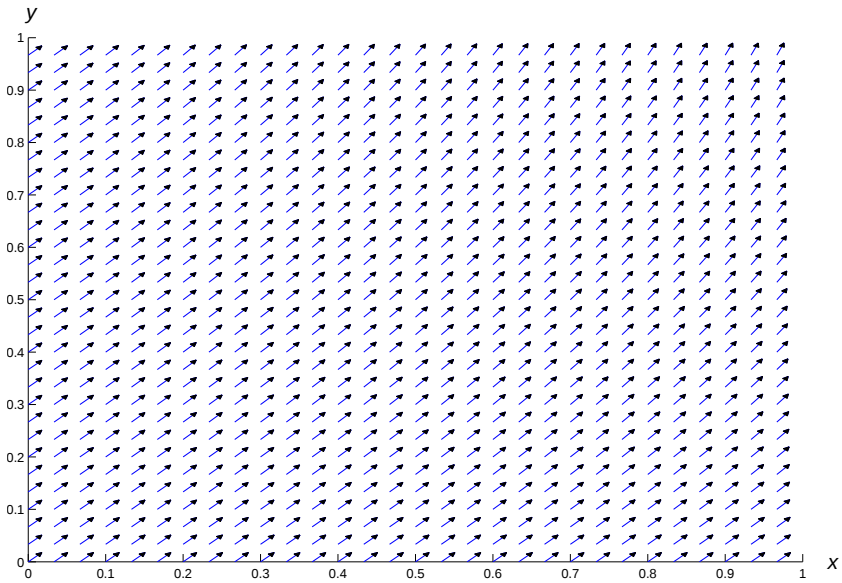


Figure 1.3E.8 : A direction field for $y' = e^{xy}$

9.


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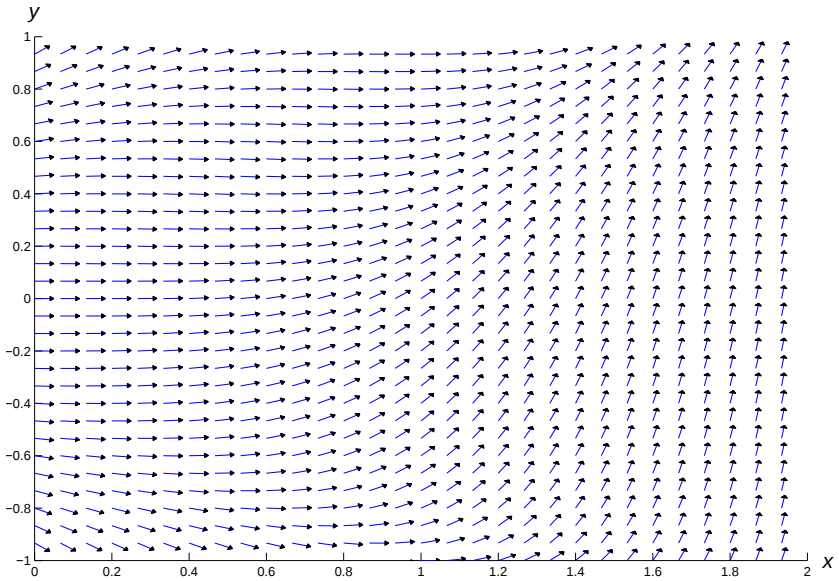



Figure 1.3E.9 : A direction field for $y' = (x - y^2)(x^2 - y)$

10.

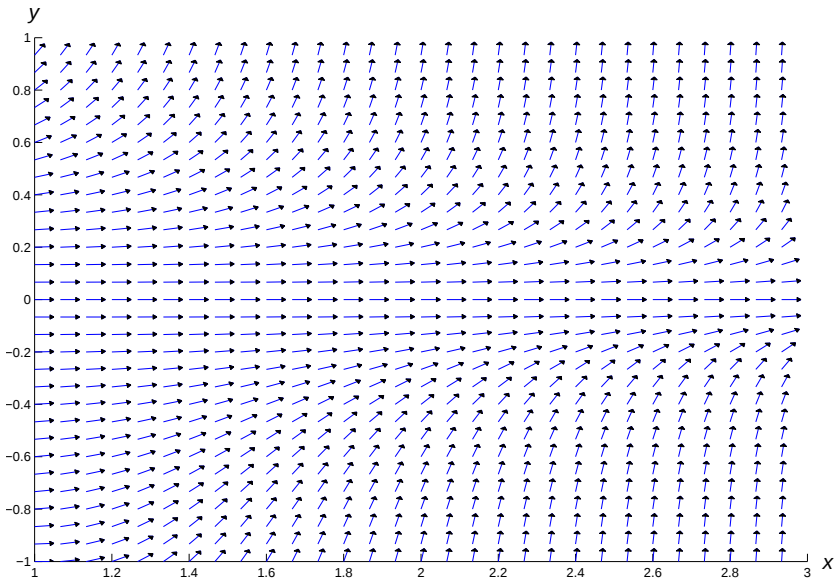



Figure 1.3E.10 : A direction field for $y' = x^3y^2 + xy^3$

11.


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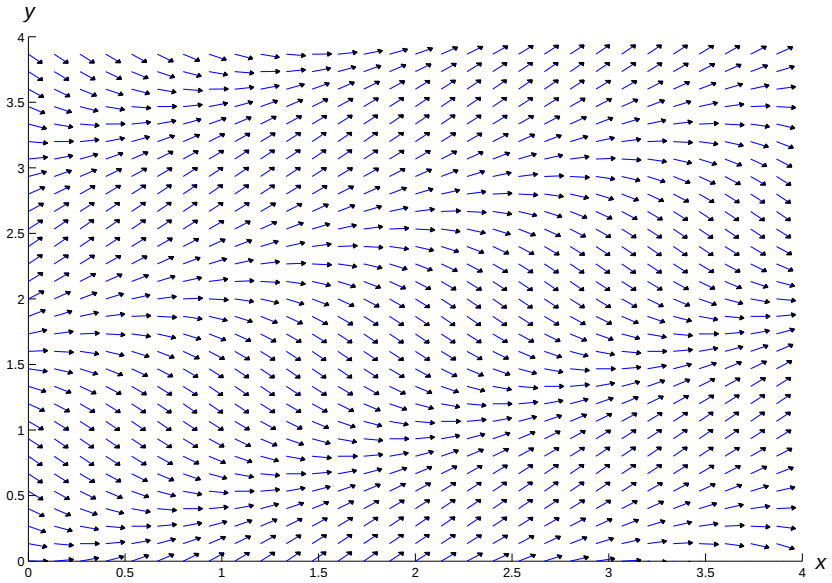


Figure 1.3E.11 : A direction field for $y' = \sin(x - 2y)$

**In Exercises 12 - 22 construct a direction field and plot some integral curves in the indicated rectangular region.**

12. $y' = y(y-1); \quad \{-1 \le x \le 2,\ -2 \le y \le 2\}$

13. $y' = 2 - 3xy; \quad \{-1 \le x \le 4,\ -4 \le y \le 4\}$

14. $y' = xy(y-1); \quad \{-2 \le x \le 2,\ -4 \le y \le 4\}$

15. $y' = 3x + y; \quad \{-2 \le x \le 2,\ 0 \le y \le 4\}$

16. $y' = y - x^3; \quad \{-2 \le x \le 2,\ -2 \le y \le 2\}$

17. $y' = 1 - x^2 - y^2; \quad \{-2 \le x \le 2,\ -2 \le y \le 2\}$

18. $y' = x(y^2 - 1); \quad \{-3 \le x \le 3,\ -3 \le y \le 2\}$

19. $y' = \frac{x}{y(y^2-1)}; \quad \{-2 \le x \le 2,\ -2 \le y \le 2\}$

20. $y' = \frac{xy^2}{y-1}; \quad \{-2 \le x \le 2,\ -1 \le y \le 4\}$

21. $y' = \frac{x(y^2-1)}{y}; \quad \{-1 \le x \le 1,\ -2 \le y \le 2\}$

22. $y' = -\frac{x^2+y^2}{1-x^2-y^2}; \quad \{-2 \le x \le 2,\ -2 \le y \le 2\}$

23. By suitably renaming the constants and dependent variables in the equations

$$T' = -k(T - T_m) \tag{A}$$

and

$$G' = -\lambda G + r \tag{B}$$

discussed in Section 1.2 in connection with Newton's law of cooling and absorption of glucose in the body, we can write both as

$$y' = -ay + b, \tag{C}$$

where $a$ is a positive constant and $b$ is an arbitrary constant. Thus, (A) is of the form (C) with $y = T$, $a = k$, and $b = kT_m$, and (B) is of the form (C) with $y = G$, $a = \lambda$, and $b = r$. We'll encounter equations of the form (C) in many other applications in Chapter 2.

Choose a positive $a$ and an arbitrary $b$. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T,\ c \le y \le d\}$$

of the $ty$-plane. Vary $T$, $c$, and $d$ until you discover a common property of all the solutions of (C). Repeat this experiment with various choices of $a$ and $b$ until you can state this property precisely in terms of $a$ and $b$.

24. By suitably renaming the constants and dependent variables in the equations

$$P' = aP(1 - \alpha P) \tag{A}$$

and

$$I' = rI(S - I) \tag{B}$$

discussed in Section 1.1 in connection with Verhulst's population model and the spread of an epidemic, we can write both in the form

$$y' = ay - by^2, \tag{C}$$

where $a$ and $b$ are positive constants. Thus, (A) is of the form (C) with $y = P$, $a = a$, and $b = a\alpha$, and (B) is of the form (C) with $y = I$, $a = rS$, and $b = r$. In Chapter 2 we'll encounter equations of the form (C) in other applications..

Choose positive numbers $a$ and $b$. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T,\ 0 \le y \le d\}$$

of the $ty$-plane. Vary $T$ and $d$ until you discover a common property of all solutions of (C) with $y(0) > 0$. Repeat this experiment with various choices of $a$ and $b$ until you can state this property precisely in terms of $a$ and $b$.

Choose positive numbers $a$ and $b$. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T,\ c \le y \le 0\}$$

of the $ty$-plane. Vary $a$, $b$, $T$ and $c$ until you discover a common property of all solutions of (C) with $y(0) < 0$.

You can verify your results later by doing Exercise 2.2.27.

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# CHAPTER OVERVIEW

## 2: First Order Equations

In this chapter, we study first-order differential equations for which there are general methods of solution.


2.1: Linear First Order Equations

2.1E: Linear First Order Equations (Exercises)

2.2: Separable Equations

2.2E: Separable Equations (Exercises)

2.3: Existence and Uniqueness of Solutions of Nonlinear Equations

2.3E: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises)

2.4: Transformation of Nonlinear Equations into Separable Equations

2.4E: Transformation of Nonlinear Equations into Separable Equations (Exercises)

2.5: Exact Equations

2.5E: Exact Equations (Exercises)

2.6: Integrating Factors

2.6E: Integrating Factors (Exercises)



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# 2.1: Linear First Order Equations

A first order differential equation is said to be **linear** if it can be written as

$$y' + p(x)y = f(x). \tag{2.1.1}$$

A first order differential equation that cannot be written like this is **nonlinear**. We say that Equation 2.1.1 is homogeneous if $f \equiv 0$; otherwise it is **nonhomogeneous**. Since $y \equiv 0$ is obviously a solution of the homgeneous equation

$$y' + p(x)y = 0,$$

we call it the **trivial solution**. Any other solution is **nontrivial**.

**Example 2.1.1**

The first order equations

$$\begin{aligned} x^2y' + 3y &= x^2 \\ xy' - 8x^2y &= \sin x \\ xy' + (\ln x)y &= 0 \\ y' &= x^2y - 2 \end{aligned}$$

are not in the form in Equation 2.1.1, but they are linear, since they can be rewritten as

$$\begin{aligned} y' + \frac{3}{x^2}y &= 1 \\ y' - 8xy &= \frac{\sin x}{x} \\ y' + \frac{\ln x}{x}y &= 0 \\ y' - x^2y &= -2 \end{aligned}$$

**Example 2.1.2**

Here are some nonlinear first order equations:

$$\begin{aligned} xy' + 3y^2 &= 2x && \text{(because } y \text{ is squared)} \\ yy' &= 3 && \text{(because of the product } yy') \\ y' + xe^y &= 12 && \text{(because of } e^y) \end{aligned}$$

## General Solution of a Linear First Order Equation

To motivate a definition that we'll need, consider the simple linear first order equation

$$y' = \frac{1}{x^2}. \tag{2.1.2}$$

From calculus we know that $y$ satisfies this equation if and only if

$$y = -\frac{1}{x} + c, \tag{2.1.3}$$

where $c$ is an arbitrary constant. We call $c$ a **parameter** and say that Equation 2.1.3 defines a **one–parameter family** of functions. For each real number $c$, the function defined by Equation 2.1.3 is a solution of Equation 2.1.2 on $(-\infty, 0)$ and $(0, \infty)$; moreover, every solution of Equation 2.1.2 on either of these intervals is of the form Equation 2.1.3 for some choice of $c$. We say that Equation 2.1.3 is **the general solution** of Equation 2.1.2.

We'll see that a similar situation occurs in connection with any first order linear equation

$$y' + p(x)y = f(x); \tag{2.1.4}$$

that is, if $p$ and $f$ are continuous on some open interval $(a, b)$ then there's a unique formula $y = y(x, c)$ analogous to Equation 2.1.3 that involves $x$ and a parameter $c$ and has the these properties:

- For each fixed value of $c$, the resulting function of $x$ is a solution of Equation 2.1.4 on $(a, b)$.
- If $y$ is a solution of Equation 2.1.4 on $(a, b)$, then $y$ can be obtained from the formula by choosing $c$ appropriately.

We'll call $y = y(x, c)$ the **general solution** of Equation 2.1.4.

When this has been established, it will follow that an equation of the form

$$P_0(x)y' + P_1(x)y = F(x) \tag{2.1.5}$$

has a general solution on any open interval $(a, b)$ on which $P_0$, $P_1$, and $F$ are all continuous and $P_0$ has no zeros, since in this case we can rewrite Equation 2.1.5 in the form Equation 2.1.4 with $p = P_1/P_0$ and $f = F/P_0$, which are both continuous on $(a, b)$.

To avoid awkward wording in examples and exercises, we will not specify the interval $(a, b)$ when we ask for the general solution of a specific linear first order equation. Let's agree that this always means that we want the general solution on every open interval on which $p$ and $f$ are continuous if the equation is of the form Equation 2.1.4, or on which $P_0$, $P_1$, and $F$ are continuous and $P_0$ has no zeros, if the equation is of the form Equation 2.1.5. We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if $P_0$, $P_1$, and $F$ are all continuous on an open interval $(a, b)$, but $P_0$ **does** have a zero in $(a, b)$, then Equation 2.1.5 may fail to have a general solution on $(a, b)$ in the sense just defined. Since this isn't a major point that needs to be developed in depth, we will not discuss it further; however, see **Exercise 2.1.44** for an example.

## Homogeneous Linear First Order Equations

We begin with the problem of finding the general solution of a homogeneous linear first order equation. The next example recalls a familiar result from calculus.

### Example 2.1.3

Let $a$ be a constant.

a. Find the general solution of

$$y' - ay = 0. \tag{2.1.6}$$

b. Solve the initial value problem

$$y' - ay = 0, \quad y(x_0) = y_0.$$

**Solution a**

(a) You already know from calculus that if $c$ is any constant, then $y = ce^{ax}$ satisfies Equation 2.1.6. However, let's pretend you've forgotten this, and use this problem to illustrate a general method for solving a homogeneous linear first order equation.

We know that Equation 2.1.6 has the trivial solution $y \equiv 0$. Now suppose $y$ is a nontrivial solution of Equation 2.1.6. Then, since a differentiable function must be continuous, there must be some open interval $I$ on which $y$ has no zeros. We rewrite Equation 2.1.6 as

$$\frac{y'}{y} = a$$

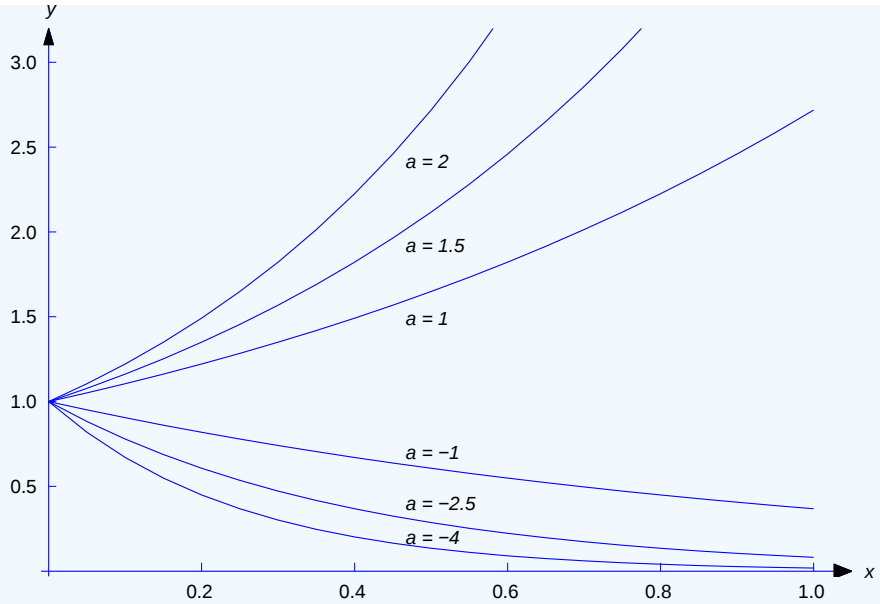


Figure 2.1.1 : Solutions of $y' - ay = 0$ , $y(0) = 1$

for $x$ in $I$. Integrating this shows that

$$\ln|y| = ax + k, \quad \text{so} \quad |y| = e^k e^{ax},$$

where $k$ is an arbitrary constant. Since $e^{ax}$ can never equal zero, $y$ has no zeros, so $y$ is either always positive or always negative. Therefore we can rewrite $y$ as

$$y = ce^{ax} \tag{2.1.7}$$

where

$$c = \begin{cases} e^k & \text{if} y > 0, \\ -e^k & \text{if} y < 0. \end{cases}$$

This shows that every nontrivial solution of Equation 2.1.6 is of the form $y = ce^{ax}$ for some nonzero constant $c$. Since setting $c = 0$ yields the trivial solution, **all** solutions of Equation 2.1.6 are of the form Equation 2.1.7. Conversely, Equation 2.1.7 is a solution of Equation 2.1.6 for every choice of $c$, since differentiating Equation 2.1.7 yields $y' = ace^{ax} = ay$ .

**Solution b**

Imposing the initial condition $y(x_0) = y_0$ yields $y_0 = ce^{ax_0}$ , so $c = y_0 e^{-ax_0}$ and

$$y = y_0 e^{-ax_0} e^{ax} = y_0 e^{a(x-x_0)}.$$

Figure 2.1.1 show the graphs of this function with $x_0 = 0$, $y_0 = 1$, and various values of $a$.

## Example 2.1.4

a. Find the general solution of

$$xy' + y = 0. \tag{2.1.8}$$

b. Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3. \tag{2.1.9}$$

**Solution a**

We rewrite Equation 2.1.8 as

$$y' + \frac{1}{x}y = 0, \tag{2.1.10}$$

where $x$ is restricted to either $(-\infty, 0)$ or $(0, \infty)$. If $y$ is a nontrivial solution of Equation 2.1.10, there must be some open interval I on which $y$ has no zeros. We can rewrite Equation 2.1.10 as

$$\frac{y'}{y} = -\frac{1}{x}$$

for $x$ in $I$. Integrating shows that

$$\ln|y| = -\ln|x| + k, \quad so \quad |y| = \frac{e^k}{|x|}.$$

Since a function that satisfies the last equation can't change sign on either $(-\infty, 0)$ or $(0, \infty)$, we can rewrite this result more simply as

$$y = \frac{c}{x} \tag{2.1.11}$$

where

$$c = \begin{cases} e^k & \text{if} y > 0, \\ -e^k & \text{if} y < 0. \end{cases}$$

We have now shown that every solution of Equation 2.1.10 is given by Equation 2.1.11 for some choice of $c$. (Even though we assumed that $y$ was nontrivial to derive Equation 2.1.11, we can get the trivial solution by setting $c = 0$ in Equation 2.1.11.) Conversely, any function of the form Equation 2.1.11 is a solution of Equation 2.1.10, since differentiating Equation 2.1.11 yields

$$y' = -\frac{c}{x^2},$$

and substituting this and Equation 2.1.11 into Equation 2.1.10 yields

$$\begin{aligned} y' + \frac{1}{x}y &= -\frac{c}{x^2} + \frac{1}{x}\frac{c}{x} \\ &= -\frac{c}{x^2} + \frac{c}{x^2} = 0. \end{aligned}$$

Figure 2.1.2 shows the graphs of some solutions corresponding to various values of $c$

**Solution b**

Imposing the initial condition $y(1) = 3$ in Equation 2.1.11 yields $c = 3$. Therefore the solution of Equation 2.1.9 is

$$y = \frac{3}{x}.$$

The interval of validity of this solution is $(0, \infty)$.

The results in Examples $2.1.3a$ and $2.1.4b$ are special cases of the next theorem.

**Theorem 2.1.1**

If $p$ is continuous on $(a, b)$, then the general solution of the homogeneous equation

$$y' + p(x)y = 0 \tag{2.1.12}$$

on $(a, b)$ is

$$y = ce^{-P(x)},$$

where

$$P(x) = \int p(x)\,dx \tag{2.1.13}$$

is any antiderivative of $p$ on $(a,b)$; that is ,

$$P'(x)=p(x), \quad a<x<b \tag{2.1.14}$$

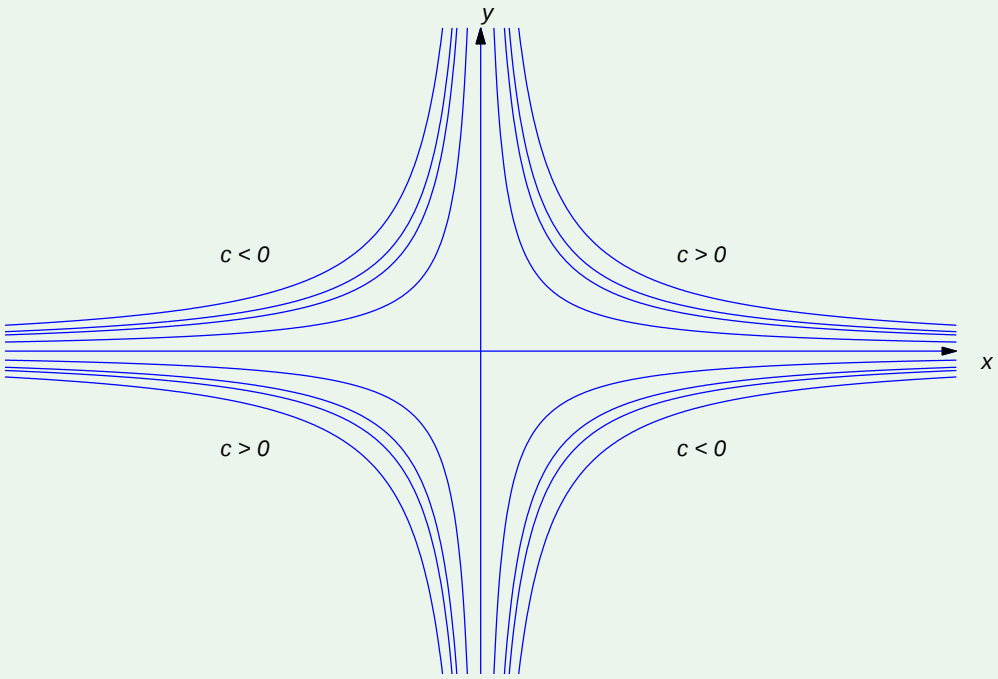



Figure 2.1.2 : Solutions of $xy'+y=0$ on $(0,\infty)$ and $(-\infty,0)$

**Proof**

If $y=ce^{-P(x)}$ , differentiating $y$ and using Equation 2.1.14 shows that

$$y'=-P'(x)ce^{-P(x)}=-p(x)ce^{-P(x)}=-p(x)y,$$

so $y'+p(x)y=0$ ; that is, $y$ is a solution of Equation 2.1.12, for any choice of $c$.

Now we'll show that any solution of Equation 2.1.12 can be written as $y=ce^{-P(x)}$ for some constant $c$. The trivial solution can be written this way, with $c=0$. Now suppose $y$ is a nontrivial solution. Then there's an open subinterval $I$ of $(a,b)$ on which $y$ has no zeros. We can rewrite Equation 2.1.12 as

$$\frac{y'}{y}=-p(x) \tag{2.1.15}$$

for $x$ in $I$. Integrating Equation 2.1.15 and recalling Equation 2.1.13 yields

$$\ln|y|=-P(x)+k,$$

where $k$ is a constant. This implies that

$$|y|=e^k e^{-P(x)}.$$

Since $P$ is defined for all $x$ in $(a,b)$ and an exponential can never equal zero, we can take $I=(a,b)$, so $y$ has zeros on $(a,b)$ $(a,b)$, so we can rewrite the last equation as $y=ce^{-P(x)}$ , where

$$c=\begin{cases} e^k & \text{if } y>0 \text{ on } (a,b), \\ -e^k & \text{if } y<0 \text{ on } (a,b). \end{cases}$$

REMARK: Rewriting a first order differential equation so that one side depends only on $y$ and $y'$ and the other depends only on $x$ is called separation of variables. We did this in Examples 2.1.3 and 2.1.4 , and in rewriting Equation 2.1.12 and Equation 2.1.15. We will apply this method to nonlinear equations in Section 2.2.

## Linear Nonhomogeneous First Order Equations

We'll now solve the nonhomogeneous equation

$$y' + p(x)y = f(x). \tag{2.1.16}$$

When considering this equation we call

$$y' + p(x)y = 0$$

the **complementary equation**.

We'll find solutions of Equation 2.1.16 in the form $y = uy_1$, where $y_1$ is a nontrivial solution of the complementary equation and $u$ is to be determined. This method of using a solution of the complementary equation to obtain solutions of a nonhomogeneous equation is a special case of a method called **variation of parameters**, which you'll encounter several times in this book. (Obviously, $u$ can't be constant, since if it were, the left side of Equation 2.1.16 would be zero. Recognizing this, the early users of this method viewed $u$ as a "parameter" that varies; hence, the name "variation of parameters.")

If

$$y = uy_1, \quad \text{then} \quad y' = u'y_1 + uy_1'.$$

Substituting these expressions for $y$ and $y'$ into Equation 2.1.16 yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x),$$

which reduces to

$$u'y_1 = f(x), \tag{2.1.17}$$

since $y_1$ is a solution of the complementary equation; that is,

$$y_1' + p(x)y_1 = 0.$$

In the proof of Theorem 2.2.1 we saw that $y_1$ has no zeros on an interval where $p$ is continuous. Therefore we can divide Equation 2.1.17 through by $y_1$ to obtain

$$u' = f(x)/y_1(x).$$

We can integrate this (introducing a constant of integration), and multiply the result by $y_1$ to get the general solution of Equation 2.1.16. Before turning to the formal proof of this claim, let's consider some examples.

### Example 2.1.5

Find the general solution of

$$y' + 2y = x^3e^{-2x}. \tag{2.1.18}$$

By applying a of Example 2.1.3 with $a = -2$, we see that $y_1 = e^{-2x}$ is a solution of the complementary equation $y' + 2y = 0$. Therefore we seek solutions of Equation 2.1.18 in the form $y = ue^{-2x}$, so that

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y' + 2y = u'e^{-2x} - 2ue^{-2x} + 2ue^{-2x} = u'e^{-2x}. \tag{2.1.19}$$

Therefore $y$ is a solution of Equation 2.1.18 if and only if

$$u'e^{-2x} = x^3e^{-2x} \quad \text{or, equivalently,} \quad u' = x^3.$$

Therefore

$$u = \frac{x^4}{4} + c,$$

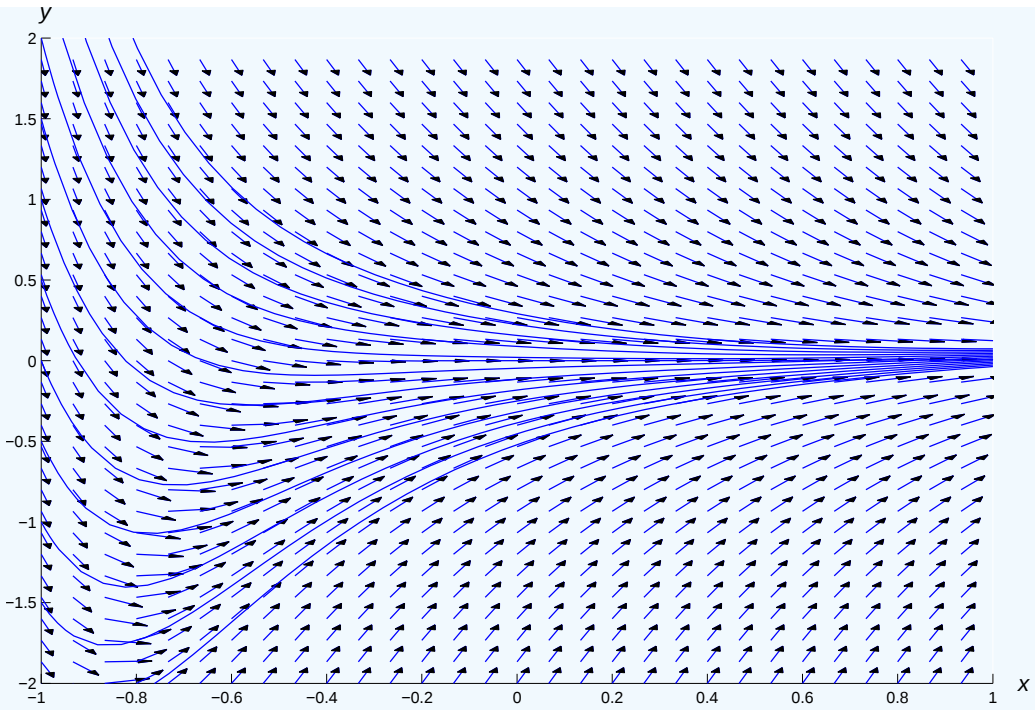


Figure 2.1.3 : A direction field and integral curves for $y' + 2y = x^2 e^{-2x}$

and

$$y = ue^{-2x} = e^{-2x}\left(\frac{x^4}{4} + c\right)$$

is the general solution of Equation 2.1.18.

Figure 2.1.3 shows a direction field and some integral curves for Equation 2.1.18.

## Example 2.1.6

Find the general solution

$$y' + (\cot x)y = x \csc x. \tag{2.1.20}$$

Solve the initial value problem

$$y' + (\cot x)y = x \csc x, \quad y(\pi/2) = 1. \tag{2.1.21}$$

### Solution

Here $p(x) = \cot x$ and $f(x) = x \csc x$ are both continuous except at the points $x = r\pi$, where $r$ is an integer. Therefore we seek solutions of Equation 2.1.20 on the intervals $(r\pi, (r+1)\pi)$. We need a nontrival solution $y_1$ of the complementary equation; thus, $y_1$ must satisfy $y_1' + (\cot x)y_1 = 0$ , which we rewrite as

$$\frac{y_1'}{y_1} = -\cot x = -\frac{\cos x}{\sin x}. \tag{2.1.22}$$

Integrating this yields

$$\ln|y_1| = -\ln|\sin x|,$$

where we take the constant of integration to be zero since we need only **one** function that satisfies Equation 2.1.22. Clearly $y_1 = 1/\sin x$ is a suitable choice. Therefore we seek solutions of Equation 2.1.20 in the form

$$y = \frac{u}{\sin x},$$

so that

$$y' = \frac{u'}{\sin x} - \frac{u\cos x}{\sin^2 x} \tag{2.1.23}$$

and

$$\begin{aligned} y' + (\cot x)y &= \tfrac{u'}{\sin x} - \tfrac{u\cos x}{\sin^2 x} + \tfrac{u\cot x}{\sin x} \\ &= \tfrac{u'}{\sin x} - \tfrac{u\cos x}{\sin^2 x} + \tfrac{u\cos x}{\sin^2 x} \\ &= \tfrac{u'}{\sin x}. \end{aligned} \tag{2.1.24}$$

Therefore $y$ is a solution of Equation 2.1.20 if and only if

$$u'/\sin x = x\csc x = x/\sin x \quad \text{or, equivalently,} \quad u' = x.$$

Integrating this yields

$$u = \frac{x^2}{2} + c, \quad \text{and} \quad y = \frac{u}{\sin x} = \frac{x^2}{2\sin x} + \frac{c}{\sin x}. \tag{2.1.25}$$

is the general solution of Equation 2.1.20 on every interval $(r\pi, (r+1)\pi)$ ( $r$ =integer).

b. Imposing the initial condition $y(\pi/2) = 1$ in Equation 2.1.25 yields

$$1 = \frac{\pi^2}{8} + c \text{ or} c = 1 - \frac{\pi^2}{8}.$$

Thus,

$$y = \frac{x^2}{2\sin x} + \frac{(1 - \pi^2/8)}{\sin x}$$

is a solution of Equation 2.1.20. The interval of validity of this solution is $(0, \pi)$; Figure 2.1.4 shows its graph.

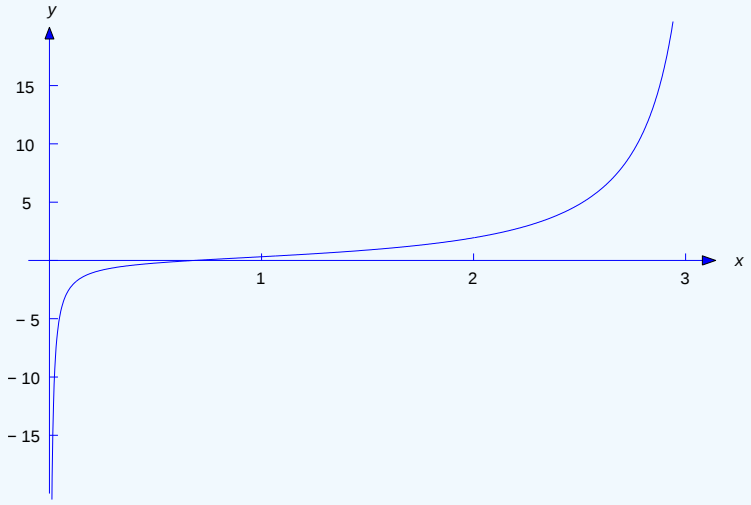



Figure 2.1.4 : Solution of $y' + (\cot x)y = x\csc x, y(\pi/2) = 1$

REMARK: It wasn't necessary to do the computations 2.1.23 and 2.1.24 in Example 2.1.6 , since we showed in the discussion preceding Example 2.1.5 that if $y = uy_1$ where $y_1' + p(x)y_1 = 0$ , then $y' + p(x)y = u'y_1$ . We did these computations so you would see this happen in this specific example. We recommend that you include these "unnecesary" computations in doing exercises, until you're confident that you really understand the method. After that, omit them.

We summarize the method of variation of parameters for solving

$$y' + p(x)y = f(x) \tag{2.1.26}$$

as follows:

a. Find a function $y_1$ such that

$$\frac{y_1'}{y_1} = -p(x).$$

For convenience, take the constant of integration to be zero.

b. Write

$$y = uy_1 \tag{2.1.27}$$

to remind yourself of what you're doing.

c. Write $u'y_1 = f$ and solve for $u'$; thus, $u' = f/y_1$.

d. Integrate $u'$ to obtain $u$, with an arbitrary constant of integration.

e. Substitute $u$ into Equation 2.1.27 to obtain $y$.

To solve an equation written as

$$P_0(x)y' + P_1(x)y = F(x),$$

we recommend that you divide through by $P_0(x)$ to obtain an equation of the form Equation 2.1.26 and then follow this procedure.

## Solutions in Integral Form

Sometimes the integrals that arise in solving a linear first order equation can't be evaluated in terms of elementary functions. In this case the solution must be left in terms of an integral.

### Example 2.1.7

Find the general solution of

$$y' - 2xy = 1.$$

Solve the initial value problem

$$y' - 2xy = 1, \quad y(0) = y_0. \tag{2.1.28}$$

**Solution**

a. To apply variation of parameters, we need a nontrivial solution $y_1$ of the complementary equation; thus, $y_1' - 2xy_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = 2x.$$

Integrating this and taking the constant of integration to be zero yields

$$\ln|y_1| = x^2, \quad \text{so} \quad |y_1| = e^{x^2}.$$

We choose $y_1 = e^{x^2}$ and seek solutions of Equation 2.1.28 in the form $y = ue^{x^2}$, where

$$u'e^{x^2} = 1, \quad \text{so} \quad u' = e^{-x^2}.$$

Therefore

$$u = c + \int e^{-x^2}\,dx,$$

but we can't simplify the integral on the right because there's no elementary function with derivative equal to $e^{-x^2}$. Therefore the best available form for the general solution of Equation 2.1.28 is

$$y = ue^{x^2} = e^{x^2}\left(c + \int e^{-x^2}\,dx\right). \tag{2.1.29}$$

b. Since the initial condition in Equation 2.1.28 is imposed at $x_0 = 0$, it is convenient to rewrite Equation 2.1.29 as

$$y = e^{x^2}\left(c + \int_0^x e^{-t^2}\,dt\right), \quad \text{since} \quad \int_0^0 e^{-t^2}\,dt = 0$$

Setting $x = 0$ and $y = y_0$ here shows that $c = y_0$. Therefore the solution of the initial value problem is

$$y = e^{x^2}\left(y_0 + \int_0^x e^{-t^2}\,dt\right). \tag{2.1.30}$$

For a given value of $y_0$ and each fixed $x$, the integral on the right can be evaluated by numerical methods. An alternate procedure is to apply the numerical integration procedures discussed in Chapter 3 directly to the initial value problem Equation 2.1.28. Figure 2.1.5 shows graphs of of Equation 2.1.30 for several values of $y_0$.

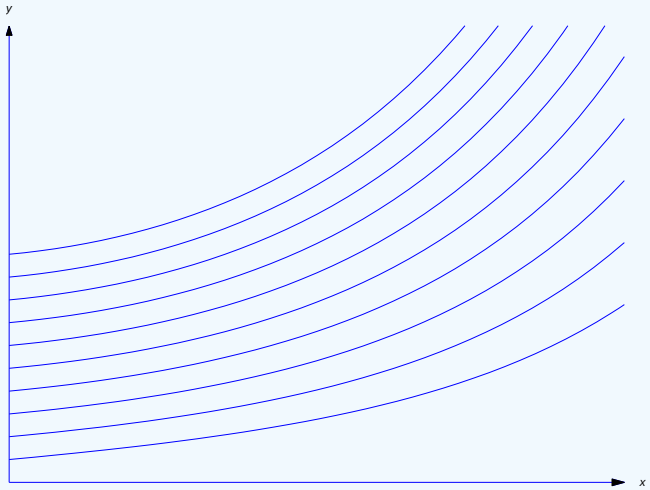



Figure 2.1.5 : Solution of $y' - 2xy = 1, y(0) = y_0$

## An Existence and Uniqueness Theorem

The method of variation of parameters leads to this theorem.

### Theorem 2.1.2 Existence and Uniqueness Theorem

Suppose $p$ and $f$ are continuous on an open interval $(a, b)$, and let $y_1$ be any nontrivial solution of the complementary equation

$$y' + p(x)y = 0$$

on $(a, b)$. Then:

a. The general solution of the nonhomogeneous equation

$$y' + p(x)y = f(x) \tag{2.1.31}$$

on $(a, b)$ is

$$y = y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right). \tag{2.1.32}$$

b. If $x_0$ is an arbitrary point in $(a, b)$ and $y_0$ is an arbitrary real number , then the initial value problem

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has the unique solution

$$y = y_1(x)\left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^x \frac{f(t)}{y_1(t)}\,dt\right)$$

on $(a, b)$.

**Proof**

(a) To show that Equation 2.1.32 is the general solution of Equation 2.1.31 on $(a,b)$, we must prove that:

i. If $c$ is any constant, the function $y$ in Equation 2.1.32 is a solution of Equation 2.1.31 on $(a,b)$.
ii. If $y$ is a solution of Equation 2.1.31 on $(a,b)$ then $y$ is of the form Equation 2.1.32 for some constant $c$.

To prove (i), we first observe that any function of the form Equation 2.1.32 is defined on $(a,b)$, since $p$ and $f$ are continuous on $(a,b)$. Differentiating Equation 2.1.32 yields

$$y' = y_1'(x)\left(c + \int f(x)/y_1(x)\,dx\right) + f(x).$$

Since $y_1' = -p(x)y_1$ , this and Equation 2.1.32 imply that

$$\begin{aligned} y' &= -p(x)y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right) + f(x) \\ &= -p(x)y(x) + f(x), \end{aligned}$$

which implies that $y$ is a solution of Equation 2.1.31.

To prove (ii), suppose $y$ is a solution of Equation 2.1.31 on $(a,b)$. From the proof of Theorem 2.1.1, we know that $y_1$ has no zeros on $(a,b)$, so the function $u = y/y_1$ is defined on $(a,b)$. Moreover, since

$$y' = -py + f \quad \text{and} \quad y_1' = -py_1,$$

$$\begin{aligned} u' &= \frac{y_1y' - y_1'y}{y_1^2} \\ &= \frac{y_1(-py+f) - (-py_1)y}{y_1^2} = \frac{f}{y_1}. \end{aligned}$$

Integrating $u' = f/y_1$ yields

$$u = \left(c + \int f(x)/y_1(x)\,dx\right),$$

which implies Equation 2.1.32, since $y = uy_1$ .

(b) We've proved (a), where $\int f(x)/y_1(x)\,dx$ in Equation 2.1.32 is an arbitrary antiderivative of $f/y_1$. Now it is convenient to choose the antiderivative that equals zero when $x = x_0$ , and write the general solution of Equation 2.1.31 as

$$y = y_1(x)\left(c + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt\right).$$

Since

$$y(x_0) = y_1(x_0)\left(c + \int_{x_0}^{x_0} \frac{f(t)}{y_1(t)}\,dt\right) = cy_1(x_0),$$

we see that $y(x_0) = y_0$ if and only if $c = y_0/y_1(x_0)$ .

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# 2.1E: Linear First Order Equations (Exercises)

## Q2.1.1

In **Exercises 2.1.1-2.1.5** find the general solution.

1. $y' + ay = 0$ ( $a$=constant)
2. $y' + 3x^2 y = 0$
3. $xy' + (\ln x)y = 0$
4. $xy' + 3y = 0$
5. $x^2 y' + y = 0$

## Q2.1.2

In **Exercises 2.1.6-2.1.11** solve the initial value problem.

6. $y' + \left(\frac{1+x}{x}\right) y = 0, \quad y(1) = 1$
7. $xy' + \left(1 + \frac{1}{\ln x}\right) y = 0, \quad y(e) = 1$
8. $xy' + (1 + x\cot x)y = 0, \quad y\left(\frac{\pi}{2}\right) = 2$
9. $y' - \left(\frac{2x}{1+x^2}\right) y = 0, \quad y(0) = 2$
10. $y' + \frac{k}{x} y = 0, \quad y(1) = 3 \quad (k = \text{constant})$
11. $y' + (\tan kx)y = 0, \quad y(0) = 2 \quad (k = \text{constant})$

## Q2.1.3

In **Exercises 2.1.12-2.1.15** find the general solution. Also, plot a direction field and some integral curves on the rectangular region $\{-2 \le x \le 2, -2 \le y \le 2\}$ .

12. $y' + 3y = 1$
13. $y' + \left(\frac{1}{x} - 1\right) y = -\frac{2}{x}$
14. $y' + 2xy = xe^{-x^2}$
15. $y' + \frac{2x}{1+x^2} y = \frac{e^{-x}}{1+x^2}$

## Q2.1.4

In **Exercises 2.1.16-2.1.24** find the general solution.

16. $y' + \frac{1}{x} y = \frac{7}{x^2} + 3$
17. $y' + \frac{4}{x-1} y = \frac{1}{(x-1)^5} + \frac{\sin x}{(x-1)^4}$
18. $xy' + (1 + 2x^2)y = x^3 e^{-x^2}$
19. $xy' + 2y = \frac{2}{x^2} + 1$
20. $y' + (\tan x)y = \cos x$
21. $(1 + x)y' + 2y = \frac{\sin x}{1+x}$
22. $(x - 2)(x - 1)y' - (4x - 3)y = (x - 2)^3$
23. $y' + (2\sin x \cos x)y = e^{-\sin^2 x}$
24. $x^2 y' + 3xy = e^x$

## Q2.1.5

In **Exercises 2.1.25-2.1.29** solve the initial value problem and sketch the graph of the solution.

25. $y' + 7y = e^{3x}, \quad y(0) = 0$

26. $(1+x^2)y' + 4xy = \frac{2}{1+x^2}, \quad y(0) = 1$

27. $xy' + 3y = \frac{2}{x(1+x^2)}, \quad y(-1) = 0$

28. $y' + (\cot x)y = \cos x, \quad y\left(\frac{\pi}{2}\right) = 1$

29. $y' + \frac{1}{x}y = \frac{2}{x^2} + 1, \quad y(-1) = 0$

## Q2.1.6

In **Exercises 2.1.30-2.1.37**, solve the initial value problem.

30. $(x-1)y' + 3y = \frac{1}{(x-1)^3} + \frac{\sin x}{(x-1)^2}, \quad y(0) = 1$

31. $xy' + 2y = 8x^2, \quad y(1) = 3$

32. $xy' - 2y = -x^2, \quad y(1) = 1$

33. $y' + 2xy = x, \quad y(0) = 3$

34. $(x-1)y' + 3y = \frac{1+(x-1)\sec^2 x}{(x-1)^3}, \quad y(0) = -1$

35. $(x+2)y' + 4y = \frac{1+2x^2}{x(x+2)^3}, \quad y(-1) = 2$

36. $(x^2-1)y' - 2xy = x(x^2-1), \quad y(0) = 4$

37. $(x^2-5)y' - 2xy = -2x(x^2-5), \quad y(2) = 7$

## Q2.1.7

In **Excercises 2.1.28-2.1.42** solve the initial value problem and leave the answer in a form involving a definite integral. (You can solve these problems numerically by methods discussed in Chapter 3.)

38. $y' + 2xy = x^2, \quad y(0) = 3$

39. $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2$

40. $y' + y = \frac{e^{-x}\tan x}{x}, \quad y(1) = 0$

41. $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1$

42. $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2$

43. Experiments indicate that glucose is absorbed by the body at a rate proportional to the amount of glucose present in the bloodstream. Let $\lambda$ denote the (positive) constant of proportionality. Now suppose glucose is injected into a patient's bloodstream at a constant rate of $r$ units per unit of time. Let $G = G(t)$ be the number of units in the patient's bloodstream at time $t > 0$. Then

$$G' = -\lambda G + r,$$

where the first term on the right is due to the absorption of the glucose by the patient's body and the second term is due to the injection. Determine $G$ for $t > 0$, given that $G(0) = G_0$. Also, find $\lim_{t\to\infty} G(t)$.

44.

(a) Plot a direction field and some integral curves for

$$xy' - 2y = -1 \tag{A}$$

on the rectangular region $\{-1 \le x \le 1, -.5 \le y \le 1.5\}$. What do all the integral curves have in common?

(b) Show that the general solution of (A) on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{1}{2} + cx^2.$$

(c) Show that $y$ is a solution of (A) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} \frac{1}{2} + c_1 x^2, & x \geq 0, \\ \frac{1}{2} + c_2 x^2, & x < 0, \end{cases}$$

where $c_1$ and $c_2$ are arbitrary constants.

(d) Conclude from c that all solutions of (A) on $(-\infty, \infty)$ are solutions of the initial value problem

$$xy' - 2y = -1, \quad y(0) = \frac{1}{2}.$$

(e) Use (b) to show that if $x_0 \neq 0$ and $y_0$ is arbitrary, then the initial value problem

$$xy' - 2y = -1, \quad y(x_0) = y_0$$

has infinitely many solutions on ( $-\infty, \infty$). Explain why this doesn't contradict Theorem 2.1.1.

45. Suppose $f$ is continuous on an open interval $(a, b)$ and $\alpha$ is a constant.

(a) Derive a formula for the solution of the initial value problem

$$y' + \alpha y = f(x), \quad y(x_0) = y_0, \tag{A}$$

where $x_0$ is in $(a, b)$ and $y_0$ is an arbitrary real number.

(b) Suppose $(a, b) = (a, \infty)$, $\alpha > 0$ and $\lim_{x \to \infty} f(x) = L$. Show that if $y$ is the solution of (A), then $\lim_{x \to \infty} y(x) = L/\alpha$.

46. Assume that all functions in this exercise are defined on a common interval $(a, b)$.

(a) Prove: If $y_1$ and $y_2$ are solutions of

$$y' + p(x)y = f_1(x)$$

and

$$y' + p(x)y = f_2(x)$$

respectively, and $c_1$ and $c_2$ are constants, then $y = c_1 y_1 + c_2 y_2$ is a solution of

$$y' + p(x)y = c_1 f_1(x) + c_2 f_2(x).$$

(This is the **principle of superposition.)**

(b) Use (a) to show that if $y_1$ and $y_2$ are solutions of the nonhomogeneous equation

$$y' + p(x)y = f(x), \quad \text{(A)}$$

then $y_1 - y_2$ is a solution of the homogeneous equation

$$y' + p(x)y = 0. \quad \text{(B)}$$

(c) Use (a) to show that if $y_1$ is a solution of (A) and $y_2$ is a solution of (B), then $y_1 + y_2$ is a solution of (A).

47. Some nonlinear equations can be transformed into linear equations by changing the dependent variable. Show that if

$$g'(y)y' + p(x)g(y) = f(x)$$

where $y$ is a function of $x$ and $g$ is a function of $y$, then the new dependent variable $z = g(y)$ satisfies the linear equation

$$z' + p(x)z = f(x).$$

48. Solve by the method discussed in Exercise 47.

(a) $(\sec^2 y)y' - 3\tan y = -1$

(b) $e^{y^2}\left(2yy'+\frac{2}{x}\right)=\frac{1}{x^2}$

(c) $\frac{xy'}{y}+2\ln y=4x^2$

(d) $\frac{y'}{(1+y)^2}-\frac{1}{x(1+y)}=-\frac{3}{x^2}$

49. We've shown that if $p$ and $f$ are continuous on $(a,b)$ then every solution of

$$y'+p(x)y=f(x) \tag{A}$$

on $(a,b)$ can be written as $y=uy_1$, where $y_1$ is a nontrivial solution of the complementary equation for (A) and $u'=f/y_1$. Now suppose $f$, $f'$, ..., $f^{(m)}$ and $p$, $p'$, ..., $p^{(m-1)}$ are continuous on $(a,b)$, where $m$ is a positive integer, and define

$$\begin{aligned} f_0&=f,\\ f_j&=f'_{j-1}+pf_{j-1},\quad 1\le j\le m. \end{aligned}$$

Show that

$$u^{(j+1)}=\frac{f_j}{y_1},\quad 0\le j\le m.$$

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## 2.2: Separable Equations

A first order differential equation is **separable** if it can be written as

$$h(y)y' = g(x), \tag{2.2.1}$$

where the left side is a product of $y'$ and a function of $y$ and the right side is a function of $x$. Rewriting a separable differential equation in this form is called **separation of variables.** In Section 2.1, we used separation of variables to solve homogeneous linear equations. In this section we'll apply this method to nonlinear equations.

To see how to solve Equation 2.2.1, let's first assume that $y$ is a solution. Let $G(x)$ and $H(y)$ be antiderivatives of $g(x)$ and $h(y)$; that is,

$$H'(y) = h(y) \quad \text{and} \quad G'(x) = g(x). \tag{2.2.2}$$

Then, from the chain rule,

$$\frac{d}{dx}H(y(x)) = H'(y(x))y'(x) = h(y)y'(x).$$

Therefore Equation 2.2.1 is equivalent to

$$\frac{d}{dx}H(y(x)) = \frac{d}{dx}G(x).$$

Integrating both sides of this equation and combining the constants of integration yields

$$H(y(x)) = G(x) + c. \tag{2.2.3}$$

Although we derived this equation on the assumption that $y$ is a solution of Equation 2.2.1, we can now view it differently: Any differentiable function $y$ that satisfies Equation 2.2.3 for some constant $c$ is a solution of Equation 2.2.1. To see this, we differentiate both sides of Equation 2.2.3, using the chain rule on the left, to obtain

$$H'(y(x))y'(x) = G'(x),$$

which is equivalent to

$$h(y(x))y'(x) = g(x)$$

because of Equation 2.2.2.

In conclusion, to solve Equation 2.2.1 it suffices to find functions $G = G(x)$ and $H = H(y)$ that satisfy Equation 2.2.2. Then any differentiable function $y = y(x)$ that satisfies Equation 2.2.3 is a solution of Equation 2.2.1.

### Example 2.2.1

Solve the equation

$$y' = x(1 + y^2).$$

**Solution**

Separating variables yields

$$\frac{y'}{1 + y^2} = x.$$

Integrating yields

$$\tan^{-1} y = \frac{x^2}{2} + c$$

Therefore

$$y = \tan\left(\frac{x^2}{2} + c\right).$$

## Example 2.2.2

a. Solve the equation

$$y' = -\frac{x}{y}. \tag{2.2.4}$$

b. Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = 1. \tag{2.2.5}$$

c. Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = -2. \tag{2.2.6}$$

### Solution a

Separating variables in Equation 2.2.4 yields

$$yy' = -x.$$

Integrating yields

$$\frac{y^2}{2} = -\frac{x^2}{2} + c, \quad \text{or equivalently} \quad x^2 + y^2 = 2c.$$

The last equation shows that $c$ must be positive if $y$ is to be a solution of Equation 2.2.4 on an open interval. Therefore we let $2c = a^2$ (with $a > 0$) and rewrite the last equation as

$$x^2 + y^2 = a^2. \tag{2.2.7}$$

This equation has two differentiable solutions for $y$ in terms of $x$:

$$y = \sqrt{a^2 - x^2}, \quad -a < x < a, \tag{2.2.8}$$

and

$$y = -\sqrt{a^2 - x^2}, \quad -a < x < a. \tag{2.2.9}$$

The solution curves defined by Equation 2.2.8 are semicircles above the $x$-axis and those defined by Equation 2.2.9 are semicircles below the $x$-axis (Figure 2.2.1 ).

### Solution b

The solution of Equation 2.2.5 is positive when $x = 1$; hence, it is of the form Equation 2.2.8. Substituting $x = 1$ and $y = 1$ into Equation 2.2.7 to satisfy the initial condition yields $a^2 = 2$; hence, the solution of Equation 2.2.5 is

$$y = \sqrt{2 - x^2}, \quad -\sqrt{2} < x < \sqrt{2}.$$

### Solution c

The solution of Equation 2.2.6 is negative when $x = 1$ and is therefore of the form Equation 2.2.9. Substituting $x = 1$ and $y = -2$ into Equation 2.2.7 to satisfy the initial condition yields $a^2 = 5$. Hence, the solution of Equation 2.2.6 is

$$y = -\sqrt{5 - x^2}, \quad -\sqrt{5} < x < \sqrt{5}.$$

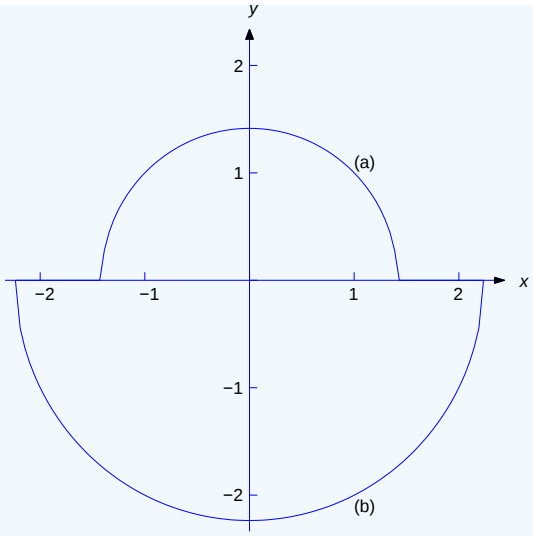



Figure 2.2.1 : (a) $y=\sqrt{2-x^2}\,,\ -\sqrt{2}<x<\sqrt{2}\,;$ (b) $y=-\sqrt{5-x^2}\,,\ -\sqrt{5}<x<\sqrt{5}$

## Implicit Solutions of Separable Equations

In Examples 2.2.1 and 2.2.2 we were able to solve the equation $H(y)=G(x)+c$ to obtain explicit formulas for solutions of the given separable differential equations. As we'll see in the next example, this isn't always possible. In this situation we must broaden our definition of a solution of a separable equation. The next theorem provides the basis for this modification. We omit the proof, which requires a result from advanced calculus called as the **implicit function theorem**.

### Theorem 2.2.1 : Implicit Function Theorem

Suppose $g=g(x)$ is continuous on $(a,b)$ and $h=h(y)$ are continuous on $(c,d)$. Let $G$ be an antiderivative of $g$ on $(a,b)$ and let $H$ be an antiderivative of $h$ on $(c,d)$. Let $x_0$ be an arbitrary point in $(a,b)$, let $y_0$ be a point in $(c,d)$ such that $h(y_0)\neq 0$, and define

$$c=H(y_0)-G(x_0). \tag{2.2.10}$$

Then there's a function $y=y(x)$ defined on some open interval $(a_1,b_1)$, where $a\le a_1<x_0<b_1\le b$, such that $y(x_0)=y_0$ and

$$H(y)=G(x)+c \tag{2.2.11}$$

for $a_1<x<b_1$ . Therefore $y$ is a solution of the initial value problem

$$h(y)y'=g(x),\quad y(x_0)=y_0. \tag{2.2.12}$$

It's convenient to say that Equation 2.2.11 with $c$ arbitrary is an **implicit solution** of $h(y)y'=g(x)$. Curves defined by Equation 2.2.11 are integral curves of $h(y)y'=g(x)$. If $c$ satisfies Equation 2.2.10, we'll say that Equation 2.2.11 is an **implicit solution of the initial value problem** Equation 2.2.12. However, keep these points in mind:

- For some choices of $c$ there may not be any differentiable functions $y$ that satisfy Equation 2.2.11.
- The function $y$ in Equation 2.2.11 (not Equation 2.2.11 itself) is a solution of $h(y)y'=g(x)$.

### Example 2.2.3

a. Find implicit solutions of

$$y'=\frac{2x+1}{5y^4+1}. \tag{2.2.13}$$

b. Find an implicit solution of

$$y' = \frac{2x+1}{5y^4+1}, \quad y(2) = 1. \tag{2.2.14}$$

**Solution a**

Separating variables yields

$$(5y^4+1)y' = 2x+1.$$

Integrating yields the implicit solution

$$y^5 + y = x^2 + x + c. \tag{2.2.15}$$

of Equation 2.2.13.

**Solution b**

Imposing the initial condition $y(2) = 1$ in Equation 2.2.15 yields $1+1 = 4+2+c$ , so $c = -4$. Therefore

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem Equation 2.2.14. Although more than one differentiable function $y = y(x)$ satisfies Equation 2.2.13 near $x = 1$, it can be shown that there is only one such function that satisfies the initial condition $y(1) = 2$. Figure 2.2.2 shows a direction field and some integral curves for Equation 2.2.13.

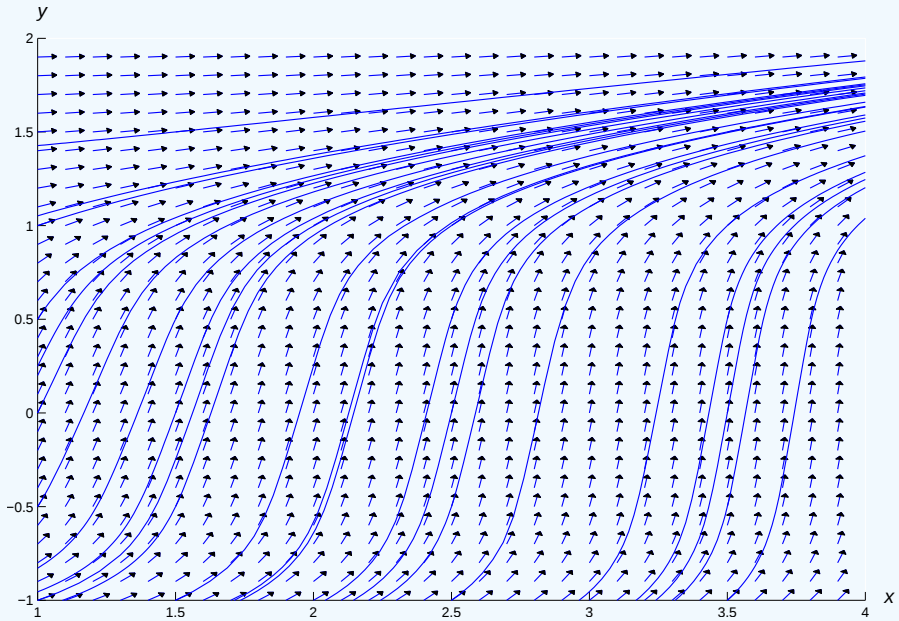



Figure 2.2.2 : A direction field and integral curves for $y' = \frac{2x+1}{5y^4+1}$

## Constant Solutions of Separable Equations

An equation of the form

$$y' = g(x)p(y)$$

is separable, since it can be rewritten as

$$\frac{1}{p(y)}y' = g(x).$$

However, the division by $p(y)$ is not legitimate if $p(y) = 0$ for some values of $y$. The next two examples show how to deal with this problem.

**Example 2.2.4**

Find all solutions of

$$y' = 2xy^2. \tag{2.2.16}$$

**Solution**

Here we must divide by $p(y) = y^2$ to separate variables. This isn't legitimate if $y$ is a solution of Equation 2.2.16 that equals zero for some value of $x$. One such solution can be found by inspection: $y \equiv 0$. Now suppose $y$ is a solution of Equation 2.2.16 that isn't identically zero. Since $y$ is continuous there must be an interval on which $y$ is never zero. Since division by $y^2$ is legitimate for $x$ in this interval, we can separate variables in Equation 2.2.16 to obtain

$$\frac{y'}{y^2} = 2x.$$

Integrating this yields

$$-\frac{1}{y} = x^2 + c,$$

which is equivalent to

$$y = -\frac{1}{x^2 + c}. \tag{2.2.17}$$

We've now shown that if $y$ is a solution of Equation 2.2.16 that is not identically zero, then $y$ must be of the form Equation 2.2.17. By substituting Equation 2.2.17 into Equation 2.2.16, you can verify that Equation 2.2.17 is a solution of Equation 2.2.16. Thus, solutions of Equation 2.2.16 are $y \equiv 0$ and the functions of the form Equation 2.2.17. Note that the solution $y \equiv 0$ isn't of the form Equation 2.2.17 for any value of $c$.

Figure 2.2.3 shows a direction field and some integral curves for Equation 2.2.16

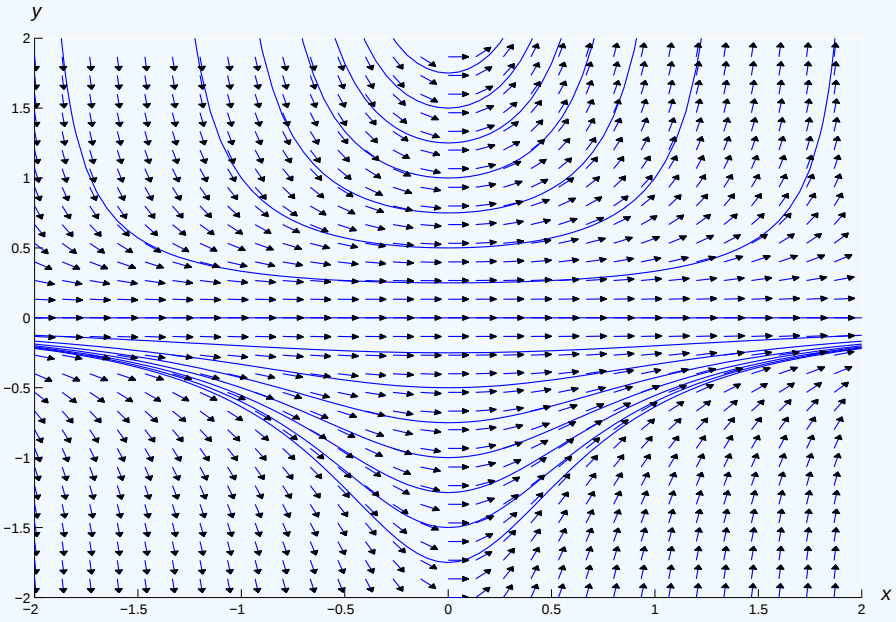



Figure 2.2.3 : A direction field and integral curves for $y' = 2xy^2$

## Example 2.2.5

Find all solutions of

$$y' = \frac{1}{2}x(1 - y^2). \tag{2.2.18}$$

Here we must divide by $p(y) = 1 - y^2$ to separate variables. This isn't legitimate if $y$ is a solution of Equation 2.2.18 that equals $\pm 1$ for some value of $x$. Two such solutions can be found by inspection: $y \equiv 1$ and $y \equiv -1$. Now suppose $y$ is a

solution of Equation 2.2.18 such that $1-y^2$ isn't identically zero. Since $1-y^2$ is continuous there must be an interval on which $1-y^2$ is never zero. Since division by $1-y^2$ is legitimate for $x$ in this interval, we can separate variables in Equation 2.2.18 to obtain

$$\frac{2y'}{y^2-1}=-x.$$

A partial fraction expansion on the left yields

$$\left[\frac{1}{y-1}-\frac{1}{y+1}\right]y'=-x,$$

and integrating yields

$$\ln\left|\frac{y-1}{y+1}\right|=-\frac{x^2}{2}+k;$$

hence,

$$\left|\frac{y-1}{y+1}\right|=e^k e^{-x^2/2}.$$

Since $y(x)\neq\pm1$ for $x$ on the interval under discussion, the quantity $(y-1)/(y+1)$ cannot change sign in this interval. Therefore we can rewrite the last equation as

$$\frac{y-1}{y+1}=ce^{-x^2/2},$$

where $c=\pm e^k$, depending upon the sign of $(y-1)/(y+1)$ on the interval. Solving for $y$ yields

$$y=\frac{1+ce^{-x^2/2}}{1-ce^{-x^2/2}}. \tag{2.2.19}$$

We've now shown that if $y$ is a solution of Equation 2.2.18 that is not identically equal to $\pm1$, then $y$ must be as in Equation 2.2.19. By substituting Equation 2.2.19 into Equation 2.2.18 you can verify that Equation 2.2.19 is a solution of Equation 2.2.18. Thus, the solutions of Equation 2.2.18 are $y\equiv1$, $y\equiv-1$ and the functions of the form Equation 2.2.19. Note that the constant solution $y\equiv1$ can be obtained from this formula by taking $c=0$; however, the other constant solution, $y\equiv-1$, cannot be obtained in this way.

Figure 2.2.4 shows a direction field and some integrals for Equation 2.2.18.

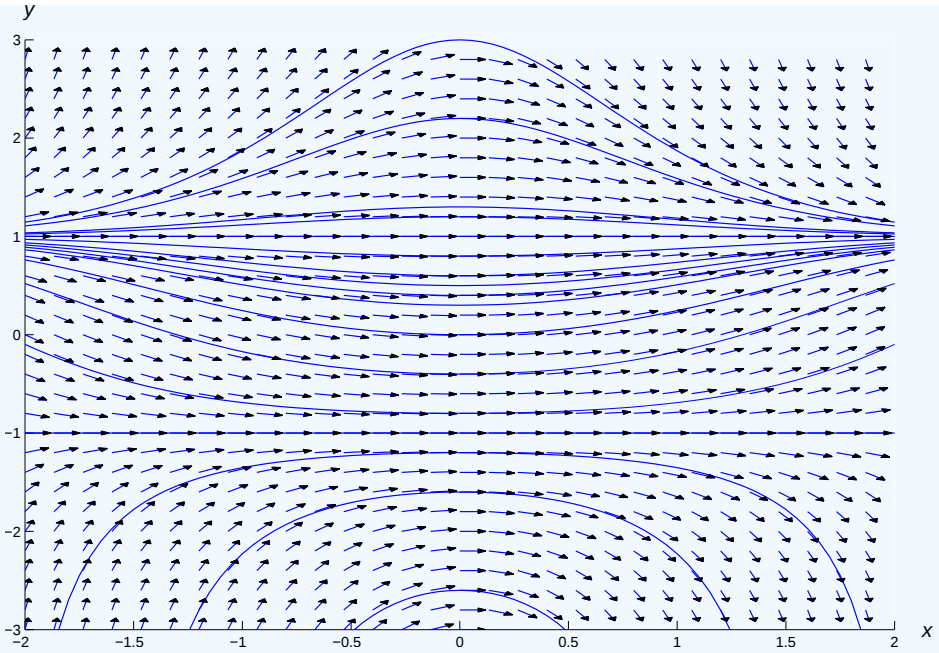



Figure 2.2.4 : A direction field and integral curves for $y' = \frac{x(1-y^2)}{2}$

## Differences Between Linear and Nonlinear Equations

Theorem 2.1.2 states that if $p$ and $f$ are continuous on $(a, b)$ then every solution of

$$y' + p(x)y = f(x)$$

on $(a, b)$ can be obtained by choosing a value for the constant $c$ in the general solution, and if $x_0$ is any point in $(a, b)$ and $y_0$ is arbitrary, then the initial value problem

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has a solution on $(a, b)$.

The not true for nonlinear equations. First, we saw in Examples 2.2.4 and 2.2.5 that a nonlinear equation may have solutions that cannot be obtained by choosing a specific value of a constant appearing in a one-parameter family of solutions. Second, it is in general impossible to determine the interval of validity of a solution to an initial value problem for a nonlinear equation by simply examining the equation, since the interval of validity may depend on the initial condition. For instance, in Example 2.2.2 we saw that the solution of

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(x_0) = y_0$$

is valid on $(-a, a)$, where $a = \sqrt{x_0^2 + y_0^2}$ .

### Example 2.2.6

Solve the initial value problem

$$y' = 2xy^2, \quad y(0) = y_0$$

and determine the interval of validity of the solution.

**Solution**

First suppose $y_0 \neq 0$. From Example 2.2.4 , we know that $y$ must be of the form

$$y = -\frac{1}{x^2 + c}. \tag{2.2.20}$$

Imposing the initial condition shows that $c = -1/y_0$. Substituting this into Equation 2.2.20 and rearranging terms yields the solution

$$y = \frac{y_0}{1 - y_0 x^2}.$$

This is also the solution if $y_0 = 0$. If $y_0 < 0$, the denominator isn't zero for any value of $x$, so the the solution is valid on $(-\infty, \infty)$. If $y_0 > 0$, the solution is valid only on $(-1/\sqrt{y_0}, 1/\sqrt{y_0})$.

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# 2.2E: Separable Equations (Exercises)

## Q2.2.1

In **Exercises 2.2.1-2.2.6** find all solutions.

1. $y' = \frac{3x^2+2x+1}{y-2}$
2. $(\sin x)(\sin y) + (\cos y)y' = 0$
3. $xy' + y^2 + y = 0$
4. $y' \ln|y| + x^2 y = 0$
5. $(3y^3 + 3y\cos y + 1)y' + \frac{(2x+1)y}{1+x^2} = 0$
6. $x^2 y y' = (y^2 - 1)^{3/2}$

## Q2.2.2

In **Exercises 2.2.7-2.2.10** find all solutions. Also, plot a direction field and some integral curves on the indicated rectangular region.

7. $y' = x^2(1+y^2);\ \{-1 \le x \le 1,\ -1 \le y \le 1\}$
8. $y'(1+x^2) + xy = 0;\ \{-2 \le x \le 2,\ -1 \le y \le 1\}$
9. $y' = (x-1)(y-1)(y-2);\ \{-2 \le x \le 2,\ -3 \le y \le 3\}$
10. $(y-1)^2 y' = 2x + 3;\ \{-2 \le x \le 2,\ -2 \le y \le 5\}$

## Q2.2.3

In **Exercises 2.2.11** and *2.2.12* solve the initial value problem.

11. $y' = \frac{x^2+3x+2}{y-2}, \quad y(1) = 4$
12. $y' + x(y^2 + y) = 0, \quad y(2) = 1$

## Q2.2.4

In **Exercises 2.2.13-2.2.16** solve the initial value problem and graph the solution.

13. $(3y^2 + 4y)y' + 2x + \cos x = 0, \quad y(0) = 1$
14. $y' + \frac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0$
15. $y' + 2x(y+1) = 0, \quad y(0) = 2$
16. $y' = 2xy(1+y^2), \quad y(0) = 1$

## Q2.2.5

In **Exercises 2.2.17-2.2.23** solve the initial value problem and find the interval of validity of the solution.

17. $y'(x^2+2) + 4x(y^2 + 2y + 1) = 0, \quad y(1) = -1$
18. $y' = -2x(y^2 - 3y + 2), \quad y(0) = 3$
19. $y' = \frac{2x}{1+2y}, \quad y(2) = 0$ &
20. $y' = 2y - y^2, \quad y(0) = 1$
21. $x + yy' = 0, \quad y(3) = -4$
22. $y' + x^2(y+1)(y-2)^2 = 0, \quad y(4) = 2$
23. $(x+1)(x-2)y' + y = 0, \quad y(1) = -3$

## Q2.2.6

24. Solve $y' = \frac{(1+y^2)}{(1+x^2)}$ explicitly.

25. Solve $y'\sqrt{1-x^2} + \sqrt{1-y^2} = 0$ explicitly.

26. Solve $y' = \frac{\cos x}{\sin y}, \quad y(\pi) = \frac{\pi}{2}$ explicitly.

27. Solve the initial value problem

$$y' = ay - by^2, \quad y(0) = y_0.$$

Discuss the behavior of the solution if a $y_0 \geq 0$; b $y_0 < 0$.

28. The population $P = P(t)$ of a species satisfies the logistic equation

$$P' = aP(1 - \alpha P)$$

and $P(0) = P_0 > 0$. Find $P$ for $t > 0$, and find $\lim_{t\to\infty} P(t)$.

29. An epidemic spreads through a population at a rate proportional to the product of the number of people already infected and the number of people susceptible, but not yet infected. Therefore, if $S$ denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time $t$, then

$$I' = rI(S - I),$$

where $r$ is a positive constant. Assuming that $I(0) = I_0$, find $I(t)$ for $t > 0$, and show that $\lim_{t\to\infty} I(t) = S$.

30. The result of *Exercise 2.2.29* is discouraging: if any susceptible member of the group is initially infected, then in the long run all susceptible members are infected! On a more hopeful note, suppose the disease spreads according to the model of *Exercise 2.2.29*, but there's a medication that cures the infected population at a rate proportional to the number of infected individuals. Now the equation for the number of infected individuals becomes

$$I' = rI(S - I) - qI \tag{A}$$

where $q$ is a positive constant.

a. Choose $r$ and $S$ positive. By plotting direction fields and solutions of (A) on suitable rectangular grids

$$R = \{0 \leq t \leq T,\ 0 \leq I \leq d\}$$

in the $(t, I)$-plane, verify that if $I$ is any solution of (A) such that $I(0) > 0$, then $\lim_{t\to\infty} I(t) = S - q/r$ if $q < rS$ and $\lim_{t\to\infty} I(t) = 0$ if $q \geq rS$.

b. To verify the experimental results of (a), use separation of variables to solve (A) with initial condition $I(0) = I_0 > 0$, and find $\lim_{t\to\infty} I(t)$.

31. Consider the differential equation

$$y' = ay - by^2 - q, \tag{A}$$

where $a$, $b$ are positive constants, and $q$ is an arbitrary constant. Suppose $y$ denotes a solution of this equation that satisfies the initial condition $y(0) = y_0$.

a. Choose $a$ and $b$ positive and $q < a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids

$$R = \{0 \leq t \leq T,\ c \leq y \leq d\} \tag{B}$$

in the $(t, y)$-plane, discover that there are numbers $y_1$ and $y_2$ with $y_1 < y_2$ such that if $y_0 > y_1$ then $\lim_{t\to\infty} y(t) = y_2$, and if $y_0 < y_1$ then $y(t) = -\infty$ for some finite value of $t$. (What happens if $y_0 = y_1$ ?)

b. Choose $a$ and $b$ positive and $q = a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids of the form (B), discover that there's a number $y_1$ such that if $y_0 \geq y_1$ then $\lim_{t\to\infty} y(t) = y_1$, while if $y_0 < y_1$ then $y(t) = -\infty$ for some finite value of $t$.

c. Choose positive $a$, $b$ and $q > a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids of the form (B), discover that no matter what $y_0$ is, $y(t) = -\infty$ for some finite value of $t$.

d. Verify your results experiments analytically. Start by separating variables in (A) to obtain

$$\frac{y'}{ay-by^2-q}=1.$$

To decide what to do next you'll have to use the quadratic formula. This should lead you to see why there are three cases. Take it from there! Because of its role in the transition between these three cases, $q_0=a^2/4b$ is called a **bifurcation value** of $q$. In general, if $q$ is a parameter in any differential equation, $q_0$ is said to be a bifurcation value of $q$ if the nature of the solutions of the equation with $q<q_0$ is qualitatively different from the nature of the solutions with $q>q_0$.

32. By plotting direction fields and solutions of

$$y'=qy-y^3,$$

convince yourself that $q_0=0$ is a bifurcation value of $q$ for this equation. Explain what makes you draw this conclusion.

33. Suppose a disease spreads according to the model of **Exercise 2.2.29**, but there's a medication that cures the infected population at a constant rate of $q$ individuals per unit time, where $q>0$. Then the equation for the number of infected individuals becomes

$$I'=rI(S-I)-q.$$

Assuming that $I(0)=I_0>0$, use the results of **Exercise 2.2.31** to describe what happens as $t\to\infty$.

34. Assuming that $p\not\equiv 0$, state conditions under which the linear equation

$$y'+p(x)y=f(x)$$

is separable. If the equation satisfies these conditions, solve it by separation of variables and by the method developed in Section 2.1.

## Q2.2.7

Solve the equations in **Exercises 2.2.35-2.2.38** using variation of parameters followed by separation of variables.

35. $y'+y=\frac{2xe^{-x}}{1+ye^x}$ &

36. $xy'-2y=\frac{x^6}{y+x^2}$

37. $y'-y=\frac{(x+1)e^{4x}}{(y+e^x)^2}$ &

38. $y'-2y=\frac{xe^{2x}}{1-ye^{-2x}}$

39. Use variation of parameters to show that the solutions of the following equations are of the form $y=uy_1$, where $u$ satisfies a separable equation $u'=g(x)p(u)$. Find $y_1$ and $g$ for each equation.

a. $xy'+y=h(x)p(xy)$
b. $xy'-y=h(x)p\left(\frac{y}{x}\right)$
c. $y'+y=h(x)p(e^xy)$
d. $xy'+ry=h(x)p(x^ry)$
e. $y'+\frac{v'(x)}{v(x)}y=h(x)p\left(v(x)y\right)$

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# 2.3: Existence and Uniqueness of Solutions of Nonlinear Equations

Although there are methods for solving some nonlinear equations, it is impossible to find useful formulas for the solutions of most. Whether we are looking for exact solutions or numerical approximations, it is useful to know conditions that imply the existence and uniqueness of solutions of initial value problems for nonlinear equations. In this section we state such a condition and illustrate it with examples.

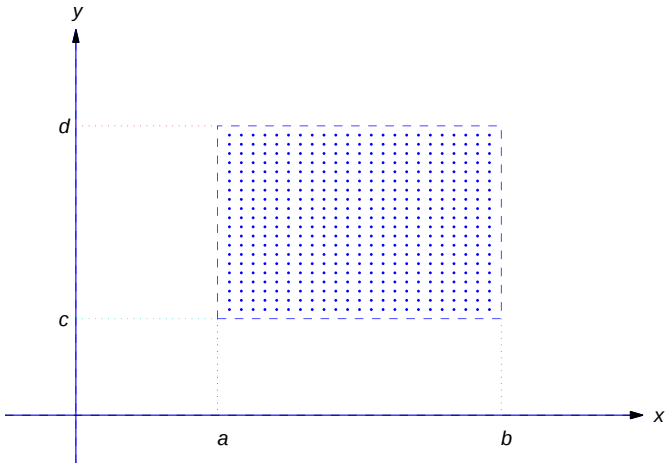



Figure 2.3.1 : An open rectangle

Some terminology: an **open rectangle** $R$ is a set of points $(x, y)$ such that

$$a < x < b \quad \text{and} \quad c < y < d$$

(Figure 2.3.1 ). We'll denote this set by $R : \{a < x < b, c < y < d\}$ . "Open" means that the boundary rectangle (indicated by the dashed lines in Figure 2.3.1 ) is not included in $R$.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for first order nonlinear differential equations. We omit the proof, which is beyond the scope of this book.

**Theorem 2.3.1 : existence and uniqueness**

a. If $f$ is continuous on an open rectangle

$$R : \{a < x < b, c < y < d\}$$

that contains $(x_0, y_0)$ then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2.3.1}$$

has at least one solution on some open subinterval of $(a, b)$ that contains $x_0$.

b. If both $f$ and $f_y$ are continuous on $R$ then Equation 2.3.1 has a unique solution on some open subinterval of $(a, b)$ that contains $x_0$

It's important to understand exactly what Theorem 2.3.1 says.

- (a) is an **existence theorem**. It guarantees that a solution exists on some open interval that contains $x_0$, but provides no information on how to find the solution, or to determine the open interval on which it exists. Moreover, (a) provides no information on the number of solutions that Equation 2.3.1 may have. It leaves open the possibility that Equation 2.3.1 may have two or more solutions that differ for values of $x$ arbitrarily close to $x_0$. We will see in Example 2.3.6 that this can happen.
- (b) is a **uniqueness theorem**. It guarantees that Equation 2.3.1 has a unique solution on some open interval (a,b) that contains $x_0$. However, if $(a, b) \neq (-\infty, \infty)$ , Equation 2.3.1 may have more than one solution on a larger interval that contains $(a, b)$. For example, it may happen that $b < \infty$ and all solutions have the same values on $(a, b)$, but two solutions $y_1$ and $y_2$ are defined on some interval $(a, b_1)$ with $b_1 > b$ , and have different values for $b < x < b_1$ ; thus, the graphs of the $y_1$ and $y_2$ "branch off" in different directions at $x = b$. (See Example 2.3.7 and Figure 2.3.3 ). In this case, continuity implies that $y_1(b) = y_2(b)$ (call their common value $y$), and $y_1$ and $y_2$ are both solutions of the initial value problem

$$y = f(x,y), \quad y(b) = \bar{y} \tag{2.3.2}$$

that differ on every open interval that contains $b$. Therefore $f$ or $f_y$ must have a discontinuity at some point in each open rectangle that contains $(b,y)$, since if this were not so, 2.3.2 would have a unique solution on some open interval that contains $b$. We leave it to you to give a similar analysis of the case where $a > -\infty$.

### Example 2.3.1

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0. \tag{2.3.3}$$

Since

$$f(x,y) = \frac{x^2 - y^2}{1 + x^2 + y^2} \quad \text{and} \quad f_y(x,y) = -\frac{2y(1 + 2x^2)}{(1 + x^2 + y^2)^2}$$

are continuous for all $(x,y)$, Theorem 2.3.1 implies that if $(x_0, y_0)$ is arbitrary, then Equation 2.3.3 has a unique solution on some open interval that contains $x_0$.

### Example 2.3.2

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{x^2 + y^2}, \quad y(x_0) = y_0. \tag{2.3.4}$$

Here

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad f_y(x,y) = -\frac{4x^2 y}{(x^2 + y^2)^2}$$

are continuous everywhere except at $(0,0)$. If $(x_0, y_0) \neq (0,0)$, there's an open rectangle $R$ that contains $(x_0, y_0)$ that does not contain $(0,0)$. Since $f$ and $f_y$ are continuous on $R$, Theorem 2.3.1 implies that if $(x_0, y_0) \neq (0,0)$ then Equation 2.3.4 has a unique solution on some open interval that contains $x_0$.

### Example 2.3.3

Consider the initial value problem

$$y' = \frac{x + y}{x - y}, \quad y(x_0) = y_0. \tag{2.3.5}$$

Here

$$f(x,y) = \frac{x + y}{x - y} \quad \text{and} \quad f_y(x,y) = \frac{2x}{(x - y)^2}$$

are continuous everywhere except on the line $y = x$. If $y_0 \neq x_0$, there's an open rectangle $R$ that contains $(x_0, y_0)$ that does not intersect the line $y = x$. Since $f$ and $f_y$ are continuous on $R$, Theorem 2.3.1 implies that if $y_0 \neq x_0$, Equation 2.3.5 has a unique solution on some open interval that contains $x_0$.

### Example 2.3.4

In Example 2.2.4, we saw that the solutions of

$$y' = 2xy^2 \tag{2.3.6}$$

are

$$y \equiv 0 \quad \text{and} \quad y = -\frac{1}{x^2 + c},$$

where $c$ is an arbitrary constant. In particular, this implies that no solution of Equation 2.3.6 other than $y \equiv 0$ can equal zero for any value of $x$. Show that Theorem 2.3.1$b$ implies this.

We'll obtain a contradiction by assuming that Equation 2.3.6 has a solution $y_1$ that equals zero for some value of $x$, but is not identically zero. If $y_1$ has this property, there's a point $x_0$ such that $y_1(x_0) = 0$, but $y_1(x) \neq 0$ for some value of $x$ in every open interval that contains $x_0$. This means that the initial value problem

$$y' = 2xy^2, \quad y(x_0) = 0 \tag{2.3.7}$$

has two solutions $y \equiv 0$ and $y = y_1$ that differ for some value of $x$ on every open interval that contains $x_0$. This contradicts Theorem 2.3.1 (b), since in Equation 2.3.6 the functions

$$f(x, y) = 2xy^2 \quad \text{and} \quad f_y(x, y) = 4xy.$$

are both continuous for all $(x, y)$, which implies that Equation 2.3.7 has a unique solution on some open interval that contains $x_0$.

## Example 2.3.5

Consider the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(x_0) = y_0. \tag{2.3.8}$$

a. For what points $(x_0, y_0)$ does Theorem 2.3.1$a$ imply that Equation 2.3.8 has a solution?
b. For what points $(x_0, y_0)$ does Theorem 2.3.1$b$ imply that Equation 2.3.8 has a unique solution on some open interval that contains $x_0$?

### Solution a

Since

$$f(x, y) = \frac{10}{3}xy^{2/5}$$

is continuous for all $(x, y)$, Theorem 2.3.1 implies that Equation 2.3.8 has a solution for every $(x_0, y_0)$.

### Solution b

Here

$$f_y(x, y) = \frac{4}{3}xy^{-3/5}$$

is continuous for all $(x, y)$ with $y \neq 0$. Therefore, if $y_0 \neq 0$ there's an open rectangle on which both $f$ and $f_y$ are continuous, and Theorem 2.3.1 implies that Equation 2.3.8 has a unique solution on some open interval that contains $x_0$.

If $y = 0$ then $f_y(x, y)$ is undefined, and therefore discontinuous; hence, Theorem 2.3.1 does not apply to Equation 2.3.8 if $y_0 = 0$.

## Example 2.3.6

Example 2.3.5 leaves open the possibility that the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 0 \tag{2.3.9}$$

has more than one solution on every open interval that contains $x_0 = 0$. Show that this is true.

### Solution

By inspection, $y \equiv 0$ is a solution of the differential equation

$$y' = \frac{10}{3}xy^{2/5}. \tag{2.3.10}$$

Since $y \equiv 0$ satisfies the initial condition $y(0) = 0$, it is a solution of Equation 2.3.9.

Now suppose $y$ is a solution of Equation 2.3.10 that is not identically zero. Separating variables in Equation 2.3.10 yields

$$y^{-2/5}y' = \frac{10}{3}x$$

on any open interval where $y$ has no zeros. Integrating this and rewriting the arbitrary constant as $5c/3$ yields

$$\frac{5}{3}y^{3/5} = \frac{5}{3}(x^2 + c).$$

Therefore

$$y = (x^2 + c)^{5/3}. \tag{2.3.11}$$

Since we divided by $y$ to separate variables in Equation 2.3.10, our derivation of Equation 2.3.11 is legitimate only on open intervals where $y$ has no zeros. However, Equation 2.3.11 actually defines $y$ for all $x$, and differentiating Equation 2.3.11 shows that

$$y' = \frac{10}{3}x(x^2 + c)^{2/3} = \frac{10}{3}xy^{2/5}, \ -\infty < x < \infty$$

Therefore Equation 2.3.11 satisfies Equation 2.3.10 on $(-\infty, \infty)$ even if $c \le 0$, so that $y(\sqrt{|c|}) = y(-\sqrt{|c|}) = 0$. In particular, taking $c = 0$ in Equation 2.3.11 yields

$$y = x^{10/3}$$

as a second solution of Equation 2.3.9. Both solutions are defined on $(-\infty, \infty)$, and they differ on every open interval that contains $x_0 = 0$ (Figure 2.3.2 ). In fact, there are **four** distinct solutions of Equation 2.3.9 defined on $(-\infty, \infty)$ that differ from each other on every open interval that contains $x_0 = 0$. Can you identify the other two?

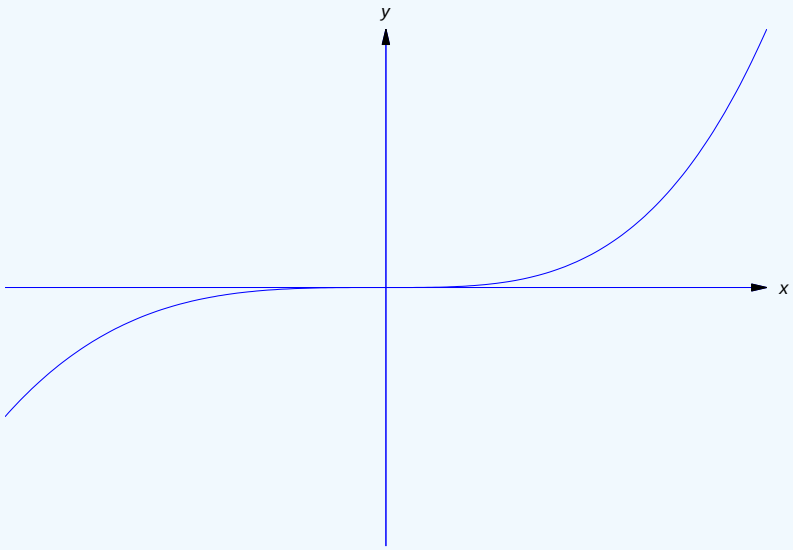



Figure 2.3.2 : Two solutions ( $y = 0$ and $y = x^{1/2}$ ) of Equation 2.3.9 that differ on every interval containing $x_0 = 0$

## Example 2.3.7

From Example 2.3.5 , the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1 \tag{2.3.12}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find a solution and determine the largest open interval $(a, b)$ on which it is unique.

### Solution

Let $y$ be any solution of Equation 2.3.12. Because of the initial condition $y(0) = -1$ and the continuity of $y$, there's an open interval $I$ that contains $x_0 = 0$ on which $y$ has no zeros, and is consequently of the form Equation 2.3.11. Setting $x = 0$ and $y = -1$ in Equation 2.3.11 yields $c = -1$, so

$$y = (x^2 - 1)^{5/3} \tag{2.3.13}$$

for $x$ in $I$. Therefore every solution of Equation 2.3.12 differs from zero and is given by Equation 2.3.13 on $(-1, 1)$; that is, Equation 2.3.13 is the unique solution of Equation 2.3.12 on $(-1, 1)$. This is the largest open interval on which Equation 2.3.12 has a unique solution. To see this, note that Equation 2.3.13 is a solution of Equation 2.3.12 on $(-\infty, \infty)$. From **Exercise 2.2.15**, there are infinitely many other solutions of Equation 2.3.12 that differ from Equation 2.3.13 on every open interval larger than $(-1, 1)$. One such solution is

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 \le x \le 1, \\ 0, & |x| > 1. \end{cases}$$

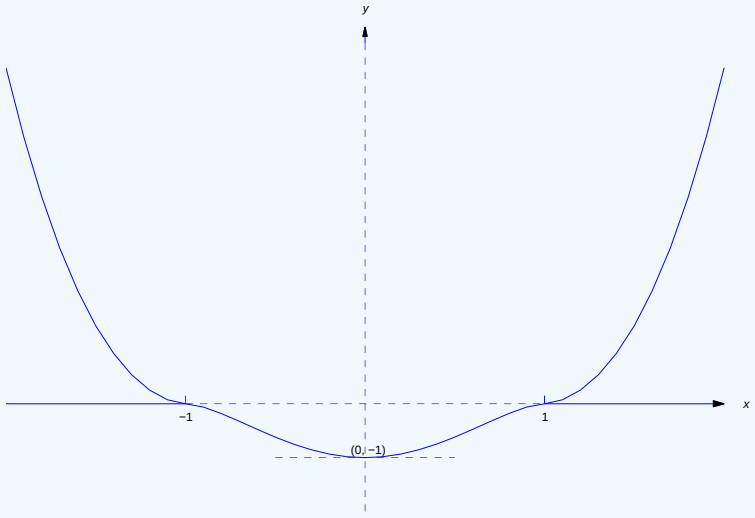



Figure 2.3.3 : Two solutions of Equation 2.3.12 on (−1,1) that coincide on (−1, 1), but on no larger open interval. (right)

## Example 2.3.8

From Example 2.3.5 ), the initial value problem

$$y' = \frac{10}{3} x y^{2/5}, \quad y(0) = 1 \tag{2.3.14}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it is unique.

### Solution

Let $y$ be any solution of Equation 2.3.14. Because of the initial condition $y(0) = 1$ and the continuity of $y$, there's an open interval $I$ that contains $x_0 = 0$ on which $y$ has no zeros, and is consequently of the form Equation 2.3.11. Setting $x = 0$ and $y = 1$ in Equation 2.3.11 yields $c = 1$, so

$$y = (x^2 + 1)^{5/3} \tag{2.3.15}$$

for $x$ in $I$. Therefore every solution of Equation 2.3.14 differs from zero and is given by Equation 2.3.15 on $(-\infty, \infty)$; that is, Equation 2.3.15 is the unique solution of Equation 2.3.14 on $(-\infty, \infty)$. Figure 2.3.4 ) shows the graph of this solution.

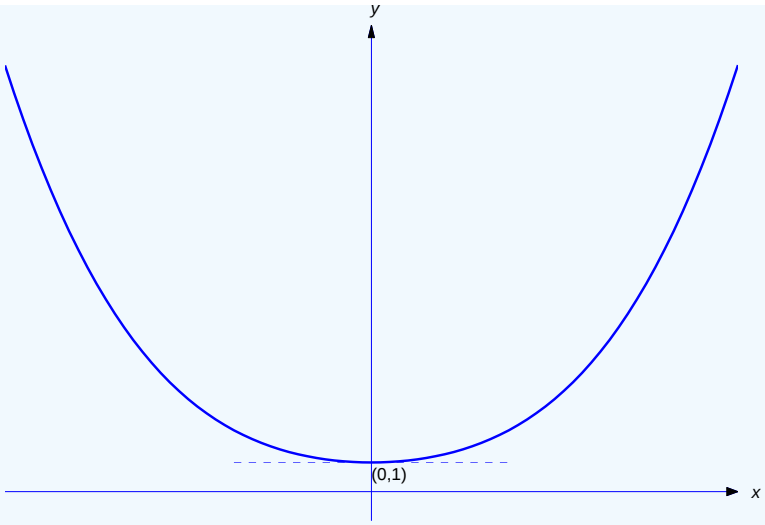



Figure 2.3.4 : The unique solution of Equation 2.3.14.

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# 2.3E: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises)

## Q2.3.1

In **Exercises 2.3.1-2.3.13**, find all $(x_0, y_0)$ for which Theorem 2.3.1 implies that the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ has (a) a solution and (b) a unique solution on some open interval that contains $x_0$.

1. $y' = \frac{x^2+y^2}{\sin x}$
2. $y' = \frac{e^x+y}{x^2+y^2}$
3. $y' = \tan xy$
4. $y' = \frac{x^2+y^2}{\ln xy}$
5. $y' = (x^2+y^2)y^{1/3}$
6. $y' = 2xy$
7. $y' = \ln(1+x^2+y^2)$
8. $y' = \frac{2x+3y}{x-4y}$
9. $y' = (x^2+y^2)^{1/2}$
10. $y' = x(y^2-1)^{2/3}$
11. $y' = (x^2+y^2)^2$
12. $y' = (x+y)^{1/2}$
13. $y' = \frac{\tan y}{x-1}$

## Q2.3.2

14. Apply Theorem 2.3.1 to the initial value problem

$$y' + p(x)y = q(x), \quad y(x_0) = y_0$$

for a linear equation, and compare the conclusions that can be drawn from it to those that follow from Theorem 2.1.2.

15.

a. Verify that the function

$$y = \begin{cases} (x^2-1)^{5/3}, & -1 < x < 1, \\ 0, & |x| \geq 1, \end{cases}$$

is a solution of the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1$$

on $(-\infty, \infty)$. HINT: You'll need the definition

$$y'(\overline{x}) = \lim_{x \to \overline{x}} \frac{y(x) - y(\overline{x})}{x - \overline{x}}$$

to verify that $y$ satisfies the differential equation at $\overline{x} = \pm 1$.

b. Verify that if $\epsilon_i = 0$ or $1$ for $i = 1, 2$ and $a, b > 1$, then the function

$$y=\begin{cases}\epsilon_1(x^2-a^2)^{5/3}, & -\infty<x<-a,\\ 0, & -a\le x\le -1,\\ (x^2-1)^{5/3}, & -1<x<1,\\ 0, & 1\le x\le b,\\ \epsilon_2(x^2-b^2)^{5/3}, & b<x<\infty,\end{cases}$$

is a solution of the initial value problem of a on $(-\infty,\infty)$.

16. Use the ideas developed in **Exercise 2.3.15** to find infinitely many solutions of the initial value problem

$$y'=y^{2/5},\quad y(0)=1$$

on $(-\infty,\infty)$.

17. Consider the initial value problem

$$y'=3x(y-1)^{1/3},\quad y(x_0)=y_0. \tag{A}$$

a. For what points $(x_0,y_0)$ does Theorem 2.3.1 imply that (A) has a solution?
b. For what points $(x_0,y_0)$ does Theorem 2.3.1 imply that (A) has a unique solution on some open interval that contains $x_0$?

18. Find nine solutions of the initial value problem

$$y'=3x(y-1)^{1/3},\quad y(0)=1$$

that are all defined on $(-\infty,\infty)$ and differ from each other for values of $x$ in every open interval that contains $x_0=0$.

19. From Theorem 2.3.1, the initial value problem

$$y'=3x(y-1)^{1/3},\quad y(0)=9$$

has a unique solution on an open interval that contains $x_0=0$. Find the solution and determine the largest open interval on which it is unique.

20.

a. From Theorem 2.3.1, the initial value problem

$$y'=3x(y-1)^{1/3},\quad y(3)=-7 \tag{A}$$

has a unique solution on some open interval that contains $x_0=3$. Determine the largest such open interval, and find the solution on this interval.
b. Find infinitely many solutions of (A), all defined on $(-\infty,\infty)$.

21. Prove:

a. If

$$f(x,y_0)=0,\quad a<x<b, \tag{A}$$

and$x_0$ is in $(a,b)$, then $y\equiv y_0$ is a solution of

$$y'=f(x,y),\quad y(x_0)=y_0$$

on $(a,b)$.
b. If $f$ and $f_y$ are continuous on an open rectangle that contains $(x_0,y_0)$ and (A) holds, no solution of $y'=f(x,y)$ other than $y\equiv y_0$ can equal $y_0$ at any point in $(a,b)$.

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# 2.4: Transformation of Nonlinear Equations into Separable Equations

In Section 2.1, we found that the solutions of a linear nonhomogeneous equation

$$y' + p(x)y = f(x)$$

are of the form $y = uy_1$, where $y_1$ is a nontrivial solution of the complementary equation

$$y' + p(x)y = 0 \tag{2.4.1}$$

and $u$ is a solution of

$$u'y_1(x) = f(x).$$

Note that this last equation is separable, since it can be rewritten as

$$u' = \frac{f(x)}{y_1(x)}.$$

In this section we'll consider nonlinear differential equations that are not separable to begin with, but can be solved in a similar fashion by writing their solutions in the form $y = uy_1$, where $y_1$ is a suitably chosen known function and $u$ satisfies a separable equation. We'llsay in this case that we **transformed** the given equation into a separable equation.

## Bernoulli Equations

A **Bernoulli equation** is an equation of the form

$$y' + p(x)y = f(x)y^r, \tag{2.4.2}$$

where $r$ can be any real number other than 0 or 1. (Note that Equation 2.4.2 is linear if and only if $r = 0$ or $r = 1$.) We can transform Equation 2.4.2 into a separable equation by variation of parameters: if $y_1$ is a nontrivial solution of Equation 2.4.1, substituting $y = uy_1$ into Equation 2.4.2 yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x)(uy_1)^r,$$

which is equivalent to the separable equation

$$u'y_1(x) = f(x)(y_1(x))^r u^r \quad \text{or} \quad \frac{u'}{u^r} = f(x)(y_1(x))^{r-1},$$

since $y_1' + p(x)y_1 = 0$.

**Example 2.4.1**

Solve the Bernoulli equation

$$y' - y = xy^2. \tag{2.4.3}$$

**Solution**

Since $y_1 = e^x$ is a solution of $y' - y = 0$, we look for solutions of Equation 2.4.3 in the form $y = ue^x$, where

$$u'e^x = xu^2e^{2x} \quad \text{or equivalently} \quad u' = xu^2e^x.$$

Separating variables yields

$$\frac{u'}{u^2} = xe^x,$$

and integrating yields

$$-\frac{1}{u} = (x-1)e^x + c.$$

Hence,

$$u = -\frac{1}{(x-1)e^x + c}$$

and

$$y = -\frac{1}{x - 1 + ce^{-x}}.$$

Figure 2.4.1 shows direction field and some integral curves of Equation 2.4.3.

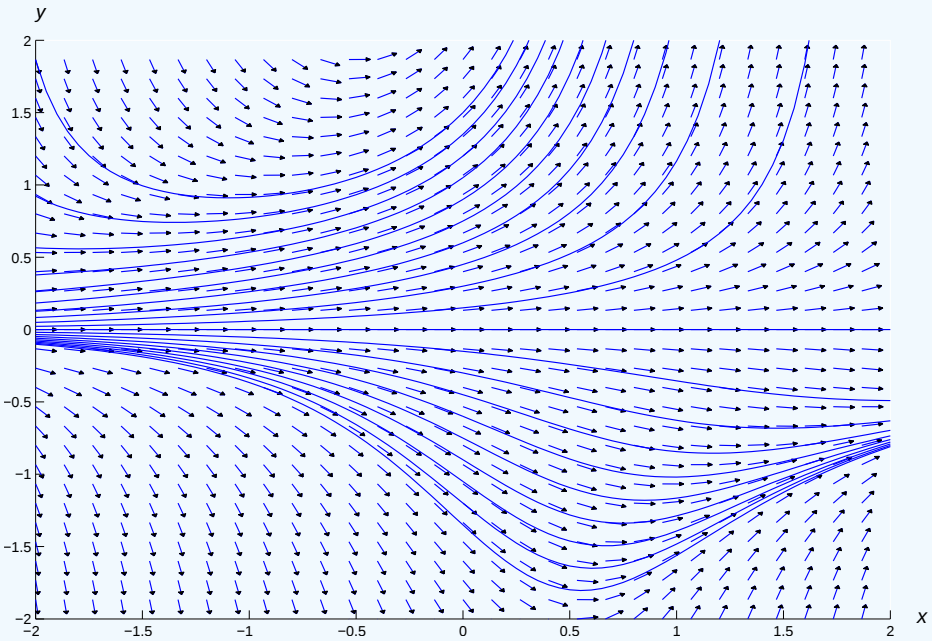



Figure 2.4.1: A direction field and integral curves for $y' - y = xy^2$

## Other Nonlinear Equations That Can be Transformed Into Separable Equations

We've seen that the nonlinear Bernoulli equation can be transformed into a separable equation by the substitution $y = uy_1$ if $y_1$ is suitably chosen. Now let's discover a sufficient condition for a nonlinear first order differential equation

$$y' = f(x, y) \tag{2.4.4}$$

to be transformable into a separable equation in the same way. Substituting $y = uy_1$ into Equation 2.4.4 yields

$$u'y_1(x) + uy_1'(x) = f(x, uy_1(x)),$$

which is equivalent to

$$u'y_1(x) = f(x, uy_1(x)) - uy_1'(x). \tag{2.4.5}$$

If

$$f(x, uy_1(x)) = q(u)y_1'(x)$$

for some function $q$, then Equation 2.4.5 becomes

$$u'y_1(x) = (q(u) - u)y_1'(x), \tag{2.4.6}$$

which is separable. After checking for constant solutions $u \equiv u_0$ such that $q(u_0) = u_0$, we can separate variables to obtain

$$\frac{u'}{q(u) - u} = \frac{y_1'(x)}{y_1(x)}.$$

## Homogeneous Nonlinear Equations

In the text we will consider only the most widely studied class of equations for which the method of the preceding paragraph works. Other types of equations appear in Exercises 2.4.44- 2.4.51.

The differential equation Equation 2.4.4 is said to be **homogeneous** if $x$ and $y$ occur in $f$ in such a way that $f(x,y)$ depends only on the ratio $y/x$; that is, Equation 2.4.4 can be written as

$$y' = q(y/x), \tag{2.4.7}$$

where $q = q(u)$ is a function of a single variable. For example,

$$y' = \frac{y + xe^{-y/x}}{x} = \frac{y}{x} + e^{-y/x}$$

and

$$y' = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$$

are of the form Equation 2.4.7, with

$$q(u) = u + e^{-u} \quad \text{and} \quad q(u) = u^2 + u - 1,$$

respectively. The general method discussed above can be applied to Equation 2.4.7 with $y_1 = x$ (and therefore $y_1' = 1$). Thus, substituting $y = ux$ in Equation 2.4.7 yields

$$u'x + u = q(u),$$

and separation of variables (after checking for constant solutions $u \equiv u_0$ such that $q(u_0) = u_0$) yields

$$\frac{u'}{q(u) - u} = \frac{1}{x}.$$

Before turning to examples, we point out something that you may've have already noticed: the definition of **homogeneous equation** given here is not the same as the definition given in Section 2.1, where we said that a linear equation of the form

$$y' + p(x)y = 0$$

is homogeneous. We make no apology for this inconsistency, since we didn't create it historically, **homogeneous** has been used in these two inconsistent ways. The one having to do with linear equations is the most important. This is the only section of the book where the meaning defined here will apply.

Since $y/x$ is in general undefined if $x = 0$, we'll consider solutions of nonhomogeneous equations only on open intervals that do not contain the point $x = 0$.

**Example 2.4.2**

Solve

$$y' = \frac{y + xe^{-y/x}}{x}. \tag{2.4.8}$$

**Solution**

Substituting $y = ux$ into Equation 2.4.8 yields

$$u'x + u = \frac{ux + xe^{-ux/x}}{x} = u + e^{-u}.$$

Simplifying and separating variables yields

$$e^u u' = \frac{1}{x}.$$

Integrating yields $e^u = \ln|x| + c$. Therefore $u = \ln(\ln|x| + c)$ and $y = ux = x\ln(\ln|x| + c)$.

Figure 2.4.2 shows a direction field and integral curves for Equation 2.4.8.

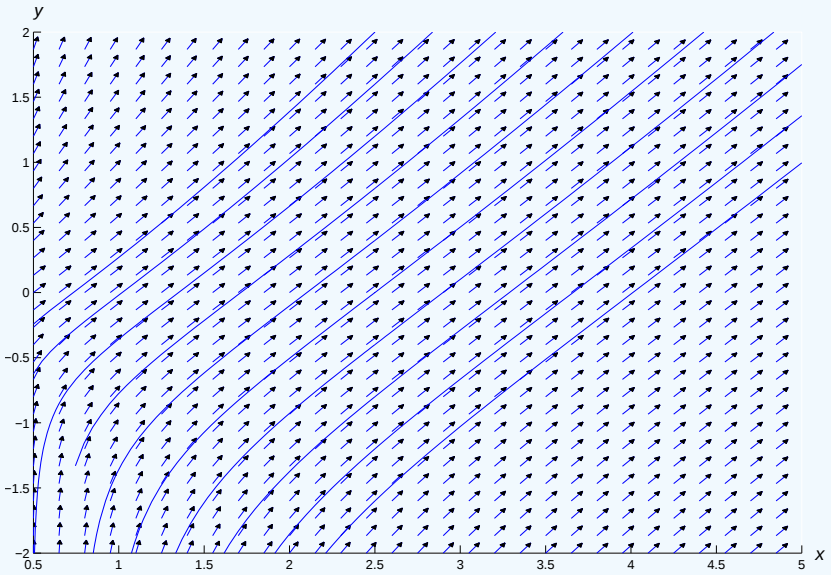



Figure 2.4.2: A direction field and some integral curves for $y' = \dfrac{y + xe^{-y/x}}{x}$

**Example 2.4.3**

a. Solve

$$x^2y' = y^2 + xy - x^2. \tag{2.4.9}$$

b. Solve the initial value problem

$$x^2y' = y^2 + xy - x^2, \quad y(1) = 2. \tag{2.4.10}$$

**Solution a**

We first find solutions of Equation 2.4.9 on open intervals that don't contain $x = 0$. We can rewrite Equation 2.4.9 as

$$y' = \frac{y^2 + xy - x^2}{x^2}$$

for $x$ in any such interval. Substituting $y = ux$ yields

$$u'x + u = \frac{(ux)^2 + x(ux) - x^2}{x^2} = u^2 + u - 1,$$

so

$$u'x = u^2 - 1. \tag{2.4.11}$$

By inspection this equation has the constant solutions $u \equiv 1$ and $u \equiv -1$. Therefore $y = x$ and $y = -x$ are solutions of Equation 2.4.9. If $u$ is a solution of Equation 2.4.11 that does not assume the values $\pm 1$ on some interval, separating variables yields

$$\frac{u'}{u^2 - 1} = \frac{1}{x},$$

or, after a partial fraction expansion,

$$\frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right]u' = \frac{1}{x}.$$

Multiplying by 2 and integrating yields

$$\ln\left|\frac{u-1}{u+1}\right| = 2\ln|x| + k,$$

or

$$\left|\frac{u-1}{u+1}\right| = e^k x^2,$$

which holds if

$$\frac{u-1}{u+1} = cx^2 \tag{2.4.12}$$

where $c$ is an arbitrary constant. Solving for $u$ yields

$$u = \frac{1+cx^2}{1-cx^2}.$$

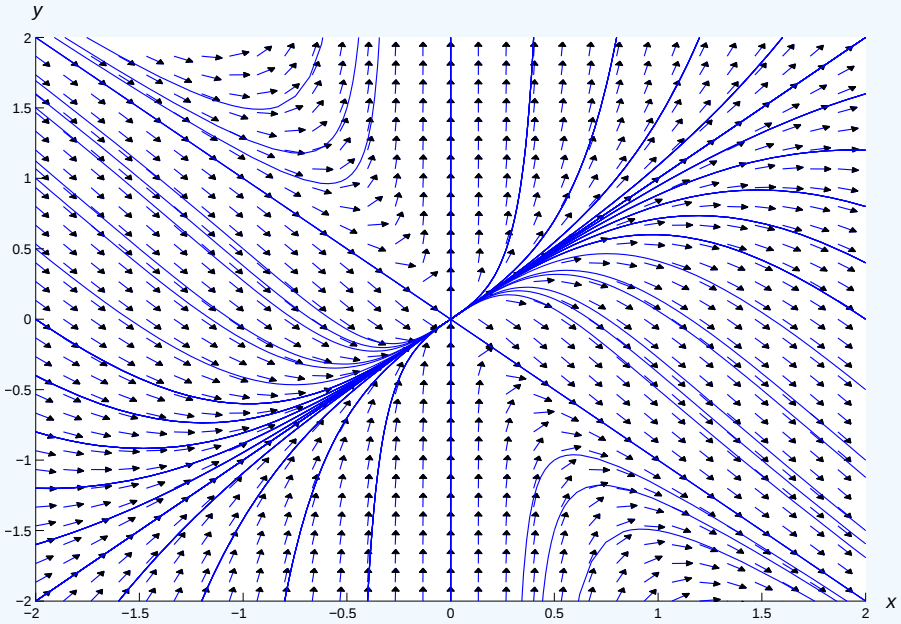



Figure 2.4.3: A direction field and integral curves for $x^2y' = y^2 + xy - x^2$

Therefore

$$y = ux = \frac{x(1+cx^2)}{1-cx^2} \tag{2.4.13}$$

is a solution of Equation 2.4.10 for any choice of the constant $c$. Setting $c = 0$ in Equation 2.4.13 yields the solution $y = x$. However, the solution $y = -x$ can't be obtained from Equation 2.4.13. Thus, the solutions of Equation 2.4.9 on intervals that don't contain $x = 0$ are $y = -x$ and functions of the form Equation 2.4.13.

The situation is more complicated if $x = 0$ is the open interval. First, note that $y = -x$ satisfies Equation 2.4.9 on $(-\infty, \infty)$. If $c_1$ and $c_2$ are arbitrary constants, the function

$$y = \begin{cases} \frac{x(1+c_1x^2)}{1-c_1x^2}, & a < x < 0 \\ \frac{x(1+c_2x^2}{1-c_2x^2}, & 0 \le x < b \end{cases} \tag{2.4.14}$$

is a solution of Equation 2.4.9 on $(a, b)$, where

$$a = \begin{cases} -\frac{1}{\sqrt{c_1}} & \text{if } c_1 > 0, \\ -\infty & \text{if } c_1 \le 0, \end{cases} \quad \text{and} \quad b = \begin{cases} \frac{1}{\sqrt{c_2}} & \text{if } c_2 > 0, \\ \infty & \text{if } c_2 \le 0. \end{cases}$$

We leave it to you to verify this. To do so, note that if $y$ is any function of the form Equation 2.4.13 then $y(0)=0$ and $y'(0)=1$.

Figure 2.4.3 shows a direction field and some integral curves for Equation 2.4.9.

### Solution b

We could obtain $c$ by imposing the initial condition $y(1)=2$ in Equation 2.4.13, and then solving for $c$. However, it is easier to use Equation 2.4.12. Since $u=y/x$, the initial condition $y(1)=2$ implies that $u(1)=2$. Substituting this into Equation 2.4.12 yields $c=1/3$. Hence, the solution of Equation 2.4.10 is

$$y=\frac{x(1+x^2/3)}{1-x^2/3}.$$

The interval of validity of this solution is $(-\sqrt{3},\sqrt{3})$. However, the largest interval on which Equation 2.4.10 has a unique solution is $(0,\sqrt{3})$. To see this, note from Equation 2.4.14 that any function of the form

$$y=\begin{cases}\dfrac{x(1+cx^2)}{1-cx^2}, & a<x\le 0\\[2ex] \dfrac{x(1+x^2/3)}{1-x^2/3} & 0\le x<\sqrt{3}\end{cases} \tag{2.4.15}$$

is a solution of Equation 2.4.10 on $(a,\sqrt{3})$, where $a=-1/\sqrt{c}$ if $c>0$ or $a=-\infty$ if $c\le 0$. Why does this not contradict Theorem 2.3.1?

Figure 2.4.4 shows several solutions of the initial value problem Equation 2.4.10. Note that these solutions coincide on $(0,\sqrt{3})$.

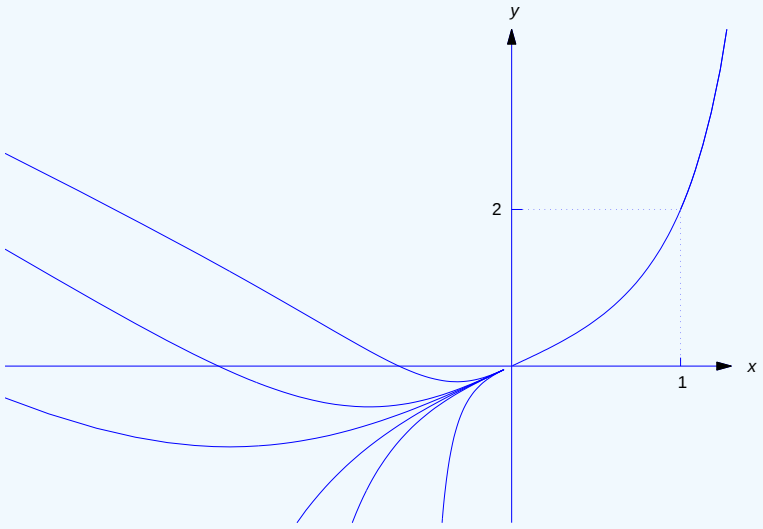



Figure 2.4.4: Solutions of $x^2y'=y^2+xy-x^2$, $y(1)=2$

In the last two examples we were able to solve the given equations explicitly. However, this is not always possible, as you'll see in the exercises.

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# 2.4E: Transformation of Nonlinear Equations into Separable Equations (Exercises)

## Q2.4.1

In **Exercises 2.4.1-2.4.4** solve the given Bernoulli equation.

1. $y' + y = y^2$
2. $7xy' - 2y = -\frac{x^2}{y^6}$
3. $x^2y' + 2y = 2e^{1/x}y^{1/2}$
4. $(1+x^2)y' + 2xy = \frac{1}{(1+x^2)y}$

## Q2.4.2

In **Exercises 2.4.5** and **2.4.6** find all solutions. Also, plot a direction field and some integral curves on the indicated rectangular region.

5. $y' - xy = x^3y^3; \quad \{-3 \le x \le 3, -2 \le y \le 2\}$
6. $y' - \frac{1+x}{3x}y = y^4; \quad \{-2 \le x \le 2, -2 \le y \le 2\}$

## Q2.4.3

In **Exercises 2.4.7-2.4.11** solve the initial value problem.

7. $y' - 2y = xy^3, \quad y(0) = 2\sqrt{2}$
8. $y' - xy = xy^{3/2}, \quad y(1) = 4$
9. $xy' + y = x^4y^4, \quad y(1) = 1/2$
10. $y' - 2y = 2y^{1/2}, \quad y(0) = 1$
11. $y' - 4y = \frac{48x}{y^2}, \quad y(0) = 1$

## Q2.4.4

In **Exercises 2.4.12** and **2.4.13** solve the initial value problem and graph the solution.

12. $x^2y' + 2xy = y^3, \quad y(1) = 1/\sqrt{2}$
13. $y' - y = xy^{1/2}, \quad y(0) = 4$

## Q2.4.5

14. You may have noticed that the logistic equation

$$P' = aP(1 - \alpha P)$$

from Verhulst's model for population growth can be written in Bernoulli form as

$$P' - aP = -a\alpha P^2.$$

This isn't particularly interesting, since the logistic equation is separable, and therefore solvable by the method studied in Section 2.2. So let's consider a more complicated model, where $a$ is a positive constant and $\alpha$ is a positive continuous function of $t$ on $[0, \infty)$. The equation for this model is

$$P' - aP = -a\alpha(t)P^2,$$

a non-separable Bernoulli equation.

a. Assuming that $P(0) = P_0 > 0$, find $P$ for $t > 0$.
b. Verify that your result reduces to the known results for the Malthusian model where $\alpha = 0$, and the Verhulst model where $\alpha$ is a nonzero constant.

c. Assuming that

$$\lim_{t\to\infty} e^{-at}\int_0^t \alpha(\tau)e^{a\tau}\,d\tau = L$$

exists (finite or infinite), find $\lim_{t\to\infty} P(t)$.

## Q2.4.6

In **Exercises 2.4.15-2.4.18** solve the equation explicitly.

15. $y' = \frac{y+x}{x}$
16. $y' = \frac{y^2+2xy}{x^2}$
17. $xy^3y' = y^4 + x^4$
18. $y' = \frac{y}{x} + \sec\frac{y}{x}$

## Q2.4.7

In **Exercises 2.4.19-2.4.21** solve the equation explicitly. Also, plot a direction field and some integral curves on the indicated rectangular region.

19. $x^2y' = xy + x^2 + y^2; \quad \{-8 \le x \le 8, -8 \le y \le 8\}$
20. $xyy' = x^2 + 2y^2; \quad \{-4 \le x \le 4, -4 \le y \le 4\}$
21. $y' = \frac{2y^2+x^2e^{-(y/x)^2}}{2xy}; \quad \{-8 \le x \le 8, -8 \le y \le 8\}$

## Q2.4.8

In **Exercises 2.4.22-2.4.27** solve the initial value problem.

22. $y' = \frac{xy+y^2}{x^2}, \quad y(-1) = 2$
23. $y' = \frac{x^3+y^3}{xy^2}, \quad y(1) = 3$
24. $xyy' + x^2 + y^2 = 0, \quad y(1) = 2$
25. $y' = \frac{y^2-3xy-5x^2}{x^2}, \quad y(1) = -1$
26. $x^2y' = 2x^2 + y^2 + 4xy, \quad y(1) = 1$
27. $xyy' = 3x^2 + 4y^2, \quad y(1) = \sqrt{3}$

## Q2.4.9

In **Exercises 2.4.28-2.4.34** solve the given homogeneous equation implicitly.

28. $y' = \frac{x+y}{x-y}$
29. $(y'x - y)(\ln|y| - \ln|x|) = x$
30. $y' = \frac{y^3+2xy^2+x^2y+x^3}{x(y+x)^2}$
31. $y' = \frac{x+2y}{2x+y}$
32. $y' = \frac{y}{y-2x}$
33. $y' = \frac{xy^2+2y^3}{x^3+x^2y+xy^2}$
34. $y' = \frac{x^3+x^2y+3y^3}{x^3+3xy^2}$

## Q2.4.10

35.

a. Find a solution of the initial value problem

$$x^2y' = y^2 + xy - 4x^2, \quad y(-1) = 0 \tag{A}$$

on the interval $(-\infty, 0)$. Verify that this solution is actually valid on $(-\infty, \infty)$.

b. Use Theorem 2.3.1 to show that (A) has a unique solution on $(-\infty, 0)$.

c. Plot a direction field for the differential equation in (A) on a square

$$\{-r \le x \le r, -r \le y \le r\},$$

where $r$ is any positive number. Graph the solution you obtained in (a) on this field.

d. Graph other solutions of (A) that are defined on $(-\infty, \infty)$.

e. Graph other solutions of (A) that are defined only on intervals of the form $(-\infty, a)$, where is a finite positive number.

36.

a. Solve the equation

$$xyy' = x^2 - xy + y^2 \tag{A}$$

implicitly.

b. Plot a direction field for (A) on a square

$$\{0 \le x \le r, 0 \le y \le r\}$$

where $r$ is any positive number.

c. Let $K$ be a positive integer. (You may have to try several choices for $K$.) Graph solutions of the initial value problems

$$xyy' = x^2 - xy + y^2, \quad y(r/2) = \frac{kr}{K},$$

for $k = 1, 2, \ldots, K$. Based on your observations, find conditions on the positive numbers $x_0$ and $y_0$ such that the initial value problem

$$xyy' = x^2 - xy + y^2, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution (i) on $(0, \infty)$ or (ii) only on an interval $(a, \infty)$, where $a > 0$?

d. What can you say about the graph of the solution of (B) as $x \to \infty$? (Again, assume that $x_0 > 0$ and $y_0 > 0$.)

37.

a. Solve the equation

$$y' = \frac{2y^2 - xy + 2x^2}{xy + 2x^2} \tag{A}$$

implicitly.

b. Plot a direction field for (A) on a square

$$\{-r \le x \le r, -r \le y \le r\}$$

where $r$ is any positive number. By graphing solutions of (A), determine necessary and sufficient conditions on $(x_0, y_0)$ such that (A) has a solution on (i) $(-\infty, 0)$ or (ii) $(0, \infty)$ such that $y(x_0) = y_0$.

38. Follow the instructions of **Exercise 2.4.37** for the equation

$$y' = \frac{xy + x^2 + y^2}{xy}.$$

39. Pick any nonlinear homogeneous equation $y' = q(y/x)$ you like, and plot direction fields on the square $\{-r \le x \le r, \ -r \le y \le r\}$, where $r > 0$. What happens to the direction field as you vary $r$? Why?

40. Prove: If $ad-bc\neq 0$ , the equation

$$y'=\frac{ax+by+\alpha}{cx+dy+\beta}$$

can be transformed into the homogeneous nonlinear equation

$$\frac{dY}{dX}=\frac{aX+bY}{cX+dY}$$

by the substitution $x=X-X_0,\ y=Y-Y_0$ , where $X_0$ and $Y_0$ are suitably chosen constants.

## Q2.4.11

In **Exercises 2.4.21-2.4.43** use a method suggested by **Exercise 2.4.40** to solve the given equation implicitly.

41. $y'=\frac{-6x+y-3}{2x-y-1}$

42. $y'=\frac{2x+y+1}{x+2y-4}$

43. $y'=\frac{-x+3y-14}{x+y-2}$

## Q2.4.12

In **Exercises 2.4.44-2.4.51** find a function $y_1$ such that the substitution $y=uy_1$ transforms the given equation into a separable equation of the form (2.4.6). Then solve the given equation explicitly.

44. $3xy^2y'=y^3+x$

45. $xyy'=3x^6+6y^2$

46. $x^3y'=2(y^2+x^2y-x^4)$

47. $y'=y^2e^{-x}+4y+2e^x$

48. $y'=\frac{y^2+y\tan x+\tan^2 x}{\sin^2 x}$

49. $x(\ln x)^2y'=-4(\ln x)^2+y\ln x+y^2$

50. $2x(y+2\sqrt{x})y'=(y+\sqrt{x})^2$

51. $(y+e^{x^2})y'=2x(y^2+ye^{x^2}+e^{2x^2}$

## Q2.4.13

52. Solve the initial value problem

$$y'+\frac{2}{x}y=\frac{3x^2y^2+6xy+2}{x^2(2xy+3)},\quad y(2)=2.$$

53. Solve the initial value problem

$$y'+\frac{3}{x}y=\frac{3x^4y^2+10x^2y+6}{x^3(2x^2y+5)},\quad y(1)=1.$$

54. Prove: If $y$ is a solution of a homogeneous nonlinear equation $y'=q(y/x)$, so is $y_1=y(ax)/a$, where $a$ is any nonzero constant.

55. A **generalized Riccati equation** is of the form

$$y'=P(x)+Q(x)y+R(x)y^2. \tag{A}$$

(If $R\equiv -1$ , (A) is a **Riccati equation**.) Let $y_1$ be a known solution and $y$ an arbitrary solution of (A). Let $z=y-y_1$ . Show that $z$ is a solution of a Bernoulli equation with $n=2$.

## Q2.4.14

In **Exercises 2.4.56-2.4.59**, given that $y_1$ is a solution of the given equation, use the method suggested by **Exercise 2.4.55** to find other solutions.

56. $y' = 1 + x - (1 + 2x)y + xy^2 \ ; y_1 = 1$

57. $y' = e^{2x} + (1 - 2e^x)y + y^2 \ ; y_1 = e^x$

58. $xy' = 2 - x + (2x - 2)y - xy^2 \ ; y_1 = 1$

59. $xy' = x^3 + (1 - 2x^2)y + xy^2 \ ; y_1 = x$

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# 2.5: Exact Equations

In this section it is convenient to write first order differential equations in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{2.5.1}$$

This equation can be interpreted as

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0, \tag{2.5.2}$$

where $x$ is the independent variable and $y$ is the dependent variable, or as

$$M(x,y)\,\frac{dx}{dy} + N(x,y) = 0, \tag{2.5.3}$$

where $y$ is the independent variable and $x$ is the dependent variable. Since the solutions of Equation 2.5.2 and Equation 2.5.3 will often have to be left in implicit form we will say that $F(x,y) = c$ is an implicit solution of Equation 2.5.1 if every differentiable function $y = y(x)$ that satisfies $F(x,y) = c$ is a solution of Equation 2.5.2 and every differentiable function $x = x(y)$ that satisfies $F(x,y) = c$ is a solution of Equation 2.5.3

Here are some examples:

Table 2.5.1: Examples of Exact Differential Equations in three forms

| Equation 2.5.1 | Equation 2.5.2 | Equation 2.5.3 |
|---|---|---|
| $3x^2y^2\,dx + 2x^3y\,dy = 0$ | $3x^2y^2 + 2x^3y\,\frac{dy}{dx} = 0$ | $3x^2y^2\,\frac{dx}{dy} + 2x^3y = 0$ |
| $(x^2+y^2)\,dx + 2xy\,dy = 0$ | $(x^2+y^2) + 2xy\,\frac{dy}{dx} = 0$ | $(x^2+y^2)\,\frac{dx}{dy} + 2xy = 0$ |
| $3y\sin x\,dx - 2xy\cos x\,dy = 0$ | $3y\sin x - 2xy\cos x\,\frac{dy}{dx} = 0$ | $3y\sin x\,\frac{dx}{dy} - 2xy\cos x = 0$ |

Note that a separable equation can be written as Equation 2.5.1 as

$$M(x)\,dx + N(y)\,dy = 0.$$

we will develop a method for solving Equation 2.5.1 under appropriate assumptions on $M$ and $N$. This method is an extension of the method of separation of variables. Before stating it we consider an example.

**Example 2.5.1**

Show that

$$x^4y^3 + x^2y^5 + 2xy = c \tag{2.5.4}$$

is an implicit solution of

$$(4x^3y^3 + 2xy^5 + 2y)\,dx + (3x^4y^2 + 5x^2y^4 + 2x)\,dy = 0. \tag{2.5.5}$$

**Solution**

Regarding $y$ as a function of $x$ and differentiating Equation 2.5.4 implicitly with respect to $x$ yields

$$(4x^3y^3 + 2xy^5 + 2y) + (3x^4y^2 + 5x^2y^4 + 2x)\,\frac{dy}{dx} = 0.$$

Similarly, regarding $x$ as a function of $y$ and differentiating Equation 2.5.4 implicitly with respect to $y$ yields

$$(4x^3y^3 + 2xy^5 + 2y)\frac{dx}{dy} + (3x^4y^2 + 5x^2y^4 + 2x) = 0.$$

Therefore Equation 2.5.4 is an implicit solution of Equation 2.5.5 in either of its two possible interpretations.

You may think Example 2.5.1 is pointless, since concocting a differential equation that has a given implicit solution is not particularly interesting. However, it illustrates the next important theorem, which we will prove by using implicit differentiation, as in Example 2.5.1.

**Theorem 2.5.1**

If $F = F(x, y)$ has continuous partial derivatives $F_x$ and $F_y$, then

$$F(x, y) = c \tag{2.5.6}$$

(with $c$ as a constant) is an implicit solution of the differential equation

$$F_x(x, y)\,dx + F_y(x, y)\,dy = 0. \tag{2.5.7}$$

**Proof**

Regarding $y$ as a function of $x$ and differentiating Equation 2.5.6implicitly with respect to $x$ yields

$$F_x(x, y) + F_y(x, y)\,\frac{dy}{dx} = 0.$$

On the other hand, regarding $x$ as a function of $y$ and differentiating Equation 2.5.6implicitly with respect to $y$ yields

$$F_x(x, y)\,\frac{dx}{dy} + F_y(x, y) = 0.$$

Thus, Equation 2.5.6is an implicit solution of Equation 2.5.7in either of its two possible interpretations.

We will say that the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{2.5.8}$$

is **exact** on an an open rectangle $R$ if there's a function $F = F(x, y)$ such $F_x$ and $F_y$ are continuous, and

$$F_x(x, y) = M(x, y) \quad \text{and} \quad F_y(x, y) = N(x, y) \tag{2.5.9}$$

for all $(x, y)$ in $R$. This usage of "exact" is related to its usage in calculus, where the expression

$$F_x(x, y)\,dx + F_y(x, y)\,dy$$

(obtained by substituting Equation 2.5.9 into the left side of Equation 2.5.8) is the **exact differential of** $F$.

Example 2.5.1 shows that it is easy to solve Equation 2.5.8 if it is exact **and** we know a function $F$ that satisfies Equation 2.5.9. The important questions are:

- Question 1. Given an equation Equation 2.5.8, how can we determine whether it is exact?
- Question 2. If Equation 2.5.8 is exact, how do we find a function $F$ satisfying Equation 2.5.9?

To discover the answer to Question 1, assume that there's a function $F$ that satisfies Equation 2.5.9 on some open rectangle $R$, and in addition that $F$ has continuous mixed partial derivatives $F_{xy}$ and $F_{yx}$. Then a theorem from calculus implies that

$$F_{xy} = F_{yx}. \tag{2.5.10}$$

If $F_x = M$ and $F_y = N$, differentiating the first of these equations with respect to $y$ and the second with respect to $x$ yields

$$F_{xy} = M_y \quad \text{and} \quad F_{yx} = N_x. \tag{2.5.11}$$

From Equation 2.5.10 and Equation 2.5.11, we conclude that a necessary condition for exactness is that $M_y = N_x$. This motivates the next theorem, which we state without proof.

**Theorem 2.5.2: The Exactness Condition**

Suppose $M$ and $N$ are continuous and have continuous partial derivatives $M_y$ and $N_x$ on an open rectangle $R$. Then

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is exact on $R$ if and only if

$$M_y(x,y) = N_x(x,y) \tag{2.5.12}$$

for all $(x,y)$ in $R$..

To help you remember the exactness condition, observe that the coefficients of $dx$ and $dy$ are differentiated in Equation 2.5.12 with respect to the "opposite" variables; that is, the coefficient of $dx$ is differentiated with respect to $y$, while the coefficient of $dy$ is differentiated with respect to $x$.

**Example 2.5.2**

Show that the equation

$$3x^2y\,dx + 4x^3\,dy = 0$$

is not exact on any open rectangle.

**Solution**

Here

$$M(x,y) = 3x^2y \quad \text{and} \quad N(x,y) = 4x^3$$

so

$$M_y(x,y) = 3x^2 \quad \text{and} N_x(x,y) = 12x^2.$$

Therefore $M_y = N_x$ on the line $x = 0$, but not on any open rectangle, so there's no function $F$ such that $F_x(x,y) = M(x,y)$ and $F_y(x,y) = N(x,y)$ for all $(x,y)$ on any open rectangle.

The next example illustrates two possible methods for finding a function $F$ that satisfies the condition $F_x = M$ and $F_y = N$ if $M\,dx + N\,dy = 0$ is exact.

**Example 2.5.3**

Solve

$$(4x^3y^3 + 3x^2)\,dx + (3x^4y^2 + 6y^2)\,dy = 0. \tag{2.5.13}$$

**Solution (Method 1)**

Here

$$M(x,y) = 4x^3y^3 + 3x^2, \quad N(x,y) = 3x^4y^2 + 6y^2,$$

and

$$M_y(x,y) = N_x(x,y) = 12x^3y^2$$

for all $(x,y)$. Therefore Theorem 2.5.2 implies that there's a function $F$ such that

$$F_x(x,y) = M(x,y) = 4x^3y^3 + 3x^2 \tag{2.5.14}$$

and

$$F_y(x,y) = N(x,y) = 3x^4y^2 + 6y^2 \tag{2.5.15}$$

for all $(x,y)$. To find $F$, we integrate Equation 2.5.14 with respect to $x$ to obtain

$$F(x,y) = x^4y^3 + x^3 + \phi(y), \tag{2.5.16}$$

where $\phi(y)$ is the "constant" of integration. (Here $\phi$ is "constant" in that it is independent of $x$, the variable of integration.) If $\phi$ is any differentiable function of $y$ then $F$ satisfies Equation 2.5.14. To determine $\phi$ so that $F$ also satisfies Equation 2.5.15, assume that $\phi$ is differentiable and differentiate $F$ with respect to $y$. This yields

$$F_y(x,y) = 3x^4y^2 + \phi'(y).$$

Comparing this with Equation 2.5.15 shows that

$$\phi'(y) = 6y^2.$$

We integrate this with respect to $y$ and take the constant of integration to be zero because we are interested only in finding **some** $F$ that satisfies Equation 2.5.14 and Equation 2.5.15. This yields

$$\phi(y) = 2y^3.$$

Substituting this into Equation 2.5.16 yields

$$F(x,y) = x^4y^3 + x^3 + 2y^3. \tag{2.5.17}$$

Now Theorem 2.5.1 implies that

$$x^4y^3 + x^3 + 2y^3 = c$$

is an implicit solution of Equation 2.5.13. Solving this for $y$ yields the explicit solution

$$y = \left(\frac{c - x^3}{2 + x^4}\right)^{1/3}.$$

### Solution (Method 2)

Instead of first integrating Equation 2.5.14 with respect to $x$, we could begin by integrating Equation 2.5.15 with respect to $y$ to obtain

$$F(x,y) = x^4y^3 + 2y^3 + \psi(x), \tag{2.5.18}$$

where $\psi$ is an arbitrary function of $x$. To determine $\psi$, we assume that $\psi$ is differentiable and differentiate $F$ with respect to $x$, which yields

$$F_x(x,y) = 4x^3y^3 + \psi'(x).$$

Comparing this with Equation 2.5.14 shows that

$$\psi'(x) = 3x^2.$$

Integrating this and again taking the constant of integration to be zero yields

$$\psi(x) = x^3.$$

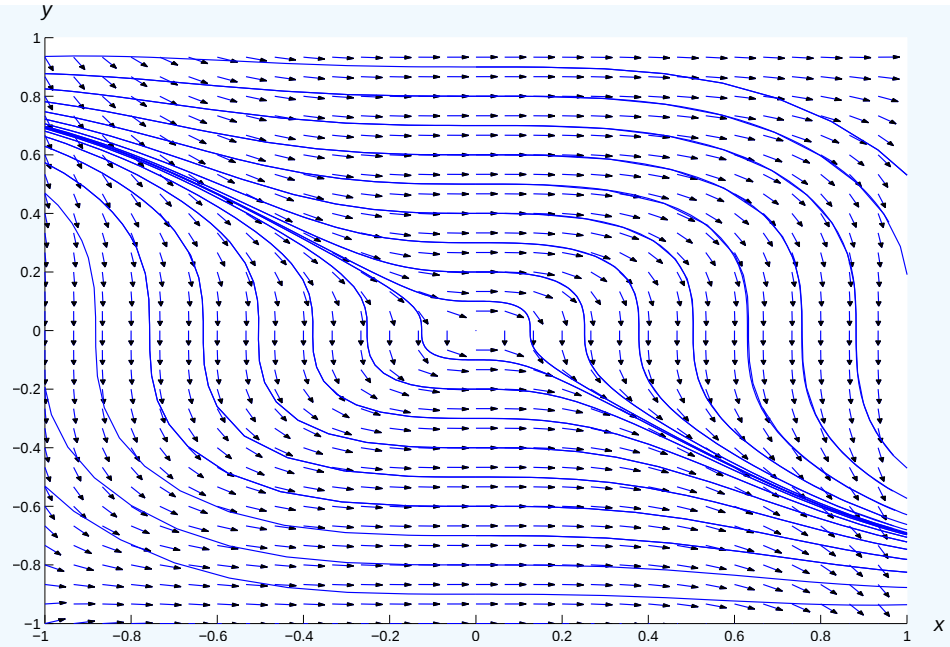



Figure 2.5.1: A direction field and integral curves for $(4x^3y^3+3x^2)\,dx+(3x^4y^2+6y^2)\,dy=0$

Substituting this into Equation 2.5.18 yields Equation 2.5.17.

Figure 2.5.1 shows a direction field and some integral curves of Equation 2.5.13.

Here's a summary of the procedure used in Method 1 of this example. You should summarize procedure used in Method 2.

### HOWTO: Procedure For Solving An Exact Equation (Method 1)

- **Step 1.** Check that the equation

$$M(x,y)\,dx+N(x,y)\,dy=0 \tag{2.5.19}$$

  satisfies the exactness condition $M_y=N_x$. If not, don't go further with this procedure.
- **Step 2.** Integrate

$$\frac{\partial F(x,y)}{\partial x}=M(x,y)$$

  with respect to $x$ to obtain

$$F(x,y)=G(x,y)+\phi(y), \tag{2.5.20}$$

  where $G$ is an antiderivative of $M$ with respect to $x$, and $\phi$ is an unknown function of $y$.
- **Step 3.** Differentiate Equation 2.5.20 with respect to $y$ to obtain

$$\frac{\partial F(x,y)}{\partial y}=\frac{\partial G(x,y)}{\partial y}+\phi'(y).$$

- **Step 4.** Equate the right side of this equation to $N$ and solve for $\phi'$; thus,

$$\frac{\partial G(x,y)}{\partial y}+\phi'(y)=N(x,y),\quad \text{so}\quad \phi'(y)=N(x,y)-\frac{\partial G(x,y)}{\partial y}.$$

- **Step 5.** Integrate $\phi'$ with respect to $y$, taking the constant of integration to be zero, and substitute the result in Equation 2.5.20 to obtain $F(x,y)$.
- **Step 6.** Set $F(x,y)=c$ to obtain an implicit solution of Equation 2.5.19. If possible, solve for $y$ explicitly as a function of $x$.

It's a common mistake to omit Step 6 in the procedure above. However, it is important to include this step, since F isn't itself a solution of Equation 2.5.19. Many equations can be conveniently solved by either of the two methods used in Example 2.5.3. However, sometimes the integration required in one approach is more difficult than in the other. In such cases we choose the approach that requires the easier integration.

**Example 2.5.4**

Solve the equation

$$\left(ye^{xy}\tan x + e^{xy}\sec^2 x\right)dx + xe^{xy}\tan x\,dy = 0 \tag{2.5.21}$$

**Solution**

We leave it to you to check that $M_y = N_x$ on any open rectangle where $\tan x$ and $\sec x$ are defined. Here we must find a function F such that

$$F_x(x,y) = ye^{xy}\tan x + e^{xy}\sec^2 x \tag{2.5.22}$$

and

$$F_y(x,y) = xe^{xy}\tan x. \tag{2.5.23}$$

It's difficult to integrate Equation 2.5.22 with respect to $x$, but easy to integrate Equation 2.5.23 with respect to $y$. This yields

$$F(x,y) = e^{xy}\tan x + \psi(x). \tag{2.5.24}$$

Differentiating this with respect to $x$ yields

$$F_x(x,y) = ye^{xy}\tan x + e^{xy}\sec^2 x + \psi'(x).$$

Comparing this with Equation 2.5.22 shows that $\psi'(x) = 0$. Hence, $\psi$ is a constant, which we can take to be zero in Equation 2.5.24, and

$$e^{xy}\tan x = c,$$

is an implicit solution of Equation 2.5.21.

Attempting to apply our procedure to an differential equation that is not exact will lead to failure in Step 4, since the function

$$N - \frac{\partial G}{\partial y}$$

will not be independent of $x$ if $M_y \neq N_x$, and therefore cannot be the derivative of a function of $y$ alone. Example 2.5.5 illustrates this.

**Example 2.5.5**

Verify that the equation

$$3x^2y^2\,dx + 6x^3y\,dy = 0 \tag{2.5.25}$$

is not exact, and show that the procedure for solving exact equations fails when applied to Equation 2.5.25.

**Solution**

Here

$$M_y(x,y) = 6x^2y \quad \text{and} \quad N_x(x,y) = 18x^2y,$$

so Equation 2.5.25 is not exact. Nevertheless, let us try to find a function $F$ such that

$$F_x(x,y) = 3x^2y^2 \tag{2.5.26}$$

and

$$F_y(x,y) = 6x^3y. \tag{2.5.27}$$

Integrating Equation 2.5.26 with respect to $x$ yields

$$F(x,y) = x^3y^2 + \phi(y),$$

and differentiating this with respect to $y$ yields

$$F_y(x,y) = 2x^3y + \phi'(y).$$

For this equation to be consistent with Equation 2.5.27,

$$6x^3y = 2x^3y + \phi'(y),$$

or

$$\phi'(y) = 4x^3y.$$

This is a contradiction, since $\phi'$ must be independent of $x$. Therefore the procedure fails.

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# 2.5E: Exact Equations (Exercises)

## Q2.5.1

In **Exercises 2.5.1-2.5.17** determine which equations are exact and solve them.

1. $6x^2y^2\,dx+4x^3y\,dy=0$
2. $(3y\cos x+4xe^x+2x^2e^x)\,dx+(3\sin x+3)\,dy=0$
3. $14x^2y^3\,dx+21x^2y^2\,dy=0$
4. $(2x-2y^2)\,dx+(12y^2-4xy)\,dy=0$
5. $(x+y)^2\,dx+(x+y)^2\,dy=0$
6. $(4x+7y)\,dx+(3x+4y)\,dy=0$
7. $(-2y^2\sin x+3y^3-2x)\,dx+(4y\cos x+9xy^2)\,dy=0$
8. $(2x+y)\,dx+(2y+2x)\,dy=0$
9. $(3x^2+2xy+4y^2)\,dx+(x^2+8xy+18y)\,dy=0$
10. $(2x^2+8xy+y^2)\,dx+(2x^2+xy^3/3)\,dy=0$
11. $\left(\frac{1}{x}+2x\right)\,dx+\left(\frac{1}{y}+2y\right)\,dy=0$
12. $(y\sin xy+xy^2\cos xy)\,dx+(x\sin xy+xy^2\cos xy)\,dy=0$
13. $\frac{x\,dx}{(x^2+y^2)^{3/2}}+\frac{y\,dy}{(x^2+y^2)^{3/2}}=0$
14. $\left(e^x(x^2y^2+2xy^2)+6x\right)\,dx+(2x^2ye^x+2)\,dy=0$
15. $\left(x^2e^{x^2+y}(2x^2+3)+4x\right)\,dx+(x^3e^{x^2+y}-12y^2)\,dy=0$
16. $\left(e^{xy}(x^4y+4x^3)+3y\right)\,dx+(x^5e^{xy}+3x)\,dy=0$
17. $(3x^2\cos xy-x^3y\sin xy+4x)\,dx+(8y-x^4\sin xy)\,dy=0$

## Q2.5.2

In **Exercises 2.5.18-2.5.22** solve the initial value problem.

18. $(4x^3y^2-6x^2y-2x-3)\,dx+(2x^4y-2x^3)\,dy=0,\quad y(1)=3$
19. $(-4y\cos x+4\sin x\cos x+\sec^2 x)\,dx+(4y-4\sin x)\,dy=0,\quad y(\pi/4)=0$
20. $(y^3-1)e^x\,dx+3y^2(e^x+1)\,dy=0,\quad y(0)=0$
21. $(\sin x-y\sin x-2\cos x)\,dx+\cos x\,dy=0,\quad y(0)=1$
22. $(2x-1)(y-1)\,dx+(x+2)(x-3)\,dy=0,\quad y(1)=-1$

## Q2.5.3

23. Solve the exact equation

$$(7x+4y)\,dx+(4x+3y)\,dy=0.$$

Plot a direction field and some integral curves for this equation on the rectangle

$$\{-1\le x\le 1,-1\le y\le 1\}.$$

24. Solve the exact equation

$$e^x(x^4y^2+4x^3y^2+1)\,dx+(2x^4ye^x+2y)\,dy=0.$$

Plot a direction field and some integral curves for this equation on the rectangle

$$\{-2 \le x \le 2, -1 \le y \le 1\}.$$

25. Plot a direction field and some integral curves for the exact equation

$$(x^3y^4 + x)\,dx + (x^4y^3 + y)\,dy = 0$$

on the rectangle $\{-1 \le x \le 1, -1 \le y \le 1\}$ . (See **Exercise 2.5.37(a)**).

26. Plot a direction field and some integral curves for the exact equation

$$(3x^2 + 2y)\,dx + (2y + 2x)\,dy = 0$$

on the rectangle $\{-2 \le x \le 2, -2 \le y \le 2\}$ . (See E**xercise 2.5.37(b)**).

27.

a. Solve the exact equation

$$(x^3y^4 + 2x)\,dx + (x^4y^3 + 3y)\,dy = 0 \tag{A}$$

implicitly.

b. For what choices of $(x_0, y_0)$ does Theorem 2.3.1 imply that the initial value problem

$$(x^3y^4 + 2x)\,dx + (x^4y^3 + 3y)\,dy = 0, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution on an open interval $(a, b)$ that contains $x_0$?

c. Plot a direction field and some integral curves for (A) on a rectangular region centered at the origin. What is the interval of validity of the solution of (B)?

28.

a. Solve the exact equation

$$(x^2 + y^2)\,dx + 2xy\,dy = 0 \tag{A}$$

implicitly.

b. For what choices of $(x_0, y_0)$ does Theorem 2.3.1 imply that the initial value problem

$$(x^2 + y^2)\,dx + 2xy\,dy = 0, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution $y = y(x)$ on some open interval $(a, b)$ that contains $x_0$?

c. Plot a direction field and some integral curves for (A). From the plot determine, the interval $(a, b)$ of b, the monotonicity properties (if any) of the solution of (B), and and $\lim_{x \to a+} y(x)$ and $\lim_{x \to b-} y(x)$.

29. Find all functions $M$ such that the equation is exact.

a. $M(x, y)\,dx + (x^2 - y^2)\,dy = 0$
b. $M(x, y)\,dx + 2xy \sin x \cos y\,dy = 0$
c. $M(x, y)\,dx + (e^x - e^y \sin x)\,dy = 0$

30. Find all functions $N$ such that the equation is exact.

a. $(x^3y^2 + 2xy + 3y^2)\,dx + N(x, y)\,dy = 0$
b. $(\ln xy + 2y \sin x)\,dx + N(x, y)\,dy = 0$
c. $(x \sin x + y \sin y)\,dx + N(x, y)\,dy = 0$

31. Suppose $M, N$, and their partial derivatives are continuous on an open rectangle $R$, and $G$ is an antiderivative of $M$ with respect to $x$; that is,

$$\frac{\partial G}{\partial x} = M.$$

Show that if $M_y \ne N_x$ in $R$ then the function

$$N - \frac{\partial G}{\partial y}$$

is not independent of $x$.

32. Prove: If the equations $M_1\,dx + N_1\,dy = 0$ and $M_2\,dx + N_2\,dy = 0$ are exact on an open rectangle $R$, so is the equation

$$(M_1 + M_2)\,dx + (N_1 + N_2)\,dy = 0.$$

33. Find conditions on the constants $A$, $B$, $C$, and $D$ such that the equation

$$(Ax + By)\,dx + (Cx + Dy)\,dy = 0$$

is exact.

34. Find conditions on the constants $A$, $B$, $C$, $D$, $E$, and $F$ such that the equation

$$(Ax^2 + Bxy + Cy^2)\,dx + (Dx^2 + Exy + Fy^2)\,dy = 0$$

is exact.

35. Suppose $M$ and $N$ are continuous and have continuous partial derivatives $M_y$ and $N_x$ that satisfy the exactness condition $M_y = N_x$ on an open rectangle $R$. Show that if $(x, y)$ is in $R$ and

$$F(x, y) = \int_{x_0}^{x} M(s, y_0)\,ds + \int_{y_0}^{y} N(x, t)\,dt,$$

then $F_x = M$ and $F_y = N$.

36. Under the assumptions of **Exercise 2.5.35**, show that

$$F(x, y) = \int_{y_0}^{y} N(x_0, s)\,ds + \int_{x_0}^{x} M(t, y)\,dt.$$

37. Use the method suggested by **Exercise 2.5.35**, with $(x_0, y_0) = (0, 0)$, to solve the these exact equations:

a. $(x^3y^4 + x)\,dx + (x^4y^3 + y)\,dy = 0$
b. $(x^2 + y^2)\,dx + 2xy\,dy = 0$
c. $(3x^2 + 2y)\,dx + (2y + 2x)\,dy = 0$

38. Solve the initial value problem

$$y' + \frac{2}{x}y = -\frac{2xy}{x^2 + 2x^2y + 1}, \quad y(1) = -2.$$

39. Solve the initial value problem

$$y' - \frac{3}{x}y = \frac{2x^4(4x^3 - 3y)}{3x^5 + 3x^3 + 2y}, \quad y(1) = 1.$$

40. Solve the initial value problem

$$y' + 2xy = -e^{-x^2}\left(\frac{3x + 2ye^{x^2}}{2x + 3ye^{x^2}}\right), \quad y(0) = -1.$$

41. Rewrite the separable equation

$$h(y)y' = g(x) \tag{A}$$

as an exact equation

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{B}$$

Show that applying the method of this section to (B) yields the same solutions that would be obtained by applying the method of separation of variables to (A)

42. Suppose all second partial derivatives of $M = M(x, y)$ and $N = N(x, y)$ are continuous and $M\,dx + N\,dy = 0$ and $-N\,dx + M\,dy = 0$ are exact on an open rectangle $R$. Show that $M_{xx} + M_{yy} = N_{xx} + N_{yy} = 0$ on $R$.

43. Suppose all second partial derivatives of $F = F(x, y)$ are continuous and $F_{xx} + F_{yy} = 0$ on an open rectangle $R$. (A function with these properties is said to be **harmonic**; see also **Exercise 2.5.42**.) Show that $-F_y\,dx + F_x\,dy = 0$ is exact on $R$, and therefore there's a function $G$ such that $G_x = -F_y$ and $G_y = F_x$ in $R$. (A function $G$ with this property is said to be a **harmonic conjugate** of $F$.)

44. Verify that the following functions are harmonic, and find all their harmonic conjugates. (See **Exercise 2.5.43**.)

a. $x^2 - y^2$
b. $e^x \cos y$
c. $x^3 - 3xy^2$
d. $\cos x \cosh y$
e. $\sin x \cosh y$

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# 2.6: Integrating Factors

In the previous section on Exact Equations we saw that if $M$, $N$, $M_y$ and $N_x$ are continuous and $M_y = N_x$ on an open rectangle $R$ then

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.6.1}$$

is exact on $R$. Sometimes an equation that isn't exact can be made exact by multiplying it by an appropriate function. For example,

$$(3x + 2y^2)\,dx + 2xy\,dy = 0 \tag{2.6.2}$$

is not exact, since $M_y(x,y) = 4y \neq N_x(x,y) = 2y$ in Equation 2.6.2. However, multiplying Equation 2.6.2 by $x$ yields

$$(3x^2 + 2xy^2)\,dx + 2x^2y\,dy = 0, \tag{2.6.3}$$

which is exact, since $M_y(x,y) = N_x(x,y) = 4xy$ in Equation 2.6.3. Solving Equation 2.6.3 by the procedure given in Section 2.5 yields the implicit solution

$$x^3 + x^2y^2 = c.$$

A function $\mu = \mu(x,y)$ is an **integrating factor** for Equation 2.6.1 if

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0 \tag{2.6.4}$$

is exact. If we know an integrating factor $\mu$ for Equation 2.6.1, we can solve the exact equation Equation 2.6.4 by the method of Section 2.5. It would be nice if we could say that Equation 2.6.1 and Equation 2.6.4 always have the same solutions, but this isn't so. For example, a solution $y = y(x)$ of Equation 2.6.4 such that $\mu(x, y(x)) = 0$ on some interval $a < x < b$ could fail to be a solution of 2.6.1 (**Exercise 2.6.1**), while Equation 2.6.1 may have a solution $y = y(x)$ such that $\mu(x, y(x))$ isn't even defined (**Exercise 2.6.2**). Similar comments apply if $y$ is the independent variable and $x$ is the dependent variable in Equation 2.6.1 and Equation 2.6.4. However, if $\mu(x,y)$ is defined and nonzero for all $(x,y)$, Equation 2.6.1 and Equation 2.6.4 are equivalent; that is, they have the same solutions.

## Finding Integrating Factors

By applying Theorem 2.5.2 (with $M$ and $N$ replaced by $\mu M$ and $\mu N$), we see that Equation 2.6.4 is exact on an open rectangle $R$ if $\mu M$, $\mu N$, $(\mu M)_y$, and $(\mu N)_x$ are continuous and

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

or, equivalently

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

on $R$. It's better to rewrite the last equation as

$$\mu(M_y - N_x) = \mu_x N - \mu_y M, \tag{2.6.5}$$

which reduces to the known result for exact equations; that is, if $M_y = N_x$ then Equation 2.6.5 holds with $\mu = 1$, so Equation 2.6.1 is exact.

You may think Equation 2.6.5 is of little value, since it involves **partial** derivatives of the unknown integrating factor $\mu$, and we haven't studied methods for solving such equations. However, we'll now show that Equation 2.6.5 is useful if we restrict our search to integrating factors that are products of a function of $x$ and a function of $y$; that is, $\mu(x,y) = P(x)Q(y)$. We're not saying that **every** equation $M\,dx + N\,dy = 0$ has an integrating factor of this form; rather, we are saying that **some** equations have such integrating factors.We'llnow develop a way to determine whether a given equation has such an integrating factor, and a method for finding the integrating factor in this case.

If $\mu(x,y) = P(x)Q(y)$, then $\mu_x(x,y) = P'(x)Q(y)$ and $\mu_y(x,y) = P(x)Q'(y)$, so Equation 2.6.5 becomes

$$P(x)Q(y)(M_y - N_x) = P'(x)Q(y)N - P(x)Q'(y)M, \tag{2.6.6}$$

or, after dividing through by $P(x)Q(y)$,

$$M_y - N_x = \frac{P'(x)}{P(x)}N - \frac{Q'(y)}{Q(y)}M. \tag{2.6.7}$$

Now let

$$p(x) = \frac{P'(x)}{P(x)} \quad \text{and} \quad q(y) = \frac{Q'(y)}{Q(y)},$$

so Equation 2.6.7 becomes

$$M_y - N_x = p(x)N - q(y)M. \tag{2.6.8}$$

We obtained Equation 2.6.8 by **assuming** that $M\,dx + N\,dy = 0$ has an integrating factor $\mu(x,y) = P(x)Q(y)$. However, we can now view Equation 2.6.7 differently: If there are functions $p = p(x)$ and $q = q(y)$ that satisfy Equation 2.6.8 and we define

$$P(x) = \pm e^{\int p(x)\,dx} \quad \text{and} \quad Q(y) = \pm e^{\int q(y)\,dy}, \tag{2.6.9}$$

then reversing the steps that led from Equation 2.6.6 to Equation 2.6.8 shows that $\mu(x,y) = P(x)Q(y)$ is an integrating factor for $M\,dx + N\,dy = 0$ . In using this result, we take the constants of integration in Equation 2.6.9 to be zero and choose the signs conveniently so the integrating factor has the simplest form.

There's no simple general method for ascertaining whether functions $p = p(x)$ and $q = q(y)$ satisfying Equation 2.6.8 exist. However, the next theorem gives simple sufficient conditions for the given equation to have an integrating factor that depends on only one of the independent variables $x$ and $y$, and for finding an integrating factor in this case.

**Theorem 2.6.1**

Let $M$, $N$, $M_y$, and $N_x$ be continuous on an open rectangle $R$. Then:

a. If $(M_y - N_x)/N$ is independent of $y$ on $R$ and we define

$$p(x) = \frac{M_y - N_x}{N}$$

then

$$\mu(x) = \pm e^{\int p(x)\,dx} \tag{2.6.10}$$

is an integrating factor for

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.6.11}$$

on $R$.

b. If $(N_x - M_y)/M$ is independent of $x$ on $R$ and we define

$$q(y) = \frac{N_x - M_y}{M},$$

then

$$\mu(y) = \pm e^{\int q(y)\,dy} \tag{2.6.12}$$

is an integrating factor for Equation 2.6.11 on $R$.

**Proof**

(a) If $(M_y - N_x)/N$ is independent of $y$, then Equation 2.6.8 holds with $p = (M_y - N_x)/N$ and $q \equiv 0$ . Therefore

$$P(x) = \pm e^{\int p(x)\,dx} \quad \text{and} \quad Q(y) = \pm e^{\int q(y)\,dy} = \pm e^0 = \pm 1,$$

so Equation 2.6.10 is an integrating factor for Equation 2.6.11 on $R$.

(b) If $(N_x - M_y)/M$ is independent of $x$ then eqrefeq:2.6.8 holds with $p \equiv 0$ and $q = (N_x - M_y)/M$ , and a similar argument shows that Equation 2.6.12 is an integrating factor for Equation 2.6.11 on $R$.

The next two examples show how to apply Theorem 2.6.1 .

**Example 2.6.1**

Find an integrating factor for the equation

$$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + (3x^2y^2 + 4y)\,dy = 0 \tag{2.6.13}$$

and solve the equation.

**Solution**

In Equation 2.6.13

$$M = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x,\ N = 3x^2y^2 + 4y,$$

and

$$M_y - N_x = (6xy^2 - 6x^3y^2 - 8xy) - 6xy^2 = -6x^3y^2 - 8xy,$$

so Equation 2.6.13 isn't exact. However,

$$\frac{M_y - N_x}{N} = -\frac{6x^3y^2 + 8xy}{3x^2y^2 + 4y} = -2x$$

is independent of $y$, so Theorem 2.6.1 (a) applies with $p(x) = -2x$. Since

$$\int p(x)\,dx = -\int 2x\,dx = -x^2,$$

$\mu(x) = e^{-x^2}$ is an integrating factor. Multiplying Equation 2.6.13 by $\mu$ yields the exact equation

$$e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + e^{-x^2}(3x^2y^2 + 4y)\,dy = 0. \tag{2.6.14}$$

To solve this equation, we must find a function $F$ such that

$$F_x(x,y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) \tag{2.6.15}$$

and

$$F_y(x,y) = e^{-x^2}(3x^2y^2 + 4y). \tag{2.6.16}$$

Integrating Equation 2.6.16 with respect to $y$ yields

$$F(x,y) = e^{-x^2}(x^2y^3 + 2y^2) + \psi(x). \tag{2.6.17}$$

Differentiating this with respect to $x$ yields

$$F_x(x,y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2) + \psi'(x).$$

Comparing this with Equation 2.6.15 shows that $\psi'(x) = 2xe^{-x^2}$; therefore, we can let $\psi(x) = -e^{-x^2}$ in Equation 2.6.17 and conclude that

$$e^{-x^2}\left(y^2(x^2y + 2) - 1\right) = c$$

is an implicit solution of Equation 2.6.14. It is also an implicit solution of Equation 2.6.13.

Figure 2.6.1 shows a direction field and some integral curves for Equation 2.6.13

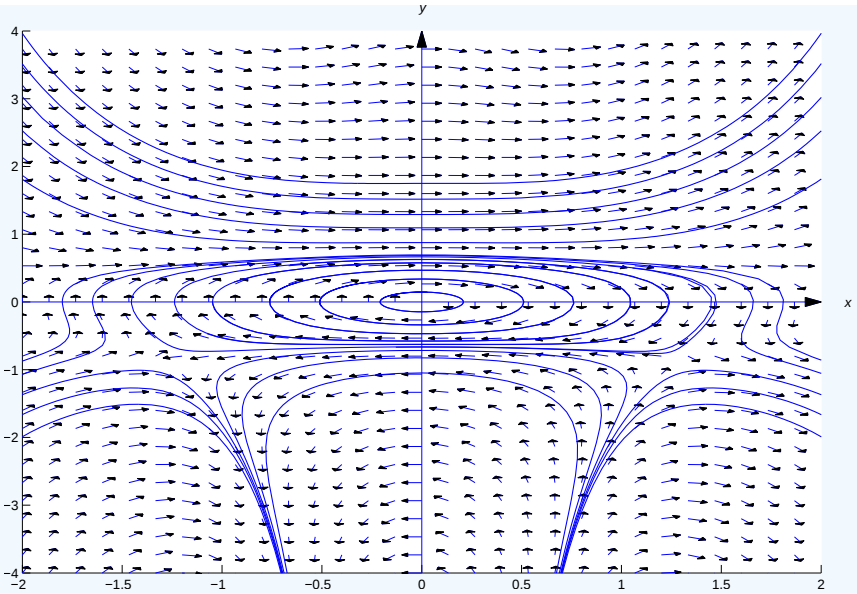


Figure 2.6.1 : A direction field and integral curves for $(2xy^3-2x^3y^3-4xy^2+2x)\,dx+(3x^2y^2+4y)\,dy=0$

## Example 2.6.2

Find an integrating factor for

$$2xy^3dx+(3x^2y^2+x^2y^3+1)dy=0 \tag{2.6.18}$$

and solve the equation.

### Solution

In Equation 2.6.18,

$$M=2xy^3,\quad N=3x^2y^2+x^2y^3+1,$$

and

$$M_y-N_x=6x^2-(6xy^2+2xy^3)=-2xy^3,$$

so Equation 2.6.18 isn't exact. Moreover,

$$\frac{M_y-N_x}{N}=-\frac{2xy^3}{3x^2y^2+x^2y^2+1}$$

is not independent of $y$, so Theorem 2.6.1(a) does not apply. However, Theorem 2.6.1(b) does apply, since

$$\frac{N_x-M_y}{M}=\frac{2xy^3}{2xy^3}=1$$

is not independent of $x$, so we can take $q(y)=1$. Since

$$\int q(y)dy=\int dy=y,$$

$\mu(y)=e^y$ is an integrating factor. Multiplying Equation 2.6.18 by $\mu$ yields the exact equation.

$$2xy^3e^ydx+(3x^2y^2+x^2y^3+1)e^ydy=0. \tag{2.6.19}$$

To solve this equation, we must find a function $F$ such that

$$F_x(x,y)=2xy^3e^y \tag{2.6.20}$$

and

$$F_y(x,y) = (3x^2y^2 + x^2y^3 + 1)e^y. \tag{2.6.21}$$

Integrating Equation 2.6.20 with respect to $x$ yields

$$F(x,y) = x^2y^3e^y + \phi(y) \tag{2.6.22}$$

Differentiating this with respect to $y$ yields

$$F_y = (3x^2y^2 + x^2y^3)e^y + \phi'(y)$$

and comparing this with Equation 2.6.21 shows that φ 0 (y) = e y . Therefore we set φ(y) = e y in Equation 2.6.22 and conclude that

$$(x^2y^3 + 1)e^y = c$$

is an implicit solution of 2.6.19. It is also an implicit solution of 2.6.18. Figure 2.6.2 shows a direction field and some integral curves for 2.6.18.

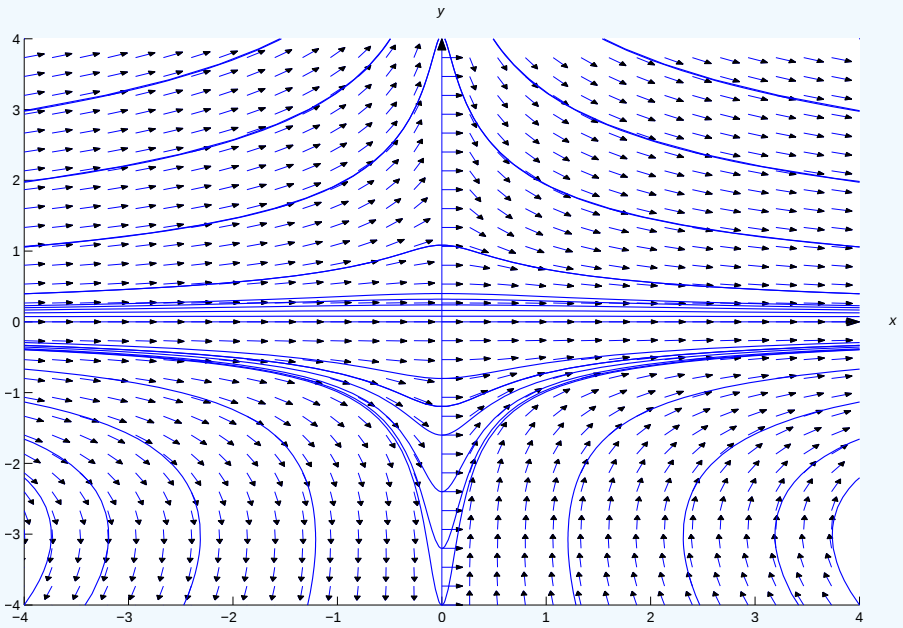



Figure 2.6.2 : A direction field and integral curves for $(2xy^3e^y dx + (3x^2y^2 + x^2y^3 + 1)e^y dy = 0$

Theorem 2.6.1 does not apply in the next example, but the more general argument that led to Theorem 2.6.1 provides an integrating factor.

## Example 2.6.3

Find an integrating factor for

$$(3xy + 6y^2)dx + (2x^2 + 9xy)dy = 0 \tag{2.6.23}$$

and solve the equation.

**Solution**

In Equation 2.6.23

$$M = 3xy + 6y^2, \quad N = 2x^2 + 9xy,$$

and

$$M_y - N_x = (3x + 12y) - (4x + 9y) = -x + 3y.$$

Therefore

$$\frac{M_y - N_x}{M} = \frac{-x+3y}{3xy+6y^2} \quad \text{and} \quad \frac{N_x - M_y}{N} = \frac{x-3y}{2x^2+9xy}$$

so Theorem 2.6.1 does not apply. Following the more general argument that led to Theorem 2.6.1 , we look for functions $p = p(x)$ and $q = q(y)$ such that

$$M_y - N_x = p(x)N - q(y)M;$$

that is,

$$-x+3y = p(x)(2x^2+9xy) - q(y)(3xy+6y^2).$$

Since the left side contains only first degree terms in $x$ and $y$, we rewrite this equation as

$$xp(x)(2x+9y) - yq(y)(3x+6y) = -x+3y.$$

This will be an identity if

$$xp(x) = A \quad \text{and} yq(y) = B, \tag{2.6.24}$$

where $A$ and $B$ are constants such that

$$-x+3y = A(2x+9y) - B(3x+6y),$$

or, equivalently,

$$-x+3y = (2A-3B)x + (9A-6B)y.$$

Equating the coefficients of x and y on both sides shows that the last equation holds for all $(x, y)$ if

$$2A-3B = -1$$
$$9A-6B = 3$$

which has the solution A = 1, B = 1. Therefore Equation 2.6.24 implies that

$$p(x) = \frac{1}{x} \quad \text{and} \quad q(y) = \frac{1}{y}.$$

Since

$$\int p(x)dx = \ln|x| \quad \text{and} \quad \int q(y)dy = \ln|y|,$$

we can let $P(x) = x$ and $Q(y) = y$; hence, $\mu(x, y) = xy$ is an integrating factor. Multiplying Equation 2.6.23 by $\mu$ yields the exact equation

$$(3x^2y^2+6xy^3)dx + (2x^3y+9x^2y^2)dy = 0.$$

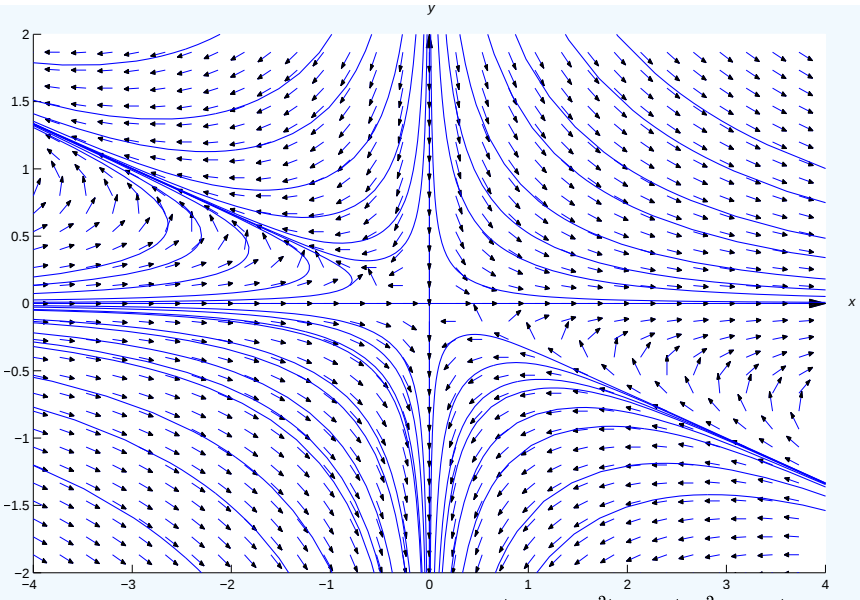


Figure 2.6.3 : A direction field and integral curves for $(3xy+6y^2)dx+(2x^2+9xy)dy=0$

We leave it to you to use the method of Section 2.5 to show that this equation has the implicit solution

$$x^3y^2+3x^2y^3=c. \tag{2.6.25}$$

This is also an implicit solution of Equation 2.6.23. Since x ≡ 0 and y ≡ 0 satisfy Equation 2.6.25, you should check to see that x ≡ 0 and y ≡ 0 are also solutions of Equation 2.6.23. (Why is it necesary to check this?) Figure 2.6.3 shows a direction field and integral curves for Equation 2.6.23. See **Exercise 2.6.28** for a general discussion of equations like Equation 2.6.23.

## Example 2.6.4

The separable equation

$$-ydx+(x+x^6)dy=0 \tag{2.6.26}$$

can be converted to the exact equation

$$-\frac{dx}{x+x^6}+\frac{dy}{y}=0 \tag{2.6.27}$$

by multiplying through by the integrating factor

$$\mu(x,y)=\frac{1}{y(x+x^6)}.$$

However, to solve Equation 2.6.27 by the method of Section 2.5 we would have to evaluate the nasty integral

$$\int\frac{dx}{x+x^6}.$$

Instead, we solve Equation 2.6.26 explicitly for $y$ by finding an integrating factor of the form $\mu(x,y)=x^ay^b$.

### Solution

In Equation 2.6.26

$$M=-y,\ N=x+x^6,$$

and

$$M_y-N_x=-1-(1+6x^5)=-2-6x^5.$$

We look for functions $p=p(x)$ and $q=q(y)$ such that

$$M_y - N_x = p(x)N - q(y)M;$$

that is,

$$-2 - 6x^5 = p(x)(x + x^6) + q(y)y. \tag{2.6.28}$$

The right side will contain the term $-6x^5$ if $p(x) = -6/x$. Then Equation 2.6.28 becomes

$$-2 - 6x^5 = -6 - 6x^5 + q(y)y,$$

so $q(y) = 4/y$. Since

$$\int p(x)\,dx = -\int \frac{6}{x}\,dx = -6\ln|x| = \ln\frac{1}{x^6},$$

and

$$\int q(y)\,dy = \int \frac{4}{y}\,dy = 4\ln|y| = \ln y^4,$$

we can take $P(x) = x^{-6}$ and $Q(y) = y^4$, which yields the integrating factor $\mu(x, y) = x^{-6}y^4$. Multiplying Equation 2.6.26 by $\mu$ yields the exact equation

$$-\frac{y^5}{x^6}\,dx + \left(\frac{y^4}{x^5} + y^4\right)\,dy = 0.$$

We leave it to you to use the method of the Section 2.5 to show that this equation has the implicit solution

$$\left(\frac{y}{x}\right)^5 + y^5 = k.$$

Solving for $y$ yields

$$y = k^{1/5}x(1 + x^5)^{-1/5},$$

which we rewrite as

$$y = cx(1 + x^5)^{-1/5}$$

by renaming the arbitrary constant. This is also a solution of Equation 2.6.26.

Figure 2.6.4 shows a direction field and some integral curves for Equation 2.6.26.

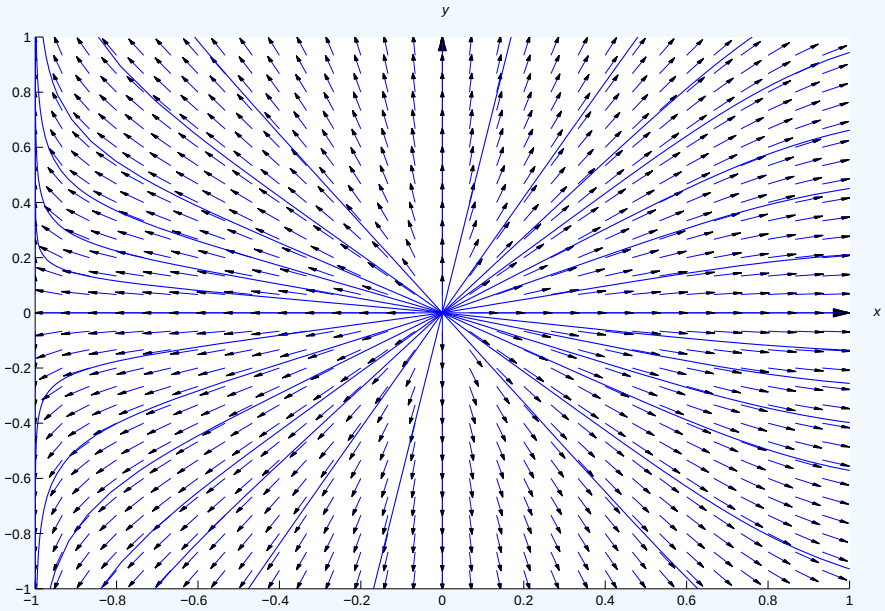



Figure 2.6.4 : A direction field and integral curves for $-ydx + (x + x^6)dy = 0$


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# 2.6E: Integrating Factors (Exercises)

## Q2.6.1

1.

a. Verify that $\mu(x,y)=y$ is an integrating factor for

$$ydx+\left(2x+\frac{1}{y}\right)dy=0 \tag{A}$$

on any open rectangle that does not intersect the $x$ axis or, equivalently, that

$$y^2dx+(2xy+1)dy=0 \tag{B}$$

is exact on any such rectangle.

b. Verify that $y\equiv 0$ is a solution of (B), but not of (A).

c. Show that

$$y(xy+1)=c \tag{C}$$

is an implicit solution of (B), and explain why every differentiable function $y=y(x)$ other than $y\equiv 0$ that satisfies (C) is also a solution of (A).

2.

a. Verify that $\mu(x,y)=1/(x-y)^2$ is an integrating factor for

$$-y^2dx+x^2dy=0 \tag{A}$$

on any open rectangle that does not intersect the line $y=x$ or, equivalently, that

$$-\frac{y^2}{(x-y)^2}dx+\frac{x^2}{(x-y)^2}dy=0 \tag{B}$$

is exact on any such rectangle.

b. Use Theorem 2.2.1 to show that

$$\frac{xy}{(x-y)}=c \tag{C}$$

is an implicit solution of (B), and explain why it is also an implicit solution of (A)

c. Verify that $y=x$ is a solution of (A), even though it can't be obtained from (C).

## Q2.6.2

In **Exercises 2.6.3-2.6.16** find an integrating factor; that is a function of only one variable, and solve the given equation.

3. $ydx-xdy=0$

4. $3x^2ydx+2x^3dy=0$

5. $2y^3dx+3y^2dy=0$

6. $(5xy+2y+5)dx+2xdy=0$

7. $(xy+x+2y+1)\,dx+(x+1)\,dy=0$

8. $(27xy^2+8y^3)\,dx+(18x^2y+12xy^2)\,dy=0$

9. $(6xy^2+2y)\,dx+(12x^2y+6x+3)\,dy=0$

10. $y^2\,dx+\left(xy^2+3xy+\frac{1}{y}\right)\,dy=0$

11. $(12x^3y+24x^2y^2)\,dx+(9x^4+32x^3y+4y)\,dy=0$

12. $(x^2y+4xy+2y)\,dx+(x^2+x)\,dy=0$

13. $-y\,dx+(x^4-x)\,dy=0$

14. $\cos x\cos y\,dx+(\sin x\cos y-\sin x\sin y+y)\,dy=0$

15. $(2xy+y^2)\,dx+(2xy+x^2-2x^2y^2-2xy^3)\,dy=0$

16. $y\sin y\,dx+x(\sin y-y\cos y)\,dy=0$

## Q2.6.3

In **Exercises 2.6.17-2.6.23** find an integrating factor of the form $\mu(x,y)=P(x)Q(y)$ and solve the given equation.

17. $y(1+5\ln|x|)\,dx+4x\ln|x|\,dy=0$

18. $(\alpha y+\gamma xy)\,dx+(\beta x+\delta xy)\,dy=0$

19. $(3x^2y^3-y^2+y)\,dx+(-xy+2x)\,dy=0$

20. $2y\,dx+3(x^2+x^2y^3)\,dy=0$

21. $(a\cos xy-y\sin xy)\,dx+(b\cos xy-x\sin xy)\,dy=0$

22. $x^4y^4\,dx+x^5y^3\,dy=0$

23. $y(x\cos x+2\sin x)\,dx+x(y+1)\sin x\,dy=0$

## Q2.6.4

In **Exercises 2.6.24-2.6.27** find an integrating factor and solve the equation. Plot a direction field and some integral curves for the equation in the indicated rectangular region.

24. $(x^4y^3+y)\,dx+(x^5y^2-x)\,dy=0;\quad\{-1\le x\le 1,-1\le y\le 1\}$

25. $(3xy+2y^2+y)\,dx+(x^2+2xy+x+2y)\,dy=0;\quad\{-2\le x\le 2,-2\le y\le 2\}$

26. $(12xy+6y^3)\,dx+(9x^2+10xy^2)\,dy=0;\quad\{-2\le x\le 2,-2\le y\le 2\}$

27. $(3x^2y^2+2y)\,dx+2x\,dy=0;\quad\{-4\le x\le 4,-4\le y\le 4\}$

## Q2.6.5

28. Suppose $a$, $b$, $c$, and $d$ are constants such that $ad-bc\ne 0$ , and let $m$ and $n$ be arbitrary real numbers. Show that

$$(ax^my+by^{n+1})\,dx+(cx^{m+1}+dxy^n)\,dy=0$$

has an integrating factor $\mu(x,y)=x^\alpha y^\beta$.

29. Suppose $M$, $N$, $M_x$, and $N_y$ are continuous for all $(x,y)$, and $\mu=\mu(x,y)$ is an integrating factor for

$$M(x,y)dx+N(x,y)dy=0. \tag{A}$$

Assume that $\mu_x$ and $\mu_y$ are continuous for all $(x,y)$, and suppose $y=y(x)$ is a differentiable function such that $\mu(x,y(x))=0$ and $\mu_x(x,y(x))\ne 0$ for all $x$ in some interval $I$. Show that $y$ is a solution of (A) on $I$.

30. According to Theorem 2.1.2, the general solution of the linear nonhomogeneous equation

$$y'+p(x)y=f(x) \tag{A}$$

is

$$y=y_1x\left(c+\int f(x)/y_1(x)dx\right), \tag{B}$$

where $y_1$ is any nontrivial solution of the complementary equation $y'+p(x)y=0$. In this exercise we obtain this conclusion in a different way. You may find it instructive to apply the method suggested here to solve some of the exercises in Section 2.1.

a. Rewrite (A) as

$$[p(x)y-f(x)]dx+dy=0, \tag{C}$$

and show that $\mu = \pm e^{\int p(x)\,dx}$ is an integrating factor for (C).

b. Multiply (A) through by $\mu = \pm e^{\int p(x)\,dx}$ and verify that the resulting equation can be rewritten as

$$(\mu(x)y)' = \mu(x)f(x).$$

Then integrate both sides of this equation and solve for $y$ to show that the general solution of (A) is

$$y = \frac{1}{\mu(x)}\left(c + \int f(x)\mu(x)\,dx\right).$$

Why is this form of the general solution equivalent to (B)?

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# CHAPTER OVERVIEW

## 3: Numerical Methods

In this chapter we study numerical methods for solving a first order differential equation $y' = f(x, y)$.


3.1: Euler's Method

3.1E: Euler’s Method (Exercises)

3.2: The Improved Euler Method and Related Methods

3.2E: The Improved Euler Method and Related Methods (Exercises)

3.3: The Runge-Kutta Method

3.3E: The Runge-Kutta Method (Exercises)


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# 3.1: Euler's Method

If an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{3.1.1}$$

cannot be solved analytically, it is necessary to resort to numerical methods to obtain useful approximations to a solution of Equation 3.1.1. we will consider such methods in this chapter.

We're interested in computing approximate values of the solution of Equation 3.1.1 at equally spaced points $x_0$, $x_1$, …, $x_n = b$ in an interval $[x_0, b]$. Thus,

$$x_i = x_0 + ih, \quad i = 0, 1, \ldots, n,$$

where

$$h = \frac{b - x_0}{n}.$$

we will denote the approximate values of the solution at these points by $y_0$, $y_1$, …, $y_n$; thus, $y_i$ is an approximation to $y(x_i)$. we will call

$$e_i = y(x_i) - y_i$$

the **error at the *i*th step**. Because of the initial condition $y(x_0) = y_0$, we will always have $e_0 = 0$. However, in general $e_i \neq 0$ if $i > 0$.

We encounter two sources of error in applying a numerical method to solve an initial value problem:

- The formulas defining the method are based on some sort of approximation. Errors due to the inaccuracy of the approximation are called **truncation errors**.
- Computers do arithmetic with a fixed number of digits, and therefore make errors in evaluating the formulas defining the numerical methods. Errors due to the computer's inability to do exact arithmetic are called **roundoff errors.**

Since a careful analysis of roundoff error is beyond the scope of this book, we will consider only truncation errors.

## Euler's Method

The simplest numerical method for solving Equation 3.1.1 is **Euler's method**. This method is so crude that it is seldom used in practice; however, its simplicity makes it useful for illustrative purposes. Euler's method is based on the assumption that the tangent line to the integral curve of Equation 3.1.1 at $(x_i, y(x_i))$ approximates the integral curve over the interval $[x_i, x_{i+1}]$. Since the slope of the integral curve of Equation 3.1.1 at $(x_i, y(x_i))$ is $y'(x_i) = f(x_i, y(x_i))$, the equation of the tangent line to the integral curve at $(x_i, y(x_i))$ is

$$y = y(x_i) + f(x_i, y(x_i))(x - x_i). \tag{3.1.2}$$

Setting $x = x_{i+1} = x_i + h$ in Equation 3.1.2 yields

$$y_{i+1} = y(x_i) + hf(x_i, y(x_i)) \tag{3.1.3}$$

as an approximation to $y(x_{i+1})$. Since $y(x_0) = y_0$ is known, we can use Equation 3.1.3 with $i = 0$ to compute

$$y_1 = y_0 + hf(x_0, y_0).$$

However, setting $i = 1$ in Equation 3.1.3 yields

$$y_2 = y(x_1) + hf(x_1, y(x_1)),$$

which isn't useful, since we **don't know** $y(x_1)$. Therefore we replace $y(x_1)$ by its approximate value $y_1$ and redefine

$$y_2 = y_1 + hf(x_1, y_1).$$

Having computed $y_2$, we can compute

$$y_3 = y_2 + hf(x_2, y_2).$$

In general, Euler's method starts with the known value $y(x_0) = y_0$ and computes $y_1$, $y_2$, …, $y_n$ successively by with the formula

$$y_{i+1} = y_i + hf(x_i, y_i), \quad 0 \le i \le n-1. \tag{3.1.4}$$

The next example illustrates the computational procedure indicated in Euler's method.

**Example 3.1.1**

Use Euler's method with $h = 0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1 \tag{3.1.5}$$

at $x = 0.1, 0.2, 0.3$

**Solution**

We rewrite Equation 3.1.5 as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form Equation 3.1.1, with

$$f(x, y) = -2y + x^3 e^{-2x},\ x_0 = 0,\ \text{and}\ y_0 = 1.$$

Euler's method yields

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + (0.1)f(0, 1) = 1 + (0.1)(-2) = 0.8, \\ y_2 &= y_1 + hf(x_1, y_1) \\ &= 0.8 + (0.1)f(0.1, 0.8) = 0.8 + (0.1)\left(-2(0.8) + (0.1)^3 e^{-0.2}\right) = 0.640081873, \\ y_3 &= y_2 + hf(x_2, y_2) \\ &= 0.640081873 + (0.1)\left(-2(0.640081873) + (0.2)^3 e^{-0.4}\right) = 0.512601754. \end{aligned}$$

We've written the details of these computations to ensure that you understand the procedure. However, in the rest of the examples as well as the exercises in this chapter, we will assume that you can use a programmable calculator or a computer to carry out the necessary computations.

## Examples Illustrating The Error in Euler's Method

**Example 3.1.2**

Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0$, 0.1, 0.2, 0.3, …, 1.0. Compare these approximate values with the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4), \tag{3.1.6}$$

which can be obtained by the method of Section 2.1. (Verify.)

Table 3.1.1 shows the values of the exact solution Equation 3.1.6 at the specified points, and the approximate values of the solution at these points obtained by Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$. In examining this table, keep in mind that the approximate values in the column corresponding to $h = 0.05$ are actually the results of 20 steps with Euler's method. We haven't listed the estimates of the solution obtained for $x = 0.05$, 0.15, …, since there's nothing to compare them with in the column corresponding to $h = 0.1$. Similarly, the approximate values in the column corresponding to $h = 0.025$ are actually the results of 40 steps with Euler's method.

Table 3.1.1 : Numerical solution of $y' + 2y = x^3 e^{-2x},\ y(0) = 1$, by Euler's method.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 0.0 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 |
| 0.1 | 0.800000000 | 0.810005655 | 0.814518349 | 0.818751221 |
| 0.2 | 0.640081873 | 0.656266437 | 0.663635953 | 0.670588174 |
| 0.3 | 0.512601754 | 0.532290981 | 0.541339495 | 0.549922980 |
| 0.4 | 0.411563195 | 0.432887056 | 0.442774766 | 0.452204669 |
| 0.5 | 0.332126261 | 0.353785015 | 0.363915597 | 0.373627557 |
| 0.6 | 0.270299502 | 0.291404256 | 0.301359885 | 0.310952904 |
| 0.7 | 0.222745397 | 0.242707257 | 0.252202935 | 0.261398947 |
| 0.8 | 0.186654593 | 0.205105754 | 0.213956311 | 0.222570721 |
| 0.9 | 0.159660776 | 0.176396883 | 0.184492463 | 0.192412038 |
| 1.0 | 0.139778910 | 0.154715925 | 0.162003293 | 0.169169104 |

You can see from Table 3.1.1 that decreasing the step size improves the accuracy of Euler's method. For example,

$$y_{exact}(1) - y_{approx}(1) \approx \begin{cases} 0.0293 \text{with} h = 0.1, \\ 0.0144 \text{ with } h = 0.05, \\ 0.0071 \text{ with } h = 0.025. \end{cases}$$

Based on this scanty evidence, you might guess that the error in approximating the exact solution at a **fixed value of** $x$ by Euler's method is roughly halved when the step size is halved. You can find more evidence to support this conjecture by examining Table 3.1.2 , which lists the approximate values of $y_{exact} - y_{approx}$ at $x = 0.1, 0.2, \ldots, 1.0$.

Table 3.1.2 : Errors in approximate solutions of $y' + 2y = x^3 e^{-2x},\ y(0) = 1$, obtained by Euler's method.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.0.25$ |
|---|---|---|---|
| 0.1 | 0.0187 | 0.0087 | 0.0042 |
| 0.2 | 0.0305 | 0.0143 | 0.0069 |
| 0.3 | 0.0373 | 0.0176 | 0.0085 |
| 0.4 | 0.0406 | 0.0193 | 0.0094 |
| 0.5 | 0.0415 | 0.0198 | 0.0097 |
| 0.6 | 0.0406 | 0.0195 | 0.0095 |
| 0.7 | 0.0386 | 0.0186 | 0.0091 |
| 0.8 | 0.0359 | 0.0174 | 0.0086 |
| 0.9 | 0.0327 | 0.0160 | 0.0079 |
| 1.0 | 0.0293 | 0.0144 | 0.0071 |

## Example 3.1.3

Tables 3.1.2 and 3.1.4 show analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2,\ y(0) = 1, \tag{3.1.7}$$

except in this case we cannot solve Equation 3.1.7 exactly. The results in the "Exact" column were obtained by using a more accurate numerical method known as the **Runge-Kutta** method with a small step size. They are exact to eight decimal places.

Table 3.1.3 : Numerical solution of $y' = -2y^2 + xy + x^2,\ y(0) = 1$ , by Euler's method.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 0.0 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 |
| 0.1 | 0.800000000 | 0.821375000 | 0.829977007 | 0.837584494 |
| 0.2 | 0.681000000 | 0.707795377 | 0.719226253 | 0.729641890 |
| 0.3 | 0.605867800 | 0.633776590 | 0.646115227 | 0.657580377 |
| 0.4 | 0.559628676 | 0.587454526 | 0.600045701 | 0.611901791 |
| 0.5 | 0.535376972 | 0.562906169 | 0.575556391 | 0.587575491 |
| 0.6 | 0.529820120 | 0.557143535 | 0.569824171 | 0.581942225 |
| 0.7 | 0.541467455 | 0.568716935 | 0.581435423 | 0.593629526 |
| 0.8 | 0.569732776 | 0.596951988 | 0.609684903 | 0.621907458 |
| 0.9 | 0.614392311 | 0.641457729 | 0.654110862 | 0.666250842 |
| 1.0 | 0.675192037 | 0.701764495 | 0.714151626 | 0.726015790 |

Since we think it is important in evaluating the accuracy of the numerical methods that we will be studying in this chapter, we often include a column listing values of the exact solution of the initial value problem, even if the directions in the example or exercise don't specifically call for it. If quotation marks are included in the heading, the values were obtained by applying the Runge-Kutta method in a way that's explained in Section 3.3. If quotation marks are not included, the values were obtained from a known formula for the solution. In either case, the values are exact to eight places to the right of the decimal point.

Table 3.1.4 : Errors in approximate solutions of $y' = -2y^2 + xy + x^2,\ y(0) = 1$ , obtained by Euler's method.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ |
|---|---|---|---|
| 0.1 | 0.0376 | 0.0162 | 0.0076 |
| 0.2 | 0.0486 | 0.0218 | 0.0104 |
| 0.3 | 0.0517 | 0.0238 | 0.0115 |
| 0.4 | 0.0523 | 0.0244 | 0.0119 |
| 0.5 | 0.0522 | 0.0247 | 0.0121 |
| 0.6 | 0.0521 | 0.0248 | 0.0121 |
| 0.7 | 0.0522 | 0.0249 | 0.0122 |
| 0.8 | 0.0522 | 0.0250 | 0.0122 |
| 0.9 | 0.0519 | 0.0248 | 0.0121 |
| 1.0 | 0.0508 | 0.0243 | 0.0119 |

## Truncation Error in Euler's Method

Consistent with the results indicated in Tables 3.1.1 - 3.1.4 , we will now show that under reasonable assumptions on $f$ there's a constant $K$ such that the error in approximating the solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0,$$

at a given point $b > x_0$ by Euler's method with step size $h = (b - x_0)/n$ satisfies the inequality

$$|y(b) - y_n| \le Kh,$$

where $K$ is a constant independent of $n$.

There are two sources of error (not counting roundoff) in Euler's method:

1. The error committed in approximating the integral curve by the tangent line Equation 3.1.2 over the interval $[x_i, x_{i+1}]$.
2. The error committed in replacing $y(x_i)$ by $y_i$ in Equation 3.1.2 and using Equation 3.1.4 rather than Equation 3.1.2 to compute $y_{i+1}$.

Euler's method assumes that $y_{i+1}$ defined in Equation 3.1.2 is an approximation to $y(x_{i+1})$. We call the error in this approximation the **local truncation error at the $i$th step**, and denote it by $T_i$; thus,

$$T_i = y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i)). \tag{3.1.8}$$

we will now use **Taylor's theorem** to estimate $T_i$, assuming for simplicity that $f$, $f_x$, and $f_y$ are continuous and bounded for all $(x, y)$. Then $y''$ exists and is bounded on $[x_0, b]$. To see this, we differentiate

$$y'(x) = f(x, y(x))$$

to obtain

$$\begin{aligned} y''(x) = & \quad f_x(x, y(x)) + f_y(x, y(x))y'(x) \\ = & \quad f_x(x, y(x)) + f_y(x, y(x))f(x, y(x)). \end{aligned}$$

Since we assumed that $f$, $f_x$ and $f_y$ are bounded, there's a constant $M$ such that

$$|f_x(x, y(x)) + f_y(x, y(x))y'(x)| \le M \quad x_0 < x < b$$

which implies that

$$|y''(x)| \le M, \quad x_0 < x < b \tag{3.1.9}$$

Since $x_{i+1} = x_i + h$ , Taylor's theorem implies that

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(\tilde{x}_i),$$

where $\tilde{x}_i$ is some number between $x_i$ and $x_{i+1}$. Since $y'(x_i) = f(x_i, y(x_i))$ this can be written as

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2}y''(\tilde{x}_i),$$

or, equivalently,

$$y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i)) = \frac{h^2}{2}y''(\tilde{x}_i).$$

Comparing this with Equation 3.1.8 shows that

$$T_i = \frac{h^2}{2}y''(\tilde{x}_i).$$

Recalling Equation 3.1.9, we can establish the bound

$$|T_i| \le \frac{Mh^2}{2}, \quad 1 \le i \le n. \tag{3.1.10}$$

Although it may be difficult to determine the constant $M$, what is important is that there's an $M$ such that Equation 3.1.10 holds. We say that the local truncation error of Euler's method is **of order** $h^2$, which we write as $O(h^2)$.

Note that the magnitude of the local truncation error in Euler's method is determined by the second derivative $y''$ of the solution of the initial value problem. Therefore the local truncation error will be larger where $|y''|$ is large, or smaller where $|y''|$ is small.

Since the local truncation error for Euler's method is $O(h^2)$, it is reasonable to expect that halving $h$ reduces the local truncation error by a factor of 4. This is true, but halving the step size also requires twice as many steps to approximate the solution at a given point. To analyze the overall effect of truncation error in Euler's method, it is useful to derive an equation relating the errors

$$e_{i+1} = y(x_{i+1}) - y_{i+1} \quad \text{and} \quad e_i = y(x_i) - y_i.$$

To this end, recall that

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + T_i \tag{3.1.11}$$

and

$$y_{i+1} = y_i + hf(x_i, y_i). \tag{3.1.12}$$

Subtracting Equation 3.1.12 from Equation 3.1.11 yields

$$e_{i+1} = e_i + h\left[f(x_i, y(x_i)) - f(x_i, y_i)\right] + T_i. \tag{3.1.13}$$

The last term on the right is the local truncation error at the $i$th step. The other terms reflect the way errors made at **previous steps** affect $e_{i+1}$. Since $|T_i| \le Mh^2/2$, we see from Equation 3.1.13 that

$$|e_{i+1}| \le |e_i| + h|f(x_i, y(x_i)) - f(x_i, y_i)| + \frac{Mh^2}{2}. \tag{3.1.14}$$

Since we assumed that $f_y$ is continuous and bounded, the mean value theorem implies that

$$f(x_i, y(x_i)) - f(x_i, y_i) = f_y(x_i, y_i^*)(y(x_i) - y_i) = f_y(x_i, y_i^*)e_i,$$

where $y_i^*$ is between $y_i$ and $y(x_i)$. Therefore

$$|f(x_i, y(x_i)) - f(x_i, y_i)| \le R|e_i|$$

for some constant $R$. From this and Equation 3.1.14,

$$|e_{i+1}| \le (1 + Rh)|e_i| + \frac{Mh^2}{2}, \quad 0 \le i \le n-1. \tag{3.1.15}$$

For convenience, let $C = 1 + Rh$. Since $e_0 = y(x_0) - y_0 = 0$, applying Equation 3.1.15 repeatedly yields

$$\begin{aligned}
|e_1| &\le \frac{Mh^2}{2} \\
|e_2| &\le C|e_1| + \frac{Mh^2}{2} \le (1 + C)\frac{Mh^2}{2} \\
|e_3| &\le C|e_2| + \frac{Mh^2}{2} \le (1 + C + C^2)\frac{Mh^2}{2} \\
&\vdots \\
|e_n| &\le C|e_{n-1}| + \frac{Mh^2}{2} \le (1 + C + \cdots + C^{n-1})\frac{Mh^2}{2}.
\end{aligned} \tag{3.1.16}$$

Recalling the formula for the sum of a geometric series, we see that

$$1 + C + \cdots + C^{n-1} = \frac{1 - C^n}{1 - C} = \frac{(1 + Rh)^n - 1}{Rh}$$

(since $C = 1 + Rh$). From this and Equation 3.1.16,

$$|y(b) - y_n| = |e_n| \le \frac{(1 + Rh)^n - 1}{R}\frac{Mh}{2}. \tag{3.1.17}$$

Since Taylor's theorem implies that

$$1 + Rh < e^{rh}$$

(verify),

$$(1+Rh)^n < e^{nRh} = e^{R(b-x_0)} \quad (\text{since } nh = b - x_0).$$

This and Equation 3.1.17 imply that

$$|y(b) - y_n| \le Kh, \tag{3.1.18}$$

with

$$K = M\frac{e^{R(b-x_0)} - 1}{2R}.$$

Because of Equation 3.1.18 we say that the **global truncation error of Euler's method is of order** $h$, which we write as $O(h)$.

## Semilinear Equations and Variation of Parameters

An equation that can be written in the form

$$y' + p(x)y = h(x, y) \tag{3.1.19}$$

with $p \not\equiv 0$ is said to be **semilinear**. (Of course, Equation 3.1.19 is linear if $h$ is independent of $y$.) One way to apply Euler's method to an initial value problem

$$y' + p(x)y = h(x, y), \quad y(x_0) = y_0 \tag{3.1.20}$$

for Equation 3.1.19 is to think of it as

$$y' = f(x, y), \quad y(x_0) = y_0,$$

where

$$f(x, y) = -p(x)y + h(x, y).$$

However, we can also start by applying variation of parameters to Equation 3.1.20, as in Sections 2.1 and 2.4; thus, we write the solution of Equation 3.1.20 as $y = uy_1$, where $y_1$ is a nontrivial solution of the complementary equation $y' + p(x)y = 0$. Then $y = uy_1$ is a solution of Equation 3.1.20 if and only if $u$ is a solution of the initial value problem

$$u' = h(x, uy_1(x))/y_1(x), \quad u(x_0) = y(x_0)/y_1(x_0). \tag{3.1.21}$$

We can apply Euler's method to obtain approximate values $u_0, u_1, \ldots, u_n$ of this initial value problem, and then take

$$y_i = u_i y_1(x_i)$$

as approximate values of the solution of Equation 3.1.20; we will call this procedure the **Euler semilinear method**.

The next two examples show that the Euler and Euler semilinear methods may yield drastically different results.

### Example 3.1.2

In Example 3.1.7 we had to leave the solution of the initial value problem

$$y' - 2xy = 1, \quad y(0) = 3 \tag{3.1.22}$$

in the form

$$y = e^{x^2}\left(3 + \int_0^x e^{-t^2}\,dt\right) \tag{3.1.23}$$

because it was impossible to evaluate this integral exactly in terms of elementary functions. Use step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$ to find approximate values of the solution of Equation 3.1.22 at $x = 0, 0.2, 0.4, 0.6, \ldots, 2.0$ by **(a)** Euler's method; **(b)** the Euler semilinear method.

**Solution a**

Rewriting Equation 3.1.22 as

$$y' = 1 + 2xy, \quad y(0) = 3 \tag{3.1.24}$$

and applying Euler's method with $f(x, y) = 1 + 2xy$ yields the results shown in Table 3.1.5 . Because of the large differences between the estimates obtained for the three values of $h$, it would be clear that these results are useless even if the "exact" values were not included in the table.

Table 3.1.5 : Numerical solution of $y' - 2xy = 1,\ y(0) = 3$, with Euler's method.

| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | Exact |
|---|---|---|---|---|
| 0.0 | 3.000000000 | 3.000000000 | 3.000000000 | 3.000000000 |
| 0.2 | 3.200000000 | 3.262000000 | 3.294348537 | 3.327851973 |
| 0.4 | 3.656000000 | 3.802028800 | 3.881421103 | 3.966059348 |
| 0.6 | 4.440960000 | 4.726810214 | 4.888870783 | 5.067039535 |
| 0.8 | 5.706790400 | 6.249191282 | 6.570796235 | 6.936700945 |
| 1.0 | 7.732963328 | 8.771893026 | 9.419105620 | 10.184923955 |
| 1.2 | 11.026148659 | 13.064051391 | 14.405772067 | 16.067111677 |
| 1.4 | 16.518700016 | 20.637273893 | 23.522935872 | 27.289392347 |
| 1.6 | 25.969172024 | 34.570423758 | 41.033441257 | 50.000377775 |
| 1.8 | 42.789442120 | 61.382165543 | 76.491018246 | 98.982969504 |
| 2.0 | 73.797840446 | 115.440048291 | 152.363866569 | 211.954462214 |

It's easy to see why Euler's method yields such poor results. Recall that the constant $M$ in Equation 3.1.10 – which plays an important role in determining the local truncation error in Euler's method – must be an upper bound for the values of the second derivative $y''$ of the solution of the initial value problem Equation 3.1.22 on $(0, 2)$. The problem is that $y''$ assumes very large values on this interval. To see this, we differentiate Equation 3.1.24 to obtain

$$y''(x) = 2y(x) + 2xy'(x) = 2y(x) + 2x(1 + 2xy(x)) = 2(1 + 2x^2)y(x) + 2x,$$

where the second equality follows again from Equation 3.1.24. Since Equation 3.1.23 implies that $y(x) > 3e^{x^2}$ if $x > 0$,

$$y''(x) > 6(1 + 2x^2)e^{x^2} + 2x, \quad x > 0.$$

For example, letting $x = 2$ shows that $y''(2) > 2952$.

### Solution b

Since $y_1 = e^{x^2}$ is a solution of the complementary equation $y' - 2xy = 0$, we can apply the Euler semilinear method to Equation 3.1.22, with

$$y = ue^{x^2} \quad \text{and} \quad u' = e^{-x^2}, \quad u(0) = 3.$$

The results listed in Table 3.1.6 are clearly better than those obtained by Euler's method.

Table 3.1.6 : Numerical solution of $y' - 2xy = 1,\ y(0) = 3$, by the Euler semilinear method.

| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | Exact |
|---|---|---|---|---|
| 0.0 | 3.000000000 | 3.000000000 | 3.000000000 | 3.000000000 |
| 0.2 | 3.330594477 | 3.329558853 | 3.328788889 | 3.327851973 |
| 0.4 | 3.980734157 | 3.974067628 | 3.970230415 | 3.966059348 |
| 0.6 | 5.106360231 | 5.087705244 | 5.077622723 | 5.067039535 |
| 0.8 | 7.021003417 | 6.980190891 | 6.958779586 | 6.936700945 |

| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | Exact |
|---|---|---|---|---|
| 1.0 | 10.350076600 | 10.269170824 | 10.227464299 | 10.184923955 |
| 1.2 | 16.381180092 | 16.226146390 | 16.147129067 | 16.067111677 |
| 1.4 | 27.890003380 | 27.592026085 | 27.441292235 | 27.289392347 |
| 1.6 | 51.183323262 | 50.594503863 | 50.298106659 | 50.000377775 |
| 1.8 | 101.424397595 | 100.206659076 | 99.595562766 | 98.982969504 |
| 2.0 | 217.301032800 | 214.631041938 | 213.293582978 | 211.954462214 |

We cannot give a general procedure for determining in advance whether Euler's method or the semilinear Euler method will produce better results for a given semilinear initial value problem Equation 3.1.19. As a rule of thumb, the Euler semilinear method will yield better results than Euler's method if $|u''|$ is small on $[x_0, b]$, while Euler's method yields better results if $|u''|$ is large on $[x_0, b]$. In many cases the results obtained by the two methods don't differ appreciably. However, we propose the an intuitive way to decide which is the better method: Try both methods with multiple step sizes, as we did in Example [example:3.1.4}, and accept the results obtained by the method for which the approximations change less as the step size decreases.

## Example 3.1.5

Applying Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to the initial value problem

$$y' - 2y = \frac{x}{1+y^2}, \quad y(1) = 7 \tag{3.1.25}$$

on $[1, 2]$ yields the results in Table 3.1.7 .

Table 3.1.7 : Numerical solution of $y' - 2y = x/(1+y^2),\ y(1) = 7$, by Euler's method.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 1.0 | 7.000000000 | 7.000000000 | 7.000000000 | 7.000000000 |
| 1.1 | 8.402000000 | 8.471970569 | 8.510493955 | 8.551744786 |
| 1.2 | 10.083936450 | 10.252570169 | 10.346014101 | 10.446546230 |
| 1.3 | 12.101892354 | 12.406719381 | 12.576720827 | 12.760480158 |
| 1.4 | 14.523152445 | 15.012952416 | 15.287872104 | 15.586440425 |
| 1.5 | 17.428443554 | 18.166277405 | 18.583079406 | 19.037865752 |
| 1.6 | 20.914624471 | 21.981638487 | 22.588266217 | 23.253292359 |
| 1.7 | 25.097914310 | 26.598105180 | 27.456479695 | 28.401914416 |
| 1.8 | 30.117766627 | 32.183941340 | 33.373738944 | 34.690375086 |
| 1.9 | 36.141518172 | 38.942738252 | 40.566143158 | 42.371060528 |
| 2.0 | 43.369967155 | 47.120835251 | 49.308511126 | 51.752229656 |

Applying the Euler semilinear method with

$$y = ue^{2x} \quad \text{and} \quad u' = \frac{xe^{-2x}}{1+u^2e^{4x}}, \quad u(1) = 7e^{-2}$$

yields the results in Table 3.1.8 . Since the latter are clearly less dependent on step size than the former, we conclude that the Euler semilinear method is better than Euler's method for Equation 3.1.25. This conclusion is supported by comparing the approximate results obtained by the two methods with the "exact" values of the solution.

Table 3.1.8 : Numerical solution of $y' - 2y = x/(1+y^2),\ y(1) = 7$, by the Euler semilinear method.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 1.0 | 7.000000000 | 7.000000000 | 7.000000000 | 7.000000000 |
| 1.1 | 8.552262113 | 8.551993978 | 8.551867007 | 8.551744786 |
| 1.2 | 10.447568674 | 10.447038547 | 10.446787646 | 10.446546230 |
| 1.3 | 12.762019799 | 12.761221313 | 12.760843543 | 12.760480158 |
| 1.4 | 15.588535141 | 15.587448600 | 15.586934680 | 15.586440425 |
| 1.5 | 19.040580614 | 19.039172241 | 19.038506211 | 19.037865752 |
| 1.6 | 23.256721636 | 23.254942517 | 23.254101253 | 23.253292359 |
| 1.7 | 28.406184597 | 28.403969107 | 28.402921581 | 28.401914416 |
| 1.8 | 34.695649222 | 34.692912768 | 34.691618979 | 34.690375086 |
| 1.9 | 42.377544138 | 42.374180090 | 42.372589624 | 42.371060528 |
| 2.0 | 51.760178446 | 51.756054133 | 51.754104262 | 51.752229656 |

## Example 3.1.6

Applying Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to the initial value problem

$$y' + 3x^2 y = 1 + y^2, \quad y(2) = 2 \tag{3.1.26}$$

on $[2, 3]$ yields the results in Table 3.1.9 . Applying the Euler semilinear method with

$$y = ue^{-x^3} \quad \text{and} \quad u' = e^{x^3}(1 + u^2 e^{-2x^3}), \quad u(2) = 2e^8$$

yields the results in Table 3.1.10 . Noting the close agreement among the three columns of Table 3.1.9 (at least for larger values of $x$) and the lack of any such agreement among the columns of Table 3.1.10 , we conclude that Euler's method is better than the Euler semilinear method for Equation 3.1.26. Comparing the results with the exact values supports this conclusion.

Table 3.1.9 Numerical solution of $y' + 3x^2 y = 1 + y^2, \quad y(2) = 2$ , by Euler's method.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 2.0 | 2.000000000 | 2.000000000 | 2.000000000 | 2.000000000 |
| 2.1 | 0.100000000 | 0.493231250 | 0.609611171 | 0.701162906 |
| 2.2 | 0.068700000 | 0.122879586 | 0.180113445 | 0.236986800 |
| 2.3 | 0.069419569 | 0.070670890 | 0.083934459 | 0.103815729 |
| 2.4 | 0.059732621 | 0.061338956 | 0.063337561 | 0.068390786 |
| 2.5 | 0.056871451 | 0.056002363 | 0.056249670 | 0.057281091 |
| 2.6 | 0.050560917 | 0.051465256 | 0.051517501 | 0.051711676 |
| 2.7 | 0.048279018 | 0.047484716 | 0.047514202 | 0.047564141 |
| 2.8 | 0.042925892 | 0.043967002 | 0.043989239 | 0.044014438 |
| 2.9 | 0.042148458 | 0.040839683 | 0.040857109 | 0.040875333 |
| 3.0 | 0.035985548 | 0.038044692 | 0.038058536 | 0.038072838: |

Table 3.1.10 : Numerical solution of $y' + 3x^2 y = 1 + y^2, \quad y(2) = 2$ , by the Euler semilinear method.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | $h = .0125$ |
|---|---|---|---|---|
| 2.0 | 2.000000000 | 2.000000000 | 2.000000000 | 2.000000000 |
| 2.1 | 0.708426286 | 0.702568171 | 0.701214274 | 0.701162906 |
| 2.2 | 0.214501852 | 0.222599468 | 0.228942240 | 0.236986800 |
| 2.3 | 0.069861436 | 0.083620494 | 0.092852806 | 0.103815729 |
| 2.4 | 0.032487396 | 0.047079261 | 0.056825805 | 0.068390786 |
| 2.5 | 0.021895559 | 0.036030018 | 0.045683801 | 0.057281091 |
| 2.6 | 0.017332058 | 0.030750181 | 0.040189920 | 0.051711676 |
| 2.7 | 0.014271492 | 0.026931911 | 0.036134674 | 0.047564141 |
| 2.8 | 0.011819555 | 0.023720670 | 0.032679767 | 0.044014438 |
| 2.9 | 0.009776792 | 0.020925522 | 0.029636506 | 0.040875333 |
| 3.0 | 0.008065020 | 0.018472302 | 0.026931099 | 0.038072838 |

In the next two sections we will study other numerical methods for solving initial value problems, called the **improved Euler method**, the **midpoint method**, **Heun's method** and the **Runge- Kutta method**. If the initial value problem is semilinear as in Equation 3.1.19, we also have the option of using variation of parameters and then applying the given numerical method to the initial value problem Equation 3.1.21 for $u$. By analogy with the terminology used here, we will call the resulting procedure **the improved Euler semilinear method**, the **midpoint semilinear method**, **Heun's semilinear method** or **the Runge- Kutta semilinear method**, as the case may be.

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# 3.1E: Euler's Method (Exercises)

## Q3.1.1

You may want to save the results of these exercises, since we will revisit in the next two sections. In **Exercises 3.1.1-3.1.5** use Euler's method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$ , where $x_0$ is the point where the initial condition is imposed and $i = 1, 2, 3$. The purpose of these exercises is to familiarize you with the computational procedure of Euler's method.

1. $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$

2. $y' = y + \sqrt{x^2 + y^2}, \quad y(0) = 1; \quad h = 0.1$

3. $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$

4. $y' = \frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$

5. $y' + x^2 y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$

## Q3.1.2

6. Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$ , which can be obtained by the method of Section 2.1. Present your results in a table like **Table 3.1.1**.

7. Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9 \ln x + x^3 + 2),$$

which can be obtained by the method of Section 2.1. Present your results in a table like **Table 3.1.1**.

8. Use Euler's method with step sizes $h = 0.05$, $h = 0.025$, and $h = 0.0125$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2$$

at $x = 1.0, 1.05, 1.10, 1.15, \ldots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}$$

obtained in Example [example:2.4.3}. Present your results in a table like **Table 3.1.1**.

9. In Example [example:2.2.3} it was shown that

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \tag{A}$$

Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 2.0, 2.1, 2.2, 2.3, \ldots, 3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = y^5 + y - x^2 - x + 4$$

for each value of $(x, y)$ appearing in the first table.

10. You can see from **Example 2.5.1** that

$$x^4y^3 + x^2y^5 + 2xy = 4$$

is an implicit solution of the initial value problem

$$y' = -\frac{4x^3y^3 + 2xy^5 + 2y}{3x^4y^2 + 5x^2y^4 + 2x}, \quad y(1) = 1. \tag{A}$$

Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = x^4y^3 + x^2y^5 + 2xy - 4$$

for each value of $(x, y)$ appearing in the first table.

11. Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$(3y^2 + 4y)y' + 2x + \cos x = 0, \quad y(0) = 1; \quad \text{(Exercise 2.2.13)}$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$.

12. Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0 \quad \text{(Exercise 2.2.14)}$$

at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$.

13. Use Euler's method and the Euler semilinear method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{-3x}, \quad y(0) = 6$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{-3x}(7x + 6)$, which can be obtained by the method of Section 2.1. Do you notice anything special about the results? Explain.

## Q3.1.3

The linear initial value problems in **Exercises 3.1.14–3.1.19** can't be solved exactly in terms of known elementary functions. In each exercise, use Euler's method and the Euler semilinear methods with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14. $y' - 2y = \frac{1}{1+x^2}, \quad y(2) = 2$ ; $h = 0.1, 0.05, 0.025$on $[2, 3]$

15. $y' + 2xy = x^2, \quad y(0) = 3 \quad$ (Exercise 2.1.38); $\quad h = 0.2, 0.1, 0.05$on $[0, 2]$

16. $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2; \quad$ (Exercise 2.1.39); $\quad h = 0.2, 0.1, 0.05$on $[1, 3]$

17. $y' + y = \frac{e^{-x}\tan x}{x}, \quad y(1) = 0; \quad$ (Exercise 2.1.40); $\quad h = 0.05, 0.025, 0.0125$on $[1, 1.5]$

18. $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1; \quad$ (Exercise 2.1.41); $\quad h = 0.2, 0.1, 0.05$on $[0, 2]$

19. $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2; \quad$ (Exercise 2.1.42); $\quad h = 0.05, 0.025, 0.0125$on $[1, 1.5]$

## Q3.1.4

In **Exercises 3.1.20-3.1.22**, use Euler's method and the Euler semilinear method with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

20. $y'+3y=xy^2(y+1), \quad y(0)=1$ ; $h=0.1, 0.05, 0.025$on $[0,1]$
21. $y'-4y=\frac{x}{y^2(y+1)}, \quad y(0)=1$ ; $h=0.1, 0.05, 0.025$on $[0,1]$
22. $y'+2y=\frac{x^2}{1+y^2}, \quad y(2)=1$ ; $h=0.1, 0.05, 0.025$on $[2,3]$

## Q3.1.5

23. Numerical Quadrature. The fundamental theorem of calculus says that if $f$ is continuous on a closed interval $[a,b]$ then it has an antiderivative $F$ such that $F'(x)=f(x)$ on $[a,b]$ and

$$\int_a^b f(x)\,dx = F(b)-F(a). \tag{A}$$

This solves the problem of evaluating a definite integral if the integrand $f$ has an antiderivative that can be found and evaluated easily. However, if $f$ doesn't have this property, (A) doesn't provide a useful way to evaluate the definite integral. In this case we must resort to approximate methods. There's a class of such methods called **numerical quadrature**, where the approximation takes the form

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} c_i f(x_i), \tag{B}$$

where $a=x_0<x_1<\cdots<x_n=b$ are suitably chosen points and $c_0$, $c_1$, …, $c_n$ are suitably chosen constants. We call (B) a **quadrature formula**.

a. Derive the quadrature formula

$$\int_a^b f(x)\,dx \approx h\sum_{i=0}^{n-1} f(a+ih) \tag{C}$$

where $h=(b-a)/n$ by applying Euler's method to the initial value problem

$$y'=f(x), \quad y(a)=0.$$

b. The quadrature formula (C) is sometimes called **the left rectangle rule**. Draw a figure that justifies this terminology.
c. For several choices of $a$, $b$, and $A$, apply (C) to $f(x)=A$ with $n=10, 20, 40, 80, 160, 320$Compare your results with the exact answers and explain what you find.
d. For several choices of $a$, $b$, $A$, and $B$, apply (C) to $f(x)=A+Bx$ with $n=10$, 20, 40, 80, 160, 320. Compare your results with the exact answers and explain what you find.

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# 3.2: The Improved Euler Method and Related Methods

In Section 3.1, we saw that the global truncation error of Euler's method is $O(h)$, which would seem to imply that we can achieve arbitrarily accurate results with Euler's method by simply choosing the step size sufficiently small. However, this is not a good idea, for two reasons. First, after a certain point decreasing the step size will increase roundoff errors to the point where the accuracy will deteriorate rather than improve. The second and more important reason is that in most applications of numerical methods to an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{3.2.1}$$

the expensive part of the computation is the evaluation of $f$. Therefore we want methods that give good results for a given number of such evaluations. This is what motivates us to look for numerical methods better than Euler's.

To clarify this point, suppose we want to approximate the value of $e$ by applying Euler's method to the initial value problem

$$y' = y, \quad y(0) = 1$$

(with solution $y = e^x$) on $[0, 1]$, with $h = 1/12$, $1/24$, and $1/48$, respectively. Since each step in Euler's method requires one evaluation of $f$, the number of evaluations of $f$ in each of these attempts is $n = 12$, 24, and 48, respectively. In each case we accept $y_n$ as an approximation to $e$. The second column of Table 3.2.1 shows the results. The first column of the table indicates the number of evaluations of $f$ required to obtain the approximation, and the last column contains the value of $e$ rounded to ten significant figures.

In this section we will study the **improved Euler method**, which requires two evaluations of $f$ at each step. We've used this method with $h = 1/6$, $1/12$, and $1/24$. The required number of evaluations of $f$ were 12, 24, and 48, as in the three applications of Euler's method; however, you can see from the third column of Table 3.2.1 that the approximation to $e$ obtained by the improved Euler method with only 12 evaluations of $f$ is better than the approximation obtained by Euler's method with 48 evaluations.

In Section 3.3, we will study the **Runge- Kutta method**, which requires four evaluations of $f$ at each step. We've used this method with $h = 1/3$, $1/6$, and $1/12$. The required number of evaluations of $f$ were again 12, 24, and 48, as in the three applications of Euler's method and the improved Euler method; however, you can see from the fourth column of Table 3.2.1 that the approximation to $e$ obtained by the Runge-Kutta method with only 12 evaluations of $f$ is better than the approximation obtained by the improved Euler method with 48 evaluations.

Table 3.2.1 : Approximations to $e$ obtained by three numerical methods.

| $n$ | Euler | Improved Euler | Runge-Kutta | Exact |
|---|---|---|---|---|
| 12 | 2.613035290 | 2.715327371 | 2.718280809 | 2.718281828 |
| 24 | 2.663731258 | 2.717519565 | 2.718281763 | 2.718281828 |
| 48 | 2.690496599 | 2.718088247 | 2.718281824 | 2.718281828 |

## The Improved Euler Method

The **improved Euler method** for solving the initial value problem Equation 3.2.1 is based on approximating the integral curve of Equation 3.2.1 at $(x_i, y(x_i))$ by the line through $(x_i, y(x_i))$ with slope

$$m_i = \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{2};$$

that is, $m_i$ is the average of the slopes of the tangents to the integral curve at the endpoints of $[x_i, x_{i+1}]$. The equation of the approximating line is therefore

$$y = y(x_i) + \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{2}(x - x_i). \tag{3.2.2}$$

Setting $x = x_{i+1} = x_i + h$ in Equation 3.2.2 yields

$$y_{i+1} = y(x_i) + \frac{h}{2}\left(f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))\right) \tag{3.2.3}$$

as an approximation to $y(x_{i+1})$. As in our derivation of Euler's method, we replace $y(x_i)$ (unknown if $i > 0$) by its approximate value $y_i$; then Equation 3.2.3 becomes

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y(x_{i+1}))\right).$$

However, this still will not work, because we do not know $y(x_{i+1})$, which appears on the right. We overcome this by replacing $y(x_{i+1})$ by $y_i + hf(x_i, y_i)$, the value that the Euler method would assign to $y_{i+1}$. Thus, the improved Euler method starts with the known value $y(x_0) = y_0$ and computes $y_1$, $y_2$, ..., $y_n$ successively with the formula

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))\right). \tag{3.2.4}$$

The computation indicated here can be conveniently organized as follows: given $y_i$, compute

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + h, y_i + hk_{1i}\right), \\ y_{i+1} &= y_i + \frac{h}{2}(k_{1i} + k_{2i}). \end{aligned}$$

The improved Euler method requires two evaluations of $f(x, y)$ per step, while Euler's method requires only one. However, we will see at the end of this section that if $f$ satisfies appropriate assumptions, the local truncation error with the improved Euler method is $O(h^3)$, rather than $O(h^2)$ as with Euler's method. Therefore the global truncation error with the improved Euler method is $O(h^2)$; however, we will not prove this.

We note that the magnitude of the local truncation error in the improved Euler method and other methods discussed in this section is determined by the third derivative $y'''$ of the solution of the initial value problem. Therefore the local truncation error will be larger where $|y'''|$ is large, or smaller where $|y'''|$ is small.

The next example, which deals with the initial value problem considered in Example 3.2.1 , illustrates the computational procedure indicated in the improved Euler method.

### Example 3.2.1

Use the improved Euler method with $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1 \tag{3.2.5}$$

at $x = 0.1, 0.2, 0.3$

**Solution**

As in Example 3.1.1, we rewrite Equation 3.2.5 as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form Equation 3.2.1, with

$$f(x, y) = -2y + x^3 e^{-2x}, x_0 = 0, \text{and } y_0 = 1.$$

The improved Euler method yields

$$
\begin{aligned}
k_{10} &= f(x_0, y_0) = f(0,1) = -2,\\
k_{20} &= f(x_1, y_0 + hk_{10}) = f(0.1, 1 + (0.1)(-2))\\
&= f(0.1, 0.8) = -2(0.8) + (0.1)^3 e^{-0.2} = -1.599181269,\\
y_1 &= y_0 + \frac{h}{2}(k_{10} + k_{20}),\\
&= 1 + (0.05)(-2 - 1.599181269) = 0.820040937,\\
k_{11} &= f(x_1, y_1) = f(0.1, 0.820040937) = -2(0.820040937) + (0.1)^3 e^{-0.2} = -1.639263142,\\
k_{21} &= f(x_2, y_1 + hk_{11}) = f(0.2, 0.820040937 + 0.1(-1.639263142)),\\
&= f(0.2, 0.656114622) = -2(0.656114622) + (.2)^3 e^{-0.4} = -1.306866684,\\
y_2 &= y_1 + \frac{h}{2}(k_{11} + k_{21}),\\
&= .820040937 + (.05)(-1.639263142 - 1.306866684) = 0.672734445,\\
k_{12} &= f(x_2, y_2) = f(.2, .672734445) = -2(.672734445) + (.2)^3 e^{-.4} = -1.340106330,\\
k_{22} &= f(x_3, y_2 + hk_{12}) = f(.3, .672734445 + .1(-1.340106330)),\\
&= f(.3, .538723812) = -2(.538723812) + (.3)^3 e^{-.6} = -1.062629710,\\
y_3 &= y_2 + \frac{h}{2}(k_{12} + k_{22})\\
&= .672734445 + (.05)(-1.340106330 - 1.062629710) = 0.552597643.
\end{aligned}
$$

## Example 3.2.2

Table 3.2.2 shows results of using the improved Euler method with step sizes $h = 0.1$ and $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0$, 0.1, 0.2, 0.3, …, 1.0. For comparison, it also shows the corresponding approximate values obtained with Euler's method in [example:3.1.2}, and the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4).$$

The results obtained by the improved Euler method with $h = 0.1$ are better than those obtained by Euler's method with $h = 0.05$.

| | **Euler** | | **Improved Euler** | | **Exact** |
|---|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.1$ | $h = 0.05$ | **Exact** |
| 0.0 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 |
| 0.1 | 0.800000000 | 0.810005655 | 0.820040937 | 0.819050572 | 0.818751221 |
| 0.2 | 0.640081873 | 0.656266437 | 0.672734445 | 0.671086455 | 0.670588174 |
| 0.3 | 0.512601754 | 0.532290981 | 0.552597643 | 0.550543878 | 0.549922980 |
| 0.4 | 0.411563195 | 0.432887056 | 0.455160637 | 0.452890616 | 0.452204669 |
| 0.5 | 0.332126261 | 0.353785015 | 0.376681251 | 0.374335747 | 0.373627557 |
| 0.6 | 0.270299502 | 0.291404256 | 0.313970920 | 0.311652239 | 0.310952904 |
| 0.7 | 0.222745397 | 0.242707257 | 0.264287611 | 0.262067624 | 0.261398947 |

| | Euler | | Improved Euler | | Exact |
|---|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.1$ | $h = 0.05$ | Exact |
| 0.8 | 0.186654593 | 0.205105754 | 0.225267702 | 0.223194281 | 0.222570721 |
| 0.9 | 0.159660776 | 0.176396883 | 0.194879501 | 0.192981757 | 0.192412038 |
| 1.0 | 0.139778910 | 0.154715925 | 0.171388070 | 0.169680673 | 0.169169104 |

Table 3.2.2 : Numerical solution of $y' + 2y = x^3 e^{-2x},\ y(0) = 1$ , by Euler's method and the improved Euler method.

## Example 3.2.3

Table 3.2.3 shows analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2,\ y(0) = 1.$$

We applied Euler's method to this problem in Example 3.2.3 .

Table 3.2.3 : Numerical solution of $y' = -2y^2 + xy + x^2,\ y(0) = 1$ , by Euler's method and the improved Euler method.

| | Euler | | Improved Euler | | Exact |
|---|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.1$ | $h = 0.05$ | Exact |
| 0.0 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 |
| 0.1 | 0.800000000 | 0.821375000 | 0.840500000 | 0.838288371 | 0.837584494 |
| 0.2 | 0.681000000 | 0.707795377 | 0.733430846 | 0.730556677 | 0.729641890 |
| 0.3 | 0.605867800 | 0.633776590 | 0.661600806 | 0.658552190 | 0.657580377 |
| 0.4 | 0.559628676 | 0.587454526 | 0.615961841 | 0.612884493 | 0.611901791 |
| 0.5 | 0.535376972 | 0.562906169 | 0.591634742 | 0.588558952 | 0.587575491 |
| 0.6 | 0.529820120 | 0.557143535 | 0.586006935 | 0.582927224 | 0.581942225 |
| 0.7 | 0.541467455 | 0.568716935 | 0.597712120 | 0.594618012 | 0.593629526 |
| 0.8 | 0.569732776 | 0.596951988 | 0.626008824 | 0.622898279 | 0.621907458 |
| 0.9 | 0.614392311 | 0.641457729 | 0.670351225 | 0.667237617 | 0.666250842 |
| 1.0 | 0.675192037 | 0.701764495 | 0.730069610 | 0.726985837 | 0.726015790 |

## Example 3.2.4

Use step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$ to find approximate values of the solution of

$$y' - 2xy = 1, \quad y(0) = 3 \tag{3.2.6}$$

at $x = 0, 0.2, 0.4, 0.6, \ldots, 2.0$ by:

a. the improved Euler method;
b. the improved Euler semilinear method.

We used Euler's method and the Euler semilinear method on this problem in Example 3.1.4.

### Solution a

Rewriting Equation 3.2.6 as

$$y' = 1 + 2xy, \quad y(0) = 3$$

and applying the improved Euler method with $f(x, y) = 1 + 2xy$ yields the results shown in Table 3.2.4 .

### Solution b

Since $y_1 = e^{x^2}$ is a solution of the complementary equation $y' - 2xy = 0$, we can apply the improved Euler semilinear method to Equation 3.2.6, with

$$y = ue^{x^2} \quad \text{and} \quad u' = e^{-x^2}, \quad u(0) = 3.$$

The results listed in Table 3.2.5 are clearly better than those obtained by the improved Euler method.

Table 3.2.4 : Numerical solution of $y' - 2xy = 1,\ y(0) = 3$, by the improved Euler method.

| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | Exact |
|---|---|---|---|---|
| 0.0 | 3.000000000 | 3.000000000 | 3.000000000 | 3.000000000 |
| 0.2 | 3.328000000 | 3.328182400 | 3.327973600 | 3.327851973 |
| 0.4 | 3.964659200 | 3.966340117 | 3.966216690 | 3.966059348 |
| 0.6 | 5.057712497 | 5.065700515 | 5.066848381 | 5.067039535 |
| 0.8 | 6.900088156 | 6.928648973 | 6.934862367 | 6.936700945 |
| 1.0 | 10.065725534 | 10.154872547 | 10.177430736 | 10.184923955 |
| 1.2 | 15.708954420 | 15.970033261 | 16.041904862 | 16.067111677 |
| 1.4 | 26.244894192 | 26.991620960 | 27.210001715 | 27.289392347 |
| 1.6 | 46.958915746 | 49.096125524 | 49.754131060 | 50.000377775 |
| 1.8 | 89.982312641 | 96.200506218 | 98.210577385 | 98.982969504 |
| 2.0 | 184.563776288 | 203.151922739 | 209.464744495 | 211.954462214 |

Table 3.2.5 : Numerical solution of $y' - 2xy = 1,\ y(0) = 3$, by the improved Euler semilinear method.

| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | Exact |
|---|---|---|---|---|
| 0.0 | 3.000000000 | 3.000000000 | 3.000000000 | 3.000000000 |
| 0.2 | 3.326513400 | 3.327518315 | 3.327768620 | 3.327851973 |
| 0.4 | 3.963383070 | 3.965392084 | 3.965892644 | 3.966059348 |
| 0.6 | 5.063027290 | 5.066038774 | 5.066789487 | 5.067039535 |
| 0.8 | 6.931355329 | 6.935366847 | 6.936367564 | 6.936700945 |
| 1.0 | 10.178248417 | 10.183256733 | 10.184507253 | 10.184923955 |
| 1.2 | 16.059110511 | 16.065111599 | 16.066611672 | 16.067111677 |
| 1.4 | 27.280070674 | 27.287059732 | 27.288809058 | 27.289392347 |
| 1.6 | 49.989741531 | 49.997712997 | 49.999711226 | 50.000377775 |
| 1.8 | 98.971025420 | 98.979972988 | 98.982219722 | 98.982969504 |
| 2.0 | 211.941217796 | 211.951134436 | 211.953629228 | 211.954462214 |

## A Family of Methods with O(h³) Local Truncation Error

We will now derive a class of methods with $O(h^3)$ local truncation error for solving Equation 3.2.1. For simplicity, we assume that $f$, $f_x$, $f_y$, $f_{xx}$, $f_{yy}$, and $f_{xy}$ are continuous and bounded for all $(x, y)$. This implies that if $y$ is the solution of Equation 3.2.1 then $y''$ and $y'''$ are bounded (**Exercise 3.2.31**).

We begin by approximating the integral curve of Equation 3.2.1 at $(x_i, y(x_i))$ by the line through $(x_i, y(x_i))$ with slope

$$m_i = \sigma y'(x_i) + \rho y'(x_i + \theta h),$$

where $\sigma$, $\rho$, and $\theta$ are constants that we will soon specify; however, we insist at the outset that $0 < \theta \leq 1$ , so that

$$x_i < x_i + \theta h \leq x_{i+1}.$$

The equation of the approximating line is

$$\begin{aligned} y &= y(x_i) + m_i(x - x_i) \\ &= y(x_i) + [\sigma y'(x_i) + \rho y'(x_i + \theta h)]\,(x - x_i). \end{aligned} \tag{3.2.7}$$

Setting $x = x_{i+1} = x_i + h$ in Equation 3.2.7 yields

$$\hat{y}_{i+1} = y(x_i) + h\,[\sigma y'(x_i) + \rho y'(x_i + \theta h)]$$

as an approximation to $y(x_{i+1})$.

To determine $\sigma$, $\rho$, and $\theta$ so that the error

$$\begin{aligned} E_i &= y(x_{i+1}) - \hat{y}_{i+1} \\ &= y(x_{i+1}) - y(x_i) - h\,[\sigma y'(x_i) + \rho y'(x_i + \theta h)] \end{aligned} \tag{3.2.8}$$

in this approximation is $O(h^3)$, we begin by recalling from Taylor's theorem that

$$y(x_{i+1}) = y(x_i) + h y'(x_i) + \frac{h^2}{2} y''(x_i) + \frac{h^3}{6} y'''(\hat{x}_i),$$

where $\hat{x}_i$ is in $(x_i, x_{i+1})$. Since $y'''$ is bounded this implies that

$$y(x_{i+1}) - y(x_i) - h y'(x_i) - \frac{h^2}{2} y''(x_i) = O(h^3).$$

Comparing this with Equation 3.2.8 shows that $E_i = O(h^3)$ if

$$\sigma y'(x_i) + \rho y'(x_i + \theta h) = y'(x_i) + \frac{h}{2} y''(x_i) + O(h^2). \tag{3.2.9}$$

However, applying Taylor's theorem to $y'$ shows that

$$y'(x_i + \theta h) = y'(x_i) + \theta h y''(x_i) + \frac{(\theta h)^2}{2} y'''(\bar{x}_i),$$

where $\bar{x}_i$ is in $(x_i, x_i + \theta h)$ . Since $y'''$ is bounded, this implies that

$$y'(x_i + \theta h) = y'(x_i) + \theta h y''(x_i) + O(h^2).$$

Substituting this into Equation 3.2.9 and noting that the sum of two $O(h^2)$ terms is again $O(h^2)$ shows that $E_i = O(h^3)$ if

$$(\sigma + \rho) y'(x_i) + \rho\theta h y''(x_i) = y'(x_i) + \frac{h}{2} y''(x_i),$$

which is true if

$$\sigma + \rho = 1 \quad \text{and} \quad \rho\theta = \frac{1}{2}. \tag{3.2.10}$$

Since $y' = f(x, y)$, we can now conclude from Equation 3.2.8 that

$$y(x_{i+1}) = y(x_i) + h\,[\sigma f(x_i, y_i) + \rho f(x_i + \theta h, y(x_i + \theta h))] + O(h^3) \tag{3.2.11}$$

if $\sigma$, $\rho$, and $\theta$ satisfy Equation 3.2.10. However, this formula would not be useful even if we knew $y(x_i)$ exactly (as we would for $i=0$), since we still wouldn't know $y(x_i+\theta h)$ exactly. To overcome this difficulty, we again use Taylor's theorem to write

$$y(x_i+\theta h)=y(x_i)+\theta h y'(x_i)+\frac{h^2}{2}y''(\tilde{x}_i),$$

where $\tilde{x}_i$ is in $(x_i, x_i+\theta h)$. Since $y'(x_i)=f(x_i,y(x_i))$ and $y''$ is bounded, this implies that

$$|y(x_i+\theta h)-y(x_i)-\theta h f(x_i,y(x_i))|\le Kh^2 \tag{3.2.12}$$

for some constant $K$. Since $f_y$ is bounded, the mean value theorem implies that

$$|f(x_i+\theta h,u)-f(x_i+\theta h,v)|\le M|u-v|$$

for some constant $M$. Letting

$$u=y(x_i+\theta h) \quad \text{and} \quad v=y(x_i)+\theta h f(x_i,y(x_i))$$

and recalling Equation 3.2.12 shows that

$$f(x_i+\theta h,y(x_i+\theta h))=f(x_i+\theta h,y(x_i)+\theta h f(x_i,y(x_i)))+O(h^2).$$

Substituting this into Equation 3.2.11 yields

$$\begin{aligned} y(x_{i+1}) &= y(x_i)+h\left[\sigma f(x_i,y(x_i))+\right.\\ &\qquad \left.\rho f(x_i+\theta h,y(x_i)+\theta h f(x_i,y(x_i)))\right]+O(h^3). \end{aligned}$$

This implies that the formula

$$y_{i+1}=y_i+h\left[\sigma f(x_i,y_i)+\rho f(x_i+\theta h,y_i+\theta h f(x_i,y_i))\right]$$

has $O(h^3)$ local truncation error if $\sigma$, $\rho$, and $\theta$ satisfy Equation 3.2.10. Substituting $\sigma=1-\rho$ and $\theta=1/2\rho$ here yields

$$y_{i+1}=y_i+h\left[(1-\rho)f(x_i,y_i)+\rho f\left(x_i+\frac{h}{2\rho},y_i+\frac{h}{2\rho}f(x_i,y_i)\right)\right]. \tag{3.2.13}$$

The computation indicated here can be conveniently organized as follows: given $y_i$, compute

$$\begin{aligned} k_{1i} &= f(x_i,y_i),\\ k_{2i} &= f\left(x_i+\frac{h}{2\rho},y_i+\frac{h}{2\rho}k_{1i}\right),\\ y_{i+1} &= y_i+h[(1-\rho)k_{1i}+\rho k_{2i}]. \end{aligned}$$

Consistent with our requirement that $0<\theta<1$, we require that $\rho\ge 1/2$. Letting $\rho=1/2$ in Equation 3.2.13 yields the improved Euler method Equation 3.2.4. Letting $\rho=3/4$ yields **Heun's method**,

$$y_{i+1}=y_i+h\left[\frac{1}{4}f(x_i,y_i)+\frac{3}{4}f\left(x_i+\frac{2}{3}h,y_i+\frac{2}{3}hf(x_i,y_i)\right)\right],$$

which can be organized as

$$\begin{aligned} k_{1i} &= f(x_i,y_i),\\ k_{2i} &= f\left(x_i+\frac{2h}{3},y_i+\frac{2h}{3}k_{1i}\right),\\ y_{i+1} &= y_i+\frac{h}{4}(k_{1i}+3k_{2i}). \end{aligned}$$

Letting $\rho=1$ yields the **midpoint method**,

$$y_{i+1}=y_i+hf\left(x_i+\frac{h}{2},y_i+\frac{h}{2}f(x_i,y_i)\right),$$

which can be organized as

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{1i}\right), \\ y_{i+1} &= y_i + hk_{2i}. \end{aligned}$$

Examples involving the midpoint method and Heun's method are given in **Exercises 3.2.23 - 3.3.30**.

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# 3.2E: The Improved Euler Method and Related Methods (Exercises)

Most of the following numerical exercises involve initial value problems considered in the exercises in Section 3.1. You'll find it instructive to compare the results that you obtain here with the corresponding results that you obtained in Section 3.1.

## Q3.2.1

In **Exercises 3.2.1–3.2.5** use the improved Euler method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$ , where $x_0$ is the point where the initial condition is imposed and $i = 1, 2, 3$.

1. $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$

2. $y' = y + \sqrt{x^2 + y^2}, \quad y(0) = 1; \quad h = 0.1$

3. $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$

4. $y' = \frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$

5. $y' + x^2 y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$

## Q3.2.2

6. Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$ , which can be obtained by the method of Section 2.1. Present your results in a table like **Table 3.2.2**.

7. Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9 \ln x + x^3 + 2)$$

which can be obtained by the method of Section 2.1. Present your results in a table like **Table 3.2.2**.

8. Use the improved Euler method with step sizes $h = 0.05$, $h = 0.025$, and $h = 0.0125$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2,$$

at $x = 1.0, 1.05, 1.10, 1.15, \ldots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}$$

obtained in Example [example:2.4.3}. Present your results in a table like **Table 3.2.2**.

9. In Example [example:3.2.2} it was shown that

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \tag{A}$$

Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 2.0, 2.1, 2.2, 2.3, \ldots, 3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = y^5 + y - x^2 - x + 4$$

for each value of $(x, y)$ appearing in the first table.

10. You can see from **Example 2.5.1** that

$$x^4y^3 + x^2y^5 + 2xy = 4$$

is an implicit solution of the initial value problem

$$y' = -\frac{4x^3y^3 + 2xy^5 + 2y}{3x^4y^2 + 5x^2y^4 + 2x}, \quad y(1) = 1. \tag{A}$$

Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 1.0, 1.14, 1.2, 1.3, \ldots, 2.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = x^4y^3 + x^2y^5 + 2xy - 4$$

for each value of $(x, y)$ appearing in the first table.

11. Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$(3y^2 + 4y)y' + 2x + \cos x = 0, \quad y(0) = 1 \quad \text{(Exercise 2.2.13)}$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$.

12. Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0 \quad \text{(Exercise 2.2.14)}$$

at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$.

13. Use the improved Euler method and the improved Euler semilinear method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = e^{-3x}(1 - 2x), \quad y(0) = 2,$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{-3x}(2 + x - x^2)$, which can be obtained by the method of Section 2.1. Do you notice anything special about the results? Explain.

## Q3.2.3

The linear initial value problems in **Exercises 3.2.14-3.2.19** can't be solved exactly in terms of known elementary functions. In each exercise use the improved Euler and improved Euler semilinear methods with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14. $y' - 2y = \frac{1}{1+x^2}, \quad y(2) = 2$ ; $h = 0.1, 0.05, 0.025$on $[2, 3]$

15. $y' + 2xy = x^2, \quad y(0) = 3$ ; $h = 0.2, 0.1, 0.05$on $[0, 2]$ **(Exercise 2.1.38)**

16. $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2$ , $h = 0.2, 0.1, 0.05$on $[1, 3]$ **(Exercise 2.1.39)**

17. $y' + y = \frac{e^{-x}\tan x}{x}, \quad y(1) = 0$ ; $h = 0.05, 0.025, 0.0125$on $[1, 1.5]$**(Exercise 2.1.40)**,

18. $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1$ ; $h = 0.2, 0.1, 0.05$on $[0, 2]$ **(Exercise 2.1.41)**

19. $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2$ ; $h = 0.05, 0.025, 0.0125$on $[1, 1.5]$**(Exercise 2.1.42)**

## Q3.2.4

In **Exercises 3.2.20-3.2.22** use the improved Euler method and the improved Euler semilinear method with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

20. $y'+3y=xy^2(y+1), \quad y(0)=1\ ; h=0.1, 0.05, 0.025 \text{on } [0,1]$
21. $y'-4y=\frac{x}{y^2(y+1)}, \quad y(0)=1\ ; h=0.1, 0.05, 0.025 \text{on } [0,1]$
22. $y'+2y=\frac{x^2}{1+y^2}, \quad y(2)=1\ ; h=0.1, 0.05, 0.025 \text{on } [2,3]$

## Q3.2.5

23. Do Exercise 3.2E.7 with “improved Euler method” replaced by “midpoint method.”
24. Do Exercise 3.2E.7 with “improved Euler method” replaced by “Heun’s method.”
25. Do Exercise 3.2E.8 with “improved Euler method” replaced by “midpoint method.”
26. Do Exercise 3.2E.8 with “improved Euler method” replaced by “Heun’s method.”
27. Do Exercise 3.2E.11 with “improved Euler method” replaced by “midpoint method.”
28. Do Exercise 3.2E.11 with “improved Euler method” replaced by “Heun’s method.”
29. Do Exercise 3.2E.12 with “improved Euler method” replaced by “midpoint method.”
30. Do Exercise 3.2E.12 with “improved Euler method” replaced by “Heun’s method.”
31. Show that if $f$, $f_x$, $f_y$, $f_{xx}$, $f_{yy}$, and $f_{xy}$ are continuous and bounded for all $(x,y)$ and $y$ is the solution of the initial value problem

$$y'=f(x,y), \quad y(x_0)=y_0,$$

then $y''$ and $y'''$ are bounded.

32. Numerical Quadrature (see **Exercise 3.1.23**).

a. Derive the quadrature formula

$$\int_a^b f(x)\,dx \approx 0.5h(f(a)+f(b))+h\sum_{i=1}^{n-1} f(a+ih) \tag{A}$$

(where $h=(b-a)/n$) by applying the improved Euler method to the initial value problem

$$y'=f(x), \quad y(a)=0.$$

b. The quadrature formula (A) is called **the trapezoid rule**. Draw a figure that justifies this terminology.
c. For several choices of $a$, $b$, $A$, and $B$, apply (A) to $f(x)=A+Bx$ , with $n=10, 20, 40, 80, 160, 320$Compare your results with the exact answers and explain what you find.
d. For several choices of $a$, $b$, $A$, $B$, and $C$, apply (A) to $f(x)=A+Bx+Cx^2$ , with $n=10$, 20, 40, 80, 160, 320. Compare your results with the exact answers and explain what you find.

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# 3.3: The Runge-Kutta Method

In general, if $k$ is any positive integer and $f$ satisfies appropriate assumptions, there are numerical methods with local truncation error $O(h^{k+1})$ for solving an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{3.3.1}$$

Moreover, it can be shown that a method with local truncation error $O(h^{k+1})$ has global truncation error $O(h^k)$. In Sections 3.1 and 3.2 we studied numerical methods where $k = 1$ and $k = 2$. We'll skip methods for which $k = 3$ and proceed to the **Runge-Kutta** method, the most widely used method, for which $k = 4$. The magnitude of the local truncation error is determined by the fifth derivative $y^{(5)}$ of the solution of the initial value problem. Therefore the local truncation error will be larger where $|y^{(5)}|$ is large, or smaller where $|y^{(5)}|$ is small. The Runge-Kutta method computes approximate values $y_1$, $y_2$, ..., $y_n$ of the solution of Equation 3.3.1 at $x_0$, $x_0 + h$, ..., $x_0 + nh$ as follows: Given $y_i$, compute

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{1i}\right), \\ k_{3i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{2i}\right), \\ k_{4i} &= f(x_i + h, y_i + hk_{3i}), \end{aligned}$$

and

$$y_{i+1} = y_i + \frac{h}{6}(k_{1i} + 2k_{2i} + 2k_{3i} + k_{4i}).$$

The next example, which deals with the initial value problem considered in Examples and Example 3.3.1 , illustrates the computational procedure indicated in the Runge-Kutta method.

## Example 3.3.1

Use the Runge-Kutta method with $h = 0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1, \tag{3.3.2}$$

at $x = 0.1, 0.2$.

**Solution**

Again we rewrite Equation 3.3.2 as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form Equation 3.3.1, with

$$f(x, y) = -2y + x^3 e^{-2x}, \ x_0 = 0, \text{ and } y_0 = 1.$$

The Runge-Kutta method yields

$$
\begin{aligned}
k_{10} &= f(x_0, y_0) = f(0, 1) = -2,\\
k_{20} &= f(x_0 + h/2, y_0 + hk_{10}/2) = f(.05, 1 + (.05)(-2))\\
&= f(.05, .9) = -2(.9) + (.05)^3 e^{-.1} = -1.799886895,\\
k_{30} &= f(x_0 + h/2, y_0 + hk_{20}/2) = f(.05, 1 + (.05)(-1.799886895))\\
&= f(.05, .910005655) = -2(.910005655) + (.05)^3 e^{-.1} = -1.819898206,\\
k_{40} &= f(x_0 + h, y_0 + hk_{30}) = f(.1, 1 + (.1)(-1.819898206))\\
&= f(.1, .818010179) = -2(.818010179) + (.1)^3 e^{-.2} = -1.635201628,\\
y_1 &= y_0 + \frac{h}{6}(k_{10} + 2k_{20} + 2k_{30} + k_{40}),\\
&= 1 + \frac{.1}{6}(-2 + 2(-1.799886895) + 2(-1.819898206) - 1.635201628) = .818753803,\\
k_{11} &= f(x_1, y_1) = f(.1, .818753803) = -2(.818753803)) + (.1)^3 e^{-.2} = -1.636688875,\\
k_{21} &= f(x_1 + h/2, y_1 + hk_{11}/2) = f(.15, .818753803 + (.05)(-1.636688875))\\
&= f(.15, .736919359) = -2(.736919359) + (.15)^3 e^{-.3} = -1.471338457,\\
k_{31} &= f(x_1 + h/2, y_1 + hk_{21}/2) = f(.15, .818753803 + (.05)(-1.471338457))\\
&= f(.15, .745186880) = -2(.745186880) + (.15)^3 e^{-.3} = -1.487873498,\\
k_{41} &= f(x_1 + h, y_1 + hk_{31}) = f(.2, .818753803 + (.1)(-1.487873498))\\
&= f(.2, .669966453) = -2(.669966453) + (.2)^3 e^{-.4} = -1.334570346,\\
y_2 &= y_1 + \frac{h}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41}),\\
&= .818753803 + \frac{.1}{6}(-1.636688875 + 2(-1.471338457) + 2(-1.487873498) - 1.334570346)\\
&= .670592417.
\end{aligned}
$$

The Runge-Kutta method is sufficiently accurate for most applications.

## Example 3.3.2

Table 3.3.1 shows results of using the Runge-Kutta method with step sizes $h = 0.1$ and $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0$, 0.1, 0.2, 0.3, …, 1.0. For comparison, it also shows the corresponding approximate values obtained with the improved Euler method in Example 3.2.2, and the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4).$$

The results obtained by the Runge-Kutta method are clearly better than those obtained by the improved Euler method in fact; the results obtained by the Runge-Kutta method with $h = 0.1$ are better than those obtained by the improved Euler method with $h = 0.05$.

Table 3.3.1 : Numerical solution of $y' + 2y = x^3 e^{-2x}$, $y(0) = 1$, by the Runge-Kuttta method and the improved Euler method.

| | Improved Euler | | Runge-Kutta | | |
|---|---|---|---|---|---|
| **x** | **h=0.1** | **h=0.05** | **h=0.1** | **h-0.05** | **"exact"** |
| 0.0 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 |

| | Improved Euler | | Runge-Kutta | | |
|---|---|---|---|---|---|
| x | h=0.1 | h=0.05 | h=0.1 | h-0.05 | "exact" |
| 0.1 | 0.820040937 | 0.819050572 | 0.818753803 | 0.818751370 | 0.818751221 |
| 0.2 | 0.672734445 | 0.671086455 | 0.670592417 | 0.670588418 | 0.670588174 |
| 0.3 | 0.552597643 | 0.550543878 | 0.549928221 | 0.549923281 | 0.549922980 |
| 0.4 | 0.455160637 | 0.452890616 | 0.452210430 | 0.452205001 | 0.452204669 |
| 0.5 | 0.376681251 | 0.374335747 | 0.373633492 | 0.373627899 | 0.373627557 |
| 0.6 | 0.313970920 | 0.311652239 | 0.310958768 | 0.310953242 | 0.310952904 |
| 0.7 | 0.264287611 | 0.262067624 | 0.261404568 | 0.261399270 | 0.261398947 |
| 0.8 | 0.225267702 | 0.223194281 | 0.222575989 | 0.222571024 | 0.222570721 |
| 0.9 | 0.194879501 | 0.192981757 | 0.192416882 | 0.192412317 | 0.192412038 |
| 1.0 | 0.171388070 | 0.169680673 | 0.169173489 | 0.169169356 | 0.169169104 |

## Example 3.3.3

Table 3.3.2 shows analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2,\ y(0) = 1.$$

We applied the improved Euler method to this problem in Example 3.2.3.

Table 3.3.2 : Numerical solution of $y' = -2y^2 + xy + x^2,\ y(0) = 1$ , by the Runge-Kuttta method and the improved Euler method.

| | Improved Euler | | Runge-Kutta | | |
|---|---|---|---|---|---|
| x | h=0.1 | h=0.05 | h=0.1 | h-0.05 | "exact" |
| 0.0 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 |
| 0.1 | 0.840500000 | 0.838288371 | 0.837587192 | 0.837584759 | 0.837584494 |
| 0.2 | 0.733430846 | 0.730556677 | 0.729644487 | 0.729642155 | 0.729641890 |
| 0.3 | 0.661600806 | 0.658552190 | 0.657582449 | 0.657580598 | 0.657580377 |
| 0.4 | 0.615961841 | 0.612884493 | 0.611903380 | 0.611901969 | 0.611901791 |
| 0.5 | 0.591634742 | 0.588558952 | 0.587576716 | 0.587575635 | 0.587575491 |
| 0.6 | 0.586006935 | 0.582927224 | 0.581943210 | 0.581942342 | 0.581942225 |
| 0.7 | 0.597712120 | 0.594618012 | 0.593630403 | 0.593629627 | 0.593629526 |
| 0.8 | 0.626008824 | 0.622898279 | 0.621908378 | 0.621907553 | 0.621907458 |
| 0.9 | 0.670351225 | 0.667237617 | 0.666251988 | 0.666250942 | 0.666250842 |
| 1.0 | 0.730069610 | 0.726985837 | 0.726017378 | 0.726015908 | 0.726015790 |

## Example 3.3.4

Tables 3.3.3 and 3.3.4 show results obtained by applying the Runge-Kutta and Runge-Kutta semilinear methods to to the initial value problem

$$y' - 2xy = 1,\ y(0) = 3,$$

which we considered in Examples 3.3.3 and 3.3.4

| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
|---|---|---|---|---|
| 0.0 | 3.000000000 | 3.000000000 | 3.000000000 | 3.000000000 |
| 0.2 | 3.327846400 | 3.327851633 | 3.327851952 | 3.327851973 |
| 0.4 | 3.966044973 | 3.966058535 | 3.966059300 | 3.966059348 |
| 0.6 | 5.066996754 | 5.067037123 | 5.067039396 | 5.067039535 |
| 0.8 | 6.936534178 | 6.936690679 | 6.936700320 | 6.936700945 |
| 1.0 | 10.184232252 | 10.184877733 | 10.184920997 | 10.184923955 |
| 1.2 | 16.064344805 | 16.066915583 | 16.067098699 | 16.067111677 |
| 1.4 | 27.278771833 | 27.288605217 | 27.289338955 | 27.289392347 |
| 1.6 | 49.960553660 | 49.997313966 | 50.000165744 | 50.000377775 |
| 1.8 | 98.834337815 | 98.971146146 | 98.982136702 | 98.982969504 |
| 2.0 | 211.393800152 | 211.908445283 | 211.951167637 | 211.954462214 |

Table 3.3.4 : Numerical solution of $y' - 2xy = 1,\ y(0) = 3$, by the Runge-Kutta semilinear method.

| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
|---|---|---|---|---|
| 0.0 | 3.000000000 | 3.000000000 | 3.000000000 | 3.000000000 |
| 0.2 | 3.327853286 | 3.327852055 | 3.327851978 | 3.327851973 |
| 0.4 | 3.966061755 | 3.966059497 | 3.966059357 | 3.966059348 |
| 0.6 | 5.067042602 | 5.067039725 | 5.067039547 | 5.067039535 |
| 0.8 | 6.936704019 | 6.936701137 | 6.936700957 | 6.936700945 |
| 1.0 | 10.184926171 | 10.184924093 | 10.184923963 | 10.184923955 |
| 1.2 | 16.067111961 | 16.067111696 | 16.067111678 | 16.067111677 |
| 1.4 | 27.289389418 | 27.289392167 | 27.289392335 | 27.289392347 |
| 1.6 | 50.000370152 | 50.000377302 | 50.000377745 | 50.000377775 |
| 1.8 | 98.982955511 | 98.982968633 | 98.982969450 | 98.982969504 |
| 2.0 | 211.954439983 | 211.954460825 | 211.954462127 | 211.954462214 |

## The Case Where $x_0$ Is not The Left Endpoint

So far in this chapter we've considered numerical methods for solving an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{3.3.3}$$

on an interval $[x_0, b]$, for which $x_0$ is the left endpoint. We haven't discussed numerical methods for solving Equation 3.3.3 on an interval $[a, x_0]$, for which $x_0$ is the right endpoint. To be specific, how can we obtain approximate values $y_{-1}, y_{-2}, \ldots, y_{-n}$ of the solution of Equation 3.3.3 at $x_0 - h, \ldots, x_0 - nh$, where $h = (x_0 - a)/n$ ? Here's the answer to this question:

Question

Consider the initial value problem

$$z' = -f(-x, z), \quad z(-x_0) = y_0, \tag{3.3.4}$$

on the interval $[-x_0, -a]$, for which $-x_0$ is the left endpoint. Use a numerical method to obtain approximate values $z_1$, $z_2$, ..., $z_n$ of the solution of (3.3.4) at $-x_0 + h$, $-x_0 + 2h$, ..., $-x_0 + nh = -a$. Then $y_{-1} = z_1$, $y_{-2} = z_2$, ..., $y_{-n} = z_n$ are approximate values of the solution of (3.3.3) at $x_0 - h$, $x_0 - 2h$, ..., $x_0 - nh = a$.

The justification for this answer is sketched in **Exercise 3.3.23**. Note how easy it is to make the change the given problem Equation 3.3.3 to the modified problem Equation 3.3.4 : first replace $f$ by $-f$ and then replace $x$, $x_0$, and $y$ by $-x$, $-x_0$, and $z$, respectively.

## Example 3.3.5

Use the Runge-Kutta method with step size $h = 0.1$ to find approximate values of the solution of

$$(y-1)^2 y' = 2x + 3, \quad y(1) = 4 \tag{3.3.5}$$

at $x = 0, 0.1, 0.2, \ldots, 1$.

### Solution

We first rewrite Equation 3.3.5 in the form Equation 3.3.3 as

$$y' = \frac{2x+3}{(y-1)^2}, \quad y(1) = 4. \tag{3.3.6}$$

Since the initial condition $y(1) = 4$ is imposed at the right endpoint of the interval $[0, 1]$, we apply the Runge-Kutta method to the initial value problem

$$z' = \frac{2x-3}{(z-1)^2}, \quad z(-1) = 4 \tag{3.3.7}$$

on the interval $[-1, 0]$. (You should verify that Equation 3.3.7 is related to Equation 3.3.6 as Equation 3.3.4 is related to Equation 3.3.3.) Table [table:3.3.5} shows the results. Reversing the order of the rows in Table [table:3.3.5} and changing the signs of the values of $x$ yields the first two columns of Table [table:3.3.6}. The last column of Table [table:3.3.6} shows the exact values of $y$, which are given by

$$y = 1 + (3x^2 + 9x + 15)^{1/3}.$$

(Since the differential equation in Equation 3.3.6 is separable, this formula can be obtained by the method of Section 2.2.)

Table 3.3.5 : Numerical solution of $z' = \frac{2x-3}{(z-1)^2}$, $z(-1) = 4$, on $[-1, 0]$.

| $x$ | $z$ |
|---|---|
| -1.0 | 4.000000000 |
| -0.9 | 3.944536474 |
| -0.8 | 3.889298649 |
| -0.7 | 3.834355648 |
| -0.6 | 3.779786399 |
| -0.5 | 3.725680888 |
| -0.4 | 3.672141529 |
| -0.3 | 3.619284615 |
| -0.2 | 3.567241862 |
| -0.1 | 3.516161955 |

| $x$ | $z$ |
|---|---|
| 0.0 | 3.466212070 |

Table 3.3.6 : Numerical solution of $(y-1)^2y' = 2x+3,\ y(1) = 4$ , on $[0,1]$ .

| $x$ | $y$ | Exact |
|---|---|---|
| 0.00 | 3.466212070 | 3.466212074 |
| 0.10 | 3.516161955 | 3.516161958 |
| 0.20 | 3.567241862 | 3.567241864 |
| 0.30 | 3.619284615 | 3.619284617 |
| 0.40 | 3.672141529 | 3.672141530 |
| 0.50 | 3.725680888 | 3.725680889 |
| 0.60 | 3.779786399 | 3.779786399 |
| 0.70 | 3.834355648 | 3.834355648 |
| 0.80 | 3.889298649 | 3.889298649 |
| 0.90 | 3.944536474 | 3.944536474 |
| 1.00 | 4.000000000 | 4.000000000 |

We leave it to you to develop a procedure for handling the numerical solution of Equation 3.3.3 on an interval $[a,b]$ such that $a < x_0 < b$ (**Exercises 3.3.26** and **3.3.27**).

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# 3.3E: The Runge-Kutta Method (Exercises)

Most of the following numerical exercises involve initial value problems considered in the exercises in Sections 3.2. You'll find it instructive to compare the results that you obtain here with the corresponding results that you obtained in those sections.

## Q3.3.1

In **Exercises 3.3.1 -3.3.5** use the Runge-Kutta method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih,$ where $x_0$ is the point where the initial condition is imposed and $i = 1, 2.$

1. $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$

2. $y' = y + \sqrt{x^2 + y^2}, \quad y(0) = 1; \quad h = 0.1$

3. $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$

4. $y' = \frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$

5. $y' + x^2 y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$

## Q3.3.2

6. Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2,$$

at $x = 0$, 0.1, 0.2, 0.3, …, 1.0. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$ , which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.3.1.

7. Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0$, 1.1, 1.2, 1.3, …, 2.0. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9 \ln x + x^3 + 2),$$

which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.3.1.

8. Use the Runge-Kutta method with step sizes $h = 0.05$, $h = 0.025$, and $h = 0.0125$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2$$

at $x = 1.0$, 1.05, 1.10, 1.15 …, 1.5. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3},$$

which was obtained in Example

Example 3.3E.1 :

Add text here. For the automatic number to work, you need to add the "AutoNum" template (preferably at 2.2.3}. Present your results in a table like Table 3.3.1.

9. In Example

Example 3.3E.1 :

Add text here. For the automatic number to work, you need to add the "AutoNum" template (preferably at 2.2.3} it was shown that

$$y^5+y=x^2+x-4$$

is an implicit solution of the initial value problem

$$y'=\frac{2x+1}{5y^4+1},\quad y(2)=1. \tag{A}$$

Use the Runge-Kutta method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of (A) at $x=2.0, 2.1, 2.2, 2.3, \ldots, 3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x,y)=y^5+y-x^2-x+4$$

for each value of $(x,y)$ appearing in the first table.

10. You can see from Example

Example 3.3E.1 :

Add text here. For the automatic number to work, you need to add the "AutoNum" template (preferably at 2.5.1} that

$$x^4y^3+x^2y^5+2xy=4$$

is an implicit solution of the initial value problem

$$y'=-\frac{4x^3y^3+2xy^5+2y}{3x^4y^2+5x^2y^4+2x},\quad y(1)=1. \tag{A}$$

Use the Runge-Kutta method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of (A) at $x=1.0, 1.1, 1.2, 1.3, \ldots, 2.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x,y)=x^4y^3+x^2y^5+2xy-4$$

for each value of $(x,y)$ appearing in the first table.

11. Use the Runge-Kutta method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of the initial value problem

$$(3y^2+4y)y'+2x+\cos x=0,\quad y(0)=1 \quad \text{(Exercise 2.2.13)}$$

at $x=0, 0.1, 0.2, 0.3, \ldots, 1.0$.

12. Use the Runge-Kutta method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of the initial value problem

$$y'+\frac{(y+1)(y-1)(y-2)}{x+1}=0,\quad y(1)=0 \quad \text{(Exercise 2.2.14)}$$

at $x=1.0, 1.1, 1.2, 1.3, \ldots, 2.0$.

13. Use the Runge-Kutta method and the Runge-Kutta semilinear method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of the initial value problem

$$y'+3y=e^{-3x}(1-4x+3x^2-4x^3),\quad y(0)=-3$$

at $x=0$, $0.1$, $0.2$, $0.3$, …, $1.0$. Compare these approximate values with the values of the exact solution $y=-e^{-3x}(3-x+2x^2-x^3+x^4)$ , which can be obtained by the method of Section 2.1. Do you notice anything special about the results? Explain.

## Q3.3.3

The linear initial value problems in **Exercises 3.3.14–3.3.19** can't be solved exactly in terms of known elementary functions. In each exercise use the Runge-Kutta and the Runge-Kutta semilinear methods with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14. $y'-2y=\frac{1}{1+x^2},\quad y(2)=2\ ;\ \ h=0.1, 0.05, 0.025 \text{on } [2,3]$

15. $y'+2xy=x^2,\quad y(0)=3\,;\,h=0.2,0.1,0.05$on $[0,2]$ **(Exercise 2.1.38)**

16. $y'+\frac{1}{x}y=\frac{\sin x}{x^2},\quad y(1)=2;\ h=0.2,0.1,0.05$on $[1,3]$ **(Exercise 2.1.39)**

17. $y'+y=\frac{e^{-x}\tan x}{x},\quad y(1)=0;\ h=0.05,0.025,0.0125$on $[1,1.5]$**(Exercise 2.1.40)**

18. $y'+\frac{2x}{1+x^2}y=\frac{e^x}{(1+x^2)^2},\quad y(0)=1;\ h=0.2,0.1,0.05$on $[0,2]$ **(Exercise 2.1.41)**

19. $xy'+(x+1)y=e^{x^2},\quad y(1)=2\,;\,h=0.05,0.025,0.0125$on $[1,1.5]$**(Exercise 2.1.42)**

## Q3.3.4

In **Exercises 3.3.20–3.3.22** use the Runge-Kutta method and the Runge-Kutta semilinear method with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

20. $y'+3y=xy^2(y+1),\quad y(0)=1\,;\ h=0.1,0.05,0.025$on $[0,1]$

21. $y'-4y=\frac{x}{y^2(y+1)},\quad y(0)=1\,;\ h=0.1,0.05,0.025$on $[0,1]$

22. $y'+2y=\frac{x^2}{1+y^2},\quad y(2)=1\,;\ h=0.1,0.05,0.025$on $[2,3]$

## Q3.3.5

23. Suppose $a<x_0$, so that $-x_0<-a$. Use the chain rule to show that if $z$ is a solution of

$$z'=-f(-x,z),\quad z(-x_0)=y_0,$$

on $[-x_0,-a]$, then $y=z(-x)$ is a solution of

$$y'=f(x,y),\quad y(x_0)=y_0,$$

on $[a,x_0]$.

24. Use the Runge-Kutta method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of

$$y'=\frac{y^2+xy-x^2}{x^2},\quad y(2)=-1$$

at $x=1.1, 1.2, 1.3, \ldots 2.0$. Compare these approximate values with the values of the exact solution

$$y=\frac{x(4-3x^2)}{4+3x^2},$$

which can be obtained by referring to Example

Example 3.3E.1 :

Add text here. For the automatic number to work, you need to add the “AutoNum” template (preferably at 2.4.3}.

25. Use the Runge-Kutta method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of

$$y'=-x^2y-xy^2,\quad y(1)=1$$

at $x=0$, 0.1, 0.2, …, 1.

26. Use the Runge-Kutta method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of

$$y'+\frac{1}{x}y=\frac{7}{x^2}+3,\quad y(1)=\frac{3}{2}$$

at $x=0.5, 0.6,\ldots, 1.5$. Compare these approximate values with the values of the exact solution

$$y=\frac{7\ln x}{x}+\frac{3x}{2},$$

which can be obtained by the method discussed in Section 2.1.

27. Use the Runge-Kutta method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of

$$xy' + 2y = 8x^2, \quad y(2) = 5$$

at $x = 1.0, 1.1, 1.2, \ldots, 3.0$. Compare these approximate values with the values of the exact solution

$$y = 2x^2 - \frac{12}{x^2},$$

which can be obtained by the method discussed in Section 2.1.

28. Numerical Quadrature (see **Exercise 3.1.23**).

a. Derive the quadrature formula

$$\int_a^b f(x)\,dx \approx \frac{h}{6}(f(a) + f(b)) + \frac{h}{3}\sum_{i=1}^{n-1} f(a+ih) + \frac{2h}{3}\sum_{i=1}^{n} f\left(a + (2i-1)h/2\right) \tag{A}$$

(where $h = (b-a)/n$) by applying the Runge-Kutta method to the initial value problem

$$y' = f(x), \quad y(a) = 0.$$

This quadrature formula is called **Simpson's Rule**.

b. For several choices of $a$, $b$, $A$, $B$, $C$, and $D$ apply (A) to $f(x) = A + Bx + Cx + Dx^3$ , with $n = 10, 20, 40, 80, 160, 320$. Compare your results with the exact answers and explain what you find.

c. For several choices of $a$, $b$, $A$, $B$, $C$, $D$, and $E$ apply (A) to $f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$ , with $n = 10, 20, 40, 80, 160, 320$Compare your results with the exact answers and explain what you find.

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# CHAPTER OVERVIEW

## 4: Applications of First Order Equations

In this chapter, we consider applications of first order differential equations.


4.1: Growth and Decay

4.1E: Growth and Decay (Exercises)

4.2: Cooling and Mixing

4.2E: Cooling and Mixing (Exercises)

4.3: Elementary Mechanics

4.3E: Elementary Mechanics (Exercises)

4.4: Autonomous Second Order Equations

4.4E: Autonomous Second Order Equations (Exercises)

4.5: Applications to Curves

4.5E: Applications to Curves (Exercises)


*Thumbnail: False color time-lapse video of E. coli colony growing on microscope slide. This growth can be model with first order logistic equation. Added approximate scale bar based on the approximate length of 2.0 μm of E. coli bacteria. (CC BY-SA 4.0 International; Stewart EJ, Madden R, Paul G, Taddei F).*

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# 4.1: Growth and Decay

This section begins with a discussion of exponential growth and decay, which you have probably already seen in calculus. We consider applications to radioactive decay, carbon dating, and compound interest. We also consider more complicated problems where the rate of change of a quantity is in part proportional to the magnitude of the quantity, but is also influenced by other other factors for example, a radioactive substance is manufactured at a certain rate, but decays at a rate proportional to its mass, or a saver makes regular deposits in a savings account that draws compound interest.

Since the applications in this section deal with functions of time, we'll denote the independent variable by $t$. If $Q$ is a function of $t$, $Q'$ will denote the derivative of $Q$ with respect to $t$; thus,

$$Q' = \frac{dQ}{dt}.$$

## Exponential Growth and Decay

One of the most common mathematical models for a physical process is the **exponential model**, where it is assumed that the rate of change of a quantity $Q$ is proportional to $Q$; thus

$$Q' = aQ, \tag{4.1.1}$$

where $a$ is the constant of proportionality.

From Example 2.1.3, the general solution of Equation 4.1.1 is

$$Q = ce^{at}$$

and the solution of the initial value problem

$$Q' = aQ, \quad Q(t_0) = Q_0$$

is

$$Q = Q_0 e^{a(t-t_0)}. \tag{4.1.2}$$

Since the solutions of $Q' = aQ$ are exponential functions, we say that a quantity $Q$ that satisfies this equation **grows exponentially** if $a > 0$, or **decays exponentially** if $a < 0$ (Figure 4.1.1 ).

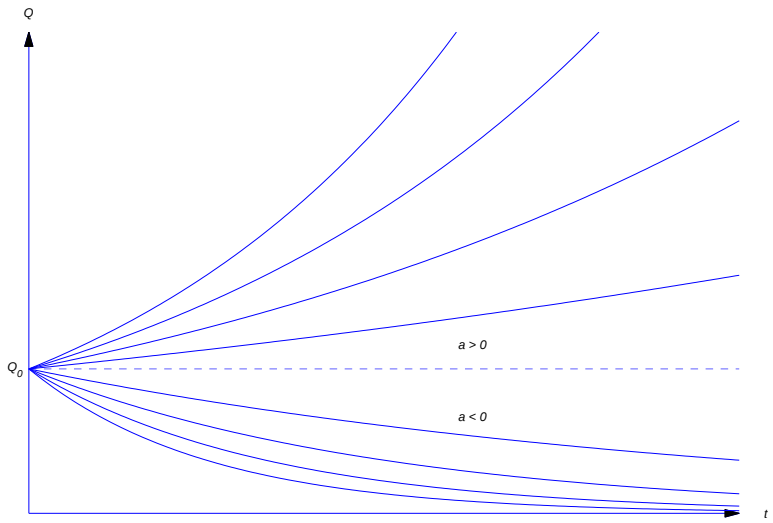



Figure 4.1.1 : Exponential growth and decay

## Radioactive Decay

Experimental evidence shows that radioactive material decays at a rate proportional to the mass of the material present. According to this model the mass $Q(t)$ of a radioactive material present at time $t$ satisfies Equation 4.1.1, where $a$ is a negative constant

whose value for any given material must be determined by experimental observation. For simplicity, we'll replace the negative constant $a$ by $-k$, where $k$ is a positive number that we'll call the **decay constant** of the material. Thus, Equation 4.1.1 becomes

$$Q' = -kQ.$$

If the mass of the material present at $t = t_0$ is $Q_0$, the mass present at time $t$ is the solution of

$$Q' = -kQ, \quad Q(t_0) = Q_0.$$

From Equation 4.1.2 with $a = -k$, the solution of this initial value problem is

$$Q = Q_0 e^{-k(t-t_0)}. \tag{4.1.3}$$

The **half–life** $\tau$ of a radioactive material is defined to be the time required for half of its mass to decay; that is, if $Q(t_0) = Q_0$, then

$$Q(\tau + t_0) = \frac{Q_0}{2}. \tag{4.1.4}$$

From Equation 4.1.3 with $t = \tau + t_0$, Equation 4.1.4 is equivalent to

$$Q_0 e^{-k\tau} = \frac{Q_0}{2},$$

so

$$e^{-k\tau} = \frac{1}{2}.$$

Taking logarithms yields

$$-k\tau = \ln \frac{1}{2} = -\ln 2,$$

so the half-life is

$$\tau = \frac{1}{k} \ln 2. \tag{4.1.5}$$

(Figure 4.1.2 ). The half-life is independent of $t_0$ and $Q_0$, since it is determined by the properties of material, not by the amount of the material present at any particular time.

### Example 4.1.1

A radioactive substance has a half-life of 1620 years.

a. If its mass is now 4 g (grams), how much will be left 810 years from now?
b. Find the time $t_1$ when 1.5 g of the substance remain.

**Solution a**

From Equation 4.1.3 with $t_0 = 0$ and $Q_0 = 4$,

$$Q = 4e^{-kt}, \tag{4.1.6}$$

where we determine $k$ from Equation 4.1.5, with $\tau$= 1620 years:

$$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{1620}.$$

Substituting this in Equation 4.1.6 yields

$$Q = 4e^{-(t \ln 2)/1620}. \tag{4.1.7}$$

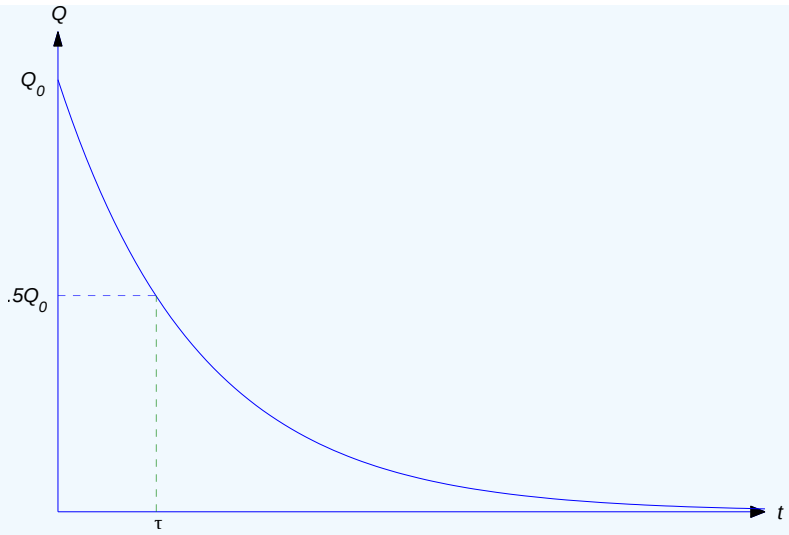


Figure 4.1.2 : Half-life of a radioactive substance

Therefore the mass left after 810 years will be

$$\begin{aligned} Q(810) &= 4e^{-(810\ln 2)/1620} = 4e^{-(\ln 2)/2} \\ &= 2\sqrt{2}\,\text{g}. \end{aligned}$$

**Solution b**

Setting $t = t_1$ in Equation 4.1.7 and requiring that $Q(t_1) = 1.5$ yields

$$\frac{3}{2} = 4e^{(-t_1 \ln 2)/1620}.$$

Dividing by 4 and taking logarithms yields

$$\ln\frac{3}{8} = -\frac{t_1 \ln 2}{1620}.$$

Since $\ln 3/8 = -\ln 8/3$,

$$t_1 = 1620\frac{\ln 8/3}{\ln 2} \approx 2292.4 \text{ years}.$$

## Interest Compounded Continuously

Suppose we deposit an amount of money $Q_0$ in an interest-bearing account and make no further deposits or withdrawals for $t$ years, during which the account bears interest at a constant annual rate $r$. To calculate the value of the account at the end of $t$ years, we need one more piece of information: how the interest is added to the account, or—as the bankers say—how it is **compounded**. If the interest is compounded annually, the value of the account is multiplied by $1+r$ at the end of each year. This means that after $t$ years the value of the account is

$$Q(t) = Q_0(1+r)^t.$$

If interest is compounded semiannually, the value of the account is multiplied by $(1+r/2)$ every 6 months. Since this occurs twice annually, the value of the account after $t$ years is

$$Q(t) = Q_0\left(1+\frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded $n$ times per year, the value of the account is multiplied $n$ times per year by $(1+r/n)$; therefore, the value of the account after $t$ years is

$$Q(t) = Q_0\left(1+\frac{r}{n}\right)^{nt}. \tag{4.1.8}$$

Thus, increasing the frequency of compounding increases the value of the account after a fixed period of time. Table 4.1.1 shows the effect of increasing the number of compoundings over $t = 5$ years on an initial deposit of $Q_0 = 100$ (dollars), at an annual interest rate of 6%.

Table 4.1.1 : The effect of compound interest

| $n$ (number of compoundings per year) | $\$100\left(1+\frac{.06}{n}\right)^{5n}$ (value in dollars after 5 years) |
|---|---|
| 1 | \$133.82 |
| 2 | \$134.39 |
| 4 | \$134.68 |
| 8 | \$134.83 |
| 364 | \$134.98 |

You can see from Table 4.1.1 that the value of the account after 5 years is an increasing function of $n$. Now suppose the maximum allowable rate of interest on savings accounts is restricted by law, but the time intervals between successive compoundings isn't; then competing banks can attract savers by compounding often. The ultimate step in this direction is to **compound continuously**, by which we mean that $n \to \infty$ in Equation 4.1.8. Since we know from calculus that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r,$$

this yields

$$\begin{aligned} Q(t) &= \lim_{n\to\infty} Q_0\left(1+\tfrac{r}{n}\right)^{nt} = Q_0\left[\lim_{n\to\infty}\left(1+\tfrac{r}{n}\right)^n\right]^t \\ &= Q_0 e^{rt}. \end{aligned}$$

Observe that $Q = Q_0 e^{rt}$ is the solution of the initial value problem

$$Q' = rQ, \quad Q(0) = Q_0;$$

that is, with continuous compounding the value of the account grows exponentially.

### Example 4.1.2

If \$150 is deposited in a bank that pays $5\frac{1}{2}\%$ annual interest compounded continuously, the value of the account after $t$ years is

$$Q(t) = 150e^{.055t}$$

dollars. (Note that it is necessary to write the interest rate as a decimal; thus, $r = .055$.) Therefore, after $t = 10$ years the value of the account is

$$Q(10) = 150e^{.55} \approx \$259.99.$$

### Example 4.1.3

We wish to accumulate \$10,000 in 10 years by making a single deposit in a savings account bearing $5\frac{1}{2}\%$ annual interest compounded continuously. How much must we deposit in the account?

**Solution**

The value of the account at time $t$ is

$$Q(t) = Q_0 e^{.055t}. \tag{4.1.9}$$

Since we want $Q(10)$ to be \$10,000, the initial deposit $Q_0$ must satisfy the equation

$$10000 = Q_0 e^{.55}, \tag{4.1.10}$$

obtained by setting $t = 10$ and $Q(10) = 10000$ in Equation 4.1.9. Solving Equation 4.1.10 for $Q_0$ yields

$$Q_0 = 10000e^{-.55} \approx \$5769.50.$$

## Mixed Growth and Decay

### Example 4.1.4

A radioactive substance with decay constant $k$ is produced at a constant rate of $a$ units of mass per unit time.

a. Assuming that $Q(0) = Q_0$, find the mass $Q(t)$ of the substance present at time $t$.
b. Find $\lim_{t\to\infty} Q(t)$.

**Solution a:**

Here

$$Q' = \text{ rate of increase of } Q - \text{ rate of decrease of } Q.$$

The rate of increase is the constant $a$. Since $Q$ is radioactive with decay constant $k$, the rate of decrease is $kQ$. Therefore

$$Q' = a - kQ.$$

This is a linear first order differential equation. Rewriting it and imposing the initial condition shows that $Q$ is the solution of the initial value problem

$$Q' + kQ = a, \quad Q(0) = Q_0. \tag{4.1.11}$$

Since $e^{-kt}$ is a solution of the complementary equation, the solutions of Equation 4.1.11 are of the form $Q = ue^{-kt}$, where $u'e^{-kt} = a$, so $u' = ae^{kt}$. Hence,

$$u = \frac{a}{k}e^{kt} + c$$

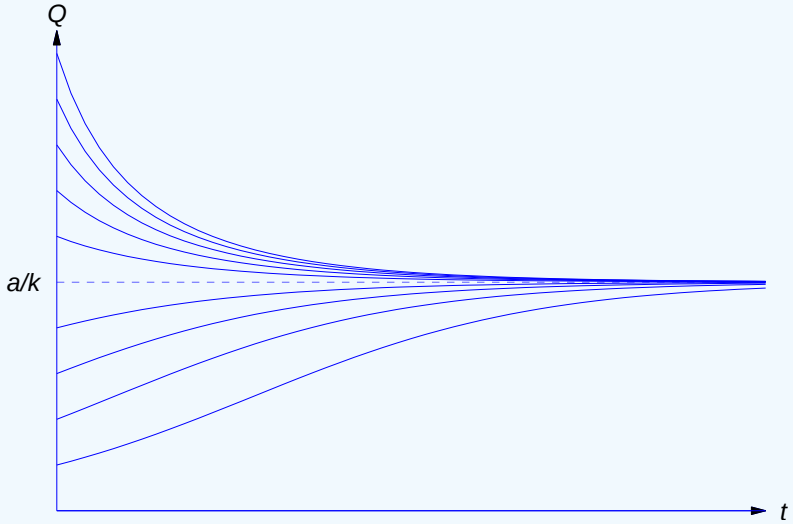



Figure 4.1.3 : $Q(t)$ approaches the steady state value $\frac{a}{k}$ as $t \to \infty$

and

$$Q = ue^{-kt} = \frac{a}{k} + ce^{-kt}.$$

Since $Q(0) = Q_0$, setting $t = 0$ here yields

$$Q_0 = \frac{a}{k} + c \quad \text{or} \quad c = Q_0 - \frac{a}{k}.$$

Therefore

$$Q = \frac{a}{k} + \left(Q_0 - \frac{a}{k}\right) e^{-kt}. \tag{4.1.12}$$

b. Since $k > 0$, $\lim_{t\to\infty} e^{-kt} = 0$, so from Equation 4.1.12

$$\lim_{t\to\infty} Q(t) = \frac{a}{k}.$$

This limit depends only on $a$ and $k$, and not on $Q_0$. We say that $a/k$ is the **steady state** value of $Q$. From Equation 4.1.12 we also see that $Q$ approaches its steady state value from above if $Q_0 > a/k$, or from below if $Q_0 < a/k$. If $Q_0 = a/k$, then $Q$ remains constant (Figure 4.1.3 ).

## Carbon Dating

The fact that $Q$ approaches a steady state value in the situation discussed in Example 4 underlies the method of **carbon dating**, devised by the American chemist and Nobel Prize Winner **W.S. Libby**.

Carbon 12 is stable, but carbon-14, which is produced by cosmic bombardment of nitrogen in the upper atmosphere, is radioactive with a half-life of about 5570 years. Libby assumed that the quantity of carbon-12 in the atmosphere has been constant throughout time, and that the quantity of radioactive carbon-14 achieved its steady state value long ago as a result of its creation and decomposition over millions of years. These assumptions led Libby to conclude that the ratio of carbon-14 to carbon-12 has been nearly constant for a long time. This constant, which we denote by $R$, has been determined experimentally.

Living cells absorb both carbon-12 and carbon-14 in the proportion in which they are present in the environment. Therefore the ratio of carbon-14 to carbon-12 in a living cell is always $R$. However, when the cell dies it ceases to absorb carbon, and the ratio of carbon-14 to carbon-12 decreases exponentially as the radioactive carbon-14 decays. This is the basis for the method of carbon dating, as illustrated in the next example.

### Example 4.1.5

An archaeologist investigating the site of an ancient village finds a burial ground where the amount of carbon-14 present in individual remains is between 42 and 44% of the amount present in live individuals. Estimate the age of the village and the length of time for which it survived.

**Solution**

Let $Q = Q(t)$ be the quantity of carbon-14 in an individual set of remains $t$ years after death, and let $Q_0$ be the quantity that would be present in live individuals. Since carbon-14 decays exponentially with half-life 5570 years, its decay constant is

$$k = \frac{\ln 2}{5570}.$$

Therefore

$$Q = Q_0 e^{-t(\ln 2)/5570}$$

if we choose our time scale so that $t_0 = 0$ is the time of death. If we know the present value of $Q$ we can solve this equation for $t$, the number of years since death occurred. This yields

$$t = -5570\frac{\ln Q/Q_0}{\ln 2}.$$

It is given that $Q = 0.42Q_0$ in the remains of individuals who died first. Therefore these deaths occurred about

$$t_1 = -5570\frac{\ln .42}{\ln 2} \approx 6971$$

years ago. For the most recent deaths, $Q = .44Q_0$; hence, these deaths occurred about

$$t_2 = -5570\frac{\ln .44}{\ln 2} \approx 6597$$

years ago. Therefore it is reasonable to conclude that the village was founded about 7000 years ago, and lasted for about 400 years.

## A Savings Program

**Example 4.1.6**

A person opens a savings account with an initial deposit of \$1000 and subsequently deposits \$50 per week. Find the value $Q(t)$ of the account at time $t > 0$, assuming that the bank pays 6% interest compounded continuously.

**Solution**

Observe that $Q$ isn't continuous, since there are 52 discrete deposits per year of \$50 each. To construct a mathematical model for this problem in the form of a differential equation, we make the simplifying assumption that the deposits are made continuously at a rate of \$2600 per year. This is essential, since solutions of differential equations are continuous functions. With this assumption, $Q$ increases continuously at the rate

$$Q' = 2600 + 0.06Q$$

and therefore $Q$ satisfies the differential equation

$$Q' - .06Q = 2600. \tag{4.1.13}$$

(Of course, we must recognize that the solution of this equation is an approximation to the true value of $Q$ at any given time. We'll discuss this further below.) Since $e^{.06t}$ is a solution of the complementary equation, the solutions of Equation 4.1.13 are of the form $Q = ue^{.06t}$, where $u'e^{.06t} = 2600$. Hence, $u' = 2600e^{-.06t}$,

$$u = -\frac{2600}{.06}e^{-0.06t} + c$$

and

$$Q = ue^{.06t} = -\frac{2600}{.06} + ce^{.06t}. \tag{4.1.14}$$

Setting $t = 0$ and $Q = 1000$ here yields

$$c = 1000 + \frac{2600}{0.06},$$

and substituting this into Equation 4.1.14 yields

$$Q = 1000e^{.06t} + \frac{2600}{.06}(e^{.06t} - 1) \tag{4.1.15}$$

where the first term is the value due to the initial deposit and the second is due to the subsequent weekly deposits.

Mathematical models must be tested for validity by comparing predictions based on them with the actual outcome of experiments. Example 6 is unusual in that we can compute the exact value of the account at any specified time and compare it with the approximate value predicted by Equation 4.1.15 (See **Exercise 4.1.21**). The following table gives a comparison for a ten year period. Each exact answer corresponds to the time of the year-end deposit, and each year is assumed to have exactly 52 weeks.

Table 4.1.3

| Year | Approximate Value of $Q$ (Example 4.1.6 ) | Exact Value of $P$ (Exercise 4.1.21) | Error $Q - P$ | Percentage Error $(Q - P)/P$ |
|---|---|---|---|---|
| 1 | \$3741.42 | \$3739.87 | \$1.55 | .0413 |
| 2 | 6652.36 | 6649.17 | 3.19 | .0479 |
| 3 | 9743.30 | 9738.37 | 4.93 | .0506 |
| 4 | 13, 025.38 | 13, 018.60 | 6.78 | .0521 |
| 5 | 16, 510.41 | 16, 501.66 | 8.75 | .0530 |

| Year | Approximate Value of $Q$ (Example 4.1.6 ) | Exact Value of $P$ (Exercise 4.1.21) | Error $Q-P$ | Percentage Error $(Q-P)/P$ |
|---|---|---|---|---|
| 6 | $20,210.94$ | $20,200.11$ | $10.83$ | $.0536$ |
| 7 | $24,140.30$ | $24,127.25$ | $13.05$ | $.0541$ |
| 8 | $28,312.63$ | $28,297.23$ | $15.40$ | $.0544$ |
| 9 | $32,742.97$ | $32,725.07$ | $17.90$ | $.0547$ |
| 10 | $37,447.27$ | $37,426.72$ | $20.55$ | $.0549$ |

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# 4.1E: Growth and Decay (Exercises)

## Q4.1.1

1. The half-life of a radioactive substance is 3200 years. Find the quantity $Q(t)$ of the substance left at time $t > 0$ if $Q(0) = 20$ g.

2. The half-life of a radioactive substance is 2 days. Find the time required for a given amount of the material to decay to 1/10 of its original mass.

3. A radioactive material loses 25% of its mass in 10 minutes. What is its half-life?

4. A tree contains a known percentage $p_0$ of a radioactive substance with half-life $\tau$. When the tree dies the substance decays and isn't replaced. If the percentage of the substance in the fossilized remains of such a tree is found to be $p_1$, how long has the tree been dead?

5. If $t_p$ and $t_q$ are the times required for a radioactive material to decay to $1/p$ and $1/q$ times its original mass (respectively), how are $t_p$ and $t_q$ related?

6. Find the decay constant $k$ for a radioactive substance, given that the mass of the substance is $Q_1$ at time $t_1$ and $Q_2$ at time $t_2$.

7. A process creates a radioactive substance at the rate of 2 g/hr and the substance decays at a rate proportional to its mass, with constant of proportionality $k = .1(\text{hr})^{-1}$. If $Q(t)$ is the mass of the substance at time $t$, find $\lim_{t\to\infty} Q(t)$.

8. A bank pays interest continuously at the rate of 6%. How long does it take for a deposit of $Q_0$ to grow in value to $2Q_0$?

9. At what rate of interest, compounded continuously, will a bank deposit double in value in 8 years?

10. A savings account pays 5% per annum interest compounded continuously. The initial deposit is $Q_0$ dollars. Assume that there are no subsequent withdrawals or deposits.

a. How long will it take for the value of the account to triple?
b. What is $Q_0$ if the value of the account after 10 years is \$100,000 dollars?

11. A candymaker makes 500 pounds of candy per week, while his large family eats the candy at a rate equal to $Q(t)/10$ pounds per week, where $Q(t)$ is the amount of candy present at time $t$.

a. Find $Q(t)$ for $t > 0$ if the candymaker has 250 pounds of candy at $t = 0$.
b. Find $\lim_{t\to\infty} Q(t)$.

12. Suppose a substance decays at a yearly rate equal to half the square of the mass of the substance present. If we start with 50 g of the substance, how long will it be until only 25 g remain?

13. A super bread dough increases in volume at a rate proportional to the volume $V$ present. If $V$ increases by a factor of 10 in 2 hours and $V(0) = V_0$, find $V$ at any time $t$. How long will it take for $V$ to increase to $100V_0$?

14. A radioactive substance decays at a rate proportional to the amount present, and half the original quantity $Q_0$ is left after 1500 years. In how many years would the original amount be reduced to $3Q_0/4$? How much will be left after 2000 years?

15. A wizard creates gold continuously at the rate of 1 ounce per hour, but an assistant steals it continuously at the rate of 5% of however much is there per hour. Let $W(t)$ be the number of ounces that the wizard has at time $t$. Find $W(t)$ and $\lim_{t\to\infty} W(t)$ if $W(0) = 1$.

16. A process creates a radioactive substance at the rate of 1 g/hr, and the substance decays at an hourly rate equal to 1/10 of the mass present (expressed in grams). Assuming that there are initially 20 g, find the mass $S(t)$ of the substance present at time $t$, and find $\lim_{t\to\infty} S(t)$.

17. A tank is empty at $t = 0$. Water is added to the tank at the rate of 10 gal/min, but it leaks out at a rate (in gallons per minute) equal to the number of gallons in the tank. What is the smallest capacity the tank can have if this process is to continue forever?

18. A person deposits \$25,000 in a bank that pays 5% per year interest, compounded continuously. The person continuously withdraws from the account at the rate of \$750 per year. Find $V(t)$, the value of the account at time $t$ after the initial deposit.

19. A person has a fortune that grows at rate proportional to the square root of its worth. Find the worth $W$ of the fortune as a function of $t$ if it was \$1 million 6 months ago and is \$4 million today.

20. Let $p = p(t)$ be the quantity of a product present at time $t$. The product is manufactured continuously at a rate proportional to $p$, with proportionality constant 1/2, and it is consumed continuously at a rate proportional to $p^2$, with proportionality constant 1/8. Find $p(t)$ if $p(0) = 100$.

21.

a. In the situation of Example 4.1.6 find the exact value P(t) of the person's account after t years, where t is an integer. Assume that each year has exactly 52 weeks, and include the year-end deposit in the computation.

HINT: At time t the initial \$1000 has been on deposit for $t$ years. There have been $52t$ deposits of \$50 each. The first \$50 has been on deposit for $t - 1/52$ years, the second for $t - 2/52$ years ... in general, the **j** th \$50 has been on deposit for $t - j/52$ years ( $1 \le j \le 52t$ ). Find the present value of each \$50 deposit assuming 6% interest compounded continuously, and use the formula

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} (x \neq 1)$$

to find their total value.

b. Let

$$p(t) = \frac{Q(t) - P(t)}{P(t)}$$

be the relative error after $t$ years. Find

$$p(\infty) = \lim_{t \to \infty} p(t).$$

22. A homebuyer borrows $P_0$ dollars at an annual interest rate $r$, agreeing to repay the loan with equal monthly payments of $M$ dollars per month over $N$ years.

a. Derive a differential equation for the loan principal (amount that the homebuyer owes) $P(t)$ at time $t > 0$, making the simplifying assumption that the homebuyer repays the loan continuously rather than in discrete steps. (See Example 4.1.6.)

b. Solve the equation derived in (a).

c. Use the result of (b) to determine an approximate value for $M$ assuming that each year has exactly 12 months of equal length.

d. It can be shown that the exact value of $M$ is given by

$$M = \frac{rP_0}{12} \left(1 - (1 + r/12)^{-12N}\right)^{-1}.$$

Compare the value of $M$ obtained from the answer in (c) to the exact value if (i) $P_0 = \$50,000$, \(r=7{1\over2}\)%, $N = 20$ (ii) $P_0 = \$150,000$, $r = 9.0\%$, $N = 30$.

23. Assume that the homebuyer of **Exercise 4.1.22** elects to repay the loan continuously at the rate of $\alpha M$ dollars per month, where $\alpha$ is a constant greater than 1. (This is called **accelerated payment**.)

a. Determine the time $T(\alpha)$ when the loan will be paid off and the amount $S(\alpha)$ that the homebuyer will save.
b. Suppose $P_0 = \$50,000$, $r = 8\%$, and $N = 15$. Compute the savings realized by accelerated payments with $\alpha = 1.05, 1.10$, and $1.15$.

24. A benefactor wishes to establish a trust fund to pay a researcher's salary for $T$ years. The salary is to start at $S_0$ dollars per year and increase at a fractional rate of $a$ per year. Find the amount of money $P_0$ that the benefactor must deposit in a trust fund paying interest at a rate $r$ per year. Assume that the researcher's salary is paid continuously, the interest is compounded continuously, and the salary increases are granted continuously.

25. A radioactive substance with decay constant $k$ is produced at the rate of

$$\frac{at}{1 + btQ(t)}$$

units of mass per unit time, where $a$ and $b$ are positive constants and $Q(t)$ is the mass of the substance present at time $t$; thus, the rate of production is small at the start and tends to slow when $Q$ is large.

a. Set up a differential equation for $Q$.

b. Choose your own positive values for $a$, $b$, $k$, and $Q_0 = Q(0)$. Use a numerical method to discover what happens to $Q(t)$ as $t \to \infty$. (Be precise, expressing your conclusions in terms of $a$, $b$, $k$. However, no proof is required.)

26. Follow the instructions of **Exercise 4.1.25**, assuming that the substance is produced at the rate of $at/(1+bt(Q(t))^2)$ units of mass per unit of time.

27. Follow the instructions of **Exercise 4.1.25**, assuming that the substance is produced at the rate of $at/(1+bt)$ units of mass per unit of time.

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# 4.2: Cooling and Mixing

## Newton's Law of Cooling

Newton's law of cooling states that if an object with temperature $T(t)$ at time $t$ is in a medium with temperature $T_m(t)$, the rate of change of $T$ at time $t$ is proportional to $T(t)-T_m(t)$ ; thus, $T$ satisfies a differential equation of the form

$$T' = -k(T - T_m). \tag{4.2.1}$$

Here $k > 0$, since the temperature of the object must decrease if $T > T_m$ , or increase if $T < T_m$ . We'll call $k$ the **temperature decay constant of the medium**.

For simplicity, in this section we'll assume that the medium is maintained at a constant temperature $T_m$. This is another example of building a simple mathematical model for a physical phenomenon. Like most mathematical models it has its limitations. For example, it is reasonable to assume that the temperature of a room remains approximately constant if the cooling object is a cup of coffee, but perhaps not if it is a huge cauldron of molten metal. (For more on this see **Exercise 4.2.17**.)

To solve Equation 4.2.1, we rewrite it as

$$T' + kT = kT_m.$$

Since $e^{-kt}$ is a solution of the complementary equation, the solutions of this equation are of the form $T = ue^{-kt}$, where $u'e^{-kt} = kT_m$ , so $u' = kT_m e^{kt}$ . Hence,

$$u = T_m e^{kt} + c,$$

so

$$T = ue^{-kt} = T_m + ce^{-kt}.$$

If $T(0) = T_0$ , setting $t = 0$ here yields $c = T_0 - T_m$ , so

$$T = T_m + (T_0 - T_m)e^{-kt}. \tag{4.2.2}$$

Note that $T - T_m$ decays exponentially, with decay constant $k$.

### Example 4.2.1

A ceramic insulator is baked at $400^\circ$C and cooled in a room in which the temperature is $25^\circ$C. After 4 minutes the temperature of the insulator is $200^\circ$C. What is its temperature after 8 minutes?

**Solution**

Here $T_0 = 400$ and $T_m = 25$, so Equation 4.2.2 becomes

$$T = 25 + 375e^{-kt}. \tag{4.2.3}$$

We determine $k$ from the stated condition that $T(4) = 200$; that is,

$$200 = 25 + 375e^{-4k};$$

hence,

$$e^{-4k} = \frac{175}{375} = \frac{7}{15}.$$

Taking logarithms and solving for $k$ yields

$$k = -\frac{1}{4}\ln\frac{7}{15} = \frac{1}{4}\ln\frac{15}{7}.$$

Substituting this into Equation 4.2.3 yields

$$T = 25 + 375e^{-\frac{t}{4}\ln\frac{15}{7}}$$

(Figure 4.2.1 ). Therefore the temperature of the insulator after 8 minutes is

$$\begin{aligned} T(8) &= 25 + 375e^{-2\ln\frac{15}{7}} \\ &= 25 + 375\left(\tfrac{7}{15}\right)^2 \approx 107^\circ\text{C}. \end{aligned}$$

## Example 4.2.2

An object with temperature $72^\circ$F is placed outside, where the temperature is $-20^\circ$F. At 11:05 the temperature of the object is $60^\circ$F and at 11:07 its temperature is $50^\circ$F. At what time was the object placed outside?

### Solution

Let $T(t)$ be the temperature of the object at time $t$. For convenience, we choose the origin $t_0 = 0$ of the time scale to be 11:05 so that $T_0 = 60$. We must determine the time $\tau$ when $T(\tau) = 72$. Substituting $T_0 = 60$ and $T_m = -20$ into Equation 4.2.2 yields

$$T = -20 + (60 - (-20))\, e^{-kt}$$

or

$$T = -20 + 80e^{-kt}. \tag{4.2.4}$$

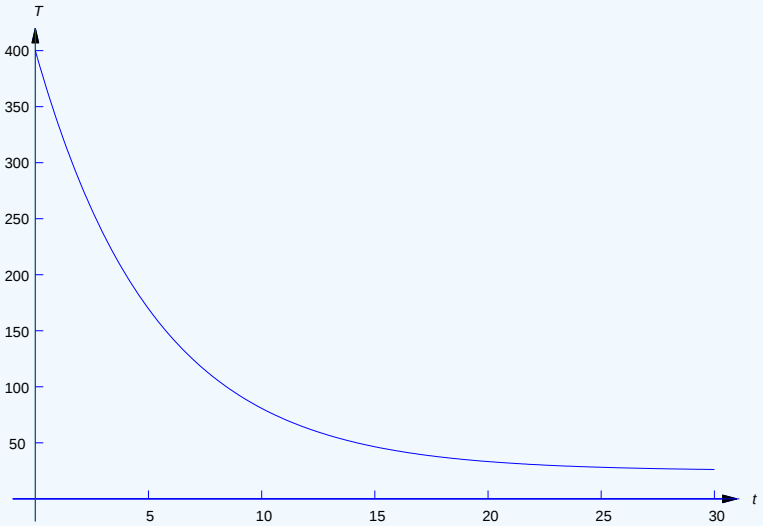



Figure 4.2.1 : $T = 25 + 375e^{-(t/4)\ln 15/7}$

We obtain $k$ from the stated condition that the temperature of the object is $50^\circ$F at 11:07. Since 11:07 is $t = 2$ on our time scale, we can determine $k$ by substituting $T = 50$ and $t = 2$ into Equation 4.2.4 to obtain

$$50 = -20 + 80e^{-2k}$$

This is shown in Figure 4.2.2 .

Hence,

$$e^{-2k} = \frac{70}{80} = \frac{7}{8}.$$

Taking logarithms and solving for $k$ yields

$$k = -\frac{1}{2}\ln\frac{7}{8} = \frac{1}{2}\ln\frac{8}{7}.$$

Substituting this into Equation 4.2.4 yields

$$T = -20 + 80e^{-\frac{t}{2}\ln\frac{8}{7}},$$

and the condition $T(\tau) = 72$ implies that

$$72 = -20 + 80e^{-\frac{\tau}{2}\ln\frac{8}{7}};$$

hence,

$$e^{-\frac{\tau}{2}\ln\frac{8}{7}} = \frac{92}{80} = \frac{23}{20}.$$

Taking logarithms and solving for $\tau$ yields

$$\tau = -\frac{2\ln\frac{23}{20}}{\ln\frac{8}{7}} \approx -2.09 \text{ min}.$$

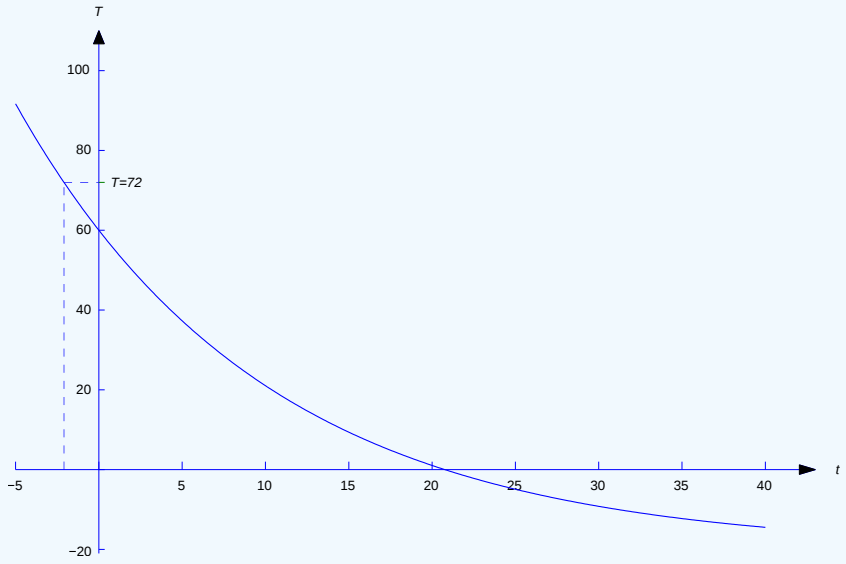



Figure 4.2.2 : $-20 + 80e^{-\frac{t}{2}\ln\frac{8}{7}}$

Therefore the object was placed outside about 2 minutes and 5 seconds before 11:05; that is, at 11:02:55.

## Mixing Problems

In the next two examples a saltwater solution with a given concentration (weight of salt per unit volume of solution) is added at a specified rate to a tank that initially contains saltwater with a different concentration. The problem is to determine the quantity of salt in the tank as a function of time. This is an example of a **mixing problem**. To construct a tractable mathematical model for mixing problems we assume in our examples (and most exercises) that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Exercises 4.2.22 and 4.2.23 deal with situations where this isn't so, but the distribution of salt becomes approximately uniform as $t \to \infty$.

### Example 4.2.3

A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains 1/2 pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate (Figure 4.2.3 ).

a. Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $Q(t)$.
b. Find $\lim_{t\to\infty} Q(t)$.

#### Solution a

To find a differential equation for $Q$, we must use the given information to derive an expression for $Q'$. But $Q'$ is the rate of change of the quantity of salt in the tank changes with respect to time; thus, if **rate in** denotes the rate at which salt enters the tank and **rate out** denotes the rate by which it leaves, then

$$Q' = \text{rate in} - \text{rate out}. \tag{4.2.5}$$

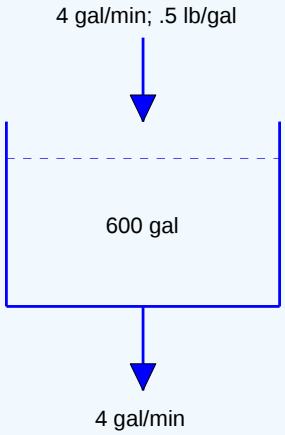



Figure 4.2.3 : A mixing problem

The rate in is

$$\left(\frac{1}{2}\ \text{lb/gal}\right) \times (4\ \text{gal/min}) = 2\ \text{lb/min}.$$

Determining the rate out requires a little more thought. We're removing 4 gallons of the mixture per minute, and there are always 600 gallons in the tank; that is, we are removing $1/150$ of the mixture per minute. Since the salt is evenly distributed in the mixture, we are also removing $1/150$ of the salt per minute. Therefore, if there are $Q(t)$ pounds of salt in the tank at time $t$, the rate out at any time $t$ is $Q(t)/150$. Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned}
\text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) \\
&= (\text{lb/gal}) \times (\text{gal/min}) \\
&= \frac{Q(t)}{600} \times 4 \\
&= \frac{Q(t)}{150}.
\end{aligned}$$

We can now write Equation 4.2.5 as

$$Q' = 2 - \frac{Q}{150}.$$

This first order equation can be rewritten as

$$Q' + \frac{Q}{150} = 2.$$

Since $e^{-t/150}$ is a solution of the complementary equation, the solutions of this equation are of the form $Q = ue^{-t/150}$, where $u'e^{-t/150} = 2$, so $u' = 2e^{t/150}$. Hence,

$$u = 300e^{t/150} + c,$$

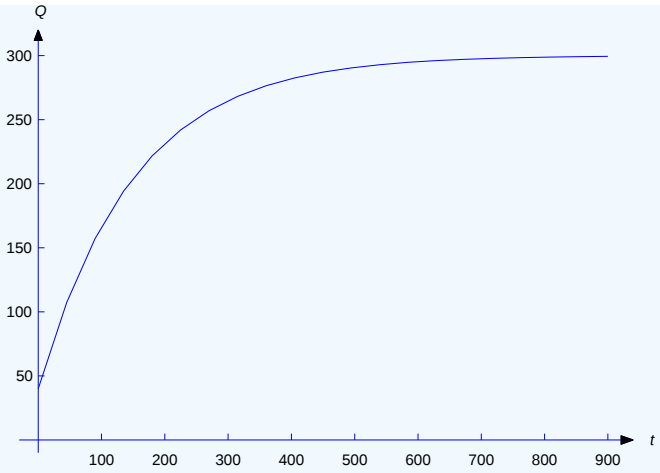


Figure 4.2.4 : $Q = 300 - 260e^{-t/150}$

so

$$Q = ue^{-t/150} = 300 + ce^{-t/150} \tag{4.2.6}$$

(Figure 4.2.4 ). Since $Q(0) = 40$, $c = -260$; therefore,

$$Q = 300 - 260e^{-t/150}.$$

### Solution b

From Equation 4.2.6, we see that that $\lim_{t\to\infty} Q(t) = 300$ for any value of $Q(0)$. This is intuitively reasonable, since the incoming solution contains 1/2 pound of salt per gallon and there are always 600 gallons of water in the tank.

## Example 4.2.4

A 500-liter tank initially contains 10 g of salt dissolved in 200 liters of water. Starting at $t_0 = 0$, water that contains 1/4 g of salt per liter is poured into the tank at the rate of 4 liters/min and the mixture is drained from the tank at the rate of 2 liters/min (Figure 4.2.5 ). Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t$ prior to the time when the tank overflows and find the concentration $K(t)$ (g/liter) of salt in the tank at any such time.

### Solution

We first determine the amount $W(t)$ of solution in the tank at any time $t$ prior to overflow. Since $W(0) = 200$ and we are adding 4 liters/min while removing only 2 liters/min, there's a net gain of 2 liters/min in the tank; therefore,

$$W(t) = 2t + 200.$$

Since $W(150) = 500$ liters (capacity of the tank), this formula is valid for $0 \le t \le 150$.

Now let $Q(t)$ be the number of grams of salt in the tank at time $t$, where $0 \le t \le 150$. As in Example 4.2.3

$$Q' = \text{rate in} - \text{rate out}. \tag{4.2.7}$$

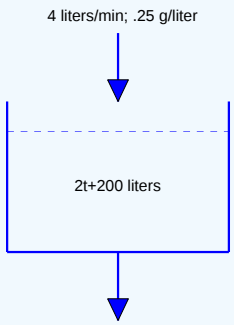


Figure 4.2.5 : Another mixing problem


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The rate in is

$$\left(\frac{1}{4}\text{ g/liter}\right)\times(4\text{ liters/min})=1\text{ g/min}. \tag{4.2.8}$$

To determine the rate out, we observe that since the mixture is being removed from the tank at the constant rate of 2 liters/min and there are $2t+200$ liters in the tank at time $t$, the fraction of the mixture being removed per minute at time $t$ is

$$\frac{2}{2t+200}=\frac{1}{t+100}.$$

We're removing this same fraction of the salt per minute. Therefore, since there are $Q(t)$ grams of salt in the tank at time $t$,

$$\text{rate out}=\frac{Q(t)}{t+100}. \tag{4.2.9}$$

Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned}\text{rate out}&=(\text{concentration})\times(\text{rate of flow out})\\&=(\text{g/liter})\times(\text{liters/min})\\&=\frac{Q(t)}{2t+200}\times 2=\frac{Q(t)}{t+100}.\end{aligned}$$

Substituting Equation 4.2.8 and Equation 4.2.9 into Equation 4.2.7 yields

$$Q'=1-\frac{Q}{t+100},\quad\text{so}\quad Q'+\frac{1}{t+100}Q=1. \tag{4.2.10}$$

By separation of variables, $1/(t+100)$ is a solution of the complementary equation, so the solutions of Equation 4.2.10 are of the form

$$Q=\frac{u}{t+100},\quad\text{where}\quad\frac{u'}{t+100}=1,\quad\text{so}\quad u'=t+100.$$

Hence,

$$u=\frac{(t+100)^2}{2}+c. \tag{4.2.11}$$

Since $Q(0)=10$ and $u=(t+100)Q$, Equation 4.2.11 implies that

$$(100)(10)=\frac{(100)^2}{2}+c,$$

so

$$c=100(10)-\frac{(100)^2}{2}=-4000$$

and therefore

$$u=\frac{(t+100)^2}{2}-4000.$$

Hence,

$$Q=\frac{u}{t+200}=\frac{t+100}{2}-\frac{4000}{t+100}.$$

Now let $K(t)$ be the concentration of salt at time $t$. Then

$$K(t)=\frac{1}{4}-\frac{2000}{(t+100)^2}$$

This is shown in Figure 4.2.6 .

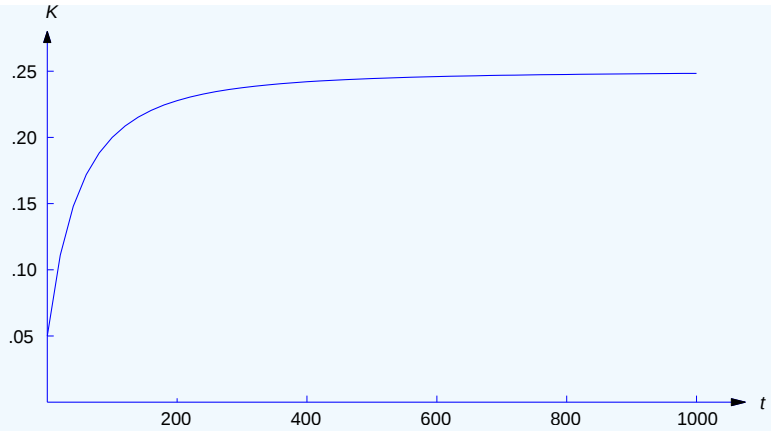


Figure 4.2.6 : $K(t) = \frac{1}{4} - \frac{2000}{(t+100)^2}$

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# 4.2E: Cooling and Mixing (Exercises)

## Q4.2.1

1. A thermometer is moved from a room where the temperature is $70^\circ$F to a freezer where the temperature is $12^\circ F$. After $30$ seconds the thermometer reads $40^\circ$F. What does it read after $2$ minutes?

2. A fluid initially at $100^\circ$C is placed outside on a day when the temperature is $-10^\circ$C, and the temperature of the fluid drops $20^\circ$C in one minute. Find the temperature $T(t)$ of the fluid for $t > 0$.

3. At 12:00 pm a thermometer reading $10^\circ$F is placed in a room where the temperature is $70^\circ$F. It reads $56^\circ$ when it is placed outside, where the temperature is $5^\circ$F, at 12:03. What does it read at 12:05 pm?

4. A thermometer initially reading $212^\circ$F is placed in a room where the temperature is $70^\circ$F. After $2$ minutes the thermometer reads $125^\circ$F.

   a. What does the thermometer read after $4$ minutes?
   b. When will the thermometer read $72^\circ$F?
   c. When will the thermometer read $69^\circ$F?

5. An object with initial temperature $150^\circ$C is placed outside, where the temperature is $35^\circ$C. Its temperatures at 12:15 and 12:20 are $120^\circ$C and $90^\circ$C, respectively.

   a. At what time was the object placed outside?
   b. When will its temperature be $40^\circ$C?

6. An object is placed in a room where the temperature is $20^\circ$C. The temperature of the object drops by $5^\circ$C in $4$ minutes and by $7^\circ$C in $8$ minutes. What was the temperature of the object when it was initially placed in the room?

7. A cup of boiling water is placed outside at 1:00 pm. One minute later the temperature of the water is $152^\circ$F. After another minute its temperature is $112^\circ$F. What is the outside temperature?

8. A tank initially contains $40$ gallons of pure water. A solution with $1$ gram of salt per gallon of water is added to the tank at $3$ gal/min, and the resulting solution drains out at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.

9. A tank initially contains a solution of $10$ pounds of salt in $60$ gallons of water. Water with $1/2$ pound of salt per gallon is added to the tank at $6$ gal/min, and the resulting solution leaves at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.

10. A tank initially contains $100$ liters of a salt solution with a concentration of $.1$ g/liter. A solution with a salt concentration of $.3$ g/liter is added to the tank at $5$ liters/min, and the resulting mixture is drained out at the same rate. Find the concentration $K(t)$ of salt in the tank as a function of $t$.

11. A $200$ gallon tank initially contains $100$ gallons of water with $20$ pounds of salt. A salt solution with $1/4$ pound of salt per gallon is added to the tank at $4$ gal/min, and the resulting mixture is drained out at $2$ gal/min. Find the quantity of salt in the tank as it is about to overflow.

12. Suppose water is added to a tank at $10$ gal/min, but leaks out at the rate of $1/5$ gal/min for each gallon in the tank. What is the smallest capacity the tank can have if the process is to continue indefinitely?

13. A chemical reaction in a laboratory with volume $V$ (in ft$^3$) produces $q_1$ ft$^3$/min of a noxious gas as a byproduct. The gas is dangerous at concentrations greater than $\overline{c}$, but harmless at concentrations $\leq \overline{c}$. Intake fans at one end of the laboratory pull in fresh air at the rate of $q_2$ ft$^3$/min and exhaust fans at the other end exhaust the mixture of gas and air from the laboratory at the same rate. Assuming that the gas is always uniformly distributed in the room and its initial concentration $c_0$ is at a safe level, find the smallest value of $q_2$ required to maintain safe conditions in the laboratory for all time.

14. A $1200$-gallon tank initially contains $40$ pounds of salt dissolved in $600$ gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is added to the tank at the rate of $6$ gal/min and the resulting mixture is drained from the tank at $4$ gal/min. Find the quantity $Q(t)$ of salt in the tank at any time $t > 0$ prior to overflow.

15. Tank $T_1$ initially contain $50$ gallons of pure water. Starting at $t_0 = 0$, water that contains $1$ pound of salt per gallon is poured into $T_1$ at the rate of $2$ gal/min. The mixture is drained from $T_1$ at the same rate into a second tank $T_2$, which initially contains $50$

gallons of pure water. Also starting at $t_0 = 0$, a mixture from another source that contains 2 pounds of salt per gallon is poured into $T_2$ at the rate of 2 gal/min. The mixture is drained from $T_2$ at the rate of 4 gal/min.

a. Find a differential equation for the quantity $Q(t)$ of salt in tank $T_2$ at time $t > 0$.
b. Solve the equation derived in (a) to determine $Q(t)$.
c. Find $\lim_{t\to\infty} Q(t)$.

16. Suppose an object with initial temperature $T_0$ is placed in a sealed container, which is in turn placed in a medium with temperature $T_m$. Let the initial temperature of the container be $S_0$. Assume that the temperature of the object does not affect the temperature of the container, which in turn does not affect the temperature of the medium. (These assumptions are reasonable, for example, if the object is a cup of coffee, the container is a house, and the medium is the atmosphere.)

a. Assuming that the container and the medium have distinct temperature decay constants $k$ and $k_m$ respectively, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time $t$.
b. Assuming that the container and the medium have the same temperature decay constant $k$, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time $t$.
c. Find $\lim_{t\to\infty} S(t)$ and $\lim_{t\to\infty} T(t)$.

17. In our previous examples and exercises concerning Newton's law of cooling we assumed that the temperature of the medium remains constant. This model is adequate if the heat lost or gained by the object is insignificant compared to the heat required to cause an appreciable change in the temperature of the medium. If this isn't so, we must use a model that accounts for the heat exchanged between the object and the medium. Let $T = T(t)$ and $T_m = T_m(t)$ be the temperatures of the object and the medium, respectively, and let $T_0$ and $T_{m0}$ be their initial values. Again, we assume that $T$ and $T_m$ are related by Newton's law of cooling,

$$T' = -k(T - T_m). \tag{A}$$

We also assume that the change in heat of the object as its temperature changes from $T_0$ to $T$ is $a(T - T_0)$ and that the change in heat of the medium as its temperature changes from $T_{m0}$ to $T_m$ is $a_m(T_m - T_{m0})$, where $a$ and $a_m$ are positive constants depending upon the masses and thermal properties of the object and medium, respectively. If we assume that the total heat of the system consisting of the object and the medium remains constant (that is, energy is conserved), then

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0. \tag{B}$$

a. Equation (A) involves two unknown functions $T$ and $T_m$. Use (A) and (B) to derive a differential equation involving only $T$.
b. Find $T(t)$ and $T_m(t)$ for $t > 0$.
c. Find $\lim_{t\to\infty} T(t)$ and $\lim_{t\to\infty} T_m(t)$.

18. Control mechanisms allow fluid to flow into a tank at a rate proportional to the volume $V$ of fluid in the tank, and to flow out at a rate proportional to $V^2$. Suppose $V(0) = V_0$ and the constants of proportionality are $a$ and $b$, respectively. Find $V(t)$ for $t > 0$ and find $\lim_{t\to\infty} V(t)$.

19. Identical tanks $T_1$ and $T_2$ initially contain $W$ gallons each of pure water. Starting at $t_0 = 0$, a salt solution with constant concentration $c$ is pumped into $T_1$ at $r$ gal/min and drained from $T_1$ into $T_2$ at the same rate. The resulting mixture in $T_2$ is also drained at the same rate. Find the concentrations $c_1(t)$ and $c_2(t)$ in tanks $T_1$ and $T_2$ for $t > 0$.

20. An infinite sequence of identical tanks $T_1$, $T_2$, …, $T_n$, …, initially contain $W$ gallons each of pure water. They are hooked together so that fluid drains from $T_n$ into $T_{n+1}$ $(n = 1, 2, \cdots)$. A salt solution is circulated through the tanks so that it enters and leaves each tank at the constant rate of $r$ gal/min. The solution has a concentration of $c$ pounds of salt per gallon when it enters $T_1$.

a. Find the concentration $c_n(t)$ in tank $T_n$ for $t > 0$.
b. Find $\lim_{t\to\infty} c_n(t)$ for each $n$.

21. Tanks $T_1$ and $T_2$ have capacities $W_1$ and $W_2$ liters, respectively. Initially they are both full of dye solutions with concentrations $c_1$ and $c_2$ grams per liter. Starting at $t_0 = 0$, the solution from $T_1$ is pumped into $T_2$ at a rate of $r$ liters per minute, and the solution from $T_2$ is pumped into $T_1$ at the same rate.

a. Find the concentrations $c_1(t)$ and $c_2(t)$ of the dye in $T_1$ and $T_2$ for $t > 0$.
b. Find $\lim_{t\to\infty} c_1(t)$ and $\lim_{t\to\infty} c_2(t)$.

22. Consider the mixing problem of Example 4.2.3, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \to \infty$. In this

case the differential equation for $Q$ is of the form

$$Q' + \frac{a(t)}{150}Q = 2$$

where $\lim_{t\to\infty} a(t) = 1$.

a. Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t\to\infty} Q(t)$?.
b. Use numerical methods to confirm your guess in the these cases:

$$\text{(i) } a(t) = t/(1+t) \quad \text{(ii) } a(t) = 1 - e^{-t^2} \quad \text{(iii) } a(t) = 1 - \sin(e^{-t}).$$

23. Consider the mixing problem of Example 4.2.4 in a tank with infinite capacity, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \to \infty$. In this case the differential equation for $Q$ is of the form

$$Q' + \frac{a(t)}{t+100}Q = 1$$

where $\lim_{t\to\infty} a(t) = 1$.

a. Let $K(t)$ be the concentration of salt at time $t$. Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t\to\infty} K(t)$?
b. Use numerical methods to confirm your guess in the these cases:

$$\text{(i) } a(t) = t/(1+t) \quad \text{(ii) } a(t) = 1 - e^{-t^2} \quad \text{(iii) } a(t) = 1 + \sin(e^{-t}).$$

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# 4.3: Elementary Mechanics

## Newton's Second Law of Motion

In this section we consider an object with constant mass $m$ moving along a line under a force $F$. Let $y = y(t)$ be the displacement of the object from a reference point on the line at time $t$, and let $v = v(t)$ and $a = a(t)$ be the velocity and acceleration of the object at time $t$. Thus, $v = y'$ and $a = v' = y''$, where the prime denotes differentiation with respect to $t$. Newton's second law of motion asserts that the force $F$ and the acceleration $a$ are related by the equation

$$F = ma. \tag{4.3.1}$$

**Note: Units**

In applications there are three main sets of units in use for length, mass, force, and time: the cgs, mks, and British systems. All three use the second as the unit of time. Table 4.3.1 shows the other units. Consistent with Equation 4.3.1, the unit of force in each system is defined to be the force required to impart an acceleration of (one unit of length)$/s^2$ to one unit of mass.

Table 4.3.1

| Set | Length | Force | Mass |
|---|---|---|---|
| cgs | centimeter (cm) | dyne (d) | gram (g) |
| mks | meter (m) | newton (N) | kilogram (kg) |
| British | foot (ft) | pound (lb) | slug (sl) |

If we assume that Earth is a perfect sphere with constant mass density, Newton's law of gravitation (discussed later in this section) asserts that the force exerted on an object by Earth's gravitational field is proportional to the mass of the object and inversely proportional to the square of its distance from the center of Earth. However, if the object remains sufficiently close to Earth's surface, we may assume that the gravitational force is constant and equal to its value at the surface. The magnitude of this force is $mg$, where $g$ is called the **acceleration due to gravity**. (To be completely accurate, $g$ should be called the **magnitude of the acceleration due to gravity at Earth's surface**.) This quantity has been determined experimentally. Approximate values of $g$ are

$$\begin{aligned} g &= 980 \text{ cm/s}^2 && \text{(cgs)} \\ g &= 9.8 \text{ m/s}^2 && \text{(mks)} \\ g &= 32 \text{ ft/s}^2 && \text{(British).} \end{aligned}$$

In general, the force $F$ in Equation 4.3.1 may depend upon $t$, $y$, and $y'$. Since $a = y''$, Equation 4.3.1 can be written in the form

$$my'' = F(t, y, y'), \tag{4.3.2}$$

which is a second order equation. We'll consider this equation with restrictions on $F$ later; however, since Chapter 2 dealt only with first order equations, we consider here only problems in which Equation 4.3.2 can be recast as a first order equation. This is possible if $F$ does not depend on $y$, so Equation 4.3.2 is of the form

$$my'' = F(t, y').$$

Letting $v = y'$ and $v' = y''$ yields a first order equation for $v$:

$$mv' = F(t, v). \tag{4.3.3}$$

Solving this equation yields $v$ as a function of $t$. If we know $y(t_0)$ for some time $t_0$, we can integrate $v$ to obtain $y$ as a function of $t$.

Equations of the form Equation 4.3.3 occur in problems involving motion through a resisting medium.

## Motion Through a Resisting Medium Under Constant Gravitational Force

Now we consider an object moving vertically in some medium. We assume that the only forces acting on the object are gravity and resistance from the medium. We also assume that the motion takes place close to Earth's surface and take the upward direction to be positive, so the gravitational force can be assumed to have the constant value $-mg$. We'll see that, under reasonable assumptions on the resisting force, the velocity approaches a limit as $t \to \infty$. We call this limit the **terminal velocity**.

### Example 4.3.1

An object with mass $m$ moves under constant gravitational force through a medium that exerts a resistance with magnitude proportional to the speed of the object. (Recall that the speed of an object is $|v|$, the absolute value of its velocity $v$.) Find the velocity of the object as a function of $t$, and find the terminal velocity. Assume that the initial velocity is $v_0$.

**Solution**

The total force acting on the object is

$$F = -mg + F_1, \tag{4.3.4}$$

where $-mg$ is the force due to gravity and $F_1$ is the resisting force of the medium, which has magnitude $k|v|$, where $k$ is a positive constant. If the object is moving downward ($v \le 0$), the resisting force is upward (Figure 4.3.1a ), so

$$F_1 = k|v| = k(-v) = -kv.$$

On the other hand, if the object is moving upward ($v \ge 0$), the resisting force is downward (Figure 4.3.1b , so

$$F_1 = -k|v| = -kv.$$

Thus, Equation 4.3.4 can be written as

$$F = -mg - kv, \tag{4.3.5}$$

regardless of the sign of the velocity.

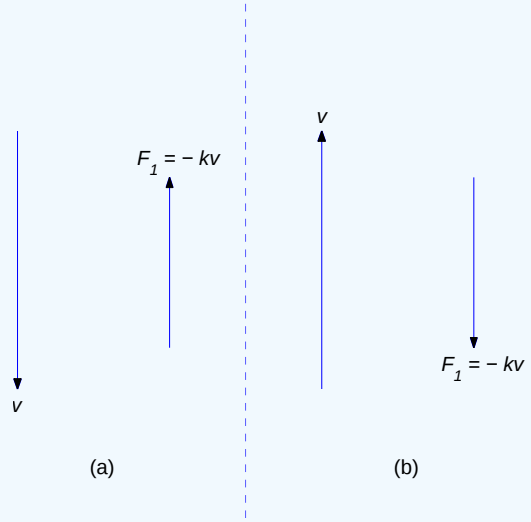



Figure 4.3.1 : Resistive forces

From Newton's second law of motion,

$$F = ma = mv',$$

so Equation 4.3.5 yields

$$mv' = -mg - kv,$$

or

$$v' + \frac{k}{m}v = -g. \tag{4.3.6}$$

Since $e^{-kt/m}$ is a solution of the complementary equation, the solutions of Equation 4.3.6 are of the form $v = ue^{-kt/m}$, where $u'e^{-kt/m} = -g$, so $u' = -ge^{kt/m}$. Hence,

$$u = -\frac{mg}{k}e^{kt/m} + c,$$

so

$$v = ue^{-kt/m} = -\frac{mg}{k} + ce^{-kt/m}. \tag{4.3.7}$$

Since $v(0) = v_0$,

$$v_0 = -\frac{mg}{k} + c,$$

so

$$c = v_0 + \frac{mg}{k}$$

and Equation 4.3.7 becomes

$$v = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}.$$

Letting $t \to \infty$ here shows that the terminal velocity is

$$\lim_{t\to\infty} v(t) = -\frac{mg}{k},$$

which is independent of the initial velocity $v_0$ (Figure 4.3.2 ).

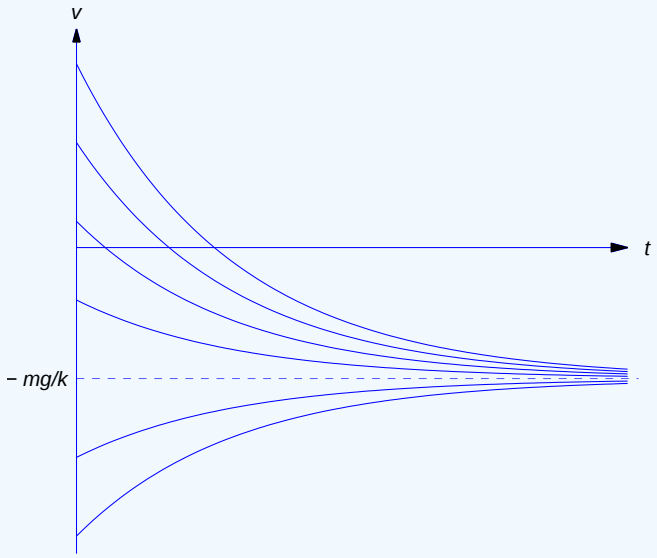



Figure 4.3.2 : Solutions of $mv' = -mg - kv$

## Example 4.3.2 : Terminal Velocity

A 960-lb object is given an initial upward velocity of 60 ft/s near the surface of Earth. The atmosphere resists the motion with a force of 3 lb for each ft/s of speed. Assuming that the only other force acting on the object is constant gravity, find its velocity $v$ as a function of $t$, and find its terminal velocity.

### Solution

Since $mg = 960$ and $g = 32$, $m = 960/32 = 30$. The atmospheric resistance is $-3v$ lb if $v$ is expressed in feet per second. Therefore

$$30v' = -960 - 3v,$$

which we rewrite as

$$v' + \frac{1}{10}v = -32.$$

Since $e^{-t/10}$ is a solution of the complementary equation, the solutions of this equation are of the form $v = ue^{-t/10}$, where $u'e^{-t/10} = -32$, so $u' = -32e^{t/10}$. Hence,

$$u = -320e^{t/10} + c,$$

so

$$v = ue^{-t/10} = -320 + ce^{-t/10}. \tag{4.3.8}$$

The initial velocity is 60 ft/s in the upward (positive) direction; hence, $v_0 = 60$. Substituting $t = 0$ and $v = 60$ in Equation 4.3.8 yields

$$60 = -320 + c,$$

so $c = 380$, and Equation 4.3.8 becomes

$$v = -320 + 380e^{-t/10} \text{ ft/s}$$

The terminal velocity is

$$\lim_{t\to\infty} v(t) = -320 \text{ ft/s}.$$

## Example 4.3.3

A 10 kg mass is given an initial velocity $v_0 \le 0$ near Earth's surface. The only forces acting on it are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the resistance is 8 N if the speed is 2 m/s, find the velocity of the object as a function of $t$, and find the terminal velocity.

### Solution

Since the object is falling, the resistance is in the upward (positive) direction. Hence,

$$mv' = -mg + kv^2, \tag{4.3.9}$$

where $k$ is a constant. Since the magnitude of the resistance is 8 N when $v = 2$ m/s,

$$k(2^2) = 8,$$

so $k = 2$ N-s$^2$/m$^2$. Since $m = 10$ and $g = 9.8$, Equation 4.3.9 becomes

$$10v' = -98 + 2v^2 = 2(v^2 - 49). \tag{4.3.10}$$

If $v_0 = -7$, then $v \equiv -7$ for all $t \ge 0$. If $v_0 \ne -7$, we separate variables to obtain

$$\frac{1}{v^2 - 49}v' = \frac{1}{5}, \tag{4.3.11}$$

which is convenient for the required partial fraction expansion

$$\frac{1}{v^2 - 49} = \frac{1}{(v-7)(v+7)} = \frac{1}{14}\left[\frac{1}{v-7} - \frac{1}{v+7}\right]. \tag{4.3.12}$$

Substituting Equation 4.3.12 into Equation 4.3.11 yields

$$\frac{1}{14}\left[\frac{1}{v-7} - \frac{1}{v+7}\right]v' = \frac{1}{5},$$

so

$$\left[\frac{1}{v-7}-\frac{1}{v+7}\right]v'=\frac{14}{5}.$$

Integrating this yields

$$\ln|v-7|-\ln|v+7|=14t/5+k.$$

Therefore

$$\left|\frac{v-7}{v+7}\right|=e^k e^{14t/5}.$$

Since Theorem 2.3.1 implies that $(v-7)/(v+7)$ cannot change sign (why?), we can rewrite the last equation as

$$\frac{v-7}{v+7}=ce^{14t/5}, \tag{4.3.13}$$

which is an implicit solution of Equation 4.3.10. Solving this for $v$ yields

$$v=-7\frac{c+e^{-14t/5}}{c-e^{-14t/5}}. \tag{4.3.14}$$

Since $v(0)=v_0$, it Equation 4.3.13 implies that

$$c=\frac{v_0-7}{v_0+7}.$$

Substituting this into Equation 4.3.14 and simplifying yields

$$v=-7\frac{v_0(1+e^{-14t/5})-7(1-e^{-14t/5})}{v_0(1-e^{-14t/5})-7(1+e^{-14t/5})}.$$

Since $v_0\le 0$, $v$ is defined and negative for all $t>0$. The terminal velocity is

$$\lim_{t\to\infty}v(t)=-7\text{ m/s},$$

independent of $v_0$. More generally, it can be shown (**Exercise 4.3.11**) that if $v$ is any solution of Equation 4.3.9 such that $v_0\le 0$ then

$$\lim_{t\to\infty}v(t)=-\sqrt{\frac{mg}{k}}.$$

This is demonstrated in Figure 4.3.3 .

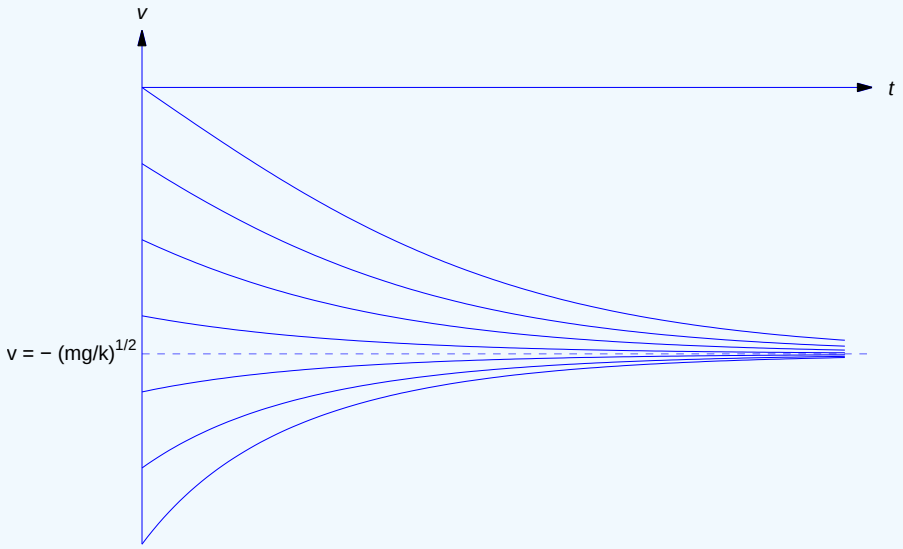



Figure 4.3.3 : Solutions of $mv'=-mg+kv^2,\ v(0)=v_0\le 0$


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## Example 4.3.4

A 10-kg mass is launched vertically upward from Earth's surface with an initial velocity of $v_0$ m/s. The only forces acting on the mass are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the atmospheric resistance is 8 N if the speed is 2 m/s, find the time $T$ required for the mass to reach maximum altitude.

### Solution

The mass will climb while $v > 0$ and reach its maximum altitude when $v = 0$. Therefore $v > 0$ for $0 \le t < T$ and $v(T) = 0$; therefore, we replace Equation 4.3.10 by

$$10v' = -98 - 2v^2. \tag{4.3.15}$$

Separating variables yields

$$\frac{5}{v^2 + 49} v' = -1,$$

and integrating this yields

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + c.$$

(Recall that $\tan^{-1} u$ is the number $\theta$ such that $-\pi/2 < \theta < \pi/2$ and $\tan\theta = u$.) Since $v(0) = v_0$,

$$c = \frac{5}{7} \tan^{-1} \frac{v_0}{7},$$

so $v$ is defined implicitly by

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + \frac{5}{7} \tan^{-1} \frac{v_0}{7}, \quad 0 \le t \le T. \tag{4.3.16}$$

Solving this for $v$ yields

$$v = 7 \tan\left( -\frac{7t}{5} + \tan^{-1} \frac{v_0}{7} \right). \tag{4.3.17}$$

Using the identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

with $A = \tan^{-1}(v_0/7)$ and $B = 7t/5$, and noting that $\tan(\tan^{-1}\theta) = \theta$, we can simplify Equation 4.3.17 to

$$v = 7 \frac{v_0 - 7\tan(7t/5)}{7 + v_0 \tan(7t/5)}.$$

Since $v(T) = 0$ and $\tan^{-1}(0) = 0$, Equation 4.3.16 implies that

$$-T + \frac{5}{7} \tan^{-1} \frac{v_0}{7} = 0.$$

Therefore

$$T = \frac{5}{7} \tan^{-1} \frac{v_0}{7}.$$

Since $\tan^{-1}(v_0/7) < \pi/2$ for all $v_0$, the time required for the mass to reach its maximum altitude is less than

$$\frac{5\pi}{14} \approx 1.122 \text{ s}$$

regardless of the initial velocity. Figure 4.3.4 shows graphs of $v$ over $[0, T]$ for various values of $v_0$.

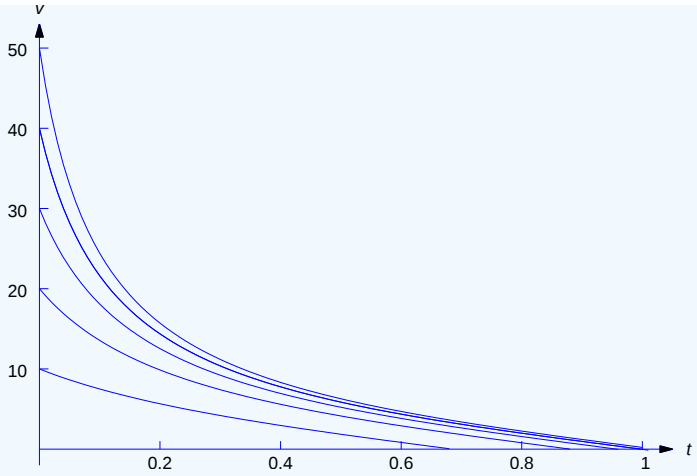


Figure 4.3.4 : Solutions of 4.3.15 for various $v_0 > 0$

## Escape Velocity

Suppose a space vehicle is launched vertically and its fuel is exhausted when the vehicle reaches an altitude $h$ above Earth, where $h$ is sufficiently large so that resistance due to Earth's atmosphere can be neglected. Let $t = 0$ be the time when burnout occurs. Assuming that the gravitational forces of all other celestial bodies can be neglected, the motion of the vehicle for $t > 0$ is that of an object with constant mass $m$ under the influence of Earth's gravitational force, which we now assume to vary inversely with the square of the distance from Earth's center; thus, if we take the upward direction to be positive then gravitational force on the vehicle at an altitude $y$ above Earth is

$$F = -\frac{K}{(y+R)^2}, \tag{4.3.18}$$

where $R$ is Earth's radius (Figure 4.3.5 ).

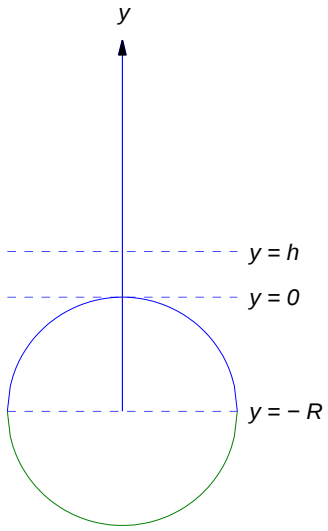


Figure 4.3.5 : Escape velocity

Since $F = -mg$ when $y = 0$, setting $y = 0$ in Equation 4.3.18 yields

$$-mg = -\frac{K}{R^2};$$

therefore $K = mgR^2$ and Equation 4.3.18 can be written more specifically as

$$F = -\frac{mgR^2}{(y+R)^2}. \tag{4.3.19}$$

From Newton's second law of motion,

$$F = m\frac{d^2y}{dt^2},$$

so Equation 4.3.19 implies that

$$\frac{d^2y}{dt^2} = -\frac{gR^2}{(y+R)^2}. \tag{4.3.20}$$

We'll show that there's a number $v_e$, called the **escape velocity**, with these properties:

1. If $v_0 \geq v_e$ then $v(t) > 0$ for all $t > 0$, and the vehicle continues to climb for all $t > 0$; that is, it "escapes" Earth. (Is it really so obvious that $\lim_{t\to\infty} y(t) = \infty$ in this case? For a proof, see **Exercise 4.3.20**.)
2. If $v_0 < v_e$ then $v(t)$ decreases to zero and becomes negative. Therefore the vehicle attains a maximum altitude $y_m$ and falls back to Earth.

Since Equation 4.3.20 is second order, we cannot solve it by methods discussed so far. However, we are concerned with $v$ rather than $y$, and $v$ is easier to find. Since $v = y'$ the chain rule implies that

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Substituting this into Equation 4.3.20 yields the first order separable equation

$$v\frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}. \tag{4.3.21}$$

When $t = 0$, the velocity is $v_0$ and the altitude is $h$. Therefore we can obtain $v$ as a function of $y$ by solving the initial value problem

$$v\frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}, \quad v(h) = v_0.$$

Integrating Equation 4.3.21 with respect to $y$ yields

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + c. \tag{4.3.22}$$

Since $v(h) = v_0$,

$$c = \frac{v_0^2}{2} - \frac{gR^2}{h+R},$$

so Equation 4.3.22 becomes

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right). \tag{4.3.23}$$

If

$$v_0 \geq \left(\frac{2gR^2}{h+R}\right)^{1/2},$$

the parenthetical expression in Equation 4.3.23 is nonnegative, so $v(y) > 0$ for $y > h$. This proves that there's an escape velocity $v_e$. We'll now prove that

$$v_e = \left(\frac{2gR^2}{h+R}\right)^{1/2}$$

by showing that the vehicle falls back to Earth if

$$v_0 < \left( \frac{2gR^2}{h+R} \right)^{1/2}. \tag{4.3.24}$$

If Equation 4.3.24 holds then the parenthetical expression in Equation 4.3.23 is negative and the vehicle will attain a maximum altitude $y_m > h$ that satisfies the equation

$$0 = \frac{gR^2}{y_m + R} + \left( \frac{v_0^2}{2} - \frac{gR^2}{h+R} \right).$$

The velocity will be zero at the maximum altitude, and the object will then fall to Earth under the influence of gravity.

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# 4.3E: Elementary Mechanics (Exercises)

Except where directed otherwise, assume that the magnitude of the gravitational force on an object with mass $m$ is constant and equal to $mg$. In exercises involving vertical motion take the upward direction to be positive.

## Q4.3.1

1. A firefighter who weighs $192$ lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to his speed, with $k = 2.5$ lb-s/ft. Assuming that he starts from rest, find his velocity as a function of time and find his terminal velocity.

2. A firefighter who weighs $192$ lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to her speed, with constant of proportionality $k$. Find $k$, given that her terminal velocity is $-16$ ft/s, and then find her velocity $v$ as a function of $t$. Assume that she starts from rest.

3. A boat weighs $64,000$ lb. Its propellor produces a constant thrust of $50,000$ lb and the water exerts a resistive force with magnitude proportional to the speed, with $k = 2000$ lb-s/ft. Assuming that the boat starts from rest, find its velocity as a function of time, and find its terminal velocity.

4. A constant horizontal force of $10$ N pushes a $20$ kg-mass through a medium that resists its motion with $.5$ N for every m/s of speed. The initial velocity of the mass is $7$ m/s in the direction opposite to the direction of the applied force. Find the velocity of the mass for $t > 0$.

5. A stone weighing $1/2$ lb is thrown upward from an initial height of $5$ ft with an initial speed of $32$ ft/s. Air resistance is proportional to speed, with $k = 1/128$ lb-s/ft. Find the maximum height attained by the stone.

6. A $3200$-lb car is moving at $64$ ft/s down a $30$-degree grade when it runs out of fuel. Find its velocity after that if friction exerts a resistive force with magnitude proportional to the square of the speed, with $k = 1$ lb-s$^2/$ft$^2$. Also find its terminal velocity.

7. A $96$ lb weight is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the speed. Find its velocity as a function of time if its terminal velocity is $-128$ ft/s.

8. An object with mass $m$ moves vertically through a medium that exerts a resistive force with magnitude proportional to the speed. Let $y = y(t)$ be the altitude of the object at time $t$, with $y(0) = y_0$. Use the results of Example 4.3.1 to show that

$$y(t) = y_0 + \frac{m}{k}(v_0 - v - gt).$$

9. An object with mass $m$ is launched vertically upward with initial velocity $v_0$ from Earth's surface ($y_0 = 0$) in a medium that exerts a resistive force with magnitude proportional to the speed. Find the time $T$ when the object attains its maximum altitude $y_m$. Then use the result of **Exercise 4.3.8** to find $y_m$.

10. An object weighing $256$ lb is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the square of the speed. The magnitude of the resisting force is $1$ lb when $|v| = 4$ ft/s. Find $v$ for $t > 0$, and find its terminal velocity.

11. An object with mass $m$ is given an initial velocity $v_0 \le 0$ in a medium that exerts a resistive force with magnitude proportional to the square of the speed. Find the velocity of the object for $t > 0$, and find its terminal velocity.

12. An object with mass $m$ is launched vertically upward with initial velocity $v_0$ in a medium that exerts a resistive force with magnitude proportional to the square of the speed.

a. Find the time $T$ when the object reaches its maximum altitude.
b. Use the result of **Exercise 4.3.11** to find the velocity of the object for $t > T$.

13. An object with mass $m$ is given an initial velocity $v_0 \le 0$ in a medium that exerts a resistive force of the form $a|v|/(1 + |v|)$, where $a$ is positive constant.

a. Set up a differential equation for the speed of the object.
b. Use your favorite numerical method to solve the equation you found in (a), to convince yourself that there's a unique number $a_0$ such that $\lim_{t\to\infty} s(t) = \infty$ if $a \le a_0$ and $\lim_{t\to\infty} s(t)$ exists (finite) if $a > a_0$. (We say that $a_0$ is the **bifurcation value** of $a$.) Try to find $a_0$ and $\lim_{t\to\infty} s(t)$ in the case where $a > a_0$.

14. An object of mass $m$ falls in a medium that exerts a resistive force $f = f(s)$, where $s = |v|$ is the speed of the object. Assume that $f(0) = 0$ and $f$ is strictly increasing and differentiable on $(0, \infty)$.

a. Write a differential equation for the speed $s = s(t)$ of the object. Take it as given that all solutions of this equation with $s(0) \geq 0$ are defined for all $t > 0$ (which makes good sense on physical grounds).
b. Show that if $\lim_{s\to\infty} f(s) \leq mg$ then $\lim_{t\to\infty} s(t) = \infty$.
c. Show that if $\lim_{s\to\infty} f(s) > mg$ then $\lim_{t\to\infty} s(t) = s_T$ (terminal speed), where $f(s_T) = mg$..

15. A 100-g mass with initial velocity $v_0 \leq 0$ falls in a medium that exerts a resistive force proportional to the fourth power of the speed. The resistance is .1 N if the speed is 3 m/s.

a. Set up the initial value problem for the velocity $v$ of the mass for $t > 0$.
b. Use **Exercise 4.3.14** (c) to determine the terminal velocity of the object.
c. To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 1]$ (seconds) of the initial value problem of (a) , with initial values $v_0 = 0$, $-2$, $-4$, …, $-12$. Present your results in graphical form similar to Figure 4.3.3.

16. A 64-lb object with initial velocity $v_0 \leq 0$ falls through a dense fluid that exerts a resistive force proportional to the square root of the speed. The resistance is 64 lb if the speed is 16 ft/s.

a. Set up the initial value problem for the velocity $v$ of the mass for $t > 0$.
b. Use **Exercise 4.3.14** (c) to determine the terminal velocity of the object.
c. To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 4]$ (seconds) of the initial value problem of (a), with initial values $v_0 = 0$, $-5$, $-10$, …, $-30$. Present your results in graphical form similar to Figure 4.3.3.

## Q4.3.2

In **Exercises 4.3.17-4.3.20**, assume that the force due to gravity is given by Newton's law of gravitation. Take the upward direction to be positive.

17. A space probe is to be launched from a space station 200 miles above Earth. Determine its escape velocity in miles/s. Take Earth's radius to be 3960 miles.

18. A space vehicle is to be launched from the moon, which has a radius of about 1080 miles. The acceleration due to gravity at the surface of the moon is about 5.31 ft/s$^2$. Find the escape velocity in miles/s.

19.

a. Show that (Equation 4.3.27) can be rewritten as

$$v^2 = \frac{h - y}{y + R} v_e^2 + v_0^2.$$

b. Show that if $v_0 = \rho v_e$ with $0 \leq \rho < 1$, then the maximum altitude $y_m$ attained by the space vehicle is

$$y_m = \frac{h + R\rho^2}{1 - \rho^2}.$$

c. By requiring that $v(y_m) = 0$, use (Equation 4.3.26) to deduce that if $v_0 < v_e$ then

$$|v| = v_e \left[ \frac{(1 - \rho^2)(y_m - y)}{y + R} \right]^{1/2},$$

where $y_m$ and $\rho$ are as defined in (b) and $y \geq h$.
d. Deduce from (c) that if $v < v_e$, the vehicle takes equal times to climb from $y = h$ to $y = y_m$ and to fall back from $y = y_m$ to $y = h$.

20. In the situation considered in the discussion of escape velocity, show that $\lim_{t\to\infty} y(t) = \infty$ if $v(t) > 0$ for all $t > 0$.

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# 4.4: Autonomous Second Order Equations

A second order differential equation that can be written as

$$y'' = F(y, y') \tag{4.4.1}$$

where $F$ is independent of $t$, is said to be **autonomous**. An autonomous second order equation can be converted into a first order equation relating $v = y'$ and $y$. If we let $v = y'$, Equation 4.4.1 becomes

$$v' = F(y, v). \tag{4.4.2}$$

Since

$$v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}, \tag{4.4.3}$$

Equation 4.4.2 can be rewritten as

$$v\frac{dv}{dy} = F(y, v). \tag{4.4.4}$$

The integral curves of Equation 4.4.4 can be plotted in the $(y, v)$ plane, which is called the **Poincaré phase plane** of Equation 4.4.1. If $y$ is a solution of Equation 4.4.1 then $y = y(t), v = y'(t)$ is a parametric equation for an integral curve of Equation 4.4.4. we will call these integral curves **trajectories** of Equation 4.4.1, and we will call Equation 4.4.4 the **phase plane equivalent** of Equation 4.4.1.

In this section we will consider autonomous equations that can be written as

$$y'' + q(y, y')y' + p(y) = 0. \tag{4.4.5}$$

Equations of this form often arise in applications of Newton's second law of motion. For example, suppose $y$ is the displacement of a moving object with mass $m$. It's reasonable to think of two types of time-independent forces acting on the object. One type - such as gravity - depends only on position. We could write such a force as $-mp(y)$. The second type - such as atmospheric resistance or friction - may depend on position and velocity. (Forces that depend on velocity are called **damping** forces.) We write this force as $-mq(y, y')y'$, where $q(y, y')$ is usually a positive function and we've put the factor $y'$ outside to make it explicit that the force is in the direction opposing the motion. In this case Newton's, second law of motion leads to Equation 4.4.5.

The phase plane equivalent of Equation 4.4.5 is

$$v\frac{dv}{dy} + q(y, v)v + p(y) = 0. \tag{4.4.6}$$

Some statements that we will be making about the properties of Equation 4.4.5 and Equation 4.4.6 are intuitively reasonable, but difficult to prove. Therefore our presentation in this section will be informal: we will just say things without proof, all of which are true if we assume that $p = p(y)$ is continuously differentiable for all $y$ and $q = q(y, v)$ is continuously differentiable for all $(y, v)$. We begin with the following statements:

**Properties**

- **Statement 1:** If $y_0$ and $v_0$ are arbitrary real numbers then Equation 4.4.5 has a unique solution on $(-\infty, \infty)$ such that $y(0) = y_0$ and $y'(0) = v_0$.
- **Statement 2:** If $y = y(t)$ is a solution of Equation 4.4.5 and $\tau$ is any constant then $y_1 = y(t - \tau)$ is also a solution of Equation 4.4.5, and $y$ and $y_1$ have the same trajectory.
- **Statement 3:** If two solutions $y$ and $y_1$ of Equation 4.4.5 have the same trajectory then $y_1(t) = y(t - \tau)$ for some constant $\tau$.
- **Statement 4:** Distinct trajectories of Equation 4.4.5 cannot intersect; that is, if two trajectories of Equation 4.4.5 intersect, they are identical.
- **Statement 5:** If the trajectory of a solution of Equation 4.4.5 is a closed curve then $(y(t), v(t))$ traverses the trajectory in a finite time $T$, and the solution is periodic with period $T$; that is, $y(t + T) = y(t)$ for all $t$ in $(-\infty, \infty)$.

If $\overline{y}$ is a constant such that $p(\overline{y}) = 0$ then $y \equiv \overline{y}$ is a constant solution of Equation 4.4.5. We say that $\overline{y}$ is an **equilibrium** of Equation 4.4.5 and $(\overline{y}, 0)$ is a **critical point** of the phase plane equivalent equation Equation 4.4.6. We say that the equilibrium and the critical point are **stable** if, for any given $\epsilon > 0$ **no matter how small**, there's a $\delta > 0$, **sufficiently small**, such that if

$$\sqrt{(y_0 - \overline{y})^2 + v_0^2} < \delta$$

then the solution of the initial value problem

$$y'' + q(y, y')y' + p(y) = 0, \quad y(0) = y_0, \quad y'(0) = v_0$$

satisfies the inequality

$$\sqrt{(y(t) - \overline{y})^2 + (v(t))^2} < \epsilon$$

for all $t > 0$. Figure 4.4.1 illustrates the geometrical interpretation of this definition in the Poincaré phase plane: if $(y_0, v_0)$ is in the smaller shaded circle (with radius $\delta$), then $(y(t), v(t))$ must be in in the larger circle (with radius $\epsilon$) for all $t > 0$.

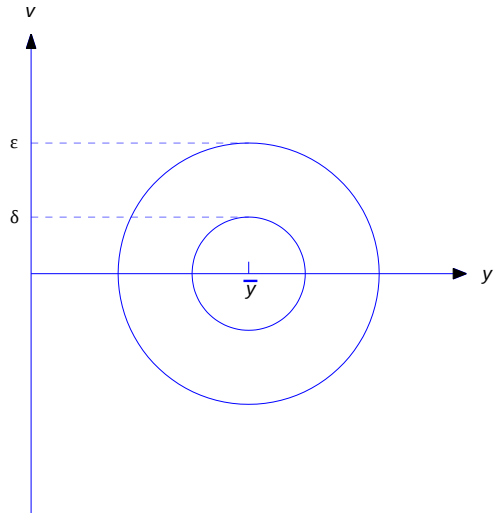



Figure 4.4.1 : Stability: if $(y_0, v_0)$ is in the smaller circle then $(y(t), v(t))$is in the larger circle for all $t > 0$

If an equilibrium and the associated critical point are not stable, we say they are **unstable**. To see if you really understand what **stable** means, try to give a direct definition of **unstable** (**Exercise 4.4.22**). we will illustrate both definitions in the following examples.

## The Undamped Case

We will begin with the case where $q \equiv 0$, so Equation 4.4.5 reduces to

$$y'' + p(y) = 0. \tag{4.4.7}$$

We say that this equation - as well as any physical situation that it may model - is **undamped**. The phase plane equivalent of Equation 4.4.7 is the separable equation

$$v\frac{dv}{dy} + p(y) = 0.$$

Integrating this yields

$$\frac{v^2}{2} + P(y) = c, \tag{4.4.8}$$

where $c$ is a constant of integration and $P(y) = \int p(y)\, dy$ is an antiderivative of $p$.

If Equation 4.4.7 is the equation of motion of an object of mass $m$, then $mv^2/2$ is the kinetic energy and $mP(y)$ is the potential energy of the object; thus, Equation 4.4.8 says that the total energy of the object remains constant, or is **conserved**. In particular, if

a trajectory passes through a given point $(y_0, v_0)$ then

$$c = \frac{v_0^2}{2} + P(y_0).$$

## Example 4.4.1 : The Undamped Spring - Mass System

Consider an object with mass $m$ suspended from a spring and moving vertically. Let $y$ be the displacement of the object from the position it occupies when suspended at rest from the spring (Figure 4.4.2 ).

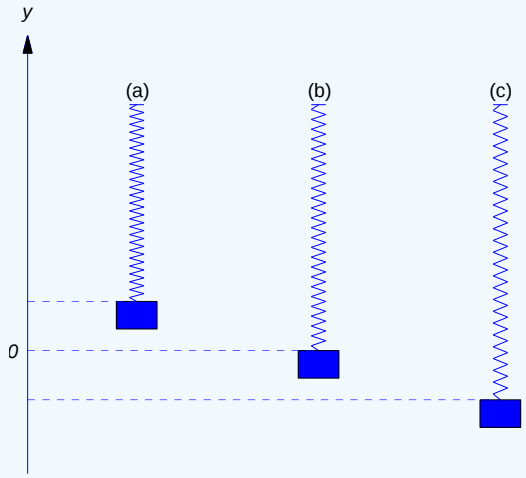



Figure 4.4.2 : (a) $y > 0$ (b) $y = 0$ (c) $y < 0$.

Assume that if the length of the spring is changed by an amount $\Delta L$ (positive or negative), then the spring exerts an opposing force with magnitude $k|\Delta L|$, where k is a positive constant. In Section 6.1, it will be shown that if the mass of the spring is negligible compared to $m$ and no other forces act on the object then Newton's second law of motion implies that

$$my'' = -ky, \tag{4.4.9}$$

which can be written in the form Equation 4.4.7 with $p(y) = ky/m$. This equation can be solved easily by a method that we will study in Section 5.2, but that method isn't available here. Instead, we will consider the phase plane equivalent of Equation 4.4.9.

From Equation 4.4.3, we can rewrite Equation 4.4.9 as the separable equation

$$mv\frac{dv}{dy} = -ky.$$

Integrating this yields

$$\frac{mv^2}{2} = -\frac{ky^2}{2} + c,$$

which implies that

$$mv^2 + ky^2 = \rho \tag{4.4.10}$$

($\rho = 2c$). This defines an ellipse in the Poincaré phase plane (Figure 4.4.3 ).

We can identify $\rho$ by setting $t = 0$ in Equation 4.4.10; thus, $\rho = mv_0^2 + ky_0^2$ , where $y_0 = y(0)$ and $v_0 = v(0)$. To determine the maximum and minimum values of $y$ we set $v = 0$ in Equation 4.4.10; thus,

$$y_{\max} = R \quad \text{and} \quad y_{\min} = -R, \quad \text{with} R = \sqrt{\frac{\rho}{k}}. \tag{4.4.11}$$

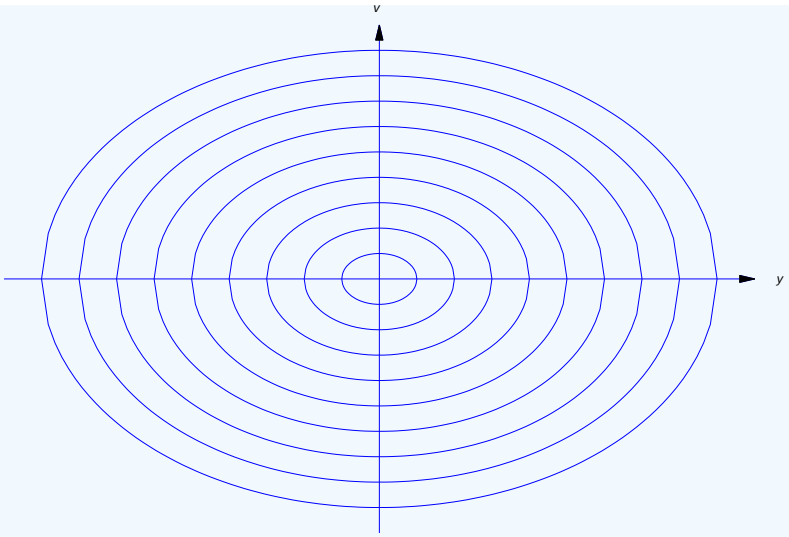



Figure 4.4.3 : Trajectories of $my'' + ky = 0$

Equation 4.4.9 has exactly one equilibrium, $\bar{y} = 0$, and it is stable. You can see intuitively why this is so: if the object is displaced in either direction from equilibrium, the spring tries to bring it back.

In this case we can find $y$ explicitly as a function of $t$. (Don't expect this to happen in more complicated problems!) If $v > 0$ on an interval $I$, Equation 4.4.10 implies that

$$\frac{dy}{dt} = v = \sqrt{\frac{\rho - ky^2}{m}}$$

on $I$. This is equivalent to

$$\frac{\sqrt{k}}{\sqrt{\rho - ky^2}}\frac{dy}{dt} = \omega_0, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{4.4.12}$$

Since

$$\begin{aligned}\int \frac{\sqrt{k}\,dy}{\sqrt{\rho - ky^2}} &= \sin^{-1}\left(\sqrt{\frac{k}{\rho}}y\right) + c \\ &= \sin^{-1}\left(\frac{y}{R}\right) + c\end{aligned}$$

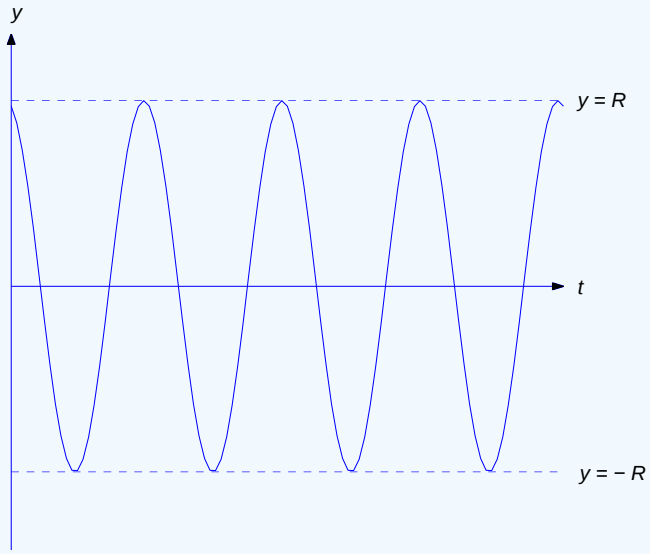



Figure 4.4.4 : $y = R\sin(\omega_0 t + \phi)$

(see Equation 4.4.11), Equation 4.4.12 implies that that there's a constant $\phi$ such that

$$\sin^{-1}\left(\frac{y}{R}\right) = \omega_0 t + \phi$$

or

$$y = R\sin(\omega_0 t + \phi)$$

for all $t$ in $I$. Although we obtained this function by assuming that $v > 0$, you can easily verify that $y$ satisfies Equation 4.4.9 for all values of $t$. Thus, the displacement varies periodically between $-R$ and $R$, with period $T = 2\pi/\omega_0$ (Figure 4.4.4 ). (If you've taken a course in elementary mechanics you may recognize this as **simple harmonic motion**.)

## Example 4.4.2 : The Undamped Pendulum

Now we consider the motion of a pendulum with mass $m$, attached to the end of a weightless rod with length $L$ that rotates on a frictionless axle (Figure 4.4.5 ). We assume that there's no air resistance.

Let $y$ be the angle measured from the rest position (vertically downward) of the pendulum, as shown in Figure 4.4.5 .

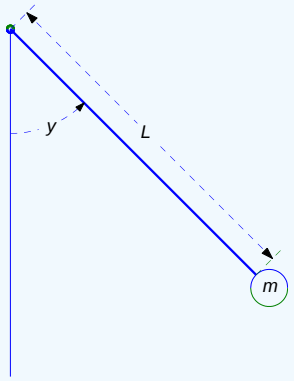



Figure 4.4.5 : The undamped pendulum

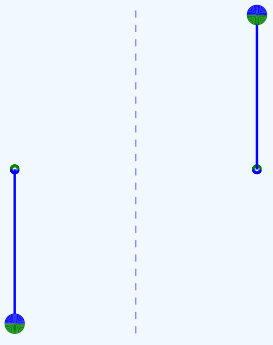
Newton's second law of motion says that the product of $m$ and the tangential acceleration equals the tangential component of the gravitational force; therefore, from Figure 4.4.5 ,

$$mLy'' = -mg\sin y,$$

or

$$y'' = -\frac{g}{L}\sin y. \tag{4.4.13}$$

Since $\sin n\pi = 0$ if $n$ is any integer, Equation 4.4.13 has infinitely many equilibria $\overline{y}_n = n\pi$. If $n$ is even, the mass is directly below the axle (Figure 4.4.6*a*) and gravity opposes any deviation from the equilibrium.



(a) Stable equilibrium (b) Unstable equilibrium

Figure 4.4.6 : (a) Stable equilibrium (b) Unstable equilibrium

However, if $n$ is odd, the mass is directly above the axle (Figure 4.4.6*b*) and gravity increases any deviation from the equilibrium. Therefore we conclude on physical grounds that $\overline{y}_{2m} = 2m\pi$ is stable and $\overline{y}_{2m+1} = (2m+1)\pi$ is unstable.

The phase plane equivalent of Equation 4.4.13 is

$$v\frac{dv}{dy} = -\frac{g}{L}\sin y,$$

where $v = y'$ is the angular velocity of the pendulum. Integrating this yields

$$\frac{v^2}{2} = \frac{g}{L}\cos y + c. \tag{4.4.14}$$

If $v = v_0$ when $y = 0$, then

$$c = \frac{v_0^2}{2} - \frac{g}{L},$$

so Equation 4.4.14 becomes

$$\begin{aligned}\frac{v^2}{2} &= \frac{v_0^2}{2} - \frac{g}{L}(1-\cos y)\\ &= \frac{v_0^2}{2} - \frac{2g}{L}\sin^2\frac{y}{2},\end{aligned}$$

which is equivalent to

$$v^2 = v_0^2 - v_c^2\sin^2\frac{y}{2}, \tag{4.4.15}$$

where

$$v_c = 2\sqrt{\frac{g}{L}}.$$

The curves defined by Equation 4.4.15 are the trajectories of Equation 4.4.13. They are periodic with period $2\pi$ in $y$, which isn't surprising, since if $y = y(t)$ is a solution of Equation 4.4.13 then so is $y_n = y(t) + 2n\pi$ for any integer $n$. Figure 4.4.7 shows trajectories over the interval $[-\pi, \pi]$.

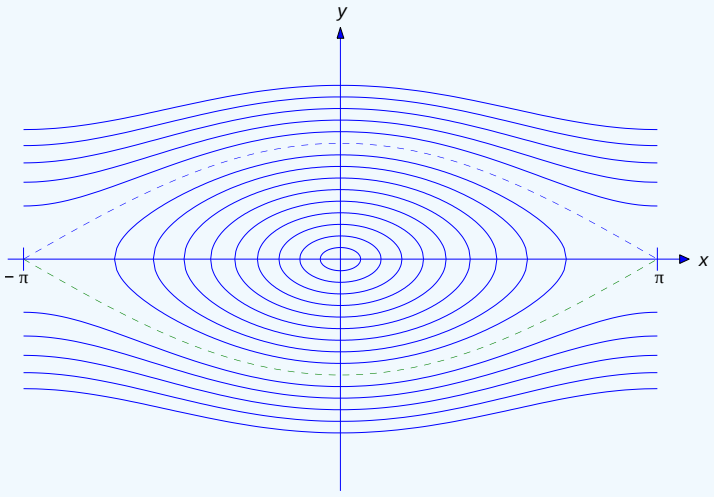



Figure 4.4.7 : Trajectories of the undamped pendulum

From Equation 4.4.15, you can see that if $|v_0| > v_c$ then $v$ is nonzero for all $t$, which means that the object whirls in the same direction forever, as in Figure 4.4.8 . The trajectories associated with this whirling motion are above the upper dashed curve and below the lower dashed curve in Figure 4.4.7 . You can also see from Equation 4.4.15 that if $0 < |v_0| < v_c$ , then $v = 0$ when $y = \pm y_{max}$ , where

$$y_{\max} = 2\sin^{-1}(|v_0|/v_c).$$

In this case the pendulum oscillates periodically between $-y_{\max}$ and $y_{\max}$, as shown in Figure 4.4.9 . The trajectories associated with this kind of motion are the ovals between the dashed curves in Figure 4.4.7 .

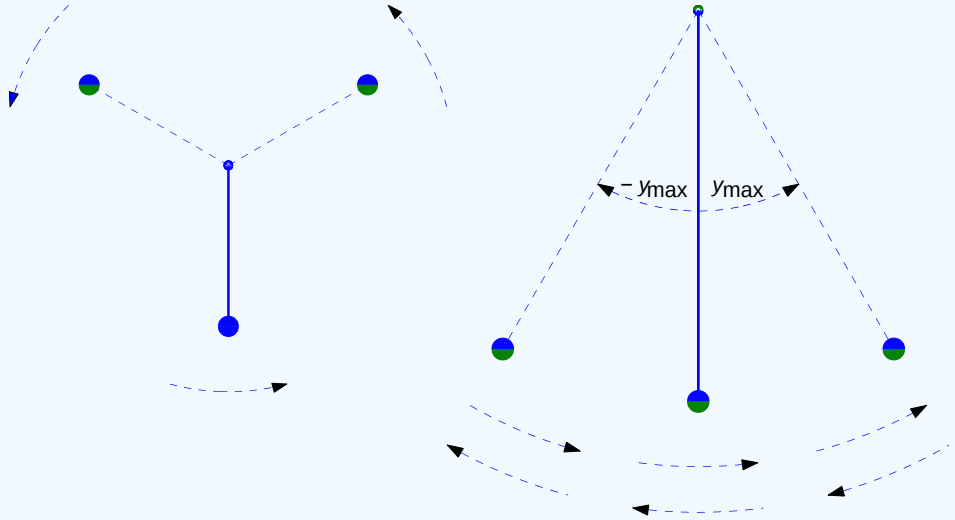



Figure 4.4.8 (left) and 4.4.9 (right): (left) The whirling undamped pendulum and (b) The oscillating undamped pendulum

It can be shown (see **Exercise 4.4.21** for a partial proof) that the period of the oscillation is

$$T = 8\int_0^{\pi/2} \frac{d\theta}{\sqrt{v_c^2 - v_0^2 \sin^2\theta}}. \tag{4.4.16}$$

Although this integral cannot be evaluated in terms of familiar elementary functions, you can see that it is finite if $|v_0| < v_c$.>

The dashed curves in Figure 4.4.7 contain four trajectories. The critical points $(\pi, 0)$ and $(-\pi, 0)$ are the trajectories of the unstable equilibrium solutions $\overline{y} = \pm\pi$ . The upper dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0,\ v(t_0) = v_c$ . If $y$ is any solution with this trajectory then

$$\lim_{t\to\infty} y(t) = \pi \quad \text{and} \quad \lim_{t\to-\infty} y(t) = -\pi.$$

The lower dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0,\ v(t_0) = -v_c$ . If $y$ is any solution with this trajectory then

$$\lim_{t\to\infty} y(t) = -\pi \quad \text{and} \quad \lim_{t\to-\infty} y(t) = \pi.$$

Consistent with this, the integral Equation 4.4.16 diverges to $\infty$ if $v_0 = \pm v_c$ . (**Exercise 4.4.21**).

Since the dashed curves separate trajectories of whirling solutions from trajectories of oscillating solutions, each of these curves is called a **separatrix**.

In general, if Equation 4.4.7 has both stable and unstable equilibria then the separatrices are the curves given by Equation 4.4.8 that pass through unstable critical points. Thus, if $(\overline{y}, 0)$ is an unstable critical point, then

$$\frac{v^2}{2} + P(y) = P(\overline{y}) \tag{4.4.17}$$

defines a separatrix passing through $(\overline{y}, 0)$.

## Stability and Instability Conditions for y'' + p(y) = 0

It can be shown (**Exercise 4.4.23**) that an equilibrium $\overline{y}$ of an undamped equation

$$y'' + p(y) = 0 \tag{4.4.18}$$

is stable if there's an open interval $(a, b)$ containing $\overline{y}$ such that

$$p(y) < 0 \text{ if } a < y < \overline{y} \text{ and } p(y) > 0 \text{ if } \overline{y} < y < b \tag{4.4.19}$$

If we regard $p(y)$ as a force acting on a unit mass, Equation 4.4.19 means that the force resists all sufficiently small displacements from $\bar{y}$.

We've already seen examples illustrating this principle. The equation Equation 4.4.9 for the undamped spring-mass system is of the form Equation 4.4.18 with $p(y) = ky/m$, which has only the stable equilibrium $\bar{y} = 0$. In this case Equation 4.4.19 holds with $a = -\infty$ and $b = \infty$. The equation Equation 4.4.13 for the undamped pendulum is of the form Equation 4.4.18 with $p(y) = (g/L)\sin y$. We've seen that $\bar{y} = 2m\pi$ is a stable equilibrium if $m$ is an integer. In this case

$$p(y) = \sin y < 0 \text{ if } (2m-1)\pi < y < 2m\pi$$

and

$$p(y) > 0 \text{ if } 2m\pi < y < (2m+1)\pi$$

It can also be shown (**Exercise 4.4.24**) that $\bar{y}$ is unstable if there's a $b > \bar{y}$ such that

$$p(y) < 0 \text{ if } \bar{y} < y < b \tag{4.4.20}$$

or an $a < \bar{y}$ such that

$$p(y) > 0 \text{ if } a < y < \bar{y} \tag{4.4.21}$$

If we regard $p(y)$ as a force acting on a unit mass, Equation 4.4.20 means that the force tends to increase all sufficiently small positive displacements from $\bar{y}$, while Equation 4.4.21 means that the force tends to increase the magnitude of all sufficiently small negative displacements from $\bar{y}$.

The undamped pendulum also illustrates this principle. We've seen that $\bar{y} = (2m+1)\pi$ is an unstable equilibrium if $m$ is an integer. In this case

$$\sin y < 0 \text{ if } (2m+1)\pi < y < (2m+2)\pi$$

so Equation 4.4.20 holds with $b = (2m+2)\pi$, and

$$\sin y > 0 \text{ if } 2m\pi < y < (2m+1)\pi$$

so Equation 4.4.21 holds with $a = 2m\pi$.

### Example 4.4.3

The equation

$$y'' + y(y-1) = 0 \tag{4.4.22}$$

is of the form Equation 4.4.18 with $p(y) = y(y-1)$. Therefore $\bar{y} = 0$ and $\bar{y} = 1$ are the equilibria of Equation 4.4.22. Since

$$\begin{aligned} y(y-1) &> 0 \qquad \text{if } y < 0 \text{ or } y > 1 \\ &< 0 \qquad \text{if } 0 < y < 1 \end{aligned}$$

$\bar{y} = 0$ is unstable and $\bar{y} = 1$ is stable.

The phase plane equivalent of Equation 4.4.22 is the separable equation

$$v\frac{dv}{dy} + y(y-1) = 0.$$

Integrating yields

$$\frac{v^2}{2} + \frac{y^3}{3} - \frac{y^2}{2} = C,$$

which we rewrite as

$$v^2 + \frac{1}{3}y^2(2y-3) = c \tag{4.4.23}$$

after renaming the constant of integration. These are the trajectories of Equation 4.4.22. If $y$ is any solution of Equation 4.4.22, the point $(y(t), v(t))$ moves along the trajectory of $y$ in the direction of increasing $y$ in the upper half plane ($v = y' > 0$ ), or in the direction of decreasing $y$ in the lower half plane ($v = y' < 0$ ).

Figure 4.4.10 shows typical trajectories. The dashed curve through the critical point $(0, 0)$, obtained by setting $c = 0$ in Equation 4.4.23, separates the $y$-$v$ plane into regions that contain different kinds of trajectories; again, we call this curve a **separatrix**. Trajectories in the region bounded by the closed loop (b) are closed curves, so solutions associated with them are periodic. Solutions associated with other trajectories are not periodic. If $y$ is any such solution with trajectory not on the separatrix, then

$$\begin{aligned} \lim_{t\to\infty} y(t) &= -\infty, & \lim_{t\to-\infty} y(t) &= -\infty, \\ \lim_{t\to\infty} v(t) &= -\infty, & \lim_{t\to-\infty} v(t) &= \infty. \end{aligned}$$

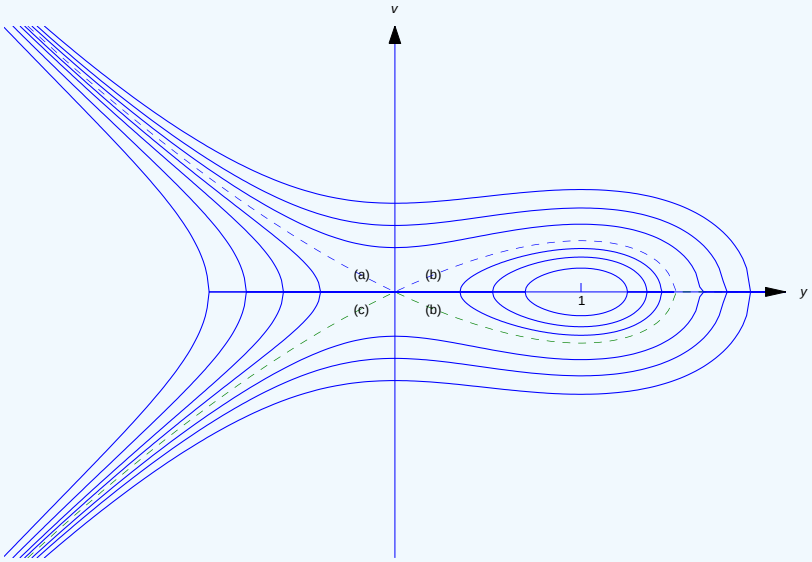



Figure 4.4.10 : Trajectories of $y'' + y(y-1) = 0$

The separatrix contains four trajectories of Equation 4.4.22. One is the point $(0, 0)$, the trajectory of the equilibrium $\overline{y} = 0$. Since distinct trajectories cannot intersect, the segments of the separatrix marked (a), b, and (c)– which don't include $(0, 0)$ – are distinct trajectories, none of which can be traversed in finite time. Solutions with these trajectories have the following asymptotic behavior:

$$\begin{aligned} \lim_{t\to\infty} y(t) &= 0, & \lim_{t\to-\infty} y(t) &= -\infty, & & \\ \lim_{t\to\infty} v(t) &= 0, & \lim_{t\to-\infty} v(t) &= \infty & &\text{(on (a))} \\ \lim_{t\to\infty} y(t) &= 0, & \lim_{t\to-\infty} y(t) &= 0, & & \\ \lim_{t\to\infty} v(t) &= 0, & \lim_{t\to-\infty} v(t) &= 0 & &\text{(on (b))} \\ \lim_{t\to\infty} y(t) &= -\infty, & \lim_{t\to-\infty} y(t) &= 0, & & \\ \lim_{t\to\infty} v(t) &= -\infty, & \lim_{t\to-\infty} v(t) &= 0 & &\text{(on (c)).} \end{aligned}$$

## The Damped Case

The phase plane equivalent of the damped autonomous equation

$$y'' + q(y, y')y' + p(y) = 0 \tag{4.4.24}$$

is

$$v\frac{dv}{dy}+q(y,v)v+p(y)=0.$$

This equation isn't separable, so we cannot solve it for $v$ in terms of $y$, as we did in the undamped case, and conservation of energy doesn't hold. (For example, energy expended in overcoming friction is lost.) However, we can study the qualitative behavior of its solutions by rewriting it as

$$\frac{dv}{dy}=-q(y,v)-\frac{p(y)}{v} \tag{4.4.25}$$

and considering the direction fields for this equation.

In the following examples we will also be showing computer generated trajectories of this equation, obtained by numerical methods. The exercises call for similar computations. The methods discussed in Chapter 3 are not suitable for this task, since $p(y)/v$ in Equation 4.4.25 is undefined on the $y$ axis of the Poincaré phase plane. Therefore we are forced to apply numerical methods briefly discussed in Section 10.1 to the system

$$\begin{aligned} y' &= v \\ v' &= -q(y,v)v-p(y), \end{aligned}$$

which is equivalent to Equation 4.4.24 in the sense defined in Section 10.1. Fortunately, most differential equation software packages enable you to do this painlessly.

In the text we will confine ourselves to the case where $q$ is constant, so Equation 4.4.24 and Equation 4.4.25 reduce to

$$y''+cy'+p(y)=0 \tag{4.4.26}$$

and

$$\frac{dv}{dy}=-c-\frac{p(y)}{v}.$$

(we will consider more general equations in the exercises.) The constant $c$ is called the **damping constant**. In situations where Equation 4.4.26 is the equation of motion of an object, $c$ is positive; however, there are situations where $c$ may be negative.

## The Damped Spring-Mass System

Earlier we considered the spring - mass system under the assumption that the only forces acting on the object were gravity and the spring's resistance to changes in its length. Now we will assume that some mechanism (for example, friction in the spring or atmospheric resistance) opposes the motion of the object with a force proportional to its velocity. In Section 6.1 it will be shown that in this case Newton's second law of motion implies that

$$my''+cy'+ky=0, \tag{4.4.27}$$

where $c>0$ is the **damping constant**. Again, this equation can be solved easily by a method that we will study in Section 5.2, but that method isn't available here. Instead, we will consider its phase plane equivalent, which can be written in the form Equation 4.4.25 as

$$\frac{dv}{dy}=-\frac{c}{m}-\frac{ky}{mv}. \tag{4.4.28}$$

(A minor note: the $c$ in Equation 4.4.26 actually corresponds to $c/m$ in this equation.) Figure 4.4.11 shows a typical direction field for an equation of this form. Recalling that motion along a trajectory must be in the direction of increasing $y$ in the upper half plane ($v>0$) and in the direction of decreasing $y$ in the lower half plane ($v<0$), you can infer that all trajectories approach the origin in clockwise fashion.

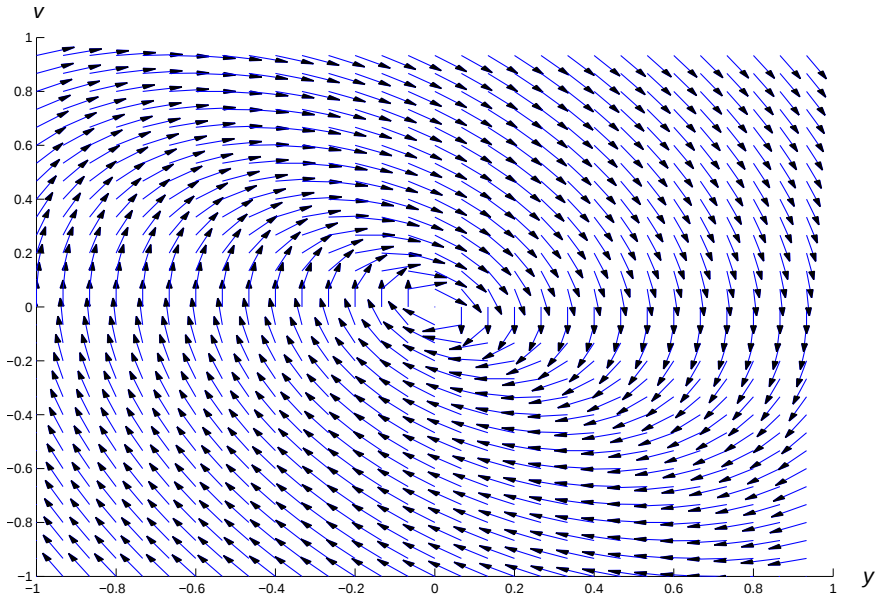



Figure 4.4.11 : A typical direction field for $my'' + cy' + ky = 0$ with $0 < c < c_1$

To confirm this, Figure 4.4.12 shows the same direction field with some trajectories filled in. All the trajectories shown there correspond to solutions of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

where

$$mv_0^2 + ky_0^2 = \rho \quad \text{(a positive constant)};$$

thus, if there were no damping ($c = 0$), all the solutions would have the same dashed elliptic trajectory, shown in Figure 4.4.14 .

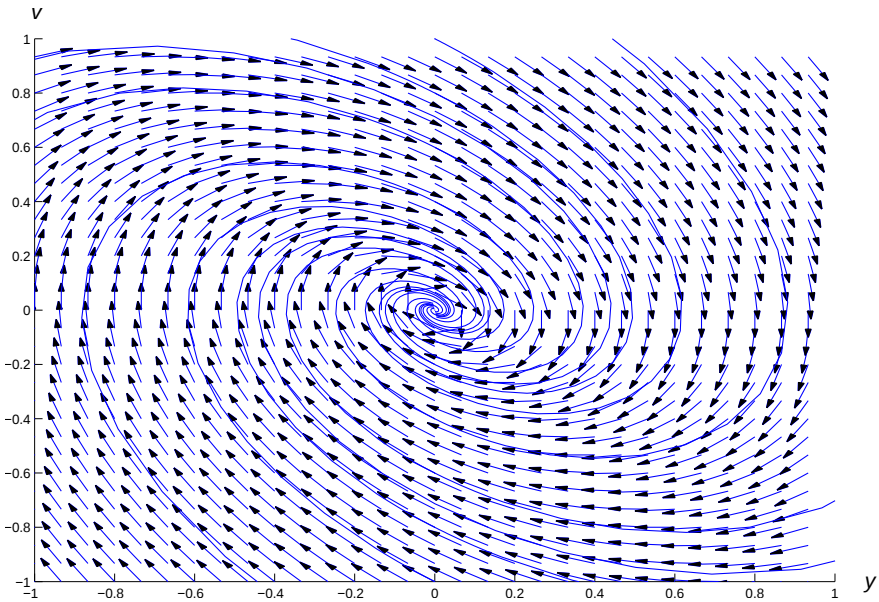



Figure 4.4.11 with some trajectories added

Solutions corresponding to the trajectories in Figure 4.4.12 cross the $y$-axis infinitely many times. The corresponding solutions are said to be **oscillatory** (Figure 4.4.13 )

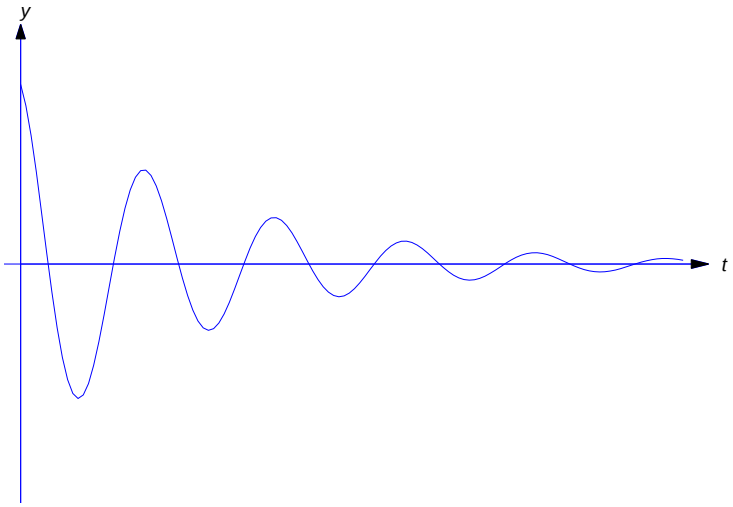



Figure 4.4.13 : An oscillatory solution of $my'' + cy' + ky = 0$

It is shown in Section 6.2 that there's a number $c_1$ such that if $0 \le c < c_1$ then all solutions of (4.4.27 ) are oscillatory, while if $c \ge c_1$ , no solutions of (4.4.27 ) have this property. (In fact, no solution not identically zero can have more than two zeros in this case.) Figure 4.4.14 shows a direction field and some integral curves for (4.4.28 ) in this case.

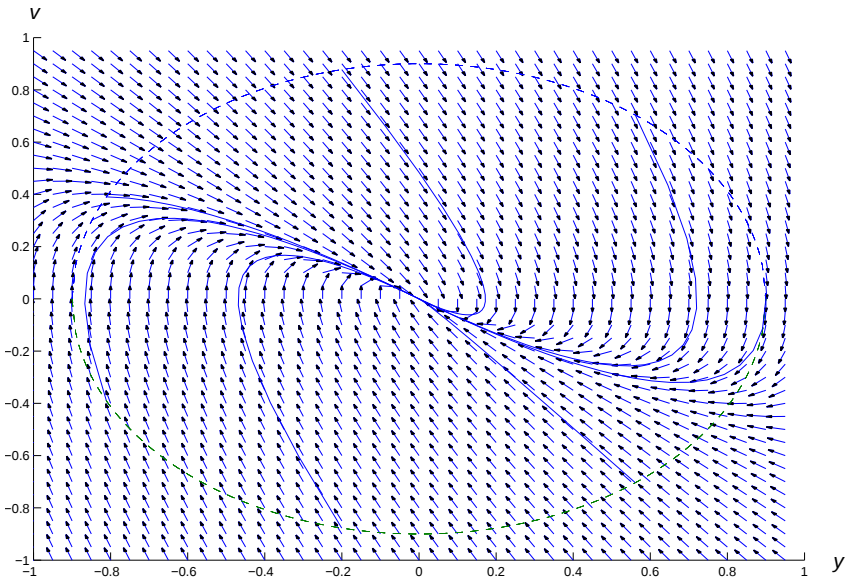



Figure 4.4.14 : A typical direction field for $my'' + cy' + ky = 0$ with $c > c_1$

## Example 4.4.4

Now we return to the pendulum. If we assume that some mechanism (for example, friction in the axle or atmospheric resistance) opposes the motion of the pendulum with a force proportional to its angular velocity, Newton's second law of motion implies that

$$mLy'' = -cy' - mg\sin y, \tag{4.4.29}$$

where $c > 0$ is the damping constant. (Again, a minor note: the $c$ in Equation 4.4.26 actually corresponds to $c/mL$ in this equation.) To plot a direction field for Equation 4.4.29 we write its phase plane equivalent as

$$\frac{dv}{dy} = -\frac{c}{mL} - \frac{g}{Lv}\sin y.$$

Figure 4.4.15 shows trajectories of four solutions of Equation 4.4.29, all satisfying $y(0) = 0$. For each $m = 0,\ 1,\ 2,\ 3$, imparting the initial velocity $v(0) = v_m$ causes the pendulum to make $m$ complete revolutions and then settle into decaying oscillation about the stable equilibrium $\bar{y} = 2m\pi$.


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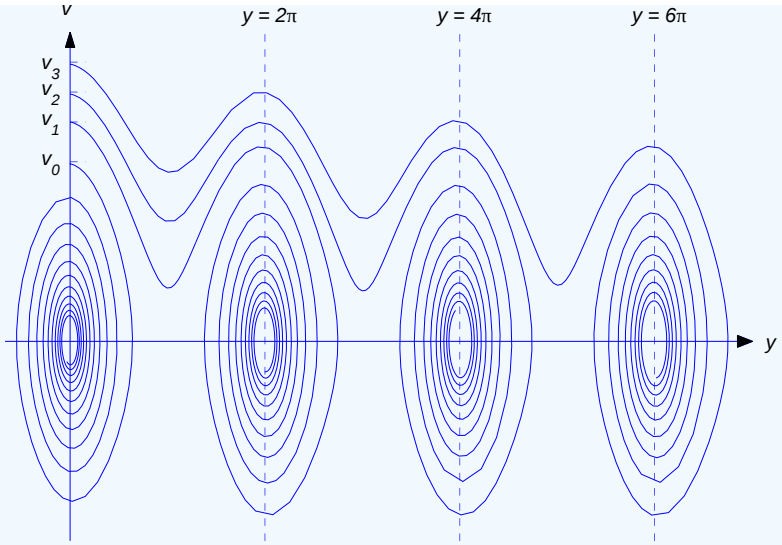



Figure 4.4.15 : Four trajectories of the damped pendulum.

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# 4.4E: Autonomous Second Order Equations (Exercises)

## Q4.4.1

In **Exercises 4.4.1-4.4.4** find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and plot some trajectories.

1. $y'' + y^3 = 0$
2. $y'' + y^2 = 0$
3. $y'' + y|y| = 0$
4. $y'' + ye^{-y} = 0$

## Q4.4.2

In **Exercises 4.4.5–4.4.8** find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and find the equations of the separatrices (that is, the curves through the unstable equilibria). Plot the separatrices and some trajectories in each of the regions of Poincaré plane determined by them.

5. $y'' - y^3 + 4y = 0$
6. $y'' + y^3 - 4y = 0$
7. $y'' + y(y^2 - 1)(y^2 - 4) = 0$
8. $y'' + y(y - 2)(y - 1)(y + 2) = 0$

## Q4.4.3

In **Exercises 4.4.9–4.4.12** plot some trajectories of the given equation for various values (positive, negative, zero) of the parameter a. Find the equilibria of the equation and classify them as stable or unstable. Explain why the phase plane plots corresponding to positive and negative values of a differ so markedly. Can you think of a reason why zero deserves to be called the **critical value** of $a$?

9. $y'' + y^2 - a = 0$
10. $y'' + y^3 - ay = 0$
11. $y'' - y^3 + ay = 0$
12. $y'' + y - ay^3 = 0$

## Q4.4.4

In **Exercises 4.4.13-4.4.18** plot trajectories of the given equation for $c = 0$ and small nonzero (positive and negative) values of $c$ to observe the effects of damping.

13. $y'' + cy' + y^3 = 0$
14. $y'' + cy' - y = 0$
15. $y'' + cy' + y^3 = 0$
16. $y'' + cy' + y^2 = 0$
17. $y'' + cy' + y|y| = 0$
18. $y'' + y(y - 1) + cy = 0$

## Q4.4.5

19. The **van der Pol equation**

$$y'' - \mu(1 - y^2)y' + y = 0, \tag{A}$$

where $\mu$ is a positive constant and $y$ is electrical current (Section 6.3), arises in the study of an electrical circuit whose resistive properties depend upon the current. The damping term $-\mu(1-y^2)y'$ works to reduce $|y|$ if $|y|<1$ or to increase $|y|$ if $|y|>1$. It can be shown that van der Pol's equation has exactly one closed trajectory, which is called a **limit cycle**. Trajectories inside the limit cycle spiral outward to it, while trajectories outside the limit cycle spiral inward to it (Figure [figure:4.4.16}). Use your favorite differential equations software to verify this for $\mu = .5, 1.1.5, 2$. Use a grid with $-4<y<4$ and $-4<v<4$.

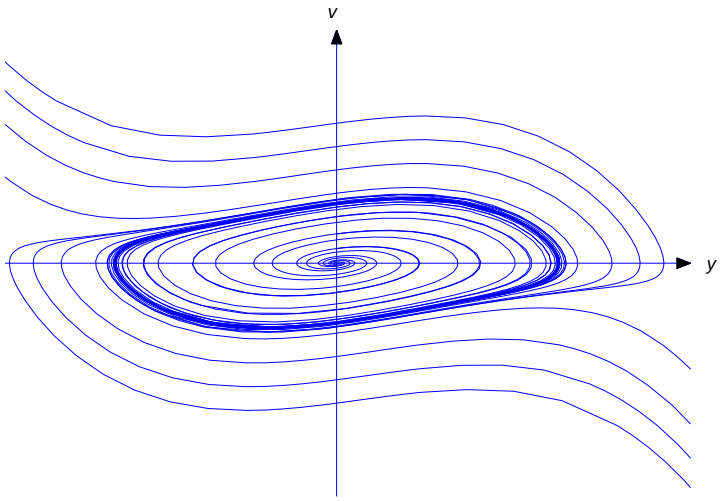



Figure 4.4E.16 : Trajectories of van der Pol's euqation

20. **Rayleigh's equation**,

$$y''-\mu(1-(y')^2/3)y'+y=0$$

also has a limit cycle. Follow the directions of **Exercise 4.4.19** for this equation.

21. In connection with Equation 4.4.16, suppose $y(0)=0$ and $y'(0)=v_0$, where $0<v_0<v_c$.

a. Let $T_1$ be the time required for $y$ to increase from zero to $y_{\max}=2\sin^{-1}(v_0/v_c)$. Show that

$$\frac{dy}{dt}=\sqrt{v_0^2-v_c^2\sin^2 y/2}, \quad 0\le t<T_1. \tag{A}$$

b. Separate variables in (A) and show that

$$T_1=\int_0^{y_{\max}}\frac{du}{\sqrt{v_0^2-v_c^2\sin^2 u/2}} \tag{B}$$

c. Substitute $\sin u/2=(v_0/v_c)\sin\theta$ in (B) to obtain

$$T_1=2\int_0^{\pi/2}\frac{d\theta}{\sqrt{v_c^2-v_0^2\sin^2\theta}}. \tag{C}$$

d. Conclude from symmetry that the time required for $(y(t),v(t))$ to traverse the trajectory

$$v^2=v_0^2-v_c^2\sin^2 y/2$$

is $T=4T_1$, and that consequently $y(t+T)=y(t)$ and $v(t+T)=v(t)$; that is, the oscillation is periodic with period $T$.

e. Show that if $v_0=v_c$, the integral in (C) is improper and diverges to $\infty$. Conclude from this that $y(t)<\pi$ for all $t$ and $\lim_{t\to\infty}y(t)=\pi$.

22. Give a direct definition of an unstable equilibrium of $y''+p(y)=0$.

23. Let $p$ be continuous for all $y$ and $p(0)=0$. Suppose there's a positive number $\rho$ such that $p(y)>0$ if $0<y\le\rho$ and $p(y)<0$ if $-\rho\le y<0$. For $0<r\le\rho$ let

$$\alpha(r)=\min\left\{\int_0^r p(x)\,dx,\ \int_{-r}^0|p(x)|\,dx\right\}\ \quad\text{and}\quad \beta(r)=\max\left\{\int_0^r p(x)\,dx,\ \int_{-r}^0|p(x)|\,dx\right\}.$$

Let $y$ be the solution of the initial value problem

$$y'' + p(y) = 0, \quad y(0) = v_0, \quad y'(0) = v_0,$$

and define $c(y_0, v_0) = v_0^2 + 2\int_0^{y_0} p(x)\,dx$.

a. Show that

$$0 < c(y_0, v_0) < v_0^2 + 2\beta(|y_0|)\quad\text{if}\quad 0 < |y_0| \le \rho.$$

b. Show that

$$v^2 + 2\int_0^y p(x)\,dx = c(y_0, v_0), \quad t > 0.$$

c. Conclude from (b) that if $c(y_0, v_0) < 2\alpha(r)$ then $|y| < r,\ t > 0$.

d. Given $\epsilon > 0$, let $\delta > 0$ be chosen so that

$$\delta^2 + 2\beta(\delta) < \max\left\{\epsilon^2/2, 2\alpha(\epsilon/\sqrt{2})\right\}.$$

Show that if $\sqrt{y_0^2 + v_0^2} < \delta$ then $\sqrt{y^2 + v^2} < \epsilon$ for $t > 0$, which implies that $\overline{y} = 0$ is a stable equilibrium of $y'' + p(y) = 0$.

e. Now let $p$ be continuous for all $y$ and $p(\overline{y}) = 0$, where $\overline{y}$ is not necessarily zero. Suppose there's a positive number $\rho$ such that $p(y) > 0$ if $\overline{y} < y \le \overline{y} + \rho$ and $p(y) < 0$ if $\overline{y} - \rho \le y < \overline{y}$. Show that $\overline{y}$ is a stable equilibrium of $y'' + p(y) = 0$.

24. Let $p$ be continuous for all $y$.

a. Suppose $p(0) = 0$ and there's a positive number $\rho$ such that $p(y) < 0$ if $0 < y \le \rho$. Let $\epsilon$ be any number such that $0 < \epsilon < \rho$. Show that if $y$ is the solution of the initial value problem

$$y'' + p(y) = 0, \quad y(0) = y_0, \quad y'(0) = 0$$

with $0 < y_0 < \epsilon$, then $y(t) \ge \epsilon$ for some $t > 0$. Conclude that $\overline{y} = 0$ is an unstable equilibrium of $y'' + p(y) = 0$.

b. Now let $p(\overline{y}) = 0$, where $\overline{y}$ isn't necessarily zero. Suppose there's a positive number $\rho$ such that $p(y) < 0$ if $\overline{y} < y \le \overline{y} + \rho$. Show that $\overline{y}$ is an unstable equilibrium of $y'' + p(y) = 0$.

c. Modify your proofs of (a) and (b) to show that if there's a positive number $\rho$ such that $p(y) > 0$ if $\overline{y} - \rho \le y < \overline{y}$, then $\overline{y}$ is an unstable equilibrium of $y'' + p(y) = 0$.

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# 4.5: Applications to Curves

We begin with two examples of families of curves generated by varying a parameter over a set of real numbers.

**Example 4.5.1**

For each value of the parameter $c$, the equation

$$y - cx^2 = 0 \tag{4.5.1}$$

defines a curve in the $xy$-plane. If $c \neq 0$, the curve is a parabola through the origin, opening upward if $c > 0$ or downward if $c < 0$. If $c = 0$, the curve is the $x$ axis (Figure 4.5.1).

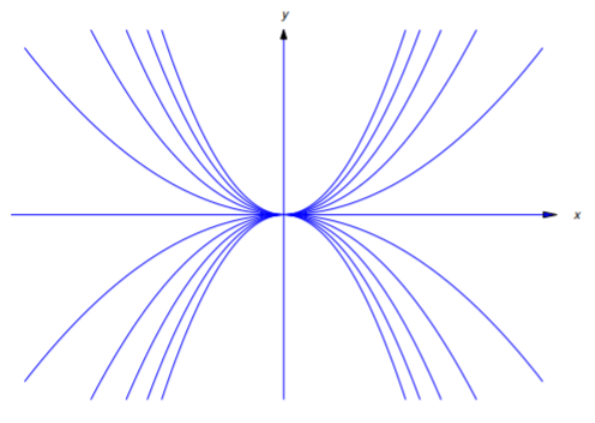



Figure 4.5.1: A family of curves defined by $y - cx^2 = 0$

**Example 4.5.2**

For each value of the parameter $c$ the equation

$$y = x + c \tag{4.5.2}$$

defines a line with slope 1(Figure 4.5.2).

**Theorem 4.5.1**

An equation that can be written in the form

$$H(x, y, c) = 0 \tag{4.5.3}$$

is said to define a *one-parameter family of curves* if, for each value of $c$ in in some nonempty set of real numbers, the set of points $(x, y)$ that satisfy Equation 4.5.3 forms a curve in the $xy$-plane.

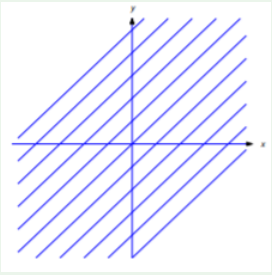


Figure 4.5.2: A family of lines defined by $y = x + c$

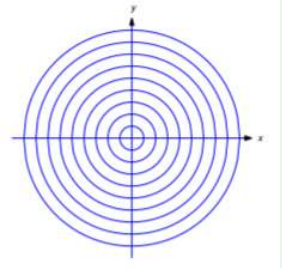


Figure 4.5.3: A family of circles defined by $x^2+y^2-c^2=0$

Equations Equation 4.5.1 and Equation 4.5.2 define one–parameter families of curves. (Although Equation 4.5.2 isn't in the form Equation 4.5.3, it can be written in this form as $y-x-c=0$ .)

**Example 4.5.3**

If $c>0$, the graph of the equation

$$x^2+y^2-c=0 \tag{4.5.4}$$

is a circle with center at $(0,0)$ and radius $\sqrt{c}$. If $c=0$, the graph is the single point $(0,0)$. (We don't regard a single point as a curve.) If $c<0$, the equation has no graph. Hence, Equation 4.5.4 defines a one–parameter family of curves for positive values of $c$. This family consists of all circles centered at $(0,0)$ (Figure 4.5.3).

**Example 4.5.4**

The equation

$$x^2+y^2+c^2=0$$

does not define a one-parameter family of curves, since no $(x,y)$ satisfies the equation if $c\neq 0$, and only the single point $(0,0)$ satisfies it if $c=0$.

Recall from Section 1.2 that the graph of a solution of a differential equation is called an *integral curve* of the equation. Solving a first order differential equation usually produces a one–parameter family of integral curves of the equation. Here we are interested in the converse problem: given a one–parameter family of curves, is there a first order differential equation for which every member of the family is an integral curve. This suggests the next definition.

**Theorem 4.5.1**

4.5.2 If every curve in a one-parameter family defined by the equation

$$H(x,y,c)=0 \tag{4.5.5}$$

is an integral curve of the first order differential equation

$$F(x,y,y')=0, \tag{4.5.6}$$

then Equation 4.5.6 is said to be a *differential equation for the family.*

To find a differential equation for a one–parameter family we differentiate its defining equation Equation 4.5.5 implicitly with respect to $x$, to obtain

$$H_x(x,y,c)+H_y(x,y,c)y'=0. \tag{4.5.7}$$

If this equation doesn't, then it is a differential equation for the family. If it does contain $c$, it may be possible to obtain a differential equation for the family by eliminating $c$ between Equation 4.5.5 and Equation 4.5.7.

**Example 4.5.5**

Find a differential equation for the family of curves defined by

$$y = cx^2. \tag{4.5.8}$$

**Solution**

Differentiating Equation 4.5.8 with respect to $x$ yields

$$y' = 2cx.$$

Therefore $c = y'/2x$, and substituting this into Equation 4.5.8 yields

$$y = \frac{xy'}{2}$$

as a differential equation for the family of curves defined by Equation 4.5.8. The graph of any function of the form $y = cx^2$ is an integral curve of this equation.

The next example shows that members of a given family of curves may be obtained by joining integral curves for more than one differential equation.

**Example 4.5.6**

a. Try to find a differential equation for the family of lines tangent to the parabola $y = x^2$.

b. Find two tangent lines to the parabola $y = x^2$ that pass through $(2, 3)$, and find the points of tangency.

**Solution**

a. The equation of the line through a given point $(x_0, y_0)$ with slope $m$ is

$$y = y_0 + m(x - x_0). \tag{4.5.9}$$

If $(x_0, y_0)$ is on the parabola, then $y_0 = x_0^2$ and the slope of the tangent line through $(x_0, x_0^2)$ is $m = 2x_0$; hence, Equation 4.5.9 becomes

$$y = x_0^2 + 2x_0(x - x_0),$$

or, equivalently,

$$y = -x_0^2 + 2x_0x. \tag{4.5.10}$$

Here $x_0$ plays the role of the constant $c$ in Definition 4.5.1; that is, varying $x_0$ over $(-\infty, \infty)$ produces the family of tangent lines to the parabola $y = x^2$.

Differentiating Equation 4.5.10 with respect to $x$ yields $y' = 2x_0$.. We can express $x_0$ in terms of $x$ and $y$ by rewriting Equation 4.5.10 as

$$x_0^2 - 2x_0x + y = 0$$

and using the quadratic formula to obtain

$$x_0 = x \pm \sqrt{x^2 - y}. \tag{4.5.11}$$

We must choose the plus sign in Equation 4.5.11 if $x < x_0$ and the minus sign if $x > x_0$; thus,

$$x_0 = \left(x + \sqrt{x^2 - y}\right) \text{ if } x < x_0.$$

and

$$x_0 = \left(x - \sqrt{x^2 - y}\right) \text{ if } x > x_0.$$

Since $y' = 2x_0$, this implies that

$$y' = 2\left(x + \sqrt{x^2 - y}\right), \quad \text{if} \quad x < x_0 \tag{4.5.12}$$

and

$$y' = 2\left(x - \sqrt{x^2 - y}\right), \quad \text{if} \quad x > x_0. \tag{4.5.13}$$

Neither Equation 4.5.12 nor Equation 4.5.13 is a differential equation for the family of tangent lines to the parabola $y = x^2$. However, if each tangent line is regarded as consisting of two *tangent half lines* joined at the point of tangency, Equation 4.5.12 is a differential equation for the family of tangent half lines on which $x$ is less than the abscissa of the point of tangency (Figure 4.5.4a), while Equation 4.5.13 is a differential equation for the family of tangent half lines on which $x$ is greater than this abscissa (Figure 4.5.4(b)). The parabola $y = x^2$ is also an integral curve of both Equation 4.5.12 and Equation 4.5.13.

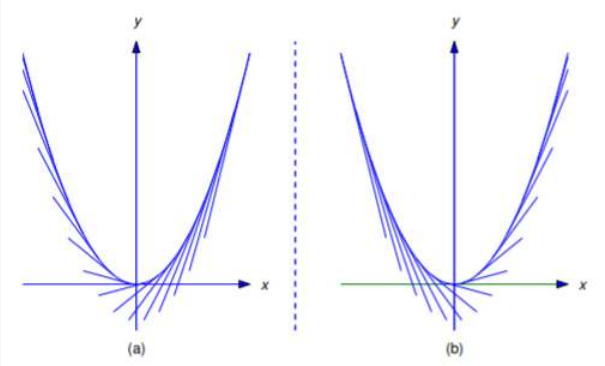



Figure 4.5.4

b. From Equation 4.5.10 the point $(x, y) = (2, 3)$ is on the tangent line through $(x_0, x_0^2)$ if and only if

$$3 = -x_0^2 + 4x_0,$$

which is equivalent to

$$x_0^2 - 4x_0 + 3 = (x_0 - 3)(x_0 - 1) = 0.$$

Letting $x_0 = 3$ in Equation 4.5.10 shows that $(2, 3)$ is on the line

$$y = -9 + 6x,$$

which is tangent to the parabola at $(x_0, x_0^2) = (3, 9)$, as shown in Figure 4.5.5

Letting $x_0 = 1$ in Equation 4.5.10 shows that $(2, 3)$ is on the line

$$y = -1 + 2x,$$

which is tangent to the parabola at $(x_0, x_0^2) = (1, 1)$, as shown in Figure 4.5.5.

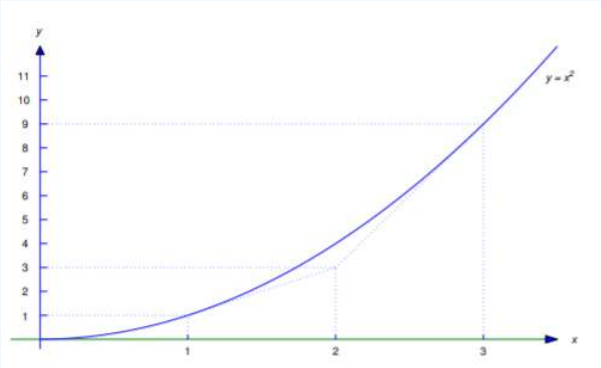



Figure 4.5.5


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## Geometric Problems

We now consider some geometric problems that can be solved by means of differential equations.

**Example 4.5.7**

Find curves $y = y(x)$ such that every point $(x_0, y(x_0))$ on the curve is the midpoint of the line segment with endpoints on the coordinate axes and tangent to the curve at $(x_0, y(x_0))$ (Figure 4.5.6).

**Solution**

The equation of the line tangent to the curve at $P = (x_0, y(x_0)$ is

$$y = y(x_0) + y'(x_0)(x - x_0).$$

If we denote the $x$ and $y$ intercepts of the tangent line by $x_I$ and $y_I$ (Figure 4.5.6), then

$$0 = y(x_0) + y'(x_0)(x_I - x_0) \tag{4.5.14}$$

and

$$y_I = y(x_0) - y'(x_0)x_0. \tag{4.5.15}$$

From Figure 4.5.6, $P$ is the midpoint of the line segment connecting $(x_I, 0)$ and $(0, y_I)$ if and only if $x_I = 2x_0$ and $y_I = 2y(x_0)$. Substituting the first of these conditions into Equation 4.5.14 or the second into Equation 4.5.15 yields

$$y(x_0) + y'(x_0)x_0 = 0.$$

Since $x_0$ is arbitrary we drop the subscript and conclude that $y = y(x)$ satisfies

$$y + xy' = 0,$$

which can be rewritten as

$$(xy)' = 0.$$

Integrating yields $xy = c$, or

$$y = \frac{c}{x}.$$

If $c = 0$ this curve is the line $y = 0$, which does not satisfy the geometric requirements imposed by the problem; thus, $c \neq 0$, and the solutions define a family of hyperbolas (Figure 4.5.7).

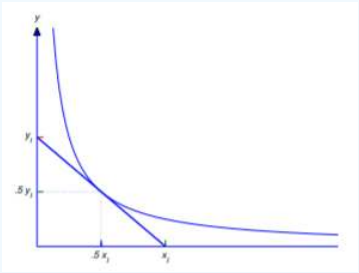


Figure 4.5.6

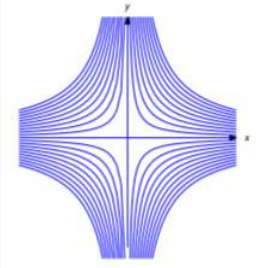


Figure 4.5.7

**Example 4.5.8**

Find curves $y = y(x)$ such that the tangent line to the curve at any point $(x_0, y(x_0))$ intersects the $x$-axis at $(x_0^2, 0)$. Figure 4.5.8 illustrates the situation in the case where the curve is in the first quadrant and $0 < x < 1$.

**Solution**

The equation of the line tangent to the curve at $(x_0, y(x_0))$ is

$$y = y(x_0) + y'(x_0)(x - x_0).$$

Since $(x_0^2, 0)$ is on the tangent line,

$$0 = y(x_0) + y'(x_0)(x_0^2 - x_0).$$

Since $x_0$ is arbitrary we drop the subscript and conclude that $y = y(x)$ satisfies

$$y + y'(x^2 - x) = 0.$$

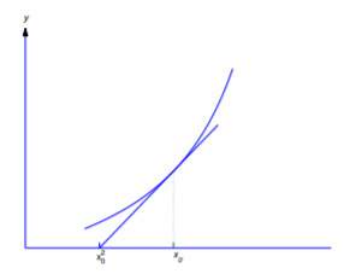


Figure 4.5.8

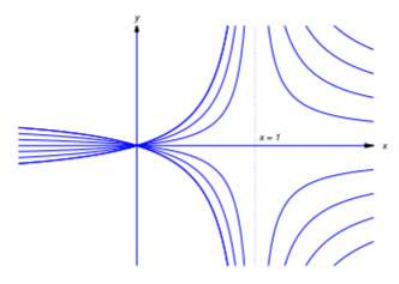


Figure 4.5.9

Therefore

$$\frac{y'}{y} = -\frac{1}{x^2 - x} = -\frac{1}{x(x-1)} = \frac{1}{x} - \frac{1}{x-1},$$

so

$$\ln|y| = \ln|x| - \ln|x-1| + k = \ln\left|\frac{x}{x-1}\right| + k,$$

and

$$y = \frac{cx}{x-1}.$$

If $c = 0$, the graph of this function is the $x$-axis. If $c \neq 0$, it is a hyperbola with vertical asymptote $x = 1$ and horizontal asymptote $y = c$. Figure 4.5.9 shows the graphs for $c \neq 0$.

## Orthogonal Trajectories

Two curves $C_1$ and $C_2$ are said to be orthogonal at a point of intersection $(x_0, y_0)$ if they have perpendicular tangents at $(x_0, y_0)$. (Figure 4.5.10). A curve is said to be an *orthogonal trajectory* of a given family of curves if it is orthogonal to every curve in the

family. For example, every line through the origin is an orthogonal trajectory of the family of circles centered at the origin. Conversely, any such circle is an orthogonal trajectory of the family of lines through the origin (Figure 4.5.11).

Orthogonal trajectories occur in many physical applications. For example, if $u = u(x, y)$ is the temperature at a point $(x, y)$, the curves defined by

$$u(x, y) = c \tag{4.5.16}$$

are called *isothermal* curves. The orthogonal trajectories of this family are called *heat-flow* lines, because at any given point the direction of maximum heat flow is perpendicular to the isothermal through the point. If $u$ represents the potential energy of an object moving under a force that depends upon $(x, y)$, the curves Equation 4.5.16 are called *equipotentials*, and the orthogonal trajectories are called *lines of force*.

From analytic geometry we know that two nonvertical lines $L_1$ and $L_2$ with slopes $m_1$ and $m_2$, respectively, are perpendicular if and only if $m_2 = -1/m_1$; therefore, the integral curves of the differential equation

$$y' = -\frac{1}{f(x, y)}$$

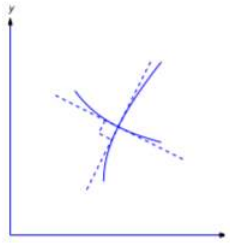


Figure 4.5.10

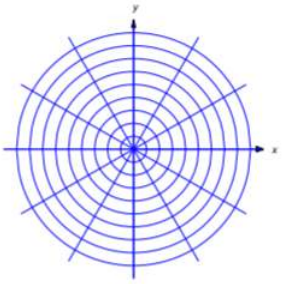


Figure 4.5.11

are orthogonal trajectories of the integral curves of the differential equation

$$y' = f(x, y),$$

because at any point $(x_0, y_0)$ where curves from the two families intersect the slopes of the respective tangent lines are

$$m_1 = f(x_0, y_0) \quad \text{and} \quad m_2 = -\frac{1}{f(x_0, y_0)}.$$

This suggests a method for finding orthogonal trajectories of a family of integral curves of a first order equation.

**Finding Orthogonal Trajectories**

- **Step 1.** Find a differential equation

  $$y' = f(x, y)$$

  for the given family.
- **Step 2.** Solve the differential equation

  $$y' = -\frac{1}{f(x, y)}$$

to find the orthogonal trajectories.

**Example 4.5.9**

Find the orthogonal trajectories of the family of circles

$$x^2+y^2=c^2 \quad (c>0). \tag{4.5.17}$$

**Solution**

To find a differential equation for the family of circles we differentiate Equation 4.5.17 implicitly with respect to $x$ to obtain

$$2x+2yy'=0,$$

or

$$y'=-\frac{x}{y}.$$

Therefore the integral curves of

$$y'=\frac{y}{x}$$

are orthogonal trajectories of the given family. We leave it to you to verify that the general solution of this equation is

$$y=kx,$$

where $k$ is an arbitrary constant. This is the equation of a nonvertical line through $(0,0)$. The $y$ axis is also an orthogonal trajectory of the given family. Therefore every line through the origin is an orthogonal trajectory of the given family Equation 4.5.17 (Figure 4.5.11). This is consistent with the theorem of plane geometry which states that a diameter of a circle and a tangent line to the circle at the end of the diameter are perpendicular.

**Example 4.5.10**

Find the orthogonal trajectories of the family of hyperbolas

$$xy=c \quad (c\neq 0) \tag{4.5.18}$$

(Figure 4.5.7).

**Solution**

Differentiating Equation 4.5.18 implicitly with respect to $x$ yields

$$y+xy'=0,$$

or

$$y'=-\frac{y}{x};$$

thus, the integral curves of

$$y'=\frac{x}{y}$$

are orthogonal trajectories of the given family. Separating variables yields

$$y'y=x$$

and integrating yields

$$y^2-x^2=k,$$

which is the equation of a hyperbola if $k\neq 0$, or of the lines $y=x$ and $y=-x$ if $k=0$ (Figure 4.5.12).

**Example 4.5.11**

Find the orthogonal trajectories of the family of circles defined by

$$(x-c)^2+y^2=c^2 \quad (c\neq 0). \tag{4.5.19}$$

These circles are centered on the $x$-axis and tangent to the $y$-axis (Figure 4.5.13a).

**Solution**

Multiplying out the left side of Equation 4.5.19 yields

$$x^2-2cx+y^2=0, \tag{4.5.20}$$

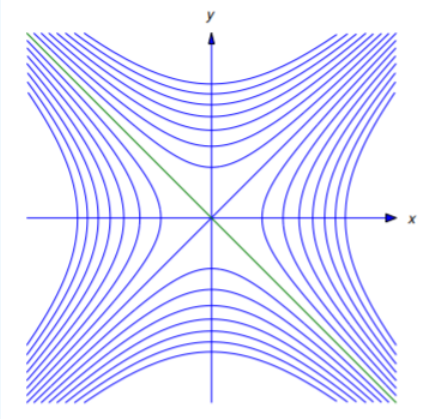



Figure 4.5.12

and differentiating this implicitly with respect to $x$ yields

$$2(x-c)+2yy'=0. \tag{4.5.21}$$

From Equation 4.5.20,

$$c=\frac{x^2+y^2}{2x},$$

so

$$x-c=x-\frac{x^2+y^2}{2x}=\frac{x^2-y^2}{2x}.$$

Substituting this into Equation 4.5.21 and solving for $y'$ yields

$$y'=\frac{y^2-x^2}{2xy}. \tag{4.5.22}$$

The curves defined by Equation 4.5.19 are integral curves of Equation 4.5.22, and the integral curves of

$$y'=\frac{2xy}{x^2-y^2}$$

are orthogonal trajectories of the family Equation 4.5.19. This is a homogeneous nonlinear equation, which we studied in Section 2.4. Substituting $y=ux$ yields

$$u'x+u=\frac{2x(ux)}{x^2-(ux)^2}=\frac{2u}{1-u^2},$$

so

$$u'x=\frac{2u}{1-u^2}-u=\frac{u(u^2+1)}{1-u^2},$$

Separating variables yields

$$\frac{1-u^2}{u(u^2+1)}u' = \frac{1}{x},$$

or, equivalently,

$$\left[\frac{1}{u} - \frac{2u}{u^2+1}\right]u' = \frac{1}{x}.$$

Therefore

$$\ln|u| - \ln(u^2+1) = \ln|x| + k.$$

By substituting $u = y/x$, we see that

$$\ln|y| - \ln|x| - \ln(x^2+y^2) + \ln(x^2) = \ln|x| + k,$$

which, since $\ln(x^2) = 2\ln|x|$, is equivalent to

$$\ln|y| - \ln(x^2+y^2) = k,$$

or

$$|y| = e^k(x^2+y^2).$$

To see what these curves are we rewrite this equation as

$$x^2 + |y|^2 - e^{-k}|y| = 0$$

and complete the square to obtain

$$x^2 + (|y| - e^{-k}/2)^2 = (e^{-k}/2)^2.$$

This can be rewritten as

$$x^2 + (y-h)^2 = h^2,$$

where

$$h = \begin{cases} \frac{e^{-k}}{2} & \text{if } y \geq 0, \\ -\frac{e^{-k}}{2} & \text{if } y \leq 0. \end{cases}$$

Thus, the orthogonal trajectories are circles centered on the $y$ axis and tangent to the $x$ axis (Figure $4.5.13$(b)). The circles for which $h > 0$ are above the $x$-axis, while those for which $h < 0$ are below.

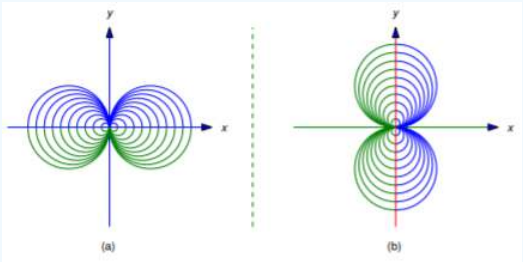



Figure 4.5.13: (a) The circles $(x-c)^2 + y^2 = c^2$ (b) The circles $x^2 + (y-b)^2 = b^2$

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# 4.5E: Applications to Curves (Exercises)

## Q4.5.1

In **Exercises 4.5.1-4.5.8** find a first order differential equation for the given family of curves.

1. $y(x^2+y^2)=c$
2. $e^{xy}=cy$
3. $\ln|xy|=c(x^2+y^2)$
4. $y=x^{1/2}+cx$
5. $y=e^{x^2}+ce^{-x^2}$
6. $y=x^3+\frac{c}{x}$
7. $y=\sin x+ce^x$
8. $y=e^x+c(1+x^2)$

## Q4.5.2

9. Show that the family of circles

$$(x-x_0)^2+y^2=1,\ -\infty<x_0<\infty,$$

can be obtained by joining integral curves of two first order differential equations. More specifically, find differential equations for the families of semicircles

$$(x-x_0)^2+y^2=1,\ x_0<x<x_0+1,\ -\infty<x_0<\infty,$$

$$(x-x_0)^2+y^2=1,\ x_0-1<x<x_0,\ -\infty<x_0<\infty.$$

10. Suppose $f$ and $g$ are differentiable for all $x$. Find a differential equation for the family of functions $y=f+cg$ ($c$=constant).

## Q4.5.3

In **Exercises 4.5.11-4.5.13** find a first order differential equation for the given family of curves.

11. Lines through a given point $(x_0,y_0)$.
12. Circles through $(-1,0)$ and $(1,0)$.
13. Circles through $(0,0)$ and $(0,2)$.

## Q4.5.4

14. Use the method Example 4.5.6 (a) to find the equations of lines through the given points tangent to the parabola $y=x^2$. Also, find the points of tangency.

a. $(5,9)$
b. $(6,11)$
c. $(-6,20)$
d. $(-3,5)$

15.

a. Show that the equation of the line tangent to the circle

$$x^2+y^2=1 \tag{A}$$

at a point $(x_0,y_0)$ on the circle is

$$y=\frac{1-x_0x}{y_0}\quad\text{if}\quad x_0\neq\pm1. \tag{B}$$

b. Show that if $y'$ is the slope of a nonvertical tangent line to the circle (A) and $(x, y)$ is a point on the tangent line then

$$(y')^2(x^2-1)-2xyy'+y^2-1=0. \tag{C}$$

c. Show that the segment of the tangent line (B) on which $(x-x_0)/y_0>0$ is an integral curve of the differential equation

$$y'=\frac{xy-\sqrt{x^2+y^2-1}}{x^2-1}, \tag{D}$$

while the segment on which $(x-x_0)/y_0<0$ is an integral curve of the differential equation

$$y'=\frac{xy+\sqrt{x^2+y^2-1}}{x^2-1}. \tag{E}$$

HINT: **Use the quadratic formula to solve (C) for $y'$. Then substitute (B) for $y$ and choose the $\pm$ sign in the quadratic formula so that the resulting expression for $y'$ reduces to the known slope $y'=-x_0/y_0$**

d. Show that the upper and lower semicircles of (A) are also integral curves of (D) and (E).

e. Find the equations of two lines through (5,5) tangent to the circle (A), and find the points of tangency.

16.

a. Show that the equation of the line tangent to the parabola

$$x=y^2 \tag{A}$$

at a point $(x_0, y_0)\neq(0,0)$ on the parabola is

$$y=\frac{y_0}{2}+\frac{x}{2y_0}. \tag{B}$$

b. Show that if $y'$ is the slope of a nonvertical tangent line to the parabola (A) and $(x, y)$ is a point on the tangent line then

$$4x^2(y')^2-4xyy'+x=0. \tag{C}$$

c. Show that the segment of the tangent line defined in (a) on which $x>x_0$ is an integral curve of the differential equation

$$y'=\frac{y+\sqrt{y^2-x}}{2x}, \tag{D}$$

while the segment on which $x<x_0$ is an integral curve of the differential equation

$$y'=\frac{y-\sqrt{y^2-x}}{2x}, \tag{E}$$

HINT: **Use the quadratic formula to solve (c) for $y'$. Then substitute (B) for y and choose the $\pm$ sign in the quadratic formula so that the resulting expression for $y'$ reduces to the known slope of $y'=\frac{1}{2y_0}$**

d. Show that the upper and lower halves of the parabola (A), given by $y=\sqrt{x}$ and $y=-\sqrt{x}$ for $x>0$, are also integral curves of (D) and (E).

17. Use the results of **Exercise 4.5.16** to find the equations of two lines tangent to the parabola $x=y^2$ and passing through the given point. Also find the points of tangency.

a. $(-5,2)$
b. $(-4,0)$
c. $(7,4)$
d. $(5,-3)$

18. Find a curve $y=y(x)$ through (1,2) such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the $x$ axis at $x_I=x_0/2$.

19. Find all curves $y=y(x)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the $x$ axis at $x_I=x_0^3$.

20. Find all curves $y=y(x)$ such that the tangent to the curve at any point passes through a given point $(x_1, y_1)$.

21. Find a curve $y = y(x)$ through $(1, -1)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the $y$ axis at $y_I = x_0^3$.

22. Find all curves $y = y(x)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the $y$ axis at $y_I = x_0$.

23. Find a curve $y = y(x)$ through $(0, 2)$ such that the normal to the curve at any point $(x_0, y(x_0))$ intersects the $x$ axis at $x_I = x_0 + 1$.

24. Find a curve $y = y(x)$ through $(2, 1)$ such that the normal to the curve at any point $(x_0, y(x_0))$ intersects the $y$ axis at $y_I = 2y(x_0)$.

## Q4.5.5

In **Exercises 4.5.25-2.5.29** find the orthogonal trajectories of the given family of curves.

25. $x^2 + 2y^2 = c^2$

26. $x^2 + 4xy + y^2 = c$

27. $y = ce^{2x}$

28. $xye^{x^2} = c$

29. $y = \frac{ce^x}{x}$

## Q4.5.6

30. Find a curve through $(-1, 3)$ orthogonal to every parabola of the form

$$y = 1 + cx^2$$

that it intersects. Which of these parabolas does the desired curve intersect?

31. Show that the orthogonal trajectories of

$$x^2 + 2axy + y^2 = c$$

satisfy

$$|y - x|^{a+1}|y + x|^{a-1} = k.$$

32. If lines $L$ and $L_1$ intersect at $(x_0, y_0)$ and $\alpha$ is the smallest angle through which $L$ must be rotated counterclockwise about $(x_0, y_0)$ to bring it into coincidence with $L_1$, we say that $\alpha$ is the **angle from $L$** to $L_1$; thus, $0 \le \alpha < \pi$. If $L$ and $L_1$ are tangents to curves $C$ and $C_1$, respectively, that intersect at $(x_0, y_0)$, we say that $C_1$ intersects $C$ at the angle $\alpha$. Use the identity

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

to show that if $C$ and $C_1$ are intersecting integral curves of

$$y' = f(x, y) \quad \text{and} \quad y' = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha} \qquad \left(\alpha \ne \frac{\pi}{2}\right),$$

respectively, then $C_1$ intersects $C$ at the angle $\alpha$.

33. Use the result of **Exercise 4.5.32** to find a family of curves that intersect every nonvertical line through the origin at the angle $\alpha = \pi/4$.

34. Use the result of **Exercise 4.5.32** to find a family of curves that intersect every circle centered at the origin at a given angle $\alpha \ne \pi/2$.

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# CHAPTER OVERVIEW

## 5: Linear Second Order Equations

In this Chapter, we study a particularly important class of second order equations. Because of their many applications in science and engineering, second order differential equation have historically been the most thoroughly studied class of differential equations. Research on the theory of second order differential equations continues to the present day. This chapter is devoted to second order equations that can be written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x).$$

Such equations are said to be *linear*. As in the case of first order linear equations, this differential equation is said to be *homogeneous* if $F \equiv 0$, or *nonhomogeneous* if $F \not\equiv 0$.


5.1: Homogeneous Linear Equations

5.1E: Homogeneous Linear Equations (Exercises)

5.2: Constant Coefficient Homogeneous Equations

5.2E: Constant Coefficient Homogeneous Equations (Exercises)

5.3: Nonhomogeneous Linear Equations

5.3E: Nonhomgeneous Linear Equations (Exercises)

5.4: The Method of Undetermined Coefficients I

5.4E: The Method of Undetermined Coefficients I (Exercises)

5.5: The Method of Undetermined Coefficients II

5.5E: The Method of Undetermined Coefficients II (Exercises)

5.6: Reduction of Order

5.6E: Reduction of Order (Exercises)

5.7: Variation of Parameters

5.7E: Variation of Parameters (Exercises)



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# 5.1: Homogeneous Linear Equations

A second order differential equation is said to be **linear** if it can be written as

$$y'' + p(x)y' + q(x)y = f(x). \tag{5.1.1}$$

We call the function $f$ on the right a **forcing function**, since in physical applications it is often related to a force acting on some system modeled by the differential equation. We say that Equation 5.1.1 is **homogeneous** if $f \equiv 0$ or **nonhomogeneous** if $f \not\equiv 0$. Since these definitions are like the corresponding definitions in Section 2.1 for the linear first order equation

$$y' + p(x)y = f(x), \tag{5.1.2}$$

it is natural to expect similarities between methods of solving Equation 5.1.1 and Equation 5.1.2. However, solving Equation 5.1.1 is more difficult than solving Equation 5.1.2. For example, while Theorem 5.1.1 gives a formula for the general solution of Equation 5.1.2 in the case where $f \equiv 0$ and Theorem 5.1.2 gives a formula for the case where $f \not\equiv 0$, there are no formulas for the general solution of Equation 5.1.1 in either case. Therefore we must be content to solve linear second order equations of special forms.

In Section 2.1 we considered the homogeneous equation $y' + p(x)y = 0$ first, and then used a nontrivial solution of this equation to find the general solution of the nonhomogeneous equation $y' + p(x)y = f(x)$. Although the progression from the homogeneous to the nonhomogeneous case isn't that simple for the linear second order equation, it is still necessary to solve the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.3}$$

in order to solve the nonhomogeneous equation Equation 5.1.1. This section is devoted to Equation 5.1.3.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for Equation 5.1.3. We omit the proof.

**Theorem 5.1.1**

Suppose $p$ and $q$ are continuous on an open interval $(a, b)$, let $x_0$ be any point in $(a, b)$, and let $k_0$ and $k_1$ be arbitrary real numbers. Then the initial value problem

$$y'' + p(x)y' + q(x)y = 0,\ y(x_0) = k_0,\ y'(x_0) = k_1$$

has a unique solution on $(a, b)$.

Since $y \equiv 0$ is obviously a solution of Equation 5.1.3 we call it the **trivial** solution. Any other solution is **nontrivial**. Under the assumptions of Theorem 5.1.1 , the only solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0,\ y(x_0) = 0,\ y'(x_0) = 0$$

on $(a, b)$ is the trivial solution (**Exercise 5.1.24**).

The next three examples illustrate concepts that we'll develop later in this section. You shouldn't be concerned with how to **find** the given solutions of the equations in these examples. This will be explained in later sections.

**Example 5.1.1**

The coefficients of $y'$ and $y$ in

$$y'' - y = 0 \tag{5.1.4}$$

are the constant functions $p \equiv 0$ and $q \equiv -1$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 5.1.1 implies that every initial value problem for Equation 5.1.4 has a unique solution on $(-\infty, \infty)$.

a. Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of Equation 5.1.4 on $(-\infty, \infty)$.
b. Verify that if $c_1$ and $c_2$ are arbitrary constants, $y = c_1e^x + c_2e^{-x}$ is a solution of Equation 5.1.4 on $(-\infty, \infty)$.
c. Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{5.1.5}$$

**Solution**:

a. If $y_1 = e^x$ then $y_1' = e^x$ and $y_1'' = e^x = y_1$, so $y_1'' - y_1 = 0$. If $y_2 = e^{-x}$, then $y_2' = -e^{-x}$ and $y_2'' = e^{-x} = y_2$, so $y_2'' - y_2 = 0$.

b. If

$$y = c_1 e^x + c_2 e^{-x} \tag{5.1.6}$$

then

$$y' = c_1 e^x - c_2 e^{-x} \tag{5.1.7}$$

and

$$y'' = c_1 e^x + c_2 e^{-x},$$

so

$$\begin{aligned} y'' - y &= (c_1 e^x + c_2 e^{-x}) - (c_1 e^x + c_2 e^{-x}) \\ &= c_1(e^x - e^x) + c_2(e^{-x} - e^{-x}) = 0 \end{aligned}$$

for all $x$. Therefore $y = c_1 e^x + c_2 e^{-x}$ is a solution of Equation 5.1.4 on $(-\infty, \infty)$.

c.

We can solve Equation 5.1.5 by choosing $c_1$ and $c_2$ in Equation 5.1.6 so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in Equation 5.1.6 and Equation 5.1.7 shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3. \end{aligned}$$

Solving these equations yields $c_1 = 2$ and $c_2 = -1$. Therefore $y = 2e^x - e^{-x}$ is the unique solution of Equation 5.1.5 on $(-\infty, \infty)$.

## Example 5.1.2

Let $\omega$ be a positive constant. The coefficients of $y'$ and $y$ in

$$y'' + \omega^2 y = 0 \tag{5.1.8}$$

are the constant functions $p \equiv 0$ and $q \equiv \omega^2$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 5.1.1 implies that every initial value problem for Equation 5.1.8 has a unique solution on $(-\infty, \infty)$.

a. Verify that $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$ are solutions of Equation 5.1.8 on $(-\infty, \infty)$.
b. Verify that if $c_1$ and $c_2$ are arbitrary constants then $y = c_1 \cos \omega x + c_2 \sin \omega x$ is a solution of Equation 5.1.8 on $(-\infty, \infty)$.
c. Solve the initial value problem

$$y'' + \omega^2 y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{5.1.9}$$

**Solution**:

a. If $y_1 = \cos \omega x$ then $y_1' = -\omega \sin \omega x$ and $y_1'' = -\omega^2 \cos \omega x = -\omega^2 y_1$, so $y_1'' + \omega^2 y_1 = 0$. If $y_2 = \sin \omega x$ then, $y_2' = \omega \cos \omega x$ and $y_2'' = -\omega^2 \sin \omega x = -\omega^2 y_2$, so $y_2'' + \omega^2 y_2 = 0$.

b. If

$$y = c_1 \cos \omega x + c_2 \sin \omega x \tag{5.1.10}$$

then

$$y' = \omega(-c_1 \sin \omega x + c_2 \cos \omega x) \tag{5.1.11}$$

and

$$y'' = -\omega^2(c_1\cos\omega x + c_2\sin\omega x),$$

so

$$\begin{aligned} y'' + \omega^2 y &= -\omega^2(c_1\cos\omega x + c_2\sin\omega x) + \omega^2(c_1\cos\omega x + c_2\sin\omega x) \\ &= c_1\omega^2(-\cos\omega x + \cos\omega x) + c_2\omega^2(-\sin\omega x + \sin\omega x) = 0 \end{aligned}$$

for all $x$. Therefore $y = c_1\cos\omega x + c_2\sin\omega x$ is a solution of Equation 5.1.8 on $(-\infty,\infty)$.

c. To solve Equation 5.1.9, we must choosing $c_1$ and $c_2$ in Equation 5.1.10 so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in Equation 5.1.10 and Equation 5.1.11 shows that $c_1 = 1$ and $c_2 = 3/\omega$. Therefore

$$y = \cos\omega x + \frac{3}{\omega}\sin\omega x$$

is the unique solution of Equation 5.1.9 on $(-\infty,\infty)$.

Theorem 5.1.1 implies that if $k_0$ and $k_1$ are arbitrary real numbers then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{5.1.12}$$

has a unique solution on an interval $(a,b)$ that contains $x_0$, provided that $P_0$, $P_1$, and $P_2$ are continuous and $P_0$ has no zeros on $(a,b)$. To see this, we rewrite the differential equation in Equation 5.1.12 as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem 5.1.1 with $p = P_1/P_0$ and $q = P_2/P_0$.

## Example 5.1.3

The equation

$$x^2y'' + xy' - 4y = 0 \tag{5.1.13}$$

has the form of the differential equation in Equation 5.1.12, with $P_0(x) = x^2$, $P_1(x) = x$, and $P_2(x) = -4$, which are are all continuous on $(-\infty,\infty)$. However, since $P(0) = 0$ we must consider solutions of Equation 5.1.13 on $(-\infty,0)$ and $(0,\infty)$. Since $P_0$ has no zeros on these intervals, Theorem 5.1.1 implies that the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(0,\infty)$ if $x_0 > 0$, or on $(-\infty,0)$ if $x_0 < 0$.

a. Verify that $y_1 = x^2$ is a solution of Equation 5.1.13 on $(-\infty,\infty)$ and $y_2 = 1/x^2$ is a solution of Equation 5.1.13 on $(-\infty,0)$ and $(0,\infty)$.
b. Verify that if $c_1$ and $c_2$ are any constants then $y = c_1x^2 + c_2/x^2$ is a solution of Equation 5.1.13 on $(-\infty,0)$ and $(0,\infty)$.
c. Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 0. \tag{5.1.14}$$

d. Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 0. \tag{5.1.15}$$

**Solution**:

a. If $y_1 = x^2$ then $y_1' = 2x$ and $y_1'' = 2$, so

$$x^2y_1'' + xy_1' - 4y_1 = x^2(2) + x(2x) - 4x^2 = 0$$

for $x$ in $(-\infty,\infty)$. If $y_2 = 1/x^2$, then $y_2' = -2/x^3$ and $y_2'' = 6/x^4$, so

$$x^2y_2'' + xy_2' - 4y_2 = x^2\left(\frac{6}{x^4}\right) - x\left(\frac{2}{x^3}\right) - \frac{4}{x^2} = 0$$

for $x$ in $(-\infty,0)$ or $(0,\infty)$.

b. If

$$y=c_1x^2+\frac{c_2}{x^2} \tag{5.1.16}$$

then

$$y'=2c_1x-\frac{2c_2}{x^3} \tag{5.1.17}$$

and

$$y''=2c_1+\frac{6c_2}{x^4},$$

so

$$\begin{aligned} x^2y''+xy'-4y &= x^2\left(2c_1+\frac{6c_2}{x^4}\right)+x\left(2c_1x-\frac{2c_2}{x^3}\right)-4\left(c_1x^2+\frac{c_2}{x^2}\right)\\ &= c_1(2x^2+2x^2-4x^2)+c_2\left(\frac{6}{x^2}-\frac{2}{x^2}-\frac{4}{x^2}\right)\\ &= c_1\cdot 0+c_2\cdot 0=0 \end{aligned}$$

for $x$ in $(-\infty,0)$ or $(0,\infty)$.

c. To solve Equation 5.1.14, we choose $c_1$ and $c_2$ in Equation 5.1.16 so that $y(1)=2$ and $y'(1)=0$. Setting $x=1$ in Equation 5.1.16 and Equation 5.1.17 shows that this is equivalent to

$$\begin{aligned} c_1+\ \ c_2 &= 2\\ 2c_1-2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1=1$ and $c_2=1$. Therefore $y=x^2+1/x^2$ is the unique solution of Equation 5.1.14 on $(0,\infty)$.

d. We can solve Equation 5.1.15 by choosing $c_1$ and $c_2$ in Equation 5.1.16 so that $y(-1)=2$ and $y'(-1)=0$. Setting $x=-1$ in Equation 5.1.16 and Equation 5.1.17 shows that this is equivalent to

$$\begin{aligned} c_1+\ \ c_2 &= 2\\ -2c_1+2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1=1$ and $c_2=1$. Therefore $y=x^2+1/x^2$ is the unique solution of Equation 5.1.15 on $(-\infty,0)$.

Although the **formulas** for the solutions of Equation 5.1.14 and Equation 5.1.15 are both $y=x^2+1/x^2$, you should not conclude that these two initial value problems have the same solution. Remember that a solution of an initial value problem is defined **on an interval that contains the initial point**; therefore, the solution of Equation 5.1.14 is $y=x^2+1/x^2$ **on the interval** $(0,\infty)$, which contains the initial point $x_0=1$, while the solution of Equation 5.1.15 is $y=x^2+1/x^2$ **on the interval** $(-\infty,0)$, which contains the initial point $x_0=-1$.

## The General Solution of a Homogeneous Linear Second Order Equation

If $y_1$ and $y_2$ are defined on an interval $(a,b)$ and $c_1$ and $c_2$ are constants, then

$$y=c_1y_1+c_2y_2$$

is a **linear combination of $y_1$** and $y_2$. For example, $y=2\cos x+7\sin x$ is a linear combination of $y_1=\cos x$ and $y_2=\sin x$, with $c_1=2$ and $c_2=7$.

The next theorem states a fact that we've already verified in Examples 5.1.1 , 5.1.2 , 5.1.3 .

**Theorem 5.1.2**

If $y_1$ and $y_2$ are solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.18}$$

on $(a, b)$, then any linear combination

$$y = c_1y_1 + c_2y_2 \tag{5.1.19}$$

of $y_1$ and $y_2$ is also a solution of (5.1.18) on $(a, b)$.

**Proof**

If

$$y = c_1y_1 + c_2y_2$$

then

$$y' = c_1y_1' + c_2y_2' \quad \text{and} \quad y'' = c_1y_1'' + c_2y_2''.$$

Therefore

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (c_1y_1'' + c_2y_2'') + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2) \\ &= c_1\left(y_1'' + p(x)y_1' + q(x)y_1\right) + c_2\left(y_2'' + p(x)y_2' + q(x)y_2\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0, \end{aligned}$$

since $y_1$ and $y_2$ are solutions of Equation 5.1.18

We say that $\{y_1, y_2\}$ is a **fundamental set of solutions of** (5.1.18) on $(a, b)$ if every solution of Equation 5.1.18 on $(a, b)$ can be written as a linear combination of $y_1$ and $y_2$ as in Equation 5.1.19. In this case we say that Equation 5.1.19 is **general solution of** (5.1.18) on $(a, b)$.

## Linear Independence

We need a way to determine whether a given set $\{y_1, y_2\}$ of solutions of Equation 5.1.18 is a fundamental set. The next definition will enable us to state necessary and sufficient conditions for this.

We say that two functions $y_1$ and $y_2$ defined on an interval $(a, b)$ are **linearly independent on** $(a, b)$ if neither is a constant multiple of the other on $(a, b)$. (In particular, this means that neither can be the trivial solution of Equation 5.1.18, since, for example, if $y_1 \equiv 0$ we could write $y_1 = 0y_2$.) We'll also say that the set $\{y_1, y_2\}$ **is linearly independent on** $(a, b)$.

**Theorem 5.1.3**

Suppose $p$ and $q$ are continuous on $(a, b)$. Then a set $\{y_1, y_2\}$ of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.20}$$

on $(a, b)$ is a fundamental set if and only if $\{y_1, y_2\}$ is linearly independent on $(a, b)$.

**Proof**

We'll present the proof of Theorem 5.1.3 in steps worth regarding as theorems in their own right. However, let's first interpret Theorem 5.1.3 in terms of Examples 5.1.1 , 5.1.2 , 5.1.3 .

**Example 5.1.4**

Since $e^x/e^{-x} = e^{2x}$ is nonconstant, Theorem 5.1.3 implies that $y = c_1e^x + c_2e^{-x}$ is the general solution of $y'' - y = 0$ on $(-\infty, \infty)$.

Since $\cos\omega x/\sin\omega x=\cot\omega x$ is nonconstant, Theorem 5.1.3 implies that $y=c_1\cos\omega x+c_2\sin\omega x$ is the general solution of $y''+\omega^2y=0$ on $(-\infty,\infty)$.

Since $x^2/x^{-2}=x^4$ is nonconstant, Theorem 5.1.3 implies that $y=c_1x^2+c_2/x^2$ is the general solution of $x^2y''+xy'-4y=0$ on $(-\infty,0)$ and $(0,\infty)$.

## The Wronskian and Abel's Formula

To motivate a result that we need in order to prove Theorem 5.1.3 , let's see what is required to prove that $\{y_1,y_2\}$ is a fundamental set of solutions of Equation 5.1.20 on $(a,b)$. Let $x_0$ be an arbitrary point in $(a,b)$, and suppose $y$ is an arbitrary solution of Equation 5.1.20 on $(a,b)$. Then $y$ is the unique solution of the initial value problem

$$y''+p(x)y'+q(x)y=0,\quad y(x_0)=k_0,\quad y'(x_0)=k_1; \tag{5.1.21}$$

that is, $k_0$ and $k_1$ are the numbers obtained by evaluating $y$ and $y'$ at $x_0$. Moreover, $k_0$ and $k_1$ can be any real numbers, since Theorem 5.1.1 implies that Equation 5.1.21 has a solution no matter how $k_0$ and $k_1$ are chosen. Therefore $\{y_1,y_2\}$ is a fundamental set of solutions of Equation 5.1.20 on $(a,b)$ if and only if it is possible to write the solution of an arbitrary initial value problem Equation 5.1.21 as $y=c_1y_1+c_2y_2$ . This is equivalent to requiring that the system

$$\begin{aligned} c_1y_1(x_0)+c_2y_2(x_0)&=k_0\\ c_1y_1'(x_0)+c_2y_2'(x_0)&=k_1 \end{aligned} \tag{5.1.22}$$

has a solution $(c_1,c_2)$ for every choice of $(k_0,k_1)$. Let's try to solve Equation 5.1.22.

Multiplying the first equation in Equation 5.1.22 by $y_2'(x_0)$ and the second by $y_2(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_2'(x_0)+c_2y_2(x_0)y_2'(x_0)&=y_2'(x_0)k_0\\ c_1y_1'(x_0)y_2(x_0)+c_2y_2'(x_0)y_2(x_0)&=y_2(x_0)k_1, \end{aligned}$$

and subtracting the second equation here from the first yields

$$(y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0))\,c_1=y_2'(x_0)k_0-y_2(x_0)k_1. \tag{5.1.23}$$

Multiplying the first equation in Equation 5.1.22 by $y_1'(x_0)$ and the second by $y_1(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_1'(x_0)+c_2y_2(x_0)y_1'(x_0)&=y_1'(x_0)k_0\\ c_1y_1'(x_0)y_1(x_0)+c_2y_2'(x_0)y_1(x_0)&=y_1(x_0)k_1, \end{aligned}$$

and subtracting the first equation here from the second yields

$$(y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0))\,c_2=y_1(x_0)k_1-y_1'(x_0)k_0. \tag{5.1.24}$$

If

$$y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)=0,$$

it is impossible to satisfy Equation 5.1.23 and Equation 5.1.24 (and therefore Equation 5.1.22) unless $k_0$ and $k_1$ happen to satisfy

$$\begin{aligned} y_1(x_0)k_1-y_1'(x_0)k_0&=0\\ y_2'(x_0)k_0-y_2(x_0)k_1&=0. \end{aligned}$$

On the other hand, if

$$y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)\neq0 \tag{5.1.25}$$

we can divide Equation 5.1.23 and Equation 5.1.24 through by the quantity on the left to obtain

$$\begin{aligned} c_1&=\frac{y_2'(x_0)k_0-y_2(x_0)k_1}{y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)}\\ c_2&=\frac{y_1(x_0)k_1-y_1'(x_0)k_0}{y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)}, \end{aligned} \tag{5.1.26}$$

no matter how $k_0$ and $k_1$ are chosen. This motivates us to consider conditions on $y_1$ and $y_2$ that imply Equation 5.1.25.

**Theorem 5.1.4**

Suppose $p$ and $q$ are continuous on $(a,b)$, let $y_1$ and $y_2$ be solutions of

$$y''+p(x)y'+q(x)y=0 \tag{5.1.27}$$

on $(a,b)$, and define

$$W=y_1y_2'-y_1'y_2. \tag{5.1.28}$$

Let $x_0$ be any point in $(a,b)$. Then

$$W(x)=W(x_0)e^{-\int_{x_0}^{x}p(t)\,dt},\quad a<x<b \tag{5.1.29}$$

Therefore either $W$ has no zeros in $(a,b)$ or $W\equiv 0$ on $(a,b)$.

**Proof**

Differentiating Equation 5.1.28yields

$$W'=y_1'y_2'+y_1y_2''-y_1'y_2'-y_1''y_2=y_1y_2''-y_1''y_2. \tag{5.1.30}$$

Since $y_1$ and $y_2$ both satisfy Equation 5.1.27,

$$y_1''=-py_1'-qy_1 \quad\text{and}\quad y_2''=-py_2'-qy_2.$$

Substituting these into Equation 5.1.30yields

$$\begin{aligned} W' &= -y_1\left(py_2'+qy_2\right)+y_2\left(py_1'+qy_1\right)\\ &= -p(y_1y_2'-y_2y_1')-q(y_1y_2-y_2y_1)\\ &= -p(y_1y_2'-y_2y_1')=-pW. \end{aligned}$$

Therefore $W'+p(x)W=0$ ; that is, $W$ is the solution of the initial value problem

$$y'+p(x)y=0,\quad y(x_0)=W(x_0).$$

We leave it to you to verify by separation of variables that this implies Equation 5.1.29. If $W(x_0)\neq 0$, Equation 5.1.29 implies that $W$ has no zeros in $(a,b)$, since an exponential is never zero. On the other hand, if $W(x_0)=0$, Equation 5.1.29implies that $W(x)=0$ for all $x$ in $(a,b)$.

The function $W$ defined in Equation 5.1.28is the **Wronskian of $\{y_1,y_2\}$**. Formula Equation 5.1.29is **Abel's formula**.

The Wronskian of $\{y_1,y_2\}$ is usually written as the determinant

$$W=\begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix}.$$

The expressions in Equation 5.1.26for $c_1$ and $c_2$ can be written in terms of determinants as

$$c_1=\frac{1}{W(x_0)}\begin{vmatrix} k_0 & y_2(x_0)\\ k_1 & y_2'(x_0)\end{vmatrix}\quad\text{and}\quad c_2=\frac{1}{W(x_0)}\begin{vmatrix} y_1(x_0) & k_0\\ y_1'(x_0) & k_1\end{vmatrix}.$$

If you've taken linear algebra you may recognize this as **Cramer's rule**.

**Example 5.1.5**

Verify Abel's formula for the following differential equations and the corresponding solutions, from Examples 5.1.1 , 5.1.2 , 5.1.3 .

a. $y''-y=0;\quad y_1=e^x,\ y_2=e^{-x}$
b. $y''+\omega^2y=0;\quad y_1=\cos\omega x,\ y_2=\sin\omega x$
c. $x^2y''+xy'-4y=0;\quad y_1=x^2,\ y_2=1/x^2$

**Solution**:

a. Since $p \equiv 0$, we can verify Abel's formula by showing that $W$ is constant, which is true, since

$$W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^x e^{-x} = -2$$

for all $x$.

b. Again, since $p \equiv 0$, we can verify Abel's formula by showing that $W$ is constant, which is true, since

$$\begin{aligned} W(x) &= \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} \\ &= \cos \omega x(\omega \cos \omega x) - (-\omega \sin \omega x) \sin \omega x \\ &= \omega(\cos^2 \omega x + \sin^2 \omega x) = \omega \end{aligned}$$

for all $x$.

c. Computing the Wronskian of $y_1 = x^2$ and $y_2 = 1/x^2$ directly yields

$$W = \begin{vmatrix} x^2 & 1/x^2 \\ 2x & -2/x^3 \end{vmatrix} = x^2\left(-\frac{2}{x^3}\right) - 2x\left(\frac{1}{x^2}\right) = -\frac{4}{x}. \tag{5.1.31}$$

To verify Abel's formula we rewrite the differential equation as

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$$

to see that $p(x) = 1/x$. If $x_0$ and $x$ are either both in $(-\infty, 0)$ or both in $(0, \infty)$ then

$$\int_{x_0}^{x} p(t)\,dt = \int_{x_0}^{x} \frac{dt}{t} = \ln\left(\frac{x}{x_0}\right),$$

so Abel's formula becomes

$$\begin{aligned} W(x) &= W(x_0)e^{-\ln(x/x_0)} = W(x_0)\frac{x_0}{x} \\ &= -\left(\frac{4}{x_0}\right)\left(\frac{x_0}{x}\right) \quad \text{from (5.1.31)} \\ &= -\frac{4}{x}, \end{aligned}$$

which is consistent with Equation 5.1.31.

The next theorem will enable us to complete the proof of Theorem 5.1.3 .

## Theorem 5.1.5

Suppose $p$ and $q$ are continuous on an open interval $(a, b)$, let $y_1$ and $y_2$ be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.32}$$

on $(a, b)$, and let $W = y_1y_2' - y_1'y_2$. Then $y_1$ and $y_2$ are linearly independent on $(a, b)$ if and only if $W$ has no zeros on $(a, b)$.

**Proof**

We first show that if $W(x_0) = 0$ for some $x_0$ in $(a, b)$, then $y_1$ and $y_2$ are linearly dependent on $(a, b)$. Let $I$ be a subinterval of $(a, b)$ on which $y_1$ has no zeros. (If there's no such subinterval, $y_1 \equiv 0$ on $(a, b)$, so $y_1$ and $y_2$ are linearly independent, and we are finished with this part of the proof.) Then $y_2/y_1$ is defined on $I$, and

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W}{y_1^2}. \tag{5.1.33}$$

However, if $W(x_0) = 0$, Theorem 5.1.4 implies that $W \equiv 0$ on $(a, b)$. Therefore Equation 5.1.33 implies that $(y_2/y_1)' \equiv 0$, so $y_2/y_1 = c$ (constant) on $I$. This shows that $y_2(x) = cy_1(x)$ for all $x$ in $I$. However, we want to show that $y_2 = cy_1(x)$ for all $x$ in $(a, b)$. Let $Y = y_2 - cy_1$ . Then $Y$ is a solution of Equation 5.1.32on $(a, b)$ such that $Y \equiv 0$ on $I$, and therefore $Y' \equiv 0$ on $I$. Consequently, if $x_0$ is chosen arbitrarily in $I$ then $Y$ is a solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0,$$

which implies that $Y \equiv 0$ on $(a, b)$, by the paragraph following Theorem 5.1.1 . (See also **Exercise 5.1.24**). Hence, $y_2 - cy_1 \equiv 0$ on $(a, b)$, which implies that $y_1$ and $y_2$ are not linearly independent on $(a, b)$.

Now suppose $W$ has no zeros on $(a, b)$. Then $y_1$ can't be identically zero on $(a, b)$ (why not?), and therefore there is a subinterval $I$ of $(a, b)$ on which $y_1$ has no zeros. Since Equation 5.1.33implies that $y_2/y_1$ is nonconstant on $I$, $y_2$ isn't a constant multiple of $y_1$ on $(a, b)$. A similar argument shows that $y_1$ isn't a constant multiple of $y_2$ on $(a, b)$, since

$$\left(\frac{y_1}{y_2}\right)' = \frac{y_1'y_2 - y_1y_2'}{y_2^2} = -\frac{W}{y_2^2}$$

on any subinterval of $(a, b)$ where $y_2$ has no zeros.

We can now complete the proof of Theorem 5.1.3 . From Theorem 5.1.5 , two solutions $y_1$ and $y_2$ of Equation 5.1.32 are linearly independent on $(a, b)$ if and only if $W$ has no zeros on $(a, b)$. From Theorem 5.1.4 and the motivating comments preceding it, $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 5.1.32 if and only if $W$ has no zeros on $(a, b)$. Therefore $\{y_1, y_2\}$ is a fundamental set for Equation 5.1.32on $(a, b)$ if and only if $\{y_1, y_2\}$ is linearly independent on $(a, b)$.

The next theorem summarizes the relationships among the concepts discussed in this section.

### Theorem 5.1.6

Suppose $p$ and $q$ are continuous on an open interval $(a, b)$ and let $y_1$ and $y_2$ be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.34}$$

on $(a, b)$. Then the following statements are equivalent; that is, they are either all true or all false.

a. The general solution of (5.1.34) on $(a, b)$ is $y = c_1y_1 + c_2y_2$ .
b. $\{y_1, y_2\}$ is a fundamental set of solutions of (5.1.34) on $(a, b)$.
c. $\{y_1, y_2\}$ is linearly independent on $(a, b)$.
d. The Wronskian of $\{y_1, y_2\}$ is nonzero at some point in $(a, b)$.
e. The Wronskian of $\{y_1, y_2\}$ is nonzero at all points in $(a, b)$.

We can apply this theorem to an equation written as

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

on an interval $(a, b)$ where $P_0$, $P_1$, and $P_2$ are continuous and $P_0$ has no zeros.dd proof here and it will automatically be hidden

### Theorem 5.1.7

Suppose $c$ is in $(a, b)$ and $\alpha$ and $\beta$ are real numbers, not both zero. Under the assumptions of Theorem 5.1.7 , suppose $y_1$ and $y_2$ are solutions of Equation 5.1.34such that

$$\alpha y_1(c) + \beta y_1'(c) = 0 \quad \text{and} \quad \alpha y_2(c) + \beta y_2'(c) = 0. \tag{5.1.35}$$

Then $\{y_1, y_2\}$ isn't linearly independent on $(a, b)$.

**Proof**

Since $\alpha$ and $\beta$ are not both zero, Equation 5.1.35implies that

$$\begin{vmatrix} y_1(c) & y_1'(c) \\ y_2(c) & y_2'(c) \end{vmatrix} = 0, \quad \text{so} \quad \begin{vmatrix} y_1(c) & y_2(c) \\ y_1'(c) & y_2'(c) \end{vmatrix} = 0$$

and Theorem 5.1.6 implies the stated conclusion.

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# 5.1E: Homogeneous Linear Equations (Exercises)

## Q5.1.1

1.

a. Verify that $y_1 = e^{2x}$ and $y_2 = e^{5x}$ are solutions of

$$y'' - 7y' + 10y = 0 \tag{A}$$

on $(-\infty, \infty)$.

b. Verify that if $c_1$ and $c_2$ are arbitrary constants then $y = c_1 e^{2x} + c_2 e^{5x}$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 1.$$

d. Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

2.

a. Verify that $y_1 = e^x \cos x$ and $y_2 = e^x \sin x$ are solutions of

$$y'' - 2y' + 2y = 0 \tag{A}$$

on $(-\infty, \infty)$.

b. Verify that if $c_1$ and $c_2$ are arbitrary constants then $y = c_1 e^x \cos x + c_2 e^x \sin x$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = -2.$$

d. Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

3.

a. Verify that $y_1 = e^x$ and $y_2 = xe^x$ are solutions of

$$y'' - 2y' + y = 0 \tag{A}$$

on $(-\infty, \infty)$.

b. Verify that if $c_1$ and $c_2$ are arbitrary constants then $y = e^x(c_1 + c_2 x)$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 7, \quad y'(0) = 4.$$

d. Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

4.

a. Verify that $y_1 = 1/(x-1)$ and $y_2 = 1/(x+1)$ are solutions of

$$(x^2 - 1)y'' + 4xy' + 2y = 0 \tag{A}$$

on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. What is the general solution of (A) on each of these intervals?

b. Solve the initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = -5, \quad y'(0) = 1.$$

What is the interval of validity of the solution?

c. Graph the solution of the initial value problem.

d. Verify Abel's formula for $y_1$ and $y_2$, with $x_0 = 0$.

5. Compute the Wronskians of the given sets of functions.

a. $\{1, e^x\}$
b. $\{e^x, e^x \sin x\}$
c. $\{x+1, x^2+2\}$
d. $\{x^{1/2}, x^{-1/3}\}$
e. $\{\frac{\sin x}{x}, \frac{\cos x}{x}\}$
f. $\{x \ln|x|, x^2 \ln|x|\}$
g. $\{e^x \cos\sqrt{x}, e^x \sin\sqrt{x}\}$

6. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$y'' + 3(x^2+1)y' - 2y = 0,$$

given that $W(\pi) = 0$.

7. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

given that $W(0) = 1$. (This is **Legendre's equation**.)

8. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

given that $W(1) = 1$. (This is **Bessel's equation**.)

9. (This exercise shows that if you know one nontrivial solution of $y'' + p(x)y' + q(x)y = 0$ , you can use Abel's formula to find another.)

Suppose $p$ and $q$ are continuous and $y_1$ is a solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

that has no zeros on $(a, b)$. Let $P(x) = \int p(x)\,dx$ be any antiderivative of $p$ on $(a, b)$.

a. Show that if $K$ is an arbitrary nonzero constant and $y_2$ satisfies

$$y_1y_2' - y_1'y_2 = Ke^{-P(x)} \tag{B}$$

on $(a, b)$, then $y_2$ also satisfies (A) on $(a, b)$, and $\{y_1, y_2\}$ is a fundamental set of solutions on (A) on $(a, b)$.

b. Conclude from (a) that if $y_2 = uy_1$ where $u' = K\frac{e^{-P(x)}}{y_1^2(x)}$ , then $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(a, b)$.

## Q5.1.2

In **Exercises 5.1.10-5.1.23** use the method suggested by **Exercise 5.1.9** to find a second solution $y_2$ that isn't a constant multiple of the solution $y_1$. Choose $K$ conveniently to simplify $y_2$.

10. $y'' - 2y' - 3y = 0$ ; $y_1 = e^{3x}$

11. $y'' - 6y' + 9y = 0$ ; $y_1 = e^{3x}$

12. $y'' - 2ay' + a^2y = 0$ ($a =$ constant); $y_1 = e^{ax}$

13. $x^2y'' + xy' - y = 0$ ; $y_1 = x$

14. $x^2y'' - xy' + y = 0$ ; $y_1 = x$

15. $x^2y'' - (2a-1)xy' + a^2y = 0$ ($a =$ nonzero constant); $x > 0$; $y_1 = x^a$

16. $4x^2y'' - 4xy' + (3 - 16x^2)y = 0$ ; $y_1 = x^{1/2}e^{2x}$

17. $(x-1)y'' - xy' + y = 0$ ; $y_1 = e^x$

18. $x^2y'' - 2xy' + (x^2+2)y = 0$ ; $y_1 = x\cos x$

19. $4x^2(\sin x)y'' - 4x(x\cos x + \sin x)y' + (2x\cos x + 3\sin x)y = 0\,;\ y_1 = x^{1/2}$

20. $(3x-1)y'' - (3x+2)y' - (6x-8)y = 0\ \ ;\ y_1 = e^{2x}$

21. $(x^2-4)y'' + 4xy' + 2y = 0\ ;\ y_1 = \frac{1}{x-2}$

22. $(2x+1)xy'' - 2(2x^2-1)y' - 4(x+1)y = 0\ \ ;y_1 = \frac{1}{x}$

23. $(x^2-2x)y'' + (2-x^2)y' + (2x-2)y = 0\ \ ;y_1 = e^x$

## Q5.1.3

24. Suppose $p$ and $q$ are continuous on an open interval $(a,b)$ and let $x_0$ be in $(a,b)$. Use Theorem 5.1.1 to show that the only solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0$$

on $(a,b)$ is the trivial solution $y \equiv 0$.

25. Suppose $P_0$, $P_1$, and $P_2$ are continuous on $(a,b)$ and let $x_0$ be in $(a,b)$. Show that if either of the following statements is true then $P_0(x) = 0$ for some $x$ in $(a,b)$.

a. The initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has more than one solution on $(a,b)$.

b. The initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0$$

has a nontrivial solution on $(a,b)$.

26. Suppose $p$ and $q$ are continuous on $(a,b)$ and $y_1$ and $y_2$ are solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

on $(a,b)$. Let

$$z_1 = \alpha y_1 + \beta y_2 \quad \text{and} \quad z_2 = \gamma y_1 + \delta y_2,$$

where $\alpha$, $\beta$, $\gamma$, and $\delta$ are constants. Show that if $\{z_1, z_2\}$ is a fundamental set of solutions of (A) on $(a,b)$ then so is $\{y_1, y_2\}$.

27. Suppose $p$ and $q$ are continuous on $(a,b)$ and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

on $(a,b)$. Let

$$z_1 = \alpha y_1 + \beta y_2 \quad \text{and} \quad z_2 = \gamma y_1 + \delta y_2,$$

where $\alpha, \beta, \gamma$, and $\delta$ are constants. Show that $\{z_1, z_2\}$ is a fundamental set of solutions of (A) on $(a,b)$ if and only if $\alpha\gamma - \beta\delta \neq 0$.

28. Suppose $y_1$ is differentiable on an interval $(a,b)$ and $y_2 = ky_1$, where $k$ is a constant. Show that the Wronskian of $\{y_1, y_2\}$ is identically zero on $(a,b)$.

29. Let

$$y_1 = x^3 \quad \text{and} \quad y_2 = \begin{cases} x^3, & x \geq 0, \\ -x^3, & x < 0. \end{cases}$$

a. Show that the Wronskian of $\{y_1, y_2\}$ is defined and identically zero on $(-\infty, \infty)$.

b. Suppose $a < 0 < b$. Show that $\{y_1, y_2\}$ is linearly independent on $(a,b)$.

c. Use **Exercise 5.1.25b** to show that these results don't contradict Theorem 5.1.5, because neither $y_1$ nor $y_2$ can be a solution of an equation

$$y'' + p(x)y' + q(x)y = 0$$

on $(a,b)$ if $p$ and $q$ are continuous on $(a,b)$.

30. Suppose $p$ and $q$ are continuous on $(a,b)$ and $\{y_1,y_2\}$ is a set of solutions of

$$y''+p(x)y'+q(x)y=0$$

on $(a,b)$ such that either $y_1(x_0)=y_2(x_0)=0$ or $y_1'(x_0)=y_2'(x_0)=0$ for some $x_0$ in $(a,b)$. Show that $\{y_1,y_2\}$ is linearly dependent on $(a,b)$.

31. Suppose $p$ and $q$ are continuous on $(a,b)$ and $\{y_1,y_2\}$ is a fundamental set of solutions of

$$y''+p(x)y'+q(x)y=0$$

on $(a,b)$. Show that if $y_1(x_1)=y_1(x_2)=0$ , where $a<x_1<x_2<b$ , then $y_2(x)=0$ for some $x$ in $(x_1,x_2)$.

32. Suppose $p$ and $q$ are continuous on $(a,b)$ and every solution of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

on $(a,b)$ can be written as a linear combination of the twice differentiable functions $\{y_1,y_2\}$. Use Theorem 5.1.1 to show that $y_1$ and $y_2$ are themselves solutions of (A) on $(a,b)$.

33. Suppose $p_1$, $p_2$, $q_1$, and $q_2$ are continuous on $(a,b)$ and the equations

$$y''+p_1(x)y'+q_1(x)y=0 \quad \text{and} \quad y''+p_2(x)y'+q_2(x)y=0$$

have the same solutions on $(a,b)$. Show that $p_1=p_2$ and $q_1=q_2$ on $(a,b)$.

34. (For this exercise you have to know about $3\times 3$ determinants.) Show that if $y_1$ and $y_2$ are twice continuously differentiable on $(a,b)$ and the Wronskian $W$ of $\{y_1,y_2\}$ has no zeros in $(a,b)$ then the equation

$$\frac{1}{W}\begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix}=0$$

can be written as

$$y''+p(x)y'+q(x)y=0, \tag{A}$$

where $p$ and $q$ are continuous on $(a,b)$ and $\{y_1,y_2\}$ is a fundamental set of solutions of (A) on $(a,b)$.

35. Use the method suggested by **Exercise 5.1.34** to find a linear homogeneous equation for which the given functions form a fundamental set of solutions on some interval.

a. $e^x\cos 2x, e^x\sin 2x$
b. $x, e^{2x}$
c. $x, x\ln x$
d. $\cos(\ln x), \sin(\ln x)$
e. $\cosh x, \sinh x$
f. $x^2-1, x^2+1$

36. Suppose $p$ and $q$ are continuous on $(a,b)$ and $\{y_1,y_2\}$ is a fundamental set of solutions of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

on $(a,b)$. Show that if $y$ is a solution of (A) on $(a,b)$, there's exactly one way to choose $c_1$ and $c_2$ so that $y=c_1y_1+c_2y_2$ on $(a,b)$.

37. Suppose $p$ and $q$ are continuous on $(a,b)$ and $x_0$ is in $(a,b)$. Let $y_1$ and $y_2$ be the solutions of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

such that

$$y_1(x_0)=1, \quad y_1'(x_0)=0 \quad \text{and} \quad y_2(x_0)=0,\ y_2'(x_0)=1.$$

(Theorem 5.1.1 implies that each of these initial value problems has a unique solution on $(a,b)$.)

a. Show that $\{y_1,y_2\}$ is linearly independent on $(a,b)$.
b. Show that an arbitrary solution $y$ of (A) on $(a,b)$ can be written as $y=y(x_0)y_1+y'(x_0)y_2$ .
c. Express the solution of the initial value problem

$$y''+p(x)y'+q(x)y=0,\quad y(x_0)=k_0,\quad y'(x_0)=k_1$$

as a linear combination of $y_1$ and $y_2$.

38. Find solutions $y_1$ and $y_2$ of the equation $y''=0$ that satisfy the initial conditions

$$y_1(x_0)=1,\quad y_1'(x_0)=0\quad\text{and}\quad y_2(x_0)=0,\quad y_2'(x_0)=1.$$

Then use **Exercise 5.1.37 (c)** to write the solution of the initial value problem

$$y''=0,\quad y(0)=k_0,\quad y'(0)=k_1$$

as a linear combination of $y_1$ and $y_2$.

39. Let $x_0$ be an arbitrary real number. Given (Example 5.1.1) that $e^x$ and $e^{-x}$ are solutions of $y''-y=0$ , find solutions $y_1$ and $y_2$ of $y''-y=0$ such that

$$y_1(x_0)=1,\quad y_1'(x_0)=0\quad\text{and}\quad y_2(x_0)=0,\ y_2'(x_0)=1.$$

Then use **Exercise 5.1.37 (c)** to write the solution of the initial value problem

$$y''-y=0,\quad y(x_0)=k_0,\quad y'(x_0)=k_1$$

as a linear combination of $y_1$ and $y_2$.

40. Let $x_0$ be an arbitrary real number. Given (Example 5.1.2) that $\cos\omega x$ and $\sin\omega x$ are solutions of $y''+\omega^2y=0$ , find solutions of $y''+\omega^2y=0$ such that

$$y_1(x_0)=1,\quad y_1'(x_0)=0\quad\text{and}\quad y_2(x_0)=0,\ y_2'(x_0)=1.$$

Then use **Exercise 5.1.37 (c)** to write the solution of the initial value problem

$$y''+\omega^2y=0,\quad y(x_0)=k_0,\quad y'(x_0)=k_1$$

as a linear combination of $y_1$ and $y_2$. Use the identities

$$\begin{aligned}\cos(A+B)&=\cos A\cos B-\sin A\sin B\\ \sin(A+B)&=\sin A\cos B+\cos A\sin B\end{aligned}$$

to simplify your expressions for $y_1$, $y_2$, and $y$.

41. Recall from **Exercise 5.1.4** that $1/(x-1)$ and $1/(x+1)$ are solutions of

$$(x^2-1)y''+4xy'+2y=0 \tag{A}$$

on $(-1,1)$. Find solutions of (A) such that

$$y_1(0)=1,\quad y_1'(0)=0\quad\text{and}\quad y_2(0)=0,\ y_2'(0)=1.$$

Then use **Exercise 5.1.37 (c)** to write the solution of initial value problem

$$(x^2-1)y''+4xy'+2y=0,\quad y(0)=k_0,\quad y'(0)=k_1$$

as a linear combination of $y_1$ and $y_2$.

42.

a. Verify that $y_1=x^2$ and $y_2=x^3$ satisfy

$$x^2y''-4xy'+6y=0 \tag{A}$$

on $(-\infty,\infty)$ and that $\{y_1,y_2\}$ is a fundamental set of solutions of (A) on $(-\infty,0)$ and $(0,\infty)$.
b. Let $a_1$, $a_2$, $b_1$, and $b_2$ be constants. Show that

$$y = \begin{cases} a_1 x^2 + a_2 x^3, & x \geq 0, \\ b_1 x^2 + b_2 x^3, & x < 0 \end{cases}$$

is a solution of (A) on $(-\infty, \infty)$ if and only if $a_1 = b_1$. From this, justify the statement that $y$ is a solution of (A) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} c_1 x^2 + c_2 x^3, & x \geq 0, \\ c_1 x^2 + c_3 x^3, & x < 0, \end{cases}$$

where $c_1$, $c_2$, and $c_3$ are arbitrary constants.

c. For what values of $k_0$ and $k_1$ does the initial value problem

$$x^2 y'' - 4xy' + 6y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

d. Show that if $x_0 \neq 0$ and $k_0, k_1$ are arbitrary constants, the initial value problem

$$x^2 y'' - 4xy' + 6y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{B}$$

has infinitely many solutions on $(-\infty, \infty)$. On what interval does (B) have a unique solution?

43.

a. Verify that $y_1 = x$ and $y_2 = x^2$ satisfy

$$x^2 y'' - 2xy' + 2y = 0 \tag{A}$$

on $(-\infty, \infty)$ and that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(-\infty, 0)$ and $(0, \infty)$.

b. Let $a_1$, $a_2$, $b_1$, and $b_2$ be constants. Show that

$$y = \begin{cases} a_1 x + a_2 x^2, & x \geq 0, \\ b_1 x + b_2 x^2, & x < 0 \end{cases}$$

is a solution of (A) on $(-\infty, \infty)$ if and only if $a_1 = b_1$ and $a_2 = b_2$. From this, justify the statement that the general solution of (A) on $(-\infty, \infty)$ is $y = c_1 x + c_2 x^2$, where $c_1$ and $c_2$ are arbitrary constants.

c. For what values of $k_0$ and $k_1$ does the initial value problem

$$x^2 y'' - 2xy' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

d. Show that if $x_0 \neq 0$ and $k_0, k_1$ are arbitrary constants then the initial value problem

$$x^2 y'' - 2xy' + 2y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(-\infty, \infty)$.

44.

a. Verify that $y_1 = x^3$ and $y_2 = x^4$ satisfy

$$x^2 y'' - 6xy' + 12y = 0 \tag{A}$$

on $(-\infty, \infty)$, and that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(-\infty, 0)$ and $(0, \infty)$.

b. Show that $y$ is a solution of (A) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} a_1 x^3 + a_2 x^4, & x \geq 0, \\ b_1 x^3 + b_2 x^4, & x < 0, \end{cases}$$

where $a_1$, $a_2$, $b_1$, and $b_2$ are arbitrary constants.

c. For what values of $k_0$ and $k_1$ does the initial value problem

$$x^2 y'' - 6xy' + 12y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

d. Show that if $x_0 \neq 0$ and $k_0, k_1$ are arbitrary constants then the initial value problem

$$x^2y'' - 6xy' + 12y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{B}$$

has infinitely many solutions on $(-\infty, \infty)$. On what interval does (B) have a unique solution?

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# 5.2: Constant Coefficient Homogeneous Equations

This section deals with homogeneous equations of the special form

$$ay'' + by' + cy = 0, \tag{5.2.1}$$

where $a$, $b$, and $c$ are constant ($a \neq 0$). When you've completed this section you'll know everything there is to know about solving such equations.

If $a$, $b$, and $c$ are real constants and $a \neq 0$, then

$$ay'' + by' + cy = F(x)$$

is said to be a **constant coefficient equation**. In this section we consider the homogeneous constant coefficient equation

$$ay'' + by' + cy = 0. \tag{5.2.2}$$

As we'll see, all solutions of Equation 5.2.2 are defined on $(-\infty, \infty)$. This being the case, we'll omit references to the interval on which solutions are defined, or on which a given set of solutions is a fundamental set, etc., since the interval will always be $(-\infty, \infty)$.

The key to solving Equation 5.2.2 is that if $y = e^{rx}$ where $r$ is a constant then the left side of Equation 5.2.2 is a multiple of $e^{rx}$; thus, if $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2e^{rx}$, so

$$ay'' + by' + cy = ar^2e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx}. \tag{5.2.3}$$

The quadratic polynomial

$$p(r) = ar^2 + br + c$$

is the **characteristic polynomial** of Equation 5.2.2, and $p(r) = 0$ is the **characteristic equation**. From Equation 5.2.3 we can see that $y = e^{rx}$ is a solution of Equation 5.2.2 if and only if $p(r) = 0$.

The roots of the characteristic equation are given by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{5.2.4}$$

**Consider Three Cases**

- Case 1: $b^2 - 4ac > 0$, so the characteristic equation has two distinct real roots.
- Case 2: $b^2 - 4ac = 0$, so the characteristic equation has a repeated real root.
- Case 3: $b^2 - 4ac < 0$, so the characteristic equation has complex roots.

In each case we'll start with an example.

## Case 1: Distinct Real Roots

**Example 5.2.1**

a. Find the general solution of

$$y'' + 6y' + 5y = 0. \tag{5.2.5}$$

b. Solve the initial value problem

$$y'' + 6y' + 5y = 0, \quad y(0) = 3,\ y'(0) = -1. \tag{5.2.6}$$

**Solution**

a. The characteristic polynomial of Equation 5.2.5 is

$$p(r) = r^2 + 6r + 5 = (r + 1)(r + 5).$$

Since $p(-1)=p(-5)=0$, $y_1=e^{-x}$ and $y_2=e^{-5x}$ are solutions of Equation 5.2.5. Since $y_2/y_1=e^{-4x}$ is nonconstant, Theorem 5.1.6 implies that the general solution of Equation 5.2.5 is

$$y=c_1e^{-x}+c_2e^{-5x}. \tag{5.2.7}$$

b. We must determine $c_1$ and $c_2$ in Equation 5.2.7 so that $y$ satisfies the initial conditions in Equation 5.2.6. Differentiating Equation 5.2.7 yields

$$y'=-c_1e^{-x}-5c_2e^{-5x}. \tag{5.2.8}$$

Imposing the initial conditions $y(0)=3$, $y'(0)=-1$ in Equation 5.2.7 and Equation 5.2.8 yields

$$\begin{aligned} c_1+\ c_2 &= 3\\ -c_1-5c_2 &= -1. \end{aligned}$$

The solution of this system is $c_1=7/2, c_2=-1/2$. Therefore the solution of Equation 5.2.6 is

$$y=\frac{7}{2}e^{-x}-\frac{1}{2}e^{-5x}.$$

Figure 5.2.1 is a graph of this solution.

If the characteristic equation has arbitrary distinct real roots $r_1$ and $r_2$, then $y_1=e^{r_1x}$ and $y_2=e^{r_2x}$ are solutions of $ay''+by'+cy=0$ . Since $y_2/y_1=e^{(r_2-r_1)x}$ is nonconstant, Theorem 5.1.6 implies that $\{y_1,y_2\}$ is a fundamental set of solutions of $ay''+by'+cy=0$ .

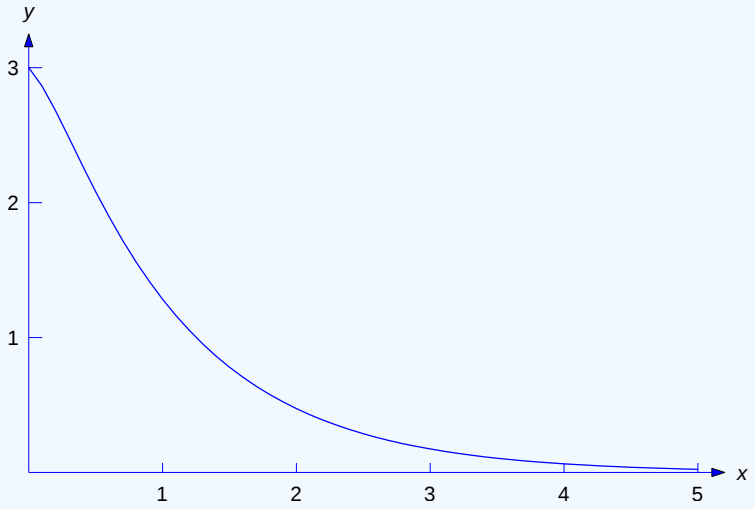


Figure 5.2.1 : $y=\frac{7}{2}e^{-x}-\frac{1}{2}e^{-5x}$

## Case 2: A Repeated Real Root

### Example 5.2.2

a. Find the general solution of

$$y''+6y'+9y=0. \tag{5.2.9}$$

b. Solve the initial value problem

$$y''+6y'+9y=0,\quad y(0)=3,\ y'(0)=-1. \tag{5.2.10}$$

**Solution**

a. The characteristic polynomial of Equation 5.2.9 is

$$p(r)=r^2+6r+9=(r+3)^2,$$

so the characteristic equation has the repeated real root $r_1 = -3$. Therefore $y_1 = e^{-3x}$ is a solution of Equation 5.2.9. Since the characteristic equation has no other roots, Equation 5.2.9 has no other solutions of the form $e^{rx}$. We look for solutions of the form $y = uy_1 = ue^{-3x}$, where $u$ is a function that we'll now determine. (This should remind you of the method of variation of parameters used in Section 2.1 to solve the nonhomogeneous equation $y' + p(x)y = f(x)$, given a solution $y_1$ of the complementary equation $y' + p(x)y = 0$. It's also a special case of a method called **reduction of order** that we'll study in Section 5.6. For other ways to obtain a second solution of Equation 5.2.9 that's not a multiple of $e^{-3x}$, see **Exercises 5.1.9, 5.1.12,** and **5.1.33**.

If $y = ue^{-3x}$, then

$$y' = u'e^{-3x} - 3ue^{-3x} \quad \text{and} \quad y'' = u''e^{-3x} - 6u'e^{-3x} + 9ue^{-3x},$$

so

$$\begin{aligned} y'' + 6y' + 9y &= e^{-3x}\left[(u'' - 6u' + 9u) + 6(u' - 3u) + 9u\right] \\ &= e^{-3x}\left[u'' - (6-6)u' + (9 - 18 + 9)u\right] = u''e^{-3x}. \end{aligned}$$

Therefore $y = ue^{-3x}$ is a solution of Equation 5.2.9 if and only if $u'' = 0$, which is equivalent to $u = c_1 + c_2x$, where $c_1$ and $c_2$ are constants. Therefore any function of the form

$$y = e^{-3x}(c_1 + c_2x) \tag{5.2.11}$$

is a solution of Equation 5.2.9. Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-3x}$ that we already knew. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = xe^{-3x}$. Since $y_2/y_1 = x$ is nonconstant, Theorem 5.1.6 implies that $\{y_1, y_2\}$ is fundamental set of solutions of Equation 5.2.9, and Equation 5.2.11 is the general solution.

b. Differentiating Equation 5.2.11 yields

$$y' = -3e^{-3x}(c_1 + c_2x) + c_2e^{-3x}. \tag{5.2.12}$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ in Equation 5.2.11 and Equation 5.2.12 yields $c_1 = 3$ and $-3c_1 + c_2 = -1$, so $c_2 = 8$. Therefore the solution of Equation 5.2.10 is

$$y = e^{-3x}(3 + 8x).$$

Figure 5.2.2 is a graph of this solution.

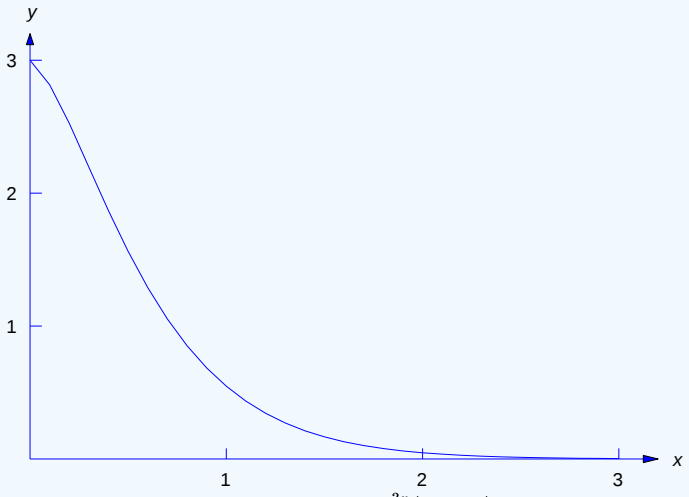


Figure 5.2.2 : $y = e^{-3x}(3 + 8x)$

If the characteristic equation of $ay'' + by' + cy = 0$ has an arbitrary repeated root $r_1$, the characteristic polynomial must be

$$p(r) = a(r - r_1)^2 = a(r^2 - 2r_1r + r_1^2).$$

Therefore

$$ar^2 + br + c = ar^2 - (2ar_1)r + ar_1^2,$$

which implies that $b=-2ar_1$ and $c=ar_1^2$. Therefore $ay''+by'+cy=0$ can be written as $a(y''-2r_1y'+r_1^2y)=0$. Since $a\neq 0$ this equation has the same solutions as

$$y''-2r_1y'+r_1^2y=0. \tag{5.2.13}$$

Since $p(r_1)=0$, t $y_1=e^{r_1x}$ is a solution of $ay''+by'+cy=0$, and therefore of Equation 5.2.13. Proceeding as in Example 5.2.1, we look for other solutions of Equation 5.2.13 of the form $y=ue^{r_1x}$; then

$$y'=u'e^{r_1x}+rue^{r_1x} \quad \text{and} \quad y''=u''e^{r_1x}+2r_1u'e^{r_1x}+r_1^2ue^{r_1x},$$

so

$$\begin{aligned} y''-2r_1y'+r_1^2y &= e^{rx}\left[(u''+2r_1u'+r_1^2u)-2r_1(u'+r_1u)+r_1^2u\right] \\ &= e^{r_1x}\left[u''+(2r_1-2r_1)u'+(r_1^2-2r_1^2+r_1^2)u\right]=u''e^{r_1x}. \end{aligned}$$

Therefore $y=ue^{r_1x}$ is a solution of Equation 5.2.13 if and only if $u''=0$, which is equivalent to $u=c_1+c_2x$, where $c_1$ and $c_2$ are constants. Hence, any function of the form

$$y=e^{r_1x}(c_1+c_2x) \tag{5.2.14}$$

is a solution of Equation 5.2.13. Letting $c_1=1$ and $c_2=0$ here yields the solution $y_1=e^{r_1x}$ that we already knew. Letting $c_1=0$ and $c_2=1$ yields the second solution $y_2=xe^{r_1x}$. Since $y_2/y_1=x$ is nonconstant, Theorem 5.1.6 implies that $\{y_1,y_2\}$ is a fundamental set of solutions of Equation 5.2.13, and Equation 5.2.14 is the general solution.

## Case 3: Complex Conjugate Roots

**Example 5.2.3**

a. Find the general solution of

$$y''+4y'+13y=0. \tag{5.2.15}$$

b. Solve the initial value problem

$$y''+4y'+13y=0, \quad y(0)=2,\ y'(0)=-3. \tag{5.2.16}$$

**Solution**

a. The characteristic polynomial of Equation 5.2.15 is

$$p(r)=r^2+4r+13=r^2+4r+4+9=(r+2)^2+9.$$

The roots of the characteristic equation are $r_1=-2+3i$ and $r_2=-2-3i$. By analogy with Case 1, it is reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions of Equation 5.2.15. This is true (see **Exercise 5.2.34**); however, there are difficulties here, since you are probably not familiar with exponential functions with complex arguments, and even if you are, it is inconvenient to work with them, since they are complex–valued. We'll take a simpler approach, which we motivate as follows: the exponential notation suggests that

$$e^{(-2+3i)x}=e^{-2x}e^{3ix} \quad \text{and} \quad e^{(-2-3i)x}=e^{-2x}e^{-3ix},$$

so even though we haven't defined $e^{3ix}$ and $e^{-3ix}$, it is reasonable to expect that every linear combination of $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ can be written as $y=ue^{-2x}$, where $u$ depends upon $x$. To determine $u$, we note that if $y=ue^{-2x}$ then

$$y'=u'e^{-2x}-2ue^{-2x} \quad \text{and} \quad y''=u''e^{-2x}-4u'e^{-2x}+4ue^{-2x},$$

so

$$\begin{aligned} y''+4y'+13y &= e^{-2x}\left[(u''-4u'+4u)+4(u'-2u)+13u\right] \\ &= e^{-2x}\left[u''-(4-4)u'+(4-8+13)u\right]=e^{-2x}(u''+9u). \end{aligned}$$

Therefore $y=ue^{-2x}$ is a solution of Equation 5.2.15 if and only if

$$u''+9u=0.$$

From Example 5.1.2, the general solution of this equation is

$$u = c_1 \cos 3x + c_2 \sin 3x.$$

Therefore any function of the form

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) \tag{5.2.17}$$

is a solution of Equation 5.2.15. Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-2x} \cos 3x$. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = e^{-2x} \sin 3x$. Since $y_2/y_1 = \tan 3x$ is nonconstant, Theorem 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 5.2.15, and Equation 5.2.17 is the general solution.

b. Imposing the condition $y(0) = 2$ in Equation 5.2.17 shows that $c_1 = 2$. Differentiating Equation 5.2.17 yields

$$y' = -2e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + 3e^{-2x}(-c_1 \sin 3x + c_2 \cos 3x),$$

and imposing the initial condition $y'(0) = -3$ here yields $-3 = -2c_1 + 3c_2 = -4 + 3c_2$ , so $c_2 = 1/3$. Therefore the solution of Equation 5.2.16 is

$$y = e^{-2x}(2 \cos 3x + \frac{1}{3} \sin 3x).$$

Figure 5.2.3 is a graph of this function.

Now suppose the characteristic equation of $ay'' + by' + cy = 0$ has arbitrary complex roots; thus, $b^2 - 4ac < 0$ and, from Equation 5.2.4, the roots are

$$r_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a}, \quad r_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a},$$

which we rewrite as

$$r_1 = \lambda + i\omega, \quad r_2 = \lambda - i\omega, \tag{5.2.18}$$

with

$$\lambda = -\frac{b}{2a}, \quad \omega = \frac{\sqrt{4ac - b^2}}{2a}.$$

Don't memorize these formulas. Just remember that $r_1$ and $r_2$ are of the form Equation 5.2.18, where $\lambda$ is an arbitrary real number and $\omega$ is positive; $\lambda$ and $\omega$ are the **real** and **imaginary parts**, respectively, of $r_1$. Similarly, $\lambda$ and $-\omega$ are the real and imaginary parts of $r_2$. We say that $r_1$ and $r_2$ are **complex conjugates**,

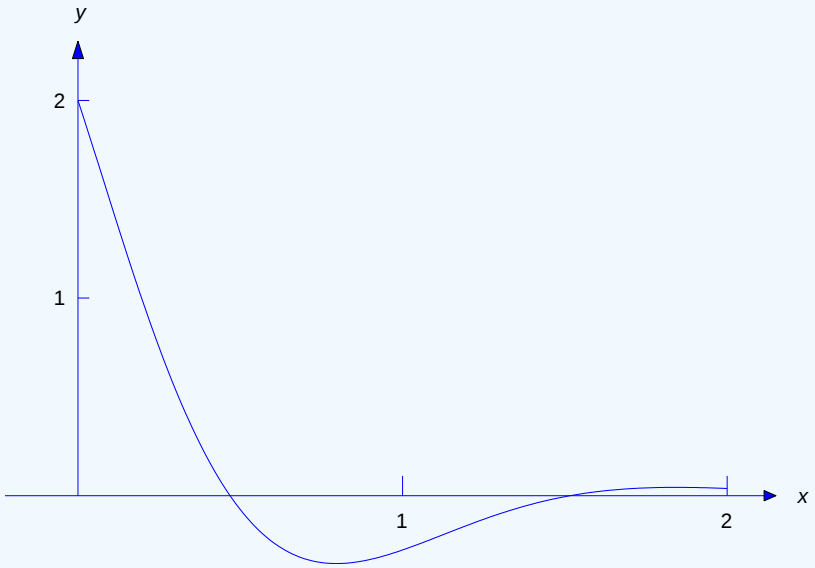



Figure 5.2.3 : $y = e^{-2x}(2 \cos 3x + \frac{1}{3} \sin 3x)$

which means that they have the same real part and their imaginary parts have the same absolute values, but opposite signs.

As in Example 5.2.3 , it is reasonable to to expect that the solutions of $ay''+by'+cy=0$ are linear combinations of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$. Again, the exponential notation suggests that

$$e^{(\lambda+i\omega)x}=e^{\lambda x}e^{i\omega x} \quad \text{and} \quad e^{(\lambda-i\omega)x}=e^{\lambda x}e^{-i\omega x},$$

so even though we haven't defined $e^{i\omega x}$ and $e^{-i\omega x}$, it is reasonable to expect that every linear combination of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$ can be written as $y=ue^{\lambda x}$, where $u$ depends upon $x$. To determine $u$ we first observe that since $r_1=\lambda+i\omega$ and $r_2=\lambda-i\omega$ are the roots of the characteristic equation, $p$ must be of the form

$$\begin{aligned} p(r) &= a(r-r_1)(r-r_2)\\ &= a(r-\lambda-i\omega)(r-\lambda+i\omega)\\ &= a\left[(r-\lambda)^2+\omega^2\right]\\ &= a(r^2-2\lambda r+\lambda^2+\omega^2). \end{aligned}$$

Therefore $ay''+by'+cy=0$ can be written as

$$a\left[y''-2\lambda y'+(\lambda^2+\omega^2)y\right]=0.$$

Since $a\neq 0$ this equation has the same solutions as

$$y''-2\lambda y'+(\lambda^2+\omega^2)y=0. \tag{5.2.19}$$

To determine $u$ we note that if $y=ue^{\lambda x}$ then

$$y'=u'e^{\lambda x}+\lambda ue^{\lambda x} \quad \text{and} \quad y''=u''e^{\lambda x}+2\lambda u'e^{\lambda x}+\lambda^2 ue^{\lambda x}.$$

Substituting these expressions into Equation 5.2.19 and dropping the common factor $e^{\lambda x}$ yields

$$(u''+2\lambda u'+\lambda^2 u)-2\lambda(u'+\lambda u)+(\lambda^2+\omega^2)u=0,$$

which simplifies to

$$u''+\omega^2 u=0.$$

From Example 5.1.2, the general solution of this equation is

$$u=c_1\cos\omega x+c_2\sin\omega x.$$

Therefore any function of the form

$$y=e^{\lambda x}(c_1\cos\omega x+c_2\sin\omega x) \tag{5.2.20}$$

is a solution of Equation 5.2.19. Letting $c_1=1$ and $c_2=0$ here yields the solution $y_1=e^{\lambda x}\cos\omega x$. Letting $c_1=0$ and $c_2=1$ yields a second solution $y_2=e^{\lambda x}\sin\omega x$. Since $y_2/y_1=\tan\omega x$ is nonconstant, so Theorem 5.1.6 implies that $\{y_1,y_2\}$ is a fundamental set of solutions of Equation 5.2.19, and Equation 5.2.20 is the general solution.

## Summary

The next theorem summarizes the results of this section.

**Theorem 5.2.1**

Let $p(r)=ar^2+br+c$ be the characteristic polynomial of

$$ay''+by'+cy=0. \tag{5.2.21}$$

Then:

a. If $p(r)=0$ has distinct real roots $r_1$ and $r_2$, then the general solution of Equation 5.2.21 is

$$y=c_1e^{r_1x}+c_2e^{r_2x}.$$

b. If $p(r) = 0$ has a repeated root $r_1$, then the general solution of Equation 5.2.21 is

$$y = e^{r_1 x}(c_1 + c_2 x).$$

c. If $p(r) = 0$ has complex conjugate roots $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ $(\text{where } \omega > 0)$, then the general solution of Equation 5.2.21 is

$$y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x).$$

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# 5.2E: Constant Coefficient Homogeneous Equations (Exercises)

## Q5.2.1

In **Exercises 5.2.1-5.2.12** find the general solution.

1. $y''+5y'-6y=0$
2. $y''-4y'+5y=0$
3. $y''+8y'+7y=0$
4. $y''-4y'+4y=0$
5. $y''+2y'+10y=0$
6. $y''+6y'+10y=0$
7. $y''-8y'+16y=0$
8. $y''+y'=0$
9. $y''-2y'+3y=0$
10. $y''+6y'+13y=0$
11. $4y''+4y'+10y=0$
12. $10y''-3y'-y=0$

## Q5.2.2

In **Exercises 5.2.13-5.2.17** solve the initial value problem.

13. $y''+14y'+50y=0, \quad y(0)=2, \quad y'(0)=-17$
14. $6y''-y'-y=0, \quad y(0)=10, \quad y'(0)=0$
15. $6y''+y'-y=0, \quad y(0)=-1, \quad y'(0)=3$
16. $4y''-4y'-3y=0, \quad y(0)=\frac{13}{12}, \quad y'(0)=\frac{23}{24}$
17. $4y''-12y'+9y=0, \quad y(0)=3, \quad y'(0)=\frac{5}{2}$

## Q5.2.3

In **Exercises 5.2.18-5.2.21** solve the initial value problem and graph the solution.

18. $y''+7y'+12y=0, \quad y(0)=-1, \quad y'(0)=0$
19. $y''-6y'+9y=0, \quad y(0)=0, \quad y'(0)=2$
20. $36y''-12y'+y=0, \quad y(0)=3, \quad y'(0)=\frac{5}{2}$
21. $y''+4y'+10y=0, \quad y(0)=3, \quad y'(0)=-2$

## Q5.2.4

22.

a. Suppose $y$ is a solution of the constant coefficient homogeneous equation

$$ay''+by'+cy=0. \tag{A}$$

Let $z(x)=y(x-x_0)$, where $x_0$ is an arbitrary real number. Show that

$$az''+bz'+cz=0.$$

b. Let $z_1(x)=y_1(x-x_0)$ and $z_2(x)=y_2(x-x_0)$, where $\{y_1,y_2\}$ is a fundamental set of solutions of (A). Show that $\{z_1,z_2\}$ is also a fundamental set of solutions of (A).

c. The statement of Theorem 5.2.1 is convenient for solving an initial value problem

$$ay'' + by' + cy = 0, \quad y(0) = k_0, \quad y'(0) = k_1,$$

where the initial conditions are imposed at $x_0 = 0$. However, if the initial value problem is

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1, \tag{B}$$

where $x_0 \neq 0$, then determining the constants in

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad y = e^{r_1 x}(c_1 + c_2 x), \text{ or } y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x)$$

(whichever is applicable) is more complicated. Use (b) to restate Theorem 5.2.1 in a form more convenient for solving (B).

## Q5.2.5

In **Exercises 5.2.23-5.2.28** use a method suggested by **Exercise 5.2.22** to solve the initial value problem.

23. $y'' + 3y' + 2y = 0, \quad y(1) = -1, \quad y'(1) = 4$

24. $y'' - 6y' - 7y = 0, \quad y(2) = -\frac{1}{3}, \quad y'(2) = -5$

25. $y'' - 14y' + 49y = 0, \quad y(1) = 2, \quad y'(1) = 11$

26. $9y'' + 6y' + y = 0, \quad y(2) = 2, \quad y'(2) = -\frac{14}{3}$

27. $9y'' + 4y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$

28. $y'' + 3y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -1$

## Q5.2.6

29. Prove: If the characteristic equation of

$$ay'' + by' + cy = 0 \tag{A}$$

has a repeated negative root or two roots with negative real parts, then every solution of (A) approaches zero as $x \to \infty$.

30. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has distinct real roots $r_1$ and $r_2$. Use a method suggested by **Exercise 5.2.22** to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

31 Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has a repeated real root $r_1$. Use a method suggested by **Exercise 5.2.22** to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

32. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has complex conjugate roots $\lambda \pm i\omega$. Use a method suggested by **Exercise 5.2.22** to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

33. Suppose the characteristic equation of

$$ay'' + by' + cy = 0 \tag{A}$$

has a repeated real root $r_1$. Temporarily, think of $e^{rx}$ as a function of two real variables $x$ and $r$.

a. Show that

$$a\frac{\partial^2}{\partial^2 x}(e^{rx}) + b\frac{\partial}{\partial x}(e^{rx}) + ce^{rx} = a(r - r_1)^2 e^{rx}. \tag{B}$$

b. Differentiate (B) with respect to $r$ to obtain

$$a\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial^2 x}(e^{rx})\right) + b\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}(e^{rx})\right) + c(xe^{rx}) = [2 + (r - r_1)x]a(r - r_1)e^{rx}. \tag{C}$$

c. Reverse the orders of the partial differentiations in the first two terms on the left side of (C) to obtain

$$a\frac{\partial^2}{\partial x^2}(xe^{rx})+b\frac{\partial}{\partial x}(xe^{rx})+c(xe^{rx})=[2+(r-r_1)x]a(r-r_1)e^{rx}. \tag{D}$$

d. Set $r=r_1$ in (B) and (D) to see that $y_1=e^{r_1x}$ and $y_2=xe^{r_1x}$ are solutions of (A)

34. In calculus you learned that $e^u$, $\cos u$, and $\sin u$ can be represented by the infinite series

$$e^u=\sum_{n=0}^{\infty}\frac{u^n}{n!}=1+\frac{u}{1!}+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots+\frac{u^n}{n!}+\cdots \tag{A}$$

$$\cos u=\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n}}{(2n)!}=1-\frac{u^2}{2!}+\frac{u^4}{4!}+\cdots+(-1)^n\frac{u^{2n}}{(2n)!}+\cdots, \tag{B}$$

and

$$\sin u=\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n+1}}{(2n+1)!}=u-\frac{u^3}{3!}+\frac{u^5}{5!}+\cdots+(-1)^n\frac{u^{2n+1}}{(2n+1)!}+\cdots \tag{C}$$

for all real values of $u$. Even though you have previously considered (A) only for real values of $u$, we can set $u=i\theta$, where $\theta$ is real, to obtain

$$e^{i\theta}=\sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}. \tag{D}$$

Given the proper background in the theory of infinite series with complex terms, it can be shown that the series in (D) converges for all real $\theta$.

a. Recalling that $i^2=-1$, write enough terms of the sequence $\{i^n\}$ to convince yourself that the sequence is repetitive:

$$1,i,-1,-i,1,i,-1,-i,1,i,-1,-i,1,i,-1,-i,\cdots.$$

Use this to group the terms in (D) as

$$\begin{aligned}e^{i\theta}&=\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{4}+\cdots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\cdots\right)\\&=\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!}+i\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!}.\end{aligned}$$

By comparing this result with (B) and (C), conclude that

$$e^{i\theta}=\cos\theta+i\sin\theta. \tag{E}$$

This is **Euler's identity**.

b. Starting from

$$e^{i\theta_1}e^{i\theta_2}=(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2),$$

collect the real part (the terms not multiplied by $i$) and the imaginary part (the terms multiplied by $i$) on the right, and use the trigonometric identities

$$\begin{aligned}\cos(\theta_1+\theta_2)&=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\\\sin(\theta_1+\theta_2)&=\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2\end{aligned}$$

to verify that

$$e^{i(\theta_1+\theta_2)}=e^{i\theta_1}e^{i\theta_2},$$

as you would expect from the use of the exponential notation $e^{i\theta}$.

c. If $\alpha$ and $\beta$ are real numbers, define

$$e^{\alpha+i\beta} = e^{\alpha}e^{i\beta} = e^{\alpha}(\cos\beta + i\sin\beta). \tag{F}$$

Show that if $z_1 = \alpha_1 + i\beta_1$ and $z_2 = \alpha_2 + i\beta_2$ then

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

d. Let $a$, $b$, and $c$ be real numbers, with $a \neq 0$. Let $z = u + iv$ where $u$ and $v$ are real-valued functions of $x$. Then we say that $z$ is a solution of

$$ay'' + by' + cy = 0 \tag{G}$$

if $u$ and $v$ are both solutions of (G). Use Theorem 5.2.1 (c) to verify that if the characteristic equation of (G) has complex conjugate roots $\lambda \pm i\omega$ then $z_1 = e^{(\lambda+i\omega)x}$ and $z_2 = e^{(\lambda-i\omega)x}$ are both solutions of (G).

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# 5.3: Nonhomogeneous Linear Equations

We'll now consider the nonhomogeneous linear second order equation

$$y''+p(x)y'+q(x)y=f(x), \tag{5.3.1}$$

where the forcing function $f$ isn't identically zero. The next theorem, an extension of Theorem 5.1.1, gives sufficient conditions for existence and uniqueness of solutions of initial value problems for Equation 5.3.1. We omit the proof, which is beyond the scope of this book.

**Theorem 5.3.1 : Uniqueness**

Suppose $p$, , $q$ and $f$ are continuous on an open interval $(a,b)$, let $x_0$ be any point in $(a,b)$, and let $k_0$ and $k_1$ be arbitrary real numbers. Then the initial value problem

$$y''+p(x)y'+q(x)y=f(x), \quad y(x_0)=k_0, \quad y'(x_0)=k_1$$

has a unique solution on $(a,b)$.

To find the general solution of Equation 5.3.1 on an interval $(a,b)$ where $p$, $q$, and $f$ are continuous, it is necessary to find the general solution of the associated homogeneous equation

$$y''+p(x)y'+q(x)y=0 \tag{5.3.2}$$

on $(a,b)$. We call Equation 5.3.2 the **complementary equation** for Equation 5.3.1.

The next theorem shows how to find the general solution of Equation 5.3.1 if we know one solution $y_p$ of Equation 5.3.1 and a fundamental set of solutions of Equation 5.3.2. We call $y_p$ a **particular solution** of Equation 5.3.1 ; it can be any solution that we can find, one way or another.

**Theorem 5.3.2**

Suppose $p$, $q$, and $f$ are continuous on $(a,b)$. Let $y_p$ be a particular solution of

$$y''+p(x)y'+q(x)y=f(x) \tag{5.3.3}$$

on $(a,b)$, and let $\{y_1,y_2\}$ be a fundamental set of solutions of the complementary equation

$$y''+p(x)y'+q(x)y=0 \tag{5.3.4}$$

on $(a,b)$. Then $y$ is a solution of (5.3.3) on $(a,b)$ if and only if

$$y=y_p+c_1y_1+c_2y_2, \tag{5.3.5}$$

where $c_1$ and $c_2$ are constants.

**Proof**

We first show that $y$ in Equation 5.3.5 is a solution of Equation 5.3.3 for any choice of the constants $c_1$ and $c_2$. Differentiating Equation 5.3.5twice yields

$$y'=y_p'+c_1y_1'+c_2y_2' \quad \text{and} \quad y''=y_p''+c_1y_1''+c_2y_2'',$$

so

$$\begin{aligned} y''+p(x)y'+q(x)y &= (y_p''+c_1y_1''+c_2y_2'')+p(x)(y_p'+c_1y_1'+c_2y_2')+q(x)(y_p+c_1y_1+c_2y_2) \\ &= (y_p''+p(x)y_p'+q(x)y_p)+c_1(y_1''+p(x)y_1'+q(x)y_1)+c_2(y_2''+p(x)y_2'+q(x)y_2) \\ &= f+c_1\cdot 0+c_2\cdot 0=f, \end{aligned}$$

since $y_p$ satisfies Equation 5.3.3and $y_1$ and $y_2$ satisfy Equation 5.3.4

Now we'll show that every solution of Equation 5.3.3has the form Equation 5.3.5for some choice of the constants $c_1$ and $c_2$. Suppose $y$ is a solution of Equation 5.3.3 We'll show that $y-y_p$ is a solution of Equation 5.3.4, and therefore of the form $y-y_p=c_1y_1+c_2y_2$ , which implies Equation 5.3.5 To see this, we compute

$$\begin{aligned}(y-y_p)''+p(x)(y-y_p)'+q(x)(y-y_p) &= (y''-y_p'')+p(x)(y'-y_p')+q(x)(y-y_p)\\ &= (y''+p(x)y'+q(x)y)-(y_p''+p(x)y_p'+q(x)y_p)\\ &= f(x)-f(x)=0,\end{aligned}$$

since $y$ and $y_p$ both satisfy Equation 5.3.3

We say that Equation 5.3.5is the **general solution of** (5.3.3) on $(a,b)$.

If $P_0$, $P_1$, and $F$ are continuous and $P_0$ has no zeros on $(a,b)$, then Theorem 5.3.2 implies that the general solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F(x) \tag{5.3.6}$$

on $(a,b)$ is $y=y_p+c_1y_1+c_2y_2$ , where $y_p$ is a particular solution of Equation 5.3.6 on $(a,b)$ and $\{y_1,y_2\}$ is a fundamental set of solutions of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0$$

on $(a,b)$. To see this, we rewrite Equation 5.3.6 as

$$y''+\frac{P_1(x)}{P_0(x)}y'+\frac{P_2(x)}{P_0(x)}y=\frac{F(x)}{P_0(x)}$$

and apply Theorem 5.3.2 with $p=P_1/P_0$, $q=P_2/P_0$, and $f=F/P_0$.

To avoid awkward wording in examples and exercises, we will not specify the interval $(a,b)$ when we ask for the general solution of a specific linear second order equation, or for a fundamental set of solutions of a homogeneous linear second order equation. Let's agree that this always means that we want the general solution (or a fundamental set of solutions, as the case may be) on every open interval on which $p$, $q$, and $f$ are continuous if the equation is of the form Equation 5.3.3, or on which $P_0$, $P_1$, $P_2$, and $F$ are continuous and $P_0$ has no zeros, if the equation is of the form Equation 5.3.6. We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if $P_0$, $P_1$, $P_2$, and $F$ are all continuous on an open interval $(a,b)$, but $P_0$ **does** have a zero in $(a,b)$, then Equation 5.3.6 may fail to have a general solution on $(a,b)$ in the sense just defined. **Exercises 5.1.42-5.1.44** illustrate this point for a homogeneous equation.

In this section we to limit ourselves to applications of Theorem 5.3.2 where we can guess at the form of the particular solution.

### Example 5.3.1

a. Find the general solution of

$$y''+y=1. \tag{5.3.7}$$

b. Solve the initial value problem

$$y''+y=1,\quad y(0)=2,\quad y'(0)=7. \tag{5.3.8}$$

**Solution**

a. We can apply Theorem 5.3.2 with $(a,b)=(-\infty,\infty)$, since the functions $p\equiv0$, $q\equiv1$, and $f\equiv1$ in Equation 5.3.7 are continuous on $(-\infty,\infty)$. By inspection we see that $y_p\equiv1$ is a particular solution of Equation 5.3.7. Since $y_1=\cos x$ and $y_2=\sin x$ form a fundamental set of solutions of the complementary equation $y''+y=0$ , the general solution of Equation 5.3.7 is

$$y=1+c_1\cos x+c_2\sin x. \tag{5.3.9}$$

b. Imposing the initial condition $y(0)=2$ in Equation 5.3.9 yields $2=1+c_1$, so $c_1=1$. Differentiating Equation 5.3.9 yields

$$y'=-c_1\sin x+c_2\cos x.$$

Imposing the initial condition $y'(0)=7$ here yields $c_2=7$, so the solution of Equation 5.3.8 is

$$y=1+\cos x+7\sin x.$$

Figure 5.3.1 is a graph of this function.

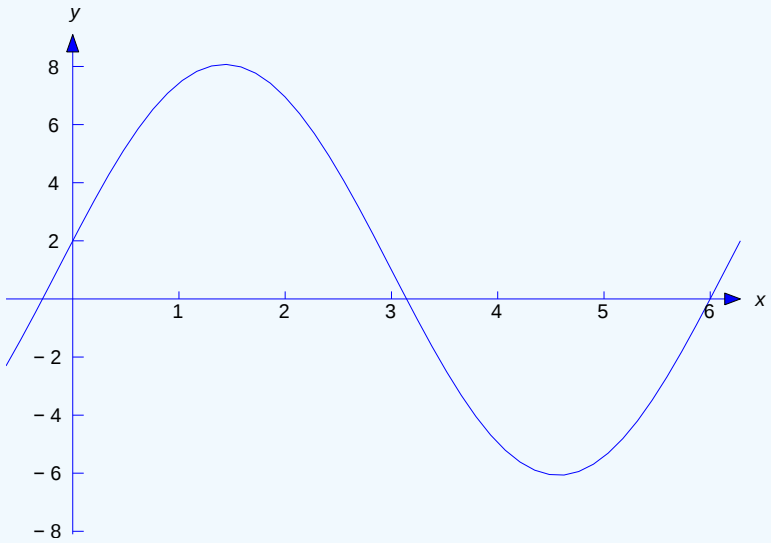



Figure 5.3.1 : $y=1+\cos x+7\sin x$

## Example 5.3.2

a. Find the general solution of

$$y''-2y'+y=-3-x+x^2. \tag{5.3.10}$$

b. Solve the initial value problem

$$y''-2y'+y=-3-x+x^2,\quad y(0)=-2,\quad y'(0)=1. \tag{5.3.11}$$

**Solution**

a. The characteristic polynomial of the complementary equation

$$y''-2y'+y=0$$

is $r^2-2r+1=(r-1)^2$, so $y_1=e^x$ and $y_2=xe^x$ form a fundamental set of solutions of the complementary equation. To guess a form for a particular solution of Equation 5.3.10, we note that substituting a second degree polynomial $y_p=A+Bx+Cx^2$ into the left side of Equation 5.3.10 will produce another second degree polynomial with coefficients that depend upon $A$, $B$, and $C$. The trick is to choose $A$, $B$, and $C$ so the polynomials on the two sides of Equation 5.3.10 have the same coefficients; thus, if

$$y_p=A+Bx+Cx^2\quad\text{then}\quad y_p'=B+2Cx\quad\text{and}\quad y_p''=2C,$$

so

$$\begin{aligned}y_p''-2y_p'+y_p &=2C-2(B+2Cx)+(A+Bx+Cx^2)\\ &=(2C-2B+A)+(-4C+B)x+Cx^2=-3-x+x^2.\end{aligned}$$

Equating coefficients of like powers of $x$ on the two sides of the last equality yields

$$C = 1$$
$$B - 4C = -1$$
$$A - 2B + 2C = -3,$$

so $C = 1$, $B = -1 + 4C = 3$, and $A = -3 - 2C + 2B = 1$. Therefore $y_p = 1 + 3x + x^2$ is a particular solution of Equation 5.3.10 and Theorem 5.3.2 implies that

$$y = 1 + 3x + x^2 + e^x(c_1 + c_2 x) \tag{5.3.12}$$

is the general solution of Equation 5.3.10.

b. Imposing the initial condition $y(0) = -2$ in Equation 5.3.12 yields $-2 = 1 + c_1$, so $c_1 = -3$. Differentiating Equation 5.3.12 yields

$$y' = 3 + 2x + e^x(c_1 + c_2 x) + c_2 e^x,$$

and imposing the initial condition $y'(0) = 1$ here yields $1 = 3 + c_1 + c_2$, so $c_2 = 1$. Therefore the solution of Equation 5.3.11 is

$$y = 1 + 3x + x^2 - e^x(3 - x).$$

Figure 5.3.2 is a graph of this solution.

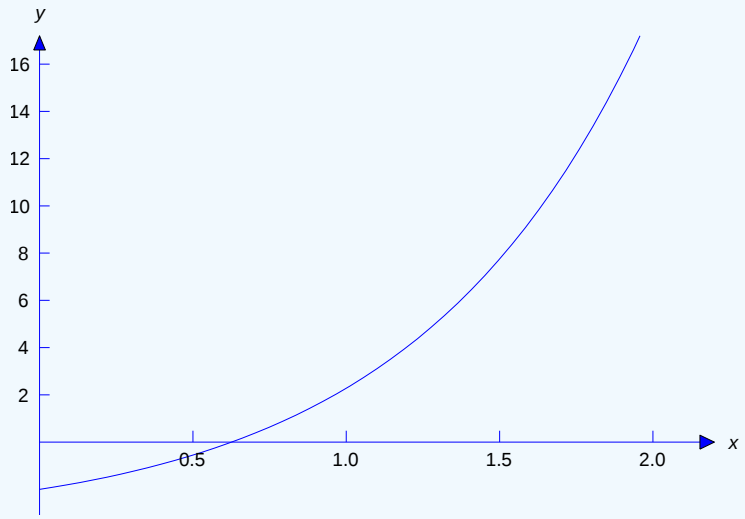



Figure 5.3.2 : $y = 1 + 3x + x^2 - e^x(3 - x)$

## Example 5.3.3

Find the general solution of

$$x^2 y'' + xy' - 4y = 2x^4 \tag{5.3.13}$$

on $(-\infty, 0)$ and $(0, \infty)$.

### Solution

In Example 5.1.3, we verified that $y_1 = x^2$ and $y_2 = 1/x^2$ form a fundamental set of solutions of the complementary equation

$$x^2 y'' + xy' - 4y = 0$$

on $(-\infty, 0)$ and $(0, \infty)$. To find a particular solution of Equation 5.3.13, we note that if $y_p = Ax^4$, where $A$ is a constant then both sides of Equation 5.3.13 will be constant multiples of $x^4$ and we may be able to choose $A$ so the two sides are equal. This is true in this example, since if $y_p = Ax^4$ then

$$x^2 y_p'' + x y_p' - 4y_p = x^2(12Ax^2) + x(4Ax^3) - 4Ax^4 = 12Ax^4 = 2x^4$$

if $A = 1/6$; therefore, $y_p = x^4/6$ is a particular solution of Equation 5.3.13 on $(-\infty, \infty)$. Theorem 5.3.2 implies that the general solution of Equation 5.3.13 on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{x^4}{6} + c_1 x^2 + \frac{c_2}{x^2}.$$

## The Principle of Superposition

The next theorem enables us to break a nonhomogeous equation into simpler parts, find a particular solution for each part, and then combine their solutions to obtain a particular solution of the original problem.

**Theorem 5.3.3 the principle of superposition**

Suppose $y_{p_1}$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on $(a, b)$ and $y_{p_2}$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on $(a, b)$. Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

on $(a, b)$.

**Proof**

If $y_p = y_{p_1} + y_{p_2}$ then

$$\begin{aligned} y_p'' + p(x)y_p' + q(x)y_p &= (y_{p_1} + y_{p_2})'' + p(x)(y_{p_1} + y_{p_2})' + q(x)(y_{p_1} + y_{p_2}) \\ &= \left(y_{p_1}'' + p(x)y_{p_1}' + q(x)y_{p_1}\right) + \left(y_{p_2}'' + p(x)y_{p_2}' + q(x)y_{p_2}\right) \\ &= f_1(x) + f_2(x). \end{aligned}$$

It's easy to generalize Theorem 5.3.3 to the equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{5.3.14}$$

where

$$f = f_1 + f_2 + \cdots + f_k;$$

thus, if $y_{p_i}$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_i(x)$$

on $(a, b)$ for $i = 1, 2, \ldots, k$, then $y_{p_1} + y_{p_2} + \cdots + y_{p_k}$ is a particular solution of Equation 5.3.14 on $(a, b)$. Moreover, by a proof similar to the proof of Theorem 5.3.3 we can formulate the principle of superposition in terms of a linear equation written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$$

that is, if $y_{p_1}$ is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x)$$

on $(a, b)$ and $y_{p_2}$ is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_2(x)$$

on $(a, b)$, then $y_{p_1} + y_{p_2}$ is a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x) + F_2(x)$$

on $(a, b)$.

**Example 5.3.4**

The function $y_{p_1} = x^4/15$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 \tag{5.3.15}$$

on $(-\infty, \infty)$ and $y_{p_2} = x^2/3$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 4x^2 \tag{5.3.16}$$

on $(-\infty, \infty)$. Use the principle of superposition to find a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 + 4x^2 \tag{5.3.17}$$

on $(-\infty, \infty)$.

**Solution**

The right side $F(x) = 2x^4 + 4x^2$ in Equation 5.3.17 is the sum of the right sides

$$F_1(x) = 2x^4 \quad \text{and} \quad F_2(x) = 4x^2.$$

in Equation 5.3.15 and Equation 5.3.16. Therefore the principle of superposition implies that

$$y_p = y_{p_1} + y_{p_2} = \frac{x^4}{15} + \frac{x^2}{3}$$

is a particular solution of Equation 5.3.17.

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# 5.3E: Nonhomgeneous Linear Equations (Exercises)

## Q5.3.1

In **Exercises 5.3.1-5.3.6** find a particular solution by the method used in Example 5.3.2. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

1. $y''+5y'-6y=22+18x-18x^2$
2. $y''-4y'+5y=1+5x$
3. $y''+8y'+7y=-8-x+24x^2+7x^3$
4. $y''-4y'+4y=2+8x-4x^2$
5. $y''+2y'+10y=4+26x+6x^2+10x^3, \quad y(0)=2, \quad y'(0)=9$
6. $y''+6y'+10y=22+20x, \quad y(0)=2,\ y'(0)=-2$

## Q5.3.2

7. Show that the method used in Example 5.3.2 will not yield a particular solution of

$$y''+y'=1+2x+x^2; \tag{A}$$

that is, (A) does'nt have a particular solution of the form $y_p=A+Bx+Cx^2$ , where $A$, $B$, and $C$ are constants.

## Q5.3.3

In **Exercises 5.3.8-5.3.13** find a particular solution by the method used in Example 5.3.3.

8. $x^2y''+7xy'+8y=\frac{6}{x}$
9. $x^2y''-7xy'+7y=13x^{1/2}$
10. $x^2y''-xy'+y=2x^3$
11. $x^2y''+5xy'+4y=\frac{1}{x^3}$
12. $x^2y''+xy'+y=10x^{1/3}$
13. $x^2y''-3xy'+13y=2x^4$

## Q5.3.4

14. Show that the method suggested for finding a particular solution in **Exercises 5.3.8-5.3.13** will not yield a particular solution of

$$x^2y''+3xy'-3y=\frac{1}{x^3}; \tag{A}$$

that is, (A) doesn't have a particular solution of the form $y_p=A/x^3$.

15. Prove: If $a$, $b$, $c$, $\alpha$, and $M$ are constants and $M\neq 0$ then

$$ax^2y''+bxy'+cy=Mx^\alpha$$

has a particular solution $y_p=Ax^\alpha$ ($A=$ constant) if and only if $a\alpha(\alpha-1)+b\alpha+c\neq 0$ .

## Q5.3.5

If $a$, $b$, $c$, and $\alpha$ are constants, then

$$\alpha(e^{\alpha x})''+b(e^{\alpha x})'+ce^{\alpha x}=(a\alpha^2+b\alpha+c)e^{\alpha x}.$$

Use this in **Exercises 5.3.16-5.3.21** to find a particular solution. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

16. $y''+5y'-6y=6e^{3x}$

17. $y''-4y'+5y=e^{2x}$

18. $y''+8y'+7y=10e^{-2x}, \quad y(0)=-2,\ y'(0)=10$

19. $y''-4y'+4y=e^{x}, \quad y(0)=2, \quad y'(0)=0$

20. $y''+2y'+10y=e^{x/2}$

21. $y''+6y'+10y=e^{-3x}$

## Q5.3.6

22. Show that the method suggested for finding a particular solution in **Exercises 5.3.16-5.3.21** will not yield a particular solution of

$$y''-7y'+12y=5e^{4x}; \tag{A}$$

that is, (A) doesn't have a particular solution of the form $y_p=Ae^{4x}$ .

23. Prove: If $\alpha$ and $M$ are constants and $M\neq 0$ then constant coefficient equation

$$ay''+by'+cy=Me^{\alpha x}$$

has a particular solution $y_p=Ae^{\alpha x}$ ($A=$ constant) if and only if $e^{\alpha x}$ isn't a solution of the complementary equation.

## Q5.3.7

If $\omega$ is a constant, differentiating a linear combination of $\cos\omega x$ and $\sin\omega x$ with respect to $x$ yields another linear combination of $\cos\omega x$ and $\sin\omega x$. In **Exercises 5.3.24-5.3.29** use this to find a particular solution of the equation. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

24. $y''-8y'+16y=23\cos x-7\sin x$

25. $y''+y'=-8\cos 2x+6\sin 2x$

26. $y''-2y'+3y=-6\cos 3x+6\sin 3x$

27. $y''+6y'+13y=18\cos x+6\sin x$

28. $y''+7y'+12y=-2\cos 2x+36\sin 2x, \quad y(0)=-3, \quad y'(0)=3$

29. $y''-6y'+9y=18\cos 3x+18\sin 3x, \quad y(0)=2, \quad y'(0)=2$

## Q5.3.8

30. Find the general solution of

$$y''+\omega_0^2y=M\cos\omega x+N\sin\omega x,$$

where $M$ and $N$ are constants and $\omega$ and $\omega_0$ are distinct positive numbers.

31. Show that the method suggested for finding a particular solution in **Exercises 5.3.24-5.3.29** will not yield a particular solution of

$$y''+y=\cos x+\sin x; \tag{A}$$

that is, (A) does not have a particular solution of the form $y_p=A\cos x+B\sin x$ .

32. Prove: If $M$, $N$ are constants (not both zero) and $\omega>0$, the constant coefficient equation

$$ay''+by'+cy=M\cos\omega x+N\sin\omega x \tag{A}$$

has a particular solution that's a linear combination of $\cos\omega x$ and $\sin\omega x$ if and only if the left side of (A) is not of the form $a(y''+\omega^2y)$, so that $\cos\omega x$ and $\sin\omega x$ are solutions of the complementary equation.

## Q5.3.9

In **Exercises 5.3.33-5.3.38** refer to the cited exercises and use the principal of superposition to find a particular solution. Then find the general solution.

33. $y''+5y'-6y=22+18x-18x^2+6e^{3x}$ (See **Exercises 5.3.1** and **5.3.16.**)

34. $y''-4y'+5y=1+5x+e^{2x}$ (See **Exercises 5.3.2** and **5.3.17.**)

35. $y''+8y'+7y=-8-x+24x^2+7x^3+10e^{-2x}$ (See **Exercises 5.3.3** and **5.3.18.**)

36. $y''-4y'+4y=2+8x-4x^2+e^{x}$ (See **Exercises 5.3.4** and **5.3.19.**)

37. $y''+2y'+10y=4+26x+6x^2+10x^3+e^{x/2}$ (See **Exercises 5.3.5** and **5.3.20.**)

38. $y''+6y'+10y=22+20x+e^{-3x}$ (See **Exercises 5.3.6** and **5.3.21.**)

## Q5.3.10

39. Prove: If $y_{p_1}$ is a particular solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F_1(x)$$

on $(a,b)$ and $y_{p_2}$ is a particular solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F_2(x)$$

on $(a,b)$, then $y_p=y_{p_1}+y_{p_2}$ is a solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F_1(x)+F_2(x)$$

on $(a,b)$.

40. Suppose $p$, $q$, and $f$ are continuous on $(a,b)$. Let $y_1$, $y_2$, and $y_p$ be twice differentiable on $(a,b)$, such that $y=c_1y_1+c_2y_2+y_p$ is a solution of

$$y''+p(x)y'+q(x)y=f$$

on $(a,b)$ for every choice of the constants $c_1, c_2$. Show that $y_1$ and $y_2$ are solutions of the complementary equation on $(a,b)$.

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# 5.4: The Method of Undetermined Coefficients I

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x} G(x), \tag{5.4.1}$$

where $\alpha$ is a constant and $G$ is a polynomial.

From Theorem 5.3.2, the general solution of Equation 5.4.1 is $y = y_p + c_1 y_1 + c_2 y_2$ , where $y_p$ is a particular solution of Equation 5.4.1 and $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation

$$ay'' + by' + cy = 0.$$

In Section 5.2 we showed how to find $\{y_1, y_2\}$. In this section we'll show how to find $y_p$. The procedure that we'll use is called **the method of undetermined coefficients**. Our first example is similar to **Exercises 5.3.16-5.3.21**.

## Example 5.4.1

Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{5.4.2}$$

Then find the general solution.

### Solution

Substituting $y_p = Ae^{2x}$ for $y$ in Equation 5.4.2 will produce a constant multiple of $Ae^{2x}$ on the left side of Equation 5.4.2, so it may be possible to choose $A$ so that $y_p$ is a solution of Equation 5.4.2. Let's try it; if $y_p = Ae^{2x}$ then

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 2Ae^{2x} = 4e^{2x}$$

if $A = 2$. Therefore $y_p = 2e^{2x}$ is a particular solution of Equation 5.4.2. To find the general solution, we note that the characteristic polynomial of the complementary equation

$$y'' - 7y' + 12y = 0 \tag{5.4.3}$$

is $p(r) = r^2 - 7r + 12 = (r-3)(r-4)$ , so $\{e^{3x}, e^{4x}\}$ is a fundamental set of solutions of Equation 5.4.3. Therefore the general solution of Equation 5.4.2 is

$$y = 2e^{2x} + c_1 e^{3x} + c_2 e^{4x}.$$

## Example 5.4.2

Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{5.4.4}$$

Then find the general solution.

### Solution

Fresh from our success in finding a particular solution of Equation 5.4.2 — where we chose $y_p = Ae^{2x}$ because the right side of Equation 5.4.2 is a constant multiple of $e^{2x}$ — it may seem reasonable to try $y_p = Ae^{4x}$ as a particular solution of Equation 5.4.4. However, this will not work, since we saw in Example 5.4.1 that $e^{4x}$ is a solution of the complementary equation Equation 5.4.3, so substituting $y_p = Ae^{4x}$ into the left side of Equation 5.4.4) produces zero on the left, no matter how we choose$A$. To discover a suitable form for $y_p$, we use the same approach that we used in Section 5.2 to find a second solution of

$$ay'' + by' + cy = 0$$

in the case where the characteristic equation has a repeated real root: we look for solutions of Equation 5.4.4 in the form $y = ue^{4x}$ , where $u$ is a function to be determined. Substituting

$$y = ue^{4x}, \quad y' = u'e^{4x} + 4ue^{4x}, \quad \text{and} \quad y'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x} \tag{5.4.5}$$

into Equation 5.4.4 and canceling the common factor $e^{4x}$ yields

$$(u''+8u'+16u)-7(u'+4u)+12u=5,$$

or

$$u''+u'=5.$$

By inspection we see that $u_p=5x$ is a particular solution of this equation, so \(y_p=5xe^{4x}\) is a particular solution of Equation 5.4.4. Therefore

$$y=5xe^{4x}+c_1e^{3x}+c_2e^{4x}$$

is the general solution.

**Example 5.4.3**

Find a particular solution of

$$y''-8y'+16y=2e^{4x}. \tag{5.4.6}$$

**Solution**

Since the characteristic polynomial of the complementary equation

$$y''-8y'+16y=0 \tag{5.4.7}$$

is $p(r)=r^2-8r+16=(r-4)^2$ , both $y_1=e^{4x}$ and $y_2=xe^{4x}$ are solutions of Equation 5.4.7. Therefore Equation 5.4.6) does not have a solution of the form $y_p=Ae^{4x}$ or $y_p=Axe^{4x}$. As in Example 5.4.2 , we look for solutions of Equation 5.4.6 in the form $y=ue^{4x}$, where $u$ is a function to be determined. Substituting from Equation 5.4.5 into Equation 5.4.6 and canceling the common factor $e^{4x}$ yields

$$(u''+8u'+16u)-8(u'+4u)+16u=2,$$

or

$$u''=2.$$

Integrating twice and taking the constants of integration to be zero shows that $u_p=x^2$ is a particular solution of this equation, so $y_p=x^2e^{4x}$ is a particular solution of Equation 5.4.4 Therefore

$$y=e^{4x}(x^2+c_1+c_2x)$$

is the general solution.

The preceding examples illustrate the following facts concerning the form of a particular solution $y_p$ of a constant coefficent equation

$$ay''+by'+cy=ke^{\alpha x},$$

where $k$ is a nonzero constant:

a. If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay''+by'+cy=0, \tag{5.4.8}$$

then $y_p=Ae^{\alpha x}$ , where $A$ is a constant. (See Example 5.4.1 ).

b. If $e^{\alpha x}$ is a solution of Equation 5.4.8 but $xe^{\alpha x}$ is not, then $y_p=Axe^{\alpha x}$ , where $A$ is a constant. (See Example 5.4.2 .)

c. If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of Equation 5.4.8, then $y_p=Ax^2e^{\alpha x}$ , where $A$ is a constant. (See Example 5.4.3 .)

See **Exercise 5.4.30** for the proofs of these facts.

In all three cases you can just substitute the appropriate form for $y_p$ and its derivatives directly into

$$ay_p''+by_p'+cy_p=ke^{\alpha x},$$

and solve for the constant $A$, as we did in Example 5.4.1 . (See **Exercises 5.4.31-5.4.33.**) However, if the equation is

$$ay''+by'+cy=ke^{\alpha x}G(x),$$

where $G$ is a polynomial of degree greater than zero, we recommend that you use the substitution $y=ue^{\alpha x}$ as we did in Examples 5.4.2 and 5.4.3 . The equation for $u$ will turn out to be

$$au''+p'(\alpha)u'+p(\alpha)u=G(x), \tag{5.4.9}$$

where $p(r)=ar^2+br+c$ is the characteristic polynomial of the complementary equation and $p'(r)=2ar+b$ (**Exercise 5.4.30**); however, you shouldn't memorize this since it is easy to derive the equation for $u$ in any particular case. Note, however, that if $e^{\alpha x}$ is a solution of the complementary equation then $p(\alpha)=0$, so Equation 5.4.9 reduces to

$$au''+p'(\alpha)u'=G(x),$$

while if both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation then $p(r)=a(r-\alpha)^2$ and $p'(r)=2a(r-\alpha)$, so $p(\alpha)=p'(\alpha)=0$ and Equation 5.4.9) reduces to

$$au''=G(x).$$

**Example 5.4.4**

Find a particular solution of

$$y''-3y'+2y=e^{3x}(-1+2x+x^2). \tag{5.4.10}$$

**Solution**

Substituting

$$y=ue^{3x},\quad y'=u'e^{3x}+3ue^{3x},\quad \text{and} y''=u''e^{3x}+6u'e^{3x}+9ue^{3x}$$

into Equation 5.4.10) and canceling $e^{3x}$ yields

$$(u''+6u'+9u)-3(u'+3u)+2u=-1+2x+x^2,$$

or

$$u''+3u'+2u=-1+2x+x^2. \tag{5.4.11}$$

As in Example 5.3.2, in order to guess a form for a particular solution of Equation 5.4.11), we note that substituting a second degree polynomial $u_p=A+Bx+Cx^2$ for $u$ in the left side of Equation 5.4.11) produces another second degree polynomial with coefficients that depend upon $A$, $B$, and $C$; thus,

$$\text{if}\quad u_p=A+Bx+Cx^2\quad\text{then}\quad u_p'=B+2Cx\quad\text{and}\quad u_p''=2C.$$

If $u_p$ is to satisfy Equation 5.4.11), we must have

$$\begin{aligned} u_p''+3u_p'+2u_p &= 2C+3(B+2Cx)+2(A+Bx+Cx^2)\\ &= (2C+3B+2A)+(6C+2B)x+2Cx^2=-1+2x+x^2. \end{aligned}$$

Equating coefficients of like powers of $x$ on the two sides of the last equality yields

$$\begin{aligned} 2C &= 1\\ 2B+6C &= 2\\ 2A+3B+2C &= -1. \end{aligned}$$

Solving these equations for $C$, $B$, and $A$ (in that order) yields $C=1/2, B=-1/2, A=-1/4$. Therefore

$$u_p=-\frac{1}{4}(1+2x-2x^2)$$

is a particular solution of Equation 5.4.11, and

$$y_p = u_p e^{3x} = -\frac{e^{3x}}{4}(1+2x-2x^2)$$

is a particular solution of Equation 5.4.10.

## Example 5.4.5

Find a particular solution of

$$y'' - 4y' + 3y = e^{3x}(6+8x+12x^2). \tag{5.4.12}$$

**Solution**

Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and } y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into Equation 5.4.12) and canceling $e^{3x}$ yields

$$(u'' + 6u' + 9u) - 4(u' + 3u) + 3u = 6 + 8x + 12x^2,$$

or

$$u'' + 2u' = 6 + 8x + 12x^2. \tag{5.4.13}$$

There's no $u$ term in this equation, since $e^{3x}$ is a solution of the complementary equation for Equation 5.4.12). (See **Exercise 5.4.30**.) Therefore Equation 5.4.13) does not have a particular solution of the form $u_p = A + Bx + Cx^2$ that we used successfully in Example 5.4.4 , since with this choice of $u_p$,

$$u_p'' + 2u_p' = 2C + (B + 2Cx)$$

can't contain the last term ($12x^2$) on the right side of Equation 5.4.13). Instead, let's try $u_p = Ax + Bx^2 + Cx^3$ on the grounds that

$$u_p' = A + 2Bx + 3Cx^2 \quad \text{and} \quad u_p'' = 2B + 6Cx$$

together contain all the powers of $x$ that appear on the right side of Equation 5.4.13).

Substituting these expressions in place of $u'$ and $u''$ in Equation 5.4.13) yields

$$(2B + 6Cx) + 2(A + 2Bx + 3Cx^2) = (2B + 2A) + (6C + 4B)x + 6Cx^2 = 6 + 8x + 12x^2.$$

Comparing coefficients of like powers of $x$ on the two sides of the last equality shows that $u_p$ satisfies Equation 5.4.13) if

$$\begin{aligned} 6C &= 12 \\ 4B + 6C &= 8 \\ 2A + 2B &= 6. \end{aligned}$$

Solving these equations successively yields $C = 2$, $B = -1$, and $A = 4$. Therefore

$$u_p = x(4 - x + 2x^2)$$

is a particular solution of Equation 5.4.13), and

$$y_p = u_p e^{3x} = xe^{3x}(4 - x + 2x^2)$$

is a particular solution of Equation 5.4.12).

## Example 5.4.6

Find a particular solution of

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2). \tag{5.4.14}$$

**Solution**

Substituting

$$y = ue^{-x/2}, \quad y' = u'e^{-x/2} - \frac{1}{2}ue^{-x/2}, \quad \text{and} \quad y'' = u''e^{-x/2} - u'e^{-x/2} + \frac{1}{4}ue^{-x/2}$$

into Equation 5.4.14 and canceling $e^{-x/2}$ yields

$$4\left(u'' - u' + \frac{u}{4}\right) + 4\left(u' - \frac{u}{2}\right) + u = 4u'' = -8 + 48x + 144x^2,$$

or

$$u'' = -2 + 12x + 36x^2, \tag{5.4.15}$$

which does not contain $u$ or $u'$ because $e^{-x/2}$ and $xe^{-x/2}$ are both solutions of the complementary equation. (See **Exercise 5.4.30**.) To obtain a particular solution of Equation 5.4.15 we integrate twice, taking the constants of integration to be zero; thus,

$$u_p' = -2x + 6x^2 + 12x^3 \quad \text{and} \quad u_p = -x^2 + 2x^3 + 3x^4 = x^2(-1 + 2x + 3x^2).$$

Therefore

$$y_p = u_p e^{-x/2} = x^2 e^{-x/2}(-1 + 2x + 3x^2)$$

is a particular solution of Equation 5.4.14.

## Summary

The preceding examples illustrate the following facts concerning particular solutions of a constant coefficent equation of the form

$$ay'' + by' + cy = e^{\alpha x}G(x),$$

where $G$ is a polynomial (see **Exercise 5.4.30**):

a. If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay'' + by' + cy = 0, \tag{5.4.16}$$

then $y_p = e^{\alpha x}Q(x)$, where $Q$ is a polynomial of the same degree as $G$. (See Example 5.4.4 ).

b. If $e^{\alpha x}$ is a solution of Equation 5.4.16 but $xe^{\alpha x}$ is not, then $y_p = xe^{\alpha x}Q(x)$, where $Q$ is a polynomial of the same degree as $G$. (See Example 5.4.5 .)

c. If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of Equation 5.4.16, then $y_p = x^2e^{\alpha x}Q(x)$, where $Q$ is a polynomial of the same degree as $G$. (See Example 5.4.6 .)

In all three cases, you can just substitute the appropriate form for $y_p$ and its derivatives directly into

$$ay_p'' + by_p' + cy_p = e^{\alpha x}G(x),$$

and solve for the coefficients of the polynomial $Q$. However, if you try this you will see that the computations are more tedious than those that you encounter by making the substitution $y = ue^{\alpha x}$ and finding a particular solution of the resulting equation for $u$. (See **Exercises 5.4.34-5.4.36**.) In Case (a) the equation for $u$ will be of the form

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x),$$

with a particular solution of the form $u_p = Q(x)$, a polynomial of the same degree as $G$, whose coefficients can be found by the method used in Example 5.4.4 . In Case (b) the equation for $u$ will be of the form

$$au'' + p'(\alpha)u' = G(x)$$

(no $u$ term on the left), with a particular solution of the form $u_p = xQ(x)$, where $Q$ is a polynomial of the same degree as $G$ whose coefficents can be found by the method used in Example 5.4.5 . In Case (c), the equation for $u$ will be of the form

$$au'' = G(x)$$

with a particular solution of the form $u_p = x^2Q(x)$ that can be obtained by integrating $G(x)/a$ twice and taking the constants of integration to be zero, as in Example 5.4.6 .

## Using the Principle of Superposition

The next example shows how to combine the method of undetermined coefficients and Theorem 5.3.3, the principle of superposition.

### Example 5.4.7

Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}. \tag{5.4.17}$$

**Solution**

In Example 5.4.1 we found that $y_{p_1} = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x},$$

and in Example 5.4.2 we found that $y_{p_2} = 5xe^{4x}$ is a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

Therefore the principle of superposition implies that $y_p = 2e^{2x} + 5xe^{4x}$ is a particular solution of Equation 5.4.17).

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# 5.4E: The Method of Undetermined Coefficients I (Exercises)

## Q5.4.1

In **Exercises 5.4.1-5.4.14** find a particular solution.

1. $y''-3y'+2y=e^{3x}(1+x)$
2. $y''-6y'+5y=e^{-3x}(35-8x)$
3. $y''-2y'-3y=e^{x}(-8+3x)$
4. $y''+2y'+y=e^{2x}(-7-15x+9x^2)$
5. $y''+4y=e^{-x}(7-4x+5x^2)$
6. $y''-y'-2y=e^{x}(9+2x-4x^2)$
7. $y''-4y'-5y=-6xe^{-x}$
8. $y''-3y'+2y=e^{x}(3-4x)$
9. $y''+y'-12y=e^{3x}(-6+7x)$
10. $2y''-3y'-2y=e^{2x}(-6+10x)$
11. $y''+2y'+y=e^{-x}(2+3x)$
12. $y''-2y'+y=e^{x}(1-6x)$
13. $y''-4y'+4y=e^{2x}(1-3x+6x^2)$
14. $9y''+6y'+y=e^{-x/3}(2-4x+4x^2)$

## Q5.4.2

In **Exercises 5.4.15-5.4.19** find the general solution.

15. $y''-3y'+2y=e^{3x}(1+x)$
16. $y''-6y'+8y=e^{x}(11-6x)$
17. $y''+6y'+9y=e^{2x}(3-5x)$
18. $y''+2y'-3y=-16xe^{x}$
19. $y''-2y'+y=e^{x}(2-12x)$

## Q5.4.3

In **Exercises 5.4.20-5.4.23** solve the initial value problem and plot the solution.

20. $y''-4y'-5y=9e^{2x}(1+x),\quad y(0)=0,\quad y'(0)=-10$
21. $y''+3y'-4y=e^{2x}(7+6x),\quad y(0)=2,\quad y'(0)=8$
22. $y''+4y'+3y=-e^{-x}(2+8x),\quad y(0)=1,\quad y'(0)=2$
23. $y''-3y'-10y=7e^{-2x},\quad y(0)=1,\quad y'(0)=-17$

## Q5.4.4

In **Exercises 5.4.24-5.4.29** use the principle of superposition to find a particular solution.

24. $y''+y'+y=xe^{x}+e^{-x}(1+2x)$
25. $y''-7y'+12y=-e^{x}(17-42x)-e^{3x}$
26. $y''-8y'+16y=6xe^{4x}+2+16x+16x^2$
27. $y''-3y'+2y=-e^{2x}(3+4x)-e^{x}$

28. $y'' - 2y' + 2y = e^x(1+x) + e^{-x}(2 - 8x + 5x^2)$
29. $y'' + y = e^{-x}(2 - 4x + 2x^2) + e^{3x}(8 - 12x - 10x^2)$

## Q5.4.5

30.

a. Prove that $y$ is a solution of the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x}G(x) \tag{A}$$

if and only if $y = ue^{\alpha x}$, where $u$ satisfies

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x) \tag{B}$$

and $p(r) = ar^2 + br + c$ is the characteristic polynomial of the complementary equation

$$ay'' + by' + cy = 0.$$

For the rest of this exercise, let $G$ be a polynomial. Give the requested proofs for the case where

$$G(x) = g_0 + g_1x + g_2x^2 + g_3x^3.$$

b. Prove that if $e^{\alpha x}$ isn't a solution of the complementary equation then (B) has a particular solution of the form $u_p = A(x)$, where $A$ is a polynomial of the same degree as $G$, as in Example 5.4.4. Conclude that (A) has a particular solution of the form $y_p = e^{\alpha x}A(x)$.

c. Show that if $e^{\alpha x}$ is a solution of the complementary equation and $xe^{\alpha x}$ isn't , then (B) has a particular solution of the form $u_p = xA(x)$, where $A$ is a polynomial of the same degree as $G$, as in Example 5.4.5. Conclude that (A) has a particular solution of the form $y_p = xe^{\alpha x}A(x)$.

d. Show that if $e^{\alpha x}$ and $xe^{\alpha x}$ are both solutions of the complementary equation then (B) has a particular solution of the form $u_p = x^2A(x)$, where $A$ is a polynomial of the same degree as $G$, and $x^2A(x)$ can be obtained by integrating $G/a$ twice, taking the constants of integration to be zero, as in Example 5.4.6. Conclude that (A) has a particular solution of the form $y_p = x^2e^{\alpha x}A(x)$.

## Q5.4.6

**Exercises 5.4.31–5.4.36** treat the equations considered in Examples 5.4.1–5.4.6. Substitute the suggested form of $y_p$ into the equation and equate the resulting coefficients of like functions on the two sides of the resulting equation to derive a set of simultaneous equations for the coefficients in $y_p$. Then solve for the coefficients to obtain $y_p$. Compare the work you've done with the work required to obtain the same results in Examples 5.4.1–5.4.6.

31. Compare with Example 5.4.1:

$$y'' - 7y' + 12y = 4e^{2x}; \quad y_p = Ae^{2x}$$

32. Compare with Example 5.4.2:

$$y'' - 7y' + 12y = 5e^{4x}; \quad y_p = Axe^{4x}$$

33. Compare with Example 5.4.3:

$$y'' - 8y' + 16y = 2e^{4x}; \quad y_p = Ax^2e^{4x}$$

34. Compare with Example 5.4.4:

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2), \quad y_p = e^{3x}(A + Bx + Cx^2)$$

35. Compare with Example 5.4.5:

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2), \quad y_p = e^{3x}(Ax + Bx^2 + Cx^3)$$

36. Compare with Example 5.4.6:

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2), \quad y_p = e^{-x/2}(Ax^2 + Bx^3 + Cx^4)$$

## Q5.4.7

37. Write $y = ue^{\alpha x}$ to find the general solution.

a. $y'' + 2y' + y = \frac{e^{-x}}{\sqrt{x}}$
b. $y'' + 6y' + 9y = e^{-3x} \ln x$
c. $y'' - 4y' + 4y = \frac{e^{2x}}{1+x}$
d. $4y'' + 4y' + y = 4e^{-x/2}\left(\frac{1}{x} + x\right)$

38. Suppose $\alpha \neq 0$ and $k$ is a positive integer. In most calculus books integrals like $\int x^k e^{\alpha x}\, dx$ are evaluated by integrating by parts $k$ times. This exercise presents another method. Let

$$y = \int e^{\alpha x} P(x)\, dx$$

with

$$P(x) = p_0 + p_1 x + \cdots + p_k x^k$$

(where $p_k \neq 0$).

a. Show that $y = e^{\alpha x} u$, where

$$u' + \alpha u = P(x). \tag{A}$$

b. Show that (A) has a particular solution of the form

$$u_p = A_0 + A_1 x + \cdots + A_k x^k,$$

where $A_k$, $A_{k-1}$, ..., $A_0$ can be computed successively by equating coefficients of $x^k, x^{k-1}, \ldots, 1$ on both sides of the equation

$$u_p' + \alpha u_p = P(x).$$

c. Conclude that

$$\int e^{\alpha x} P(x)\, dx = \left(A_0 + A_1 x + \cdots + A_k x^k\right) e^{\alpha x} + c,$$

where $c$ is a constant of integration.

39. Use the method of **Exercise 5.4.38** to evaluate the integral.

a. $\int e^x (4 + x) dx$
b. $\int e^{-x} (-1 + x^2) dx$
c. $\int x^3 e^{-2x} dx$
d. $\int e^x (1 + x)^2 dx$
e. $\int e^{3x} (-14 + 30x + 27x^2) dx$
f. $\int e^{-x} (1 + 6x^2 - 14x^3 + 3x^4) dx$

40. Use the method suggested in **Exercise 5.4.38** to evaluate $\int x^k e^{\alpha x}\, dx$, where $k$ is an arbitrary positive integer and $\alpha \neq 0$.

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# 5.5: The Method of Undetermined Coefficients II

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{5.5.1}$$

where $\lambda$ and $\omega$ are real numbers, $\omega \neq 0$, and $P$ and $Q$ are polynomials. We want to find a particular solution of Equation 5.5.1. As in Section 5.4, the procedure that we will use is called **the method of undetermined coefficients**.

## Forcing Functions Without Exponential Factors

We begin with the case where $\lambda = 0$ in Equation 5.5.1; thus, we we want to find a particular solution of

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x, \tag{5.5.2}$$

where $P$ and $Q$ are polynomials.

Differentiating $x^r\cos\omega x$ and $x^r\sin\omega x$ yields

$$\frac{d}{dx}x^r\cos\omega x = -\omega x^r\sin\omega x + rx^{r-1}\cos\omega x$$

and

$$\frac{d}{dx}x^r\sin\omega x = \quad \omega x^r\cos\omega x + rx^{r-1}\sin\omega x.$$

This implies that if

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x$$

where $A$ and $B$ are polynomials, then

$$ay_p'' + by_p' + cy_p = F(x)\cos\omega x + G(x)\sin\omega x,$$

where $F$ and $G$ are polynomials with coefficients that can be expressed in terms of the coefficients of $A$ and $B$. This suggests that we try to choose $A$ and $B$ so that $F = P$ and $G = Q$, respectively. Then $y_p$ will be a particular solution of Equation 5.5.2. The next theorem tells us how to choose the proper form for $y_p$. For the proof see *Exercise 5.5.37*.

**Theorem 5.5.1**

Suppose $\omega$ is a positive number and $P$ and $Q$ are polynomials. Let $k$ be the larger of the degrees of $P$ and $Q$. Then the equation

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x, \tag{5.5.3}$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \text{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

provided that $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation. The solutions of

$$a(y'' + \omega^2 y) = P(x)\cos\omega x + Q(x)\sin\omega x$$

for which $\cos\omega x$ and $\sin\omega x$ are solutions of the complementary equation are of the form of Equation 5.5.3, where

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \text{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}.$$

For an analog of this theorem that's applicable to Equation 5.5.1, see **Exercise 5.5.38**.

**Example 5.5.1**

Find a particular solution of

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x. \tag{5.5.4}$$

**Solution**

In Equation 5.5.4 the coefficients of $\cos 2x$ and $\sin 2x$ are both zero degree polynomials (constants). Therefore Theorem 5.5.1 implies that Equation 5.5.4 has a particular solution

$$y_p = A\cos 2x + B\sin 2x.$$

Since

$$y_p' = -2A\sin 2x + 2B\cos 2x \quad \text{and} \quad y_p'' = -4(A\cos 2x + B\sin 2x),$$

replacing $y$ by $y_p$ in Equation 5.5.4 yields

$$\begin{aligned} y_p'' - 2y_p' + y_p &= -4(A\cos 2x + B\sin 2x) - 4(-A\sin 2x + B\cos 2x) + (A\cos 2x + B\sin 2x) \\ &= (-3A - 4B)\cos 2x + (4A - 3B)\sin 2x. \end{aligned}$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ here with the corresponding coefficients on the right side of Equation 5.5.4 shows that $y_p$ is a solution of Equation 5.5.4 if

$$\begin{aligned} -3A - 4B &= 5 \\ 4A - 3B &= 10. \end{aligned}$$

Solving these equations yields $A = 1$, $B = -2$. Therefore

$$y_p = \cos 2x - 2\sin 2x$$

is a particular solution of Equation 5.5.4.

## Example 5.5.2

Find a particular solution of

$$y'' + 4y = 8\cos 2x + 12\sin 2x. \tag{5.5.5}$$

**Solution**

The procedure used in Example 5.5.1 doesn't work here; substituting $y_p = A\cos 2x + B\sin 2x$ for $y$ in Equation 5.5.5 yields

$$y_p'' + 4y_p = -4(A\cos 2x + B\sin 2x) + 4(A\cos 2x + B\sin 2x) = 0$$

for any choice of $A$ and $B$, since $\cos 2x$ and $\sin 2x$ are both solutions of the complementary equation for Equation 5.5.5. We're dealing with the second case mentioned in Theorem 5.5.1 , and should therefore try a particular solution of the form

$$y_p = x(A\cos 2x + B\sin 2x). \tag{5.5.6}$$

Then

$$\begin{aligned} y_p' &= A\cos 2x + B\sin 2x + 2x(-A\sin 2x + B\cos 2x) \quad \text{and} \\ y_p'' &= -4A\sin 2x + 4B\cos 2x - 4x(A\cos 2x + B\sin 2x) \\ &= -4A\sin 2x + 4B\cos 2x - 4y_p \text{ (see (5.5.6))}, \end{aligned}$$

so

$$y_p'' + 4y_p = -4A\sin 2x + 4B\cos 2x.$$

Therefore $y_p$ is a solution of Equation 5.5.5 if

$$-4A\sin 2x + 4B\cos 2x = 8\cos 2x + 12\sin 2x,$$

which holds if $A = -3$ and $B = 2$. Therefore

$$y_p = -x(3\cos 2x - 2\sin 2x)$$

is a particular solution of Equation 5.5.5.

## Example 5.5.3

Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x. \tag{5.5.7}$$

**Solution**

The coefficients of $\cos x$ and $\sin x$ in Equation 5.5.7 are polynomials of degree one and zero, respectively. Therefore Theorem 5.5.1 tells us to look for a particular solution of Equation 5.5.7 of the form

$$y_p = (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x. \tag{5.5.8}$$

Then

$$y_p' = (A_1 + B_0 + B_1 x)\cos x + (B_1 - A_0 - A_1 x)\sin x \tag{5.5.9}$$

and

$$y_p'' = (2B_1 - A_0 - A_1 x)\cos x - (2A_1 + B_0 + B_1 x)\sin x, \tag{5.5.10}$$

so

$$y_p'' + 3y_p' + 2y_p \quad = [A_0 + 3A_1 + 3B_0 + 2B_1 + (A_1 + 3B_1)x]\cos x + [B_0 + 3B_1 - 3A_0 - 2A_1 + (B_1 - 3A_1)x]\sin x. \tag{5.5.11}$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in Equation 5.5.7 shows that $y_p$ is a solution of Equation 5.5.7 if

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + B_0 - 2A_1 + 3B_1 &= 10. \end{aligned}$$

Solving the first two equations yields $A_1 = 2$, $B_1 = 6$. Substituting these into the last two equations yields

$$\begin{aligned} A_0 + 3B_0 &= 16 - 3A_1 - 2B_1 = -2 \\ -3A_0 + B_0 &= 10 + 2A_1 - 3B_1 = -4. \end{aligned}$$

Solving these equations yields $A_0 = 1$, $B_0 = -1$. Substituting $A_0 = 1$, $A_1 = 2$, $B_0 = -1$, $B_1 = 6$ into Equation 5.5.8 shows that

$$y_p = (1 + 2x)\cos x - (1 - 6x)\sin x$$

is a particular solution of Equation 5.5.7.

## A Useful Observation

In Equations 5.5.9, 5.5.10, and 5.5.11 the polynomials multiplying $\sin x$ can be obtained by replacing $A_0$, $A_1$, $B_0$, and $B_1$ by $B_0$, $B_1$, $-A_0$, and $-A_1$, respectively, in the polynomials multiplying $\cos x$. An analogous result applies in general, as follows (**Exercise 5.5.36**).

**Theorem 5.5.2**

If

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x,$$

where $A(x)$ and $B(x)$ are polynomials with coefficients $A_0$ …, $A_k$ and $B_0$, …, $B_k$, then the polynomials multiplying $\sin\omega x$ in

$$y_p', \quad y_p'', \quad ay_p'' + by_p' + cy_p \quad \text{and} \quad y_p'' + \omega^2 y_p$$

can be obtained by replacing $A_0$, …, $A_k$ by $B_0$, …, $B_k$ and $B_0$, …, $B_k$ by $-A_0$, …, $-A_k$ in the corresponding polynomials multiplying $\cos\omega x$.

We will not use this theorem in our examples, but we recommend that you use it to check your manipulations when you work the exercises.

**Example 5.5.4**

Find a particular solution of

$$y'' + y = (8 - 4x)\cos x - (8 + 8x)\sin x. \tag{5.5.12}$$

**Solution**

According to Theorem 5.5.1 , we should look for a particular solution of the form

$$y_p = (A_0 x + A_1 x^2)\cos x + (B_0 x + B_1 x^2)\sin x, \tag{5.5.13}$$

since $\cos x$ and $\sin x$ are solutions of the complementary equation. However, let's try

$$y_p = (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x \tag{5.5.14}$$

first, so you can see why it doesn't work. From Equation 5.5.10,

$$y_p'' = (2B_1 - A_0 - A_1 x)\cos x - (2A_1 + B_0 + B_1 x)\sin x,$$

which together with Equation 5.5.14 implies that

$$y_p'' + y_p = 2B_1 \cos x - 2A_1 \sin x.$$

Since the right side of this equation does not contain $x\cos x$ or $x\sin x$, Equation 5.5.14 can't satisfy Equation 5.5.12 no matter how we choose $A_0$, $A_1$, $B_0$, and $B_1$.

Now let $y_p$ be as in Equation 5.5.13. Then

$$\begin{aligned} y_p' &= \left[A_0 + (2A_1 + B_0)x + B_1 x^2\right]\cos x \\ &\quad + \left[B_0 + (2B_1 - A_0)x - A_1 x^2\right]\sin x \end{aligned}$$

and

$$\begin{aligned} y_p'' &= \left[2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1 x^2\right]\cos x \\ &\quad + \left[2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1 x^2\right]\sin x, \end{aligned}$$

so

$$y_p'' + y_p = (2A_1 + 2B_0 + 4B_1 x)\cos x + (2B_1 - 2A_0 - 4A_1 x)\sin x.$$

Comparing the coefficients of $\cos x$ and $\sin x$ here with the corresponding coefficients in Equation 5.5.12 shows that $y_p$ is a solution of Equation 5.5.12 if

$$\begin{aligned} 4B_1 &= -4 \\ -4A_1 &= -8 \\ 2B_0 + 2A_1 &= 8 \\ -2A_0 + 2B_1 &= -8. \end{aligned}$$

The solution of this system is $A_1 = 2$, $B_1 = -1$, $A_0 = 3$, $B_0 = 2$. Therefore

$$y_p = x\left[(3 + 2x)\cos x + (2 - x)\sin x\right]$$

is a particular solution of Equation 5.5.12.

## Forcing Functions with Exponential Factors

To find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{5.5.15}$$

when $\lambda \neq 0$, we recall from Section 5.4 that substituting $y = ue^{\lambda x}$ into Equation 5.5.15 will produce a constant coefficient equation for $u$ with the forcing function $P(x)\cos\omega x + Q(x)\sin\omega x$. We can find a particular solution $u_p$ of this equation by the procedure that we used in Examples 5.5.1 -5.5.4 . Then $y_p = u_p e^{\lambda x}$ is a particular solution of Equation 5.5.15.

### Example 5.5.5

Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right]. \tag{5.5.16}$$

Let $y = ue^{-2x}$. Then

$$\begin{aligned} y'' - 3y' + 2y &= e^{-2x}\left[(u'' - 4u' + 4u) - 3(u' - 2u) + 2u\right] \\ &= e^{-2x}(u'' - 7u' + 12u) \\ &= e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right] \end{aligned}$$

if

$$u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x. \tag{5.5.17}$$

Since $\cos 3x$ and $\sin 3x$ aren't solutions of the complementary equation

$$u'' - 7u' + 12u = 0,$$

Theorem 5.5.1 tells us to look for a particular solution of Equation 5.5.17 of the form

$$u_p = (A_0 + A_1 x)\cos 3x + (B_0 + B_1 x)\sin 3x. \tag{5.5.18}$$

Then

$$\begin{aligned} u_p' &= (A_1 + 3B_0 + 3B_1 x)\cos 3x + (B_1 - 3A_0 - 3A_1 x)\sin 3x \\ \text{and} \quad u_p'' &= (-9A_0 + 6B_1 - 9A_1 x)\cos 3x - (9B_0 + 6A_1 + 9B_1 x)\sin 3x, \end{aligned}$$

so

$$\begin{aligned} u_p'' - 7u_p' + 12u_p &= [3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x]\cos 3x \\ &\quad + [21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x]\sin 3x. \end{aligned}$$

Comparing the coefficients of $x\cos 3x$, $x\sin 3x$, $\cos 3x$, and $\sin 3x$ here with the corresponding coefficients on the right side of Equation 5.5.17 shows that $u_p$ is a solution of Equation 5.5.17 if

$$\begin{aligned} 3A_1 - 21B_1 &= 0 \\ 21A_1 + 3B_1 &= 150 \\ 3A_0 - 21B_0 - 7A_1 + 6B_1 &= 2 \\ 21A_0 + 3B_0 - 6A_1 - 7B_1 &= -34. \end{aligned} \tag{5.5.19}$$

Solving the first two equations yields $A_1 = 7$, $B_1 = 1$. Substituting these values into the last two equations of Equation 5.5.19 yields

$$\begin{aligned} 3A_0 - 21B_0 &= 2 + 7A_1 - 6B_1 = 45 \\ 21A_0 + 3B_0 &= -34 + 6A_1 + 7B_1 = 15. \end{aligned}$$

Solving this system yields $A_0 = 1$, $B_0 = -2$. Substituting $A_0 = 1$, $A_1 = 7$, $B_0 = -2$, and $B_1 = 1$ into Equation 5.5.18 shows that

$$u_p = (1 + 7x)\cos 3x - (2 - x)\sin 3x$$

is a particular solution of Equation 5.5.17. Therefore

$$y_p = e^{-2x}\left[(1 + 7x)\cos 3x - (2 - x)\sin 3x\right]$$

is a particular solution of Equation 5.5.16.

## Example 5.5.6

Find a particular solution of

$$y'' + 2y' + 5y = e^{-x}\left[(6 - 16x)\cos 2x - (8 + 8x)\sin 2x\right]. \tag{5.5.20}$$

**Solution**

Let $y = ue^{-x}$. Then

$$\begin{aligned} y'' + 2y' + 5y &= e^{-x}\left[(u'' - 2u' + u) + 2(u' - u) + 5u\right] \\ &= e^{-x}(u'' + 4u) \\ &= e^{-x}\left[(6 - 16x)\cos 2x - (8 + 8x)\sin 2x\right] \end{aligned}$$

if

$$u'' + 4u = (6 - 16x)\cos 2x - (8 + 8x)\sin 2x. \tag{5.5.21}$$

Since $\cos 2x$ and $\sin 2x$ are solutions of the complementary equation

$$u'' + 4u = 0,$$

Theorem 5.5.1 tells us to look for a particular solution of Equation 5.5.21 of the form

$$u_p = (A_0 x + A_1 x^2)\cos 2x + (B_0 x + B_1 x^2)\sin 2x.$$

Then

$$\begin{aligned} u_p' &= \left[A_0 + (2A_1 + 2B_0)x + 2B_1 x^2\right]\cos 2x \\ &\quad + \left[B_0 + (2B_1 - 2A_0)x - 2A_1 x^2\right]\sin 2x \end{aligned}$$

and

$$\begin{aligned} u_p'' &= \left[2A_1 + 4B_0 - (4A_0 - 8B_1)x - 4A_1 x^2\right]\cos 2x \\ &\quad + \left[2B_1 - 4A_0 - (4B_0 + 8A_1)x - 4B_1 x^2\right]\sin 2x, \end{aligned}$$

so

$$u_p'' + 4u_p = (2A_1 + 4B_0 + 8B_1 x)\cos 2x + (2B_1 - 4A_0 - 8A_1 x)\sin 2x.$$

Equating the coefficients of $x\cos 2x$, $x\sin 2x$, $\cos 2x$, and $\sin 2x$ here with the corresponding coefficients on the right side of Equation 5.5.21 shows that $u_p$ is a solution of Equation 5.5.21 if

$$\begin{aligned} 8B_1 &= -16 \\ -8A_1 &= -\ 8 \\ 4B_0 + 2A_1 &= 6 \\ -4A_0 + 2B_1 &= -8. \end{aligned} \tag{5.5.22}$$

The solution of this system is $A_1 = 1$, $B_1 = -2$, $B_0 = 1$, $A_0 = 1$. Therefore

$$u_p = x[(1+x)\cos 2x + (1-2x)\sin 2x]$$

is a particular solution of Equation 5.5.21, and

$$y_p = xe^{-x}\left[(1+x)\cos 2x + (1-2x)\sin 2x\right]$$

is a particular solution of Equation 5.5.20.

You can also find a particular solution of Equation 5.5.20 by substituting

$$y_p = xe^{-x}\left[(A_0 + A_1 x)\cos 2x + (B_0 + B_1 x)\sin 2x\right]$$

for $y$ in Equation 5.5.20 and equating the coefficients of $xe^{-x}\cos 2x$, $xe^{-x}\sin 2x$, $e^{-x}\cos 2x$, and $e^{-x}\sin 2x$ in the resulting expression for

$$y_p'' + 2y_p' + 5y_p$$

with the corresponding coefficients on the right side of Equation 5.5.20. (See **Exercise 5.5.38**). This leads to the same system Equation 5.5.22 of equations for $A_0$, $A_1$, $B_0$, and $B_1$ that we obtained in Example 5.5.6 . However, if you try this approach you'll see that deriving Equation 5.5.22 this way is much more tedious than the way we did it in Example 5.5.6 .

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# 5.5E: The Method of Undetermined Coefficients II (Exercises)

## Q5.5.1

In **Exercises 5.5.1-5.5.17** find a particular solution.

1. $y''+3y'+2y=7\cos x-\sin x$
2. $y''+3y'+y=(2-6x)\cos x-9\sin x$
3. $y''+2y'+y=e^x(6\cos x+17\sin x)$
4. $y''+3y'-2y=-e^{2x}(5\cos 2x+9\sin 2x)$
5. $y''-y'+y=e^x(2+x)\sin x$
6. $y''+3y'-2y=e^{-2x}\left[(4+20x)\cos 3x+(26-32x)\sin 3x\right]$
7. $y''+4y=-12\cos 2x-4\sin 2x$
8. $y''+y=(-4+8x)\cos x+(8-4x)\sin x$
9. $4y''+y=-4\cos x/2-8x\sin x/2$
10. $y''+2y'+2y=e^{-x}(8\cos x-6\sin x)$
11. $y''-2y'+5y=e^x\left[(6+8x)\cos 2x+(6-8x)\sin 2x\right]$
12. $y''+2y'+y=8x^2\cos x-4x\sin x$
13. $y''+3y'+2y=(12+20x+10x^2)\cos x+8x\sin x$
14. $y''+3y'+2y=(1-x-4x^2)\cos 2x-(1+7x+2x^2)\sin 2x$
15. $y''-5y'+6y=-e^x\left[(4+6x-x^2)\cos x-(2-4x+3x^2)\sin x\right]$
16. $y''-2y'+y=-e^x\left[(3+4x-x^2)\cos x+(3-4x-x^2)\sin x\right]$
17. $y''-2y'+2y=e^x\left[(2-2x-6x^2)\cos x+(2-10x+6x^2)\sin x\right]$

## Q5.5.2

In **Exercises 5.5.18-5.5.21** find a particular solution and graph it.

18. $y''+2y'+y=e^{-x}\left[(5-2x)\cos x-(3+3x)\sin x\right]$
19. $y''+9y=-6\cos 3x-12\sin 3x$
20. $y''+3y'+2y=(1-x-4x^2)\cos 2x-(1+7x+2x^2)\sin 2x$
21. $y''+4y'+3y=e^{-x}\left[(2+x+x^2)\cos x+(5+4x+2x^2)\sin x\right]$

## Q5.5.3

In **Exercises 5.5.22-5.5.26** solve the initial value problem.

22. $y''-7y'+6y=-e^x(17\cos x-7\sin x),\quad y(0)=4,\ y'(0)=2$
23. $y''-2y'+2y=-e^x(6\cos x+4\sin x),\quad y(0)=1,\ y'(0)=4$
24. $y''+6y'+10y=-40e^x\sin x,\quad y(0)=2,\quad y'(0)=-3$
25. $y''-6y'+10y=-e^{3x}(6\cos x+4\sin x),\quad y(0)=2,\quad y'(0)=7$
26. $y''-3y'+2y=e^{3x}\left[21\cos x-(11+10x)\sin x\right],\ y(0)=0,\quad y'(0)=6$

## Q5.5.4

In **Exercises 5.5.27-5.5.32** use the principle of superposition to find a particular solution. Where indicated, solve the initial value problem.

27. $y''-2y'-3y=4e^{3x}+e^x(\cos x-2\sin x)$

28. $y''+y=4\cos x-2\sin x+xe^x+e^{-x}$
29. $y''-3y'+2y=xe^x+2e^{2x}+\sin x$
30. $y''-2y'+2y=4xe^x\cos x+xe^{-x}+1+x^2$
31. $y''-4y'+4y=e^{2x}(1+x)+e^{2x}(\cos x-\sin x)+3e^{3x}+1+x$
32. $y''-4y'+4y=6e^{2x}+25\sin x,\quad y(0)=5,\ y'(0)=3$

## Q5.5.5

In **Exercises 5.5.33-5.5.35** solve the initial value problem and graph the solution.

33. $y''+4y=-e^{-2x}\left[(4-7x)\cos x+(2-4x)\sin x\right],\ y(0)=3,\quad y'(0)=1$
34. $y''+4y'+4y=2\cos 2x+3\sin 2x+e^{-x},\quad y(0)=-1,\ y'(0)=2$
35. $y''+4y=e^x(11+15x)+8\cos 2x-12\sin 2x,\quad y(0)=3,\ y'(0)=5$

## Q5.5.6

36.

a. Verify that if

$$y_p=A(x)\cos\omega x+B(x)\sin\omega x$$

where $A$ and $B$ are twice differentiable, then

$$\begin{aligned} y_p' &= (A'+\omega B)\cos\omega x+(B'-\omega A)\sin\omega x \quad \text{and}\\ y_p'' &= (A''+2\omega B'-\omega^2A)\cos\omega x+(B''-2\omega A'-\omega^2B)\sin\omega x. \end{aligned}$$

b. Use the results of (a) to verify that

$$\begin{aligned} ay_p''+by_p'+cy_p= &\left[(c-a\omega^2)A+b\omega B+2a\omega B'+bA'+aA''\right]\cos\omega x+\\ &\left[-b\omega A+(c-a\omega^2)B-2a\omega A'+bB'+aB''\right]\sin\omega x. \end{aligned}$$

c. Use the results of (a) to verify that

$$y_p''+\omega^2y_p=(A''+2\omega B')\cos\omega x+(B''-2\omega A')\sin\omega x.$$

d. Prove Theorem 5.5.2.

37. Let $a$, $b$, $c$, and $\omega$ be constants, with $a\neq 0$ and $\omega>0$, and let

$$P(x)=p_0+p_1x+\cdots+p_kx^k \quad\text{and}\quad Q(x)=q_0+q_1x+\cdots+q_kx^k,$$

where at least one of the coefficients $p_k$, $q_k$ is nonzero, so $k$ is the larger of the degrees of $P$ and $Q$.

a. Show that if $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation

$$ay''+by'+cy=0,$$

then there are polynomials

$$A(x)=A_0+A_1x+\cdots+A_kx^k \quad\text{and}\quad B(x)=B_0+B_1x+\cdots+B_kx^k \tag{A}$$

such that

$$\begin{aligned} (c-a\omega^2)A+b\omega B+2a\omega B'+bA'+aA'' &= P\\ -b\omega A+(c-a\omega^2)B-2a\omega A'+bB'+aB'' &= Q, \end{aligned}$$

where $(A_k,B_k)$, $(A_{k-1},B_{k-1})$, ...,$(A_0,B_0)$ can be computed successively by solving the systems

$$(c-a\omega^2)A_k+b\omega B_k \quad = p_k$$
$$-b\omega A_k+(c-a\omega^2)B_k \quad = q_k,$$

and, if $1\le r\le k$,

$$(c-a\omega^2)A_{k-r}+b\omega B_{k-r} \quad = p_{k-r}+\cdots$$
$$-b\omega A_{k-r}+(c-a\omega^2)B_{k-r} \quad = q_{k-r}+\cdots,$$

where the terms indicated by "$\cdots$" depend upon the previously computed coefficients with subscripts greater than $k-r$. Conclude from this and **Exercise 5.5.36b** that

$$y_p=A(x)\cos\omega x+B(x)\sin\omega x \tag{B}$$

is a particular solution of

$$ay''+by'+cy=P(x)\cos\omega x+Q(x)\sin\omega x.$$

b. Conclude from **Exercise 5.5.36c** that the equation

$$a(y''+\omega^2 y)=P(x)\cos\omega x+Q(x)\sin\omega x \tag{C}$$

does not have a solution of the form (B) with $A$ and $B$ as in (A). Then show that there are polynomials

$$A(x)=A_0x+A_1x^2+\cdots+A_kx^{k+1} \quad \text{and} \quad B(x)=B_0x+B_1x^2+\cdots+B_kx^{k+1}$$

such that

$$a(A''+2\omega B') \quad = P$$
$$a(B''-2\omega A') \quad = Q,$$

where the pairs $(A_k,B_k)$, $(A_{k-1},B_{k-1})$, ..., $(A_0,B_0)$ can be computed successively as follows:

$$A_k=-\frac{q_k}{2a\omega(k+1)}$$
$$B_k=\frac{p_k}{2a\omega(k+1)},$$

and, if $k\ge 1$,

$$A_{k-j} \;=-\frac{1}{2\omega}\left[\frac{q_{k-j}}{a(k-j+1)}-(k-j+2)B_{k-j+1}\right]$$
$$B_{k-j} \;=\frac{1}{2\omega}\left[\frac{p_{k-j}}{a(k-j+1)}-(k-j+2)A_{k-j+1}\right]$$

for $1\le j\le k$. Conclude that (B) with this choice of the polynomials $A$ and $B$ is a particular solution of (C).

38. Show that Theorem 5.5.1 implies the next theorem:

**Theorem 5.5E.1**

**Suppose $\omega$** is a positive number and $P$ and $Q$ are polynomials. Let $k$ be the larger of the degrees of $P$ and $Q$. Then the equation

$$ay''+by'+cy=e^{\lambda x}\left(P(x)\cos\omega x+Q(x)\sin\omega x\right)$$

has a particular solution

$$y_p=e^{\lambda x}\left(A(x)\cos\omega x+B(x)\sin\omega x\right), \tag{A}$$

where

$$A(x)=A_0+A_1x+\cdots+A_kx^k \quad \text{and} \quad B(x)=B_0+B_1x+\cdots+B_kx^k,$$

provided that $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are not solutions of the complementary equation. The equation

$$a\left[y''-2\lambda y'+(\lambda^2+\omega^2)y\right]=e^{\lambda x}\left(P(x)\cos\omega x+Q(x)\sin\omega x\right)$$

(for which $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are solutions of the complementary equation) has a particular solution of the form (A), where

$$A(x)=A_0x+A_1x^2+\cdots+A_kx^{k+1}\quad\text{and}\quad B(x)=B_0x+B_1x^2+\cdots+B_kx^{k+1}.$$

39. This exercise presents a method for evaluating the integral

$$y=\int e^{\lambda x}\left(P(x)\cos\omega x+Q(x)\sin\omega x\right)\,dx$$

where $\omega\neq0$ and

$$P(x)=p_0+p_1x+\cdots+p_kx^k,\quad Q(x)=q_0+q_1x+\cdots+q_kx^k.$$

a. Show that $y=e^{\lambda x}u$, where

$$u'+\lambda u=P(x)\cos\omega x+Q(x)\sin\omega x. \tag{A}$$

b. Show that (A) has a particular solution of the form

$$u_p=A(x)\cos\omega x+B(x)\sin\omega x,$$

where

$$A(x)=A_0+A_1x+\cdots+A_kx^k,\quad B(x)=B_0+B_1x+\cdots+B_kx^k,$$

and the pairs of coefficients $(A_k,B_k)$, $(A_{k-1},B_{k-1})$, …,$(A_0,B_0)$ can be computed successively as the solutions of pairs of equations obtained by equating the coefficients of $x^r\cos\omega x$ and $x^r\sin\omega x$ for $r=k$, $k-1$, …, $0$.

c. Conclude that

$$\int e^{\lambda x}\left(P(x)\cos\omega x+Q(x)\sin\omega x\right)\,dx=e^{\lambda x}\left(A(x)\cos\omega x+B(x)\sin\omega x\right)+c,$$

where $c$ is a constant of integration.

40. Use the method of **Exercise 5.5.39** to evaluate the integral.

a. $\int x^2\cos x\,dx$
b. $\int x^2e^x\cos x\,dx$
c. $\int xe^{-x}\sin2x\,dx$
d. $\int x^2e^{-x}\sin x\,dx$
e. $\int x^3e^x\sin x\,dx$
f. $\int e^x[x\cos x-(1+3x)\sin x]dx$
g. $\int e^{-x}[(1+x^2)\cos x+(1^2_x)\sin x]dx$

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# 5.6: Reduction of Order

In this section we give a method for finding the general solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{5.6.1}$$

if we know a nontrivial solution $y_1$ of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{5.6.2}$$

The method is called **reduction of order** because it reduces the task of solving Equation 5.6.1 to solving a first order equation. Unlike the method of undetermined coefficients, it does not require $P_0$, $P_1$, and $P_2$ to be constants, or $F$ to be of any special form.

By now you shoudn't be surprised that we look for solutions of Equation 5.6.1 in the form

$$y = uy_1 \tag{5.6.3}$$

where $u$ is to be determined so that $y$ satisfies Equation 5.6.1. Substituting Equation 5.6.3 and

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

into Equation 5.6.1 yields

$$P_0(x)(u''y_1 + 2u'y_1' + uy_1'') + P_1(x)(u'y_1 + uy_1') + P_2(x)uy_1 = F(x).$$

Collecting the coefficients of $u$, $u'$, and $u''$ yields

$$(P_0y_1)u'' + (2P_0y_1' + P_1y_1)u' + (P_0y_1'' + P_1y_1' + P_2y_1)u = F. \tag{5.6.4}$$

However, the coefficient of $u$ is zero, since $y_1$ satisfies Equation 5.6.2. Therefore Equation 5.6.4 reduces to

$$Q_0(x)u'' + Q_1(x)u' = F, \tag{5.6.5}$$

with

$$Q_0 = P_0y_1 \quad \text{and} \quad Q_1 = 2P_0y_1' + P_1y_1.$$

(It isn't worthwhile to memorize the formulas for $Q_0$ and $Q_1$!) Since Equation 5.6.5 is a linear first order equation in $u'$, we can solve it for $u'$ by variation of parameters as in Section 1.2, integrate the solution to obtain $u$, and then obtain $y$ from Equation 5.6.3.

## Example 5.6.1

a. Find the general solution of

$$xy'' - (2x+1)y' + (x+1)y = x^2, \tag{5.6.6}$$

given that $y_1 = e^x$ is a solution of the complementary equation

$$xy'' - (2x+1)y' + (x+1)y = 0. \tag{5.6.7}$$

b. As a byproduct of (a), find a fundamental set of solutions of Equation 5.6.7.

### Solution

a. If $y = ue^x$, then $y' = u'e^x + ue^x$ and $y'' = u''e^x + 2u'e^x + ue^x$, so

$$\begin{aligned} xy'' - (2x+1)y' + (x+1)y &= x(u''e^x + 2u'e^x + ue^x) - (2x+1)(u'e^x + ue^x) + (x+1)ue^x \\ &= (xu'' - u')e^x. \end{aligned}$$

Therefore $y = ue^x$ is a solution of Equation 5.6.6 if and only if

$$(xu'' - u')e^x = x^2,$$

which is a first order equation in $u'$. We rewrite it as

$$u'' - \frac{u'}{x} = xe^{-x}. \tag{5.6.8}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that Equation 5.6.8 becomes

$$z' - \frac{z}{x} = xe^{-x}. \tag{5.6.9}$$

We leave it to you to show (by separation of variables) that $z_1 = x$ is a solution of the complementary equation

$$z' - \frac{z}{x} = 0$$

for Equation 5.6.9. By applying variation of parameters as in Section 1.2, we can now see that every solution of Equation 5.6.9 is of the form

$$z = vx \quad \text{where} \quad v'x = xe^{-x}, \quad \text{so} \quad v' = e^{-x} \quad \text{and} \quad v = -e^{-x} + C_1.$$

Since $u' = z = vx$, $u$ is a solution of Equation 5.6.8 if and only if

$$u' = vx = -xe^{-x} + C_1 x.$$

Integrating this yields

$$u = (x+1)e^{-x} + \frac{C_1}{2}x^2 + C_2.$$

Therefore the general solution of Equation 5.6.6 is

$$y = ue^x = x + 1 + \frac{C_1}{2}x^2e^x + C_2e^x. \tag{5.6.10}$$

b. By letting $C_1 = C_2 = 0$ in Equation 5.6.10, we see that $y_{p_1} = x + 1$ is a solution of Equation 5.6.6. By letting $C_1 = 2$ and $C_2 = 0$, we see that $y_{p_2} = x + 1 + x^2e^x$ is also a solution of Equation 5.6.6. Since the difference of two solutions of Equation 5.6.6 is a solution of Equation 5.6.7, $y_2 = y_{p_1} - y_{p_2} = x^2e^x$ is a solution of Equation 5.6.7. Since $y_2/y_1$ is nonconstant and we already know that $y_1 = e^x$ is a solution of Equation 5.6.6, Theorem 5.1.6 implies that $\{e^x, x^2e^x\}$ is a fundamental set of solutions of Equation 5.6.7.

Although Equation 5.6.10 is a correct form for the general solution of Equation 5.6.6, it is silly to leave the arbitrary coefficient of $x^2e^x$ as $C_1/2$ where $C_1$ is an arbitrary constant. Moreover, it is sensible to make the subscripts of the coefficients of $y_1 = e^x$ and $y_2 = x^2e^x$ consistent with the subscripts of the functions themselves. Therefore we rewrite Equation 5.6.10 as

$$y = x + 1 + c_1e^x + c_2x^2e^x$$

by simply renaming the arbitrary constants. We'll also do this in the next two examples, and in the answers to the exercises.

## Example 5.6.2

a. Find the general solution of

$$x^2y'' + xy' - y = x^2 + 1,$$

given that $y_1 = x$ is a solution of the complementary equation

$$x^2y'' + xy' - y = 0. \tag{5.6.11}$$

As a byproduct of this result, find a fundamental set of solutions of Equation 5.6.11.

b. Solve the initial value problem

$$x^2y'' + xy' - y = x^2 + 1, \quad y(1) = 2,\ y'(1) = -3. \tag{5.6.12}$$

**Solution**

a. If $y = ux$, then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y''+xy'-y &= x^2(u''x+2u')+x(u'x+u)-ux \\ &= x^3u''+3x^2u'. \end{aligned}$$

Therefore $y=ux$ is a solution of Equation 5.6.12 if and only if

$$x^3u''+3x^2u'=x^2+1,$$

which is a first order equation in $u'$. We rewrite it as

$$u''+\frac{3}{x}u'=\frac{1}{x}+\frac{1}{x^3}. \tag{5.6.13}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z=u'$, so that Equation 5.6.13 becomes

$$z'+\frac{3}{x}z=\frac{1}{x}+\frac{1}{x^3}. \tag{5.6.14}$$

We leave it to you to show by separation of variables that $z_1=1/x^3$ is a solution of the complementary equation

$$z'+\frac{3}{x}z=0$$

for Equation 5.6.14. By variation of parameters, every solution of Equation 5.6.14 is of the form

$$z=\frac{v}{x^3} \quad \text{where} \quad \frac{v'}{x^3}=\frac{1}{x}+\frac{1}{x^3}, \quad \text{so} \quad v'=x^2+1 \quad \text{and} \quad v=\frac{x^3}{3}+x+C_1.$$

Since $u'=z=v/x^3$, $u$ is a solution of Equation 5.6.14 if and only if

$$u'=\frac{v}{x^3}=\frac{1}{3}+\frac{1}{x^2}+\frac{C_1}{x^3}.$$

Integrating this yields

$$u=\frac{x}{3}-\frac{1}{x}-\frac{C_1}{2x^2}+C_2.$$

Therefore the general solution of Equation 5.6.12 is

$$y=ux=\frac{x^2}{3}-1-\frac{C_1}{2x}+C_2x. \tag{5.6.15}$$

Reasoning as in the solution of Example 5.6.1*a*, we conclude that $y_1=x$ and $y_2=1/x$ form a fundamental set of solutions for Equation 5.6.11.

As we explained above, we rename the constants in Equation 5.6.15 and rewrite it as

$$y=\frac{x^2}{3}-1+c_1x+\frac{c_2}{x}. \tag{5.6.16}$$

b. Differentiating Equation 5.6.16 yields

$$y'=\frac{2x}{3}+c_1-\frac{c_2}{x^2}. \tag{5.6.17}$$

Setting $x=1$ in Equation 5.6.16 and Equation 5.6.17 and imposing the initial conditions $y(1)=2$ and $y'(1)=-3$ yields

$$\begin{aligned} c_1+c_2 &= \frac{8}{3} \\ c_1-c_2 &= -\frac{11}{3}. \end{aligned}$$

Solving these equations yields $c_1=-1/2$, $c_2=19/6$. Therefore the solution of Equation 5.6.12 is

$$y=\frac{x^2}{3}-1-\frac{x}{2}+\frac{19}{6x}.$$

Using reduction of order to find the general solution of a homogeneous linear second order equation leads to a homogeneous linear first order equation in $u'$ that can be solved by separation of variables. The next example illustrates this.

**Example 5.6.3**

Find the general solution and a fundamental set of solutions of

$$x^2y'' - 3xy' + 3y = 0, \tag{5.6.18}$$

given that $y_1 = x$ is a solution.

**Solution**

If $y = ux$ then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(u''x + 2u') - 3x(u'x + u) + 3ux \\ &= x^3u'' - x^2u'. \end{aligned}$$

Therefore $y = ux$ is a solution of Equation 5.6.18 if and only if

$$x^3u'' - x^2u' = 0.$$

Separating the variables $u'$ and $x$ yields

$$\frac{u''}{u'} = \frac{1}{x},$$

so

$$\ln|u'| = \ln|x| + k, \quad \text{or equivalently} \quad u' = C_1x.$$

Therefore

$$u = \frac{C_1}{2}x^2 + C_2,$$

so the general solution of Equation 5.6.18 is

$$y = ux = \frac{C_1}{2}x^3 + C_2x,$$

which we rewrite as

$$y = c_1x + c_2x^3.$$

Therefore $\{x, x^3\}$ is a fundamental set of solutions of Equation 5.6.18.

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# 5.6E: Reduction of Order (Exercises)

## Q5.6.1

In **Exercises 5.6.1-5.6.17** find the general solution, given that $y_1$ satisfies the complementary equation. As a byproduct, find a fundamental set of solutions of the complementary equation.

1. $(2x+1)y''-2y'-(2x+3)y=(2x+1)^2;\quad y_1=e^{-x}$
2. $x^2y''+xy'-y=\frac{4}{x^2};\quad y_1=x$
3. $x^2y''-xy'+y=x;\quad y_1=x$
4. $y''-3y'+2y=\frac{1}{1+e^{-x}};\quad y_1=e^{2x}$
5. $y''-2y'+y=7x^{3/2}e^x;\quad y_1=e^x$
6. $4x^2y''+(4x-8x^2)y'+(4x^2-4x-1)y=4x^{1/2}e^x(1+4x);\quad y_1=x^{1/2}e^x$
7. $y''-2y'+2y=e^x\sec x;\quad y_1=e^x\cos x$
8. $y''+4xy'+(4x^2+2)y=8e^{-x(x+2)};\quad y_1=e^{-x^2}$
9. $x^2y''+xy'-4y=-6x-4;\quad y_1=x^2$
10. $x^2y''+2x(x-1)y'+(x^2-2x+2)y=x^3e^{2x};\quad y_1=xe^{-x}$
11. $x^2y''-x(2x-1)y'+(x^2-x-1)y=x^2e^x;\quad y_1=xe^x$
12. $(1-2x)y''+2y'+(2x-3)y=(1-4x+4x^2)e^x;\quad y_1=e^x$
13. $x^2y''-3xy'+4y=4x^4;\quad y_1=x^2$
14. $2xy''+(4x+1)y'+(2x+1)y=3x^{1/2}e^{-x};\quad y_1=e^{-x}$
15. $xy''-(2x+1)y'+(x+1)y=-e^x;\quad y_1=e^x$
16. $4x^2y''-4x(x+1)y'+(2x+3)y=4x^{5/2}e^{2x};\quad y_1=x^{1/2}$
17. $x^2y''-5xy'+8y=4x^2;\quad y_1=x^2$

## Q5.6.2

In **Exercises 5.6.18-5.6.30** find a fundamental set of solutions, given that $y_1$ is a solution.

18. $xy''+(2-2x)y'+(x-2)y=0;\quad y_1=e^x$
19. $x^2y''-4xy'+6y=0;\quad y_1=x^2$
20. $x^2(\ln|x|)^2y''-(2x\ln|x|)y'+(2+\ln|x|)y=0;\quad y_1=\ln|x|$
21. $4xy''+2y'+y=0;\quad y_1=\sin\sqrt{x}$
22. $xy''-(2x+2)y'+(x+2)y=0;\quad y_1=e^x$
23. $x^2y''-(2a-1)xy'+a^2y=0;\quad y_1=x^a$
24. $x^2y''-2xy'+(x^2+2)y=0;\quad y_1=x\sin x$
25. $xy''-(4x+1)y'+(4x+2)y=0;\quad y_1=e^{2x}$
26. $4x^2(\sin x)y''-4x(x\cos x+\sin x)y'+(2x\cos x+3\sin x)y=0;\quad y_1=x^{1/2}$
27. $4x^2y''-4xy'+(3-16x^2)y=0;\quad y_1=x^{1/2}e^{2x}$
28. $(2x+1)xy''-2(2x^2-1)y'-4(x+1)y=0;\quad y_1=1/x$
29. $(x^2-2x)y''+(2-x^2)y'+(2x-2)y=0;\quad y_1=e^x$
30. $xy''-(4x+1)y'+(4x+2)y=0;\quad y_1=e^{2x}$

## Q5.6.3

In **Exercises 5.6.31-5.6.33** solve the initial value problem, given that $y_1$ satisfies the complementary equation.

31. $x^2y''-3xy'+4y=4x^4, \quad y(-1)=7, \quad y'(-1)=-8; \quad y_1=x^2$

32. $(3x-1)y''-(3x+2)y'-(6x-8)y=0, \quad y(0)=2,\ y'(0)=3; \quad y_1=e^{2x}$

33. $(x+1)^2y''-2(x+1)y'-(x^2+2x-1)y=(x+1)^3e^x, \quad y(0)=1, \quad y'(0)=-1; \quad y_1=(x+1)e^x$

## Q5.6.4

In **Exercises 5.6.34** and **5.6.35** solve the initial value problem and graph the solution, given that $y_1$ satisfies the complementary equation.

34. $x^2y''+2xy'-2y=x^2, \quad y(1)=\frac{5}{4},\ y'(1)=\frac{3}{2}; \quad y_1=x$

35. $(x^2-4)y''+4xy'+2y=x+2, \quad y(0)=-\frac{1}{3}, \quad y'(0)=-1; \quad y_1=\frac{1}{x-2}$

## Q5.6.5

36. Suppose $p_1$ and $p_2$ are continuous on $(a,b)$. Let $y_1$ be a solution of

$$y''+p_1(x)y'+p_2(x)y=0 \tag{A}$$

that has no zeros on $(a,b)$, and let $x_0$ be in $(a,b)$. Use reduction of order to show that $y_1$ and

$$y_2(x)=y_1(x)\int_{x_0}^{x}\frac{1}{y_1^2(t)}\exp\left(-\int_{x_0}^{t}p_1(s)\,ds\right)dt$$

form a fundamental set of solutions of (A) on $(a,b)$.

37. The nonlinear first order equation

$$y'+y^2+p(x)y+q(x)=0 \tag{A}$$

is a **Riccati equation**. (See **Exercise 2.4.55**.) Assume that $p$ and $q$ are continuous.

a. Show that $y$ is a solution of (A) if and only if $y=z'/z$, where

$$z''+p(x)z'+q(x)z=0. \tag{B}$$

b. Show that the general solution of (A) is

$$y=\frac{c_1z_1'+c_2z_2'}{c_1z_1+c_2z_2}, \tag{C}$$

where $\{z_1,z_2\}$ is a fundamental set of solutions of (B) and $c_1$ and $c_2$ are arbitrary constants.

c. Does the formula (C) imply that the first order equation (A) has a two–parameter family of solutions? Explain your answer.

38. Use a method suggested by *Exercise 5.6.37* to find all solutions. of the equation.

a. $y'+y^2+k^2=0$
b. $y'+y^2-3y+2=0$
c. $y'+y^2+5y-6=0$
d. $y'+y^2+8y+7=0$
e. $y'+y^2+14y+50=0$
f. $6y'+6y^2-y-1=0$
g. $36y'+36y^2-12y+1=0$

39. Use a method suggested by **Exercise 5.6.37** and reduction of order to find all solutions of the equation, given that $y_1$ is a solution.

a. $x^2(y'+y^2)-x(x+2)y+x+2=0; \quad y_1=1/x$
b. $y'+y^2+4xy+4x^2+2=0; \quad y_1=-2x$
c. $(2x+1)(y'+y^2)-2y-(2x+3)=0; \quad y_1=-1$

d. $(3x-1)(y'+y^2)-(3x+2)y-6x+8=0; \quad y_1=2$
e. $x^2(y'+y^2)+xy+x^2-\frac{1}{4}=0; \quad y_1=-\tan x-\frac{1}{2x}$
f. $x^2(y'+y^2)-7xy+7=0; \quad y_1=1/x$

40. The nonlinear first order equation

$$y'+r(x)y^2+p(x)y+q(x)=0 \tag{A}$$

is the **generalized Riccati equation**. (See **Exercise 2.4.55**.) Assume that $p$ and $q$ are continuous and $r$ is differentiable.

a. Show that $y$ is a solution of (A) if and only if $y=z'/rz$, where

$$z''+\left[p(x)-\frac{r'(x)}{r(x)}\right]z'+r(x)q(x)z=0. \tag{B}$$

b. Show that the general solution of (A) is

$$y=\frac{c_1z_1'+c_2z_2'}{r(c_1z_1+c_2z_2)},$$

where $\{z_1,z_2\}$ is a fundamental set of solutions of (B) and $c_1$ and $c_2$ are arbitrary constants.

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# 5.7: Variation of Parameters

In this section we give a method called **variation of parameters** for finding a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{5.7.1}$$

if we know a fundamental set $\{y_1, y_2\}$ of solutions of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{5.7.2}$$

Having found a particular solution $y_p$ by this method, we can write the general solution of Equation 5.7.1 as

$$y = y_p + c_1y_1 + c_2y_2.$$

Since we need only one nontrivial solution of Equation 5.7.2 to find the general solution of Equation 5.7.1 by reduction of order, it is natural to ask why we are interested in variation of parameters, which requires two linearly independent solutions of Equation 5.7.2 to achieve the same goal. Here's the answer:

- If we already know two linearly independent solutions of Equation 5.7.2 then variation of parameters will probably be simpler than reduction of order.
- Variation of parameters generalizes naturally to a method for finding particular solutions of higher order linear equations (Section 9.4) and linear systems of equations (Section 10.7), while reduction of order doesn't.
- Variation of parameters is a powerful theoretical tool used by researchers in differential equations. Although a detailed discussion of this is beyond the scope of this book, you can get an idea of what it means from **Exercises 5.7.37–5.7.39**.

We'll now derive the method. As usual, we consider solutions of Equation 5.7.1 and Equation 5.7.2 on an interval $(a, b)$ where $P_0$, $P_1$, $P_2$, and $F$ are continuous and $P_0$ has no zeros. Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation Equation 5.7.2. We look for a particular solution of Equation 5.7.1 in the form

$$y_p = u_1y_1 + u_2y_2 \tag{5.7.3}$$

where $u_1$ and $u_2$ are functions to be determined so that $y_p$ satisfies Equation 5.7.1. You may not think this is a good idea, since there are now two unknown functions to be determined, rather than one. However, since $u_1$ and $u_2$ have to satisfy only one condition (that $y_p$ is a solution of Equation 5.7.1), we can impose a second condition that produces a convenient simplification, as follows.

Differentiating Equation 5.7.3 yields

$$y_p' = u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2. \tag{5.7.4}$$

As our second condition on $u_1$ and $u_2$ we require that

$$u_1'y_1 + u_2'y_2 = 0. \tag{5.7.5}$$

Then Equation 5.7.4 becomes

$$y_p' = u_1y_1' + u_2y_2'; \tag{5.7.6}$$

that is, Equation 5.7.5 permits us to differentiate $y_p$ (once!) as if $u_1$ and $u_2$ are constants. Differentiating Equation 5.7.4 yields

$$y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'. \tag{5.7.7}$$

(There are no terms involving $u_1''$ and $u_2''$ here, as there would be if we hadn't required Equation 5.7.5.) Substituting Equation 5.7.3, Equation 5.7.6, and Equation 5.7.7 into Equation 5.7.1 and collecting the coefficients of $u_1$ and $u_2$ yields

$$u_1(P_0y_1'' + P_1y_1' + P_2y_1) + u_2(P_0y_2'' + P_1y_2' + P_2y_2) + P_0(u_1'y_1' + u_2'y_2') = F.$$

As in the derivation of the method of reduction of order, the coefficients of $u_1$ and $u_2$ here are both zero because $y_1$ and $y_2$ satisfy the complementary equation. Hence, we can rewrite the last equation as

$$P_0(u_1'y_1' + u_2'y_2') = F. \tag{5.7.8}$$

Therefore $y_p$ in Equation 5.7.3 satisfies Equation 5.7.1 if

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \tfrac{F}{P_0}, \end{aligned} \tag{5.7.9}$$

where the first equation is the same as Equation 5.7.5 and the second is from Equation 5.7.8.

We'll now show that you can always solve Equation 5.7.9 for $u_1'$ and $u_2'$. (The method that we use here will always work, but simpler methods usually work when you're dealing with specific equations.) To obtain $u_1'$, multiply the first equation in Equation 5.7.9 by $y_2'$ and the second equation by $y_2$. This yields

$$\begin{aligned} u_1'y_1y_2' + u_2'y_2y_2' &= 0 \\ u_1'y_1'y_2 + u_2'y_2'y_2 &= \frac{Fy_2}{P_0}. \end{aligned}$$

Subtracting the second equation from the first yields

$$u_1'(y_1y_2' - y_1'y_2) = -\frac{Fy_2}{P_0}. \tag{5.7.10}$$

Since $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 5.7.2 on $(a, b)$, Theorem 5.1.6 implies that the Wronskian $y_1y_2' - y_1'y_2$ has no zeros on $(a, b)$. Therefore we can solve Equation 5.7.10 for $u_1'$, to obtain

$$u_1' = -\frac{Fy_2}{P_0(y_1y_2' - y_1'y_2)}. \tag{5.7.11}$$

We leave it to you to start from Equation 5.7.9 and show by a similar argument that

$$u_2' = \frac{Fy_1}{P_0(y_1y_2' - y_1'y_2)}. \tag{5.7.12}$$

We can now obtain $u_1$ and $u_2$ by integrating $u_1'$ and $u_2'$. The constants of integration can be taken to be zero, since any choice of $u_1$ and $u_2$ in Equation 5.7.3 will suffice.

You should not memorize Equation 5.7.11 and Equation 5.7.12. On the other hand, you don't want to rederive the whole procedure for every specific problem. We recommend the a compromise:

a. Write

$$y_p = u_1y_1 + u_2y_2 \tag{5.7.13}$$

to remind yourself of what you're doing.

b. Write the system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \tfrac{F}{P_0} \end{aligned} \tag{5.7.14}$$

for the specific problem you're trying to solve.

c. Solve Equation 5.7.14 for $u_1'$ and $u_2'$ by any convenient method.

d. Obtain $u_1$ and $u_2$ by integrating $u_1'$ and $u_2'$, taking the constants of integration to be zero.

e. Substitute $u_1$ and $u_2$ into Equation 5.7.13 to obtain $y_p$.

**Example 5.7.1**

Find a particular solution $y_p$ of

$$x^2y'' - 2xy' + 2y = x^{9/2}, \tag{5.7.15}$$

given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation

$$x^2y'' - 2xy' + 2y = 0.$$

Then find the general solution of Equation 5.7.15.

**Solution**

We set

$$y_p = u_1 x + u_2 x^2,$$

where

$$\begin{aligned} u_1' x + \ u_2' x^2 &= 0 \\ u_1' \ + 2u_2' x \ &= \frac{x^{9/2}}{x^2} = x^{5/2}. \end{aligned}$$

From the first equation, $u_1' = -u_2' x$. Substituting this into the second equation yields $u_2' x = x^{5/2}$, so $u_2' = x^{3/2}$ and therefore $u_1' = -u_2' x = -x^{5/2}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -\frac{2}{7} x^{7/2} \quad \text{and} \quad u_2 = \frac{2}{5} x^{5/2}.$$

Therefore

$$y_p = u_1 x + u_2 x^2 = -\frac{2}{7} x^{7/2} x + \frac{2}{5} x^{5/2} x^2 = \frac{4}{35} x^{9/2},$$

and the general solution of Equation 5.7.15 is

$$y = \frac{4}{35} x^{9/2} + c_1 x + c_2 x^2.$$

**Example 5.7.2**

Find a particular solution $y_p$ of

$$(x-1)y'' - xy' + y = (x-1)^2, \tag{5.7.16}$$

given that $y_1 = x$ and $y_2 = e^x$ are solutions of the complementary equation

$$(x-1)y'' - xy' + y = 0.$$

Then find the general solution of Equation 5.7.16.

**Solution**

We set

$$y_p = u_1 x + u_2 e^x,$$

where

$$\begin{aligned} u_1' x + u_2' e^x \ &= 0 \\ u_1' \ + u_2' e^x \ &= \frac{(x-1)^2}{x-1} = x - 1. \end{aligned}$$

Subtracting the first equation from the second yields $-u_1'(x-1) = x-1$, so $u_1' = -1$. From this and the first equation, $u_2' = -xe^{-x} u_1' = xe^{-x}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -x \quad \text{and} \quad u_2 = -(x+1)e^{-x}.$$

Therefore

$$y_p = u_1 x + u_2 e^x = (-x)x + (-(x+1)e^{-x})e^x = -x^2 - x - 1,$$

so the general solution of Equation 5.7.16 is

$$y = y_p + c_1 x + c_2 e^x = -x^2 - x - 1 + c_1 x + c_2 e^x = -x^2 - 1 + (c_1 - 1)x + c_2 e^x. \tag{5.7.17}$$

However, since $c_1$ is an arbitrary constant, so is $c_1 - 1$; therefore, we improve the appearance of this result by renaming the constant and writing the general solution as

$$y = -x^2 - 1 + c_1 x + c_2 e^x. \tag{5.7.18}$$

There's nothing **wrong** with leaving the general solution of Equation 5.7.16 in the form Equation 5.7.17; however, we think you'll agree that Equation 5.7.18 is preferable. We can also view the transition from Equation 5.7.17 to Equation 5.7.18 differently. In this example the particular solution $y_p = -x^2 - x - 1$ contained the term $-x$, which satisfies the complementary equation. We can drop this term and redefine $y_p = -x^2 - 1$, since $-x^2 - x - 1$ is a solution of Equation 5.7.16 and $x$ is a solution of the complementary equation; hence, $-x^2 - 1 = (-x^2 - x - 1) + x$ is also a solution of Equation 5.7.16. In general, it is always legitimate to drop linear combinations of $\{y_1, y_2\}$ from particular solutions obtained by variation of parameters. (See **Exercise 5.7.36** for a general discussion of this question.) We'll do this in the following examples and in the answers to exercises that ask for a particular solution. Therefore, don't be concerned if your answer to such an exercise differs from ours only by a solution of the complementary equation.

### Example 5.7.3

Find a particular solution of

$$y'' + 3y' + 2y = \frac{1}{1 + e^x}. \tag{5.7.19}$$

Then find the general solution.

**Solution**

The characteristic polynomial of the complementary equation

$$y'' + 3y' + 2y = 0 \tag{5.7.20}$$

is $p(r) = r^2 + 3r + 2 = (r + 1)(r + 2)$, so $y_1 = e^{-x}$ and $y_2 = e^{-2x}$ form a fundamental set of solutions of Equation 5.7.20. We look for a particular solution of Equation 5.7.19 in the form

$$y_p = u_1 e^{-x} + u_2 e^{-2x},$$

where

$$\begin{aligned} u_1' e^{-x} + u_2' e^{-2x} &= 0 \\ -u_1' e^{-x} - 2u_2' e^{-2x} &= \frac{1}{1 + e^x}. \end{aligned}$$

Adding these two equations yields

$$-u_2' e^{-2x} = \frac{1}{1 + e^x}, \quad \text{so} \quad u_2' = -\frac{e^{2x}}{1 + e^x}.$$

From the first equation,

$$u_1' = -u_2' e^{-x} = \frac{e^x}{1 + e^x}.$$

Integrating by means of the substitution $v = e^x$ and taking the constants of integration to be zero yields

$$u_1 = \int \frac{e^x}{1 + e^x}\, dx = \int \frac{dv}{1 + v} = \ln(1 + v) = \ln(1 + e^x)$$

and

$$\begin{aligned} u_2 &= -\int \frac{e^{2x}}{1 + e^x}\, dx = -\int \frac{v}{1 + v}\, dv = \int \left[\frac{1}{1 + v} - 1\right] dv \\ &= \ln(1 + v) - v = \ln(1 + e^x) - e^x. \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= u_1 e^{-x} + u_2 e^{-2x} \\ &= [\ln(1+e^x)]e^{-x} + [\ln(1+e^x) - e^x]\, e^{-2x}, \end{aligned}$$

so

$$y_p = \left(e^{-x} + e^{-2x}\right) \ln(1+e^x) - e^{-x}.$$

Since the last term on the right satisfies the complementary equation, we drop it and redefine

$$y_p = \left(e^{-x} + e^{-2x}\right) \ln(1+e^x).$$

The general solution of Equation 5.7.19 is

$$y = y_p + c_1 e^{-x} + c_2 e^{-2x} = \left(e^{-x} + e^{-2x}\right) \ln(1+e^x) + c_1 e^{-x} + c_2 e^{-2x}.$$

## Example 5.7.4

Solve the initial value problem

$$(x^2-1)y'' + 4xy' + 2y = \frac{2}{x+1}, \quad y(0) = -1, \quad y'(0) = -5, \tag{5.7.21}$$

given that

$$y_1 = \frac{1}{x-1} \quad \text{and} \quad y_2 = \frac{1}{x+1}$$

are solutions of the complementary equation

$$(x^2-1)y'' + 4xy' + 2y = 0.$$

### Solution

We first use variation of parameters to find a particular solution of

$$(x^2-1)y'' + 4xy' + 2y = \frac{2}{x+1}$$

on $(-1, 1)$ in the form

$$y_p = \frac{u_1}{x-1} + \frac{u_2}{x+1},$$

where

$$\frac{u_1'}{x-1} + \frac{u_2'}{x+1} = 0 \tag{5.7.22}$$

$$-\frac{u_1'}{(x-1)^2} - \frac{u_2'}{(x+1)^2} = \frac{2}{(x+1)(x^2-1)}$$

Multiplying the first equation by $1/(x-1)$ and adding the result to the second equation yields

$$\left[\frac{1}{x^2-1} - \frac{1}{(x+1)^2}\right] u_2' = \frac{2}{(x+1)(x^2-1)}. \tag{5.7.23}$$

Since

$$\left[\frac{1}{x^2-1} - \frac{1}{(x+1)^2}\right] = \frac{(x+1)-(x-1)}{(x+1)(x^2-1)} = \frac{2}{(x+1)(x^2-1)},$$

Equation 5.7.23 implies that $u_2' = 1$. From Equation 5.7.22,

$$u_1' = -\frac{x-1}{x+1} u_2' = -\frac{x-1}{x+1}.$$

Integrating and taking the constants of integration to be zero yields

$$\begin{aligned} u_1 &= -\int \frac{x-1}{x+1}\, dx = -\int \frac{x+1-2}{x+1}\, dx \\ &= \int \left[\frac{2}{x+1} - 1\right] dx = 2\ln(x+1) - x \end{aligned}$$

and

$$u_2 = \int dx = x.$$

Therefore

$$\begin{aligned} y_p &= \frac{u_1}{x-1} + \frac{u_2}{x+1} = [2\ln(x+1) - x]\frac{1}{x-1} + x\frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + x\left[\frac{1}{x+1} - \frac{1}{x-1}\right] = \frac{2\ln(x+1)}{x-1} - \frac{2x}{(x+1)(x-1)}. \end{aligned}$$

However, since

$$\frac{2x}{(x+1)(x-1)} = \left[\frac{1}{x+1} + \frac{1}{x-1}\right]$$

is a solution of the complementary equation, we redefine

$$y_p = \frac{2\ln(x+1)}{x-1}.$$

Therefore the general solution of Equation 5.7.24 is

$$y = \frac{2\ln(x+1)}{x-1} + \frac{c_1}{x-1} + \frac{c_2}{x+1}. \tag{5.7.24}$$

Differentiating this yields

$$y' = \frac{2}{x^2-1} - \frac{2\ln(x+1)}{(x-1)^2} - \frac{c_1}{(x-1)^2} - \frac{c_2}{(x+1)^2}.$$

Setting $x = 0$ in the last two equations and imposing the initial conditions $y(0) = -1$ and $y'(0) = -5$ yields the system

$$\begin{aligned} -c_1 + c_2 &= -1 \\ -2 - c_1 - c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = 2,\ c_2 = 1$ . Substituting these into Equation 5.7.24 yields

$$\begin{aligned} y &= \frac{2\ln(x+1)}{x-1} + \frac{2}{x-1} + \frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1} \end{aligned}$$

as the solution of Equation 5.7.21. Figure 5.7.1 is a graph of the solution.

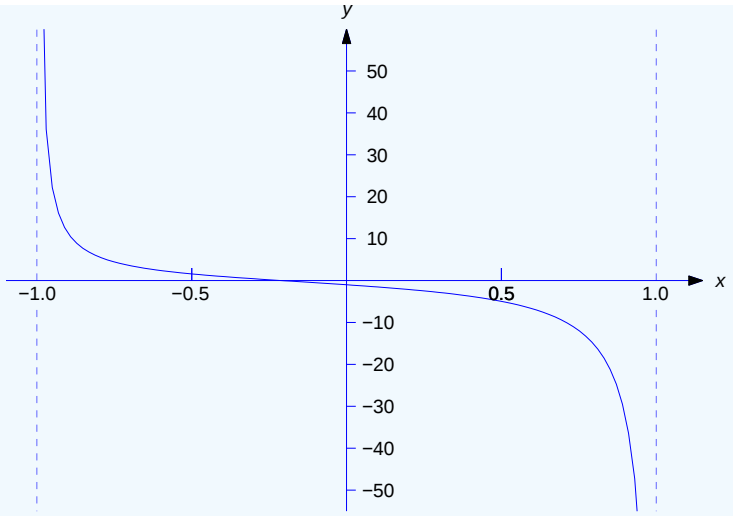



Figure 5.7.1 : $y = \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1}$

We've now considered three methods for solving nonhomogeneous linear equations: undetermined coefficients, reduction of order, and variation of parameters. It's natural to ask which method is best for a given problem. The method of undetermined coefficients should be used for constant coefficient equations with forcing functions that are linear combinations of polynomials multiplied by functions of the form $e^{\alpha x}$, $e^{\lambda x}\cos\omega x$, or $e^{\lambda x}\sin\omega x$. Although the other two methods can be used to solve such problems, they will be more difficult except in the most trivial cases, because of the integrations involved.

If the equation isn't a constant coefficient equation or the forcing function isn't of the form just specified, the method of undetermined coefficients does not apply and the choice is necessarily between the other two methods. The case could be made that reduction of order is better because it requires only one solution of the complementary equation while variation of parameters requires two. However, variation of parameters will probably be easier if you already know a fundamental set of solutions of the complementary equation.

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# 5.7E: Variation of Parameters (Exercises)

## Q5.7.1

In **Exercises 5.7.1–5.7.6** use variation of parameters to find a particular solution.

1. $y''+9y=\tan 3x$
2. $y''+4y=\sin 2x\sec^2 2x$
3. $y''-3y'+2y=\frac{4}{1+e^{-x}}$
4. $y''-2y'+2y=3e^x\sec x$
5. $y''-2y'+y=14x^{3/2}e^x$
6. $y''-y=\frac{4e^{-x}}{1-e^{-2x}}$

## Q5.7.2

In **Exercises 5.7.7-5.7.29** use variation of parameters to find a particular solution, given the solutions $y_1, y_2$ of the complementary equation.

7. $x^2y''+xy'-y=2x^2+2;\quad y_1=x,\quad y_2=\frac{1}{x}$
8. $xy''+(2-2x)y'+(x-2)y=e^{2x};\quad y_1=e^x,\quad y_2=\frac{e^x}{x}$
9. $4x^2y''+(4x-8x^2)y'+(4x^2-4x-1)y=4x^{1/2}e^x,\quad x>0\ ;y_1=x^{1/2}e^x,\ y_2=x^{-1/2}e^x$
10. $y''+4xy'+(4x^2+2)y=4e^{-x(x+2)};\quad y_1=e^{-x^2},\quad y_2=xe^{-x^2}$
11. $x^2y''-4xy'+6y=x^{5/2},\ x>0;\quad y_1=x^2,\ y_2=x^3$
12. $x^2y''-3xy'+3y=2x^4\sin x;\quad y_1=x,\ y_2=x^3$
13. $(2x+1)y''-2y'-(2x+3)y=(2x+1)^2e^{-x};\quad y_1=e^{-x},\quad y_2=xe^x$
14. $4xy''+2y'+y=\sin\sqrt{x};\quad y_1=\cos\sqrt{x},\quad y_2=\sin\sqrt{x}$
15. $xy''-(2x+2)y'+(x+2)y=6x^3e^x;\quad y_1=e^x,\quad y_2=x^3e^x$
16. $x^2y''-(2a-1)xy'+a^2y=x^{a+1};\quad y_1=x^a,\quad y_2=x^a\ln x$
17. $x^2y''-2xy'+(x^2+2)y=x^3\cos x;\quad y_1=x\cos x,\quad y_2=x\sin x$
18. $xy''-y'-4x^3y=8x^5;\quad y_1=e^{x^2},\ y_2=e^{-x^2}$
19. $(\sin x)y''+(2\sin x-\cos x)y'+(\sin x-\cos x)y=e^{-x};\quad y_1=e^{-x},\quad y_2=e^{-x}\cos x$
20. $4x^2y''-4xy'+(3-16x^2)y=8x^{5/2};\quad y_1=\sqrt{x}e^{2x},\ y_2=\sqrt{x}e^{-2x}$
21. $4x^2y''-4xy'+(4x^2+3)y=x^{7/2};\quad y_1=\sqrt{x}\sin x,\ y_2=\sqrt{x}\cos x$
22. $x^2y''-2xy'-(x^2-2)y=3x^4;\quad y_1=xe^x,\ y_2=xe^{-x}$
23. $x^2y''-2x(x+1)y'+(x^2+2x+2)y=x^3e^x;\quad y_1=xe^x,\quad y_2=x^2e^x$
24. $x^2y''-xy'-3y=x^{3/2};\quad y_1=1/x,\quad y_2=x^3$
25. $x^2y''-x(x+4)y'+2(x+3)y=x^4e^x;\quad y_1=x^2,\quad y_2=x^2e^x$
26. $x^2y''-2x(x+2)y'+(x^2+4x+6)y=2xe^x;\quad y_1=x^2e^x,\quad y_2=x^3e^x$
27. $x^2y''-4xy'+(x^2+6)y=x^4;\quad y_1=x^2\cos x,\quad y_2=x^2\sin x$
28. $(x-1)y''-xy'+y=2(x-1)^2e^x;\quad y_1=x,\quad y_2=e^x$
29. $4x^2y''-4x(x+1)y'+(2x+3)y=x^{5/2}e^x;\quad y_1=\sqrt{x},\quad y_2=\sqrt{x}e^x$

## Q5.7.3

In **Exercises 5.7.30–5.7.32** use variation of parameters to solve the initial value problem, given $y_1$, $y_2$ are solutions of the complementary equation.

30. $(3x-1)y''-(3x+2)y'-(6x-8)y=(3x-1)^2e^{2x}, \quad y(0)=1,\ y'(0)=2 \ ; y_1=e^{2x},\ y_2=xe^{-x}$

31. $(x-1)^2y''-2(x-1)y'+2y=(x-1)^2, \quad y(0)=3, \quad y'(0)=-6 \ ;$

$y_1=x-1\,, y_2=x^2-1$

32. $(x-1)^2y''-(x^2-1)y'+(x+1)y=(x-1)^3e^x, \quad y(0)=4, \quad y'(0)=-6 \ ;$

$y_1=(x-1)e^x, \quad y_2=x-1$

## Q5.7.4

In **Exercises 5.7.33-5.7.35** use variation of parameters to solve the initial value problem and graph the solution, given that $y_1$, $y_2$ are solutions of the complementary equation.

33. $(x^2-1)y''+4xy'+2y=2x, \quad y(0)=0,\ y'(0)=-2; \quad y_1=\frac{1}{x-1},\ y_2=\frac{1}{x+1}$

34. $x^2y''+2xy'-2y=-2x^2, \quad y(1)=1,\ y'(1)=-1; \quad y_1=x,\ y_2=\frac{1}{x^2}$

35. $(x+1)(2x+3)y''+2(x+2)y'-2y=(2x+3)^2, \quad y(0)=0, \quad y'(0)=0 \ ; y_1=x+2, \quad y_2=\frac{1}{x+1}$

## Q5.7.5

36. Suppose

$$y_p=\overline{y}+a_1y_1+a_2y_2$$

is a particular solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F(x), \tag{A}$$

where $y_1$ and $y_2$ are solutions of the complementary equation

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0.$$

Show that $\overline{y}$ is also a solution of (A).

37. Suppose $p$, $q$, and $f$ are continuous on $(a,b)$ and let $x_0$ be in $(a,b)$. Let $y_1$ and $y_2$ be the solutions of

$$y''+p(x)y'+q(x)y=0$$

such that

$$y_1(x_0)=1, \quad y_1'(x_0)=0, \quad y_2(x_0)=0, \quad y_2'(x_0)=1.$$

Use variation of parameters to show that the solution of the initial value problem

$$y''+p(x)y'+q(x)y=f(x), \quad y(x_0)=k_0,\ y'(x_0)=k_1,$$

is

$$\begin{aligned} y(x) &= k_0y_1(x)+k_1y_2(x) \\ &\quad +\int_{x_0}^{x}\left(y_1(t)y_2(x)-y_1(x)y_2(t)\right)f(t)\exp\left(\int_{x_0}^{t}p(s)\,ds\right)dt. \end{aligned}$$

HINT: **Use Abel's formula for the Wronskian of $\{y_1,y_2\}$, and integrate $u_1'$ and $u_2'$ from $x_0$ to $x$.**

Show also that

$$\begin{aligned} y'(x) &= k_0y_1'(x)+k_1y_2'(x) \\ &\quad +\int_{x_0}^{x}\left(y_1(t)y_2'(x)-y_1'(x)y_2(t)\right)f(t)\exp\left(\int_{x_0}^{t}p(s)\,ds\right)dt. \end{aligned}$$

38. Suppose $f$ is continuous on an open interval that contains $x_0 = 0$. Use variation of parameters to find a formula for the solution of the initial value problem

$$y'' - y = f(x), \quad y(0) = k_0, \quad y'(0) = k_1.$$

39. Suppose $f$ is continuous on $(a, \infty)$, where $a < 0$, so $x_0 = 0$ is in $(a, \infty)$.

a. Use variation of parameters to find a formula for the solution of the initial value problem

$$y'' + y = f(x), \quad y(0) = k_0, \quad y'(0) = k_1.$$

HINT: **You will need the addition formulas for the sine and cosine.**

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B \end{aligned}$$

For the rest of this exercise assume that the improper integral $\int_0^\infty f(t)\,dt$ is absolutely convergent.

b. Show that if $y$ is a solution of

$$y'' + y = f(x) \tag{A}$$

on $(a, \infty)$, then

$$\lim_{x\to\infty} (y(x) - A_0 \cos x - A_1 \sin x) = 0 \tag{B}$$

and

$$\lim_{x\to\infty} (y'(x) + A_0 \sin x - A_1 \cos x) = 0, \tag{C}$$

where

$$A_0 = k_0 - \int_0^\infty f(t) \sin t\, dt \quad \text{and} \quad A_1 = k_1 + \int_0^\infty f(t) \cos t\, dt.$$

HINT: Recall from calculus that if $\int_0^\infty f(t)dt$ *converges absolutely, then* $\lim_{x\to\infty} \int_x^\infty |f(t)|dt = 0$.

c. Show that if $A_0$ and $A_1$ are arbitrary constants, then there's a unique solution of $y'' + y = f(x)$ on $(a, \infty)$ that satisfies (B) and (C).

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# CHAPTER OVERVIEW

## 6: Applications of Linear Second Order Equations

In this Chapter, we study applications of linear second order equations.


6.1: Spring Problems I

6.1E: Spring Problems I (Exercises)

6.2: Spring Problems II

6.2E: Spring Problems II (Exercises)


*Thumbnail: A series RLC network: a resistor, an inductor, and a capacitor. Image used with permission (CC BY-SA 3.0; Spinningspark at Wikipedia)*

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# 6.1: Spring Problems I

We consider the motion of an object of mass $m$, suspended from a spring of negligible mass. We say that the spring–mass system is in **equilibrium** when the object is at rest and the forces acting on it sum to zero. The position of the object in this case is the **equilibrium position**. We define $y$ to be the displacement of the object from its equilibrium position (Figure 6.1.1 ), measured positive upward.

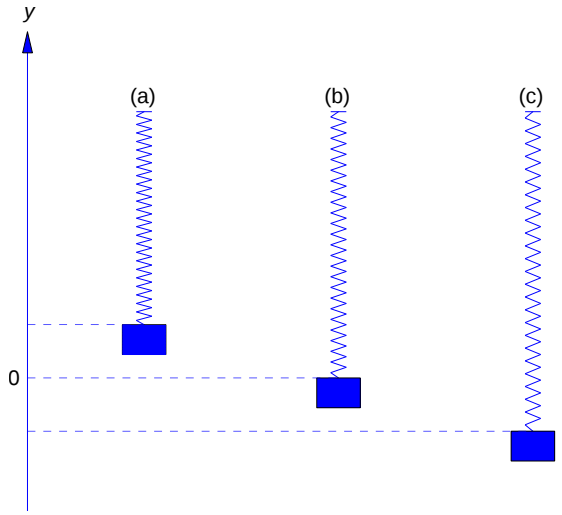



Figure 6.1.1 : (a) $y > 0$ (b) $y = 0$, (c) $y < 0$

Our model accounts for the following kinds of forces acting on the object:

- The force $-mg$, due to gravity.
- A force $F_s$ exerted by the spring resisting change in its length. The **natural length** of the spring is its length with no mass attached. We assume that the spring obeys **Hooke's law**: If the length of the spring is changed by an amount $\Delta L$ from its natural length, then the spring exerts a force $F_s = k\Delta L$, where $k$ is a positive number called the **spring constant**. If the spring is stretched then $\Delta L > 0$ and $F_s > 0$, so the spring force is upward, while if the spring is compressed then $\Delta L < 0$ and $F_s < 0$, so the spring force is downward.
- A **damping force** $F_d = -cy'$ that resists the motion with a force proportional to the velocity of the object. It may be due to air resistance or friction in the spring. However, a convenient way to visualize a damping force is to assume that the object is rigidly attached to a piston with negligible mass immersed in a cylinder (called a **dashpot**) filled with a viscous liquid (Figure 6.1.1 ). As the piston moves, the liquid exerts a damping force. We say that the motion is **undamped** if $c = 0$, or **damped** if $c > 0$.
- An external force $F$, other than the force due to gravity, that may vary with $t$, but is independent of displacement and velocity. We say that the motion is **free** if $F \equiv 0$, or **forced** if $F \not\equiv 0$.

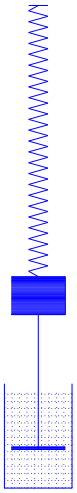


Figure 6.1.2 : A spring – mass system with damping

From Newton's second law of motion,

$$my'' = -mg + F_d + F_s + F = -mg - cy' + F_s + F. \tag{6.1.1}$$

We must now relate $F_s$ to $y$. In the absence of external forces the object stretches the spring by an amount $\Delta l$ to assume its equilibrium position (Figure 6.1.3 ). Since the sum of the forces acting on the object is then zero, Hooke's Law implies that $mg = k\Delta l$. If the object is displaced $y$ units from its equilibrium position, the total change in the length of the spring is $\Delta L = \Delta l - y$ , so Hooke's law implies that

$$F_s = k\Delta L = k\Delta l - ky.$$

Substituting this into Equation 6.1.1 yields

$$my'' = -mg - cy' + k\Delta L - ky + F.$$

Since $mg = k\Delta l$ this can be written as

$$my'' + cy' + ky = F. \tag{6.1.2}$$

We call this **the equation of motion**.

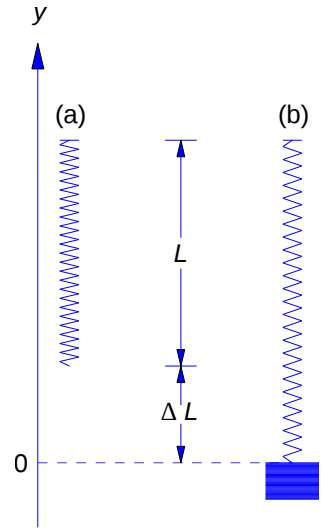



Figure 6.1.3 : (a) Natural length of spring (b) Spring stretched by mass

## Simple Harmonic Motion

Throughout the rest of this section we'll consider spring–mass systems without damping; that is, $c = 0$. We'll consider systems with damping in the next section. We first consider the case where the motion is also free; that is, $F$=0. We begin with an example.

## Example 6.1.1

An object stretches a spring 6 inches in equilibrium.

a. Set up the equation of motion and find its general solution.
b. Find the displacement of the object for $t > 0$ if it is initially displaced 18 inches above equilibrium and given a downward velocity of 3 ft/s.

**Solution a**:

Setting $c = 0$ and $F = 0$ in Equation 6.1.2 yields the equation of motion

$$my'' + ky = 0,$$

which we rewrite as

$$y'' + \frac{k}{m}y = 0. \tag{6.1.3}$$

Although we would need the weight of the object to obtain $k$ from the equation $mg = k\Delta l$ we can obtain $k/m$ from $\Delta l$ alone; thus, $k/m = g/\Delta l$. Consistent with the units used in the problem statement, we take $g = 32$ ft/s$^2$. Although $\Delta l$ is stated in inches, we must convert it to feet to be consistent with this choice of $g$; that is, $\Delta l = 1/2$ ft. Therefore

$$\frac{k}{m} = \frac{32}{1/2} = 64$$

and Equation 6.1.3 becomes

$$y'' + 64y = 0. \tag{6.1.4}$$

The characteristic equation of Equation 6.1.4 is

$$r^2 + 64 = 0,$$

which has the zeros $r = \pm 8i$. Therefore the general solution of Equation 6.1.4 is

$$y = c_1 \cos 8t + c_2 \sin 8t. \tag{6.1.5}$$

**Solution b**:

The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus,

$$y(0) = \frac{3}{2} \quad \text{and} \quad y'(0) = -3.$$

Differentiating Equation 6.1.5 yields

$$y' = -8c_1 \sin 8t + 8c_2 \cos 8t. \tag{6.1.6}$$

Setting $t = 0$ in Equation 6.1.5 and Equation 6.1.6 and imposing the initial conditions shows that $c_1 = 3/2$ and $c_2 = -3/8$. Therefore

$$y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t,$$

where $y$ is in feet (Figure 6.1.4 ).

We'll now consider the equation

$$my'' + ky = 0$$

where $m$ and $k$ are arbitrary positive numbers. Dividing through by $m$ and defining $\omega_0 = \sqrt{k/m}$ yields

$$y'' + \omega_0^2 y = 0.$$

The general solution of this equation is

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t. \tag{6.1.7}$$

We can rewrite this in a more useful form by defining

$$R = \sqrt{c_1^2 + c_2^2}, \tag{6.1.8}$$

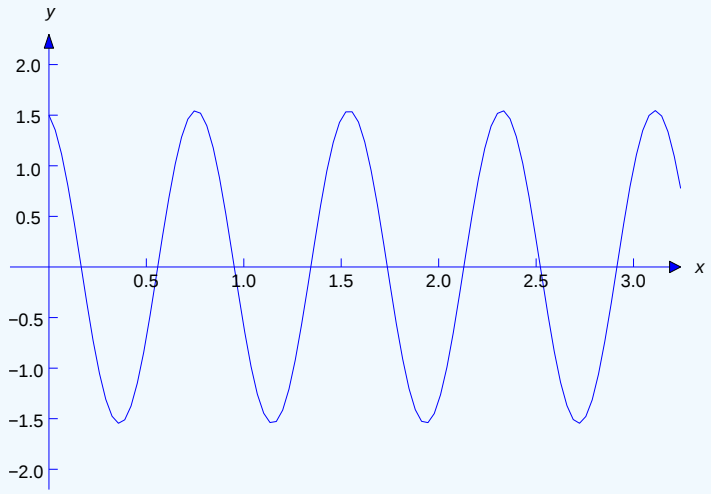



Figure 6.1.4 : $y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t$

and

$$c_1 = R\cos\phi \quad \text{and} \quad c_2 = R\sin\phi. \tag{6.1.9}$$

Substituting from Equation 6.1.9 into Equation 6.1.7 and applying the identity

$$\cos\omega_0 t\cos\phi + \sin\omega_0 t\sin\phi = \cos(\omega_0 t - \phi)$$

yields

$$y = R\cos(\omega_0 t - \phi). \tag{6.1.10}$$

From Equation 6.1.8 and Equation 6.1.9 we see that the $R$ and $\phi$ can be interpreted as polar coordinates of the point with rectangular coordinates $(c_1, c_2)$ (Figure 6.1.5 ). Given $c_1$ and $c_2$, we can compute $R$ from Equation 6.1.8. From Equation 6.1.8 and Equation 6.1.9, we see that $\phi$ is related to $c_1$ and $c_2$ by

$$\cos\phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \quad \text{and} \quad \sin\phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}.$$

There are infinitely many angles $\phi$, differing by integer multiples of $2\pi$, that satisfy these equations. We will always choose $\phi$ so that $-\pi \le \phi < \pi$.

The motion described by Equation 6.1.7 or Equation 6.1.10 is **simple harmonic motion**. We see from either of these equations that the motion is periodic, with period

$$T = 2\pi/\omega_0.$$

This is the time required for the object to complete one full cycle of oscillation (for example, to move from its highest position to its lowest position and back to its highest position). Since the highest and lowest positions of the object are $y = R$ and $y = -R$, we say that $R$ is the **amplitude** of the oscillation. The angle $\phi$ in Equation 6.1.10 is the **phase angle**. It's measured in radians. Equation Equation 6.1.10 is the **amplitude–phase form** of the displacement. If $t$ is in seconds then $\omega_0$ is in radians per second (rad/s); it is the **frequency** of the motion. It is also called the **natural frequency** of the spring–mass system without damping.

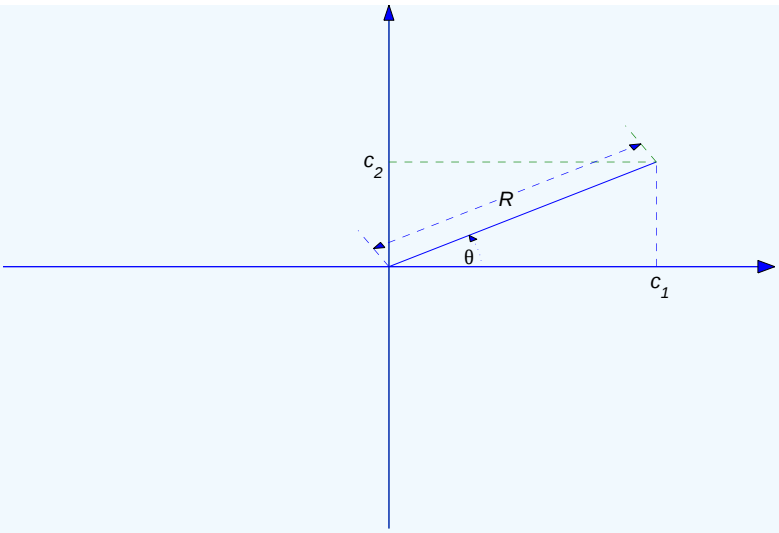



Figure 6.1.5 : $R=\sqrt{c_1^2+c_2^2}$ ; $c_1=R\cos\phi$; $c_2=R\sin\phi$

## Example 6.1.2

We found the displacement of the object in Example 6.1.1 to be

$$y=\frac{3}{2}\cos 8t-\frac{3}{8}\sin 8t.$$

Find the frequency, period, amplitude, and phase angle of the motion.

**Solution**:

The frequency is $\omega_0=8$ rad/s, and the period is $T=2\pi/\omega_0=\pi/4$ s. Since $c_1=3/2$ and $c_2=-3/8$, the amplitude is

$$R=\sqrt{c_1^2+c_2^2}=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{3}{8}\right)^2}=\frac{3}{8}\sqrt{17}.$$

The phase angle is determined by

$$\cos\phi=\frac{\frac{3}{2}}{\frac{3}{8}\sqrt{17}}=\frac{4}{\sqrt{17}} \tag{6.1.11}$$

and

$$\sin\phi=\frac{-\frac{3}{8}}{\frac{3}{8}\sqrt{17}}=-\frac{1}{\sqrt{17}}. \tag{6.1.12}$$

Using a calculator, we see from Equation 6.1.11that

$$\phi\approx\pm.245 \text{ rad}.$$

Since $\sin\phi<0$ (see Equation 6.1.12), the minus sign applies here; that is,

$$\phi\approx-.245 \text{ rad}.$$

## Example 6.1.3

The natural length of a spring is 1 m. An object is attached to it and the length of the spring increases to 102 cm when the object is in equilibrium. Then the object is initially displaced downward 1 cm and given an upward velocity of 14 cm/s. Find the displacement for $t>0$. Also, find the natural frequency, period, amplitude, and phase angle of the resulting motion. Express the answers in cgs units.

**Solution**:

In cgs units $g = 980$ cm/s$^2$. Since $\Delta l = 2$ cm, $\omega_0^2 = g/\Delta l = 490$. Therefore

$$y'' + 490y = 0, \quad y(0) = -1, \quad y'(0) = 14.$$

The general solution of the differential equation is

$$y = c_1 \cos 7\sqrt{10}t + c_2 \sin 7\sqrt{10}t,$$

so

$$y' = 7\sqrt{10}\left(-c_1 \sin 7\sqrt{10}t + c_2 \cos 7\sqrt{10}t\right).$$

Substituting the initial conditions into the last two equations yields $c_1 = -1$ and $c_2 = 2/\sqrt{10}$. Hence,

$$y = -\cos 7\sqrt{10}t + \frac{2}{\sqrt{10}} \sin 7\sqrt{10}t.$$

The frequency is $7\sqrt{10}$ rad/s, and the period is $T = 2\pi/(7\sqrt{10})$ s. The amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + \left(\frac{2}{\sqrt{10}}\right)^2} = \sqrt{\frac{7}{5}}\text{cm}$$

The phase angle is determined by

$$\cos\phi = \frac{c_1}{R} = -\sqrt{\frac{5}{7}} \quad \text{and} \quad \sin\phi = \frac{c_2}{R} = \sqrt{\frac{2}{7}}$$

Therefore $\phi$ is in the second quadrant and

$$\phi = \cos^{-1}\left(-\sqrt{\frac{5}{7}}\right) \approx 2.58\text{rad}$$

## Undamped Forced Oscillation

In many mechanical problems a device is subjected to periodic external forces. For example, soldiers marching in cadence on a bridge cause periodic disturbances in the bridge, and the engines of a propeller driven aircraft cause periodic disturbances in its wings. In the absence of sufficient damping forces, such disturbances – even if small in magnitude – can cause structural breakdown if they are at certain critical frequencies. To illustrate, this we'll consider the motion of an object in a spring–mass system without damping, subject to an external force

$$F(t) = F_0 \cos\omega t$$

where $F_0$ is a constant. In this case the equation of motion Equation 6.1.2 is

$$my'' + ky = F_0 \cos\omega t,$$

which we rewrite as

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos\omega t \tag{6.1.13}$$

with $\omega_0 = \sqrt{k/m}$. We'll see from the next two examples that the solutions of Equation 6.1.13 with $\omega \neq \omega_0$ behave very differently from the solutions with $\omega = \omega_0$.

### Example 6.1.4

Solve the initial value problem

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos\omega t, \quad y(0) = 0, \quad y'(0) = 0, \tag{6.1.14}$$

given that $\omega \neq \omega_0$.

**Solution**:

We first obtain a particular solution of Equation 6.1.13 by the method of undetermined coefficients. Since $\omega \neq \omega_0$, $\cos \omega t$ isn't a solution of the complementary equation

$$y'' + \omega_0^2 y = 0.$$

Therefore Equation 6.1.13 has a particular solution of the form

$$y_p = A \cos \omega t + B \sin \omega t.$$

Since

$$y_p'' = -\omega^2 (A \cos \omega t + B \sin \omega t),$$

$$y_p'' + \omega_0^2 y_p = \frac{F_0}{m} \cos \omega t$$

if and only if

$$(\omega_0^2 - \omega^2)\,(A \cos \omega t + B \sin \omega t) = \frac{F_0}{m} \cos \omega t.$$

This holds if and only if

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad B = 0,$$

so

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

The general solution of Equation 6.1.13 is

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t, \tag{6.1.15}$$

so

$$y' = \frac{-\omega F_0}{m(\omega_0^2 - \omega^2)} \sin \omega t + \omega_0 (-c_1 \sin \omega_0 t + c_2 \cos \omega_0 t).$$

The initial conditions $y(0) = 0$ and $y'(0) = 0$ in Equation 6.1.14 imply that

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad c_2 = 0.$$

Substituting these into Equation 6.1.15 yields

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t). \tag{6.1.16}$$

It is revealing to write this in a different form. We start with the trigonometric identities

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$

Subtracting the second identity from the first yields

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin \alpha \sin \beta \tag{6.1.17}$$

Now let

$$\alpha - \beta = \omega t \quad \text{and} \quad \alpha + \beta = \omega_0 t, \tag{6.1.18}$$

so that

$$\alpha = \frac{(\omega_0 + \omega)t}{2} \quad \text{and} \quad \beta = \frac{(\omega_0 - \omega)t}{2}. \tag{6.1.19}$$

Substituting Equation 6.1.18and Equation 6.1.19into Equation 6.1.17yields

$$\cos \omega t - \cos \omega_0 t = 2 \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2},$$

and substituting this into Equation 6.1.16yields

$$y = R(t) \sin \frac{(\omega_0 + \omega)t}{2}, \tag{6.1.20}$$

where

$$R(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}. \tag{6.1.21}$$

From Equation 6.1.20we can regard $y$ as a sinusoidal variation with frequency $(\omega_0 + \omega)/2$ and variable amplitude $|R(t)|$. In Figure 6.1.6 the dashed curve above the $t$ axis is $y = |R(t)|$, the dashed curve below the $t$ axis is $y = -|R(t)|$, and the displacement $y$ appears as an oscillation bounded by them. The oscillation of $y$ for $t$ on an interval between successive zeros of $R(t)$ is called a **beat**.

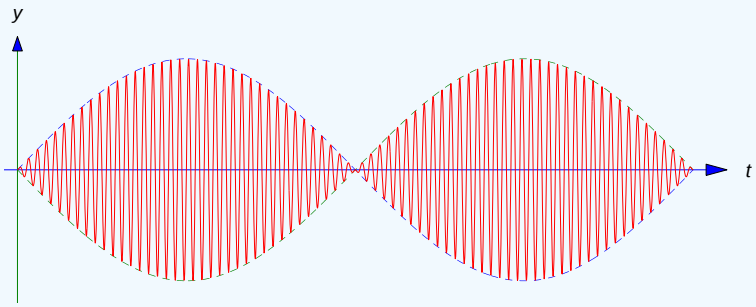



Figure 6.1.6 : Undamped oscillation with beats

You can see from Equation 6.1.20and Equation 6.1.21that

$$|y(t)| \le \frac{2|F_0|}{m|\omega_0^2 - \omega^2|};$$

moreover, if $\omega + \omega_0$ is sufficiently large compared with $\omega - \omega_0$, then $|y|$ assumes values close to (perhaps equal to) this upper bound during each beat. However, the oscillation remains bounded for all $t$. (This assumes that the spring can withstand deflections of this size and continue to obey Hooke's law.) The next example shows that this isn't so if $\omega = \omega_0$.

## Example 6.1.5

Find the general solution of

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t. \tag{6.1.22}$$

**Solution**:

We first obtain a particular solution $y_p$ of Equation 6.1.22. Since $\cos \omega_0 t$ is a solution of the complementary equation, the form for $y_p$ is

$$y_p = t(A \cos \omega_0 t + B \sin \omega_0 t). \tag{6.1.23}$$

Then

$$y_p' = A \cos \omega_0 t + B \sin \omega_0 t + \omega_0 t(-A \sin \omega_0 t + B \cos \omega_0 t)$$

and

$$y_p'' = 2\omega_0(-A \sin \omega_0 t + B \cos \omega_0 t) - \omega_0^2 t(A \cos \omega_0 t + B \sin \omega_0 t). \tag{6.1.24}$$

From Equation 6.1.23and Equation 6.1.24, we see that $y_p$ satisfies Equation 6.1.22if

$$-2A\omega_0 \sin\omega_0 t + 2B\omega_0 \cos\omega_0 t = \frac{F_0}{m}\cos\omega_0 t;$$

that is, if

$$A = 0 \quad \text{and} \quad B = \frac{F_0}{2m\omega_0}.$$

Therefore

$$y_p = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t$$

is a particular solution of Equation 6.1.22 The general solution of Equation 6.1.22is

$$y = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t + c_1\cos\omega_0 t + c_2\sin\omega_0 t.$$

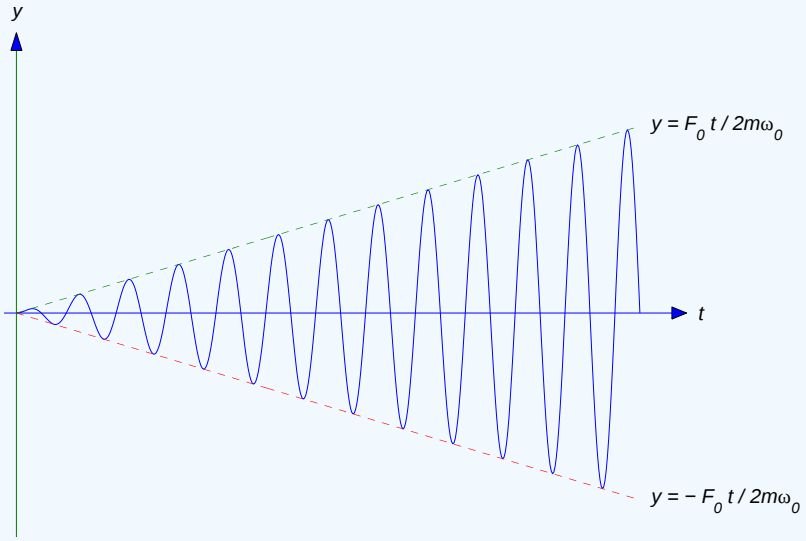



Figure 6.1.7 : Undamped displacement due to resonance.

The graph of $y_p$ is shown in Figure 6.1.7 , where it can be seen that $y_p$ oscillates between the dashed lines

$$y = \frac{F_0 t}{2m\omega_0} \quad \text{and} \quad y = -\frac{F_0 t}{2m\omega_0}$$

with increasing amplitude that approaches $\infty$ as $t \to \infty$. Of course, this means that the spring must eventually fail to obey Hooke's law or break.

This phenomenon of unbounded displacements of a spring–mass system in response to a periodic forcing function at its natural frequency is called **resonance**. More complicated mechanical structures can also exhibit resonance–like phenomena. For example, rhythmic oscillations of a suspension bridge by wind forces or of an airplane wing by periodic vibrations of reciprocating engines can cause damage or even failure if the frequencies of the disturbances are close to critical frequencies determined by the parameters of the mechanical system in question.

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# 6.1E: Spring Problems I (Exercises)

In the following exercises assume that there's no damping.

## Q6.1.1

1. An object stretches a spring $4$ inches in equilibrium. Find and graph its displacement for $t > 0$ if it is initially displaced $36$ inches above equilibrium and given a downward velocity of $2$ ft/s.

2. An object stretches a string $1.2$ inches in equilibrium. Find its displacement for $t > 0$ if it is initially displaced $3$ inches below equilibrium and given a downward velocity of $2$ ft/s.

3. A spring with natural length $.5$ m has length $50.5$ cm with a mass of $2$ gm suspended from it. The mass is initially displaced $1.5$ cm below equilibrium and released with zero velocity. Find its displacement for $t > 0$.

4. An object stretches a spring $6$ inches in equilibrium. Find its displacement for $t > 0$ if it is initially displaced $3$ inches above equilibrium and given a downward velocity of $6$ inches/s. Find the frequency, period, amplitude and phase angle of the motion.

5. An object stretches a spring $5$ cm in equilibrium. It is initially displaced $10$ cm above equilibrium and given an upward velocity of $.25$ m/s. Find and graph its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

6. A $10$ kg mass stretches a spring $70$ cm in equilibrium. Suppose a $2$ kg mass is attached to the spring, initially displaced $25$ cm below equilibrium, and given an upward velocity of $2$ m/s. Find its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

7. A weight stretches a spring $1.5$ inches in equilibrium. The weight is initially displaced $8$ inches above equilibrium and given a downward velocity of $4$ ft/s. Find its displacement for $t > 0$.

8. A weight stretches a spring $6$ inches in equilibrium. The weight is initially displaced $6$ inches above equilibrium and given a downward velocity of $3$ ft/s. Find its displacement for $t > 0$.

9. A spring–mass system has natural frequency $7\sqrt{10}$ rad/s. The natural length of the spring is $.7$ m. What is the length of the spring when the mass is in equilibrium?

10. A $64$ lb weight is attached to a spring with constant $k = 8$ lb/ft and subjected to an external force $F(t) = 2\sin t$. The weight is initially displaced $6$ inches above equilibrium and given an upward velocity of $2$ ft/s. Find its displacement for $t > 0$.

11. A unit mass hangs in equilibrium from a spring with constant $k = 1/16$. Starting at $t = 0$, a force $F(t) = 3\sin t$ is applied to the mass. Find its displacement for $t > 0$.

12. A $4$ lb weight stretches a spring $1$ ft in equilibrium. An external force $F(t) = .25\sin 8t$ lb is applied to the weight, which is initially displaced $4$ inches above equilibrium and given a downward velocity of $1$ ft/s. Find and graph its displacement for $t > 0$.

13. A $2$ lb weight stretches a spring $6$ inches in equilibrium. An external force $F(t) = \sin 8t$ lb is applied to the weight, which is released from rest $2$ inches below equilibrium. Find its displacement for $t > 0$.

14. A $10$ gm mass suspended on a spring moves in simple harmonic motion with period $4$ s. Find the period of the simple harmonic motion of a $20$ gm mass suspended from the same spring.

15. A $6$ lb weight stretches a spring $6$ inches in equilibrium. Suppose an external force $F(t) = \frac{3}{16}\sin\omega t + \frac{3}{8}\cos\omega t$ lb is applied to the weight. For what value of $\omega$ will the displacement be unbounded? Find the displacement if $\omega$ has this value. Assume that the motion starts from equilibrium with zero initial velocity.

16. A $6$ lb weight stretches a spring $4$ inches in equilibrium. Suppose an external force $F(t) = 4\sin\omega t - 6\cos\omega t$ lb is applied to the weight. For what value of $\omega$ will the displacement be unbounded? Find and graph the displacement if $\omega$ has this value. Assume that the motion starts from equilibrium with zero initial velocity.

17. A mass of one kg is attached to a spring with constant $k = 4$ N/m. An external force $F(t) = -\cos\omega t - 2\sin\omega t$ n is applied to the mass. Find the displacement $y$ for $t > 0$ if $\omega$ equals the natural frequency of the spring–mass system. Assume that the mass is initially displaced $3$ m above equilibrium and given an upward velocity of $450$ cm/s.

18. An object is in simple harmonic motion with frequency $\omega_0$, with $y(0) = y_0$ and $y'(0) = v_0$. Find its displacement for $t > 0$. Also, find the amplitude of the oscillation and give formulas for the sine and cosine of the initial phase angle.

19. Two objects suspended from identical springs are set into motion. The period of one object is twice the period of the other. How are the weights of the two objects related?

20. Two objects suspended from identical springs are set into motion. The weight of one object is twice the weight of the other. How are the periods of the resulting motions related?

21. Two identical objects suspended from different springs are set into motion. The period of one motion is $3$ times the period of the other. How are the two spring constants related?

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# 6.2: Spring Problems II

## Free Vibrations with Damping

In this section we consider the motion of an object in a spring–mass system with damping. We start with unforced motion, so the equation of motion is

$$my'' + cy' + ky = 0. \tag{6.2.1}$$

Now suppose the object is displaced from equilibrium and given an initial velocity. Intuition suggests that if the damping force is sufficiently weak the resulting motion will be oscillatory, as in the undamped case considered in the previous section, while if it is sufficiently strong the object may just move slowly toward the equilibrium position without ever reaching it. We'll now confirm these intuitive ideas mathematically. The characteristic equation of Equation 6.2.1 is

$$mr^2 + cr + k = 0.$$

The roots of this equation are

$$r_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}. \tag{6.2.2}$$

We saw in Section 5.2 that the form of the solution of Equation 6.2.1 depends upon whether $c^2 - 4mk$ is positive, negative, or zero. We'll now consider these three cases.

## Underdamped Motion

We say the motion is **underdamped** if $c < \sqrt{4mk}$. In this case $r_1$ and $r_2$ in Equation 6.2.2 are complex conjugates, which we write as

$$r_1 = -\frac{c}{2m} - \omega_1\, i \quad \text{and} \quad r_2 = -\frac{c}{2m} + \omega_1\, i,$$

where

$$\omega_1 = \frac{\sqrt{4mk - c^2}}{2m}.$$

The general solution of Equation 6.2.1 in this case is

$$y = e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

By the method used in Section 6.1 to derive the amplitude–phase form of the displacement of an object in simple harmonic motion, we can rewrite this equation as

$$y = Re^{-ct/2m} \cos(\omega_1 t - \phi), \tag{6.2.3}$$

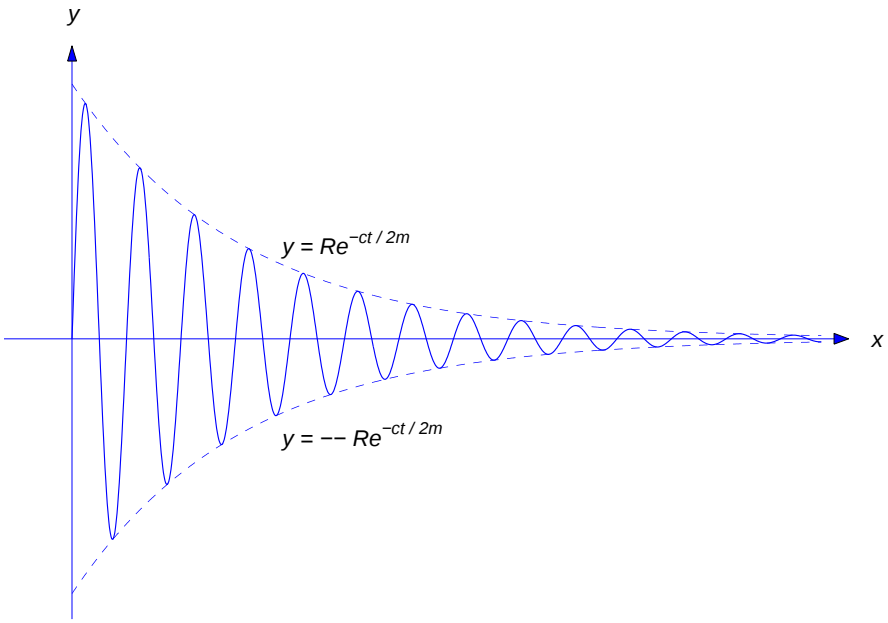



Figure 6.2.1 : Underdamped motion

where

$$R=\sqrt{c_1^2+c_2^2},\quad R\cos\phi=c_1,\quad \text{and}\quad R\sin\phi=c_2.$$

The factor $Re^{-ct/2m}$ in Equation 6.2.3 is called the **time–varying amplitude** of the motion, the quantity $\omega_1$ is called the **frequency**, and $T=2\pi/\omega_1$ (which is the period of the cosine function in Equation 6.2.3 is called the **quasi–period**. A typical graph of Equation 6.2.3 is shown in Figure 6.2.1 . As illustrated in that figure, the graph of $y$ oscillates between the dashed exponential curves $y=\pm Re^{-ct/2m}$ .

## Overdamped Motion

We say the motion is **overdamped** if $c>\sqrt{4mk}$. In this case the zeros $r_1$ and $r_2$ of the characteristic polynomial are real, with $r_1<r_2<0$ (see 6.2.2), and the general solution of 6.2.1 is

$$y=c_1e^{r_1t}+c_2e^{r_2t}.$$

Again $\lim_{t\to\infty}y(t)=0$ as in the underdamped case, but the motion isn't oscillatory, since $y$ can't equal zero for more than one value of $t$ unless $c_1=c_2=0$ . (**Exercise 6.2.23**.)

## Critically Damped Motion

We say the motion is **critically damped** if $c=\sqrt{4mk}$. In this case $r_1=r_2=-c/2m$ and the general solution of Equation 6.2.1 is

$$y=e^{-ct/2m}(c_1+c_2t).$$

Again $\lim_{t\to\infty}y(t)=0$ and the motion is nonoscillatory, since $y$ can't equal zero for more than one value of $t$ unless $c_1=c_2=0$ . (**Exercise 6.2.22**).

### Example 6.2.1

Suppose a 64 lb weight stretches a spring 6 inches in equilibrium and a dashpot provides a damping force of $c$ lb for each ft/sec of velocity.

a. Write the equation of motion of the object and determine the value of $c$ for which the motion is critically damped.
b. Find the displacement $y$ for $t>0$ if the motion is critically damped and the initial conditions are $y(0)=1$ and $y'(0)=20$.
c. Find the displacement $y$ for $t>0$ if the motion is critically damped and the initial conditions are $y(0)=1$ and $y'(0)=-20$.

**Solution a**

Here $m = 2$ slugs and $k = 64/.5 = 128$ lb/ft. Therefore the equation of motion Equation 6.2.1 is

$$2y'' + cy' + 128y = 0. \tag{6.2.4}$$

The characteristic equation is

$$2r^2 + cr + 128 = 0,$$

which has roots

$$r = \frac{-c \pm \sqrt{c^2 - 8 \cdot 128}}{4}.$$

Therefore the damping is critical if

$$c = \sqrt{8 \cdot 128} = 32 \text{ lb-sec/ft}.$$

**Solution b**

Setting $c = 32$ in Equation 6.2.4 and cancelling the common factor 2 yields

$$y'' + 16y + 64y = 0.$$

The characteristic equation is

$$r^2 + 16r + 64y = (r+8)^2 = 0.$$

Hence, the general solution is

$$y = e^{-8t}(c_1 + c_2 t). \tag{6.2.5}$$

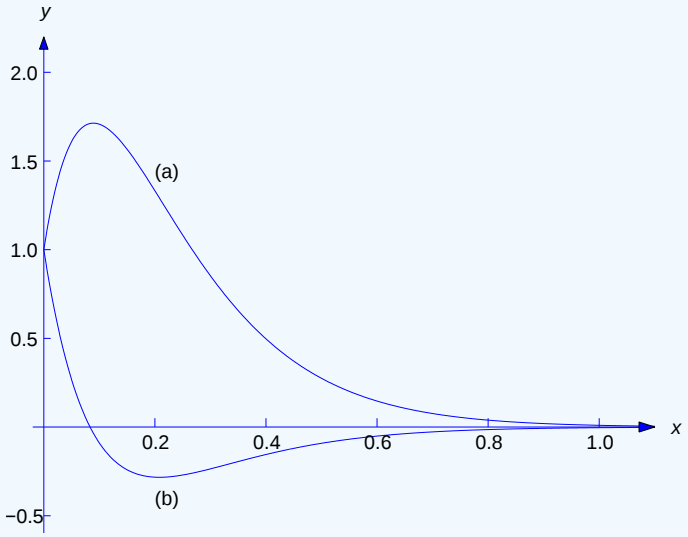



Figure 6.2.2 : (a) $y = e^{-8t}(1 + 28t)$ and (b) $y = e^{-8t}(1 - 12t)$

Differentiating this yields

$$y' = -8y + c_2 e^{-8t}. \tag{6.2.6}$$

6.2.2 Imposing the initial conditions $y(0) = 1$ and $y'(0) = 20$ in the last two equations shows that $1 = c_1$ and $20 = -8 + c_2$ . Hence, the solution of the initial value problem is

$$y = e^{-8t}(1 + 28t).$$

Therefore the object approaches equilibrium from above as $t \to \infty$. There's no oscillation.

**Solution c**

Imposing the initial conditions $y(0)=1$ and $y'(0)=-20$ in Equation 6.2.5 and Equation 6.2.6 yields $1=c_1$ and $-20=-8+c_2$. Hence, the solution of this initial value problem is

$$y=e^{-8t}(1-12t).$$

Therefore the object moves downward through equilibrium just once, and then approaches equilibrium from below as $t\to\infty$. Again, there's no oscillation. The solutions of these two initial value problems are graphed in Figure 6.2.2 .

## Example 6.2.2

Find the displacement of the object in Example 6.2.1 if the damping constant is $c=4$ lb–sec/ft and the initial conditions are $y(0)=1.5$ ft and $y'(0)=-3$ ft/sec.

**Solution**

With $c=4$, the equation of motion Equation 6.2.4 becomes

$$y''+2y'+64y=0 \tag{6.2.7}$$

after cancelling the common factor 2. The characteristic equation

$$r^2+2r+64=0$$

has complex conjugate roots

$$r=\frac{-2\pm\sqrt{4-4\cdot 64}}{2}=-1\pm 3\sqrt{7}i.$$

Therefore the motion is underdamped and the general solution of Equation 6.2.7 is

$$y=e^{-t}(c_1\cos 3\sqrt{7}t+c_2\sin 3\sqrt{7}t).$$

Differentiating this yields

$$y'=-y+3\sqrt{7}e^{-t}(-c_1\sin 3\sqrt{7}t+c_2\cos 3\sqrt{7}t).$$

Imposing the initial conditions $y(0)=1.5$ and $y'(0)=-3$ in the last two equations yields $1.5=c_1$ and $-3=-1.5+3\sqrt{7}c_2$. Hence, the solution of the initial value problem is

$$y=e^{-t}\left(\frac{3}{2}\cos 3\sqrt{7}t-\frac{1}{2\sqrt{7}}\sin 3\sqrt{7}t\right). \tag{6.2.8}$$

The amplitude of the function in parentheses is

$$R=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{1}{2\sqrt{7}}\right)^2}=\sqrt{\frac{9}{4}+\frac{1}{4\cdot 7}}=\sqrt{\frac{64}{4\cdot 7}}=\frac{4}{\sqrt{7}}$$

Therefore we can rewrite Equation 6.2.8 as

$$y=\frac{4}{\sqrt{7}}e^{-t}\cos(3\sqrt{7}t-\phi),$$

where

$$\cos\phi=\frac{3}{2R}=\frac{3\sqrt{7}}{8}\quad\text{and}\quad\sin\phi=-\frac{1}{2\sqrt{7}R}=-\frac{1}{8}.$$

Therefore $\phi\cong -0.125$ radians.

## Example 6.2.3

Let the damping constant in Example 6.2.1 be $c=40$ lb–sec/ft. Find the displacement $y$ for $t>0$ if $y(0)=1$ and $y'(0)=1$.

**Solution**

With $c = 40$, the equation of motion Equation 6.2.4 reduces to

$$y'' + 20y' + 64y = 0 \tag{6.2.9}$$

after cancelling the common factor 2. The characteristic equation

$$r^2 + 20r + 64 = (r+16)(r+4) = 0$$

has the roots $r_1 = -4$ and $r_2 = -16$. Therefore the general solution of Equation 6.2.9 is

$$y = c_1 e^{-4t} + c_2 e^{-16t}. \tag{6.2.10}$$

Differentiating this yields

$$y' = -4e^{-4t} - 16c_2 e^{-16t}.$$

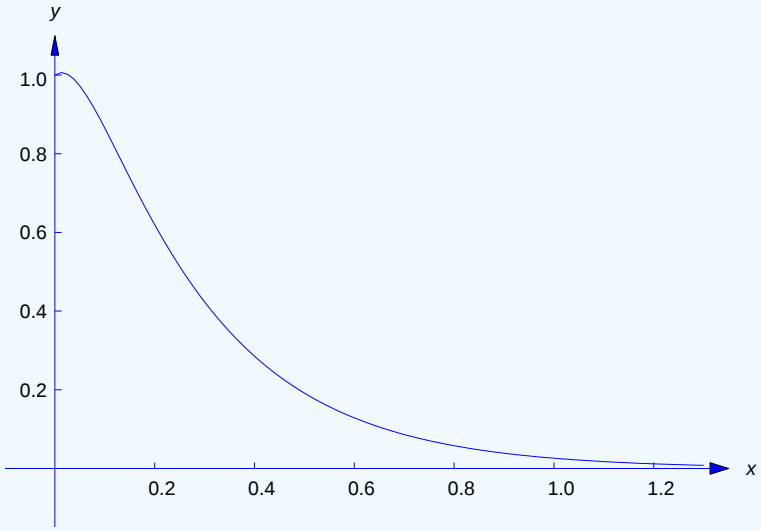



Figure 6.2.3 : $y = \frac{17}{12}e^{-4t} - \frac{5}{12}e^{-16t}$

The last two equations and the initial conditions $y(0) = 1$ and $y'(0) = 1$ imply that

$$\begin{aligned} c_1 \quad + \quad c_2 \quad &= 1 \\ -4c_1 \quad - \quad 16c_2 \quad &= 1. \end{aligned}$$

The solution of this system is $c_1 = 17/12$, $c_2 = -5/12$. Substituting these into Equation 6.2.10 yields

$$y = \frac{17}{12}e^{-4t} - \frac{5}{12}e^{-16t}$$

as the solution of the given initial value problem (Figure 6.2.3 ).

## Forced Vibrations with Damping

Now we consider the motion of an object in a spring-mass system with damping, under the influence of a periodic forcing function $F(t) = F_0 \cos \omega t$, so that the equation of motion is

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{6.2.11}$$

In Section 6.1 we considered this equation with $c = 0$ and found that the resulting displacement $y$ assumed arbitrarily large values in the case of resonance (that is, when $\omega = \omega_0 = \sqrt{k/m}$ ). Here we'll see that in the presence of damping the displacement remains bounded for all $t$, and the initial conditions have little effect on the motion as $t \to \infty$. In fact, we'll see that for large $t$ the displacement is closely approximated by a function of the form

$$y = R\cos(\omega t - \phi), \tag{6.2.12}$$

where the amplitude $R$ depends upon $m$, $c$, $k$, $F_0$, and $\omega$. We're interested in the following question:

QUESTION

Assuming that $m$, $c$, $k$, and $F_0$ are held constant, what value of $\omega$ produces the largest amplitude $R$ in Equation 6.2.12, and what is this largest amplitude?

To answer this question, we must solve Equation 6.2.11 and determine $R$ in terms of $F_0, \omega_0, \omega$, and $c$. We can obtain a particular solution of Equation 6.2.11 by the method of undetermined coefficients. Since $\cos \omega t$ does not satisfy the complementary equation

$$my'' + cy' + ky = 0,$$

we can obtain a particular solution of Equation 6.2.11 in the form

$$y_p = A\cos\omega t + B\sin\omega t. \tag{6.2.13}$$

Differentiating this yields

$$y_p' = \quad \omega(-A\sin\omega t + B\cos\omega t)$$

and

$$y_p'' = -\omega^2(A\cos\omega t + B\sin\omega t).$$

From the last three equations,

$$my_p'' + cy_p' + ky_p = (-m\omega^2 A + c\omega B + kA)\cos\omega t + (-m\omega^2 B - c\omega A + kB)\sin\omega t,$$

so $y_p$ satisfies Equation 6.2.11 if

$$\begin{aligned} (k - m\omega^2)A + \quad c\omega B \quad &= F_0 \\ -c\omega A \quad + (k - m\omega^2)B \quad &= 0. \end{aligned}$$

Solving for $A$ and $B$ and substituting the results into Equation 6.2.13 yields

$$y_p = \frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2}\left[(k - m\omega^2)\cos\omega t + c\omega\sin\omega t\right],$$

which can be written in amplitude–phase form as

(6.2.14)

where

$$\cos\phi = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \text{and} \quad \sin\phi = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}. \tag{6.2.15}$$

To compare this with the undamped forced vibration that we considered in Section 6.1 it is useful to write

$$k - m\omega^2 = m\left(\frac{k}{m} - \omega^2\right) = m(\omega_0^2 - \omega^2), \tag{6.2.16}$$

where $\omega_0 = \sqrt{k/m}$ is the natural angular frequency of the undamped simple harmonic motion of an object with mass $m$ on a spring with constant $k$. Substituting Equation 6.2.16 into Equation 6.2.14 yields

$$y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}\cos(\omega t - \phi). \tag{6.2.17}$$

The solution of an initial value problem

$$my'' + cy' + ky = F_0\cos\omega t, \quad y(0) = y_0, \quad y'(0) = v_0,$$

is of the form $y = y_c + y_p$ , where $y_c$ has one of the three forms

$$y_c = e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t),$$
$$y_c = e^{-ct/2m}(c_1 + c_2 t),$$
$$y_c = c_1 e^{r_1 t} + c_2 e^{r_2 t} \ (r_1, r_2 < 0).$$

In all three cases $\lim_{t\to\infty} y_c(t) = 0$ for any choice of $c_1$ and $c_2$. For this reason we say that $y_c$ is the **transient component** of the solution $y$. The behavior of $y$ for large $t$ is determined by $y_p$, which we call the **steady state component** of $y$. Thus, for large $t$ the motion is like simple harmonic motion at the frequency of the external force.

The amplitude of $y_p$ in Equation 6.2.17 is

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}, \tag{6.2.18}$$

which is finite for all $\omega$; that is, the presence of damping precludes the phenomenon of resonance that we encountered in studying undamped vibrations under a periodic forcing function. We'll now find the value $\omega_{\max}$ of $\omega$ for which $R$ is maximized. This is the value of $\omega$ for which the function

$$\rho(\omega) = m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2$$

in the denominator of Equation 6.2.18 attains its minimum value. By rewriting this as

$$\rho(\omega) = m^2(\omega_0^4 + \omega^4) + (c^2 - 2m^2\omega_0^2)\omega^2, \tag{6.2.19}$$

you can see that $\rho$ is a strictly increasing function of $\omega^2$ if

$$c \ge \sqrt{2m^2\omega_0^2} = \sqrt{2mk}.$$

(Recall that $\omega_0^2 = k/m$). Therefore $\omega_{\max} = 0$ if this inequality holds. From Equation 6.2.15, you can see that $\phi = 0$ if $\omega = 0$. In this case, Equation 6.2.14 reduces to

$$y_p = \frac{F_0}{\sqrt{m^2\omega_0^4}} = \frac{F_0}{k},$$

which is consistent with Hooke's law: if the mass is subjected to a constant force $F_0$, its displacement should approach a constant $y_p$ such that $ky_p = F_0$ . Now suppose $c < \sqrt{2mk}$. Then, from Equation 6.2.19,

$$\rho'(\omega) = 2\omega(2m^2\omega^2 + c^2 - 2m^2\omega_0^2),$$

and $\omega_{\max}$ is the value of $\omega$ for which the expression in parentheses equals zero; that is,

$$\omega_{\max} = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}} = \sqrt{\frac{k}{m}\left(1 - \frac{c^2}{2km}\right)}.$$

(To see that $\rho(\omega_{\max})$ is the minimum value of $\rho(\omega)$, note that $\rho'(\omega) < 0$ if $\omega < \omega_{\max}$ and $\rho'(\omega) > 0$ if $\omega > \omega_{\max}$.) Substituting $\omega = \omega_{\max}$ in Equation 6.2.18 and simplifying shows that the maximum amplitude $R_{\max}$ is

$$R_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}} \quad \text{if} \quad c < \sqrt{2mk}.$$

We summarize our results as follows.

**Theorem 6.2.1**

Suppose we consider the amplitude $R$ of the steady state component of the solution of

$$my'' + cy' + ky = F_0 \cos \omega t$$

as a function of $\omega$.

a. If $c \ge \sqrt{2mk}$, the maximum amplitude is $R_{\max} = F_0/k$ and it is attained when $\omega = \omega_{\max} = 0$ .

b. If $c < \sqrt{2mk}$, the maximum amplitude is

$$R_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}}, \tag{6.2.20}$$

and it is attained when

$$\omega = \omega_{\max} = \sqrt{\frac{k}{m}\left(1 - \frac{c^2}{2km}\right)}. \tag{6.2.21}$$

Note that $R_{\max}$ and $\omega_{\max}$ are continuous functions of $c$, for $c \geq 0$, since Equation 6.2.20 and Equation 6.2.21 reduce to $R_{\max} = F_0/k$ and $\omega_{\max} = 0$ if $c = \sqrt{2km}$.

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# 6.2E: Spring Problems II (Exercises)

## Q6.2.1

1. A 64 lb object stretches a spring 4 ft in equilibrium. It is attached to a dashpot with damping constant $c = 8$ lb-sec/ft. The object is initially displaced 18 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement and time–varying amplitude for $t > 0$.

2. A 16 lb weight is attached to a spring with natural length 5 ft. With the weight attached, the spring measures 8.2 ft. The weight is initially displaced 3 ft below equilibrium and given an upward velocity of 2 ft/sec. Find and graph its displacement for $t > 0$ if the medium resists the motion with a force of one lb for each ft/sec of velocity. Also, find its time–varying amplitude.

3. An 8 lb weight stretches a spring 1.5 inches. It is attached to a dashpot with damping constant $c = 8$ lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given an upward velocity of 6 ft/sec. Find and graph its displacement for $t > 0$.

4. A 96 lb weight stretches a spring 3.2 ft in equilibrium. It is attached to a dashpot with damping constant $c$=18 lb-sec/ft. The weight is initially displaced 15 inches below equilibrium and given a downward velocity of 12 ft/sec. Find its displacement for $t > 0$.

5. A 16 lb weight stretches a spring 6 inches in equilibrium. It is attached to a damping mechanism with constant $c$. Find all values of $c$ such that the free vibration of the weight has infinitely many oscillations.

6. An 8 lb weight stretches a spring .32 ft. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 4 ft/sec. Find its displacement for $t > 0$ if the medium exerts a damping force of 1.5 lb for each ft/sec of velocity.

7. A 32 lb weight stretches a spring 2 ft in equilibrium. It is attached to a dashpot with constant $c = 8$ lb-sec/ft. The weight is initially displaced 8 inches below equilibrium and released from rest. Find its displacement for $t > 0$.

8. A mass of 20 gm stretches a spring 5 cm. The spring is attached to a dashpot with damping constant 400 dyne sec/cm. Determine the displacement for $t > 0$ if the mass is initially displaced 9 cm above equilibrium and released from rest.

9. A 64 lb weight is suspended from a spring with constant $k = 25$ lb/ft. It is initially displaced 18 inches above equilibrium and released from rest. Find its displacement for $t > 0$ if the medium resists the motion with 6 lb of force for each ft/sec of velocity.

10. A 32 lb weight stretches a spring 1 ft in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/sec. Find its displacement for $t > 0$ if the medium resists the motion with a force equal to 3 times the speed in ft/sec.

11. An 8 lb weight stretches a spring 2 inches. It is attached to a dashpot with damping constant $c = 4$ lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement for $t > 0$.

12. A 2 lb weight stretches a spring .32 ft. The weight is initially displaced 4 inches below equilibrium and given an upward velocity of 5 ft/sec. The medium provides damping with constant $c = 1/8$ lb-sec/ft. Find and graph the displacement for $t > 0$.

13. An 8 lb weight stretches a spring 8 inches in equilibrium. It is attached to a dashpot with damping constant $c = .5$ lb-sec/ft and subjected to an external force $F(t) = 4\cos 2t$ lb. Determine the steady state component of the displacement for $t > 0$.

14. A 32 lb weight stretches a spring 1 ft in equilibrium. It is attached to a dashpot with constant $c = 12$ lb-sec/ft. The weight is initially displaced 8 inches above equilibrium and released from rest. Find its displacement for $t > 0$.

15. A mass of one kg stretches a spring 49 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 4 N for each m/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.

16. A mass of 100 grams stretches a spring 98 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 600 dynes for each cm/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.

17. A 192 lb weight is suspended from a spring with constant $k = 6$ lb/ft and subjected to an external force $F(t) = 8\cos 3t$ lb. Find the steady state component of the displacement for $t > 0$ if the medium resists the motion with a force equal to 8 times the speed in ft/sec.

18. A 2 gm mass is attached to a spring with constant 20 dyne/cm. Find the steady state component of the displacement if the mass is subjected to an external force $F(t) = 3\cos 4t - 5\sin 4t$ dynes and a dashpot supplies 4 dynes of damping for each cm/sec of velocity.

19. A 96 lb weight is attached to a spring with constant 12 lb/ft. Find and graph the steady state component of the displacement if the mass is subjected to an external force $F(t) = 18\cos t - 9\sin t$ lb and a dashpot supplies 24 lb of damping for each ft/sec of velocity.

20. A mass of one kg stretches a spring 49 cm in equilibrium. It is attached to a dashpot that supplies a damping force of 4 N for each m/sec of speed. Find the steady state component of its displacement if it is subjected to an external force $F(t) = 8\sin 2t - 6\cos 2t$ N.

21. A mass $m$ is suspended from a spring with constant $k$ and subjected to an external force $F(t) = \alpha\cos\omega_0 t + \beta\sin\omega_0 t$ , where $\omega_0$ is the natural frequency of the spring–mass system without damping. Find the steady state component of the displacement if a dashpot with constant $c$ supplies damping.

22. Show that if $c_1$ and $c_2$ are not both zero then

$$y = e^{r_1 t}(c_1 + c_2 t)$$

can't equal zero for more than one value of $t$.

23. Show that if $c_1$ and $c_2$ are not both zero then

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

can't equal zero for more than one value of $t$.

24. Find the solution of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0,\ y'(0) = v_0,$$

given that the motion is underdamped, so the general solution of the equation is

$$y = e^{-ct/2m}(c_1\cos\omega_1 t + c_2\sin\omega_1 t).$$

25. Find the solution of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0,\ y'(0) = v_0,$$

given that the motion is overdamped, so the general solution of the equation is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}\ (r_1, r_2 < 0).$$

26. Find the solution of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0,\ y'(0) = v_0,$$

given that the motion is critically damped, so that the general solution of the equation is of the form

$$y = e^{r_1 t}(c_1 + c_2 t)\ (r_1 < 0).$$

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# CHAPTER OVERVIEW

## 7: Series Solutions of Linear Second Order Equations

In this Chapter, we study a class of second order differential equations that occur in many applications, but cannot be solved in closed form in terms of elementary functions.


7.1: Review of Power Series

7.1E: Review of Power Series (Exercises)

7.2: Series Solutions Near an Ordinary Point I

7.2E: Series Solutions Near an Ordinary Point I (Exercises)

7.3: Series Solutions Near an Ordinary Point II

7.3E: Series Solutions Near an Ordinary Point II (Exercises)

Prelude to Series Solutions of Linear Second Order Equations


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# 7.1: Review of Power Series

Many applications give rise to differential equations with solutions that can't be expressed in terms of elementary functions such as polynomials, rational functions, exponential and logarithmic functions, and trigonometric functions. The solutions of some of the most important of these equations can be expressed in terms of power series. We'll study such equations in this chapter. In this section we review relevant properties of power series. We'll omit proofs, which can be found in any standard calculus text.

**Definition 7.2.1**

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n, \tag{7.1.1}$$

where $x_0$ and $a_0$, $a_1$, ..., $a_n$, ...are constants, is called a **power series in** $x-x_0$. We say that the power series Equation 7.1.1 **converges** for a given $x$ if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x-x_0)^n$$

exists; otherwise, we say that the power series **diverges** for the given $x$.

A power series in $x-x_0$ must converge if $x=x_0$, since the positive powers of $x-x_0$ are all zero in this case. This may be the only value of $x$ for which the power series converges. However, the next theorem shows that if the power series converges for some $x\neq x_0$ then the set of all values of $x$ for which it converges forms an interval.

**Theorem 7.2.2**

For any power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

exactly one of the these statements is true:

I. The power series converges only for $x=x_0$.
II. The power series converges for all values of $x$.
III. There's a positive number $R$ such that the power series converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$.

In case (iii) we say that $R$ is the **radius of convergence** of the power series. For convenience, we include the other two cases in this definition by defining $R=0$ in case (i) and $R=\infty$ in case (ii). We define the **open interval of convergence** of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ to be

$$(x_0-R,x_0+R) \quad \text{if} \quad 0<R<\infty, \quad \text{or} \quad (-\infty,\infty) \quad \text{if} \quad R=\infty.$$

If $R$ is finite, no general statement can be made concerning convergence at the endpoints $x=x_0\pm R$ of the open interval of convergence; the series may converge at one or both points, or diverge at both.

Recall from calculus that a series of constants $\sum_{n=0}^{\infty}\alpha_n$ is said to **converge absolutely** if the series of absolute values $\sum_{n=0}^{\infty}|\alpha_n|$ converges. It can be shown that a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ with a positive radius of convergence $R$ converges absolutely in its open interval of convergence; that is, the series

$$\sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

of absolute values converges if $|x-x_0|<R$. However, if $R<\infty$, the series may fail to converge absolutely at an endpoint $x_0\pm R$, even if it converges here.

The next theorem provides a useful method for determining the radius of convergence of a power series. It's derived in calculus by applying the ratio test to the corresponding series of absolute values. For related theorems see **Exercises 7.2.2** and **7.2.4**.

**Theorem 7.2.3**

Suppose there's an integer $N$ such that $a_n \neq 0$ if $n \geq N$ and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L,$$

where $0 \leq L \leq \infty$. Then the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $R = 1/L$, which should be interpreted to mean that $R = 0$ if $L = \infty$, or $R = \infty$ if $L = 0$.

**Example 7.2.1**

Find the radius of convergence of the series:

a. $\displaystyle\sum_{n=0}^{\infty} n!x^n$

b. $\displaystyle\sum_{n=10}^{\infty} (-1)^n \frac{x^n}{n!}$

c. $\displaystyle\sum_{n=0}^{\infty} 2^n n^2 (x-1)^n.$

**Solution a**

Here $a_n = n!$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty.$$

Hence, $R = 0$.

**Solution b**

Here $a_n = (1)^n/n!$ for $n \geq N = 10$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Hence, $R = \infty$.

**Solution c**

Here $a_n = 2^n n^2$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}(n+1)^2}{2^n n^2} = 2\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 2.$$

Hence, $R = 1/2$.

## Taylor Series

If a function $f$ has derivatives of all orders at a point $x = x_0$, then the **Taylor series of $f$ about** $x_0$ is defined by

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

In the special case where $x_0 = 0$, this series is also called the **Maclaurin series of** $f$.

Taylor series for most of the common elementary functions converge to the functions on their open intervals of convergence. For example, you are probably familiar with the following Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty \tag{7.1.2}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty \tag{7.1.3}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad -\infty < x < \infty \tag{7.1.4}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad -1 < x < 1 \tag{7.1.5}$$

## Differentiation of Power Series

A power series with a positive radius of convergence defines a function

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

on its open interval of convergence. We say that the series **represents** $f$ on the open interval of convergence. A function $f$ represented by a power series may be a familiar elementary function as in Equations 7.1.2 - 7.1.5; however, it often happens that $f$ isn't a familiar function, so the series actually **defines** $f$.

The next theorem shows that a function represented by a power series has derivatives of all orders on the open interval of convergence of the power series, and provides power series representations of the derivatives.

### Theorem 7.2.4 : A Power Series

A power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with positive radius of convergence $R$ has derivatives of all orders in its open interval of convergence, and successive derivatives can be obtained by repeatedly differentiating term by term; that is,

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \tag{7.1.6}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}, \tag{7.1.7}$$

$$\vdots \tag{7.1.8}$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x - x_0)^{n-k}. \tag{7.1.9}$$

Moreover, all of these series have the same radius of convergence $R$.

### Example 7.2.2

Let $f(x) = \sin x$. From Equation 7.1.3,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

From Equation 7.1.6,

$$f'(x)=\sum_{n=0}^{\infty}(-1)^n\frac{d}{dx}\left[\frac{x^{2n+1}}{(2n+1)!}\right]=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!},$$

which is the series Equation 7.1.4 for $\cos x$.

## Uniqueness of Power Series

The next theorem shows that if $f$ is **defined** by a power series in $x-x_0$ with a positive radius of convergence, then the power series is the Taylor series of $f$ about $x_0$.

**Theorem 7.2.5**

If the power series

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

has a positive radius of convergence, then

$$a_n=\frac{f^{(n)}(x_0)}{n!}; \tag{7.1.10}$$

that is, $\sum_{n=0}^{\infty}a_n(x-x_0)^n$ is the Taylor series of $f$ about $x_0$.

This result can be obtained by setting $x=x_0$ in Equation 7.1.9, which yields

$$f^{(k)}(x_0)=k(k-1)\cdots1\cdot a_k=k!a_k.$$

This implies that

$$a_k=\frac{f^{(k)}(x_0)}{k!}.$$

Except for notation, this is the same as Equation 7.1.10.

The next theorem lists two important properties of power series that follow from Theorem 7.2.4 .

**Theorem 7.2.6**

a. If

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=\sum_{n=0}^{\infty}b_n(x-x_0)^n$$

for all $x$ in an open interval that contains $x_0$, then $a_n=b_n$ for $n=0,1,2,\ldots$.

b. If

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=0$$

for all $x$ in an open interval that contains $x_0$, then $a_n=0$ for $n=0,1,2,\ldots$.

To obtain (a) we observe that the two series represent the same function $f$ on the open interval; hence, Theorem 7.2.4 implies that

$$a_n=b_n=\frac{f^{(n)}(x_0)}{n!},\quad n=0,1,2,\ldots.$$

(b) can be obtained from (a) by taking $b_n=0$ for $n=0,1,2,\ldots$.

## Taylor Polynomials

If $f$ has $N$ derivatives at a point $x_0$, we say that

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is the **$N$-th Taylor polynomial of $f$ about $x_0$**. This definition and Theorem 7.2.4 imply that if

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

where the power series has a positive radius of convergence, then the Taylor polynomials of $f$ about $x_0$ are given by

$$T_N(x) = \sum_{n=0}^{N} a_n(x - x_0)^n.$$

In numerical applications, we use the Taylor polynomials to approximate $f$ on subintervals of the open interval of convergence of the power series. For example, Equation 7.1.2 implies that the Taylor polynomial $T_N$ of $f(x) = e^x$ is

$$T_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}.$$

The solid curve in Figure 7.2.1 is the graph of $y = e^x$ on the interval $[0, 5]$. The dotted curves in Figure 7.2.1 are the graphs of the Taylor polynomials $T_1$, ..., $T_6$ of $y = e^x$ about $x_0 = 0$. From this figure, we conclude that the accuracy of the approximation of $y = e^x$ by its Taylor polynomial $T_N$ improves as $N$ increases.

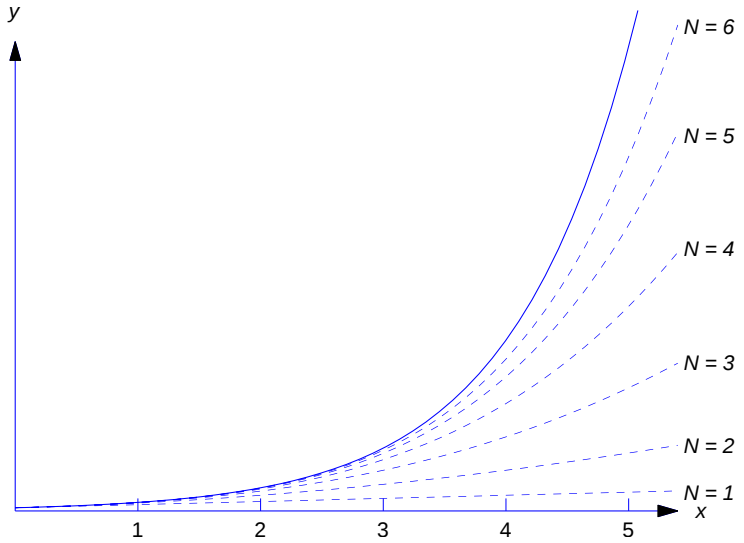



Figure 7.2.1 : Approximation of $y = e^x$ by Taylor polynomials about $x = 0$

## Shifting the Summation Index

In Definition 7.2.1 of a power series in $x - x_0$, the $n$-th term is a constant multiple of $(x - x_0)^n$. This isn't true in Equation 7.1.6, Equation 7.1.7, and Equation 7.1.9, where the general terms are constant multiples of $(x - x_0)^{n-1}$, $(x - x_0)^{n-2}$, and $(x - x_0)^{n-k}$, respectively. However, these series can all be rewritten so that their $n$-th terms are constant multiples of $(x - x_0)^n$. For example, letting $n = k + 1$ in the series in Equation 7.1.6 yields

$$f'(x) = \sum_{k=0}^{\infty} (k+1)a_{k+1}(x - x_0)^k, \tag{7.1.11}$$

where we start the new summation index $k$ from zero so that the first term in Equation 7.1.11 (obtained by setting $k = 0$) is the same as the first term in Equation 7.1.6 (obtained by setting $n = 1$). However, the sum of a series is independent of the symbol

used to denote the summation index, just as the value of a definite integral is independent of the symbol used to denote the variable of integration. Therefore we can replace $k$ by $n$ in Equation 7.1.11 to obtain

$$f'(x)=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_0)^n, \tag{7.1.12}$$

where the general term is a constant multiple of $(x-x_0)^n$.

It isn't really necessary to introduce the intermediate summation index $k$. We can obtain Equation 7.1.12 directly from Equation 7.1.6 by replacing $n$ by $n+1$ in the general term of Equation 7.1.6 and subtracting 1 from the lower limit of Equation 7.1.6. More generally, we use the following procedure for shifting indices.

### Shifting the Summation Index in a Power Series

For any integer $k$, the power series

$$\sum_{n=n_0}^{\infty} b_n(x-x_0)^{n-k}$$

can be rewritten as

$$\sum_{n=n_0-k}^{\infty} b_{n+k}(x-x_0)^n$$

that is, replacing $n$ by $n+k$ in the general term and subtracting k from the lower limit of summation leaves the series unchanged.

### Example 7.2.3

Rewrite the following power series from Equation 7.1.7 and Equation 7.1.9 so that the general term in each is a constant multiple of $(x-x_0)^n$:

$$(a)\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}\quad (b)\sum_{n=k}^{\infty}n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}.$$

**Solution a**

Replacing $n$ by $n+2$ in the general term and subtracting 2 from the lower limit of summation yields

$$\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n.$$

**Solution b**

Replacing $n$ by $n+k$ in the general term and subtracting $k$ from the lower limit of summation yields

$$\sum_{n=k}^{\infty}n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}=\sum_{n=0}^{\infty}(n+k)(n+k-1)\cdots(n+1)a_{n+k}(x-x_0)^n.$$

### Example 7.2.4

Given that

$$f(x)=\sum_{n=0}^{\infty}a_nx^n,$$

write the function $xf''$ as a power series in which the general term is a constant multiple of $x^n$.

**Solution**

From Theorem 7.2.4 with $x_0 = 0$,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$xf''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}.$$

Replacing $n$ by $n+1$ in the general term and subtracting 1 from the lower limit of summation yields

$$xf''(x) = \sum_{n=1}^{\infty} (n+1)na_{n+1}x^n.$$

We can also write this as

$$xf''(x) = \sum_{n=0}^{\infty} (n+1)na_{n+1}x^n,$$

since the first term in this last series is zero. (We'll see later that sometimes it is useful to include zero terms at the beginning of a series.)

## Linear Combinations of Power Series

If a power series is multiplied by a constant, then the constant can be placed inside the summation; that is,

$$c\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} ca_n(x-x_0)^n.$$

Two power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

with positive radii of convergence can be added term by term at points common to their open intervals of convergence; thus, if the first series converges for $|x-x_0| < R_1$ and the second converges for $|x-x_0| < R_2$, then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-x_0)^n$$

for $|x-x_0| < R$, where $R$ is the smaller of $R_1$ and $R_2$. More generally, linear combinations of power series can be formed term by term; for example,

$$c_1 f(x) + c_2 f(x) = \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n)(x-x_0)^n.$$

### Example 7.2.5

Find the Maclaurin series for $\cosh x$ as a linear combination of the Maclaurin series for $e^x$ and $e^{-x}$.

**Solution**

By definition,

$$\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}.$$

Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!},$$

it follows that

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{2}[1+(-1)^n]\frac{x^n}{n!}. \tag{7.1.13}$$

Since

$$\frac{1}{2}[1+(-1)^n] = \begin{cases} 1 & \text{if } n = 2m, \text{ an even integer,} \\ 0 & \text{if } n = 2m+1, \text{ an odd integer,} \end{cases}$$

we can rewrite Equation 7.1.13 more simply as

$$\cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}.$$

This result is valid on $(-\infty, \infty)$, since this is the open interval of convergence of the Maclaurin series for $e^x$ and $e^{-x}$.

## Example 7.2.6

Suppose

$$y = \sum_{n=0}^{\infty} a_n x^n$$

on an open interval $I$ that contains the origin.

a. Express

$$(2-x)y'' + 2y$$

as a power series in $x$ on $I$.

b. Use the result of (a) to find necessary and sufficient conditions on the coefficients $\{a_n\}$ for $y$ to be a solution of the homogeneous equation

$$(2-x)y'' + 2y = 0 \tag{7.1.14}$$

on $I$.

### Solution a

From Equation 7.1.7 with $x_0 = 0$,

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$\begin{aligned} (2-x)y'' + 2y &= 2y'' - xy' + 2y \\ &= \textstyle\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n. \end{aligned} \tag{7.1.15}$$

To combine the three series we shift indices in the first two to make their general terms constant multiples of $x^n$; thus,

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n \tag{7.1.16}$$

and

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n = \sum_{n=0}^{\infty}(n+1)na_{n+1}x^n, \tag{7.1.17}$$

where we added a zero term in the last series so that when we substitute from Equation 7.1.16 and Equation 7.1.17 into Equation 7.1.15 all three series will start with $n=0$; thus,

$$(2-x)y'' + 2y = \sum_{n=0}^{\infty}[2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n]x^n. \tag{7.1.18}$$

**Solution b**

From Equation 7.1.18 we see that $y$ satisfies Equation 7.1.14 on $I$ if

$$2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n = 0, \quad n = 0, 1, 2, \ldots. \tag{7.1.19}$$

Conversely, Theorem 7.2.5 b implies that if $y = \sum_{n=0}^{\infty} a_n x^n$ satisfies Equation 7.1.14 on $I$, then Equation 7.1.19 holds.

## Example 7.2.7

Suppose

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

on an open interval $I$ that contains $x_0 = 1$. Express the function

$$(1+x)y'' + 2(x-1)^2 y' + 3y \tag{7.1.20}$$

as a power series in $x-1$ on $I$.

**Solution**

Since we want a power series in $x-1$, we rewrite the coefficient of $y''$ in Equation 7.1.20 as $1+x = 2+(x-1)$ , so Equation 7.1.20 becomes

$$2y'' + (x-1)y'' + 2(x-1)^2 y' + 3y.$$

From Equation 7.1.6 and Equation 7.1.7 with $x_0 = 1$,

$$y' = \sum_{n=1}^{\infty} na_n (x-1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2}.$$

Therefore

$$\begin{aligned} 2y'' &= \sum_{n=2}^{\infty} 2n(n-1)a_n (x-1)^{n-2}, \\ (x-1)y'' &= \sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-1}, \\ 2(x-1)^2 y' &= \sum_{n=1}^{\infty} 2na_n (x-1)^{n+1}, \\ 3y &= \sum_{n=0}^{\infty} 3a_n (x-1)^n. \end{aligned}$$

Before adding these four series we shift indices in the first three so that their general terms become constant multiples of $(x-1)^n$. This yields

$$2y'' = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n, \tag{7.1.21}$$

$$(x-1)y'' = \sum_{n=0}^{\infty} (n+1)na_{n+1}(x-1)^n, \tag{7.1.22}$$

$$2(x-1)^2y' = \sum_{n=1}^{\infty} 2(n-1)a_{n-1}(x-1)^n, \tag{7.1.23}$$

$$3y = \sum_{n=0}^{\infty} 3a_n(x-1)^n, \tag{7.1.24}$$

where we added initial zero terms to the series in Equation 7.1.22 and Equation 7.1.23. Adding Equations 7.1.21–7.1.24 yields

$$\begin{aligned}(1+x)y''+2(x-1)^2y'+3y &= 2y''+(x-1)y''+2(x-1)^2y'+3y\\ &= \sum_{n=0}^{\infty} b_n(x-1)^n,\end{aligned}$$

where

$$b_0 = 4a_2+3a_0, \tag{7.1.25}$$

$$b_n = 2(n+2)(n+1)a_{n+2}+(n+1)na_{n+1}+2(n-1)a_{n-1}+3a_n,\ n\ge 1. \tag{7.1.26}$$

The formula Equation 7.1.25 for $b_0$ can't be obtained by setting $n=0$ in Equation 7.1.26, since the summation in Equation 7.1.23 begins with $n=1$, while those in Equation 7.1.21, Equation 7.1.22, and Equation 7.1.24 begin with $n=0$.

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# 7.1E: Review of Power Series (Exercises)

## Q7.1.1

1.For each power series use Theorem 7.1.3 to find the radius of convergence $R$. If $R>0$, find the open interval of convergence.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n}(x-1)^n$

b. $\sum_{n=0}^{\infty} 2^n n(x-2)^n$

c. $\sum_{n=0}^{\infty} \frac{n!}{9^n}x^n$

d. $\sum_{n=0}^{\infty} \frac{n(n+1)}{16^n}(x-2)^n$

e. $\sum_{n=0}^{\infty} (-1)^n \frac{7^n}{n!}x^n$

f. $\sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}(n+1)^2}(x+7)^n$

2. Suppose there's an integer $M$ such that $b_m \neq 0$ for $m \geq M$, and

$$\lim_{m\to\infty} \left|\frac{b_{m+1}}{b_m}\right| = L,$$

where $0 \leq L \leq \infty$. Show that the radius of convergence of

$$\sum_{m=0}^{\infty} b_m (x-x_0)^{2m}$$

is $R = 1/\sqrt{L}$, which is interpreted to mean that $R=0$ if $L=\infty$ or $R=\infty$ if $L=0$.

3. For each power series, use the result of **Exercise 7.1.2** to find the radius of convergence $R$. If $R>0$, find the open interval of convergence.

a. $\sum_{m=0}^{\infty} (-1)^m (3m+1)(x-1)^{2m+1}$

b. $\sum_{m=0}^{\infty} (-1)^m \frac{m(2m+1)}{2^m}(x+2)^{2m}$

c. $\sum_{m=0}^{\infty} \frac{m!}{(2m)!}(x-1)^{2m}$

d. $\sum_{m=0}^{\infty} (-1)^m \frac{m!}{9^m}(x+8)^{2m}$

e. $\sum_{m=0}^{\infty} (-1)^m \frac{(2m-1)}{3^m}x^{2m+1}$

f. $\sum_{m=0}^{\infty} (x-1)^{2m}$

4. Suppose there's an integer $M$ such that $b_m \neq 0$ for $m \geq M$, and

$$\lim_{m\to\infty} \left|\frac{b_{m+1}}{b_m}\right| = L,$$

where $0 \leq L \leq \infty$. Let $k$ be a positive integer. Show that the radius of convergence of

$$\sum_{m=0}^{\infty} b_m (x-x_0)^{km}$$

is $R = 1/\sqrt[k]{L}$, which is interpreted to mean that $R = 0$ if $L = \infty$ or $R = \infty$ if $L = 0$.

5. For each power series use the result of **Exercise 7.1.4** to find the radius of convergence $R$. If $R > 0$, find the open interval of convergence.

a. $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{(27)^m}(x-3)^{3m+2}$

b. $\displaystyle\sum_{m=0}^{\infty} \frac{x^{7m+6}}{m}$

c. $\displaystyle\sum_{m=0}^{\infty} \frac{9^m(m+1)}{(m+2)}(x-3)^{4m+2}$

d. $\displaystyle\sum_{m=0}^{\infty} (-1)^m \frac{2^m}{m!}x^{4m+3}$

e. $\displaystyle\sum_{m=0}^{\infty} \frac{m!}{(26)^m}(x+1)^{4m+3}$

f. $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m m(m+1)}(x-1)^{3m+1}$

6. Graph $y = \sin x$ and the Taylor polynomial

$$T_{2M+1}(x) = \sum_{n=0}^{M} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

on the interval $(-2\pi, 2\pi)$ for $M = 1, 2, 3, \ldots$, until you find a value of $M$ for which there's no perceptible difference between the two graphs.

7. Graph $y = \cos x$ and the Taylor polynomial

$$T_{2M}(x) = \sum_{n=0}^{M} \frac{(-1)^n x^{2n}}{(2n)!}$$

on the interval $(-2\pi, 2\pi)$ for $M = 1, 2, 3, \ldots$, until you find a value of $M$ for which there's no perceptible difference between the two graphs.

8. Graph $y = 1/(1-x)$ and the Taylor polynomial

$$T_N(x) = \sum_{n=0}^{N} x^n$$

on the interval $[0, .95]$ for $N = 1, 2, 3, \ldots$, until you find a value of $N$ for which there's no perceptible difference between the two graphs. Choose the scale on the $y$-axis so that $0 \le y \le 20$.

9. Graph $y = \cosh x$ and the Taylor polynomial

$$T_{2M}(x) = \sum_{n=0}^{M} \frac{x^{2n}}{(2n)!}$$

on the interval $(-5, 5)$ for $M = 1, 2, 3, \ldots$, until you find a value of $M$ for which there's no perceptible difference between the two graphs. Choose the scale on the $y$-axis so that $0 \le y \le 75$.

10. Graph $y = \sinh x$ and the Taylor polynomial

$$T_{2M+1}(x) = \sum_{n=0}^{M} \frac{x^{2n+1}}{(2n+1)!}$$

on the interval $(-5, 5)$ for $M = 0, 1, 2, \ldots$, until you find a value of $M$ for which there's no perceptible difference between the two graphs. Choose the scale on the $y$-axis so that $-75 \le y \le 75$.

## Q7.1.2

In **Exercises 7.1.11-7.1.15** find a power series solution \(y(x)=\sum_{n=0}^{\infty} a_{n}x^{n} \nonumber \].

11. $(2+x)y''+xy'+3y$
12. $(1+3x^2)y''+3x^2y'-2y$
13. $(1+2x^2)y''+(2-3x)y'+4y$
14. $(1+x^2)y''+(2-x)y'+3y$
15. $(1+3x^2)y''-2xy'+4y$

## Q7.1.3

16. Suppose $y(x)=\sum_{n=0}^{\infty} a_n(x+1)^n$ on an open interval that contains $x_0 = -1$. Find a power series in $x+1$ for

$$xy''+(4+2x)y'+(2+x)y.$$

17. Suppose $y(x)=\sum_{n=0}^{\infty} a_n(x-2)^n$ on an open interval that contains $x_0 = 2$. Find a power series in $x-2$ for

$$x^2y''+2xy'-3xy.$$

18. Do the following experiment for various choices of real numbers $a_0$ and $a_1$.

a. Use differential equations software to solve the initial value problem

$$(2-x)y''+2y=0, \quad y(0)=a_0, \quad y'(0)=a_1,$$

numerically on $(-1.95, 1.95)$ Choose the most accurate method your software package provides. (See Section 10.1 for a brief discussion of one such method.)

b. For $N = 2, 3, 4, \ldots$, compute $a_2, \ldots, a_N$ from Equation 7.1.18 and graph

$$T_N(x)=\sum_{n=0}^{N} a_n x^n$$

and the solution obtained in (a) on the same axes. Continue increasing $N$ until it is obvious that there's no point in continuing. (This sounds vague, but you'll know when to stop.)

19. Follow the directions of **Exercise 7.1.18** for the initial value problem

$$(1+x)y''+2(x-1)^2y'+3y=0, \quad y(1)=a_0, \quad y'(1)=a_1,$$

on the interval $(0, 2)$. Use Equations 7.1.24 and 7.1.25 to compute $\{a_n\}$.

20. Suppose the series $\sum_{n=0}^{\infty} a_n x^n$ converges on an open interval $(-R, R)$, let $r$ be an arbitrary real number, and define

$$y(x)=x^r\sum_{n=0}^{\infty} a_n x^n=\sum_{n=0}^{\infty} a_n x^{n+r}$$

on $(0, R)$. Use Theorem 7.1.4 and the rule for differentiating the product of two functions to show that

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1},\\
y''(x) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2},\\
&\vdots\\
y^{(k)}(x) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)\cdots(n+r-k)a_n x^{n+r-k}
\end{aligned}
$$

on $(0, R)$

## Q7.1.4

21. $x^2(1-x)y''+x(4+x)y'+(2-x)y$
22. $x^2(1+x)y''+x(1+2x)y'-(4+6x)y$
23. $x^2(1+x)y''-x(1-6x-x^2)y'+(1+6x+x^2)y$
24. $x^2(1+3x)y''+x(2+12x+x^2)y'+2x(3+x)y$
25. $x^2(1+2x^2)y''+x(4+2x^2)y'+2(1-x^2)y$
26. $x^2(2+x^2)y''+2x(5+x^2)y'+2(3-x^2)y$

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# 7.2: Series Solutions Near an Ordinary Point I

Many physical applications give rise to second order homogeneous linear differential equations of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \tag{7.2.1}$$

where $P_0$, $P_1$, and $P_2$ are polynomials. Usually the solutions of these equations can't be expressed in terms of familiar elementary functions. Therefore we'll consider the problem of representing solutions of Equation 7.2.1 with series.

We assume throughout that $P_0$, $P_1$ and $P_2$ have no common factors. Then we say that $x_0$ is an **ordinary point** of Equation 7.2.1 if $P_0(x_0) \neq 0$, or a **singular point** if $P_0(x_0) = 0$. For Legendre's equation,

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \tag{7.2.2}$$

$x_0 = 1$ and $x_0 = -1$ are singular points and all other points are ordinary points. For Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

$x_0 = 0$ is a singular point and all other points are ordinary points. If $P_0$ is a nonzero constant as in Airy's equation,

$$y'' - xy = 0, \tag{7.2.3}$$

then every point is an ordinary point.

Since polynomials are continuous everywhere, $P_1/P_0$ and $P_2/P_0$ are continuous at any point $x_0$ that isn't a zero of $P_0$. Therefore, if $x_0$ is an ordinary point of Equation 7.2.1 and $a_0$ and $a_1$ are arbitrary real numbers, then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = a_0, \quad y'(x_0) = a_1 \tag{7.2.4}$$

has a unique solution on the largest open interval that contains $x_0$ and does not contain any zeros of $P_0$. To see this, we rewrite the differential equation in Equation 7.2.4 as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem 5.1.1 with $p = P_1/P_0$ and $q = P_2/P_0$. In this section and the next we consider the problem of representing solutions of Equation 7.2.1 by power series that converge for values of $x$ near an ordinary point $x_0$.

We state the next theorem without proof.

**Theorem 7.3.1**

Suppose $P_0$, $P_1$, and $P_2$ are polynomials with no common factor and $P_0$ isn't identically zero. Let $x_0$ be a point such that $P_0(x_0) \neq 0$, and let $\rho$ be the distance from $x_0$ to the nearest zero of $P_0$ in the complex plane. (If $P_0$ is constant, then $\rho = \infty$.) Then every solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{7.2.5}$$

can be represented by a power series

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{7.2.6}$$

that converges at least on the open interval $(x_0 - \rho, x_0 + \rho)$. ( If $P_0$ is nonconstant, so that $\rho$ is necessarily finite, then the open interval of convergence of (7.2.6) may be larger than $(x_0 - \rho, x_0 + \rho)$. If $P_0$ is constant then $\rho = \infty$ and $(x_0 - \rho, x_0 + \rho) = (-\infty, \infty)$ .

We call Equation 7.2.6 a **power series solution in $x - x_0$** of Equation 7.2.5. We'll now develop a method for finding power series solutions of Equation 7.2.5. For this purpose we write Equation 7.2.5 as $Ly = 0$, where

$$Ly = P_0y'' + P_1y' + P_2y. \tag{7.2.7}$$

Theorem 7.3.1 implies that every solution of $Ly = 0$ on $(x_0 - \rho, x_0 + \rho)$ can be written as

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n.$$

Setting $x=x_0$ in this series and in the series

$$y'=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1}$$

shows that $y(x_0)=a_0$ and $y'(x_0)=a_1$. Since every initial value problem Equation 7.2.4 has a unique solution, this means that $a_0$ and $a_1$ can be chosen arbitrarily, and $a_2$, $a_3$, …are uniquely determined by them.

To find $a_2$, $a_3$, …, we write $P_0$, $P_1$, and $P_2$ in powers of $x-x_0$, substitute

$$\begin{aligned} y&=\sum_{n=0}^{\infty}a_n(x-x_0)^n,\\ y'&=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1},\\ y''&=\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2} \end{aligned}$$

into Equation 7.2.7, and collect the coefficients of like powers of $x-x_0$. This yields

$$Ly=\sum_{n=0}^{\infty}b_n(x-x_0)^n, \tag{7.2.8}$$

where $\{b_0,b_1,\dots,b_n,\dots\}$ are expressed in terms of $\{a_0,a_1,\dots,a_n,\dots\}$ and the coefficients of $P_0$, $P_1$, and $P_2$, written in powers of $x-x_0$. Since Equation 7.2.8 and (a) of Theorem 7.1.6 imply that $Ly=0$ if and only if $b_n=0$ for $n\ge 0$, all power series solutions in $x-x_0$ of $Ly=0$ can be obtained by choosing $a_0$ and $a_1$ arbitrarily and computing $a_2$, $a_3$, …, successively so that $b_n=0$ for $n\ge 0$. For simplicity, we call the power series obtained this way **the power series in $x-x_0$** for the general solution of $Ly=0$, without explicitly identifying the open interval of convergence of the series.

### Example 7.3.1

Let $x_0$ be an arbitrary real number. Find the power series in $x-x_0$ for the general solution of Equations like Equations 7.2.11, 7.2.12, and 7.2.13, which define a given coefficient in the sequence $\{a_n\}$ in terms of one or more coefficients with lesser indices are called **recurrence relations**. When we use a recurrence relation to compute terms of a sequence we are computing **recursively**.

In the remainder of this section we consider the problem of finding power series solutions in $x-x_0$ for equations of the form

$$\left(1+\alpha(x-x_0)^2\right)y''+\beta(x-x_0)y'+\gamma y=0. \tag{7.2.9}$$

Many important equations that arise in applications are of this form with $x_0=0$, including Legendre's equation Equation 7.2.2, Airy's equation Equation 7.2.3, **Chebyshev's equation**,

$$(1-x^2)y''-xy'+\alpha^2y=0,$$

and **Hermite's equation**,

$$y''-2xy'+2\alpha y=0.$$

Since

$$P_0(x)=1+\alpha(x-x_0)^2$$

in Equation 7.2.9, the point $x_0$ is an ordinary point of Equation 7.2.9, and Theorem 7.3.1 implies that the solutions of Equation 7.2.9 can be written as power series in $x-x_0$ that converge on the interval $(x_0-1/\sqrt{|\alpha|},x_0+1/\sqrt{|\alpha|})$ if $\alpha\ne 0$, or on $(-\infty,\infty)$ if $\alpha=0$. We'll see that the coefficients in these power series can be obtained by methods similar to the one used in Example 7.3.1 .

To simplify finding the coefficients, we introduce some notation for products:

$$\prod_{j=r}^{s} b_j = b_r b_{r+1} \cdots b_s \quad \text{if} \quad s \geq r.$$

Thus,

$$\prod_{j=2}^{7} b_j = b_2 b_3 b_4 b_5 b_6 b_7,$$

$$\prod_{j=0}^{4} (2j+1) = (1)(3)(5)(7)(9) = 945,$$

and

$$\prod_{j=2}^{2} j^2 = 2^2 = 4.$$

We define

$$\prod_{j=r}^{s} b_j = 1 \quad \text{if} \quad s < r,$$

no matter what the form of $b_j$.

## Example 7.3.2

Find the power series in $x$ for the general solution of

$$y'' + y = 0. \tag{7.2.10}$$

**Solution**

Here

$$Ly = y'' + y.$$

If

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

so

$$Ly = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} + \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

To collect coefficients of like powers of $x - x_0$, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n + \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

with

$$b_n = (n+2)(n+1) a_{n+2} + a_n.$$

Therefore $Ly = 0$ if and only if

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0, \tag{7.2.11}$$

where $a_0$ and $a_1$ are arbitrary. Since the indices on the left and right sides of Equation 7.2.11 differ by two, we write Equation 7.2.11 separately for $n$ even $(n = 2m)$ and $n$ odd $(n = 2m+1)$. This yields

$$a_{2m+2} = \frac{-a_{2m}}{(2m+2)(2m+1)}, \quad m \geq 0, \tag{7.2.12}$$

and

$$a_{2m+3} = \frac{-a_{2m+1}}{(2m+3)(2m+2)}, \quad m \geq 0. \tag{7.2.13}$$

Computing the coefficients of the even powers of $x - x_0$ from Equation 7.2.12 yields

$$\begin{aligned} a_2 &= -\frac{a_0}{2 \cdot 1} \\ a_4 &= -\frac{a_2}{4 \cdot 3} = -\frac{1}{4 \cdot 3}\left(-\frac{a_0}{2 \cdot 1}\right) = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}, \\ a_6 &= -\frac{a_4}{6 \cdot 5} = -\frac{1}{6 \cdot 5}\left(\frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}\right) = -\frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}, \end{aligned}$$

and, in general,

$$a_{2m} = (-1)^m \frac{a_0}{(2m)!}, \quad m \geq 0. \tag{7.2.14}$$

Computing the coefficients of the odd powers of $x - x_0$ from Equation 7.2.13 yields

$$\begin{aligned} a_3 &= -\frac{a_1}{3 \cdot 2} \\ a_5 &= -\frac{a_3}{5 \cdot 4} = -\frac{1}{5 \cdot 4}\left(-\frac{a_1}{3 \cdot 2}\right) = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, \\ a_7 &= -\frac{a_5}{7 \cdot 6} = -\frac{1}{7 \cdot 6}\left(\frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}\right) = -\frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}, \end{aligned}$$

and, in general,

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!} \quad m \geq 0. \tag{7.2.15}$$

Thus, the general solution of Equation 7.2.10 can be written as

$$y = \sum_{m=0}^{\infty} a_{2m}(x - x_0)^{2m} + \sum_{m=0}^{\infty} a_{2m+1}(x - x_0)^{2m+1},$$

or, from Equation 7.2.14 and Equation 7.2.15, as

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m+1}}{(2m+1)!}. \tag{7.2.16}$$

If we recall from calculus that

$$\sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m}}{(2m)!} = \cos(x - x_0) \quad \text{and} \quad \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m+1}}{(2m+1)!} = \sin(x - x_0),$$

then Equation 7.2.16 becomes

$$y = a_0 \cos(x - x_0) + a_1 \sin(x - x_0),$$

which should look familiar.

$$(1+2x^2)y''+6xy'+2y=0. \tag{7.2.17}$$

**Solution**

Here

$$Ly=(1+2x^2)y''+6xy'+2y.$$

If

$$y=\sum_{n=0}^{\infty}a_nx^n$$

then

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1} \quad \text{and} \quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2},$$

so

$$\begin{aligned} Ly &= (1+2x^2)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+6x\sum_{n=1}^{\infty}na_nx^{n-1}+2\sum_{n=0}^{\infty}a_nx^n \\ &= \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=0}^{\infty}\left[2n(n-1)+6n+2\right]a_nx^n \\ &= \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+2\sum_{n=0}^{\infty}(n+1)^2a_nx^n. \end{aligned}$$

To collect coefficients of $x^n$, we shift the summation index in the first sum. This yields

$$Ly=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+2\sum_{n=0}^{\infty}(n+1)^2a_nx^n=\sum_{n=0}^{\infty}b_nx^n,$$

with

$$b_n=(n+2)(n+1)a_{n+2}+2(n+1)^2a_n, \quad n\ge 0.$$

To obtain solutions of Equation 7.2.17, we set $b_n=0$ for $n\ge 0$. This is equivalent to the recurrence relation

$$a_{n+2}=-2\frac{n+1}{n+2}a_n, \quad n\ge 0. \tag{7.2.18}$$

Since the indices on the left and right differ by two, we write Equation 7.2.18 separately for $n=2m$ and $n=2m+1$, as in Example 7.3.1 . This yields

$$a_{2m+2}=-2\frac{2m+1}{2m+2}a_{2m}=-\frac{2m+1}{m+1}a_{2m}, \quad m\ge 0, \tag{7.2.19}$$

and

$$a_{2m+3}=-2\frac{2m+2}{2m+3}a_{2m+1}=-4\frac{m+1}{2m+3}a_{2m+1}, \quad m\ge 0. \tag{7.2.20}$$

Computing the coefficients of even powers of $x$ from Equation 7.2.19 yields

$$
\begin{aligned}
a_2 &= -\frac{1}{1}a_0,\\
a_4 &= -\frac{3}{2}a_2 = \left(-\frac{3}{2}\right)\left(-\frac{1}{1}\right)a_0 = \frac{1\cdot 3}{1\cdot 2}a_0,\\
a_6 &= -\frac{5}{3}a_4 = -\frac{5}{3}\left(\frac{1\cdot 3}{1\cdot 2}\right)a_0 = -\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}a_0,\\
a_8 &= -\frac{7}{4}a_6 = -\frac{7}{4}\left(-\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\right)a_0 = \frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 3\cdot 4}a_0.
\end{aligned}
$$

In general,

$$
a_{2m} = (-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!}a_0, \quad m \ge 0. \tag{7.2.21}
$$

(Note that Equation 7.2.21 is correct for $m = 0$ because we defined $\prod_{j=1}^{0} b_j = 1$ for any $b_j$.)

Computing the coefficients of odd powers of $x$ from Equation 7.2.20 yields

$$
\begin{aligned}
a_3 &= -4\,\frac{1}{3}a_1,\\
a_5 &= -4\,\frac{2}{5}a_3 = -4\,\frac{2}{5}\left(-4\frac{1}{3}\right)a_1 = 4^2\frac{1\cdot 2}{3\cdot 5}a_1,\\
a_7 &= -4\,\frac{3}{7}a_5 = -4\,\frac{3}{7}\left(4^2\frac{1\cdot 2}{3\cdot 5}\right)a_1 = -4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}a_1,\\
a_9 &= -4\,\frac{4}{9}a_7 = -4\,\frac{4}{9}\left(4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)a_1 = 4^4\frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9}a_1.
\end{aligned}
$$

In general,

$$
a_{2m+1} = \frac{(-1)^m 4^m m!}{\prod_{j=1}^{m}(2j+1)}a_1, \quad m \ge 0. \tag{7.2.22}
$$

From Equation 7.2.21 and Equation 7.2.22,

$$
y = a_0\sum_{m=0}^{\infty}(-1)^m\frac{\prod_{j=1}^{m}(2j-1)}{m!}x^{2m} + a_1\sum_{m=0}^{\infty}(-1)^m\frac{4^m m!}{\prod_{j=1}^{m}(2j+1)}x^{2m+1}.
$$

is the power series in $x$ for the general solution of Equation 7.2.17. Since $P_0(x) = 1 + 2x^2$ has no real zeros, Theorem 5.1.1 implies that every solution of Equation 7.2.17 is defined on $(-\infty, \infty)$. However, since $P_0(\pm i/\sqrt{2}) = 0$, Theorem 7.3.1 implies only that the power series converges in $(-1/\sqrt{2}, 1/\sqrt{2})$ for any choice of $a_0$ and $a_1$.

The results in Examples 7.3.1 and 7.3.2 are consequences of the following general theorem.

### Theorem 7.3.2

The coefficients $\{a_n\}$ in any solution $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ of

$$
\left(1 + \alpha(x - x_0)^2\right)y'' + \beta(x - x_0)y' + \gamma y = 0 \tag{7.2.23}
$$

satisfy the recurrence relation

$$
a_{n+2} = -\frac{p(n)}{(n+2)(n+1)}a_n, \quad n \ge 0, \tag{7.2.24}
$$

where

$$
p(n) = \alpha n(n-1) + \beta n + \gamma. \tag{7.2.25}
$$

Moreover, the coefficients of the even and odd powers of $x-x_0$ can be computed separately as

$$a_{2m+2}=-\frac{p(2m)}{(2m+2)(2m+1)}a_{2m},\quad m\geq 0 \tag{7.2.26}$$

$$a_{2m+3}=-\frac{p(2m+1)}{(2m+3)(2m+2)}a_{2m+1},\quad m\geq 0, \tag{7.2.27}$$

where $a_0$ and $a_1$ are arbitrary.

**Proof**

Here

$$Ly=(1+\alpha(x-x_0)^2)y''+\beta(x-x_0)y'+\gamma y.$$

If

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n,$$

then

$$y'=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1}\quad\text{and}\quad y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}.$$

Hence,

$$\begin{aligned}Ly &= \textstyle\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}+\sum_{n=0}^{\infty}\left[\alpha n(n-1)+\beta n+\gamma\right]a_n(x-x_0)^n\\ &= \textstyle\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}+\sum_{n=0}^{\infty}p(n)a_n(x-x_0)^n,\end{aligned}$$

from Equation 7.2.25 To collect coefficients of powers of $x-x_0$ , we shift the summation index in the first sum. This yields

$$Ly=\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+p(n)a_n\right](x-x_0)^n.$$

Thus, $Ly=0$ if and only if

$$(n+2)(n+1)a_{n+2}+p(n)a_n=0,\quad n\geq 0,$$

which is equivalent to Equation 7.2.24 Writing Equation 7.2.24separately for the cases where $n=2m$ and $n=2m+1$ yields Equation 7.2.26and Equation 7.2.27.

## Example 7.3.3

Find the power series in $x-1$ for the general solution of

$$(2+4x-2x^2)y''-12(x-1)y'-12y=0. \tag{7.2.28}$$

**Solution**

We must first write the coefficient $P_0(x)=2+4x-x^2$ in powers of $x-1$. To do this, we write $x=(x-1)+1$ in $P_0(x)$ and then expand the terms, collecting powers of $x-1$; thus,

$$\begin{aligned}2+4x-2x^2 &= 2+4[(x-1)+1]-2[(x-1)+1]^2\\ &= 4-2(x-1)^2.\end{aligned}$$

Therefore we can rewrite Equation 7.2.28as

$$\left(4-2(x-1)^2\right)y''-12(x-1)y'-12y=0,$$

or, equivalently,

$$\left(1-\frac{1}{2}(x-1)^2\right)y''-3(x-1)y'-3y=0.$$

This is of the form Equation 7.2.23 with $\alpha=-1/2$, $\beta=-3$, and $\gamma=-3$. Therefore, from Equation 7.2.25

$$p(n)=-\frac{n(n-1)}{2}-3n-3=-\frac{(n+2)(n+3)}{2}.$$

Hence, Theorem7.3.2 implies that

$$\begin{aligned}a_{2m+2}&=-\frac{p(2m)}{(2m+2)(2m+1)}a_{2m}\\&=\frac{(2m+2)(2m+3)}{2(2m+2)(2m+1)}a_{2m}=\frac{2m+3}{2(2m+1)}a_{2m},\quad m\ge0\end{aligned}$$

and

$$\begin{aligned}a_{2m+3}&=-\frac{p(2m+1)}{(2m+3)(2m+2)}a_{2m+1}\\&=\frac{(2m+3)(2m+4)}{2(2m+3)(2m+2)}a_{2m+1}=\frac{m+2}{2(m+1)}a_{2m+1},\quad m\ge0.\end{aligned}$$

We leave it to you to show that

$$a_{2m}=\frac{2m+1}{2^m}a_0\quad\text{and}\quad a_{2m+1}=\frac{m+1}{2^m}a_1,\quad m\ge0,$$

which implies that the power series in $x-1$ for the general solution of Equation 7.2.28 is

$$y=a_0\sum_{m=0}^{\infty}\frac{2m+1}{2^m}(x-1)^{2m}+a_1\sum_{m=0}^{\infty}\frac{m+1}{2^m}(x-1)^{2m+1}.$$

In the examples considered so far we were able to obtain closed formulas for coefficients in the power series solutions. In some cases this is impossible, and we must settle for computing a finite number of terms in the series. The next example illustrates this with an initial value problem.

## Example 7.3.4

Compute $a_0$, $a_1$, …, $a_7$ in the series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of the initial value problem

$$(1+2x^2)y''+10xy'+8y=0,\quad y(0)=2,\quad y'(0)=-3. \tag{7.2.29}$$

**Solution**

Since $\alpha=2$, $\beta=10$, and $\gamma=8$ in Equation 7.2.29,

$$p(n)=2n(n-1)+10n+8=2(n+2)^2.$$

Therefore

$$a_{n+2}=-2\frac{(n+2)^2}{(n+2)(n+1)}a_n=-2\frac{n+2}{n+1}a_n,\quad n\ge0.$$

Writing this equation separately for $n=2m$ and $n=2m+1$ yields

$$a_{2m+2}=-2\frac{(2m+2)}{2m+1}a_{2m}=-4\frac{m+1}{2m+1}a_{2m},\quad m\ge0 \tag{7.2.30}$$

$$a_{2m+3}=-2\frac{2m+3}{2m+2}a_{2m+1}=-\frac{2m+3}{m+1}a_{2m+1},\quad m\ge0. \tag{7.2.31}$$

Starting with $a_0 = y(0) = 2$, we compute $a_2, a_4$, and $a_6$ from Equation 7.2.30:

$$\begin{aligned} a_2 &= -4\,\frac{1}{1}2 = -8, \\ a_4 &= -4\,\frac{2}{3}(-8) = \frac{64}{3}, \\ a_6 &= -4\,\frac{3}{5}\left(\frac{64}{3}\right) = -\frac{256}{5}. \end{aligned}$$

Starting with $a_1 = y'(0) = -3$, we compute $a_3, a_5$ and $a_7$ from Equation 7.2.31:

$$\begin{aligned} a_3 &= -\frac{3}{1}(-3) = 9, \\ a_5 &= -\frac{5}{2}9 = -\frac{45}{2}, \\ a_7 &= -\frac{7}{3}\left(-\frac{45}{2}\right) = \frac{105}{2}. \end{aligned}$$

Therefore the solution of Equation 7.2.29 is

$$y = 2 - 3x - 8x^2 + 9x^3 + \frac{64}{3}x^4 - \frac{45}{2}x^5 - \frac{256}{5}x^6 + \frac{105}{2}x^7 + \cdots .$$

## Using Technology

Computing coefficients recursively as in Example 7.3.4 is tedious. We recommend that you do this kind of computation by writing a short program to implement the appropriate recurrence relation on a calculator or computer. You may wish to do this in verifying examples and doing exercises (identified by the symbol $C$) in this chapter that call for numerical computation of the coefficients in series solutions. We obtained the answers to these exercises by using software that can produce answers in the form of rational numbers. However, it is perfectly acceptable - and more practical - to get your answers in decimal form. You can always check them by converting our fractions to decimals.

If you're interested in actually using series to compute numerical approximations to solutions of a differential equation, then whether or not there's a simple closed form for the coefficients is essentially irrelevant. For computational purposes it is usually more efficient to start with the given coefficients $a_0 = y(x_0)$ and $a_1 = y'(x_0)$, compute $a_2$, …, $a_N$ recursively, and then compute approximate values of the solution from the Taylor polynomial

$$T_N(x) = \sum_{n=0}^{N} a_n (x - x_0)^n.$$

The trick is to decide how to choose $N$ so the approximation $y(x) \approx T_N(x)$ is sufficiently accurate on the subinterval of the interval of convergence that you're interested in. In the computational exercises in this and the next two sections, you will often be asked to obtain the solution of a given problem by numerical integration with software of your choice (see Section 10.1 for a brief discussion of one such method), and to compare the solution obtained in this way with the approximations obtained with $T_N$ for various values of $N$. This is a typical textbook kind of exercise, designed to give you insight into how the accuracy of the approximation $y(x) \approx T_N(x)$ behaves as a function of $N$ and the interval that you're working on. In real life, you would choose one or the other of the two methods (numerical integration or series solution). If you choose the method of series solution, then a practical procedure for determining a suitable value of $N$ is to continue increasing $N$ until the maximum of $|T_N - T_{N-1}|$ on the interval of interest is within the margin of error that you're willing to accept.

In doing computational problems that call for numerical solution of differential equations you should choose the most accurate numerical integration procedure your software supports, and experiment with the step size until you're confident that the numerical results are sufficiently accurate for the problem at hand.

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# 7.2E: Series Solutions Near an Ordinary Point I (Exercises)

## Q7.2.1

In **Exercises 7.2.1-7.2.8** find the power series in $x$ for the general solution.

1. $(1+x^2)y''+6xy'+6y=0$
2. $(1+x^2)y''+2xy'-2y=0$
3. $(1+x^2)y''-8xy'+20y=0$
4. $(1-x^2)y''-8xy'-12y=0$
5. $(1+2x^2)y''+7xy'+2y=0$
6. $(1+x^2)y''+2xy'+\frac{1}{4}y=0$
7. $(1-x^2)y''-5xy'-4y=0$
8. $(1+x^2)y''-10xy'+28y=0$

## Q7.2.2

9.

a. Find the power series in $x$ for the general solution of $y''+xy'+2y=0$ .
b. For several choices of $a_0$ and $a_1$, use differential equations software to solve the initial value problem

$$y''+xy'+2y=0, \quad y(0)=a_0, \quad y'(0)=a_1, \tag{A}$$

numerically on $(-5,5)$.
c. For fixed $r$ in $\{1,2,3,4,5\}$graph

$$T_N(x)=\sum_{n=0}^{N} a_n x^n$$

and the solution obtained in (a) on $(-r,r)$. Continue increasing $N$ until there's no perceptible difference between the two graphs.

10. Follow the directions of Exercise [exer:7.2.9} for the differential equation

$$y''+2xy'+3y=0.$$

## Q7.2.3

In **Exercises 7.2.11-7.2.13** find $a_0,\dots,a_N$ for $N$ at least 7 in the power series solution $y=\sum_{n=0}^{\infty} a_n x^n$ of the initial value problem.

11. $(1+x^2)y''+xy'+y=0, \quad y(0)=2, \quad y'(0)=-1$
12. $(1+2x^2)y''-9xy'-6y=0, \quad y(0)=1, \quad y'(0)=-1$
13. $(1+8x^2)y''+2y=0, \quad y(0)=2, \quad y'(0)=-1$

## Q7.2.4

14. Do the next experiment for various choices of real numbers $a_0$, $a_1$, and $r$, with $0<r<1/\sqrt{2}$.

a. Use differential equations software to solve the initial value problem

$$(1-2x^2)y''-xy'+3y=0, \quad y(0)=a_0, \quad y'(0)=a_1, \tag{A}$$

numerically on $(-r,r)$.
b. For $N=2$, 3, 4, …, compute $a_2$, …, $a_N$ in the power series solution $y=\sum_{n=0}^{\infty} a_n x^n$ of (A), and graph

$$T_N(x) = \sum_{n=0}^{N} a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing $N$ until there's no perceptible difference between the two graphs.

15. Do (a) and (b) for several values of $r$ in $(0, 1)$:

a. Use differential equations software to solve the initial value problem

$$(1+x^2)y'' + 10xy' + 14y = 0, \quad y(0) = 5, \quad y'(0) = 1, \tag{A}$$

numerically on $(-r, r)$.

b. For $N = 2, 3, 4, \ldots$, compute $a_2, \ldots, a_N$ in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of (A), and graph

$$T_N(x) = \sum_{n=0}^{N} a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing $N$ until there's no perceptible difference between the two graphs. What happens to the required $N$ as $r \to 1$?

c. Try (a) and (b) with $r = 1.2$. Explain your results.

## Q7.2.5

In **Exercises 7.2.16-7.2.20** find the power series in $x - x_0$ for the general solution.

16. $y'' - y = 0; \quad x_0 = 3$

17. $y'' - (x-3)y' - y = 0; \quad x_0 = 3$

18. $(1 - 4x + 2x^2)y'' + 10(x-1)y' + 6y = 0; \quad x_0 = 1$

19. $(11 - 8x + 2x^2)y'' - 16(x-2)y' + 36y = 0; \quad x_0 = 2$

20. $(5 + 6x + 3x^2)y'' + 9(x+1)y' + 3y = 0; \quad x_0 = -1$

## Q7.2.6

In **Exercises 7.2.21-7.2.26** find $a_0, \ldots a_N$ for $N$ at least 7 in the power series $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for the solution of the initial value problem. Take $x_0$ to be the point where the initial conditions are imposed.

21. $(x^2 - 4)y'' - xy' - 3y = 0, \quad y(0) = -1, \quad y'(0) = 2$

22. $y'' + (x-3)y' + 3y = 0, \quad y(3) = -2, \quad y'(3) = 3$

23. $(5 - 6x + 3x^2)y'' + (x-1)y' + 12y = 0, \quad y(1) = -1, \quad y'(1) = 1$

24. $(4x^2 - 24x + 37)y'' + y = 0, \quad y(3) = 4, \quad y'(3) = -6$

25. $(x^2 - 8x + 14)y'' - 8(x-4)y' + 20y = 0, \quad y(4) = 3, \quad y'(4) = -4$

26. $(2x^2 + 4x + 5)y'' - 20(x+1)y' + 60y = 0, \quad y(-1) = 3, \quad y'(-1) = -3$

## Q7.2.7

27.

a. Find a power series in $x$ for the general solution of

$$(1+x^2)y'' + 4xy' + 2y = 0. \tag{A}$$

b. Use (a) and the formula

$$\frac{1}{1-r} = 1 + r + r^2 + \cdots + r^n + \cdots \quad (-1 < r < 1)$$

for the sum of a geometric series to find a closed form expression for the general solution of (A) on $(-1, 1)$.

c. Show that the expression obtained in (b) is actually the general solution of of (A) on $(-\infty,\infty)$.

28. Use Theorem 7.2.2 to show that the power series in $x$ for the general solution of

$$(1+\alpha x^2)y''+\beta xy'+\gamma y=0$$

is

$$y=a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}p(2j)\right]\frac{x^{2m}}{(2m)!}+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}p(2j+1)\right]\frac{x^{2m+1}}{(2m+1)!}.$$

29. Use **Exercise 7.2.28** to show that all solutions of

$$(1+\alpha x^2)y''+\beta xy'+\gamma y=0$$

are polynomials if and only if

$$\alpha n(n-1)+\beta n+\gamma=\alpha(n-2r)(n-2s-1),$$

where $r$ and $s$ are nonnegative integers.

30.

a. Use Exercise [exer:7.2.28} to show that the power series in $x$ for the general solution of

$$(1-x^2)y''-2bxy'+\alpha(\alpha+2b-1)y=0$$

is $y=a_0y_1+a_1y_2$ , where

$$y_1=\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j-\alpha)(2j+\alpha+2b-1)\right]\frac{x^{2m}}{(2m)!}$$

and

$$y_2=\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j+1-\alpha)(2j+\alpha+2b)\right]\frac{x^{2m+1}}{(2m+1)!}$$

b. Suppose $2b$ isn't a negative odd integer and $k$ is a nonnegative integer. Show that $y_1$ is a polynomial of degree $2k$ such that $y_1(-x)=y_1(x)$ if $\alpha=2k$, while $y_2$ is a polynomial of degree $2k+1$ such that $y_2(-x)=-y_2(-x)$ if $\alpha=2k+1$ . Conclude that if $n$ is a nonnegative integer, then there's a polynomial $P_n$ of degree $n$ such that $P_n(-x)=(-1)^nP_n(x)$ and

$$(1-x^2)P_n''-2bxP_n'+n(n+2b-1)P_n=0. \tag{A}$$

c. Show that (A) implies that

$$[(1-x^2)^bP_n']'=-n(n+2b-1)(1-x^2)^{b-1}P_n,$$

and use this to show that if $m$ and $n$ are nonnegative integers, then

$$[(1-x^2)^bP_n']P_m-[(1-x^2)^bP_m']P_n=[m(m+2b-1)-n(n+2b-1)](1-x^2)^{b-1}P_mP_n \tag{B}$$

d. Now suppose $b>0$. Use (B) and integration by parts to show that if $m\neq n$, then

$$\int_{-1}^{1}(1-x^2)^{b-1}P_m(x)P_n(x)\,dx=0.$$

(We say that $P_m$ and $P_n$ are **orthogonal on** $(-1,1)$ **with respect to the weighting function** $(1-x^2)^{b-1}$ .)

31.

a. Use **Exercise 7.2.28** to show that the power series in $x$ for the general solution of Hermite's equation

$$y''-2xy'+2\alpha y=0$$

is $y=a_0y_1+a_1y_1$ , where

$$y_1 = \sum_{m=0}^{\infty} \left[ \prod_{j=0}^{m-1} (2j-\alpha) \right] \frac{2^m x^{2m}}{(2m)!}$$

and

$$y_2 = \sum_{m=0}^{\infty} \left[ \prod_{j=0}^{m-1} (2j+1-\alpha) \right] \frac{2^m x^{2m+1}}{(2m+1)!}$$

b. Suppose $k$ is a nonnegative integer. Show that $y_1$ is a polynomial of degree $2k$ such that $y_1(-x) = y_1(x)$ if $\alpha = 2k$, while $y_2$ is a polynomial of degree $2k+1$ such that $y_2(-x) = -y_2(-x)$ if $\alpha = 2k+1$ . Conclude that if $n$ is a nonnegative integer then there's a polynomial $P_n$ of degree $n$ such that $P_n(-x) = (-1)^n P_n(x)$ and

$$P_n'' - 2xP_n' + 2nP_n = 0. \tag{A}$$

c. Show that (A) implies that

$$[e^{-x^2} P_n']' = -2ne^{-x^2} P_n,$$

and use this to show that if $m$ and $n$ are nonnegative integers, then

$$[e^{-x^2} P_n']' P_m - [e^{-x^2} P_m']' P_n = 2(m-n)e^{-x^2} P_m P_n. \tag{B}$$

d. Use (B) and integration by parts to show that if $m \neq n$, then

$$\int_{-\infty}^{\infty} e^{-x^2} P_m(x) P_n(x)\, dx = 0.$$

(We say that $P_m$ and $P_n$ are **orthogonal on** $(-\infty, \infty)$ **with respect to the weighting function** $e^{-x^2}$.)

32. Consider the equation

$$\left(1+\alpha x^3\right) y'' + \beta x^2 y' + \gamma x y = 0, \tag{A}$$

and let $p(n) = \alpha n(n-1) + \beta n + \gamma$ . (The special case $y'' - xy = 0$ of (A) is Airy's equation.)

a. Modify the argument used to prove Theorem [thmtype:7.2.2} to show that

$$y = \sum_{n=0}^{\infty} a_n x^n$$

is a solution of (A) if and only if $a_2 = 0$ and

$$a_{n+3} = -\frac{p(n)}{(n+3)(n+2)} a_n, \quad n \geq 0.$$

b. Show from (a) that $a_n = 0$ unless $n = 3m$ or $n = 3m+1$ for some nonnegative integer $m$, and that

$$a_{3m+3} = -\frac{p(3m)}{(3m+3)(3m+2)} a_{3m}, \quad m \geq 0,$$

and

$$a_{3m+4} = -\frac{p(3m+1)}{(3m+4)(3m+3)} a_{3m+1}, \quad m \geq 0,$$

where $a_0$ and $a_1$ may be specified arbitrarily.

c. Conclude from (b) that the power series in $x$ for the general solution of (A) is

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \left[ \prod_{j=0}^{m-1} \frac{p(3j)}{3j+2} \right] \frac{x^{3m}}{3^m m!}$$
$$+ a_1 \sum_{m=0}^{\infty} (-1)^m \left[ \prod_{j=0}^{m-1} \frac{p(3j+1)}{3j+4} \right] \frac{x^{3m+1}}{3^m m!}.$$

## Q7.2.8

In **Exercises 7.2.33-7.2.37** use the method of **Exercise 7.2.32** to find the power series in $x$ for the general solution.

33. $y''-xy=0$

34. $(1-2x^3)y''-10x^2y'-8xy=0$

35. $(1+x^3)y''+7x^2y'+9xy=0$

36. $(1-2x^3)y''+6x^2y'+24xy=0$

37. $(1-x^3)y''+15x^2y'-63xy=0$

## Q7.2.9

38. Consider the equation

$$\left(1+\alpha x^{k+2}\right)y''+\beta x^{k+1}y'+\gamma x^k y=0, \tag{A}$$

where $k$ is a positive integer, and let $p(n)=\alpha n(n-1)+\beta n+\gamma$ .

a. Modify the argument used to prove Theorem 7.2.2 to show that

$$y=\sum_{n=0}^{\infty}a_n x^n$$

is a solution of (A) if and only if $a_n=0$ for $2\le n\le k+1$ and

$$a_{n+k+2}=-\frac{p(n)}{(n+k+2)(n+k+1)}a_n,\quad n\ge 0.$$

b. Show from (a) that $a_n=0$ unless $n=(k+2)m$ or $n=(k+2)m+1$ for some nonnegative integer $m$, and that

$$a_{(k+2)(m+1)}=-\frac{p\left((k+2)m\right)}{(k+2)(m+1)[(k+2)(m+1)-1]}a_{(k+2)m},\quad m\ge 0,$$

and

$$a_{(k+2)(m+1)+1}=-\frac{p\left((k+2)m+1\right)}{[(k+2)(m+1)+1](k+2)(m+1)}a_{(k+2)m+1},\quad m\ge 0,$$

where $a_0$ and $a_1$ may be specified arbitrarily.

c. Conclude from (b) that the power series in $x$ for the general solution of (A) is

$$y=a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\frac{p((k+2)j)}{(k+2)(j+1)-1}\right]\frac{x^{(k+2)m}}{(k+2)^m m!}$$
$$+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\frac{p((k+2)j+1)}{(k+2)(j+1)+1}\right]\frac{x^{(k+2)m+1}}{(k+2)^m m!}.$$

## Q7.2.10

In **Exercises 7.2.39-7.2.44** use the method of **Exercise 7.2.38** to find the power series in $x$ for the general solution.

39. $(1+2x^5)y''+14x^4y'+10x^3y=0$

40. $y''+x^2y=0$

41. $y''+x^6y'+7x^5y=0$

42. $(1+x^8)y''-16x^7y'+72x^6y=0$

43. $(1-x^6)y''-12x^5y'-30x^4y=0$

44. $y''+x^5y'+6x^4y=0$

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# 7.3: Series Solutions Near an Ordinary Point II

In this section we continue to find series solutions

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

of initial value problems

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0,\quad y(x_0)=a_0,\quad y'(x_0)=a_1, \tag{7.3.1}$$

where $P_0$, $P_1$, and $P_2$ are polynomials and $P_0(x_0)\neq 0$, so $x_0$ is an ordinary point of Equation 7.3.1. However, here we consider cases where the differential equation in Equation 7.3.1 is not of the form

$$\left(1+\alpha(x-x_0)^2\right)y''+\beta(x-x_0)y'+\gamma y=0,$$

so Theorem 7.2.2 does not apply, and the computation of the coefficients $\{a_n\}$ is more complicated. For the equations considered here it is difficult or impossible to obtain an explicit formula for $a_n$ in terms of $n$. Nevertheless, we can calculate as many coefficients as we wish. The next three examples illustrate this.

## Example 7.4.1

Find the coefficients $a_0, \ldots, a_7$ in the series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of the initial value problem

$$(1+x+2x^2)y''+(1+7x)y'+2y=0,\quad y(0)=-1,\quad y'(0)=-2. \tag{7.3.2}$$

**Solution**

Here

$$Ly=(1+x+2x^2)y''+(1+7x)y'+2y.$$

The zeros $(-1\pm i\sqrt{7})/4$ of $P_0(x)=1+x+2x^2$ have absolute value $1/\sqrt{2}$, so Theorem 7.2.2 implies that the series solution converges to the solution of Equation 7.3.2 on $(-1/\sqrt{2},1/\sqrt{2})$. Since

$$y=\sum_{n=0}^{\infty}a_nx^n,\quad y'=\sum_{n=1}^{\infty}na_nx^{n-1}\quad\text{and}\quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2},$$

$$\begin{aligned}Ly\ &=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}+2\sum_{n=2}^{\infty}n(n-1)a_nx^n\\&\quad+\sum_{n=1}^{\infty}na_nx^{n-1}+7\sum_{n=1}^{\infty}na_nx^n+2\sum_{n=0}^{\infty}a_nx^n.\end{aligned}$$

Shifting indices so the general term in each series is a constant multiple of $x^n$ yields

$$\begin{aligned}Ly\ &=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)na_{n+1}x^n+2\sum_{n=0}^{\infty}n(n-1)a_nx^n\\&\quad+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+7\sum_{n=0}^{\infty}na_nx^n+2\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}b_nx^n,\end{aligned}$$

where

$$b_n=(n+2)(n+1)a_{n+2}+(n+1)^2a_{n+1}+(n+2)(2n+1)a_n.$$

Therefore $y=\sum_{n=0}^{\infty}a_nx^n$ is a solution of $Ly=0$ if and only if

$$a_{n+2}=-\frac{n+1}{n+2}\,a_{n+1}-\frac{2n+1}{n+1}\,a_n,\ n\geq 0. \tag{7.3.3}$$

From the initial conditions in Equation 7.3.2, $a_0=y(0)=-1$ and $a_1=y'(0)=-2$. Setting $n=0$ in Equation 7.3.3 yields

$$a_2 = -\frac{1}{2}a_1 - a_0 = -\frac{1}{2}(-2) - (-1) = 2.$$

Setting $n = 1$ in Equation 7.3.3 yields

$$a_3 = -\frac{2}{3}a_2 - \frac{3}{2}a_1 = -\frac{2}{3}(2) - \frac{3}{2}(-2) = \frac{5}{3}.$$

We leave it to you to compute $a_4, a_5, a_6, a_7$ from Equation 7.3.3 and show that

$$y = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \frac{61}{8}x^6 - \frac{443}{56}x^7 + \cdots.$$

We also leave it to you (Exercise [exer:7.3.13}) to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^{N} a_n x^n$ converge to the solution of Equation 7.3.2 on $(-1/\sqrt{2}, 1/\sqrt{2})$.

## Example 7.4.2

Find the coefficients $a_0, \ldots, a_5$ in the series solution

$$y = \sum_{n=0}^{\infty} a_n (x+1)^n$$

of the initial value problem

$$(3+x)y'' + (1+2x)y' - (2-x)y = 0, \quad y(-1) = 2, \quad y'(-1) = -3. \tag{7.3.4}$$

Since the desired series is in powers of $x+1$ we rewrite the differential equation in Equation 7.3.4 as $Ly = 0$, with

$$Ly = (2 + (x+1))\, y'' - (1 - 2(x+1))\, y' - (3 - (x+1))\, y.$$

Since

$$y = \sum_{n=0}^{\infty} a_n (x+1)^n, \quad y' = \sum_{n=1}^{\infty} n a_n (x+1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x+1)^{n-2},$$

$$\begin{aligned} Ly \;&= 2\sum_{n=2}^{\infty} n(n-1) a_n (x+1)^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n (x+1)^{n-1} \\ &\quad - \sum_{n=1}^{\infty} n a_n (x+1)^{n-1} + 2\sum_{n=1}^{\infty} n a_n (x+1)^n \\ &\quad - 3\sum_{n=0}^{\infty} a_n (x+1)^n + \sum_{n=0}^{\infty} a_n (x+1)^{n+1}. \end{aligned}$$

Shifting indices so that the general term in each series is a constant multiple of $(x+1)^n$ yields

$$\begin{aligned} Ly \;&= 2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x+1)^n + \sum_{n=0}^{\infty} (n+1) n a_{n+1} (x+1)^n \\ &\quad - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x+1)^n + \sum_{n=0}^{\infty} (2n-3) a_n (x+1)^n + \sum_{n=1}^{\infty} a_{n-1} (x+1)^n \\ &= \sum_{n=0}^{\infty} b_n (x+1)^n, \end{aligned}$$

where

$$b_0 = 4a_2 - a_1 - 3a_0$$

and

$$b_n = 2(n+2)(n+1)a_{n+2} + (n^2-1)a_{n+1} + (2n-3)a_n + a_{n-1}, \quad n \ge 1.$$

Therefore $y = \sum_{n=0}^{\infty} a_n (x+1)^n$ is a solution of $Ly = 0$ if and only if

$$a_2 = \frac{1}{4}(a_1 + 3a_0) \tag{7.3.5}$$

and

$$a_{n+2} = -\frac{1}{2(n+2)(n+1)}\left[(n^2-1)a_{n+1} + (2n-3)a_n + a_{n-1}\right], \quad n \ge 1. \tag{7.3.6}$$

From the initial conditions in Equation 7.3.4, $a_0 = y(-1) = 2$ and $a_1 = y'(-1) = -3$. We leave it to you to compute $a_2$, …, $a_5$ with Equation 7.3.5 and Equation 7.3.6 and show that the solution of Equation 7.3.4 is

$$y = -2 - 3(x+1) + \frac{3}{4}(x+1)^2 - \frac{5}{12}(x+1)^3 + \frac{7}{48}(x+1)^4 - \frac{1}{60}(x+1)^5 + \cdots.$$

We also leave it to you (Exercise [exer:7.3.14]) to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^{N} a_n x^n$ converge to the solution of Equation 7.3.4 on the interval of convergence of the power series solution.

## Example 7.4.3

Find the coefficients $a_0$, …, $a_5$ in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$y'' + 3xy' + (4+2x^2)y = 0, \quad y(0) = 2, \quad y'(0) = -3. \tag{7.3.7}$$

**Solution**

Here

$$Ly = y'' + 3xy' + (4+2x^2)y.$$

Since

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},$$

$$Ly = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3\sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n + 2\sum_{n=0}^{\infty} a_n x^{n+2}.$$

Shifting indices so that the general term in each series is a constant multiple of $x^n$ yields

$$Ly = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(3n+4)a_n x^n + 2\sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{n=0}^{\infty} b_n x^n$$

where

$$b_0 = 2a_2 + 4a_0, \quad b_1 = 6a_3 + 7a_1,$$

and

$$b_n = (n+2)(n+1)a_{n+2} + (3n+4)a_n + 2a_{n-2}, \quad n \ge 2.$$

Therefore $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ if and only if

$$a_2 = -2a_0, \quad a_3 = -\frac{7}{6}a_1, \tag{7.3.8}$$

and

$$a_{n+2} = -\frac{1}{(n+2)(n+1)}\left[(3n+4)a_n + 2a_{n-2}\right], \quad n \ge 2. \tag{7.3.9}$$

From the initial conditions in Equation 7.3.7, $a_0 = y(0) = 2$ and $a_1 = y'(0) = -3$. We leave it to you to compute $a_2$, …, $a_5$ with Equation 7.3.8 and Equation 7.3.9 and show that the solution of Equation 7.3.7 is

$$y = 2 - 3x - 4x^2 + \frac{7}{2}x^3 + 3x^4 - \frac{79}{40}x^5 + \cdots.$$

We also leave it to you (Exercise [exer:7.3.15}) to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^{N} a_n x^n$ converge to the solution of Equation 7.3.9 on the interval of convergence of the power series solution.

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# 7.3E: Series Solutions Near an Ordinary Point II (Exercises)

## Q7.3.1

In **Exercises 7.3.1-7.3.12** find the coefficients $a_0,\ldots,\ a_N$ for $N$ at least 7 in the series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of the initial value problem.

1. $(1+3x)y''+xy'+2y=0,\quad y(0)=2,\quad y'(0)=-3$
2. $(1+x+2x^2)y''+(2+8x)y'+4y=0,\quad y(0)=-1,\quad y'(0)=2$
3. $(1-2x^2)y''+(2-6x)y'-2y=0,\quad y(0)=1,\quad y'(0)=0$
4. $(1+x+3x^2)y''+(2+15x)y'+12y=0,\quad y(0)=0,\quad y'(0)=1$
5. $(2+x)y''+(1+x)y'+3y=0,\quad y(0)=4,\quad y'(0)=3$
6. $(3+3x+x^2)y''+(6+4x)y'+2y=0,\quad y(0)=7,\quad y'(0)=3$
7. $(4+x)y''+(2+x)y'+2y=0,\quad y(0)=2,\quad y'(0)=5$
8. $(2-3x+2x^2)y''-(4-6x)y'+2y=0,\quad y(1)=1,\quad y'(1)=-1$
9. $(3x+2x^2)y''+10(1+x)y'+8y=0,\quad y(-1)=1,\quad y'(-1)=-1$
10. $(1-x+x^2)y''-(1-4x)y'+2y=0,\quad y(1)=2,\quad y'(1)=-1$
11. $(2+x)y''+(2+x)y'+y=0,\quad y(-1)=-2,\quad y'(-1)=3$
12. $x^2y''-(6-7x)y'+8y=0,\quad y(1)=1,\quad y'(1)=-2$

## Q7.3.2

13. Do the following experiment for various choices of real numbers $a_0$, $a_1$, and $r$, with $0<r<1/\sqrt{2}$.

    a. Use differential equations software to solve the initial value problem

    $$(1+x+2x^2)y''+(1+7x)y'+2y=0,\quad y(0)=a_0,\quad y'(0)=a_1, \tag{A}$$

    numerically on $(-r,r)$. (See Example 7.3.1.)

    b. For $N=2,\ 3,\ 4,\ \ldots$, compute $a_2,\ \ldots,\ a_N$ in the power series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of (A), and graph

    $$T_N(x)=\sum_{n=0}^{N}a_nx^n$$

    and the solution obtained in (a) on $(-r,r)$. Continue increasing $N$ until there's no perceptible difference between the two graphs.

14. Do the following experiment for various choices of real numbers $a_0$, $a_1$, and $r$, with $0<r<2$.

    a. Use differential equations software to solve the initial value problem

    $$(3+x)y''+(1+2x)y'-(2-x)y=0,\quad y(-1)=a_0,\quad y'(-1)=a_1, \tag{A}$$

    numerically on $(-1-r,-1+r)$. (See Example 7.3.2). Why this interval?)

    b. For $N=2,\ 3,\ 4,\ \ldots$, compute $a_2,\ldots,a_N$ in the power series solution

    $$y=\sum_{n=0}^{\infty}a_n(x+1)^n$$

    of (A), and graph

    $$T_N(x)=\sum_{n=0}^{N}a_n(x+1)^n$$

and the solution obtained in (a) on $(-1-r,-1+r)$. Continue increasing $N$ until there's no perceptible difference between the two graphs.

15. Do the following experiment for several choices of $a_0$, $a_1$, and $r$, with $r>0$.

a. Use differential equations software to solve the initial value problem

$$y''+3xy'+(4+2x^2)y=0,\quad y(0)=a_0,\quad y'(0)=a_1, \tag{A}$$

numerically on $(-r,r)$. (See Example 7.3.3.)

b. Find the coefficients $a_0$, $a_1$, ..., $a_N$ in the power series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of (A), and graph

$$T_N(x)=\sum_{n=0}^{N}a_nx^n$$

and the solution obtained in (a) on $(-r,r)$. Continue increasing $N$ until there's no perceptible difference between the two graphs.

16. Do the following experiment for several choices of $a_0$ and $a_1$.

a. Use differential equations software to solve the initial value problem

$$(1-x)y''-(2-x)y'+y=0,\quad y(0)=a_0,\quad y'(0)=a_1, \tag{A}$$

numerically on $(-r,r)$.

b. Find the coefficients $a_0$, $a_1$, ..., $a_N$ in the power series solution $y=\sum_{n=0}^{N}a_nx^n$ of (A), and graph

$$T_N(x)=\sum_{n=0}^{N}a_nx^n$$

and the solution obtained in (a) on $(-r,r)$. Continue increasing $N$ until there's no perceptible difference between the two graphs. What happens as you let $r\to 1$?

17. Follow the directions of **Exercise 7.3.16** for the initial value problem

$$(1+x)y''+3y'+32y=0,\quad y(0)=a_0,\quad y'(0)=a_1.$$

18. Follow the directions of **Exercise 7.3.16** for the initial value problem

$$(1+x^2)y''+y'+2y=0,\quad y(0)=a_0,\quad y'(0)=a_1.$$

## Q7.3.3

In **Exercises 7.3.19-7.3.28** find the coefficients $a_0,\dots a_N$ for $N$ at least 7 in the series solution

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

of the initial value problem. Take $x_0$ to be the point where the initial conditions are imposed.

19. $(2+4x)y''-4y'-(6+4x)y=0,\quad y(0)=2,\quad y'(0)=-7$
20. $(1+2x)y''-(1-2x)y'-(3-2x)y=0,\quad y(1)=1,\quad y'(1)=-2$
21. $(5+2x)y''-y'+(5+x)y=0,\quad y(-2)=2,\quad y'(-2)=-1$
22. $(4+x)y''-(4+2x)y'+(6+x)y=0,\quad y(-3)=2,\quad y'(-3)=-2$
23. $(2+3x)y''-xy'+2xy=0,\quad y(0)=-1,\quad y'(0)=2$
24. $(3+2x)y''+3y'-xy=0,\quad y(-1)=2,\quad y'(-1)=-3$
25. $(3+2x)y''-3y'-(2+x)y=0,\quad y(-2)=-2,\quad y'(-2)=3$
26. $(10-2x)y''+(1+x)y=0,\quad y(2)=2,\quad y'(2)=-4$
27. $(7+x)y''+(8+2x)y'+(5+x)y=0,\quad y(-4)=1,\quad y'(-4)=2$

28. $(6+4x)y''+(1+2x)y=0, \quad y(-1)=-1, \quad y'(-1)=2$

## Q7.3.4

29. Show that the coefficients in the power series in $x$ for the general solution of

$$(1+\alpha x+\beta x^2)y''+(\gamma+\delta x)y'+\epsilon y=0$$

satisfy the recurrrence relation

$$a_{n+2}=-\frac{\gamma+\alpha n}{n+2}\,a_{n+1}-\frac{\beta n(n-1)+\delta n+\epsilon}{(n+2)(n+1)}\,a_n.$$

30.

a. Let $\alpha$ and $\beta$ be constants, with $\beta\neq 0$. Show that $y=\sum_{n=0}^{\infty}a_nx^n$ is a solution of

$$(1+\alpha x+\beta x^2)y''+(2\alpha+4\beta x)y'+2\beta y=0 \tag{A}$$

if and only if

$$a_{n+2}+\alpha a_{n+1}+\beta a_n=0, \quad n\geq 0. \tag{B}$$

An equation of this form is called a **second order homogeneous linear difference equation**. The polynomial $p(r)=r^2+\alpha r+\beta$ is called the **characteristic polynomial** of (B). If $r_1$ and $r_2$ are the zeros of $p$, then $1/r_1$ and $1/r_2$ are the zeros of

$$P_0(x)=1+\alpha x+\beta x^2$$

b. Suppose $p(r)=(r-r_1)(r-r_2)$ where $r_1$ and $r_2$ are real and distinct, and let $\rho$ be the smaller of the two numbers $\{1/|r_1|,1/|r_2|\}$ Show that if $c_1$ and $c_2$ are constants then the sequence

$$a_n=c_1r_1^n+c_2r_2^n, \quad n\geq 0$$

satisfies (B). Conclude from this that any function of the form

$$y=\sum_{n=0}^{\infty}(c_1r_1^n+c_2r_2^n)x^n$$

is a solution of (A) on $(-\rho,\rho)$.

c. Use (b) and the formula for the sum of a geometric series to show that the functions

$$y_1=\frac{1}{1-r_1x} \quad \text{and} \quad y_2=\frac{1}{1-r_2x}$$

form a fundamental set of solutions of (A) on $(-\rho,\rho)$.

d. Show that $\{y_1,y_2\}$ is a fundamental set of solutions of (A) on any interval that does'nt contain either $1/r_1$ or $1/r_2$.

e. Suppose $p(r)=(r-r_1)^2$ , and let $\rho=1/|r_1|$. Show that if $c_1$ and $c_2$ are constants then the sequence

$$a_n=(c_1+c_2n)r_1^n, \quad n\geq 0$$

satisfies (B). Conclude from this that any function of the form

$$y=\sum_{n=0}^{\infty}(c_1+c_2n)r_1^nx^n$$

is a solution of (A) on $(-\rho,\rho)$.

f. Use (e) and the formula for the sum of a geometric series to show that the functions

$$y_1=\frac{1}{1-r_1x} \quad \text{and} \quad y_2=\frac{x}{(1-r_1x)^2}$$

form a fundamental set of solutions of (A) on $(-\rho,\rho)$.

g. Show that $\{y_1,y_2\}$ is a fundamental set of solutions of (A) on any interval that does not contain $1/r_1$.

31. Use the results of **Exercise 7.3.30** to find the general solution of the given equation on any interval on which polynomial multiplying $y''$ has no zeros.

a. $(1+3x+2x^2)y''+(6+8x)y'+4y=0$
b. $(1-5x+6x^2)y''-(10-24x)y'+12y=0$
c. $(1-4x+4x^2)y''-(8-16x)y'+8y=0$
d. $(4+4x+x^2)y''+(8+4x)y'+2y=0$
e. $(4+8x+3x^2)y''+(16+12x)y'+6y=0$

## Q7.3.5

In **Exercises 7.3.32-7.3.38** find the coefficients $a_0,\dots,a_N$ for $N$ at least 7 in the series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of the initial value problem.

32. $y''+2xy'+(3+2x^2)y=0,\quad y(0)=1,\quad y'(0)=-2$
33. $y''-3xy'+(5+2x^2)y=0,\quad y(0)=1,\quad y'(0)=-2$
34. $y''+5xy'-(3-x^2)y=0,\quad y(0)=6,\quad y'(0)=-2$
35. $y''-2xy'-(2+3x^2)y=0,\quad y(0)=2,\quad y'(0)=-5$
36. $y''-3xy'+(2+4x^2)y=0,\quad y(0)=3,\quad y'(0)=6$
37. $2y''+5xy'+(4+2x^2)y=0,\quad y(0)=3,\quad y'(0)=-2$
38. $3y''+2xy'+(4-x^2)y=0,\quad y(0)=-2,\quad y'(0)=3$

## Q7.3.6

39. Find power series in $x$ for the solutions $y_1$ and $y_2$ of

$$y''+4xy'+(2+4x^2)y=0$$

such that $y_1(0)=1$, $y_1'(0)=0$, $y_2(0)=0$, $y_2'(0)=1$, and identify $y_1$ and $y_2$ in terms of familiar elementary functions.

## Q7.3.7

In **Exercises 7.3.40-7.3.49** find the coefficients $a_0,\dots,a_N$ for $N$ at least 7 in the series solution

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

of the initial value problem. Take $x_0$ to be the point where the initial conditions are imposed.

40. $(1+x)y''+x^2y'+(1+2x)y=0,\quad y(0)-2,\quad y'(0)=3$
41. $y''+(1+2x+x^2)y'+2y=0,\quad y(0)=2,\quad y'(0)=3$
42. $(1+x^2)y''+(2+x^2)y'+xy=0,\quad y(0)=-3,\quad y'(0)=5$
43. $(1+x)y''+(1-3x+2x^2)y'-(x-4)y=0,\quad y(1)=-2,\quad y'(1)=3$
44. $y''+(13+12x+3x^2)y'+(5+2x),\quad y(-2)=2,\quad y'(-2)=-3$
45. $(1+2x+3x^2)y''+(2-x^2)y'+(1+x)y=0,\quad y(0)=1,\quad y'(0)=-2$
46. $(3+4x+x^2)y''-(5+4x-x^2)y'-(2+x)y=0,\quad y(-2)=2,\quad y'(-2)=-1$
47. $(1+2x+x^2)y''+(1-x)y=0,\quad y(0)=2,\quad y'(0)=-1$
48. $(x-2x^2)y''+(1+3x-x^2)y'+(2+x)y=0,\quad y(1)=1,\quad y'(1)=0$
49. $(16-11x+2x^2)y''+(10-6x+x^2)y'-(2-x)y,\quad y(3)=1,\quad y'(3)=-2$

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# Prelude to Series Solutions of Linear Second Order Equations

In this Chapter, we study a class of second order differential equations that occur in many applications, but cannot be solved in closed form in terms of elementary functions. Here are some examples:

1. **Bessel's equation** which occurs in problems displaying cylindrical symmetry, such as diffraction of light through a circular aperture, propagation of electromagnetic radiation through a coaxial cable, and vibrations of a circular drum head.

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

2. **Airy's equation** which occurs in astronomy and quantum physics.

$$y'' - xy = 0,$$

3. **Legendre's equation** which occurs in problems displaying spherical symmetry, particularly in electromagnetism.

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

These equations and others considered in this chapter can be written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \tag{1}$$

where $P_0$, $P_1$, and $P_2$ are polynomials with no common factor. For most equations that occur in applications, these polynomials are of degree two or less. We'll impose this restriction, although the methods that we'll develop can be extended to the case where the coefficient functions are polynomials of arbitrary degree, or even power series that converge in some circle around the origin in the complex plane.

Since Equation 1 does not in general have closed form solutions, we seek series representations for solutions. We'll see that if $P_0(0) \neq 0$ then solutions of (A) can be written as power series

$$y = \sum_{n=0}^{\infty} a_n x^n$$

that converge in an open interval centered at $x = 0$.

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# CHAPTER OVERVIEW

## 8: Laplace Transforms

IN THIS CHAPTER we study the method of Laplace transforms, which illustrates one of the basic problem solving techniques in mathematics: transform a difficult problem into an easier one, solve the latter, and then use its solution to obtain a solution of the original problem. The method discussed here transforms an initial value problem for a constant coefficient equation into an algebraic equation whose solution can then be used to solve the initial value problem. In some cases this method is merely an alternative procedure for solvingproblems that can be solved equally well by methods that we considered previously; however, in other cases the method of Laplace transforms is more efficient than the methods previously discussed. This is especially true in physical problems dealing with discontinuous forcing functions.


8.1: Introduction to the Laplace Transform

8.1E: Introduction to the Laplace Transform (Exercises)

8.2: The Inverse Laplace Transform

8.2E: The Inverse Laplace Transform (Exercises)

8.3: Solution of Initial Value Problems

8.3E: Solution of Initial Value Problems (Exercises)

8.4: A Brief Table of Laplace Transforms



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# 8.1: Introduction to the Laplace Transform

## Definition of the Laplace Transform

To define the Laplace transform, we first recall the definition of an improper integral. If $g$ is integrable over the interval $[a, T]$ for every $T > a$, then the **improper integral of $g$ over** $[a, \infty)$ is defined as

$$\int_a^\infty g(t)\,dt = \lim_{T\to\infty} \int_a^T g(t)\,dt. \tag{8.1.1}$$

We say that the improper integral **converges** if the limit in Equation 8.1.1 exists; otherwise, we say that the improper integral **diverges** or **does not exist**. Here's the definition of the Laplace transform of a function $f$.

**Defintion 8.1.1 : Laplace Transform**

Let $f$ be defined for $t \geq 0$ and let $s$ be a real number. Then the **Laplace transform** of $f$ is the function $F$ defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \tag{8.1.2}$$

for those values of $s$ for which the improper integral converges.

It is important to keep in mind that the variable of integration in Equation 8.1.2 is $t$, while $s$ is a parameter independent of $t$. We use $t$ as the independent variable for $f$ because in applications the Laplace transform is usually applied to functions of time.

The Laplace transform can be viewed as an operator $\mathcal{L}$ that transforms the function $f = f(t)$ into the function $F = F(s)$. Thus, Equation 8.1.2 can be expressed as

$$F = \mathcal{L}(f).$$

The functions $f$ and $F$ form a **transform pair**, which we'll sometimes denote by

$$f(t) \leftrightarrow F(s).$$

It can be shown that if $F(s)$ is defined for $s = s_0$ then it is defined for all $s > s_0$.

## Computation of Some Simple Laplace Transforms

**Example 8.1.1**

Find the Laplace transform of $f(t) = 1$.

**Solution**

From Equation 8.1.2 with $f(t) = 1$,

$$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{T\to\infty} \int_0^T e^{-st}\,dt.$$

If $s \neq 0$ then

$$\int_0^T e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^T = \frac{1 - e^{-sT}}{s}. \tag{8.1.3}$$

Therefore

$$\lim_{T\to\infty} \int_0^T e^{-st}\,dt = \begin{cases} \frac{1}{s}, & s > 0, \\ \infty, & s < 0. \end{cases} \tag{8.1.4}$$

If $s = 0$ the integrand reduces to the constant 1, and

$$\lim_{T\to\infty}\int_0^T 1\,dt = \lim_{T\to\infty}\int_0^T 1\,dt = \lim_{T\to\infty} T = \infty.$$

Therefore $F(0)$ is undefined, and

$$F(s) = \int_0^\infty e^{-st}\,dt = \frac{1}{s}, \quad s > 0.$$

This result can be written in operator notation as

$$\mathcal{L}(1) = \frac{1}{s}, \quad s > 0,$$

or as the transform pair

$$1 \leftrightarrow \frac{1}{s}, \quad s > 0.$$

**Note**

It is convenient to combine the steps of integrating from 0 to $T$ and letting $T \to \infty$. Therefore, instead of writing Equation 8.1.3 and 8.1.4 as separate steps we write

$$\int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \begin{cases} \frac{1}{s}, & s > 0 \\ \infty, & s < 0 \end{cases}$$

We'll follow this practice throughout this chapter.

**Example 8.1.2**

Find the Laplace transform of $f(t) = t$.

**Solution**

From Equation 8.1.2 with $f(t) = t$,

$$F(s) = \int_0^\infty e^{-st}\,t\,dt. \tag{8.1.5}$$

If $s \neq 0$, integrating by parts yields

$$\int_0^\infty e^{-st}\,t\,dt = -\frac{te^{-st}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = -\left[\frac{t}{s} + \frac{1}{s^2}\right]e^{-st}\Big|_0^\infty$$
$$= \begin{cases} \frac{1}{s^2}, & s > 0, \\ \infty, & s < 0. \end{cases}$$

If $s = 0$, the integral in Equation 8.1.5 becomes

$$\int_0^\infty t\,dt = \frac{t^2}{2}\Big|_0^\infty = \infty.$$

Therefore $F(0)$ is undefined and

$$F(s) = \frac{1}{s^2}, \quad s > 0.$$

This result can also be written as

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad s > 0,$$

or as the transform pair

$$t \leftrightarrow \frac{1}{s^2}, \quad s > 0.$$

### Example 8.1.3

Find the Laplace transform of $f(t) = e^{at}$, where $a$ is a constant.

**Solution**

From Equation 8.1.2 with $f(t) = e^{at}$,

$$F(s) = \int_0^\infty e^{-st} e^{at}\, dt.$$

Combining the exponentials yields

$$F(s) = \int_0^\infty e^{-(s-a)t}\, dt.$$

However, we know from Example 8.1.1 that

$$\int_0^\infty e^{-st}\, dt = \frac{1}{s}, \quad s > 0.$$

Replacing $s$ by $s - a$ here shows that

$$F(s) = \frac{1}{s-a}, \quad s > a.$$

This can also be written as

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad s > a, \text{ or } e^{at} \leftrightarrow \frac{1}{s-a}, \quad s > a.$$

### Example 8.1.4

[Find the Laplace transforms of $f(t) = \sin \omega t$ and $g(t) = \cos \omega t$, where $\omega$ is a constant.

**Solution**

Define

$$F(s) = \int_0^\infty e^{-st} \sin \omega t\, dt \tag{8.1.6}$$

and

$$G(s) = \int_0^\infty e^{-st} \cos \omega t\, dt. \tag{8.1.7}$$

If $s > 0$, integrating Equation 8.1.6 by parts yields

$$F(s) = -\frac{e^{-st}}{s} \sin \omega t \Big|_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t\, dt,$$

so

$$F(s) = \frac{\omega}{s} G(s). \tag{8.1.8}$$

If $s > 0$, integrating Equation 8.1.7 by parts yields

$$G(s) = -\frac{e^{-st} \cos \omega t}{s} \Big|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t\, dt,$$

so

$$G(s) = \frac{1}{s} - \frac{\omega}{s} F(s).$$

Now substitute from Equation 8.1.8 into this to obtain

$$G(s) = \frac{1}{s} - \frac{\omega^2}{s^2} G(s).$$

Solving this for $G(s)$ yields

$$G(s) = \frac{s}{s^2 + \omega^2}, \quad s > 0.$$

This and Equation 8.1.8 imply that

$$F(s) = \frac{\omega}{s^2 + \omega^2}, \quad s > 0.$$

## Tables of Laplace Transforms

Extensive tables of Laplace transforms have been compiled and are commonly used in applications. The brief table of Laplace transforms in the Appendix will be adequate for our purposes.

### Example 8.1.5

Use the table of Laplace transforms to find $\mathcal{L}(t^3 e^{4t})$.

**Solution**

The table includes the transform pair

$$t^n e^{at} \leftrightarrow \frac{n!}{(s-a)^{n+1}}.$$

Setting $n = 3$ and $a = 4$ here yields

$$\mathcal{L}(t^3 e^{4t}) = \frac{3!}{(s-4)^4} = \frac{6}{(s-4)^4}.$$

We'll sometimes write Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms.

## Linearity of the Laplace Transform

The next theorem presents an important property of the Laplace transform.

### Theorem 8.1.2 Linearity Property

Suppose $\mathcal{L}(f_i)$ is defined for $s > s_i$, $1 \le i \le n)$. Let $s_0$ be the largest of the numbers $s_1$, $s_2$, ...,$s_n$, and let $c_1$, $c_2$,..., $c_n$ be constants. Then

$$\mathcal{L}(c_1 f_1 + c_2 f_2 + \cdots + c_n f_n) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2) + \cdots + c_n \mathcal{L}(f_n) \text{ for } s > s_0.$$

**Proof**

We give the proof for the case where $n = 2$. If $s > s_0$ then

$$\begin{aligned} \mathcal{L}(c_1 f_1 + c_2 f_2) &= \int_0^\infty e^{-st} \left(c_1 f_1(t) + c_2 f_2(t)\right) dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt \\ &= c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2). \end{aligned}$$

## Example 8.1.6

Use Theorem 8.1.2 and the known Laplace transform

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

to find $\mathcal{L}(\cosh bt)$ $(b \neq 0)$.

**Solution**

By definition,

$$\cosh bt = \frac{e^{bt} + e^{-bt}}{2}.$$

Therefore

$$\begin{aligned} \mathcal{L}(\cosh bt) &= \mathcal{L}\left(\tfrac{1}{2}e^{bt} + \tfrac{1}{2}e^{-bt}\right) \\ &= \tfrac{1}{2}\mathcal{L}(e^{bt}) + \tfrac{1}{2}\mathcal{L}(e^{-bt}) \qquad \text{(linearity property)} \\ &= \tfrac{1}{2}\,\tfrac{1}{s-b} + \tfrac{1}{2}\,\tfrac{1}{s+b}, \end{aligned} \tag{8.1.9}$$

where the first transform on the right is defined for $s > b$ and the second for $s > -b$; hence, both are defined for $s > |b|$. Simplifying the last expression in Equation 8.1.9 yields

$$\mathcal{L}(\cosh bt) = \frac{s}{s^2 - b^2}, \quad s > |b|.$$

The next theorem enables us to start with known transform pairs and derive others. (For other results of this kind, see **Exercises 8.1.6** and **8.1.13**.)

## Theorem 8.1.3 First Shifting Theorem

If

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{8.1.10}$$

is the Laplace transform of $f(t)$ for $s > s_0$, then $F(s-a)$ is the Laplace transform of $e^{at}f(t)$ for $s > s_0 + a$.

**Proof**

Replacing $s$ by $s-a$ in Equation 8.1.10 yields

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt \tag{8.1.11}$$

if $s - a > s_0$; that is, if $s > s_0 + a$. However, Equation 8.1.11 can be rewritten as

$$F(s-a) = \int_0^\infty e^{-st}\left(e^{at}f(t)\right)\, dt,$$

which implies the conclusion.

## Example 8.1.7

Use Theorem 8.1.3 and the known Laplace transforms of $1$, $t$, $\cos \omega t$, and $\sin \omega t$ to find

$$\mathcal{L}(e^{at}), \quad \mathcal{L}(te^{at}), \quad \mathcal{L}(e^{\lambda t}\sin \omega t), \text{ and } \mathcal{L}(e^{\lambda t}\cos \omega t).$$

**Solution**

In the following table the known transform pairs are listed on the left and the required transform pairs listed on the right are obtained by applying Theorem 8.1.3 .

Table 8.1.1

| $f(t) \leftrightarrow F(s)$ | $e^{at}f(t) \leftrightarrow F(s-a)$ |
|---|---|
| $1 \leftrightarrow \frac{1}{s}, \quad s>0$ | $e^{at} \leftrightarrow \frac{1}{(s-a)}, \quad s>a$ |
| $t \leftrightarrow \frac{1}{s^2}, \quad s>0$ | $te^{at} \leftrightarrow \frac{1}{(s-a)^2}, \quad s>a$ |
| $\sin \omega t \leftrightarrow \frac{\omega}{s^2+\omega^2}, \quad s>0$ | $e^{\lambda t}\sin \omega t \leftrightarrow \frac{\omega}{(s-\lambda)^2+\omega^2}, \quad s>\lambda$ |
| $\cos \omega t \leftrightarrow \frac{s}{s^2+\omega^2}, \quad s>0$ | $e^{\lambda t}\sin \omega t \leftrightarrow \frac{s-\lambda}{(s-\lambda)^2+\omega^2}, \quad s>\lambda$ |

## Existence of Laplace Transforms

Not every function has a Laplace transform. For example, it can be shown (**Exercise 8.1.3**) that

$$\int_0^\infty e^{-st}e^{t^2}\,dt = \infty$$

for every real number $s$. Hence, the function $f(t) = e^{t^2}$ does not have a Laplace transform.

Our next objective is to establish conditions that ensure the existence of the Laplace transform of a function. We first review some relevant definitions from calculus.

Recall that a limit

$$\lim_{t\to t_0} f(t)$$

exists if and only if the one-sided limits

$$\lim_{t\to t_0-} f(t) \quad \text{and} \quad \lim_{t\to t_0+} f(t)$$

both exist and are equal; in this case,

$$\lim_{t\to t_0} f(t) = \lim_{t\to t_0-} f(t) = \lim_{t\to t_0+} f(t).$$

Recall also that $f$ is continuous at a point $t_0$ in an open interval $(a, b)$ if and only if

$$\lim_{t\to t_0} f(t) = f(t_0),$$

which is equivalent to

$$\lim_{t\to t_0+} f(t) = \lim_{t\to t_0-} f(t) = f(t_0). \tag{8.1.12}$$

For simplicity, we define

$$f(t_0+) = \lim_{t\to t_0+} f(t) \quad \text{and} \quad f(t_0-) = \lim_{t\to t_0-} f(t),$$

so Equation 8.1.12 can be expressed as

$$f(t_0+) = f(t_0-) = f(t_0).$$

If $f(t_0+)$ and $f(t_0-)$ have finite but distinct values, we say that $f$ has a **jump discontinuity** at $t_0$, and

$$f(t_0+) - f(t_0-)$$

is called the **jump** in $f$ at $t_0$ (Figure 8.1.1 ).

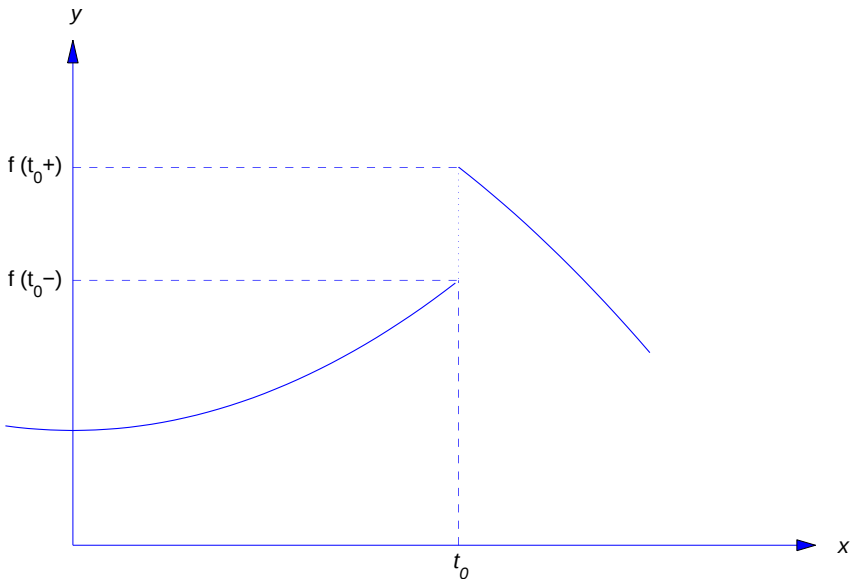


Figure 8.1.1 : A jump discontinuity.

If $f(t_0+)$ and $f(t_0-)$ are finite and equal, but either $f$ isn't defined at $t_0$ or it is defined but

$$f(t_0) \neq f(t_0+) = f(t_0-),$$

we say that $f$ has a **removable discontinuity** at $t_0$ (Figure 8.1.2 ). This terminology is appropriate since a function $f$ with a removable discontinuity at $t_0$ can be made continuous at $t_0$ by defining (or redefining)

$$f(t_0) = f(t_0+) = f(t_0-).$$

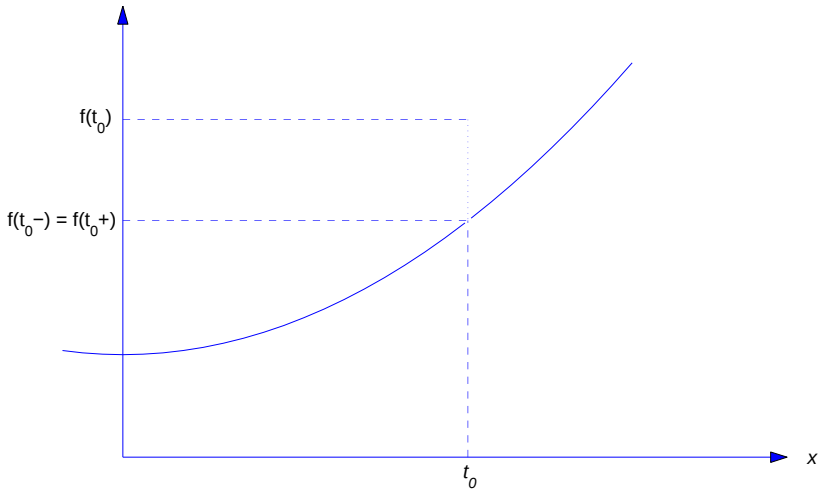


Figure 8.1.2

**Note**

We know from calculus that a definite integral is not affected by changing the values of its integrand at isolated points. Therefore, redefining a function f to make it continuous at removable discontinuities does not change $\mathcal{L}(f)$.

**Definition 8.1.4 : Piecewise Continuous**

- A function $f$ is said to be **piecewise continuous** on a finite closed interval $[0, T]$ if $f(0+)$ and $f(T-)$ are finite and $f$ is continuous on the open interval $(0, T)$ except possibly at finitely many points, where $f$ may have jump discontinuities or removable discontinuities.
- A function $f$ is said to be **piecewise continuous** on the infinite interval $[0, \infty)$ if it is piecewise continuous on $[0, T]$ for every $T > 0$.

Figure 8.1.3 shows the graph of a typical piecewise continuous function.

It is shown in calculus that if a function is piecewise continuous on a finite closed interval then it is integrable on that interval. But if $f$ is piecewise continuous on $[0,\infty)$, then so is $e^{-st}f(t)$, and therefore

$$\int_0^T e^{-st}f(t)\,dt$$

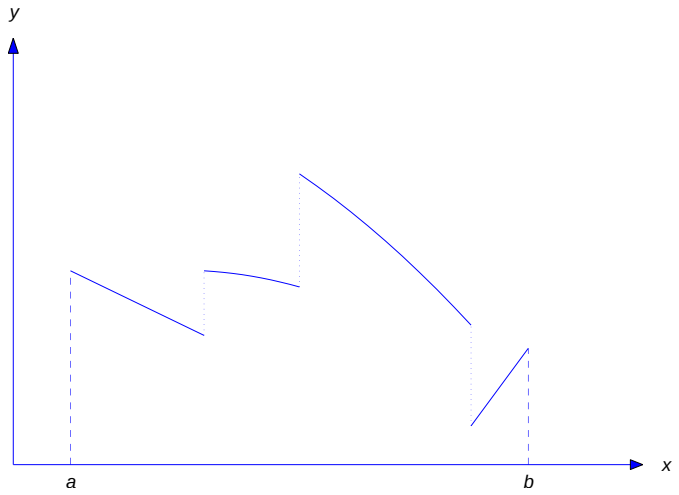


Figure 8.1.3

exists for every $T>0$. However, piecewise continuity alone does not guarantee that the improper integral

$$\int_0^\infty e^{-st}f(t)\,dt=\lim_{T\to\infty}\int_0^T e^{-st}f(t)\,dt \tag{8.1.13}$$

converges for $s$ in some interval $(s_0,\infty)$. For example, we noted earlier that Equation 8.1.13 diverges for all $s$ if $f(t)=e^{t^2}$. Stated informally, this occurs because $e^{t^2}$ increases too rapidly as $t\to\infty$. The next definition provides a constraint on the growth of a function that guarantees convergence of its Laplace transform for $s$ in some interval $(s_0,\infty)$.

**Definition 8.1.5 : of exponential order**

A function $f$ is said to be **of exponential order** $s_0$ if there are constants $M$ and $t_0$ such that

$$|f(t)|\le Me^{s_0t},\quad t\ge t_0. \tag{8.1.14}$$

In situations where the specific value of $s_0$ is irrelevant we say simply that $f$ is **of exponential order**.

The next theorem gives useful sufficient conditions for a function $f$ to have a Laplace transform. The proof is sketched in **Exercise 8.1.10**.

**Theorem 8.1.6**

If $f$ is piecewise continuous on $[0,\infty)$ and of exponential order $s_0$, then $\mathcal{L}(f)$ is defined for $s>s_0$ .

**Note**

We emphasize that the conditions of Theorem 8.1.6 are sufficient, but **not necessary**, for $f$ to have a Laplace transform. For example, **Exercise 8.1.14(c)** shows that $f$ may have a Laplace transform even though $f$ isn't of exponential order

**Example 8.1.8**

If $f$ is bounded on some interval $[t_0,\infty)$, say

$$|f(t)| \le M, \quad t \ge t_0,$$

then Equation 8.1.14 holds with $s_0 = 0$, so $f$ is of exponential order zero. Thus, for example, $\sin \omega t$ and $\cos \omega t$ are of exponential order zero, and Theorem 8.1.6 implies that $\mathcal{L}(\sin \omega t)$ and $\mathcal{L}(\cos \omega t)$ exist for $s > 0$. This is consistent with the conclusion of Example 8.1.4 .

## Example 8.1.9

It can be shown that if $\lim_{t \to \infty} e^{-s_0 t} f(t)$ exists and is finite then $f$ is of exponential order $s_0$ (**Exercise 8.1.9**). If $\alpha$ is any real number and $s_0 > 0$ then $f(t) = t^\alpha$ is of exponential order $s_0$, since

$$\lim_{t \to \infty} e^{-s_0 t} t^\alpha = 0,$$

by L'Hôpital's rule. If $\alpha \ge 0$, $f$ is also continuous on $[0, \infty)$. Therefore **Exercise 8.1.9** and Theorem 8.1.6 imply that $\mathcal{L}(t^\alpha)$ exists for $s \ge s_0$ . However, since $s_0$ is an arbitrary positive number, this really implies that $\mathcal{L}(t^\alpha)$ exists for all $s > 0$. This is consistent with the results of Example 8.1.2 and **Exercises 8.1.6** and **8.1.8**.

## Example 8.1.10

Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -3e^{-t}, & t \ge 1. \end{cases}$$

**Solution**

Since $f$ is defined by different formulas on $[0, 1)$ and $[1, \infty)$, we write

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}(1)\, dt + \int_1^\infty e^{-st}(-3e^{-t})\, dt.$$

Since

$$\int_0^1 e^{-st}\, dt = \begin{cases} \frac{1 - e^{-s}}{s} & s \ne 0 \\ 1 & s = 0 \end{cases}$$

and

$$\int_1^\infty e^{-st}(-3e^{-t})\, dt = -3 \int_1^\infty e^{-(s+1)t}\, dt = -\frac{3e^{-(s+1)}}{s+1}, \quad s > -1,$$

it follows that

$$F(s) = \begin{cases} \frac{1 - e^{-s}}{s} - 3\frac{e^{-(s+1)}}{s+1} & s > -1, s \ne 0 \\ 1 - \frac{3}{e} & s = 0 \end{cases}$$

This is consistent with Theorem 8.1.6 , since

$$|f(t)| \le 3e^{-t}, \quad t \ge 1,$$

and therefore $f$ is of exponential order $s_0 = -1$ .

## Note

In Section 8.4 we'll develop a more efficient method for finding Laplace transforms of piecewise continuous functions.

## Example 8.1.11

We stated earlier that

$$\int_0^\infty e^{-st}e^{t^2}\,dt = \infty$$

for all $s$, so Theorem 8.1.6 implies that $f(t) = e^{t^2}$ is not of exponential order, since

$$\lim_{t\to\infty} \frac{e^{t^2}}{Me^{s_0 t}} = \lim_{t\to\infty} \frac{1}{M} e^{t^2 - s_0 t} = \infty,$$

so

$$e^{t^2} > Me^{s_0 t}$$

for sufficiently large values of $t$, for any choice of $M$ and $s_0$ (**Exercise 8.1.3**).

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# 8.1E: Introduction to the Laplace Transform (Exercises)

## Q8.1.1

1. Find the Laplace transforms of the following functions by evaluating the integral $F(s) = \int_0^\infty e^{-st} f(t)\, dt$ .

a. $t$
b. $te^{-t}$
c. $\sinh bt$
d. $e^{2t} - 3e^t$
e. $t^2$

2. Use the table of Laplace transforms to find the Laplace transforms of the following functions.

a. $\cosh t \sin t$
b. $\sin^2 t$
c. $\cos^2 2t$
d. $\cosh^2 t$
e. $t \sinh 2t$
f. $\sin t \cos t$
g. $\sin\left(t + \frac{\pi}{4}\right)$
h. $\cos 2t - \cos 3t$
i. $\sin 2t + \cos 4t$

3. Show that $\int_0^\infty e^{-st} e^{t^2}\, dt = \infty$ for every real number $s$.

4. Graph the following piecewise continuous functions and evaluate $f(t+)$, $f(t-)$, and $f(t)$ at each point of discontinuity.

a. $f(t) = \begin{cases} -t, & 0 \le t < 2, \\ t-4, & 2 \le t < 3, \\ 1, & t \ge 3. \end{cases}$

b. $f(t) = \begin{cases} t^2 + 2, & 0 \le t < 1, \\ 4, & t = 1, \\ t, & t > 1. \end{cases}$

c. $f(t) = \begin{cases} \sin t, & 0 \le t < \pi/2, \\ 2\sin t, & \pi/2 \le t < \pi, \\ \cos t, & t \ge \pi. \end{cases}$

d. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 2, & t = 1, \\ 2 - t, & 1 \le t < 2, \\ 3, & t = 2, \\ 6, & t > 2. \end{cases}$

5. Find the Laplace transform:

a. $f(t) = \begin{cases} e^{-t}, & 0 \le t < 1, \\ e^{-2t}, & t \ge 1. \end{cases}$

b. $f(t) = \begin{cases} 1, & 0 \le t < 4, \\ t, & t \ge 4. \end{cases}$

c. $f(t)=\begin{cases} t, & 0\le t<1,\\ 1, & t\ge 1.\end{cases}$

d. $f(t)=\begin{cases} te^t, & 0\le t<1,\\ e^t, & t\ge 1.\end{cases}$

6. Prove that if $f(t)\leftrightarrow F(s)$ then $t^k f(t)\leftrightarrow (-1)^k F^{(k)}(s)$. HINT: **Assume that it's permissible to differentiate the integral $\int_0^\infty e^{-st}f(t)dt$ with respect to $s$ under the integral sign.**

7. Use the known Laplace transforms

$$\mathcal{L}(e^{\lambda t}\sin\omega t)=\frac{\omega}{(s-\lambda)^2+\omega^2}\quad\text{and}\quad \mathcal{L}(e^{\lambda t}\cos\omega t)=\frac{s-\lambda}{(s-\lambda)^2+\omega^2}$$

and the result of **Exercise 8.1.6** to find $\mathcal{L}(te^{\lambda t}\cos\omega t)$ and $\mathcal{L}(te^{\lambda t}\sin\omega t)$.

8. Use the known Laplace transform $\mathcal{L}(1)=1/s$ and the result of **Exercise 8.1.6** to show that

$$\mathcal{L}(t^n)=\frac{n!}{s^{n+1}},\quad n=\text{ integer.}$$

9. Exponential order:

a. Show that if $\lim_{t\to\infty}e^{-s_0t}f(t)$ exists and is finite then $f$ is of exponential order $s_0$.

b. Show that if $f$ is of exponential order $s_0$ then $\lim_{t\to\infty}e^{-st}f(t)=0$ for all $s>s_0$.

c. Show that if $f$ is of exponential order $s_0$ and $g(t)=f(t+\tau)$ where $\tau>0$, then $g$ is also of exponential order $s_0$.

10. Recall the next theorem from calculus.

**Theorem 8.1E.1**

Let $g$ be integrable on $[0,T]$ for every $T>0$. Suppose there's a function $w$ defined on some interval $[\tau,\infty)$ (with $\tau\ge 0$) such that $|g(t)|\le w(t)$ for $t\ge\tau$ and $\int_\tau^\infty w(t)\,dt$ converges. Then $\int_0^\infty g(t)\,dt$ converges.

Use Theorem 8.1E.1 to show that if $f$ is piecewise continuous on $[0,\infty)$ and of exponential order $s_0$, then $f$ has a Laplace transform $F(s)$ defined for $s>s_0$.

11. Prove: If $f$ is piecewise continuous and of exponential order then $\lim_{s\to\infty}F(s)=0$.

12. Prove: If $f$ is continuous on $[0,\infty)$ and of exponential order $s_0>0$, then

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)=\frac{1}{s}\mathcal{L}(f),\quad s>s_0.$$

HINT: **Use integration by parts to evaluate the transform on the left.**

13. Suppose $f$ is piecewise continuous and of exponential order, and that $\lim_{t\to 0+}f(t)/t$ exists. Show that

$$\mathcal{L}\left(\frac{f(t)}{t}\right)=\int_s^\infty F(r)\,dr.$$

HINT: **Use the results of Exercises 8.1.6 and 8.1.11.**

14. Suppose $f$ is piecewise continuous on $[0,\infty)$.

a. Prove: If the integral $g(t)=\int_0^t e^{-s_0\tau}f(\tau)\,d\tau$ satisfies the inequality $|g(t)|\le M$ $(t\ge 0)$, then $f$ has a Laplace transform $F(s)$ defined for $s>s_0$. HINT: **Use integration by parts to show that**

$$\int_0^T e^{-st} f(t)dt = e^{-(s-s_0)T} g(T) + (s-s_0) \int_0^T e^{-(s-s_0)t} g(t)dt$$

b. Show that if $\mathcal{L}(f)$ exists for $s = s_0$ then it exists for $s > s_0$. Show that the function $f(t) = te^{t^2} \cos(e^{t^2})$ has a Laplace transform defined for $s > 0$, even though $f$ isn't of exponential order.

c. Show that the function $f(t) = te^{t^2} \cos(e^{t^2})$ has a Laplace transform defined for $s > 0$, even though $f$ isn't of exponential order.

15. Use the table of Laplace transforms and the result of **Exercise 8.1.13** to find the Laplace transforms of the following functions.

a. $\dfrac{\sin \omega t}{t} \quad (\omega > 0)$

b. $\dfrac{\cos \omega t - 1}{t} \quad (\omega > 0)$

c. $\dfrac{e^{at} - e^{bt}}{t}$

d. $\dfrac{\cosh t - 1}{t}$

e. $\dfrac{\sinh^2 t}{t}$

16. The **gamma function** is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx,$$

which can be shown to converge if $\alpha > 0$.

a. Use integration by parts to show that

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \alpha > 0.$$

b. Show that $\Gamma(n+1) = n!$ if $n = 1, 2, 3, \ldots$.

c. From (b) and the table of Laplace transforms,

$$\mathcal{L}(t^\alpha) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad s > 0,$$

if $\alpha$ is a nonnegative integer. Show that this formula is valid for any $\alpha > -1$. HINT: **Change the variable of integration in the integral for** $\Gamma(\alpha+1)$.

17. Suppose $f$ is continuous on $[0, T]$ and $f(t+T) = f(t)$ for all $t \geq 0$. (We say in this case that $f$ is **periodic with period** $T$.)

a. Conclude from Theorem 8.1.6 that the Laplace transform of $f$ is defined for $s > 0$.

b. Show that

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt, \quad s > 0.$$

HINT: **Write**

$$F(s) = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)dt$$

**Then show that**

$$\int_{nT}^{(n+1)T} e^{-st} f(t)dt = e^{-nsT} \int_0^T e^{-st} f(t)dt$$

**and recall the formula for the sum of a geometric series.**

18. Use the formula given in **Exercise 8.1.17b** to find the Laplace transforms of the given periodic functions:

a. $f(t)=\begin{cases} t, & 0\le t<1,\\ 2-t, & 1\le t<2,\end{cases}$ $\qquad f(t+2)=f(t), \quad t\ge 0$

b. $f(t)=\begin{cases} 1, & 0\le t<\frac{1}{2},\\ -1, & \frac{1}{2}\le t<1,\end{cases}$ $\qquad f(t+1)=f(t), \quad t\ge 0$

c. $f(t)=|\sin t|$

d. $f(t)=\begin{cases} \sin t, & 0\le t<\pi,\\ 0, & \pi\le t<2\pi,\end{cases}$ $\qquad f(t+2\pi)=f(t)$

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# 8.2: The Inverse Laplace Transform

## Definition of the Inverse Laplace Transform

In Section 8.1 we defined the Laplace transform of $f$ by

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt.$$

We'll also say that $f$ is an **inverse Laplace Transform** of $F$, and write

$$f = \mathcal{L}^{-1}(F).$$

To solve differential equations with the Laplace transform, we must be able to obtain $f$ from its transform $F$. There's a formula for doing this, but we can't use it because it requires the theory of functions of a complex variable. Fortunately, we can use the table of Laplace transforms to find inverse transforms that we'll need.

### Example 8.2.1

Use the table of Laplace transforms to find

a. $$\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right)$$

b. $$\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right).$$

**Solution a**

Setting $b = 1$ in the transform pair

$$\sinh bt \leftrightarrow \frac{b}{s^2 - b^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) = \sinh t.$$

**Solution b**

Setting $\omega = 3$ in the transform pair

$$\cos \omega t \leftrightarrow \frac{s}{s^2 + \omega^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t.$$

The next theorem enables us to find inverse transforms of linear combinations of transforms in the table. We omit the proof.

### Theorem 8.2.1 : Linearity Property

If $F_1$, $F_2$, …, $F_n$ are Laplace transforms and $c_1$, $c_2$, …, $c_n$ are constants, then

$$\mathcal{L}^{-1}(c_1F_1 + c_2F_2 + \cdots + c_nF_n) = c_1\mathcal{L}^{-1}(F_1) + c_2\mathcal{L}^{-1}(F_2) + \cdots + c_n\mathcal{L}^{-1}F_n.$$

### Example 8.2.2

Find

$$\mathcal{L}^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right).$$

**Solution**

From the table of Laplace transforms in Section 8.8,,

$$e^{at}\leftrightarrow\frac{1}{s-a}\quad\text{and}\quad\sin\omega t\leftrightarrow\frac{\omega}{s^2+\omega^2}.$$

Theorem 8.2.1 with $a=-5$ and $\omega=\sqrt{3}$ yields

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right)&=8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)+7\mathcal{L}^{-1}\left(\frac{1}{s^2+3}\right)\\&=8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)+\frac{7}{\sqrt{3}}\mathcal{L}^{-1}\left(\frac{\sqrt{3}}{s^2+3}\right)\\&=8e^{-5t}+\frac{7}{\sqrt{3}}\sin\sqrt{3}t.\end{aligned}$$

## Example 8.2.3

Find

$$\mathcal{L}^{-1}\left(\frac{3s+8}{s^2+2s+5}\right).$$

**Solution**

Completing the square in the denominator yields

$$\frac{3s+8}{s^2+2s+5}=\frac{3s+8}{(s+1)^2+4}.$$

Because of the form of the denominator, we consider the transform pairs

$$e^{-t}\cos 2t\leftrightarrow\frac{s+1}{(s+1)^2+4}\quad\text{and}\quad e^{-t}\sin 2t\leftrightarrow\frac{2}{(s+1)^2+4},$$

and write

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{3s+8}{(s+1)^2+4}\right)&=\mathcal{L}^{-1}\left(\frac{3s+3}{(s+1)^2+4}\right)+\mathcal{L}^{-1}\left(\frac{5}{(s+1)^2+4}\right)\\&=3\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right)+\frac{5}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right)\\&=e^{-t}(3\cos 2t+\frac{5}{2}\sin 2t).\end{aligned}$$

## Note

We'll often write inverse Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms in Section 8.8.

### Inverse Laplace Transforms of Rational Functions

Using the Laplace transform to solve differential equations often requires finding the inverse transform of a rational function

$$F(s)=\frac{P(s)}{Q(s)},$$

where $P$ and $Q$ are polynomials in $s$ with no common factors. Since it can be shown that $\lim_{s\to\infty} F(s) = 0$ if $F$ is a Laplace transform, we need only consider the case where $\text{degree}(P) < \text{degree}(Q)$. To obtain $\mathcal{L}^{-1}(F)$, we find the partial fraction expansion of $F$, obtain inverse transforms of the individual terms in the expansion from the table of Laplace transforms, and use the linearity property of the inverse transform. The next two examples illustrate this.

### Example 8.2.4

Find the inverse Laplace transform of

$$F(s) = \frac{3s+2}{s^2-3s+2}. \tag{8.2.1}$$

**Solution**

(Method 1)

Factoring the denominator in Equation 8.2.1 yields

$$F(s) = \frac{3s+2}{(s-1)(s-2)}. \tag{8.2.2}$$

The form for the partial fraction expansion is

$$\frac{3s+2}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}. \tag{8.2.3}$$

Multiplying this by $(s-1)(s-2)$ yields

$$3s+2 = (s-2)A + (s-1)B.$$

Setting $s=2$ yields $B=8$ and setting $s=1$ yields $A=-5$. Therefore

$$F(s) = -\frac{5}{s-1} + \frac{8}{s-2}$$

and

$$\mathcal{L}^{-1}(F) = -5\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + 8\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = -5e^t + 8e^{2t}.$$

(Method 2) We don't really have to multiply Equation 8.2.3 by $(s-1)(s-2)$ to compute $A$ and $B$. We can obtain $A$ by simply ignoring the factor $s-1$ in the denominator of Equation 8.2.2 and setting $s=1$ elsewhere; thus,

$$A = \left.\frac{3s+2}{s-2}\right|_{s=1} = \frac{3\cdot 1+2}{1-2} = -5. \tag{8.2.4}$$

Similarly, we can obtain $B$ by ignoring the factor $s-2$ in the denominator of Equation 8.2.2 and setting $s=2$ elsewhere; thus,

$$B = \left.\frac{3s+2}{s-1}\right|_{s=2} = \frac{3\cdot 2+2}{2-1} = 8. \tag{8.2.5}$$

To justify this, we observe that multiplying Equation 8.2.3 by $s-1$ yields

$$\frac{3s+2}{s-2} = A + (s-1)\frac{B}{s-2},$$

and setting $s=1$ leads to Equation 8.2.4. Similarly, multiplying Equation 8.2.3 by $s-2$ yields

$$\frac{3s+2}{s-1} = (s-2)\frac{A}{s-2} + B$$

and setting $s=2$ leads to Equation 8.2.5. (It isn't necesary to write the last two equations. We wrote them only to justify the shortcut procedure indicated in Equation 8.2.4 and Equation 8.2.5.)

The shortcut employed in the second solution of Example 8.2.4 is **Heaviside's method**. The next theorem states this method formally. For a proof and an extension of this theorem, see **Exercise 8.2.10**.

**Theorem 8.2.2**

Suppose

$$F(s) = \frac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)}, \tag{8.2.6}$$

where $s_1, s_2, \ldots, s_n$ are distinct and $P$ is a polynomial of degree less than $n$. Then

$$F(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n},$$

where $A_i$ can be computed from Equation 8.2.6 by ignoring the factor $s-s_i$ and setting $s = s_i$ elsewhere.

**Example 8.2.5**

Find the inverse Laplace transform of

$$F(s) = \frac{6+(s+1)(s^2-5s+11)}{s(s-1)(s-2)(s+1)}. \tag{8.2.7}$$

**Solution**

The partial fraction expansion of Equation 8.2.7 is of the form

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{s+1}. \tag{8.2.8}$$

To find $A$, we ignore the factor $s$ in the denominator of Equation 8.2.7 and set $s = 0$ elsewhere. This yields

$$A = \frac{6+(1)(11)}{(-1)(-2)(1)} = \frac{17}{2}.$$

Similarly, the other coefficients are given by

$$B = \frac{6+(2)(7)}{(1)(-1)(2)} = -10,$$

$$C = \frac{6+3(5)}{2(1)(3)} = \frac{7}{2},$$

and

$$D = \frac{6}{(-1)(-2)(-3)} = -1.$$

Therefore

$$F(s) = \frac{17}{2}\,\frac{1}{s} - \frac{10}{s-1} + \frac{7}{2}\,\frac{1}{s-2} - \frac{1}{s+1}$$

and

$$\begin{aligned}\mathcal{L}^{-1}(F) &= \frac{17}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - 10\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{7}{2}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)\\ &= \frac{17}{2} - 10e^t + \frac{7}{2}e^{2t} - e^{-t}.\end{aligned}$$

**Note**

We didn't "multiply out" the numerator in Equation 8.2.7 before computing the coefficients in Equation 8.2.8 , since it wouldn't simplify the computations.

## Example 8.2.6

Find the inverse Laplace transform of

$$F(s)=\frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2}. \tag{8.2.9}$$

**Solution**

The form for the partial fraction expansion is

$$F(s)=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}. \tag{8.2.10}$$

Because of the repeated factor $(s+2)^2$ in Equation 8.2.9, Heaviside's method doesn't work. Instead, we find a common denominator in Equation 8.2.10. This yields

$$F(s)=\frac{A(s+2)^2+B(s+1)(s+2)+C(s+1)}{(s+1)(s+2)^2}. \tag{8.2.11}$$

If Equation 8.2.9 and Equation 8.2.11 are to be equivalent, then

$$A(s+2)^2+B(s+1)(s+2)+C(s+1)=8-(s+2)(4s+10). \tag{8.2.12}$$

The two sides of this equation are polynomials of degree two. From a theorem of algebra, they will be equal for all $s$ if they are equal for any three distinct values of $s$. We may determine $A$, $B$ and $C$ by choosing convenient values of $s$.

The left side of Equation 8.2.12 suggests that we take $s=-2$ to obtain $C=-8$, and $s=-1$ to obtain $A=2$. We can now choose any third value of $s$ to determine $B$. Taking $s=0$ yields $4A+2B+C=-12$ . Since $A=2$ and $C=-8$ this implies that $B=-6$. Therefore

$$F(s)=\frac{2}{s+1}-\frac{6}{s+2}-\frac{8}{(s+2)^2}$$

and

$$\begin{aligned}\mathcal{L}^{-1}(F) &=2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)-6\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)-8\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)\\ &=2e^{-t}-6e^{-2t}-8te^{-2t}.\end{aligned}$$

## Example 8.2.7

Find the inverse Laplace transform of

$$F(s)=\frac{s^2-5s+7}{(s+2)^3}.$$

**Solution**

The form for the partial fraction expansion is

$$F(s)=\frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{C}{(s+2)^3}.$$

The easiest way to obtain $A$, $B$, and $C$ is to expand the numerator in powers of $s+2$ . This yields

$$s^2-5s+7=[(s+2)-2]^2-5[(s+2)-2]+7=(s+2)^2-9(s+2)+21.$$

Therefore

$$F(s) = \frac{(s+2)^2 - 9(s+2) + 21}{(s+2)^3}$$
$$= \frac{1}{s+2} - \frac{9}{(s+2)^2} + \frac{21}{(s+2)^3}$$

and

$$\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 9\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) + \frac{21}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+2)^3}\right)$$
$$= e^{-2t}\left(1 - 9t + \frac{21}{2}t^2\right).$$

## Example 8.2.8

Find the inverse Laplace transform of

$$F(s) = \frac{1 - s(5+3s)}{s\left[(s+1)^2+1\right]}. \tag{8.2.13}$$

**Solution**

One form for the partial fraction expansion of $F$ is

$$F(s) = \frac{A}{s} + \frac{Bs+C}{(s+1)^2+1}. \tag{8.2.14}$$

However, we see from the table of Laplace transforms that the inverse transform of the second fraction on the right of Equation 8.2.14 will be a linear combination of the inverse transforms

$$e^{-t}\cos t \quad \text{and} \quad e^{-t}\sin t$$

of

$$\frac{s+1}{(s+1)^2+1} \quad \text{and} \quad \frac{1}{(s+1)^2+1}$$

respectively. Therefore, instead of Equation 8.2.14 we write

$$F(s) = \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2+1}. \tag{8.2.15}$$

Finding a common denominator yields

$$F(s) = \frac{A\left[(s+1)^2+1\right] + B(s+1)s + Cs}{s\left[(s+1)^2+1\right]}. \tag{8.2.16}$$

If Equation 8.2.13 and Equation 8.2.16 are to be equivalent, then

$$A\left[(s+1)^2+1\right] + B(s+1)s + Cs = 1 - s(5+3s).$$

This is true for all $s$ if it is true for three distinct values of $s$. Choosing $s = 0$, $-1$, and $1$ yields the system

$$\begin{aligned} 2A &= 1 \\ A - C &= 3 \\ 5A + 2B + C &= -7. \end{aligned}$$

Solving this system yields

$$A = \frac{1}{2}, \quad B = -\frac{7}{2}, \quad C = -\frac{5}{2}.$$

Hence, from Equation 8.2.15,

$$F(s) = \frac{1}{2s} - \frac{7}{2}\,\frac{s+1}{(s+1)^2+1} - \frac{5}{2}\,\frac{1}{(s+1)^2+1}.$$

Therefore

$$\begin{aligned}\mathcal{L}^{-1}(F) &= \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{7}{2}\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - \frac{5}{2}\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right)\\ &= \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t.\end{aligned}$$

## Example 8.2.9

Find the inverse Laplace transform of

$$F(s) = \frac{8+3s}{(s^2+1)(s^2+4)}. \tag{8.2.17}$$

### Solution

The form for the partial fraction expansion is

$$F(s) = \frac{A+Bs}{s^2+1} + \frac{C+Ds}{s^2+4}.$$

The coefficients $A$, $B$, $C$ and $D$ can be obtained by finding a common denominator and equating the resulting numerator to the numerator in Equation 8.2.17. However, since there's no first power of $s$ in the denominator of Equation 8.2.17, there's an easier way: the expansion of

$$F_1(s) = \frac{1}{(s^2+1)(s^2+4)}$$

can be obtained quickly by using Heaviside's method to expand

$$\frac{1}{(x+1)(x+4)} = \frac{1}{3}\left(\frac{1}{x+1} - \frac{1}{x+4}\right)$$

and then setting $x = s^2$ to obtain

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right).$$

Multiplying this by $8+3s$ yields

$$F(s) = \frac{8+3s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{8+3s}{s^2+1} - \frac{8+3s}{s^2+4}\right).$$

Therefore

$$\mathcal{L}^{-1}(F) = \frac{8}{3}\sin t + \cos t - \frac{4}{3}\sin 2t - \cos 2t.$$

## Using Technology

Some software packages that do symbolic algebra can find partial fraction expansions very easily. We recommend that you use such a package if one is available to you, but only after you've done enough partial fraction expansions on your own to master the technique.

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# 8.2E: The Inverse Laplace Transform (Exercises)

## Q8.2.1

1. Use the table of Laplace transforms to find the inverse Laplace transform.

a. $\dfrac{3}{(s-7)^4}$

b. $\dfrac{2s-4}{s^2-4s+13}$

c. $\dfrac{1}{s^2+4s+20}$

d. $\dfrac{2}{s^2+9}$

e. $\dfrac{s^2-1}{(s^2+1)^2}$

f. $\dfrac{1}{(s-2)^2-4}$

g. $\dfrac{12s-24}{(s^2-4s+85)^2}$

h. $\dfrac{2}{(s-3)^2-9}$

i. $\dfrac{s^2-4s+3}{(s^2-4s+5)^2}$

2. Use Theorem 8.2.1 and the table of Laplace transforms to find the inverse Laplace transform.

a. $\dfrac{2s+3}{(s-7)^4}$

b. $\dfrac{s^2-1}{(s-2)^6}$

c. $\dfrac{s+5}{s^2+6s+18}$

d. $\dfrac{2s+1}{s^2+9}$

e. $\dfrac{s}{s^2+2s+1}$

f. $\dfrac{s+1}{s^2-9}$

g. $\dfrac{s^3+2s^2-s-3}{(s+1)^4}$

h. $\dfrac{2s+3}{(s-1)^2+4}$

i. $\dfrac{1}{s}-\dfrac{s}{s^2+1}$

j. $\dfrac{3s+4}{s^2-1}$

k. $\dfrac{3}{s-1}+\dfrac{4s+1}{s^2+9}$

l. $\dfrac{3}{(s+2)^2}-\dfrac{2s+6}{s^2+4}$

3. Use Heaviside's method to find the inverse Laplace transform.

a. $\dfrac{3-(s+1)(s-2)}{(s+1)(s+2)(s-2)}$

b. $\dfrac{7+(s+4)(18-3s)}{(s-3)(s-1)(s+4)}$

c. $\dfrac{2+(s-2)(3-2s)}{(s-2)(s+2)(s-3)}$

d. $\dfrac{3-(s-1)(s+1)}{(s+4)(s-2)(s-1)}$

e. $\dfrac{3+(s-2)(10-2s-s^2)}{(s-2)(s+2)(s-1)(s+3)}$

f. $\dfrac{3+(s-3)(2s^2+s-21)}{(s-3)(s-1)(s+4)(s-2)}$

4. Find the inverse Laplace transform.

a. $\dfrac{2+3s}{(s^2+1)(s+2)(s+1)}$

b. $\dfrac{3s^2+2s+1}{(s^2+1)(s^2+2s+2)}$

c. $\dfrac{3s+2}{(s-2)(s^2+2s+5)}$

d. $\dfrac{3s^2+2s+1}{(s-1)^2(s+2)(s+3)}$

e. $\dfrac{2s^2+s+3}{(s-1)^2(s+2)^2}$

f. $\dfrac{3s+2}{(s^2+1)(s-1)^2}$

5. Use the method of Example 8.2.9 to find the inverse Laplace transform.

a. $\dfrac{3s+2}{(s^2+4)(s^2+9)}$

b. $\dfrac{-4s+1}{(s^2+1)(s^2+16)}$

c. $\dfrac{5s+3}{(s^2+1)(s^2+4)}$

d. $\dfrac{-s+1}{(4s^2+1)(s^2+1)}$

e. $\dfrac{17s-34}{(s^2+16)(16s^2+1)}$

f. $\dfrac{2s-1}{(4s^2+1)(9s^2+1)}$

6. Find the inverse Laplace transform.

a. $\dfrac{17s-15}{(s^2-2s+5)(s^2+2s+10)}$

b. $\dfrac{8s+56}{(s^2-6s+13)(s^2+2s+5)}$

c. $\dfrac{s+9}{(s^2+4s+5)(s^2-4s+13)}$

d. $\dfrac{3s-2}{(s^2-4s+5)(s^2-6s+13)}$

e. $\dfrac{3s-1}{(s^2-2s+2)(s^2+2s+5)}$

f. $\dfrac{20s+40}{(4s^2-4s+5)(4s^2+4s+5)}$

7. Find the inverse Laplace transform.

a. $\dfrac{1}{s(s^2+1)}$

b. $\dfrac{1}{(s-1)(s^2-2s+17)}$

c. $\dfrac{3s+2}{(s-2)(s^2+2s+10)}$

d. $\dfrac{34-17s}{(2s-1)(s^2-2s+5)}$

e. $\dfrac{s+2}{(s-3)(s^2+2s+5)}$

f. $\dfrac{2s-2}{(s-2)(s^2+2s+10)}$

8. Find the inverse Laplace transform.

a. $\dfrac{2s+1}{(s^2+1)(s-1)(s-3)}$

b. $\dfrac{s+2}{(s^2+2s+2)(s^2-1)}$

c. $\dfrac{2s-1}{(s^2-2s+2)(s+1)(s-2)}$

d. $\dfrac{s-6}{(s^2-1)(s^2+4)}$

e. $\dfrac{2s-3}{s(s-2)(s^2-2s+5)}$

f. $\dfrac{5s-15}{(s^2-4s+13)(s-2)(s-1)}$

9. Given that $f(t) \leftrightarrow F(s)$, find the inverse Laplace transform of $F(as-b)$, where $a>0$.

10. a. If $s_1$, $s_2$, …, $s_n$ are distinct and $P$ is a polynomial of degree less than $n$, then

$$P(s)(s-s_1)(s-s_2)\cdots(s-s_n) = A_1 s-s_1 + A_2 s-s_2 + \cdots + A_n s-s_n.$$

Multiply through by $s-s_i$ to show that $A_i$ can be obtained by ignoring the factor $s-s_i$ on the left and setting $s=s_i$ elsewhere.

b. Suppose $P$ and $Q_1$ are polynomials such that degree$(P) \le$ degree$(Q_1)$ and $Q_1(s_1) \neq 0$. Show that the coefficient of $1/(s-s_1)$ in the partial fraction expansion of

$$F(s) = P(s)(s-s_1)Q_1(s)$$

is $P(s_1)/Q_1(s_1)$.

c. Explain how the results of (a) and (b) are related.

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# 8.3: Solution of Initial Value Problems

## Laplace Transforms of Derivatives

In the rest of this chapter we'll use the Laplace transform to solve initial value problems for constant coefficient second order equations. To do this, we must know how the Laplace transform of $f'$ is related to the Laplace transform of $f$. The next theorem answers this question.

**Theorem 8.3.1**

Suppose $f$ is continuous on $[0,\infty)$ and of exponential order $s_0$, and $f'$ is piecewise continuous on $[0,\infty)$. Then $f$ and $f'$ have Laplace transforms for $s > s_0$, and

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0). \tag{8.3.1}$$

**Proof**

We know from Theorem 8.1.6 that $\mathcal{L}(f)$ is defined for $s > s_0$ . We first consider the case where $f'$ is continuous on $[0,\infty)$. Integration by parts yields

$$\begin{aligned}\int_0^T e^{-st} f'(t)\,dt &= e^{-st} f(t)\Big|_0^T + s\int_0^T e^{-st} f(t)\,dt \\ &= e^{-sT} f(T) - f(0) + s\int_0^T e^{-st} f(t)\,dt \end{aligned} \tag{8.3.2}$$

for any $T > 0$. Since $f$ is of exponential order $s_0$, $\lim_{T\to\infty} e^{-sT} f(T) = 0$ and the integral in on the right side of Equation 8.3.2converges as $T \to \infty$ if $s > s_0$ . Therefore

$$\begin{aligned}\int_0^\infty e^{-st} f'(t)\,dt &= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt \\ &= -f(0) + s\mathcal{L}(f),\end{aligned}$$

which proves Equation 8.3.1.

Suppose $T > 0$ and $f'$ is only piecewise continuous on $[0,T]$, with discontinuities at $t_1 < t_2 < \cdots < t_{n-1}$ . For convenience, let $t_0 = 0$ and $t_n = T$. Integrating by parts yields

$$\begin{aligned}\int_{t_{i-1}}^{t_i} e^{-st} f'(t)\,dt &= e^{-st} f(t)\Big|_{t_{i-1}}^{t_i} + s\int_{t_{i-1}}^{t_i} e^{-st} f(t)\,dt \\ &= e^{-st_i} f(t_i) - e^{-st_{i-1}} f(t_{i-1}) + s\int_{t_{i-1}}^{t_i} e^{-st} f(t)\,dt.\end{aligned}$$

Summing both sides of this equation from $i = 1$ to $n$ and noting that

$$\begin{aligned}&\left(e^{-st_1} f(t_1) - e^{-st_0} f(t_0)\right) + \left(e^{-st_2} f(t_2) - e^{-st_1} f(t_1)\right) + \cdots + \left(e^{-st_N} f(t_N) - e^{-st_{N-1}} f(t_{N-1})\right) \\ &= e^{-st_N} f(t_N) - e^{-st_0} f(t_0) = e^{-sT} f(T) - f(0)\end{aligned}$$

yields Equation 8.3.2, so Equation 8.3.1follows as before.

**Example 8.3.1**

In Example 8.1.4 we saw that

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}.$$

**Solution**

Applying Equation 8.3.1 with $f(t)=\cos\omega t$ shows that

$$\mathcal{L}(-\omega\sin\omega t)=s\frac{s}{s^2+\omega^2}-1=-\frac{\omega^2}{s^2+\omega^2}.$$

Therefore

$$\mathcal{L}(\sin\omega t)=\frac{\omega}{s^2+\omega^2},$$

which agrees with the corresponding result obtained in 8.1.4.

In Section 2.1 we showed that the solution of the initial value problem

$$y'=ay,\quad y(0)=y_0, \tag{8.3.3}$$

is $y=y_0e^{at}$. We'll now obtain this result by using the Laplace transform.

Let $Y(s)=\mathcal{L}(y)$ be the Laplace transform of the unknown solution of Equation 8.3.3. Taking Laplace transforms of both sides of Equation 8.3.3 yields

$$\mathcal{L}(y')=\mathcal{L}(ay),$$

which, by Theorem 8.3.1 , can be rewritten as

$$s\mathcal{L}(y)-y(0)=a\mathcal{L}(y),$$

or

$$sY(s)-y_0=aY(s).$$

Solving for $Y(s)$ yields

$$Y(s)=\frac{y_0}{s-a},$$

so

$$y=\mathcal{L}^{-1}(Y(s))=\mathcal{L}^{-1}\left(\frac{y_0}{s-a}\right)=y_0\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)=y_0e^{at},$$

which agrees with the known result.

We need the next theorem to solve second order differential equations using the Laplace transform.

**Theorem 8.3.2**

Suppose $f$ and $f'$ are continuous on $[0,\infty)$ and of exponential order $s_0$, and that $f''$ is piecewise continuous on $[0,\infty)$. Then $f$, $f'$, and $f''$ have Laplace transforms for $s>s_0$ ,

$$\mathcal{L}(f')=s\mathcal{L}(f)-f(0), \tag{8.3.4}$$

and

$$\mathcal{L}(f'')=s^2\mathcal{L}(f)-f'(0)-sf(0). \tag{8.3.5}$$

**Proof**

Theorem 8.3.1 implies that $\mathcal{L}(f')$ exists and satisfies Equation 8.3.4 for $s>s_0$ . To prove that $\mathcal{L}(f'')$ exists and satisfies Equation 8.3.5 for $s>s_0$ , we first apply Theorem 8.3.1 to $g=f'$ . Since $g$ satisfies the hypotheses of Theorem 8.3.1 , we conclude that $\mathcal{L}(g')$ is defined and satisfies

$$\mathcal{L}(g')=s\mathcal{L}(g)-g(0)$$

for $s>s_0$ . However, since $g'=f''$ , this can be rewritten as

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0).$$

Substituting Equation 8.3.4 into this yields Equation 8.3.5

## Solving Second Order Equations with the Laplace Transform

We'll now use the Laplace transform to solve initial value problems for second order equations.

### Example 8.3.2

Use the Laplace transform to solve the initial value problem

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2, \quad y'(0) = 3. \tag{8.3.6}$$

**Solution**

Taking Laplace transforms of both sides of the differential equation in Equation 8.3.6 yields

$$\mathcal{L}(y'' - 6y' + 5y) = \mathcal{L}\left(3e^{2t}\right) = \frac{3}{s-2},$$

which we rewrite as

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 5\mathcal{L}(y) = \frac{3}{s-2}. \tag{8.3.7}$$

Now denote $\mathcal{L}(y) = Y(s)$ . Theorem 8.3.2 and the initial conditions in Equation 8.3.6 imply that

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 2$$

and

$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0) = s^2Y(s) - 3 - 2s.$$

Substituting from the last two equations into Equation 8.3.7 yields

$$\left(s^2Y(s) - 3 - 2s\right) - 6\left(sY(s) - 2\right) + 5Y(s) = \frac{3}{s-2}.$$

Therefore

$$(s^2 - 6s + 5)Y(s) = \frac{3}{s-2} + (3 + 2s) + 6(-2), \tag{8.3.8}$$

so

$$(s-5)(s-1)Y(s) = \frac{3 + (s-2)(2s-9)}{s-2},$$

and

$$Y(s) = \frac{3 + (s-2)(2s-9)}{(s-2)(s-5)(s-1)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = -\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-5} + \frac{5}{2}\frac{1}{s-1},$$

and taking the inverse transform of this yields

$$y = -e^{2t} + \frac{1}{2}e^{5t} + \frac{5}{2}e^{t}$$

as the solution of Equation 8.3.6.

It isn't necessary to write all the steps that we used to obtain Equation 8.3.8. To see how to avoid this, let's apply the method of Example 8.3.2 to the general initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{8.3.9}$$

Taking Laplace transforms of both sides of the differential equation in Equation 8.3.9 yields

$$a\mathcal{L}(y'') + b\mathcal{L}(y') + c\mathcal{L}(y) = F(s). \tag{8.3.10}$$

Now let $Y(s) = \mathcal{L}(y)$ . Theorem 8.3.2 and the initial conditions in Equation 8.3.9 imply that

$$\mathcal{L}(y') = sY(s) - k_0 \quad \text{and} \quad \mathcal{L}(y'') = s^2Y(s) - k_1 - k_0 s.$$

Substituting these into Equation 8.3.10 yields

$$a\left(s^2Y(s) - k_1 - k_0 s\right) + b\left(sY(s) - k_0\right) + cY(s) = F(s). \tag{8.3.11}$$

The coefficient of $Y(s)$ on the left is the characteristic polynomial

$$p(s) = as^2 + bs + c$$

of the complementary equation for Equation 8.3.9. Using this and moving the terms involving $k_0$ and $k_1$ to the right side of Equation 8.3.11 yields

$$p(s)Y(s) = F(s) + a(k_1 + k_0 s) + bk_0. \tag{8.3.12}$$

This equation corresponds to Equation 8.3.8 of Example 8.3.2 . Having established the form of this equation in the general case, it is preferable to go directly from the initial value problem to this equation. You may find it easier to remember Equation 8.3.12 rewritten as

$$p(s)Y(s) = F(s) + a\left(y'(0) + sy(0)\right) + by(0). \tag{8.3.13}$$

### Example 8.3.3

Use the Laplace transform to solve the initial value problem

$$2y'' + 3y' + y = 8e^{-2t}, \quad y(0) = -4,\ y'(0) = 2. \tag{8.3.14}$$

**Solution**

The characteristic polynomial is

$$p(s) = 2s^2 + 3s + 1 = (2s+1)(s+1)$$

and

$$F(s) = \mathcal{L}(8e^{-2t}) = \frac{8}{s+2},$$

so Equation 8.3.13 becomes

$$(2s+1)(s+1)Y(s) = \frac{8}{s+2} + 2(2-4s) + 3(-4).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{4\left(1 - (s+2)(s+1)\right)}{(s+1/2)(s+1)(s+2)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = \frac{4}{3}\frac{1}{s+1/2} - \frac{8}{s+1} + \frac{8}{3}\frac{1}{s+2},$$

so the solution of Equation 8.3.14 is

$$y = \mathcal{L}^{-1}(Y(s)) = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$$

(Figure 8.3.1 ).

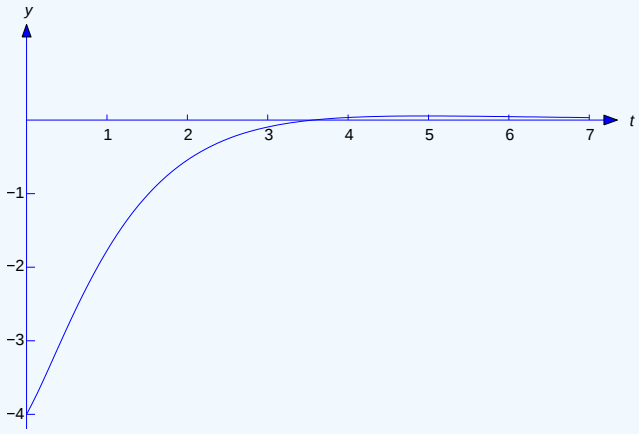



Figure 8.3.1 : $y = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$

## Example 8.3.4

Solve the initial value problem

$$y'' + 2y' + 2y = 1, \quad y(0) = -3,\ y'(0) = 1. \tag{8.3.15}$$

**Solution**

The characteristic polynomial is

$$p(s) = s^2 + 2s + 2 = (s+1)^2 + 1$$

and

$$F(s) = \mathcal{L}(1) = \frac{1}{s},$$

so Equation 8.3.13becomes

$$\left[(s+1)^2 + 1\right] Y(s) = \frac{1}{s} + 1 \cdot (1 - 3s) + 2(-3).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{1 - s(5 + 3s)}{s\left[(s+1)^2 + 1\right]}.$$

In Example 8.2.8 we found the inverse transform of this function to be

$$y = \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t$$

(Figure 8.3.2 ), which is therefore the solution of Equation 8.3.15.

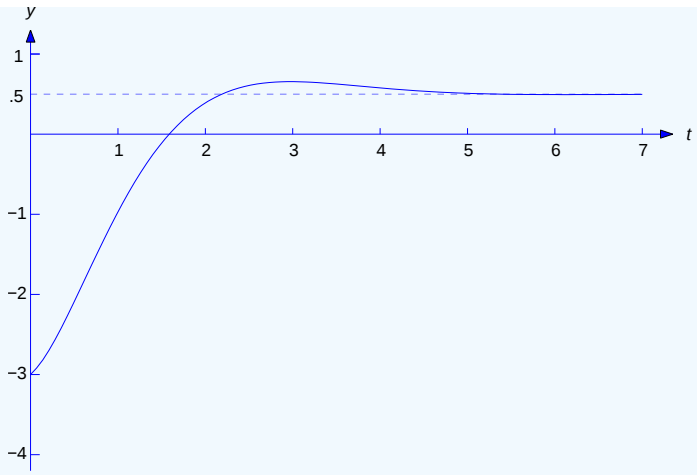



Figure 8.3.2 : $y = \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t$

**Note**

In our examples we applied Theorems 8.3.1 and 8.3.2 without verifying that the unknown function $y$ satisfies their hypotheses. This is characteristic of the formal manipulative way in which the Laplace transform is used to solve differential equations. Any doubts about the validity of the method for solving a given equation can be resolved by verifying that the resulting function $y$ is the solution of the given problem.

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# 8.3E: Solution of Initial Value Problems (Exercises)

## Q8.3.1

In **Exercises 8.3.1-8.3.31** use the Laplace transform to solve the initial value problem.

1. $y''+3y'+2y=e^t, \quad y(0)=1, \quad y'(0)=-6$
2. $y''-y'-6y=2, \quad y(0)=1, \quad y'(0)=0$
3. $y''+y'-2y=2e^{3t}, \quad y(0)=-1, \quad y'(0)=4$
4. $y''-4y=2e^{3t}, \quad y(0)=1, \quad y'(0)=-1$
5. $y''+y'-2y=e^{3t}, \quad y(0)=1, \quad y'(0)=-1$
6. $y''+3y'+2y=6e^t, \quad y(0)=1, \quad y'(0)=-1$
7. $y''+y=\sin 2t, \quad y(0)=0, \quad y'(0)=1$
8. $y''-3y'+2y=2e^{3t}, \quad y(0)=1, \quad y'(0)=-1$
9. $y''-3y'+2y=e^{4t}, \quad y(0)=1, \quad y'(0)=-2$
10. $y''-3y'+2y=e^{3t}, \quad y(0)=-1, \quad y'(0)=-4$
11. $y''+3y'+2y=2e^t, \quad y(0)=0, \quad y'(0)=-1$
12. $y''+y'-2y=-4, \quad y(0)=2, \quad y'(0)=3$
13. $y''+4y=4, \quad y(0)=0, \quad y'(0)=1$
14. $y''-y'-6y=2, \quad y(0)=1, \quad y'(0)=0$
15. $y''+3y'+2y=e^t, \quad y(0)=0, \quad y'(0)=1$
16. $y''-y=1, \quad y(0)=1, \quad y'(0)=0$
17. $y''+4y=3\sin t, \quad y(0)=1, \quad y'(0)=-1$
18. $y''+y'=2e^{3t}, \quad y(0)=-1, \quad y'(0)=4$
19. $y''+y=1, \quad y(0)=2, \quad y'(0)=0$
20. $y''+y=t, \quad y(0)=0, \quad y'(0)=2$
21. $y''+y=t-3\sin 2t, \quad y(0)=1, \quad y'(0)=-3$
22. $y''+5y'+6y=2e^{-t}, \quad y(0)=1, \quad y'(0)=3$
23. $y''+2y'+y=6\sin t-4\cos t, \quad y(0)=-1, \; y'(0)=1$
24. $y''-2y'-3y=10\cos t, \quad y(0)=2, \quad y'(0)=7$
25. $y''+y=4\sin t+6\cos t, \quad y(0)=-6, \; y'(0)=2$
26. $y''+4y=8\sin 2t+9\cos t, \quad y(0)=1, \quad y'(0)=0$
27. $y''-5y'+6y=10e^t\cos t, \quad y(0)=2, \quad y'(0)=1$
28. $y''+2y'+2y=2t, \quad y(0)=2, \quad y'(0)=-7$
29. $y''-2y'+2y=5\sin t+10\cos t, \quad y(0)=1, \; y'(0)=2$
30. $y''+4y'+13y=10e^{-t}-36e^t, \quad y(0)=0, \; y'(0)=-16$
31. $y''+4y'+5y=e^{-t}(\cos t+3\sin t), \quad y(0)=0, \quad y'(0)=4$

## Q8.3.2

32. $2y''-3y'-2y=4e^t, \quad y(0)=1, \; y'(0)=-2$
33. $6y''-y'-y=3e^{2t}, \quad y(0)=0, \; y'(0)=0$

34. $2y'' + 2y' + y = 2t, \quad y(0) = 1,\ y'(0) = -1$

35. $4y'' - 4y' + 5y = 4\sin t - 4\cos t, \quad y(0) = 0,\ y'(0) = 11/17$

36. $4y'' + 4y' + y = 3\sin t + \cos t, \quad y(0) = 2,\ y'(0) = -1$

37. $9y'' + 6y' + y = 3e^{3t}, \quad y(0) = 0,\ y'(0) = -3$

38. Suppose $a, b$, and $c$ are constants and $a \neq 0$. Let

$$y_1 = \mathcal{L}^{-1}\left(\frac{as+b}{as^2+bs+c}\right) \quad \text{and} \quad y_2 = \mathcal{L}^{-1}\left(\frac{a}{as^2+bs+c}\right).$$

Show that

$$y_1(0) = 1, \quad y_1'(0) = 0 \quad \text{and} \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

HINT: **Use the Laplace transform to solve the initial value problems**

$$ay'' + by' + cy = 0, \quad y(0) = 1, \quad y'(0) = 0$$
$$ay'' + by' + cy = 0, \quad y(0) = 0, \quad y'(0) = 1$$

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# 8.4: A Brief Table of Laplace Transforms

This page is a draft and is under active development.

Table 8.4.1

| $f(t)$ | $F(s)$ | |
|---|---|---|
| $1$ | $\frac{1}{s}$ | $(s>0)$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ | $(s>0)$ |
| $(n=\text{ integer }>0)$ | | |
| $t^p,\ p>-1$ | $\frac{\Gamma(p+1)}{s^{(p+1)}}$ | $(s>0)$ |
| $e^{at}$ | $\frac{1}{s-a}$ | $(s>a)$ |
| $t^n e^{at}$ | $\frac{n!}{(s-a)^{n+1}}$ | $(s>0)$ |
| $(n=\text{ integer }>0)$ | | |
| $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ | $(s>0)$ |
| $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ | $(s>0)$ |
| $e^{\lambda t}\cos \omega t$ | $\frac{s-\lambda}{(s-\lambda)^2+\omega^2}$ | $(s>\lambda)$ |
| $e^{\lambda t}\sin \omega t$ | $\frac{\omega}{(s-\lambda)^2+\omega^2}$ | $(s>\lambda)$ |
| $\cosh bt$ | $\frac{s}{s^2-b^2}$ | $(s>\lvert b\rvert)$ |
| $\sinh bt$ | $\frac{b}{s^2-b^2}$ | $(s>\lvert b\rvert)$ |
| $t\cos \omega t$ | $\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$ | $(s>0)$ |
| $t\sin \omega t$ | $\frac{2\omega s}{(s^2+\omega^2)^2}$ | $(s>0)$ |
| $\sin \omega t-\omega t\cos \omega t$ | $\frac{2\omega^3}{(s^2+\omega^2)^2}$ | $(s>0)$ |
| $\omega t-\sin \omega t$ | $\frac{\omega^3}{s^2(s^2+\omega^2)}$ | $(s>0)$ |
| $\frac{1}{t}\sin \omega t$ | $\arctan\left(\frac{\omega}{s}\right)$ | $(s>0)$ |
| $e^{at}f(t)$ | $F(s-a)$ | |
| $t^k f(t)$ | $(-1)^k F^{(k)}(s)$ | |
| $f(\omega t)$ | $\frac{1}{\omega}F\left(\frac{s}{\omega}\right),\quad \omega>0$ | |

| $f(t)$ | $F(s)$ | |
|---|---|---|
| $u(t-\tau)$ | $\frac{e^{-\tau s}}{s}$ | $(s>0)$ |
| $u(t-\tau)f(t-\tau)\ (\tau>0)$ | $e^{-\tau s}F(s)$ | |
| $\int_o^t f(\tau)g(t-\tau)\,d\tau$ | $F(s)\cdot G(s)$ | |
| $\delta(t-a)$ | $e^{-as}$ | $(s>0)$ |

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# CHAPTER OVERVIEW

## 9: Linear Higher Order Differential Equations

This chapter extend the results obtained in Chapter 5 for linear second order equations to linear higher order equations.


9.1: Introduction to Linear Higher Order Equations

9.1E: Introduction to Linear Higher Order Equations (Exercises)

9.2: Higher Order Constant Coefficient Homogeneous Equations

9.2E: Higher Order Constant Coefficient Homogeneous Equations (Exercises)

9.3: Undetermined Coefficients for Higher Order Equations

9.3E: Undetermined Coefficients for Higher Order Equations (Exercises)

9.4: Variation of Parameters for Higher Order Equations

9.4E: Variation of Parameters for Higher Order Equations (Exercises)


*Thumbnail: The Wronskian. In general, for an $n^{th}$ order linear differential equation, if $(n-1)$ solutions are known, the last one can be determined by using the Wronskian.*

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# 9.1: Introduction to Linear Higher Order Equations

An $n$th order differential equation is said to be **linear** if it can be written in the form

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=f(x). \tag{9.1.1}$$

We considered equations of this form with $n=1$ in Section 2.1 and with $n=2$ in Chapter 5. In this chapter $n$ is an arbitrary positive integer. In this section we sketch the general theory of linear $n$th order equations. Since this theory has already been discussed for $n=2$ in Sections 5.1 and 5.3, we'll omit proofs.

For convenience, we consider linear differential equations written as

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=F(x), \tag{9.1.2}$$

which can be rewritten as Equation 9.1.1 on any interval on which $P_0$ has no zeros, with $p_1=P_1/P_0$, ..., $p_n=P_n/P_0$ and $f=F/P_0$. For simplicity, throughout this chapter we'll abbreviate the left side of Equation 9.1.2 by $Ly$; that is,

$$Ly=P_0y^{(n)}+P_1y^{(n-1)}+\cdots+P_ny.$$

We say that the equation $Ly=F$ is **normal** on $(a,b)$ if $P_0$, $P_1$, ..., $P_n$ and $F$ are continuous on $(a,b)$ and $P_0$ has no zeros on $(a,b)$. If this is so then $Ly=F$ can be written as Equation 9.1.1 with $p_1$, ..., $p_n$ and $f$ continuous on $(a,b)$.

The next theorem is analogous to Theorem 5.3.1.

**Theorem 9.1.1**

Suppose $Ly=F$ is normal on $(a,b)$, let $x_0$ be a point in $(a,b)$, and let $k_0$, $k_1$, ..., $k_{n-1}$ be arbitrary real numbers. Then the initial value problem

$$Ly=F,\quad y(x_0)=k_0,\quad y'(x_0)=k_1,\dots,\quad y^{(n-1)}(x_0)=k_{n-1}$$

has a unique solution on $(a,b)$.

## Homogeneous Equations

Equation 9.1.2 is said to be **homogeneous** if $F\equiv0$ and **nonhomogeneous** otherwise. Since $y\equiv0$ is obviously a solution of $Ly=0$, we call it the **trivial** solution. Any other solution is **nontrivial**.

If $y_1$, $y_2$, ..., $y_n$ are defined on $(a,b)$ and $c_1$, $c_2$, ..., $c_n$ are constants, then

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n \tag{9.1.3}$$

is a **linear combination** of $\{y_1,y_2\dots,y_n\}$. It's easy to show that if $y_1$, $y_2$, ..., $y_n$ are solutions of $Ly=0$ on $(a,b)$, then so is any linear combination of $\{y_1,y_2,\dots,y_n\}$. (See the proof of Theorem 5.1.2.) We say that $\{y_1,y_2,\dots,y_n\}$ is a **fundamental set of solutions of $Ly=0$** on $(a,b)$ if every solution of $Ly=0$ on $(a,b)$ can be written as a linear combination of $\{y_1,y_2,\dots,y_n\}$, as in Equation 9.1.3. In this case we say that Equation 9.1.3 is the **general solution of $Ly=0$** on $(a,b)$.

It can be shown (**Exercises 9.1.14** and **9.1.15**) that if the equation $Ly=0$ is normal on $(a,b)$ then it has infinitely many fundamental sets of solutions on $(a,b)$. The next definition will help to identify fundamental sets of solutions of $Ly=0$.

We say that $\{y_1,y_2,\dots,y_n\}$ is **linearly independent** on $(a,b)$ if the only constants $c_1$, $c_2$, ..., $c_n$ such that

$$c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)=0,\quad a<x<b, \tag{9.1.4}$$

are $c_1=c_2=\cdots=c_n=0$. If Equation 9.1.4 holds for some set of constants $c_1$, $c_2$, ..., $c_n$ that are not all zero, then $\{y_1,y_2,\dots,y_n\}$ is **linearly dependent** on $(a,b)$

The next theorem is analogous to Theorem 5.1.3.

**Theorem 9.1.1**

If $Ly=0$ is normal on $(a,b)$, then a set $\{y_1,y_2,\dots,y_n\}$ of $n$ solutions of $Ly=0$ on $(a,b)$ is a fundamental set if and only if it is linearly independent on $(a,b)$.

## Example 9.1.1

The equation

$$x^3y''' - x^2y'' - 2xy' + 6y = 0 \tag{9.1.5}$$

is normal and has the solutions $y_1 = x^2$, $y_2 = x^3$, and $y_3 = 1/x$ on $(-\infty, 0)$ and $(0, \infty)$. Show that $\{y_1, y_2, y_3\}$ is linearly independent on $(-\infty, 0)$ and $(0, \infty)$. Then find the general solution of Equation 9.1.5 on $(-\infty, 0)$ and $(0, \infty)$.

### Solution

Suppose

$$c_1x^2 + c_2x^3 + \frac{c_3}{x} = 0 \tag{9.1.6}$$

on $(0, \infty)$. We must show that $c_1 = c_2 = c_3 = 0$. Differentiating Equation 9.1.6 twice yields the system

$$\begin{aligned} c_1x^2 + c_2x^3 + \frac{c_3}{x} &= 0 \\ 2c_1x + 3c_2x^2 - \frac{c_3}{x^2} &= 0 \\ 2c_1 + 6c_2x + \frac{2c_3}{x^3} &= 0. \end{aligned} \tag{9.1.7}$$

If Equation 9.1.7 holds for all $x$ in $(0, \infty)$, then it certainly holds at $x = 1$; therefore,

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 + 6c_2 + 2c_3 &= 0. \end{aligned} \tag{9.1.8}$$

By solving this system directly, you can verify that it has only the trivial solution $c_1 = c_2 = c_3 = 0$; however, for our purposes it is more useful to recall from linear algebra that a homogeneous linear system of $n$ equations in $n$ unknowns has only the trivial solution if its determinant is nonzero. Since the determinant of Equation 9.1.8 is

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -3 \\ 2 & 4 & 0 \end{vmatrix} = 12,$$

it follows that Equation 9.1.8 has only the trivial solution, so $\{y_1, y_2, y_3\}$ is linearly independent on $(0, \infty)$. Now Theorem 9.1.2 implies that

$$y = c_1x^2 + c_2x^3 + \frac{c_3}{x}$$

is the general solution of Equation 9.1.5 on $(0, \infty)$. To see that this is also true on $(-\infty, 0)$, assume that Equation 9.1.6 holds on $(-\infty, 0)$. Setting $x = -1$ in Equation 9.1.7 yields

$$\begin{aligned} c_1 - c_2 - c_3 &= 0 \\ -2c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 - 6c_2 - 2c_3 &= 0. \end{aligned}$$

Since the determinant of this system is

$$\begin{vmatrix} 1 & -1 & -1 \\ -2 & 3 & -1 \\ 2 & -6 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 2 & -4 & 0 \end{vmatrix} = -12,$$

it follows that $c_1 = c_2 = c_3 = 0$; that is, $\{y_1, y_2, y_3\}$ is linearly independent on $(-\infty, 0)$.

**Example 9.1.2**

The equation

$$y^{(4)}+y'''-7y''-y'+6y=0 \tag{9.1.9}$$

is normal and has the solutions $y_1=e^x$, $y_2=e^{-x}$, $y_3=e^{2x}$, and $y_4=e^{-3x}$ on $(-\infty,\infty)$. (Verify.) Show that $\{y_1,y_2,y_3,y_4\}$ is linearly independent on $(-\infty,\infty)$. Then find the general solution of Equation 9.1.9.

**Solution**

Suppose $c_1$, $c_2$, $c_3$, and $c_4$ are constants such that

$$c_1e^x+c_2e^{-x}+c_3e^{2x}+c_4e^{-3x}=0 \tag{9.1.10}$$

for all $x$. We must show that $c_1=c_2=c_3=c_4=0$ . Differentiating Equation 9.1.10 three times yields the system

$$\begin{aligned} c_1e^x+c_2e^{-x}+\ c_3e^{2x}+\ \ c_4e^{-3x} &= 0\\ c_1e^x-c_2e^{-x}+2c_3e^{2x}-\ 3c_4e^{-3x} &= 0\\ c_1e^x+c_2e^{-x}+4c_3e^{2x}+\ 9c_4e^{-3x} &= 0\\ c_1e^x-c_2e^{-x}+8c_3e^{2x}-27c_4e^{-3x} &= 0. \end{aligned} \tag{9.1.11}$$

If Equation 9.1.11 holds for all $x$, then it certainly holds for $x=0$. Therefore

$$\begin{aligned} c_1+c_2+\ c_3+\ \ c_4 &= 0\\ c_1-c_2+2c_3-\ 3c_4 &= 0\\ c_1+c_2+4c_3+\ 9c_4 &= 0\\ c_1-c_2+8c_3-27c_4 &= 0. \end{aligned}$$

The determinant of this system is

$$\begin{vmatrix} 1 & 1 & 1 & 1\\ 1 & -1 & 2 & -3\\ 1 & 1 & 4 & 9\\ 1 & -1 & 8 & -27 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1\\ 0 & -2 & 1 & -4\\ 0 & 0 & 3 & 8\\ 0 & -2 & 7 & -28 \end{vmatrix} = \begin{vmatrix} -2 & 1 & -4\\ 0 & 3 & 8\\ -2 & 7 & -28 \end{vmatrix} \tag{9.1.12}$$

$$[5pt] = \begin{vmatrix} -2 & 1 & -4\\ 0 & 3 & 8\\ 0 & 6 & -24 \end{vmatrix} = -2\begin{vmatrix} 3 & 8\\ 6 & -24 \end{vmatrix} = 240,$$

so the system has only the trivial solution $c_1=c_2=c_3=c_4=0$ . Now Theorem 9.1.2 implies that

$$y=c_1e^x+c_2e^{-x}+c_3e^{2x}+c_4e^{-3x}$$

is the general solution of Equation 9.1.9.

## The Wronskian

We can use the method used in Examples 9.1.1 and 9.1.2 to test $n$ solutions $\{y_1,y_2,\dots,y_n\}$ of any $n$th order equation $Ly=0$ for linear independence on an interval $(a,b)$ on which the equation is normal. Thus, if $c_1$, $c_2$,..., $c_n$ are constants such that

$$c_1y_1+c_2y_2+\cdots+c_ny_n=0,\quad a<b,$$

then differentiating $n-1$ times leads to the $n\times n$ system of equations

$$\begin{aligned} c_1y_1(x)+c_2y_2(x)+ \quad &\cdots \quad +c_ny_n(x)=0\\ c_1y_1'(x)+c_2y_2'(x)+ \quad &\cdots \quad +c_ny_n'(x)=0\\ &\vdots \\ c_1y_1^{(n-1)}(x)+c_2y_2^{(n-1)}(x)+ \quad &\cdots \quad +c_ny_n^{(n-1)}(x)=0 \end{aligned} \tag{9.1.13}$$

for $c_1$, $c_2$, ..., $c_n$. For a fixed $x$, the determinant of this system is

$$W(x)=\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x)\\ y_1'(x) & y_2'(x) & \cdots & y_n'(x)\\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}.$$

We call this determinant the **Wronskian** of $\{y_1,y_2,\dots,y_n\}$. If $W(x)\neq 0$ for some $x$ in $(a,b)$ then the system Equation 9.1.13 has only the trivial solution $c_1=c_2=\cdots=c_n=0$ , and Theorem 9.1.2 implies that

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n$$

is the general solution of $Ly=0$ on $(a,b)$.

The next theorem generalizes Theorem 5.1.4. The proof is sketched in **Exercises 9.1.17-9.1.20**.

### Theorem 9.1.3 Abel's Formula

Suppose the homogeneous linear $n$th order equation

$$P_0(x)y^{(n)}+P_1(x)y^{n-1}+\cdots+P_n(x)y=0 \tag{9.1.14}$$

is normal on $(a,b)$, let $y_1$, $y_2$, ..., $y_n$ be solutions of Equation 9.1.14 on $(a,b)$, and let $x_0$ be in $(a,b)$. Then the Wronskian of $\{y_1,y_2,\dots,y_n\}$ is given by

$$W(x)=W(x_0)\exp\left\{-\int_{x_0}^{x}\frac{P_1(t)}{P_0(t)}\,dt\right\},\quad a<b. \tag{9.1.15}$$

Therefore, either $W$ has no zeros in $(a,b)$ or $W\equiv 0$ on $(a,b)$.

Formula Equation 9.1.15 is **Abel's formula**.

The next theorem is analogous to Theorem 5.1.6.

### Theorem 9.1.4

Suppose $Ly=0$ is normal on $(a,b)$ and let $y_1$, $y_2$, ..., $y_n$ be $n$ solutions of $Ly=0$ on $(a,b)$. Then the following statements are equivalent; that is, they are either all true or all false:

a. The general solution of $Ly=0$ on $(a,b)$ is $y=c_1y_1+c_2y_2+\cdots+c_ny_n$.
b. $\{y_1,y_2,\dots,y_n\}$ is a fundamental set of solutions of $Ly=0$ on $(a,b)$.
c. $\{y_1,y_2,\dots,y_n\}$ is linearly independent on $(a,b)$.
d. The Wronskian of $\{y_1,y_2,\dots,y_n\}$ is nonzero at some point in $(a,b)$.
e. The Wronskian of $\{y_1,y_2,\dots,y_n\}$ is nonzero at all points in $(a,b)$.

### Example 9.1.3

In Example 9.1.1 we saw that the solutions $y_1=x^2$ , $y_2=x^3$ , and $y_3=1/x$ of

$$x^3y'''-x^2y''-2xy'+6y=0$$

are linearly independent on $(-\infty,0)$ and $(0,\infty)$. Calculate the Wronskian of $\{y_1,y_2,y_3\}$.

**Solution**

If $x\neq 0$, then

$$W(x)=\begin{vmatrix} x^2 & x^3 & \frac{1}{x} \\ 2x & 3x^2 & -\frac{1}{x^2} \\ 2 & 6x & \frac{2}{x^3} \end{vmatrix} = 2x^3 \begin{vmatrix} 1 & x & \frac{1}{x^3} \\ 2 & 3x & -\frac{1}{x^3} \\ 1 & 3x & \frac{1}{x^3} \end{vmatrix}$$

where we factored $x^2$, $x$, and 2 out of the first, second, and third rows of $W(x)$, respectively. Adding the second row of the last determinant to the first and third rows yields

$$W(x)=2x^3\begin{vmatrix} 3 & 4x & 0 \\ 2 & 3x & -\frac{1}{x^3} \\ 3 & 6x & 0 \end{vmatrix} = 2x^3\left(\frac{1}{x^3}\right)\begin{vmatrix} 3 & 4x \\ 3 & 6x \end{vmatrix} = 12x$$

Therefore $W(x)\neq 0$ on $(-\infty,0)$ and $(0,\infty)$.

## Example 9.1.4

In Example 9.1.2 we saw that the solutions $y_1=e^x$, $y_2=e^{-x}$, $y_3=e^{2x}$, and $y_4=e^{-3x}$ of

$$y^{(4)}+y'''-7y''-y'+6y=0$$

are linearly independent on every open interval. Calculate the Wronskian of $\{y_1,y_2,y_3,y_4\}$.

**Solution**

For all $x$,

$$W(x)=\begin{vmatrix} e^x & e^{-x} & e^{2x} & e^{-3x} \\ e^x & -e^{-x} & 2e^{2x} & -3e^{-3x} \\ e^x & e^{-x} & 4e^{2x} & 9e^{-3x} \\ e^x & -e^{-x} & 8e^{2x} & -27e^{-3x} \end{vmatrix}.$$

Factoring the exponential common factor from each row yields

$$W(x)=e^{-x}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{vmatrix} = 240e^{-x},$$

from Equation 9.1.12.

## Note

Under the assumptions of Theorem 9.1.4 , it isn't necessary to obtain a formula for $W(x)$. Just evaluate $W(x)$ at a convenient point in $(a,b)$, as we did in Examples 9.1.1 and 9.1.2 .

## Theorem 9.1.5

Suppose $c$ is in $(a,b)$ and $\alpha_1$, $\alpha_2$, ..., are real numbers, not all zero. Under the assumptions of Theorem 10.3.3, suppose $y_1$ and $y_2$ are solutions of Equation 5.1.35 such that

$$\alpha y_i(c)+y_i'(c)+\cdots+y_i^{(n-1)}(c)=0, \quad 1\le i\le n. \tag{9.1.16}$$

Then $\{y_1, y_2, \dots y_n\}$ isn't linearly independent on $(a, b)$.

**Proof**

Since $\alpha_1$, $\alpha_2$, ..., $\alpha_n$ are not all zero, Equation 9.1.14implies that

$$\begin{vmatrix} y_1(c) & y_1'(c) & \cdots & y_1^{(n-1)}(c) \\ y_2(c) & y_2'(c) & \cdots & y_2^{(n-1)}(c) \\ \vdots & \vdots & \ddots & \vdots \\ y_n(c) & y_n'(c) & \cdots & y_n^{(n-1)}(c) \end{vmatrix} = 0,$$

so

$$\begin{vmatrix} y_1(c) & y_2(c) & \cdots & y_n(c) \\ y_1'(c) & y_2'(c) & \cdots & y_n'(c) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(c) & y_2^{(n-1)}(c)(c) & \cdots & y_n^{(n-1)}(c)(c) \end{vmatrix} = 0$$

and Theorem 9.1.4 implies the stated conclusion.

## General Solution of a Nonhomogeneous Equation

The next theorem is analogous to Theorem 5.3.2. It shows how to find the general solution of $Ly = F$ if we know a particular solution of $Ly = F$ and a fundamental set of solutions of the **complementary equation** $Ly = 0$.

### Theorem 9.1.6

The probabilities assigned to events by a distribution function on a sample space are given by.

**Proof**

Suppose $Ly = F$ is normal on $(a, b)$. Let $y_p$ be a particular solution of $Ly = F$ on $(a, b)$, and let $\{y_1, y_2, \dots, y_n\}$ be a fundamental set of solutions of the complementary equation $Ly = 0$ on $(a, b)$. Then $y$ is a solution of $Ly = F$ on $(a, b)$ if and only if

$$y = y_p + c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

where $c_1, c_2, \dots, c_n$ are constants.

The next theorem is analogous to Theorem 5.3.2.

### Theorem 9.1.7 The Principle of Superposition

Suppose for each $i = 1, 2, \dots, r$, the function $y_{p_i}$ is a particular solution of $Ly = F_i$ on $(a, b)$. Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_r}$$

is a particular solution of

$$Ly = F_1(x) + F_2(x) + \cdots + F_r(x)$$

on $(a, b)$.

We'll apply Theorems 9.1.6 and 9.1.7 throughout the rest of this chapter.

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# 9.1E: Introduction to Linear Higher Order Equations (Exercises)

## Q9.1.1

1. Verify that the given function is the solution of the initial value problem.

a. $x^3y'''-3x^2y''+6xy'-6y=\dfrac{-24}{x}, \quad y(-1)=0\ , y'(-1)=0, \quad y''(-1)=0\ ; y=-6x-8x^2-3x^3+\frac{1}{x}$

b. $y'''-\dfrac{1}{x}y''-y'+\dfrac{1}{x}y=\dfrac{x^2-4}{x^4}, \quad y(1)=\dfrac{3}{2}, \quad y'(1)=\dfrac{1}{2}, y''(1)=1 \quad ; y=x+\dfrac{1}{2x}$

c. $xy'''-y''-xy'+y=x^2, \quad y(1)=2, \quad y'(1)=5, \quad y''(1)=-1 \quad ; y=-x^2-2+2e^{(x-1)}-e^{-(x-1)}+4x$

d. $4x^3y'''+4x^2y''-5xy'+2y=30x^2, \quad y(1)=5, \quad y'(1)=\dfrac{17}{2} \quad ; y''(1)=\dfrac{63}{4}; \quad y=2x^2\ln x-x^{1/2}+2x^{-1/2}+4x^2$

e. $x^4y^{(4)}-4x^3y'''+12x^2y''-24xy'+24y=6x^4, \quad y(1)=-2 \quad ; y'(1)=-9, \quad y''(1)=-27, \quad y'''(1)=-52\ ;$
$y=x^4\ln x+x-2x^2+3x^3-4x^4$

f. $xy^{(4)}-y'''-4xy''+4y'=96x^2, \quad y(1)=-5, \quad y'(1)=-24\ ;$
$y''(1)=-36; \quad y'''(1)=-48; \quad y=9-12x+6x^2-8x^3$

2. Solve the initial value problem

$$x^3y'''-x^2y''-2xy'+6y=0, \quad y(-1)=-4, \quad y'(-1)=-14, \quad y''(-1)=-20.$$

HINT: **See Example 9.1.1.**

3. Solve the initial value problem

$$y^{(4)}+y'''-7y''-y'+6y=0, \quad y(0)=5, \quad y'(0)=-6, \quad y''(0)=10, \quad y'''(0)-36.$$

HINT: **See Example 9.1.2.**

4. Find solutions $y_1$, $y_2$, …, $y_n$ of the equation $y^{(n)}=0$ that satisfy the initial conditions

$$y_i^{(j)}(x_0)=\begin{cases} 0, & j\neq i-1, \\ [5pt]1, & j=i-1, \end{cases} \quad 1\le i\le n.$$

5.

a. Verify that the function

$$y=c_1x^3+c_2x^2+\frac{c_3}{x}$$

satisfies

$$x^3y'''-x^2y''-2xy'+6y=0 \tag{A}$$

if $c_1$, $c_2$, and $c_3$ are constants.

b. Use (a) to find solutions $y_1$, $y_2$, and $y_3$ of (A) such that

$$\begin{aligned} y_1(1) &=1, \quad y_1'(1)=0, \quad y_1''(1)=0 \\ [5pt]y_2(1) &=0, \quad y_2'(1)=1, \quad y_2''(1)=0 \\ [5pt]y_3(1) &=0, \quad y_3'(1)=0, \quad y_3''(1)=1. \end{aligned}$$

c. Use (b) to find the solution of (A) such that

$$y(1)=k_0, \quad y'(1)=k_1, \quad y''(1)=k_2.$$

6. Verify that the given functions are solutions of the given equation, and show that they form a fundamental set of solutions of the equation on any interval on which the equation is normal.

a. $y'''+y''-y'-y=0; \quad \{e^x,\ e^{-x},\ xe^{-x}\}$

b. $y'''-3y''+7y'-5y=0; \quad \{e^x,\ e^x\cos 2x,\ e^x\sin 2x\}$.

c. $xy''' - y'' - xy' + y = 0; \quad \{e^x,\ e^{-x},\ x\}$
d. $x^2y''' + 2xy'' - (x^2+2)y = 0; \quad \{e^x/x,\ e^{-x}/x,\ 1\}$
e. $(x^2 - 2x + 2)y''' - x^2y'' + 2xy' - 2y = 0; \quad \{x,\ x^2,\ e^x\}$
f. $(2x-1)y^{(4)} - 4xy''' + (5-2x)y'' + 4xy' - 4y = 0; \quad \{x,\ e^x,\ e^{-x}, e^{2x}\}$
g. $xy^{(4)} - y''' - 4xy' + 4y' = 0; \quad \{1, x^2,\ e^{2x},\ e^{-2x}\}$

7. Find the Wronskian $W$ of a set of three solutions of

$$y''' + 2xy'' + e^xy' - y = 0,$$

given that $W(0) = 2$.

8. Find the Wronskian $W$ of a set of four solutions of

$$y^{(4)} + (\tan x)y''' + x^2y'' + 2xy = 0,$$

given that $W(\pi/4) = K$.

9.

a. Evaluate the Wronskian $W\ \{e^x,\ xe^x,\ x^2e^x\}$. Evaluate $W(0)$.
b. Verify that $y_1$, $y_2$, and $y_3$ satisfy

$$y''' - 3y'' + 3y' - y = 0. \tag{A}$$

c. Use $W(0)$ from (a) and Abel's formula to calculate $W(x)$.
d. What is the general solution of (A)?

10. Compute the Wronskian of the given set of functions.

a. $\{1,\ e^x,\ e^{-x}\}$
b. $\{e^x,\ e^x \sin x,\ e^x \cos x\}$
c. $\{2,\ x+1,\ x^2+2\}$
d. $x,\ x\ln x,\ 1/x\}$
e. $\{1,\ x,\ \frac{x^2}{2!},\ \frac{x^3}{3!}, \cdots,\ \frac{x^n}{n!}\}$
f. $\{e^x,\ e^{-x},\ x\}$
g. $\{e^x/x,\ e^{-x}/x,\ 1\}$
h. $\{x,\ x^2,\ e^x\}$
i. $\{x,\ x^3,\ 1/x,\ 1/x^2\}$
j. $\{e^x,\ e^{-x},\ x,\ e^{2x}\}$
k. $\{e^{2x},\ e^{-2x},\ 1,\ x^2\}$

11. Suppose $Ly = 0$ is normal on $(a, b)$ and $x_0$ is in $(a, b)$. Use Theorem 9.1.1 to show that $y \equiv 0$ is the only solution of the initial value problem

$$Ly = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0, \ldots, y^{(n-1)}(x_0) = 0,$$

on $(a, b)$.

12. Prove: If $y_1$, $y_2$, ..., $y_n$ are solutions of $Ly = 0$ and the functions

$$z_i = \sum_{j=1}^{n} a_{ij}y_j, \quad 1 \le i \le n,$$

form a fundamental set of solutions of $Ly = 0$, then so do $y_1$, $y_2$, ..., $y_n$.

13. Prove: If

$$y = c_1y_1 + c_2y_2 + \cdots + c_ky_k + y_p$$

is a solution of a linear equation $Ly = F$ for every choice of the constants $c_1$, $c_2$,..., $c_k$, then $Ly_i = 0$ for $1 \le i \le k$.

14. Suppose $Ly = 0$ is normal on $(a, b)$ and let $x_0$ be in $(a, b)$. For $1 \le i \le n$, let $y_i$ be the solution of the initial value problem

$$Ly_i = 0, \quad y_i^{(j)}(x_0) = \begin{cases} 0, & j \neq i-1, \\ 1, & j = i-1, \end{cases} \quad 1 \leq i \leq n,$$

where $x_0$ is an arbitrary point in $(a, b)$. Show that any solution of $Ly = 0$ on $(a, b)$, can be written as

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

with $c_j = y^{(j-1)}(x_0)$.

15. Suppose $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions of

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0$$

on $(a, b)$, and let

$$\begin{aligned} z_1 &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ z_2 &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad \vdots \\ z_n &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n, \end{aligned}$$

where the $\{a_{ij}\}$ are constants. Show that $\{z_1, z_2, \ldots, z_n\}$ is a fundamental set of solutions of (A) if and only if the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

is nonzero. HINT: **The determinant of a product of $n \times n$ matrices equals the product of the determinants.**

16. Show that $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent on $(a, b)$ if and only if at least one of the functions $y_1$, $y_2$, ..., $y_n$ can be written as a linear combination of the others on $(a, b)$.

## Q9.1.2

Take the following as a hint in **Exercises 9.1.17-9.1.19**:

By the definition of determinant,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum \pm a_{1i_1} a_{2i_2}, \cdots, a_{ni_n},$$

where the sum is over all permutations $(i_i, i_2, \cdots, i_n)$ of $(1, 2, \cdots, n)$ and the choice of $+$ or $-$ in each term depends only on the permutation associated with that term.

17. Prove: If

$$A(u_1, u_2, \ldots, u_n) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ u_1 & u_2 & \cdots & u_n \end{vmatrix},$$

then

$$A(u_1+v_1,u_2+v_2,\dots,u_n+v_n)=A(u_1,u_2,\dots,u_n)+A(v_1,v_2,\dots,v_n).$$

18. Let

$$F=\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix},$$

where $f_{ij}\ (1\le i,\ j\le n)$ is differentiable. Show that

$$F'=F_1+F_2+\cdots+F_n,$$

where $F_i$ is the determinant obtained by differentiating the $i$th row of $F$.

19. Use **Exercise 9.1.18** to show that if $W$ is the Wronskian of the $n$-times differentiable functions $y_1$, $y_2$, ..., $y_n$, then

$$W'=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}.$$

## Q9.1.3

20. Use **Exercises 9.1.17** and **9.1.19** to show that if $W$ is the Wronskian of solutions $\{y_1,y_2,\dots,y_n\}$ of the normal equation

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=0, \tag{A}$$

then $W'=-P_1W/P_0$. Derive Abel's formula (Equation 9.1.15) from this.

21. Prove Theorem 9.1.6.

22. Prove Theorem 9.1.7.

23. Show that if the Wronskian of the $n$-times continuously differentiable functions $\{y_1,y_2,\dots,y_n\}$ has no zeros in $(a,b)$, then the differential equation obtained by expanding the determinant

$$\begin{vmatrix} y & y_1 & y_2 & \cdots & y_n \\ y' & y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y^{(n)} & y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}=0,$$

in cofactors of its first column is normal and has $\{y_1,y_2,\dots,y_n\}$ as a fundamental set of solutions on $(a,b)$.

24. Use the method suggested by **Exercise 9.1.23** to find a linear homogeneous equation such that the given set of functions is a fundamental set of solutions on intervals on which the Wronskian of the set has no zeros.

a. $\{x,\ x^2-1,\ x^2+1\}$
b. $\{e^x,\ e^{-x},\ x\}$
c. $\{e^x,\ xe^{-x},\ 1\}$
d. $\{x,\ x^2,\ e^x\}$
e. $\{x,\ x^2,\ 1/x\}$
f. $\{x+1,\ e^x,\ e^{3x}\}$

g. $\{x,\ x^3,\ 1/x,\ 1/x^2\}$
h. $\{x,\ x\ln x,\ 1/x,\ x^2\}$
i. $\{e^x,\ e^{-x},\ x,\ e^{2x}\}$
j. $\{e^{2x},\ e^{-2x},\ 1,\ x^2\}$

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# 9.2: Higher Order Constant Coefficient Homogeneous Equations

If $a_0$, $a_1$, ..., $a_n$ are constants and $a_0 \neq 0$, then

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = F(x)$$

is said to be a **constant coefficient equation**. In this section we consider the homogeneous constant coefficient equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = 0. \tag{9.2.1}$$

Since Equation 9.2.1 is normal on $(-\infty, \infty)$, the theorems in Section 9.1 all apply with $(a, b) = (-\infty, \infty)$.

As in Section 5.2, we call

$$p(r) = a_0r^n + a_1r^{n-1} + \cdots + a_n \tag{9.2.2}$$

the **characteristic polynomial** of Equation 9.2.1. We saw in Section 5.2 that when $n = 2$ the solutions of Equation 9.2.1 are determined by the zeros of the characteristic polynomial. This is also true when $n > 2$, but the situation is more complicated in this case. Consequently, we take a different approach.

If $k$ is a positive integer, let $D^k$ stand for the $k$-th derivative operator; that is

$$D^ky = y^{(k)}.$$

If

$$q(r) = b_0r^m + b_1r^{m-1} + \cdots + b_m$$

is an arbitrary polynomial, define the operator

$$q(D) = b_0D^m + b_1D^{m-1} + \cdots + b_m$$

such that

$$q(D)y = (b_0D^m + b_1D^{m-1} + \cdots + b_m)y = b_0y^{(m)} + b_1y^{(m-1)} + \cdots + b_my$$

whenever $y$ is a function with $m$ derivatives. We call $q(D)$ a **polynomial operator**.

With $p$ as in Equation 9.2.2,

$$p(D) = a_0D^n + a_1D^{n-1} + \cdots + a_n,$$

so Equation 9.2.1 can be written as $p(D)y = 0$. If $r$ is a constant then

$$\begin{aligned} p(D)e^{rx} &= \left(a_0D^ne^{rx} + a_1D^{n-1}e^{rx} + \cdots + a_ne^{rx}\right) \\ &= (a_0r^n + a_1r^{n-1} + \cdots + a_n)e^{rx}; \end{aligned}$$

that is

$$p(D)(e^{rx}) = p(r)e^{rx}.$$

This shows that $y = e^{rx}$ is a solution of Equation 9.2.1 if $p(r) = 0$. In the simplest case, where $p$ has $n$ distinct real zeros $r_1$, $r_2$,..., $r_n$, this argument yields $n$ solutions

$$y_1 = e^{r_1x}, \quad y_2 = e^{r_2x}, \ldots, \quad y_n = e^{r_nx}.$$

It can be shown (**Exercise 9.2.39**) that the Wronskian of $\{e^{r_1x}, e^{r_2x}, \ldots, e^{r_nx}\}$ is nonzero if $r_1$, $r_2$, ..., $r_n$ are distinct; hence, $\{e^{r_1x}, e^{r_2x}, \ldots, e^{r_nx}\}$ is a fundamental set of solutions of $p(D)y = 0$ in this case.

**Example 9.2.1**

a. Find the general solution of

$$y''' - 6y'' + 11y' - 6y = 0. \tag{9.2.3}$$

b. and solve the initial value problem

$$y''' - 6y'' + 11y' - 6y = 0, \quad y(0) = 4, \quad y'(0) = 5, \quad y''(0) = 9. \tag{9.2.4}$$

**Solution a**

The characteristic polynomial of Equation 9.2.3 is

$$p(r) = r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3).$$

Therefore $\{e^x, e^{2x}, e^{3x}\}$ is a set of solutions of Equation 9.2.3. It is a fundamental set, since its Wronskian is

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x} \neq 0.$$

Therefore the general solution of Equation 9.2.3 is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}. \tag{9.2.5}$$

**Solution b**

We must determine $c_1$, $c_2$ and $c_3$ in Equation 9.2.5 so that $y$ satisfies the initial conditions in Equation 9.2.4. Differentiating Equation 9.2.5 twice yields

$$\begin{aligned} y' &= c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x} \\ y'' &= c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}. \end{aligned} \tag{9.2.6}$$

Setting $x = 0$ in Equation 9.2.5 and Equation 9.2.6 and imposing the initial conditions yields

$$\begin{aligned} c_1 + c_2 + c_3 &= 4 \\ c_1 + 2c_2 + 3c_3 &= 5 \\ c_1 + 4c_2 + 9c_3 &= 9. \end{aligned}$$

The solution of this system is $c_1 = 4$, $c_2 = -1$, $c_3 = 1$. Therefore the solution of Equation 9.2.4 is

$$y = 4e^x - e^{2x} + e^{3x}$$

(Figure 9.2.1 ).

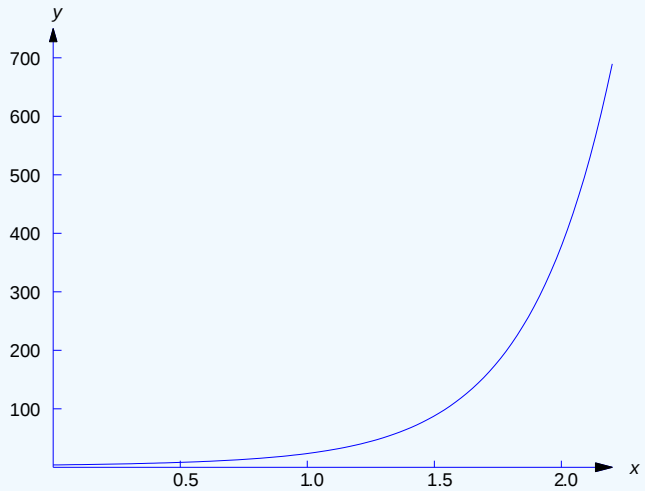



Figure 9.2.1 : $y = 4e^x - e^{2x} + e^{3x}$

Now we consider the case where the characteristic polynomial Equation 9.2.2 does not have $n$ distinct real zeros. For this purpose it is useful to define what we mean by a factorization of a polynomial operator. We begin with an example.

## Example 9.2.2

Consider the polynomial

$$p(r) = r^3 - r^2 + r - 1$$

and the associated polynomial operator

$$p(D) = D^3 - D^2 + D - 1.$$

Since $p(r)$ can be factored as

$$p(r) = (r-1)(r^2+1) = (r^2+1)(r-1),$$

it is reasonable to expect that p(D) can be factored as

$$p(D) = (D-1)(D^2+1) = (D^2+1)(D-1). \tag{9.2.7}$$

However, before we can make this assertion we must **define** what we mean by saying that two operators are equal, and what we mean by the products of operators in Equation 9.2.7. We say that two operators are equal if they apply to the same functions and always produce the same result. The definitions of the products in Equation 9.2.7 is this: if $y$ is any three-times differentiable function then

a. $(D-1)(D^2+1)y$ is the function obtained by first applying $D^2+1$ to $y$ and then applying $D-1$ to the resulting function
b. $(D^2+1)(D-1)y$ is the function obtained by first applying $D-1$ to $y$ and then applying $D^2+1$ to the resulting function.

From (a):

$$\begin{aligned}(D-1)(D^2+1)y &= (D-1)[(D^2+1)y] \\ &= (D-1)(y''+y) = D(y''+y) - (y''+y) \\ &= (y'''+y') - (y''+y) \\ &= y''' - y'' + y' - y = (D^3 - D^2 + D - 1)y.\end{aligned} \tag{9.2.8}$$

This implies that

$$(D-1)(D^2+1) = (D^3 - D^2 + D - 1).$$

From (b):

$$\begin{aligned}(D^2+1)(D-1)y &= (D^2+1)[(D-1)y] \\ &= (D^2+1)(y'-y) = D^2(y'-y) + (y'-y) \\ &= (y'''-y'') + (y'-y) \\ &= y''' - y'' + y' - y = (D^3 - D^2 + D - 1)y,\end{aligned} \tag{9.2.9}$$

$$(D^2+1)(D-1) = (D^3 - D^2 + D - 1),$$

which completes the justification of Equation 9.2.7.

## Example 9.2.3

Use the result of Example 9.2.2 to find the general solution of

$$y''' - y'' + y' - y = 0. \tag{9.2.10}$$

**Solution**

From Equation 9.2.8, we can rewrite Equation 9.2.10 as

$$(D-1)(D^2+1)y = 0,$$

which implies that any solution of $(D^2+1)y=0$ is a solution of Equation 9.2.10. Therefore $y_1=\cos x$ and $y_2=\sin x$ are solutions of Equation 9.2.10.

From Equation 9.2.9, we can rewrite Equation 9.2.10 as

$$(D^2+1)(D-1)y=0,$$

which implies that any solution of $(D-1)y=0$ is a solution of Equation 9.2.10. Therefore $y_3=e^x$ is solution of Equation 9.2.10.

The Wronskian of $\{e^x,\cos x,\sin x\}$ is

$$W(x)=\begin{vmatrix}\cos x & \sin x & e^x\\ -\sin x & \cos x & e^x\\ -\cos x & -\sin x & e^x\end{vmatrix}.$$

Since

$$W(0)=\begin{vmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & 0 & 1\end{vmatrix}=2,$$

$\{\cos x,\sin x,e^x\}$ is linearly independent and

$$y=c_1\cos x+c_2\sin x+c_3e^x$$

is the general solution of Equation 9.2.10.

## Example 9.2.4

Find the general solution of

$$y^{(4)}-16y=0. \tag{9.2.11}$$

**Solution**

The characteristic polynomial of Equation 9.2.11 is

$$\begin{aligned}p(r)&=r^4-16\\&=(r^2-4)(r^2+4)\\&=(r-2)(r+2)(r^2+4).\end{aligned}$$

By arguments similar to those used in Examples 9.2.2 and 9.2.4 , it can be shown that Equation 9.2.11 can be written as

$$(D^2+4)(D+2)(D-2)y=0$$

or

$$(D^2+4)(D-2)(D+2)y=0$$

or

$$(D-2)(D+2)(D^2+4)y=0.$$

Therefore $y$ is a solution of Equation 9.2.11 if it is a solution of any of the three equations

$$(D-2)y=0,\quad (D+2)y=0,\quad (D^2+4)y=0.$$

Hence, $\{e^{2x},e^{-2x},\cos 2x,\sin 2x\}$ is a set of solutions of Equation 9.2.11. The Wronskian of this set is

$$W(x)=\begin{vmatrix} e^{2x} & e^{-2x} & \cos 2x & \sin 2x \\ 2e^{2x} & -2e^{-2x} & -2\sin 2x & 2\cos 2x \\ 4e^{2x} & 4e^{-2x} & -4\cos 2x & -4\sin 2x \\ 8e^{2x} & -8e^{-2x} & 8\sin 2x & -8\cos 2x \end{vmatrix}.$$

Since

$$W(0)=\begin{vmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 4 & 4 & -4 & 0 \\ 8 & -8 & 0 & -8 \end{vmatrix}=-512,$$

$\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$is linearly independent, and

$$y_1=c_1e^{2x}+c_2e^{-2x}+c_3\cos 2x+c_4\sin 2x$$

is the general solution of Equation 9.2.11.

It is known from algebra that every polynomial

$$p(r)=a_0r^n+a_1r^{n-1}+\cdots+a_n$$

with real coefficients can be factored as

$$p(r)=a_0p_1(r)p_2(r)\cdots p_k(r),$$

where no pair of the polynomials $p_1$, $p_2$, …, $p_k$ has a commom factor and each is either of the form

$$p_j(r)=(r-r_j)^{m_j}, \tag{9.2.12}$$

where $r_j$ is real and $m_j$ is a positive integer, or

$$p_j(r)=\left[(r-\lambda_j)^2+\omega_j^2\right]^{m_j}, \tag{9.2.13}$$

where $\lambda_j$ and $\omega_j$ are real, $\omega_j\neq 0$, and $m_j$ is a positive integer. If Equation 9.2.12 holds then $r_j$ is a real zero of $p$, while if Equation 9.2.13 holds then $\lambda+i\omega$ and $\lambda-i\omega$ are complex conjugate zeros of $p$. In either case, $m_j$ is the **multiplicity** of the zero(s).

By arguments similar to those used in our examples, it can be shown that

$$p(D)=a_0p_1(D)p_2(D)\cdots p_k(D) \tag{9.2.14}$$

and that the order of the factors on the right can be chosen arbitrarily. Therefore, if $p_j(D)y=0$ for some $j$ then $p(D)y=0$. To see this, we simply rewrite Equation 9.2.14 so that $p_j(D)$ is applied first. Therefore the problem of finding solutions of $p(D)y=0$ with $p$ as in Equation 9.2.14reduces to finding solutions of each of these equations

$$p_j(D)y=0, \quad 1\le j\le k,$$

where $p_j$ is a power of a first degree term or of an irreducible quadratic. To find a fundamental set of solutions $\{y_1, y_2, \ldots, y_n\}$ of $p(D)y=0$, we find fundamental set of solutions of each of the equations and take $\{y_1, y_2, \ldots, y_n\}$ to be the set of all functions in these separate fundamental sets. In **Exercise 9.2.40** we sketch the proof that $\{y_1, y_2, \ldots, y_n\}$ is linearly independent, and therefore a fundamental set of solutions of $p(D)y=0$.

To apply this procedure to general homogeneous constant coefficient equations, we must be able to find fundamental sets of solutions of equations of the form

$$(D-a)^my=0$$

and

$$\left[(D-\lambda)^2+\omega^2\right]^m y=0,$$

where $m$ is an arbitrary positive integer. The next two theorems show how to do this.

**Theorem 9.2.1**

If $m$ is a positive integer, then

$$\{e^{ax}, xe^{ax}, \dots, x^{m-1}e^{ax}\} \tag{9.2.15}$$

is a fundamental set of solutions of

$$(D-a)^m y=0. \tag{9.2.16}$$

**Proof**

We'll show that if

$$f(x)=c_1+c_2x+\cdots+c_mx^{m-1}$$

is an arbitrary polynomial of degree $\le m-1$, then $y=e^{ax}f$ is a solution of Equation 9.2.16. First note that if $g$ is any differentiable function then

$$(D-a)e^{ax}g=De^{ax}g-ae^{ax}g=ae^{ax}g+e^{ax}g'-ae^{ax}g,$$

so

$$(D-a)e^{ax}g=e^{ax}g'. \tag{9.2.17}$$

Therefore

$$\begin{aligned}
(D-a)e^{ax}f &= e^{ax}f' && \text{(from (9.2.17) with } g=f)\\
(D-a)^2e^{ax}f &= (D-a)e^{ax}f'=e^{ax}f'' && \text{(from (9.2.17) with } g=f')\\
(D-a)^3e^{ax}f &= (D-a)e^{ax}f''=e^{ax}f''' && \text{(from (9.2.17) with } g=f'')\\
&\vdots\\
(D-a)^me^{ax}f &= (D-a)e^{ax}f^{(m-1)}=e^{ax}f^{(m)} && \text{(from (9.2.17) with } g=f^{(m-1)}).
\end{aligned}$$

Since $f^{(m)}=0$, the last equation implies that $y=e^{ax}f$ is a solution of Equation 9.2.16 if $f$ is any polynomial of degree $\le m-1$. In particular, each function in Equation 9.2.15 is a solution of Equation 9.2.16. To see that Equation 9.2.15 is linearly independent (and therefore a fundamental set of solutions of Equation 9.2.16), note that if

$$c_1e^{ax}+c_2xe^{ax}+c\cdots+c_{m-1}x^{m-1}e^{ax}=0$$

for all $x$ in some interval $(a,b)$, then

$$c_1+c_2x+c\cdots+c_{m-1}x^{m-1}=0$$

for all $x$ in $(a,b)$. However, we know from algebra that if this polynomial has more than $m-1$ zeros then $c_1=c_2=\cdots=c_n=0$.

**Example 9.2.5**

Find the general solution of

$$y'''+3y''+3y'+y=0. \tag{9.2.18}$$

**Solution**

The characteristic polynomial of Equation 9.2.18 is

$$p(r)=r^3+3r^2+3r+1=(r+1)^3.$$

Therefore Equation 9.2.18 can be written as

$$(D+1)^3 y = 0,$$

so Theorem 9.2.1 implies that the general solution of Equation 9.2.18 is

$$y = e^{-x}(c_1 + c_2 x + c_3 x^2).$$

The proof of the next theorem is sketched in **Exercise 9.2.41**.

**Theorem 9.2.2**

If $\omega \neq 0$ and $m$ is a positive integer, then

$$\{e^{\lambda x}\cos\omega x, xe^{\lambda x}\cos\omega x, \quad \ldots, x^{m-1}e^{\lambda x}\cos\omega x,$$
$$e^{\lambda x}\sin\omega x, xe^{\lambda x}\sin\omega x, \quad \ldots, x^{m-1}e^{\lambda x}\sin\omega x\}$$

is a fundamental set of solutions of

$$[(D-\lambda)^2 + \omega^2]^m y = 0.$$

**Example 9.2.6**

Find the general solution of

$$(D^2 + 4D + 13)^3 y = 0. \tag{9.2.19}$$

**Solution**

The characteristic polynomial of Equation 9.2.19 is

$$p(r) = (r^2 + 4r + 13)^3 = \left((r+2)^2 + 9\right)^3.$$

Therefore Equation 9.2.19 can be be written as

$$[(D+2)^2 + 9]^3 y = 0,$$

so Theorem 9.2.2 implies that the general solution of Equation 9.2.19 is

$$y = (a_1 + a_2 x + a_3 x^2)e^{-2x}\cos 3x + (b_1 + b_2 x + b_3 x^2)e^{-2x}\sin 3x.$$

**Example 9.2.7**

Find the general solution of

$$y^{(4)} + 4y''' + 6y'' + 4y' = 0. \tag{9.2.20}$$

**Solution**

The characteristic polynomial of Equation 9.2.20 is

$$\begin{aligned} p(r) &= r^4 + 4r^3 + 6r^2 + 4r \\ &= r(r^3 + 4r^2 + 6r + 4) \\ &= r(r+2)(r^2 + 2r + 2) \\ &= r(r+2)[(r+1)^2 + 1]. \end{aligned}$$

Therefore Equation 9.2.20 can be written as

$$[(D+1)^2 + 1](D+2)Dy = 0.$$

Fundamental sets of solutions of

$$\left[(D+1)^2+1\right]y=0,\quad (D+2)y=0,\quad \text{and}\quad Dy=0.$$

are given by

$$\{e^{-x}\cos x, e^{-x}\sin x\},\quad \{e^{-2x}\},\quad \text{and}\quad \{1\},$$

respectively. Therefore the general solution of Equation 9.2.20 is

$$y=e^{-x}(c_1\cos x+c_2\sin x)+c_3e^{-2x}+c_4.$$

## Example 9.2.8

Find a fundamental set of solutions of

$$[(D+1)^2+1]^2(D-1)^3(D+1)D^2y=0. \tag{9.2.21}$$

### Solution

A fundamental set of solutions of Equation 9.2.21 can be obtained by combining fundamental sets of solutions of

- $\left[(D+1)^2+1\right]^2y=0$
- $(D-1)^3y=0$
- $(D+1)y=0$
- $D^2y=0$

Fundamental sets of solutions of these equations are given by

- $\{e^{-x}\cos x, xe^{-x}\cos x, e^{-x}\sin x, xe^{-x}\sin x\}$
- $\{e^x, xe^x, x^2e^x\}$
- $\{e^{-x}\}$,
- $\{1, x\}$

respectively. These ten functions form a fundamental set of solutions of Equation 9.2.21.

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# 9.2E: Higher Order Constant Coefficient Homogeneous Equations (Exercises)

## Q9.2.1

In **Exercises 9.2.1-9.2.14** find the general solution.

1. $y''' - 3y'' + 3y' - y = 0$
2. $y^{(4)} + 8y'' - 9y = 0$
3. $y''' - y'' + 16y' - 16y = 0$
4. $2y''' + 3y'' - 2y' - 3y = 0$
5. $y''' + 5y'' + 9y' + 5y = 0$
6. $4y''' - 8y'' + 5y' - y = 0$
7. $27y''' + 27y'' + 9y' + y = 0$
8. $y^{(4)} + y'' = 0$
9. $y^{(4)} - 16y = 0$
10. $y^{(4)} + 12y'' + 36y = 0$
11. $16y^{(4)} - 72y'' + 81y = 0$
12. $6y^{(4)} + 5y''' + 7y'' + 5y' + y = 0$
13. $4y^{(4)} + 12y''' + 3y'' - 13y' - 6y = 0$
14. $y^{(4)} - 4y''' + 7y'' - 6y' + 2y = 0$

## Q9.2.2

In **Exercises 9.2.15-9.2.27** solve the initial value problem. Graph the solution for **Exercises 9.2.17-9.2.19** and **9.2.27**.

15. $y''' - 2y'' + 4y' - 8y = 0, \quad y(0) = 2, \quad y'(0) = -2, \; y''(0) = 0$
16. $y''' + 3y'' - y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = 14, \quad y''(0) = -40$
17. $y''' - y'' - y' + y = 0, \quad y(0) = -2, \quad y'(0) = 9, \quad y''(0) = 4$
18. $y''' - 2y' - 4y = 0, \quad y(0) = 6, \quad y'(0) = 3, \quad y''(0) = 22$
19. $3y''' - y'' - 7y' + 5y = 0, \quad y(0) = \frac{14}{5}, \quad y'(0) = 0, \quad y''(0) = 10$
20. $y''' - 6y'' + 12y' - 8y = 0, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = -4$
21. $2y''' - 11y'' + 12y' + 9y = 0, \quad y(0) = 6, \quad y'(0) = 3, \quad y''(0) = 13$
22. $8y''' - 4y'' - 2y' + y = 0, \quad y(0) = 4, \quad y'(0) = -3, \quad y''(0) = -1$
23. $y^{(4)} - 16y = 0, \quad y(0) = 2, \; y'(0) = 2, \; y''(0) = -2, \; y'''(0) = 0$
24. $y^{(4)} - 6y''' + 7y'' + 6y' - 8y = 0, \quad y(0) = -2, \quad y'(0) = -8, \quad y''(0) = -14, \quad y'''(0) = -62$
25. $4y^{(4)} - 13y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 3, \quad y''(0) = 1, \quad y'''(0) = 3$
26. $y^{(4)} + 2y''' - 2y'' - 8y' - 8y = 0, \quad y(0) = 5, \quad y'(0) = -2, \quad y''(0) = 6, \quad y'''(0) = 8$
27. $4y^{(4)} + 8y''' + 19y'' + 32y' + 12y = 0, \quad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -\frac{7}{2}, \quad y'''(0) = \frac{31}{4}$

## Q9.2.3

28. Find a fundamental set of solutions of the given equation, and verify that it is a fundamental set by evaluating its Wronskian at $x = 0$.

a. $(D-1)^2(D-2)y = 0$
b. $(D^2+4)(D-3)y = 0$

c. $(D^2+2D+2)(D-1)y=0$
d. $D^3(D-1)y=0$
e. $(D^2-1)(D^2+1)y=0$
f. $(D^2-2D+2)(D^2+1)y=0$

## Q9.2.4

In **Exercises 9.2.29-9.2.38** find a fundamental set of solutions.

29. $(D^2+6D+13)(D-2)^2D^3y=0$

30. $(D-1)^2(2D-1)^3(D^2+1)y=0$

31. $(D^2+9)^3D^2y=0$

32. $(D-2)^3(D+1)^2Dy=0$

33. $(D^2+1)(D^2+9)^2(D-2)y=0$

34. $(D^4-16)^2y=0$

35. $(4D^2+4D+9)^3y=0$

36. $D^3(D-2)^2(D^2+4)^2y=0$

37. $(4D^2+1)^2(9D^2+4)^3y=0$

38. $\left[(D-1)^4-16\right]y=0$

## Q9.2.5

39 It can be shown that

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n}(a_j-a_i), \tag{A}$$

where the left side is the **Vandermonde determinant** and the right side is the product of all factors of the form $(a_j-a_i)$ with $i$ and $j$ between 1 and $n$ and $i<j$.>

a. Verify (A) for $n=2$ and $n=3$.
b. Find the Wronskian of $\{e^{a_1x}, \quad e^{a_2x},\dots,e^{a_nx}\}$.

40. A theorem from algebra says that if $P_1$ and $P_2$ are polynomials with no common factors then there are polynomials $Q_1$ and $Q_2$ such that

$$Q_1P_1+Q_2P_2=1.$$

This implies that

$$Q_1(D)P_1(D)y+Q_2(D)P_2(D)y=y$$

for every function $y$ with enough derivatives for the left side to be defined.

a. Use this to show that if $P_1$ and $P_2$ have no common factors and

$$P_1(D)y=P_2(D)y=0$$

then $y=0$.
b. Suppose $P_1$ and $P_2$ are polynomials with no common factors. Let $u_1$, ..., $u_r$ be linearly independent solutions of $P_1(D)y=0$ and let $v_1$, ..., $v_s$ be linearly independent solutions of $P_2(D)y=0$. Use (a) to show that $\{u_1,\dots,u_r,\ v_1,\dots,v_s\}$is a linearly independent set.

c. Suppose the characteristic polynomial of the constant coefficient equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \tag{A}$$

has the factorization

$$p(r) = a_0 p_1(r) p_2(r) \cdots p_k(r),$$

where each $p_j$ is of the form

$$p_j(r) = (r - r_j)^{n_j} \text{ or } p_j(r) = [(r - \lambda_j)^2 + w_j^2]^{m_j} \quad (\omega_j > 0)$$

and no two of the polynomials $p_1$, $p_2$, ..., $p_k$ have a common factor. Show that we can find a fundamental set of solutions $\{y_1, y_2, \ldots, y_n\}$ of (A) by finding a fundamental set of solutions of each of the equations

$$p_j(D)y = 0, \quad 1 \le j \le k,$$

and taking $\{y_1, y_2, \ldots, y_n\}$ to be the set of all functions in these separate fundamental sets.

41.

a. Show that if

$$z = p(x)\cos\omega x + q(x)\sin\omega x, \tag{A}$$

where $p$ and $q$ are polynomials of degree $\le k$, then

$$(D^2 + \omega^2)z = p_1(x)\cos\omega x + q_1(x)\sin\omega x,$$

where $p_1$ and $q_1$ are polynomials of degree $\le k - 1$ .

b. Apply (a) $m$ times to show that if $z$ is of the form (A) where $p$ and $q$ are polynomial of degree $\le m - 1$ , then

$$(D^2 + \omega^2)^m z = 0. \tag{B}$$

c. Use Equation 9.2.17 to show that if $y = e^{\lambda x} z$ then

$$[(D - \lambda)^2 + \omega^2]^m y = e^{\lambda x}(D^2 + \omega^2)^m z.$$

d. Conclude from (b) and (c) that if $p$ and $q$ are arbitrary polynomials of degree $\le m - 1$ then

$$y = e^{\lambda x}(p(x)\cos\omega x + q(x)\sin\omega x)$$

is a solution of

$$[(D - \lambda)^2 + \omega^2]^m y = 0. \tag{C}$$

e. Conclude from (d) that the functions

$$\begin{aligned} &e^{\lambda x}\cos\omega x, xe^{\lambda x}\cos\omega x, \quad \ldots, x^{m-1}e^{\lambda x}\cos\omega x, \\ &e^{\lambda x}\sin\omega x, xe^{\lambda x}\sin\omega x, \quad \ldots, x^{m-1}e^{\lambda x}\sin\omega x \end{aligned} \tag{D}$$

are all solutions of (C).

f. Complete the proof of Theorem 9.2.2 by showing that the functions in (D) are linearly independent.

42.

a. Use the trigonometric identities

$$\begin{aligned} \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \sin(A + B) &= \cos A \sin B + \sin A \cos B \end{aligned}$$

to show that

$$(\cos A + i \sin A)(\cos B + i \sin B) = \cos(A + B) + i \sin(A + B).$$

b. Apply (a) repeatedly to show that if $n$ is a positive integer then

$$\prod_{k=1}^{n}(\cos A_k + i\sin A_k) = \cos(A_1 + A_2 + \cdots + A_n) + i\sin(A_1 + A_2 + \cdots + A_n).$$

c. Infer from (b) that if $n$ is a positive integer then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{A}$$

d. Show that (A) also holds if $n = 0$ or a negative integer. HINT: *Verify by direct calculation that*

$$(\cos\theta + i\sin\theta)^{-1} = (\cos\theta - i\sin\theta).$$

**Then replace $\theta$ by $-\theta$ in $(A)$.**

e. Now suppose $n$ is a positive integer. Infer from (A) that if

$$z_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1,$$

and

$$\zeta_k = \cos\left(\frac{(2k+1)\pi}{n}\right) + i\sin\left(\frac{(2k+1)\pi}{n}\right), \quad k = 0, 1, \ldots, n-1,$$

then

$$z_k^n = 1 \quad \text{and} \quad \zeta_k^n = -1, \quad k = 0, 1, \ldots, n-1.$$

(Why don't we also consider other integer values for $k$?)

f. Let $\rho$ be a positive number. Use (e) to show that

$$z^n - \rho = (z - \rho^{1/n}z_0)(z - \rho^{1/n}z_1)\cdots(z - \rho^{1/n}z_{n-1})$$

and

$$z^n + \rho = (z - \rho^{1/n}\zeta_0)(z - \rho^{1/n}\zeta_1)\cdots(z - \rho^{1/n}\zeta_{n-1}).$$

43. Use (e) of **Exercise 9.2.42** to find a fundamental set of solutions of the given equation.

a. $y''' - y = 0$
b. $y''' + y = 0$
c. $y^{(4)} + 64y = 0$
d. $y^{(6)} - y = 0$
e. $y^{(6)} + 64y = 0$
f. $\left[(D-1)^6 - 1\right] y = 0$
g. $y^{(5)} + y^{(4)} + y''' + y'' + y' + y = 0$

44. An equation of the form

$$a_0x^n y^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy' + a_n y = 0, \quad x > 0, \tag{A}$$

where $a_0$, $a_1$, …, $a_n$ are constants, is an **Euler** or **equidimensional** equation. Show that if

$$x = e^t \quad \text{and} \quad Y(t) = y(x(t)), \tag{B}$$

then

$$x\frac{dy}{dx} = \frac{dY}{dt}$$
$$x^2\frac{d^2y}{dx^2} = \frac{d^2Y}{dt^2} - \frac{dY}{dt}$$
$$x^3\frac{d^3y}{dx^3} = \frac{d^3Y}{dt^3} - 3\frac{d^2Y}{dt^2} + 2\frac{dY}{dt}.$$

In general, it can be shown that if $r$ is any integer $\geq 2$ then

$$x^r \frac{d^r y}{dx^r} = \frac{d^r Y}{dt^r} + A_{1r} \frac{d^{r-1} Y}{dt^{r-1}} + \cdots + A_{r-1,r} \frac{dY}{dt}$$

where $A_{1r}$, …, $A_{r-1,r}$ are integers. Use these results to show that the substitution (B) transforms (A) into a constant coefficient equation for $Y$ as a function of $t$.

45. Use **Exercise 9.2.44** to show that a function $y = y(x)$ satisfies the equation

$$a_0 x^3 y''' + a_1 x^2 y'' + a_2 x y' + a_3 y = 0, \tag{A}$$

on $(0, \infty)$ if and only if the function $Y(t) = y(e^t)$ satisfies

$$a_0 \frac{d^3 Y}{dt^3} + (a_1 - 3a_0) \frac{d^2 Y}{dt^2} + (a_2 - a_1 + 2a_0) \frac{dY}{dt} + a_3 Y = 0.$$

Assuming that $a_0$, $a_1$, $a_2$, $a_3$ are real and $a_0 \neq 0$, find the possible forms for the general solution of (A).

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# 9.3: Undetermined Coefficients for Higher Order Equations

In this section we consider the constant coefficient equation

$$a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_ny=F(x), \tag{9.3.1}$$

where $n\ge 3$ and $F$ is a linear combination of functions of the form

$$e^{\alpha x}\left(p_0+p_1x+\cdots+p_kx^k\right)$$

or

$$e^{\lambda x}\left[\left(p_0+p_1x+\cdots+p_kx^k\right)\cos\omega x+\left(q_0+q_1x+\cdots+q_kx^k\right)\sin\omega x\right].$$

From Theorem 9.1.5, the general solution of Equation 9.3.1 is $y=y_p+y_c$ , where $y_p$ is a particular solution of Equation 9.3.1 and $y_c$ is the general solution of the complementary equation

$$a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_ny=0.$$

In Section 9.2 we learned how to find $y_c$. Here we will learn how to find $y_p$ when the forcing function has the form stated above. The procedure that we use is a generalization of the method that we used in Sections 5.4 and 5.5, and is again called **method of undetermined coefficients**. Since the underlying ideas are the same as those in these section, we'll give an informal presentation based on examples.

## Forcing Functions of the Form $e^{\alpha x}(p_0+p_1x+\ldots+p_kx^k)$

We first consider equations of the form

$$a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_ny=e^{\alpha x}\left(p_0+p_1x+\cdots+p_kx^k\right).$$

**Example 9.3.1**

Find a particular solution of

$$y'''+3y''+2y'-y=e^x(21+24x+28x^2+5x^3). \tag{9.3.2}$$

**Solution**

Substituting

$$\begin{aligned} y&=ue^x,\\ y'&=e^x(u'+u),\\ y''&=e^x(u''+2u'+u),\\ y'''&=e^x(u'''+3u''+3u'+u) \end{aligned}$$

into Equation 9.3.2 and canceling $e^x$ yields

$$(u'''+3u''+3u'+u)+3(u''+2u'+u)+2(u'+u)-u=21+24x+28x^2+5x^3,$$

or

$$u'''+6u''+11u'+5u=21+24x+28x^2+5x^3. \tag{9.3.3}$$

Since the unknown $u$ appears on the left, we can see that Equation 9.3.3 has a particular solution of the form

$$u_p=A+Bx+Cx^2+Dx^3.$$

Then

$$\begin{aligned} u_p'&=B+2Cx+3Dx^2\\ u_p''&=2C+6Dx\\ u_p'''&=6D. \end{aligned}$$

Substituting from the last four equations into the left side of Equation 9.3.3 yields

$$\begin{aligned} u_p'''+6u_p''+11u_p'+5u_p &= 6D+6(2C+6Dx)+11(B+2Cx+3Dx^2) \\ &\quad +5(A+Bx+Cx^2+Dx^3) \\ &= (5A+11B+12C+6D)+(5B+22C+36D)x \\ &\quad +(5C+33D)x^2+5Dx^3. \end{aligned}$$

Comparing coefficients of like powers of $x$ on the right sides of this equation and Equation 9.3.3 shows that $u_p$ satisfies Equation 9.3.3 if

$$\begin{aligned} 5D &= 5 \\ 5C+33D &= 28 \\ 5B+22C+36D &= 24 \\ 5A+11B+12C+6D &= 21. \end{aligned}$$

Solving these equations successively yields $D=1$, $C=-1$, $B=2$, $A=1$. Therefore

$$u_p = 1+2x-x^2+x^3$$

is a particular solution of Equation 9.3.3, so

$$y_p = e^x u_p = e^x(1+2x-x^2+x^3)$$

is a particular solution of Equation 9.3.2 (Figure 9.3.1 ).

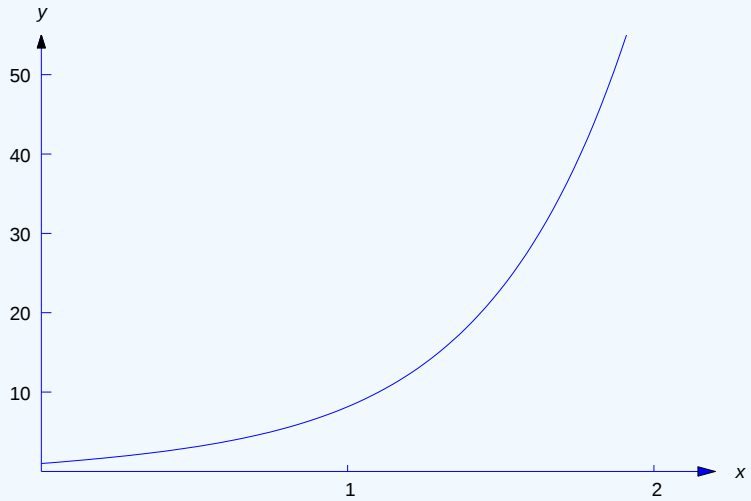



Figure 9.3.1 : $y_p = e^x(1+2x-x^2+x^3)$

## Example 9.3.2

Find a particular solution of

$$y^{(4)}-y'''-6y''+4y'+8y = e^{2x}(4+19x+6x^2). \tag{9.3.4}$$

**Solution**

Substituting

$$\begin{aligned} y &= ue^{2x}, \\ y' &= e^{2x}(u'+2u), \\ y'' &= e^{2x}(u''+4u'+4u), \\ y''' &= e^{2x}(u'''+6u''+12u'+8u), \\ y^{(4)} &= e^{2x}(u^{(4)}+8u'''+24u''+32u'+16u) \end{aligned}$$

into Equation 9.3.4 and canceling $e^{2x}$ yields

$$\begin{aligned} &(u^{(4)}+8u'''+24u''+32u'+16u)-(u'''+6u''+12u'+8u) \\ &-6(u''+4u'+4u)+4(u'+2u)+8u = 4+19x+6x^2, \end{aligned}$$

or

$$u^{(4)}+7u'''+12u'' = 4+19x+6x^2. \tag{9.3.5}$$

Since neither $u$ nor $u'$ appear on the left, we can see that Equation 9.3.5 has a particular solution of the form

$$u_p = Ax^2+Bx^3+Cx^4. \tag{9.3.6}$$

Then

$$\begin{aligned} u_p' &= 2Ax+3Bx^2+4Cx^3 \\ u_p'' &= 2A+6Bx+12Cx^2 \\ u_p''' &= 6B+24Cx \\ u_p^{(4)} &= 24C. \end{aligned}$$

Substituting $u_p''$, $u_p'''$, and $u_p^{(4)}$ into the left side of Equation 9.3.5 yields

$$\begin{aligned} u_p^{(4)}+7u_p'''+12u_p'' &= 24C+7(6B+24Cx)+12(2A+6Bx+12Cx^2) \\ &= (24A+42B+24C)+(72B+168C)x+144Cx^2. \end{aligned}$$

Comparing coefficients of like powers of $x$ on the right sides of this equation and Equation 9.3.5 shows that $u_p$ satisfies Equation 9.3.5 if

$$\begin{aligned} 144C &= 6 \\ 72B+168C &= 19 \\ 24A+42B+24C &= 4. \end{aligned}$$

Solving these equations successively yields $C = 1/24$, $B = 1/6$, $A = -1/6$. Substituting these into Equation 9.3.6 shows that

$$u_p = \frac{x^2}{24}(-4+4x+x^2)$$

is a particular solution of Equation 9.3.5, so

$$y_p = e^{2x}u_p = \frac{x^2e^{2x}}{24}(-4+4x+x^2)$$

is a particular solution of Equation 9.3.4 (Figure 9.3.2 ).

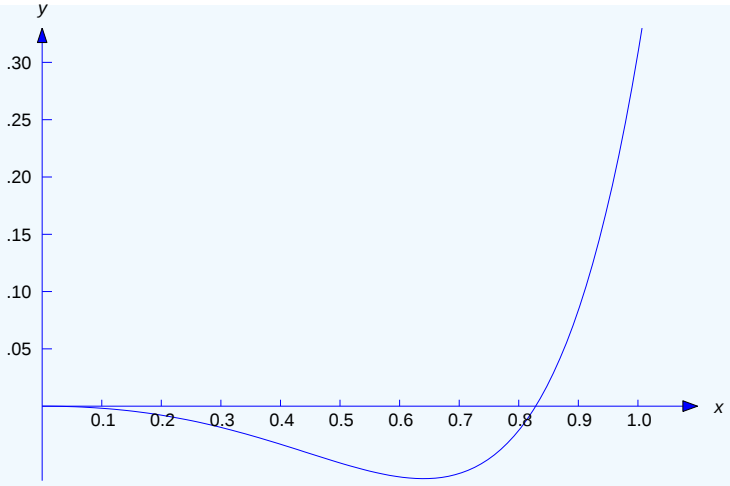



Figure 9.3.2 : $y_p = \dfrac{x^2 e^{2x}}{24}(-4+4x+x^2)$

## Forcing Functions of the Form $e^{\alpha x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$

We now consider equations of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right),$$

where $P$ and $Q$ are polynomials.

### Example 9.3.3

Find a particular solution of

$$y''' + y'' - 4y' - 4y = e^x[(5-5x)\cos x + (2+5x)\sin x]. \tag{9.3.7}$$

**Solution**

Substituting

$$\begin{aligned} y &= ue^x, \\ y' &= e^x(u'+u), \\ y'' &= e^x(u''+2u'+u), \\ y''' &= e^x(u'''+3u''+3u'+u) \end{aligned}$$

into Equation 9.3.7 and canceling $e^x$ yields

$$(u'''+3u''+3u'+u)+(u''+2u'+u)-4(u'+u)-4u = (5-5x)\cos x + (2+5x)\sin x,$$

or

$$u''' + 4u'' + u' - 6u = (5-5x)\cos x + (2+5x)\sin x. \tag{9.3.8}$$

Since $\cos x$ and $\sin x$ are not solutions of the complementary equation

$$u''' + 4u'' + u' - 6u = 0,$$

a theorem analogous to Theorem 5.5.1 implies that Equation 9.3.8 has a particular solution of the form

$$u_p = (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x. \tag{9.3.9}$$

Then

$$\begin{aligned} u_p' &= (A_1 + B_0 + B_1 x)\cos x + (B_1 - A_0 - A_1 x)\sin x, \\ u_p' &= (2B_1 - A_0 - A_1 x)\cos x - (2A_1 + B_0 + B_1 x)\sin x, \\ u_p''' &= -(3A_1 + B_0 + B_1 x)\cos x - (3B_1 - A_0 - A_1 x)\sin x, \end{aligned}$$

so

$$u_p''' + 4u_p'' + u_p' - 6u_p = -\left[10A_0 + 2A_1 - 8B_1 + 10A_1 x\right]\cos x - \left[10B_0 + 2B_1 + 8A_1 + 10B_1 x\right]\sin x.$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in Equation 9.3.8 shows that $u_p$ is a solution of Equation 9.3.8 if

$$\begin{aligned} -10A_1 &= -5 \\ -10B_1 &= 5 \\ -10A_0 - 2A_1 + 8B_1 &= 5 \\ -10B_0 - 2B_1 - 8A_1 &= 2. \end{aligned}$$

Solving the first two equations yields $A_1 = 1/2$, $B_1 = -1/2$. Substituting these into the last two equations yields

$$\begin{aligned} -10A_0 &= 5 + 2A_1 - 8B_1 = 10 \\ -10B_0 &= 2 + 2B_1 + 8A_1 = 5, \end{aligned}$$

so $A_0 = -1$, $B_0 = -1/2$. Substituting $A_0 = -1$, $A_1 = 1/2$, $B_0 = -1/2$, $B_1 = -1/2$ into Equation 9.3.9 shows that

$$u_p = -\frac{1}{2}\left[(2-x)\cos x + (1+x)\sin x\right]$$

is a particular solution of Equation 9.3.8, so

$$y_p = e^x u_p = -\frac{e^x}{2}\left[(2-x)\cos x + (1+x)\sin x\right]$$

is a particular solution of Equation 9.3.7 (Figure 9.3.3 ).

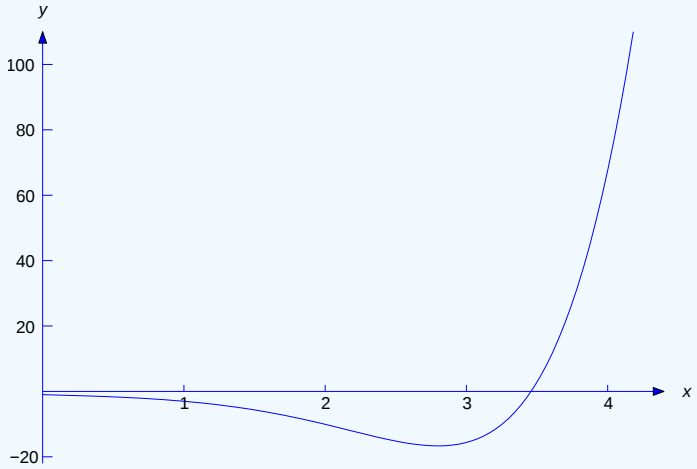



Figure 9.3.3 : $y_p = \{e^x u_p = -\frac{e^x}{2}\left[(2-x)\cos x + (1+x)\sin x\right]$

## Example 9.3.4

Find a particular solution of

$$y''' + 4y'' + 6y' + 4y = e^{-x}\left[(1-6x)\cos x - (3+2x)\sin x\right]. \tag{9.3.10}$$

**Solution**

Substituting

$$\begin{aligned} y &= ue^{-x}, \\ y' &= e^{-x}(u'-u), \\ y'' &= e^{-x}(u''-2u'+u), \\ y''' &= e^{-x}(u'''-3u''+3u'-u) \end{aligned}$$

into Equation 9.3.10 and canceling $e^{-x}$ yields

$$(u'''-3u''+3u'-u)+4(u''-2u'+u)+6(u'-u)+4u=(1-6x)\cos x-(3+2x)\sin x,$$

or

$$u'''+u''+u'+u=(1-6x)\cos x-(3+2x)\sin x. \tag{9.3.11}$$

Since $\cos x$ and $\sin x$ are solutions of the complementary equation

$$u'''+u''+u'+u=0,$$

a theorem analogous to Theorem 5.5.1 implies that Equation 9.3.11 has a particular solution of the form

$$u_p=(A_0x+A_1x^2)\cos x+(B_0x+B_1x^2)\sin x. \tag{9.3.12}$$

Then

$$\begin{aligned} u_p' = {} & [A_0+(2A_1+B_0)x+B_1x^2]\cos x+[B_0+(2B_1-A_0)x-A_1x^2]\sin x, \\ u_p'' = {} & [2A_1+2B_0-(A_0-4B_1)x-A_1x^2]\cos x \\ & +[2B_1-2A_0-(B_0+4A_1)x-B_1x^2]\sin x, \\ u_p''' = {} & -[3A_0-6B_1+(6A_1+B_0)x+B_1x^2]\cos x \\ & -[3B_0+6A_1+(6B_1-A_0)x-A_1x^2]\sin x, \end{aligned}$$

so

$$\begin{aligned} u_p'''+u_p''+u_p'+u_p = {} & -[2A_0-2B_0-2A_1-6B_1+(4A_1-4B_1)x]\cos x \\ & -[2B_0+2A_0-2B_1+6A_1+(4B_1+4A_1)x]\sin x. \end{aligned}$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in Equation 9.3.11 shows that $u_p$ is a solution of Equation 9.3.11 if

$$\begin{aligned} -4A_1+4B_1 &= -6 \\ -4A_1-4B_1 &= -2 \\ -2A_0+2B_0+2A_1+6B_1 &= 1 \\ -2A_0-2B_0-6A_1+2B_1 &= -3. \end{aligned}$$

Solving the first two equations yields $A_1=1$, $B_1=-1/2$. Substituting these into the last two equations yields

$$\begin{aligned} -2A_0+2B_0 &= 1-2A_1-6B_1=2 \\ -2A_0-2B_0 &= -3+6A_1-2B_1=4, \end{aligned}$$

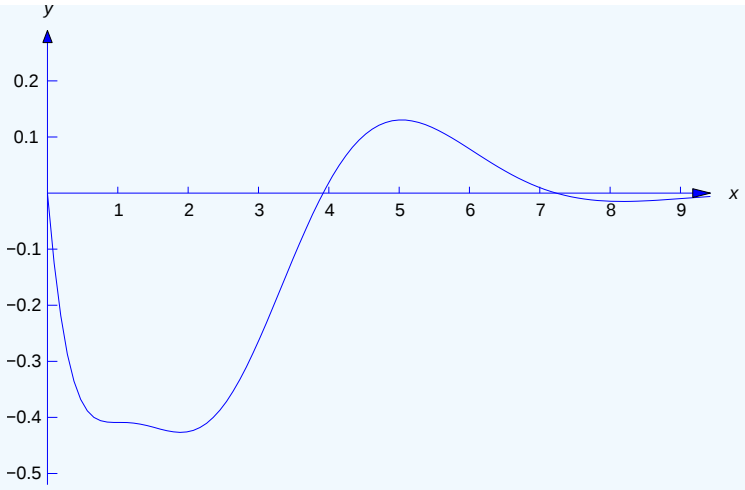



Figure 9.3.4 : $y_p = -\dfrac{xe^{-x}}{2}[(3-2x)\cos x + (1+x)\sin x]$

so $A_0 = -3/2$ and $B_0 = -1/2$. Substituting $A_0 = -3/2$, $A_1 = 1$, $B_0 = -1/2$, $B_1 = -1/2$ into Equation 9.3.12 shows that

$$u_p = -\frac{x}{2}[(3-2x)\cos x + (1+x)\sin x]$$

is a particular solution of Equation 9.3.11, so

$$y_p = e^{-x}u_p = -\frac{xe^{-x}}{2}[(3-2x)\cos x + (1+x)\sin x]$$

Figure 9.3.4 is a particular solution of Equation 9.3.10.

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# 9.3E: Undetermined Coefficients for Higher Order Equations (Exercises)

## Q9.3.1

In **Exercises 9.3.1-9.3.59** find a particular solution.

1. $y''' - 6y'' + 11y' - 6y = -e^{-x}(4 + 76x - 24x^2)$
2. $y''' - 2y'' - 5y' + 6y = e^{-3x}(32 - 23x + 6x^2)$
3. $4y''' + 8y'' - y' - 2y = -e^{x}(4 + 45x + 9x^2)$
4. $y''' + 3y'' - y' - 3y = e^{-2x}(2 - 17x + 3x^2)$
5. $y''' + 3y'' - y' - 3y = e^{x}(-1 + 2x + 24x^2 + 16x^3)$
6. $y''' + y'' - 2y = e^{x}(14 + 34x + 15x^2)$
7. $4y''' + 8y'' - y' - 2y = -e^{-2x}(1 - 15x)$
8. $y''' - y'' - y' + y = e^{x}(7 + 6x)$
9. $2y''' - 7y'' + 4y' + 4y = e^{2x}(17 + 30x)$
10. $y''' - 5y'' + 3y' + 9y = 2e^{3x}(11 - 24x^2)$
11. $y''' - 7y'' + 8y' + 16y = 2e^{4x}(13 + 15x)$
12. $8y''' - 12y'' + 6y' - y = e^{x/2}(1 + 4x)$
13. $y^{(4)} + 3y''' - 3y'' - 7y' + 6y = -e^{-x}(12 + 8x - 8x^2)$
14. $y^{(4)} + 3y''' + y'' - 3y' - 2y = -3e^{2x}(11 + 12x)$
15. $y^{(4)} + 8y''' + 24y'' + 32y' = -16e^{-2x}(1 + x + x^2 - x^3)$
16. $4y^{(4)} - 11y'' - 9y' - 2y = -e^{x}(1 - 6x)$
17. $y^{(4)} - 2y''' + 3y' - y = e^{x}(3 + 4x + x^2)$
18. $y^{(4)} - 4y''' + 6y'' - 4y' + 2y = e^{2x}(24 + x + x^4)$
19. $2y^{(4)} + 5y''' - 5y' - 2y = 18e^{x}(5 + 2x)$
20. $y^{(4)} + y''' - 2y'' - 6y' - 4y = -e^{2x}(4 + 28x + 15x^2)$
21. $2y^{(4)} + y''' - 2y' - y = 3e^{-x/2}(1 - 6x)$
22. $y^{(4)} - 5y'' + 4y = e^{x}(3 + x - 3x^2)$
23. $y^{(4)} - 2y''' - 3y'' + 4y' + 4y = e^{2x}(13 + 33x + 18x^2)$
24. $y^{(4)} - 3y''' + 4y' = e^{2x}(15 + 26x + 12x^2)$
25. $y^{(4)} - 2y''' + 2y' - y = e^{x}(1 + x)$
26. $2y^{(4)} - 5y''' + 3y'' + y' - y = e^{x}(11 + 12x)$
27. $y^{(4)} + 3y''' + 3y'' + y' = e^{-x}(5 - 24x + 10x^2)$
28. $y^{(4)} - 7y''' + 18y'' - 20y' + 8y = e^{2x}(3 - 8x - 5x^2)$
29. $y''' - y'' - 4y' + 4y = e^{-x}\left[(16 + 10x)\cos x + (30 - 10x)\sin x\right]$
30. $y''' + y'' - 4y' - 4y = e^{-x}\left[(1 - 22x)\cos 2x - (1 + 6x)\sin 2x\right]$
31. $y''' - y'' + 2y' - 2y = e^{2x}[(27 + 5x - x^2)\cos x + (2 + 13x + 9x^2)\sin x]$
32. $y''' - 2y'' + y' - 2y = -e^{x}[(9 - 5x + 4x^2)\cos 2x - (6 - 5x - 3x^2)\sin 2x]$
33. $y''' + 3y'' + 4y' + 12y = 8\cos 2x - 16\sin 2x$
34. $y''' - y'' + 2y = e^{x}[(20 + 4x)\cos x - (12 + 12x)\sin x]$

35. $y''' - 7y'' + 20y' - 24y = -e^{2x}[(13 - 8x)\cos 2x - (8 - 4x)\sin 2x]$
36. $y''' - 6y'' + 18y' = -e^{3x}[(2 - 3x)\cos 3x - (3 + 3x)\sin 3x]$
37. $y^{(4)} + 2y''' - 2y'' - 8y' - 8y = e^{x}(8\cos x + 16\sin x)$
38. $y^{(4)} - 3y''' + 2y'' + 2y' - 4y = e^{x}(2\cos 2x - \sin 2x)$
39. $y^{(4)} - 8y''' + 24y'' - 32y' + 15y = e^{2x}(15x\cos 2x + 32\sin 2x)$
40. $y^{(4)} + 6y''' + 13y'' + 12y' + 4y = e^{-x}[(4 - x)\cos x - (5 + x)\sin x]$
41. $y^{(4)} + 3y''' + 2y'' - 2y' - 4y = -e^{-x}(\cos x - \sin x)$
42. $y^{(4)} - 5y''' + 13y'' - 19y' + 10y = e^{x}(\cos 2x + \sin 2x)$
43. $y^{(4)} + 8y''' + 32y'' + 64y' + 39y = e^{-2x}[(4 - 15x)\cos 3x - (4 + 15x)\sin 3x]$
44. $y^{(4)} - 5y''' + 13y'' - 19y' + 10y = e^{x}[(7 + 8x)\cos 2x + (8 - 4x)\sin 2x]$
45. $y^{(4)} + 4y''' + 8y'' + 8y' + 4y = -2e^{-x}(\cos x - 2\sin x)$
46. $y^{(4)} - 8y''' + 32y'' - 64y' + 64y = e^{2x}(\cos 2x - \sin 2x)$
47. $y^{(4)} - 8y''' + 26y'' - 40y' + 25y = e^{2x}[3\cos x - (1 + 3x)\sin x]$
48. $y''' - 4y'' + 5y' - 2y = e^{2x} - 4e^{x} - 2\cos x + 4\sin x$
49. $y''' - y'' + y' - y = 5e^{2x} + 2e^{x} - 4\cos x + 4\sin x$
50. $y''' - y' = -2(1 + x) + 4e^{x} - 6e^{-x} + 96e^{3x}$
51. $y''' - 4y'' + 9y' - 10y = 10e^{2x} + 20e^{x}\sin 2x - 10$
52. $y''' + 3y'' + 3y' + y = 12e^{-x} + 9\cos 2x - 13\sin 2x$
53. $y''' + y'' - y' - y = 4e^{-x}(1 - 6x) - 2x\cos x + 2(1 + x)\sin x$
54. $y^{(4)} - 5y'' + 4y = -12e^{x} + 6e^{-x} + 10\cos x$
55. $y^{(4)} - 4y''' + 11y'' - 14y' + 10y = -e^{x}(\sin x + 2\cos 2x)$
56. $y^{(4)} + 2y''' - 3y'' - 4y' + 4y = 2e^{x}(1 + x) + e^{-2x}$
57. $y^{(4)} + 4y = \sinh x\cos x - \cosh x\sin x$
58. $y^{(4)} + 5y''' + 9y'' + 7y' + 2y = e^{-x}(30 + 24x) - e^{-2x}$
59. $y^{(4)} - 4y''' + 7y'' - 6y' + 2y = e^{x}(12x - 2\cos x + 2\sin x)$

## Q9.3.2

In **Exercises 9.3.60-9.3.68** find the general solution.

60. $y''' - y'' - y' + y = e^{2x}(10 + 3x)$
61. $y''' + y'' - 2y = -e^{3x}(9 + 67x + 17x^{2})$
62. $y''' - 6y'' + 11y' - 6y = e^{2x}(5 - 4x - 3x^{2})$
63. $y''' + 2y'' + y' = -2e^{-x}(7 - 18x + 6x^{2})$
64. $y''' - 3y'' + 3y' - y = e^{x}(1 + x)$
65. $y^{(4)} - 2y'' + y = -e^{-x}(4 - 9x + 3x^{2})$
66. $y''' + 2y'' - y' - 2y = e^{-2x}\left[(23 - 2x)\cos x + (8 - 9x)\sin x\right]$
67. $y^{(4)} - 3y''' + 4y'' - 2y' = e^{x}\left[(28 + 6x)\cos 2x + (11 - 12x)\sin 2x\right]$
68. $y^{(4)} - 4y''' + 14y'' - 20y' + 25y = e^{x}\left[(2 + 6x)\cos 2x + 3\sin 2x\right]$

## Q9.3.3

In **Exercises 9.3.69-9.3.74** solve the initial value problem and graph the solution.

69. $y''' - 2y'' - 5y' + 6y = 2e^x(1-6x), \quad y(0) = 2, \quad y'(0) = 7, \quad y''(0) = 9$

70. $y''' - y'' - y' + y = -e^{-x}(4-8x), \quad y(0) = 2, \quad y'(0) = 0, \quad y''(0) = 0$

71. $4y''' - 3y' - y = e^{-x/2}(2-3x), \quad y(0) = -1, \quad y'(0) = 15, \quad y''(0) = -17$

72. $y^{(4)} + 2y''' + 2y'' + 2y' + y = e^{-x}(20-12x), \; y(0) = 3, \; y'(0) = -4, \; y''(0) = 7, \; y'''(0) = -22$

73. $y''' + 2y'' + y' + 2y = 30\cos x - 10\sin x, \quad y(0) = 3, \quad y'(0) = -4, \quad y''(0) = 16$

74. $y^{(4)} - 3y''' + 5y'' - 2y' = -2e^x(\cos x - \sin x), \; y(0) = 2, \; y'(0) = 0, \; y''(0) = -1, \; y'''(0) = -5$

## Q9.3.4

75. Prove: A function $y$ is a solution of the constant coefficient nonhomogeneous equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = e^{\alpha x} G(x) \tag{A}$$

if and only if $y = ue^{\alpha x}$, where $u$ satisfies the differential equation

$$a_0 u^{(n)} + \frac{p^{(n-1)}(\alpha)}{(n-1)!} u^{(n-1)} + \frac{p^{(n-2)}(\alpha)}{(n-2)!} u^{(n-2)} + \cdots + p(\alpha) u = G(x) \tag{B}$$

and

$$p(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_n$$

is the characteristic polynomial of the complementary equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0.$$

76. Prove:

a. The equation

$$\begin{aligned} a_0 u^{(n)} \quad &+ \quad \tfrac{p^{(n-1)}(\alpha)}{(n-1)!} u^{(n-1)} + \tfrac{p^{(n-2)}(\alpha)}{(n-2)!} u^{(n-2)} + \cdots + P(\alpha) u \\ &= \quad (p_0 + p_1 x + \cdots + p_k x^k)\cos \omega x \\ &+ \quad (q_0 + q_1 x + \cdots + q_k x^k)\sin \omega x \end{aligned} \tag{A}$$

has a particular solution of the form

$$u_p = x^m \left(u_0 + u_1 x + \cdots + u_k x^k\right)\cos \omega x + \left(v_0 + v_1 x + \cdots + v_k x^k\right)\sin \omega x.$$

b. If $\lambda + i\omega$ is a zero of $p$ with multiplicity $m \geq 1$, then (A) can be written as

$$a(u'' + \omega^2 u) = \left(p_0 + p_1 x + \cdots + p_k x^k\right)\cos \omega x + \left(q_0 + q_1 x + \cdots + q_k x^k\right)\sin \omega x,$$

which has a particular solution of the form

$$u_p = U(x)\cos \omega x + V(x)\sin \omega x,$$

where

$$U(x) = u_0 x + u_1 x^2 + \cdots + u_k x^{k+1}, \; V(x) = v_0 x + v_1 x^2 + \cdots + v_k x^{k+1}$$

and

$$\begin{aligned} a(U''(x) + 2\omega V'(x)) \quad &= \quad p_0 + p_1 x + \cdots + p_k x^k \\ a(V''(x) - 2\omega U'(x)) \quad &= \quad q_0 + q_1 x + \cdots + q_k x^k. \end{aligned}$$

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# 9.4: Variation of Parameters for Higher Order Equations

## Derivation of the method

We assume throughout this section that the nonhomogeneous linear equation

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=F(x) \tag{9.4.1}$$

is normal on an interval $(a,b)$. We'll abbreviate this equation as $Ly=F$, where

$$Ly=P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y.$$

When we speak of solutions of this equation and its complementary equation $Ly=0$, we mean solutions on $(a,b)$. We'll show how to use the method of variation of parameters to find a particular solution of $Ly=F$, provided that we know a fundamental set of solutions $\{y_1,y_2,\dots,y_n\}$ of $Ly=0$.

We seek a particular solution of $Ly=F$ in the form

$$y_p=u_1y_1+u_2y_2+\cdots+u_ny_n \tag{9.4.2}$$

where $\{y_1,y_2,\dots,y_n\}$ is a known fundamental set of solutions of the complementary equation

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=0$$

and $u_1$, $u_2$, …, $u_n$ are functions to be determined. We begin by imposing the following $n-1$ conditions on $u_1,u_2,\dots,u_n$:

$$\begin{aligned}
u_1'y_1+u_2'y_2+&\quad\cdots\quad+u_n'y_n=0\\
u_1'y_1'+u_2'y_2'+&\quad\cdots\quad+u_n'y_n'=0\\
&\quad\vdots\\
u_1'y_1^{(n-2)}+u_2'y_2^{(n-2)}+&\quad\cdots\quad+u_n'y_n^{(n-2)}=0.
\end{aligned} \tag{9.4.3}$$

These conditions lead to simple formulas for the first $n-1$ derivatives of $y_p$:

$$y_p^{(r)}=u_1y_1^{(r)}+u_2y_2^{(r)}\cdots+u_ny_n^{(r)},\ 0\le r\le n-1. \tag{9.4.4}$$

These formulas are easy to remember, since they look as though we obtained them by differentiating Equation 9.4.2 $n-1$ times while treating $u_1$, $u_2$, …, $u_n$ as constants. To see that Equation 9.4.3 implies Equation 9.4.4, we first differentiate Equation 9.4.2 to obtain

$$y_p'=u_1y_1'+u_2y_2'+\cdots+u_ny_n'+u_1'y_1+u_2'y_2+\cdots+u_n'y_n,$$

which reduces to

$$y_p'=u_1y_1'+u_2y_2'+\cdots+u_ny_n'$$

because of the first equation in Equation 9.4.3. Differentiating this yields

$$y_p''=u_1y_1''+u_2y_2''+\cdots+u_ny_n''+u_1'y_1'+u_2'y_2'+\cdots+u_n'y_n',$$

which reduces to

$$y_p''=u_1y_1''+u_2y_2''+\cdots+u_ny_n''$$

because of the second equation in Equation 9.4.3. Continuing in this way yields Equation 9.4.4.

The last equation in Equation 9.4.4 is

$$y_p^{(n-1)}=u_1y_1^{(n-1)}+u_2y_2^{(n-1)}+\cdots+u_ny_n^{(n-1)}.$$

Differentiating this yields

$$y_p^{(n)}=u_1y_1^{(n)}+u_2y_2^{(n)}+\cdots+u_ny_n^{(n)}+u_1'y_1^{(n-1)}+u_2'y_2^{(n-1)}+\cdots+u_n'y_n^{(n-1)}.$$

Substituting this and Equation 9.4.4 into Equation 9.4.1 yields

$$u_1Ly_1+u_2Ly_2+\cdots+u_nLy_n+P_0(x)\left(u_1'y_1^{(n-1)}+u_2'y_2^{(n-1)}+\cdots+u_n'y_n^{(n-1)}\right)=F(x).$$

Since $Ly_i=0\ (1\le i\le n)$, this reduces to

$$u_1'y_1^{(n-1)}+u_2'y_2^{(n-1)}+\cdots+u_n'y_n^{(n-1)}=\frac{F(x)}{P_0(x)}.$$

Combining this equation with Equation 9.4.3 shows that

$$y_p=u_1y_1+u_2y_2+\cdots+u_ny_n$$

is a solution of Equation 9.4.1 if

$$\begin{aligned}
u_1'y_1+u_2'y_2+&\quad\cdots\quad+u_n'y_n=0\\
u_1'y_1'+u_2'y_2'+&\quad\cdots\quad+u_n'y_n'=0\\
&\quad\vdots\\
u_1'y_1^{(n-2)}+u_2'y_2^{(n-2)}+&\quad\cdots\quad+u_n'y_n^{(n-2)}=0\\
u_1'y_1^{(n-1)}+u_2'y_2^{(n-1)}+&\quad\cdots\quad+u_n'y_n^{(n-1)}=F/P_0,
\end{aligned}$$

which can be written in matrix form as

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n\\ y_1' & y_2' & \cdots & y_n'\\ \vdots & \vdots & \ddots & \vdots\\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)}\\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}\begin{bmatrix} u_1'\\ u_2'\\ \vdots\\ u_{n-1}'\\ u_n' \end{bmatrix}=\begin{bmatrix} 0\\ 0\\ \vdots\\ 0\\ F/P_0 \end{bmatrix}. \tag{9.4.5}$$

The determinant of this system is the Wronskian $W$ of the fundamental set of solutions $\{y_1,y_2,\dots,y_n\}$, which has no zeros on $(a,b)$, by Theorem 9.1.4. Solving Equation 9.4.5 by Cramer's rule yields

$$u_j'=(-1)^{n-j}\frac{FW_j}{P_0W},\quad 1\le j\le n, \tag{9.4.6}$$

where $W_j$ is the Wronskian of the set of functions obtained by deleting $y_j$ from $\{y_1,y_2,\dots,y_n\}$ and keeping the remaining functions in the same order. Equivalently, $W_j$ is the determinant obtained by deleting the last row and $j$-th column of $W$.

Having obtained $u_1'$, $u_2'$, $\dots$ ,$u_n'$, we can integrate to obtain $u_1$, $u_2,\dots,u_n$. As in Section 5.7, we take the constants of integration to be zero, and we drop any linear combination of $\{y_1,y_2,\dots,y_n\}$ that may appear in $y_p$.

**Note**

For efficiency, it's best to compute $W_1,W_2,\cdots,W_n$ first, and then compute $W$ by expanding in cofactors of the last row; thus,

$$W=\sum_{j=1}^{n}(-1)^{n-j}y_j^{(n-1)}W_j.$$

## Third Order Equations

If $n=3$, then

$$W=\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

Therefore

$$W_1=\begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix}, \quad W_2=\begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix}, \quad W_3=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$

and Equation 9.4.6 becomes

$$u_1'=\frac{FW_1}{P_0W}, \quad u_2'=-\frac{FW_2}{P_0W}, \quad u_3'=\frac{FW_3}{P_0W}. \tag{9.4.7}$$

## Example 9.4.1

Find a particular solution of

$$xy'''-y''-xy'+y=8x^2e^x, \tag{9.4.8}$$

given that $y_1=x$, $y_2=e^x$, and $y_3=e^{-x}$ form a fundamental set of solutions of the complementary equation. Then find the general solution of Equation 9.4.8.

### Solution

We seek a particular solution of Equation 9.4.8 of the form

$$y_p=u_1x+u_2e^x+u_3e^{-x}.$$

The Wronskian of $\{y_1,y_2,y_3\}$ is

$$W(x)=\begin{vmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix},$$

so

$$\begin{aligned} W_1&=\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}=-2, \\ W_2&=\begin{vmatrix} x & e^{-x} \\ 1 & -e^{-x} \end{vmatrix}=-e^{-x}(x+1), \\ W_3&=\begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}=e^x(x-1). \end{aligned}$$

Expanding $W$ by cofactors of the last row yields

$$W=0W_1-e^xW_2+e^{-x}W_3=0(-2)-e^x\left(-e^{-x}(x+1)\right)+e^{-x}e^x(x-1)=2x.$$

Since $F(x)=8x^2e^x$ and $P_0(x)=x$,

$$\frac{F}{P_0W}=\frac{8x^2e^x}{x\cdot 2x}=4e^x.$$

Therefore, from Equation 9.4.7

$$\begin{aligned} u_1'&=4e^xW_1=4e^x(-2)=-8e^x, \\ u_2'&=-4e^xW_2=-4e^x\left(-e^{-x}(x+1)\right)=4(x+1), \\ u_3'&=4e^xW_3=4e^x\left(e^x(x-1)\right)=4e^{2x}(x-1). \end{aligned}$$

Integrating and taking the constants of integration to be zero yields

$$u_1 = -8e^x, \quad u_2 = 2(x+1)^2, u_3 = e^{2x}(2x-3).$$

Hence,

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 + u_3y_3 \\ &= (-8e^x)x + e^x(2(x+1)^2) + e^{-x}\left(e^{2x}(2x-3)\right) \\ &= e^x(2x^2 - 2x - 1). \end{aligned}$$

Since $-e^x$ is a solution of the complementary equation, we redefine

$$y_p = 2xe^x(x-1).$$

Therefore the general solution of Equation 9.4.8 is

$$y = 2xe^x(x-1) + c_1x + c_2e^x + c_3e^{-x}.$$

## Fourth Order Equations

If $n = 4$, then

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix},$$

Therefore

$$W_1 = \begin{vmatrix} y_2 & y_3 & y_4 \\ y_2' & y_3' & y_4' \\ y_2'' & y_3'' & y_4'' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & y_3 & y_4 \\ y_1' & y_3' & y_4' \\ y_1'' & y_3'' & y_4'' \end{vmatrix},$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & y_4 \\ y_1' & y_2' & y_4' \\ y_1'' & y_2'' & y_4'' \end{vmatrix}, \quad W_4 = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix},$$

and Equation 9.4.6 becomes

$$u_1' = -\frac{FW_1}{P_0W}, \quad u_2' = \frac{FW_2}{P_0W}, \quad u_3' = -\frac{FW_3}{P_0W}, \quad u_4' = \frac{FW_4}{P_0W}. \tag{9.4.9}$$

### Example 9.4.2

Find a particular solution of

$$x^4y^{(4)} + 6x^3y''' + 2x^2y'' - 4xy' + 4y = 12x^2, \tag{9.4.10}$$

given that $y_1 = x$, $y_2 = x^2$, $y_3 = 1/x$ and $y_4 = 1/x^2$ form a fundamental set of solutions of the complementary equation. Then find the general solution of Equation 9.4.10 on $(-\infty, 0)$ and $(0, \infty)$.

**Solution**

We seek a particular solution of Equation 9.4.10 of the form

$$y_p = u_1x + u_2x^2 + \frac{u_3}{x} + \frac{u_4}{x^2}.$$

The Wronskian of $\{y_1, y_2, y_3, y_4\}$ is

$$W(x)=\begin{vmatrix} x & x^2 & 1/x & -1/x^2 \\ 1 & 2x & -1/x^2 & -2/x^3 \\ 0 & 2 & 2/x^3 & 6/x^4 \\ 0 & 0 & -6/x^4 & -24/x^5 \end{vmatrix},$$

so

$$W_1=\begin{vmatrix} x^2 & 1/x & 1/x^2 \\ 2x & -1/x^2 & -2/x^3 \\ 2 & 2/x^3 & 6/x^4 \end{vmatrix}=-\frac{12}{x^4},$$

$$W_2=\begin{vmatrix} x & 1/x & 1/x^2 \\ 1 & -1/x^2 & -2/x^3 \\ 0 & 2/x^3 & 6/x^4 \end{vmatrix}=-\frac{6}{x^5},$$

$$W_3=\begin{vmatrix} x & x^2 & 1/x^2 \\ 1 & 2x & -2/x^3 \\ 0 & 2 & 6/x^4 \end{vmatrix}=\frac{12}{x^2},$$

$$W_4=\begin{vmatrix} x & x^2 & 1/x \\ 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \end{vmatrix}=\frac{6}{x}.$$

Expanding $W$ by cofactors of the last row yields

$$\begin{aligned} W &= -0W_1+0W_2-\left(-\frac{6}{x^4}\right)W_3+\left(-\frac{24}{x^5}\right)W_4 \\ &= \frac{6}{x^4}\frac{12}{x^2}-\frac{24}{x^5}\frac{6}{x}=-\frac{72}{x^6}. \end{aligned}$$

Since $F(x)=12x^2$ and $P_0(x)=x^4$,

$$\frac{F}{P_0W}=\frac{12x^2}{x^4}\left(-\frac{x^6}{72}\right)=-\frac{x^4}{6}.$$

Therefore, from Equation 9.4.9,

$$\begin{aligned} u_1' &= -\left(-\frac{x^4}{6}\right)W_1=\frac{x^4}{6}\left(-\frac{12}{x^4}\right)=-2, \\ u_2' &= -\frac{x^4}{6}W_2=-\frac{x^4}{6}\left(-\frac{6}{x^5}\right)=\frac{1}{x}, \\ u_3' &= -\left(-\frac{x^4}{6}\right)W_3=\frac{x^4}{6}\frac{12}{x^2}=2x^2, \\ u_4' &= -\frac{x^4}{6}W_4=-\frac{x^4}{6}\frac{6}{x}=-x^3. \end{aligned}$$

Integrating these and taking the constants of integration to be zero yields

$$u_1=-2x, \quad u_2=\ln|x|, \quad u_3=\frac{2x^3}{3}, u_4=-\frac{x^4}{4}.$$

Hence,

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 + u_3y_3 + u_4y_4 \\ &= (-2x)x + (\ln|x|)x^2 + \frac{2x^3}{3}\frac{1}{x} + \left(-\frac{x^4}{4}\right)\frac{1}{x^2} \\ &= x^2\ln|x| - \frac{19x^2}{12}. \end{aligned}$$

Since $-19x^2/12$ is a solution of the complementary equation, we redefine

$$y_p = x^2\ln|x|.$$

Therefore

$$y = x^2\ln|x| + c_1x + c_2x^2 + \frac{c_3}{x} + \frac{c_4}{x^2}$$

is the general solution of Equation 9.4.10 on $(-\infty, 0)$ and $(0, \infty)$.

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# 9.4E: Variation of Parameters for Higher Order Equations (Exercises)

## Q9.4.1

In **Exercises 9.4.1-9.4.21** find a particular solution, given the fundamental set of solutions of the complementary equation.

1. $x^3y'''-x^2(x+3)y''+2x(x+3)y'-2(x+3)y=-4x^4 \;\; ;\{x,\, x^2,\, xe^x\}$
2. $y'''+6xy''+(6+12x^2)y'+(12x+8x^3)y=x^{1/2}e^{-x^2} \;\; ;\{e^{-x^2},\, xe^{-x^2},\, x^2e^{-x^2}\}$
3. $x^3y'''-3x^2y''+6xy'-6y=2x \;;\{x, x^2, x^3\}$
4. $x^2y'''+2xy''-(x^2+2)y'=2x^2 \;;\{1,\, e^x/x,\, e^{-x}/x\}$
5. $x^3y'''-3x^2(x+1)y''+3x(x^2+2x+2)y'-(x^3+3x^2+6x+6)y=x^4e^{-3x} \;\; ;\{xe^x,\, x^2e^x,\, x^3e^x\}$
6. $x(x^2-2)y'''+(x^2-6)y''+x(2-x^2)y'+(6-x^2)y=2(x^2-2)^2 \;\; ;\{e^x,\, e^{-x},\, 1/x\}$
7. $xy'''-(x-3)y''-(x+2)y'+(x-1)y=-4e^{-x} \;\; ;\{e^x,\, e^x/x,\, e^{-x}/x\}$
8. $4x^3y'''+4x^2y''-5xy'+2y=30x^2 \;;\{\sqrt{x},\, 1/\sqrt{x},\, x^2\}$
9. $x(x^2-1)y'''+(5x^2+1)y''+2xy'-2y=12x^2 \;\; ;\{x,\, 1/(x-1),\, 1/(x+1)\}$
10. $x(1-x)y'''+(x^2-3x+3)y''+xy'-y=2(x-1)^2 \;\; ;\{x,\, 1/x, e^x/x\}$
11. $x^3y'''+x^2y''-2xy'+2y=x^2 \;;\{x,\, x^2,\, 1/x\}$
12. $xy'''-y''-xy'+y=x^2 \;;\{x,\, e^x,\, e^{-x}\}$
13. $xy^{(4)}+4y'''=6\ln|x|;\{1,\, x,\, x^2,\, 1/x\}$
14. $16x^4y^{(4)}+96x^3y'''+72x^2y''-24xy'+9y=96x^{5/2};\{\sqrt{x},\, 1/\sqrt{x},\, x^{3/2},\, x^{-3/2}\}$
15. $x(x^2-6)y^{(4)}+2(x^2-12)y'''+x(6-x^2)y''+2(12-x^2)y'=2(x^2-6)^2 \;\; ;\{1,\, 1/x,\, e^x,\, e^{-x}\}$
16. $x^4y^{(4)}-4x^3y'''+12x^2y''-24xy'+24y=x^4 \;;\{x,\, x^2,\, x^3,\, x^4\}$
17. $x^4y^{(4)}-4x^3y'''+2x^2(6-x^2)y''+4x(x^2-6)y'+(x^4-4x^2+24)y=4x^5e^x \;\; ;\{xe^x,\, x^2e^x,\, xe^{-x},\, x^2e^{-x}\}$
18. $x^4y^{(4)}+6x^3y'''+2x^2y''-4xy'+4y=12x^2 \;;\{x, x^2, 1/x, 1/x^2\}$
19. $xy^{(4)}+4y'''-2xy''-4y'+xy=4e^x \;;\{e^x,\, e^{-x},\, e^x/x,\, e^{-x}/x\}$
20. $xy^{(4)}+(4-6x)y'''+(13x-18)y''+(26-12x)y'+(4x-12)y=3e^x \;;\{e^x,\, e^{2x},\, e^x/x,\, e^{2x}/x\}$
21. $x^4y^{(4)}-4x^3y'''+x^2(12-x^2)y''+2x(x^2-12)y'+2(12-x^2)y=2x^5 \;\; ;\{x,\, x^2,\, xe^x,\, xe^{-x}\}$

## Q9.4.2

In **Exercises 9.4.22-9.4.33** solve the initial value problem, given the fundamental set of solutions of the complementary equation. Graph the solution for **Exercises 9.4.22, 9.4.26, 9.4.29,** and **9.4.30.**

22. $x^3y'''-2x^2y''+3xy'-3y=4x, \quad y(1)=4, \quad y'(1)=4, \quad y''(1)=2 \;;\{x,\, x^3,\, x\ln x\}$
23. $x^3y'''-5x^2y''+14xy'-18y=x^3, \quad y(1)=0, \quad y'(1)=1, \quad y''(1)=7 \;;\{x^2,\, x^3,\, x^3\ln x\}$
24. $(5-6x)y'''+(12x-4)y''+(6x-23)y'+(22-12x)y=-(6x-5)^2e^x \quad \{y(0)=-4, \quad y'(0)=-\frac{3}{2},$
$y''(0)=-19;\{e^x,\, e^{2x},\, xe^{-x}\}$
25. $x^3y'''-6x^2y''+16xy'-16y=9x^4, \quad y(1)=2, \quad y'(1)=1, \quad y''(1)=5 \;;\{x,\, x^4,\, x^4\ln|x|\}$
26. $(x^2-2x+2)y'''-x^2y''+2xy'-2y=(x^2-2x+2)^2, \quad y(0)=0, \quad y'(0)=5 \quad ,y''(0)=0;\{x,\, x^2,\, e^x\}$
27. $x^3y'''+x^2y''-2xy'+2y=x(x+1), \quad y(-1)=-6, \quad y'(-1)=\frac{[4pt]}{6}, \quad y''(-1)=-\frac{5}{2};\{x,\, x^2,\, 1/x\}$
28. $(3x-1)y'''-(12x-1)y''+9(x+1)y'-9y=2e^x(3x-1)^2, \quad y(0)=\frac{3}{4}, \quad y'(0)=\frac{5}{4},$
$y''(0)=\frac{1}{4};\{x+1,\, e^x,\, e^{3x}\}$
29. $(x^2-2)y'''-2xy''+(2-x^2)y'+2xy=2(x^2-2)^2, \quad y(0)=1, \quad y'(0)=-5 \quad ,y''(0)=5;\{x^2,\, e^x,\, e^{-x}\}$

30. $x^4y^{(4)}+3x^3y'''-x^2y''+2xy'-2y=9x^2, \quad y(1)=-7, \quad y'(1)=-11, \quad y''(1)=-5, \quad y'''(1)=6;$
$\{x,\ x^2,\ 1/x,\ x\ln x\}$

31. $(2x-1)y^{(4)}-4xy'''+(5-2x)y''+4xy'-4y=6(2x-1)^2, \quad y(0)=\frac{55}{4}, \quad y'(0)=0, \quad y''(0)=13,$
$y'''(0)=1; \{x,\ e^x,\ e^{-x},\ e^{2x}\}$

32. $4x^4y^{(4)}+24x^3y'''+23x^2y''-xy'+y=6x, \quad y(1)=2, \quad y'(1)=0, \quad y''(1)=4,$
$y'''(1)=-\frac{37}{4}; \{x, \sqrt{x}, 1/x, 1/\sqrt{x}\}$

33. $x^4y^4+5x^3y'''-3x^2y''-6xy'+6y=[4pt]x^3, \quad y(-1)=-1,\ y'(-1)=-7\ ,$
$y''(-1)=-1, \quad y'''(-1)=-31\,; \{x,\ x^3,\ 1/x,\ 1/x^2\}$

## Q9.4.3

34. Su[4pt]ose the equation

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=F(x) \tag{A}$$

is normal on an interval $(a,b)$. Let $\{y_1,y_2,\ldots,y_n\}$ be a fundamental set of solutions of its complementary equation on $(a,b)$, let $W$ be the Wronskian of $\{y_1,y_2,\ldots,y_n\}$, and let $W_j$ be the determinant obtained by deleting the last row and the $j$-th column of $W$. Su[4pt]ose $x_0$ is in $(a,b)$, let

$$u_j(x)=(-1)^{(n-j)}\int_{x_0}^{x}\frac{F(t)W_j(t)}{P_0(t)W(t)}\,dt, \quad 1\le j\le n,$$

and define

$$y_p=u_1y_1+u_2y_2+\cdots+u_ny_n.$$

a. Show that $y_p$ is a solution of (A) and that

$$y_p^{(r)}=u_1y_1^{(r)}+u_2y_2^{(r)}\cdots+u_ny_n^{(r)}, \quad 1\le r\le n-1,$$

and

$$y_p^{(n)}=u_1y_1^{(n)}+u_2y_2^{(n)}+\cdots+u_ny_n^{(n)}+\frac{F}{P_0}.$$

HINT: **See the derivation of the method of variation of parameters at the beginning of the section.**

b. Show that $y_p$ is the solution of the initial value problem

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=F(x),$$
$$y(x_0)=0,\ y'(x_0)=0,\ldots, \quad y^{(n-1)}(x_0)=0.$$

c. Show that $y_p$ can be wri[4pt]en as

$$y_p(x)=\int_{x_0}^{x}G(x,t)F(t)\,dt,$$

where

$$G(x,t)=\frac{1}{P_0(t)W(t)}\begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t)\\ y_1'(t) & y_2'(t) & \cdots & y_n'(t)\\ \vdots & \vdots & \ddots & \vdots\\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t)\\ y_1(x) & y_2(x) & \cdots & y_n(x)\end{vmatrix},$$

which is called the **Green's function** for (A).

d. Show that

$$\frac{\partial^j G(x,t)}{\partial x^j} = \frac{1}{P_0(t)W(t)} \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t) \\ y_1^{(j)}(x) & y_2^{(j)}(x) & \cdots & y_n^{(j)}(x) \end{vmatrix}, \quad 0 \le j \le n.$$

e. Show that if $a < t < b$ then

$$\left.\frac{\partial^j G(x,t)}{\partial x^j}\right|_{x=t} = \begin{cases} 0, & 1 \le j \le n-2 \\ \frac{1}{P_0(t)}, & j = n-1 \end{cases}$$

f. Show that

$$y_p^{(j)}(x) = \begin{cases} \int_{x_0}^x \frac{\partial^j G(x,t)}{\partial x^j} F(t)dt, & 1 \le j \le n-1, \\ \frac{F(x)}{P_0(x)} + \int_{x_0}^x \frac{\partial^{(n)} G(x,t)}{\partial x^n} F(t)dt, & j = n. \end{cases}$$

## Q9.4.4

In **Exercises 9.4.35-9.4.[4pt]** use the method suggested by **Exercise 9.4.34** to find a particular solution in the form $y_p = \int_{x_0}^x G(x,t)F(t)dt$, given the indicated fundamental set of solutions. Assume that $x$ and $x_0$ are in an interval on which the equation is normal.

35. $y''' + 2y' - y' - 2y = F(x); \quad \{e^x,\ e^{-x}, e^{-2x}\}$

36. $x^3y''' + x^2y'' - 2xy' + 2y = F(x); \quad \{x,\ x^2,\ 1/x\}$

37. $x^3y''' - x^2(x+3)y'' + 2x(x+3)y' - 2(x+3)y = F(x); \{x, x^2, xe^x\}$

38. $x(1-x)y''' + (x^2 - 3x + 3)y'' + xy' - y = F(x); \quad \{x,\ 1/x,\ e^x/x\}$

39. $y^{(4)} - 5y'' + 4y = F(x); \quad \{e^x,\ e^{-x},\ e^{2x},\ e^{-2x}\}$

[4pt]. $xy^{(4)} + 4y''' = F(x); \quad \{1,\ x,\ x^2,\ 1/x\}$

[4pt]. $x^4y^{(4)} + 6x^3y''' + 2x^2y'' - 4xy' + 4y = F(x)\ ; \{x, x^2, 1/x, 1/x^2\}$

[4pt]. $xy^{(4)} - y''' - 4xy' + 4y' = F(x); \quad \{1,\ x^2,\ e^{2x}, e^{-2x}\}$

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# CHAPTER OVERVIEW

## 10: Linear Systems of Differential Equations

IN THIS CHAPTER we consider systems of differential equations involving more than one unknown function. Such systems arise in many physical applications.SECTION 10.1 presents examples of physical situations that lead to systems of differential equations. SECTION 10.2 discusses linear systems of differential equations. SECTION 10.3 deals with the basic theory of homogeneous linear systems. SECTIONS 10.4, 10.5, AND 10.6 present the theory of constant coefficient homogeneous systems. SECTION 10.7 presents the method of variation of parameters for nonhomogeneous linear systems.


10.1: Introduction to Systems of Differential Equations

10.1E: Introduction to Systems of Differential Equations (Exercises)

10.2: Linear Systems of Differential Equations

10.2E: Linear Systems of Differential Equations (Exercises)

10.3: Basic Theory of Homogeneous Linear Systems

10.3E: Basic Theory of Homogeneous Linear Systems (Exercises)

10.4: Constant Coefficient Homogeneous Systems I

10.4E: Constant Coefficient Homogeneous Systems I (Exercises)

10.5: Constant Coefficient Homogeneous Systems II

10.5E: Constant Coefficient Homogeneous Systems II (Exercises)

10.6: Constant Coefficient Homogeneous Systems III

10.6E: Constant Coefficient Homogeneous Systems III (Exercises)



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# 10.1: Introduction to Systems of Differential Equations

Many physical situations are modelled by systems of $n$ differential equations in $n$ unknown functions, where $n \geq 2$. The next three examples illustrate physical problems that lead to systems of differential equations. In these examples and throughout this chapter we'll denote the independent variable by $t$.

## Example 10.1.1 : Flow

Tanks $T_1$ and $T_2$ contain 100 gallons and 300 gallons of salt solutions, respectively. Salt solutions are simultaneously added to both tanks from external sources, pumped from each tank to the other, and drained from both tanks (Figure 10.1.1 ). A solution with 1 pound of salt per gallon is pumped into $T_1$ from an external source at 5 gal/min, and a solution with 2 pounds of salt per gallon is pumped into $T_2$ from an external source at 4 gal/min. The solution from $T_1$ is pumped into $T_2$ at 2 gal/min, and the solution from $T_2$ is pumped into $T_1$ at 3 gal/min. $T_1$ is drained at 6 gal/min and $T_2$ is drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in $T_1$ and $T_2$, respectively, at time $t > 0$. Derive a system of differential equations for $Q_1$ and $Q_2$. Assume that both mixtures are well stirred.

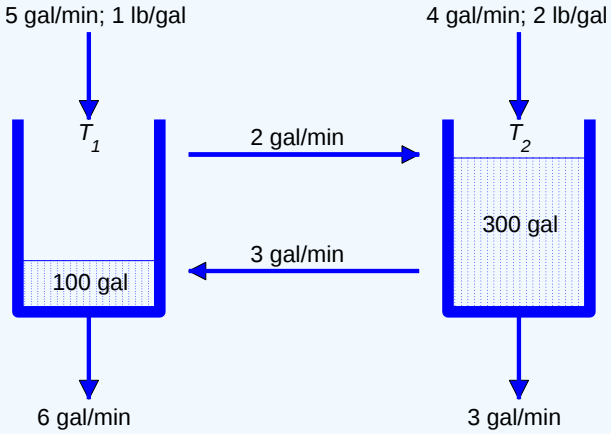



Figure 10.1.1

As in Section 4.2, let **rate in** and **rate out** denote the rates (lb/min) at which salt enters and leaves a tank; thus,

$$Q_1' = (\text{rate in})_1 - (\text{rate out})_1,$$
$$Q_2' = (\text{rate in})_2 - (\text{rate out})_2.$$

Note that the volumes of the solutions in $T_1$ and $T_2$ remain constant at 100 gallons and 300 gallons, respectively.

$T_1$ receives salt from the external source at the rate of

$$(1\ \text{lb/gal}) \times (5\text{-gal/min}) = 5\ \text{lb/min},$$

and from $T_2$ at the rate of

$$(\text{lb/gal in } T_2) \times (3\text{-gal/min}) = \frac{1}{300} Q_2 \times 3$$
$$= \frac{1}{100} Q_2\ \text{lb/min}.$$

Therefore

$$(\text{rate in})_1 = 5 + \frac{1}{100} Q_2. \tag{10.1.1}$$

Solution leaves $T_1$ at the rate of 8 gal/min, since 6 gal/min are drained and 2 gal/min are pumped to $T_2$; hence,

$$(\text{rate out})_1 = (\text{ lb/gal in } \mathrm{T}_1) \times (8\text{-gal/min}) = \frac{1}{100} Q_1 \times 8 = \frac{2}{25} Q_1. \tag{10.1.2}$$

Equations 10.1.1 and Equation 10.1.2 imply that

$$Q_1' = 5 + \frac{1}{100}Q_2 - \frac{2}{25}Q_1. \tag{10.1.3}$$

$T_2$ receives salt from the external source at the rate of

$$(2\text{ lb/gal}) \times (4\text{-gal/min}) = 8\text{ lb/min},$$

and from $T_1$ at the rate of

$$\begin{aligned}(\text{lb/gal in } T_1) \times (2\text{-gal/min}) &= \frac{1}{100}Q_1 \times 2 \\ &= \frac{1}{50}Q_1 \text{ lb/min}.\end{aligned}$$

Therefore

$$(\text{rate in})_2 = 8 + \frac{1}{50}Q_1. \tag{10.1.4}$$

Solution leaves $T_2$ at the rate of 6 gal/min, since 3 gal/min are drained and 3 gal/min are pumped to $T_1$; hence,

$$(\text{rate out})_2 = (\text{ lb/gal in } \mathrm{T}_2) \times (6\text{-gal/min}) = \frac{1}{300}Q_2 \times 6 = \frac{1}{50}Q_2. \tag{10.1.5}$$

Equations 10.1.4 and 10.1.5 imply that

$$Q_2' = 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2. \tag{10.1.6}$$

We say that Equations 10.1.3 and 10.1.6 form a **system of two first order equations in two unknowns**, and write them together as

$$\begin{aligned}Q_1' &= 5 - \frac{2}{25}Q_1 + \frac{1}{100}Q_2 \\ Q_2' &= 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2.\end{aligned}$$

## Example 10.1.2 : Coupled Springs

A mass $m_1$ is suspended from a rigid support on a spring $S_1$ and a second mass $m_2$ is suspended from the first on a spring $S_2$ (Figure 10.1.2 ). The springs obey Hooke's law, with spring constants $k_1$ and $k_2$. Internal friction causes the springs to exert damping forces proportional to the rates of change of their lengths, with damping constants $c_1$ and $c_2$. Let $y_1 = y_1(t)$ and $y_2 = y_2(t)$ be the displacements of the two masses from their equilibrium positions at time $t$, measured positive upward. Derive a system of differential equations for $y_1$ and $y_2$, assuming that the masses of the springs are negligible and that vertical external forces $F_1$ and $F_2$ also act on the objects.

### Solution

In equilibrium, $S_1$ supports both $m_1$ and $m_2$ and $S_2$ supports only $m_2$. Therefore, if $\Delta\ell_1$ and $\Delta\ell_2$ are the elongations of the springs in equilibrium then

$$(m_1 + m_2)g = k_1\Delta\ell_1 \quad \text{and} \quad m_2 g = k_2\Delta\ell_2. \tag{10.1.7}$$

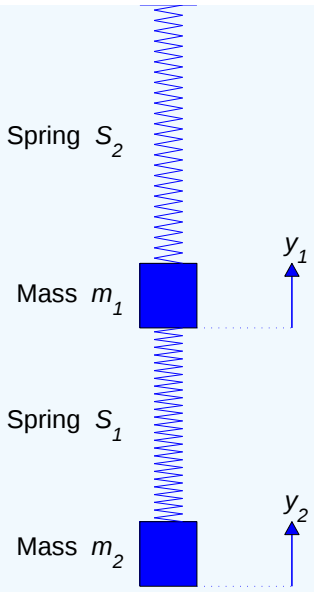


Figure 10.1.2

Let $H_1$ be the Hooke's law force acting on $m_1$, and let $D_1$ be the damping force on $m_1$. Similarly, let $H_2$ and $D_2$ be the Hooke's law and damping forces acting on $m_2$. According to Newton's second law of motion,

$$\begin{aligned} m_1y_1'' &= -m_1g + H_1 + D_1 + F_1, \\ m_2y_2'' &= -m_2g + H_2 + D_2 + F_2. \end{aligned} \tag{10.1.8}$$

When the displacements are $y_1$ and $y_2$, the change in length of $S_1$ is $-y_1 + \Delta\ell_1$ and the change in length of $S_2$ is $-y_2 + y_1 + \Delta\ell_2$. Both springs exert Hooke's law forces on $m_1$, while only $S_2$ exerts a Hooke's law force on $m_2$. These forces are in directions that tend to restore the springs to their natural lengths. Therefore

$$H_1 = k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \quad \text{and} \quad H_2 = k_2(-y_2 + y_1 + \Delta\ell_2). \tag{10.1.9}$$

When the velocities are $y_1'$ and $y_2'$, $S_1$ and $S_2$ are changing length at the rates $-y_1'$ and $-y_2' + y_1'$, respectively. Both springs exert damping forces on $m_1$, while only $S_2$ exerts a damping force on $m_2$. Since the force due to damping exerted by a spring is proportional to the rate of change of length of the spring and in a direction that opposes the change, it follows that

$$D_1 = -c_1y_1' + c_2(y_2' - y_1') \quad \text{and} \quad D_2 = -c_2(y_2' - y_1'). \tag{10.1.10}$$

From Equations 10.1.8, 10.1.9, and 10.1.10,

$$\begin{aligned} &= -m_1g + k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) - c_1y_1' + c_2(y_2' - y_1') + F_1 \\ &= -(m_1g - k_1\Delta\ell_1 + k_2\Delta\ell_2) - k_1y_1 + k_2(y_2 - y_1) - c_1y_1' + c_2(y_2' - y_1') + F_1 \end{aligned} \tag{10.1.11}$$

and

$$\begin{aligned} m_2y_2'' \quad &= -m_2g + k_2(-y_2 + y_1 + \Delta\ell_2) - c_2(y_2' - y_1') + F_2 \\ &= -(m_2g - k_2\Delta\ell_2) - k_2(y_2 - y_1) - c_2(y_2' - y_1') + F_2. \end{aligned} \tag{10.1.12}$$

From Equation 10.1.7,

$$m_1g - k_1\Delta\ell_1 + k_2\Delta\ell_2 = -m_2g + k_2\Delta\ell_2 = 0.$$

Therefore we can rewrite Equation 10.1.11 and Equation 10.1.12 as

$$\begin{aligned} m_1y_1'' &= -(c_1 + c_2)y_1' + c_2y_2' - (k_1 + k_2)y_1 + k_2y_2 + F_1 \\ m_2y_2'' &= c_2y_1' - c_2y_2' + k_2y_1 - k_2y_2 + F_2. \end{aligned}$$

## Example 10.1.3 : Gravity

Let $\mathbf{X} = \mathbf{X}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ be the position vector at time $t$ of an object with mass $m$, relative to a rectangular coordinate system with origin at Earth's center (Figure 10.1.3 ).

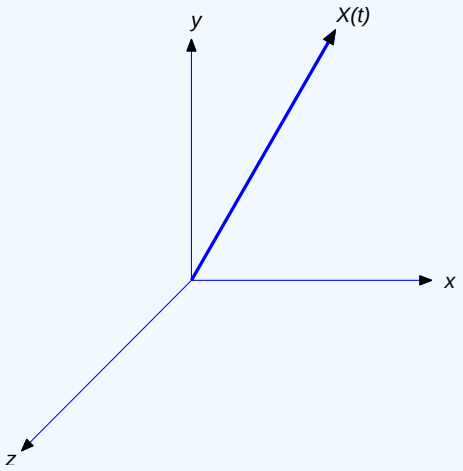



Figure 10.1.3

According to Newton's law of gravitation, Earth's gravitational force $\mathbf{F} = \mathbf{F}(x, y, z)$ on the object is inversely proportional to the square of the distance of the object from Earth's center, and directed toward the center; thus,

$$\mathbf{F} = \frac{\mathbf{K}}{||\mathbf{X}||^{\mathbf{2}}}\left(-\frac{\mathbf{X}}{||\mathbf{X}||}\right) = -\mathbf{K}\frac{\mathbf{x}\,\mathbf{i} + \mathbf{y}\,\mathbf{j} + \mathbf{z}\,\mathbf{k}}{\left(\mathbf{x^2} + \mathbf{y^2} + \mathbf{z^2}\right)^{\mathbf{3/2}}}, \tag{10.1.13}$$

where $K$ is a constant. To determine $K$, we observe that the magnitude of $\mathbf{F}$ is

$$\|\mathbf{F}\| = K\frac{\|\mathbf{X}\|}{\|\mathbf{X}\|^3} = \frac{K}{\|\mathbf{X}\|^2} = \frac{K}{(x^2 + y^2 + z^2)}.$$

Let $R$ be Earth's radius. Since $\|\mathbf{F}\| = mg$ when the object is at Earth's surface,

$$mg = \frac{K}{R^2}, \quad \text{so} \quad K = mgR^2.$$

Therefore we can rewrite Equation 10.1.13as

$$\mathbf{F} = -mgR^2\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\left(x^2 + y^2 + z^2\right)^{3/2}}.$$

Now suppose $\mathbf{F}$ is the only force acting on the object. According to Newton's second law of motion, $\mathbf{F} = m\mathbf{X}''$; that is,

$$m(x''\,\mathbf{i} + y''\,\mathbf{j} + z''\,\mathbf{k}) = -mgR^2\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\left(x^2 + y^2 + z^2\right)^{3/2}}.$$

Cancelling the common factor $m$ and equating components on the two sides of this equation yields the system

$$\begin{aligned} x'' &= -\frac{gR^2x}{(x^2 + y^2 + z^2)^{3/2}} \\ y'' &= -\frac{gR^2y}{(x^2 + y^2 + z^2)^{3/2}} \\ z'' &= -\frac{gR^2z}{(x^2 + y^2 + z^2)^{3/2}}. \end{aligned} \tag{10.1.14}$$

## Rewriting Higher Order Systems as First Order Systems

A system of the form

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2, \ldots, y_n) \\ y_2' &= g_2(t, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= g_n(t, y_1, y_2, \ldots, y_n) \end{aligned} \tag{10.1.15}$$

is called a **first order system**, since the only derivatives occurring in it are first derivatives. The derivative of each of the unknowns may depend upon the independent variable and all the unknowns, but not on the derivatives of other unknowns. When we wish to emphasize the number of unknown functions in Equation 10.1.15we will say that Equation 10.1.15is an $n \times n$ system.

Systems involving higher order derivatives can often be reformulated as first order systems by introducing additional unknowns. The next two examples illustrate this.

### Example 10.1.4

Rewrite the system

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2) y_1' + c_2 y_2' - (k_1 + k_2) y_1 + k_2 y_2 + F_1 \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + F_2 . \end{aligned} \tag{10.1.16}$$

derived in Example 10.1.2 as a system of first order equations.

**Solution**

If we define $v_1 = y_1'$ and $v_2 = y_2'$, then $v_1' = y_1''$ and $v_2' = y_2''$, so Equation 10.1.16becomes

$$\begin{aligned} m_1 v_1' &= -(c_1 + c_2) v_1 + c_2 v_2 - (k_1 + k_2) y_1 + k_2 y_2 + F_1 \\ m_2 v_2' &= c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + F_2 . \end{aligned}$$

Therefore $\{y_1, y_2, v_1, v_2\}$ satisfies the $4 \times 4$ first order system

$$\begin{aligned} y_1' &= v_1 \\ y_2' &= v_2 \\ v_1' &= \frac{1}{m_1}[-(c_1 + c_2) v_1 + c_2 v_2 - (k_1 + k_2) y_1 + k_2 y_2 + F_1] \\ v_2' &= \frac{1}{m_2}[c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + F_2] . \end{aligned} \tag{10.1.17}$$

### Note

The difference in form between Equation 10.1.15and Equation 10.1.17, due to the way in which the unknowns are denoted in the two systems, isn't important; Equation 10.1.17is a first order system, in that each equation in Equation 10.1.17expresses the first derivative of one of the unknown functions in a way that does not involve derivatives of any of the other unknowns.

### Example 10.1.5

Rewrite the system

$$\begin{aligned} x'' &= f(t, x, x', y, y', y'') \\ y''' &= g(t, x, x', y, y' y'') \end{aligned}$$

as a first order system.

**Solution**

We regard $x$, $x'$, $y$, $y'$, and $y''$ as unknown functions, and rename them

$$x = x_1,\ x' = x_2, \quad y = y_1, \quad y' = y_2, \quad y'' = y_3.$$

These unknowns satisfy the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= f(t, x_1, x_2, y_1, y_2, y_3) \\ y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= g(t, x_1, x_2, y_1, y_2, y_3). \end{aligned}$$

## Rewriting Scalar Differential Equations as Systems

In this chapter we'll refer to differential equations involving only one unknown function as **scalar** differential equations. Scalar differential equations can be rewritten as systems of first order equations by the method illustrated in the next two examples.

**Example 10.1.6**

Rewrite the equation

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0 \tag{10.1.18}$$

as a $4 \times 4$ first order system.

**Solution**

We regard $y$, $y'$, $y''$, and $y'''$ as unknowns and rename them

$$y = y_1, \quad y' = y_2, \quad y'' = y_3, \quad \text{and} \quad y''' = y_4.$$

Then $y^{(4)} = y_4'$, so Equation 10.1.18 can be written as

$$y_4' + 4y_4 + 6y_3 + 4y_2 + y_1 = 0.$$

Therefore $\{y_1, y_2, y_3, y_4\}$ satisfies the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= -4y_4 - 6y_3 - 4y_2 - y_1. \end{aligned}$$

**Example 10.1.7**

Rewrite

$$x''' = f(t, x, x', x'')$$

as a system of first order equations.

**Solution**

We regard $x$, $x'$, and $x''$ as unknowns and rename them

$$x = y_1, \quad x' = y_2, \quad \text{and} \quad x'' = y_3.$$

Then

$$y_1' = x' = y_2, \quad y_2' = x'' = y_3, \quad \text{and} \quad y_3' = x'''.$$

Therefore $\{y_1, y_2, y_3\}$ satisfies the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= f(t, y_1, y_2, y_3). \end{aligned}$$

Since systems of differential equations involving higher derivatives can be rewritten as first order systems by the method used in Examples 10.1.5 -10.1.7 , we'll consider only first order systems.

## Numerical Solution of Systems

The numerical methods that we studied in Chapter 3 can be extended to systems, and most differential equation software packages include programs to solve systems of equations. We will not go into detail on numerical methods for systems; however, for illustrative purposes we'll describe the Runge-Kutta method for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20} \end{aligned}$$

at equally spaced points $t_0$, $t_1$, ..., $t_n = b$ in an interval $[t_0, b]$. Thus,

$$t_i = t_0 + ih, \quad i = 0, 1, \ldots, n,$$

where

$$h = \frac{b - t_0}{n}.$$

We'll denote the approximate values of $y_1$ and $y_2$ at these points by $y_{10}, y_{11}, \ldots, y_{1n}$ and $y_{20}, y_{21}, \ldots, y_{2n}$. The Runge-Kutta method computes these approximate values as follows: given $y_{1i}$ and $y_{2i}$, compute

$$\begin{aligned} I_{1i} &= g_1(t_i, y_{1i}, y_{2i}), \\ J_{1i} &= g_2(t_i, y_{1i}, y_{2i}), \\ I_{2i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{1i}, y_{2i} + \frac{h}{2} J_{1i}\right), \\ J_{2i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{1i}, y_{2i} + \frac{h}{2} J_{1i}\right), \\ I_{3i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{2i}, y_{2i} + \frac{h}{2} J_{2i}\right), \\ J_{3i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{2i}, y_{2i} + \frac{h}{2} J_{2i}\right), \\ I_{4i} &= g_1(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}), \\ J_{4i} &= g_2(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}), \end{aligned}$$

and

$$\begin{aligned} y_{1,i+1} &= y_{1i} + \frac{h}{6}(I_{1i} + 2I_{2i} + 2I_{3i} + I_{4i}), \\ y_{2,i+1} &= y_{2i} + \frac{h}{6}(J_{1i} + 2J_{2i} + 2J_{3i} + J_{4i}) \end{aligned}$$

for $i = 0$, ..., $n - 1$. Under appropriate conditions on $g_1$ and $g_2$, it can be shown that the global truncation error for the Runge-Kutta method is $O(h^4)$, as in the scalar case considered in Section 3.3.

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# 10.1E: Introduction to Systems of Differential Equations (Exercises)

## Q10.1.1

1. Tanks $T_1$ and $T_2$ contain 50 gallons and 100 gallons of salt solutions, respectively. A solution with 2 pounds of salt per gallon is pumped into $T_1$ from an external source at 1 gal/min, and a solution with 3 pounds of salt per gallon is pumped into $T_2$ from an external source at 2 gal/min. The solution from $T_1$ is pumped into $T_2$ at 3 gal/min, and the solution from $T_2$ is pumped into $T_1$ at 4 gal/min. $T_1$ is drained at 2 gal/min and $T_2$ is drained at 1 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in $T_1$ and $T_2$, respectively, at time $t > 0$. Derive a system of differential equations for $Q_1$ and $Q_2$. Assume that both mixtures are well stirred.

2. Two 500 gallon tanks $T_1$ and $T_2$ initially contain 100 gallons each of salt solution. A solution with 2 pounds of salt per gallon is pumped into $T_1$ from an external source at 6 gal/min, and a solution with 1 pound of salt per gallon is pumped into $T_2$ from an external source at 5 gal/min. The solution from $T_1$ is pumped into $T_2$ at 2 gal/min, and the solution from $T_2$ is pumped into $T_1$ at 1 gal/min. Both tanks are drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in $T_1$ and $T_2$, respectively, at time $t > 0$. Derive a system of differential equations for $Q_1$ and $Q_2$ that's valid until a tank is about to overflow. Assume that both mixtures are well stirred.

3. A mass $m_1$ is suspended from a rigid support on a spring $S_1$ with spring constant $k_1$ and damping constant $c_1$. A second mass $m_2$ is suspended from the first on a spring $S_2$ with spring constant $k_2$ and damping constant $c_2$, and a third mass $m_3$ is suspended from the second on a spring $S_3$ with spring constant $k_3$ and damping constant $c_3$. Let $y_1 = y_1(t)$, $y_2 = y_2(t)$, and $y_3 = y_3(t)$ be the displacements of the three masses from their equilibrium positions at time $t$, measured positive upward. Derive a system of differential equations for $y_1$, $y_2$ and $y_3$, assuming that the masses of the springs are negligible and that vertical external forces $F_1$, $F_2$, and $F_3$ also act on the masses.

4. Let $\mathbf{X} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector of an object with mass $m$, expressed in terms of a rectangular coordinate system with origin at Earth's center (Figure 10.1.3). Derive a system of differential equations for $x$, $y$, and $z$, assuming that the object moves under Earth's gravitational force (given by Newton's law of gravitation, as in Example 10.1.3) and a resistive force proportional to the speed of the object. Let $\alpha$ be the constant of proportionality.

5. Rewrite the given system as a first order system.

a.
$$x''' = f(t, x, y, y')$$
$$y'' = g(t, y, y')$$
$$u' = f(t, u, v, v', w')$$

b.
$$v'' = g(t, u, v, v', w)$$
$$w'' = h(t, u, v, v', w, w')$$

c. $y''' = f(t, y, y', y'')$

d. $y^{(4)} = f(t, y)$

e.
$$x'' = f(t, x, y)$$
$$y'' = g(t, x, y)$$

6. Rewrite the system Equation 10.1.14 of differential equations derived in Example 10.1.3 as a first order system.

7. Formulate a version of Euler's method (Section 3.1) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20}, \end{aligned}$$

on an interval $[t_0, b]$.

8. Formulate a version of the improved Euler method (Section 3.2) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20}, \end{aligned}$$

on an interval $[t_0, b]$.

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# 10.2: Linear Systems of Differential Equations

A first order system of differential equations that can be written in the form

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t) \\ y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_n(t) \end{aligned} \tag{10.2.1}$$

is called a **linear system**.

The linear system Equation 10.2.1 can be written in matrix form as

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t)_{12}(t) & \cdots_{1n}(t) \\ a_{21}(t)_{22}(t) & \cdots_{2n}(t) \\ \vdots & \vdots \quad \ddots \quad \vdots \\ a_{n1}(t)_{n2}(t) & \cdots_{nn}(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

or more briefly as

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \tag{10.2.2}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{A(t)} = \begin{bmatrix} a_{11}(t)_{12}(t) & \cdots_{1n}(t) \\ a_{21}(t)_{22}(t) & \cdots_{2n}(t) \\ \vdots & \vdots \quad \ddots \quad \vdots \\ a_{n1}(t)_{n2}(t) & \cdots_{nn}(t) \end{bmatrix}, \quad \textbf{and} \quad \mathbf{f(t)} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

We call $A$ the **coefficient matrix** of Equation 10.2.2 and $\mathbf{f}$ the **forcing function**. We'll say that $A$ and $\mathbf{f}$ are **continuous** if their entries are continuous. If $\mathbf{f} = \mathbf{0}$, then Equation 10.2.2 is **homogeneous**; otherwise, Equation 10.2.2 is **nonhomogeneous**.

An initial value problem for Equation 10.2.2 consists of finding a solution of Equation 10.2.2 that equals a given constant vector

$$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}.$$

at some initial point $t_0$. We write this initial value problem as

$$\mathbf{y}' = \mathbf{A(t)y} + \mathbf{f(t)}, \quad \mathbf{y(t_0) = k}.$$

The next theorem gives sufficient conditions for the existence of solutions of initial value problems for Equation 10.2.2. We omit the proof.

**Theorem 10.2.1 : Existence**

Suppose the coefficient matrix $A$ and the forcing function $\mathbf{f}$ are continuous on $(a, b)$, let $t_0$ be in $(a, b)$, and let $\mathbf{k}$ be an arbitrary constant $n$-vector. Then the initial value problem

$$\mathbf{y}' = \mathbf{A(t)y} + \mathbf{f(t)}, \quad \mathbf{y(t_0) = k}$$

has a unique solution on $(a, b)$.

**Example 10.2.1**

a. Write the system

$$\begin{aligned} y_1' &= y_1 + 2y_2 + 2e^{4t} \\ y_2' &= 2y_1 + y_2 + e^{4t} \end{aligned} \tag{10.2.3}$$

in matrix form and conclude from Theorem 10.2.1 that every initial value problem for Equation 10.2.3 has a unique solution on $(-\infty, \infty)$.

b. Verify that

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \tag{10.2.4}$$

is a solution of Equation 10.2.3 for all values of the constants $c_1$ and $c_2$.

c. Find the solution of the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{y}(0) = \frac{1}{5}\begin{bmatrix} 3 \\ 22 \end{bmatrix}. \tag{10.2.5}$$

### Solution a

The system Equation 10.2.3 can be written in matrix form as

$$\mathbf{y}' = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}.$$

An initial value problem for Equation 10.2.3 can be written as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad y(t_0) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

Since the coefficient matrix and the forcing function are both continuous on $(-\infty, \infty)$, Theorem 10.2.1 implies that this problem has a unique solution on $(-\infty, \infty)$.

### Solution b

If $\mathbf{y}$ is given by Equation 10.2.4, then

$$\begin{aligned} A\mathbf{y} + \mathbf{f} &= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5}\begin{bmatrix} 22 \\ 23 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5}\begin{bmatrix} 32 \\ 28 \end{bmatrix} e^{4t} + 3c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \\ &= \mathbf{y}'. \end{aligned}$$

### Solution c

We must choose $c_1$ and $c_2$ in Equation 10.2.4 so that

$$\frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 22 \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Solving this system yields $c_1 = 1$, $c_2 = -2$, so

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

is the solution of Equation 10.2.5.

**Note**

The theory of $n \times n$ linear systems of differential equations is analogous to the theory of the scalar n-th order equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y = F(t) \tag{10.2.6}$$

as developed in Sections 9.1. For example by rewriting Equation 10.2.6 as an equivalent linear system it can be shown that Theorem 10.2.1 implies Theorem 9.1.1.

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# 10.2E: Linear Systems of Differential Equations (Exercises)

## Q10.2.1

1. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants $c_1$ and $c_2$.

a. $\begin{aligned} y_1' &= 2y_1 + 4y_2 \\ y_2' &= 4y_1 + 2y_2; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$

b. $\begin{aligned} y_1' &= -2y_1 - 2y_2 \\ y_2' &= -5y_1 + y_2; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{3t}$

c. $\begin{aligned} y_1' &= -4y_1 - 10y_2 \\ y_2' &= 3y_1 + 7y_2; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} -5 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{t}$

d. $\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= y_1 + 2y_2; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$

2. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants $c_1$, $c_2$, and $c_3$.

a. $\begin{aligned} y_1' &= -y_1 + 2y_2 + 3y_3 \\ y_2' &= y_2 + 6y_3 \\ y_3' &= -2y_3; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-2t}$

b. $\begin{aligned} y_1' &= 2y_2 + 2y_3 \\ y_2' &= 2y_1 + 2y_3 \\ y_3' &= 2y_1 + 2y_2; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}$

c. $\begin{aligned} y_1' &= -y_1 + 2y_2 + 2y_3 \\ y_2' &= 2y_1 - y_2 + 2y_3 \\ y_3' &= 2y_1 + 2y_2 - y_3; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t}$

d. $\begin{aligned} y_1' &= 3y_1 - y_2 - y_3 \\ y_2' &= -2y_1 + 3y_2 + 2y_3 \\ y_3' &= 4y_1 - y_2 - 2y_3; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$

3. Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

a. $\begin{aligned} y_1' &= y_1 + y_2, & y_1(0) &= 1 \\ y_2' &= -2y_1 + 4y_2, & y_2(0) &= 0; \end{aligned}$ $\mathbf{y} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$

b. $\begin{aligned} y_1' &= 5y_1+3y_2 \quad y_1(0) = 12 \\ y_2' &= -y_1+y_2, \quad y_2(0) = -6; \end{aligned}$ $\mathbf{y} = 3\begin{bmatrix}1\\-1\end{bmatrix}e^{2t} + 3\begin{bmatrix}3\\-1\end{bmatrix}e^{4t}$

4. Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

a. $\begin{aligned} y_1' &= 6y_1+4y_2+4y_3 \quad y_1(0) = 3 \\ y_2' &= -7y_1-2y_2-y_3, \ , \quad y_2(0) = -6 \\ y_3' &= 7y_1+4y_2+3y_3 \quad y_3(0) = 4 \end{aligned}$

$\mathbf{y} = \begin{bmatrix}1\\-1\\1\end{bmatrix}e^{6t} + 2\begin{bmatrix}1\\-2\\1\end{bmatrix}e^{2t} + \begin{bmatrix}0\\-1\\1\end{bmatrix}e^{-t}$

b. $\begin{aligned} y_1' &= 8y_1+7y_2+7y_3 \quad y_1(0) = 2 \\ y_2' &= -5y_1-6y_2-9y_3, \quad y_2(0) = -4 \\ y_3' &= 5y_1+7y_2+10y_3, \quad y_3(0) = 3 \end{aligned}$

$\mathbf{y} = \begin{bmatrix}1\\-1\\1\end{bmatrix}e^{8t} + \begin{bmatrix}0\\-1\\1\end{bmatrix}e^{3t} + \begin{bmatrix}1\\-2\\1\end{bmatrix}e^{t}$

5. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants $c_1$ and $c_2$.

a. $\begin{aligned} y_1' &= -3y_1+2y_2+3-2t \\ y_2' &= -5y_1+3y_2+6-3t \end{aligned}$

$\mathbf{y} = c_1\begin{bmatrix}2\cos t\\3\cos t-\sin t\end{bmatrix} + c_2\begin{bmatrix}2\sin t\\3\sin t+\cos t\end{bmatrix} + \begin{bmatrix}1\\t\end{bmatrix}$

b. $\begin{aligned} y_1' &= 3y_1+y_2-5e^t \\ y_2' &= -y_1+y_2+e^t \end{aligned}$

$\mathbf{y} = c_1\begin{bmatrix}-1\\1\end{bmatrix}e^{2t} + c_2\begin{bmatrix}1+t\\-t\end{bmatrix}e^{2t} + \begin{bmatrix}1\\3\end{bmatrix}e^{t}$

c. $\begin{aligned} y_1' &= -y_1-4y_2+4e^t+8te^t \\ y_2' &= -y_1-y_2+e^{3t}+(4t+2)e^t \end{aligned}$

$\mathbf{y} = c_1\begin{bmatrix}2\\1\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}-2\\1\end{bmatrix}e^{t} + \begin{bmatrix}e^{3t}\\2te^t\end{bmatrix}$

d. $\begin{aligned} y_1' &= -6y_1-3y_2+14e^{2t}+12e^t \\ y_2' &= y_1-2y_2+7e^{2t}-12e^t \end{aligned}$

$\mathbf{y} = c_1\begin{bmatrix}-3\\1\end{bmatrix}e^{-5t} + c_2\begin{bmatrix}-1\\1\end{bmatrix}e^{-3t} + \begin{bmatrix}e^{2t}+3e^t\\2e^{2t}-3e^t\end{bmatrix}$

6. Convert the linear scalar equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y(t) = F(t) \tag{A}$$

into an equivalent $n \times n$ system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t),$$

and show that $A$ and $\mathbf{f}$ are continuous on an interval $(a, b)$ if and only if (A) is normal on $(a, b)$.

7. A matrix function

$$Q(t) = \begin{bmatrix} q_{11}(t) & q_{12}(t) & \cdots & q_{1s}(t) \\ q_{21}(t) & q_{22}(t) & \cdots & q_{2s}(t) \\ \vdots & \vdots & \ddots & \vdots \\ q_{r1}(t) & q_{r2}(t) & \cdots & q_{rs}(t) \end{bmatrix}$$

is said to be **differentiable** if its entries $\{q_{ij}\}$ are differentiable. Then the **derivative** $Q'$ is defined by

$$Q(t) = \begin{bmatrix} q'_{11}(t) & q'_{12}(t) & \cdots & q'_{1s}(t) \\ q'_{21}(t) & q'_{22}(t) & \cdots & q'_{2s}(t) \\ \vdots & \vdots & \ddots & \vdots \\ q'_{r1}(t) & q'_{r2}(t) & \cdots & q'_{rs}(t) \end{bmatrix}$$

a. Prove: If $P$ and $Q$ are differentiable matrices such that $P+Q$ is defined and if $c_1$ and $c_2$ are constants, then

$$(c_1P + c_2Q)' = c_1P' + c_2Q'.$$

b. Prove: If $P$ and $Q$ are differentiable matrices such that $PQ$ is defined, then

$$(PQ)' = P'Q + PQ'.$$

8. Verify that $Y' = AY$.

a. $Y = \begin{bmatrix} e^{6t} & e^{-2t} \\ e^{6t} & -e^{-2t} \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

b. $Y = \begin{bmatrix} e^{-4t} & -2e^{3t} \\ e^{-4t} & 5e^{3t} \end{bmatrix}, \quad A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}$

c. $Y = \begin{bmatrix} -5e^{2t} & 2e^{t} \\ 3e^{2t} & -e^{t} \end{bmatrix}, \quad A = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix}$

d. $Y = \begin{bmatrix} e^{3t} & e^{t} \\ e^{3t} & -e^{t} \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

e. $Y = \begin{bmatrix} e^{t} & e^{-t} & e^{-2t} \\ e^{t} & 0 & -2e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}$

f. $Y = \begin{bmatrix} -e^{-2t} & -e^{-2t} & e^{4t} \\ 0 & e^{-2t} & e^{4t} \\ e^{-2t} & 0 & e^{4t} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

g. $Y = \begin{bmatrix} e^{3t} & e^{-3t} & 0 \\ e^{3t} & 0 & -e^{-3t} \\ e^{3t} & e^{-3t} & e^{-3t} \end{bmatrix}, \quad A = \begin{bmatrix} -9 & 6 & 6 \\ -6 & 3 & 6 \\ -6 & 6 & 3 \end{bmatrix}$

h. $Y = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix}$

9. Suppose

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix}$$

are solutions of the homogeneous system

$$\mathbf{y}' = A(t)\mathbf{y}, \tag{A}$$

and define

$$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}.$$

a. Show that $Y' = AY$.
b. Show that if $\mathbf{c}$ is a constant vector then $\mathbf{y} = Y\mathbf{c}$ is a solution of (A).
c. State generalizations of (a) and (b) for $n \times n$ systems.

10. Suppose $Y$ is a differentiable square matrix.

a. Find a formula for the derivative of $Y^2$.
b. Find a formula for the derivative of $Y^n$, where $n$ is any positive integer.
c. State how the results obtained in (a) and (b) are analogous to results from calculus concerning scalar functions.

11. It can be shown that if $Y$ is a differentiable and invertible square matrix function, then $Y^{-1}$ is differentiable.

a. Show that $(Y^{-1})' = -Y^{-1}Y'Y^{-1}$. (Hint: Differentiate the identity $Y^{-1}Y = I$.)
b. Find the derivative of $Y^{-n} = \left(Y^{-1}\right)^n$, where $n$ is a positive integer.
c. State how the results obtained in (a) and (b) are analogous to results from calculus concerning scalar functions.

12. Show that Theorem 10.2.1 implies Theorem 9.1.1. HINT: **Write the scalar function**

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x)$$

**as an $n \times n$ system of linear equations.**

13. Suppose $\mathbf{y}$ is a solution of the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$, and that the $n \times n$ matrix $P$ is invertible and differentiable on $(a, b)$. Find a matrix $B$ such that the function $\mathbf{x} = P\mathbf{y}$ is a solution of $\mathbf{x}' = B\mathbf{x}$ on $(a, b)$.

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# 10.3: Basic Theory of Homogeneous Linear Systems

In this section we consider homogeneous linear systems $\mathbf{y}' = A(t)\mathbf{y}$, where $A = A(t)$ is a continuous $n \times n$ matrix function on an interval $(a, b)$. The theory of linear homogeneous systems has much in common with the theory of linear homogeneous scalar equations, which we considered in Sections 2.1, 5.1, and 9.1.

Whenever we refer to solutions of $\mathbf{y}' = A(t)\mathbf{y}$ we'll mean solutions on $(a, b)$. Since $\mathbf{y} \equiv \mathbf{0}$ is obviously a solution of $\mathbf{y}' = A(t)\mathbf{y}$, we call it the **trivial** solution. Any other solution is **nontrivial**.

If $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$ are vector functions defined on an interval $(a, b)$ and $c_1$, $c_2$, ..., $c_n$ are constants, then

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n \tag{10.3.1}$$

is a **linear combination of** $\mathbf{y}_1$, $\mathbf{y}_2$, ...,$\mathbf{y}_n$. It's easy show that if $\mathbf{y}_1$, $\mathbf{y}_2$, ...,$\mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$, then so is any linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$. We say that $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is a **fundamental set of solutions of** $\mathbf{y}' = \boldsymbol{A(t)}\mathbf{y}$ on $(a, b)$ on if every solution of $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$ can be written as a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$, as in Equation 10.3.1. In this case we say that Equation 10.3.1 is the **general solution of** $\mathbf{y}' = \boldsymbol{A(t)}\mathbf{y}$ on $(a, b)$.

It can be shown that if $A$ is continuous on $(a, b)$ then $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on $(a, b)$. The next definition will help to characterize fundamental sets of solutions of $\mathbf{y}' = A(t)\mathbf{y}$.

We say that a set $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of $n$-vector functions is **linearly independent** on $(a, b)$ if the only constants $c_1$, $c_2$, ..., $c_n$ such that

$$c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) = 0, \quad a < t < b, \tag{10.3.2}$$

are $c_1 = c_2 = \cdots = c_n = 0$. If Equation 10.3.2 holds for some set of constants $c_1$, $c_2$, ..., $c_n$ that are not all zero, then $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is **linearly dependent** on $(a, b)$

The next theorem is analogous to Theorems 5.1.3 and 9.1.2.

**Theorem 10.3.1**

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on $(a, b)$. Then a set $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of $n$ solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$ is a fundamental set if and only if it is linearly independent on $(a, b)$.

**Example 10.3.1**

Show that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix}$$

are linearly independent on every interval $(a, b)$.

**Solution**

Suppose

$$c_1\begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix} + c_3\begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

We must show that $c_1 = c_2 = c_3 = 0$. Rewriting this equation in matrix form yields

$$\begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

Expanding the determinant of this system in cofactors of the entries of the first row yields

$$\begin{vmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} = e^t \begin{vmatrix} e^{3t} & e^{3t} \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 0 \end{vmatrix} + e^{2t} \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 1 \end{vmatrix}$$

$$= e^t(-e^{3t}) + e^{2t}(-e^{2t}) = -2e^{4t}.$$

Since this determinant is never zero, $c_1 = c_2 = c_3 = 0$ .

We can use the method in Example 10.3.1 to test $n$ solutions $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of any $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ for linear independence on an interval $(a, b)$ on which $A$ is continuous. To explain this (and for other purposes later), it is useful to write a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$ in a different way. We first write the vector functions in terms of their components as

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix}, \ldots, \quad \mathbf{y}_n = \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}.$$

If

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$$

then

$$\mathbf{y} = c_1 \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix} + c_2 \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

This shows that

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = Y\mathbf{c}, \tag{10.3.3}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

and

$$Y = [\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n] = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}; \tag{10.3.4}$$

that is, the columns of $Y$ are the vector functions $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$.

For reference below, note that

$$\begin{aligned} Y' &= [\mathbf{y}_1' \ \mathbf{y}_2' \ \cdots \ \mathbf{y}_n'] \\ &= [A\mathbf{y}_1 \ A\mathbf{y}_2 \ \cdots \ A\mathbf{y}_n] \\ &= A[\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n] = AY; \end{aligned}$$

that is, $Y$ satisfies the matrix differential equation

$$Y' = AY.$$

The determinant of $Y$,

$$W = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} \tag{10.3.5}$$

is called the **Wronskian** of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$. It can be shown (**Exercises 10.3.2** and **10.3.3**) that this definition is analogous to definitions of the Wronskian of scalar functions given in Sections 5.1 and 9.1. The next theorem is analogous to Theorems 5.1.4 and 9.1.3. The proof is sketched in **Exercise 10.3.4** for $n = 2$ and in **Exercise 10.3.5** for general $n$.

### Theorem 10.3.2 : Abel's Formula

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on $(a, b)$, let $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$, and let $t_0$ be in $(a, b)$. Then the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is given by

$$W(t) = W(t_0) \exp\left( \int_{t_0}^{t} [a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)] \, ds \right), \quad a < t < b. \tag{10.3.6}$$

Therefore, either $W$ has no zeros in $(a, b)$ or $W \equiv 0$ on $(a, b)$.

### Note

The sum of the diagonal entries of a square matrix $A$ is called the **trace** of $A$, denoted by $\operatorname{tr}(A)$. Thus, for an $n \times n$ matrix $A$,

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots a_{nn},$$

and Equation 10.3.6 can be written as

$$W(t) = W(t_0) \exp\left( \int_{t_0}^{t} \operatorname{tr}(A(s)) ds \right), \quad a < t < b.$$

The next theorem is analogous to Theorems 5.1.6 and 9.1.4.

### Theorem 10.3.3

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on $(a, b)$ and let $\mathbf{y}_1$, $\mathbf{y}_2$, ...,$\mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$. Then the following statements are equivalent; that is, they are either all true or all false:

a. The general solution of $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$ is $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$ , where $c_1$, $c_2$, ..., $c_n$ are arbitrary constants.
b. $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$.
c. $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is linearly independent on $(a, b)$.
d. The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is nonzero at some point in $(a, b)$.
e. The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is nonzero at all points in $(a, b)$.

We say that $Y$ in Equation 10.3.4 is a **fundamental matrix** for $\mathbf{y}' = A(t)\mathbf{y}$ if any (and therefore all) of the statements (**a**)-(**e**) of Theorem 10.3.2 are true for the columns of $Y$. In this case, Equation 10.3.3 implies that the general solution of $\mathbf{y}' = A(t)\mathbf{y}$ can be written as $\mathbf{y} = Y\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant $n$-vector.

## Example 10.3.2

The vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

are solutions of the constant coefficient system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y} \tag{10.3.7}$$

on $(-\infty, \infty)$. (Verify.)

a. Compute the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$ directly from the definition Equation 10.3.5
b. Verify Abel's formula Equation 10.3.6 for the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.
c. Find the general solution of Equation 10.3.7.
d. Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}. \tag{10.3.8}$$

### Solution a

From Equation 10.3.5

$$W(t) = \begin{vmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{vmatrix} = e^{2t}e^{-t} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = e^t. \tag{10.3.9}$$

### Solution b

Here

$$A = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix},$$

so $\operatorname{tr}(A) = -4 + 5 = 1$ . If $t_0$ is an arbitrary real number then Equation 10.3.6 implies that

$$W(t) = W(t_0) \exp\left( \int_{t_0}^{t} 1 \, ds \right) = \begin{vmatrix} -e^{2t_0} & -e^{-t_0} \\ 2e^{2t_0} & e^{-t_0} \end{vmatrix} e^{(t-t_0)} = e^{t_0} e^{t-t_0} = e^t,$$

which is consistent with Equation 10.3.9.

### Solution c

Since $W(t) \neq 0$, Theorem 10.3.3 implies that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions of Equation 10.3.7 and

$$Y = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix for Equation 10.3.7. Therefore the general solution of Equation 10.3.7 is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1 \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{10.3.10}$$

### Solution d

Setting $t = 0$ in Equation 10.3.10 and imposing the initial condition in Equation 10.3.8 yields

$$c_1\begin{bmatrix}-1\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}4\\-5\end{bmatrix}.$$

Thus,

$$\begin{aligned}-c_1-c_2&=\ \ 4\\2c_1+c_2&=-5.\end{aligned}$$

The solution of this system is $c_1=-1$, $c_2=-3$. Substituting these values into Equation 10.3.10yields

$$\mathbf{y}=-\begin{bmatrix}-e^{2t}\\2e^{2t}\end{bmatrix}-3\begin{bmatrix}-e^{-t}\\e^{-t}\end{bmatrix}=\begin{bmatrix}e^{2t}+3e^{-t}\\-2e^{2t}-3e^{-t}\end{bmatrix}$$

as the solution of Equation 10.3.8.

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# 10.3E: Basic Theory of Homogeneous Linear Systems (Exercises)

## Q10.3.1

1. Prove: If $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$, then any linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ is also a solution of $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$.

2. In Section 5.1 the Wronskian of two solutions $y_1$ and $y_2$ of the scalar second order equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{A}$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

a. Rewrite (A) as a system of first order equations and show that $W$ is the Wronskian (as defined in this section) of two solutions of this system.

b. Apply Equation 10.3.6 to the system derived in (a), and show that

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x} \frac{P_1(s)}{P_0(s)}\,ds\right\},$$

which is the form of Abel's formula given in Theorem 9.1.3.

3. In Section 9.1 the Wronskian of $n$ solutions $y_1$, $y_2$, …, $y_n$ of the $n-$th order equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0 \tag{A}$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

a. Rewrite (A) as a system of first order equations and show that $W$ is the Wronskian (as defined in this section) of $n$ solutions of this system.

b. Apply Equation 10.3.6 to the system derived in (a), and show that

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x} \frac{P_1(s)}{P_0(s)}\,ds\right\},$$

which is the form of Abel's formula given in Theorem 9.1.3.

4. Suppose

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix}$$

are solutions of the $2 \times 2$ system $\mathbf{y}' = A\mathbf{y}$ on $(a, b)$, and let

$$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \quad \text{and} \quad W = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}$$

thus, $W$ is the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.

a. Deduce from the definition of determinant that

$$W' = \begin{vmatrix} y_{11}' & y_{12}' \\ y_{21} & y_{22} \end{vmatrix} + \begin{vmatrix} y_{11} & y_{12} \\ y_{21}' & y_{22}' \end{vmatrix}.$$

b. Use the equation $Y' = A(t)Y$ and the definition of matrix multiplication to show that

$$[y_{11}' \quad y_{12}'] = a_{11}[y_{11} \quad y_{12}] + a_{12}[y_{21} \quad y_{22}]$$

and

$$[y_{21}' \quad y_{22}'] = a_{21}[y_{11} \quad y_{12}] + a_{22}[y_{21} \quad y_{22}].$$

c. Use properties of determinants to deduce from (a) and (a) that

$$\begin{vmatrix} y_{11}' & y_{12}' \\ y_{21} & y_{22} \end{vmatrix} = a_{11}W \quad \text{and} \quad \begin{vmatrix} y_{11} & y_{12} \\ y_{21}' & y_{22}' \end{vmatrix} = a_{22}W.$$

d. Conclude from (c) that

$$W' = (a_{11} + a_{22})W,$$

and use this to show that if $a < t_0 < b$ then

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} [a_{11}(s) + a_{22}(s)]\, ds\right) \quad a < t < b.$$

5. Suppose the $n \times n$ matrix $A = A(t)$ is continuous on $(a, b)$. Let

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix},$$

where the columns of $Y$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$. Let

$$r_i = [y_{i1}\ y_{i2}\ \ldots\ y_{in}]$$

be the $i$th row of $Y$, and let $W$ be the determinant of $Y$.

a. Deduce from the definition of determinant that

$$W' = W_1 + W_2 + \cdots + W_n,$$

where, for $1 \le m \le n$, the $i$th row of $W_m$ is $r_i$ if $i \ne m$, and $r_m'$ if $i = m$.

b. Use the equation $Y' = AY$ and the definition of matrix multiplication to show that

$$r_m' = a_{m1}r_1 + a_{m2}r_2 + \cdots + a_{mn}r_n.$$

c. Use properties of determinants to deduce from (b) that

$$\det(W_m) = a_{mm}W.$$

d. Conclude from (a) and (c) that

$$W' = (a_{11} + a_{22} + \cdots + a_{nn})W,$$

and use this to show that if $a < t_0 < b$ then

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} [a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)]\, ds\right), \quad a < t < b.$$

6. Suppose the $n \times n$ matrix $A$ is continuous on $(a, b)$ and $t_0$ is a point in $(a, b)$. Let $Y$ be a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$.

a. Show that $Y(t_0)$ is invertible.
b. Show that if $\mathbf{k}$ is an arbitrary $n$-vector then the solution of the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y}, \quad \mathbf{y}(t_0) = \mathbf{k}$$

is

$$\mathbf{y} = Y(t)Y^{-1}(t_0)\mathbf{k}.$$

7. Let

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}.$$

a. Verify that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$.
b. Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{k}. \tag{A}$$

c. Use the result of **Exercise 10.3.6** (b) to find a formula for the solution of (A) for an arbitrary initial vector $\mathbf{k}$.

8. Repeat **Exercise 10.3.7** with

$$A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -2e^{3t} \\ 5e^{3t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 10 \\ -4 \end{bmatrix}.$$

9. Repeat **Exercise 10.3.7** with

$$A = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} -5e^{2t} \\ 3e^{2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} -19 \\ 11 \end{bmatrix}.$$

10. Repeat **Exercise 10.3.7** with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}.$$

11. Let

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-t} \\ -3e^{-t} \\ 7e^{-t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 2 \\ -7 \\ 20 \end{bmatrix}.$$

a. Verify that $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$.
b. Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{k}. \tag{A}$$

c. Use the result of **Exercise 10.3.6** (b) to find a formula for the solution of (A) for an arbitrary initial vector $\mathbf{k}$.

12. Repeat **Exercise 10.3.11** with

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} -e^{-2t} \\ 0 \\ e^{-2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-2t} \\ e^{-2t} \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ -9 \\ 12 \end{bmatrix}.$$

13. Repeat **Exercise 10.3.11** with

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} e^{t} \\ e^{t} \\ 0 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix}.$$

14. Suppose $Y$ and $Z$ are fundamental matrices for the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$. Then some of the four matrices $YZ^{-1}$, $Y^{-1}Z$, $Z^{-1}Y$, $ZY^{-1}$ are necessarily constant. Identify them and prove that they are constant.

15. Suppose the columns of an $n \times n$ matrix $Y$ are solutions of the $n \times n$ system $\mathbf{y}' = A\mathbf{y}$ and $C$ is an $n \times n$ constant matrix.

a. Show that the matrix $Z = YC$ satisfies the differential equation $Z' = AZ$.
b. Show that $Z$ is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ if and only if $C$ is invertible and $Y$ is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$.

16. Suppose the $n \times n$ matrix $A = A(t)$ is continuous on $(a, b)$ and $t_0$ is in $(a, b)$. For $i = 1, 2, \ldots, n$, let $\mathbf{y}_i$ be the solution of the initial value problem $\mathbf{y}_i' = A(t)\mathbf{y}_i$, $\mathbf{y}_i(t_0) = \mathbf{e}_i$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

that is, the $j$th component of $\mathbf{e}_i$ is 1 if $j = i$, or 0 if $j \neq i$.

a. Show that $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on $(a, b)$.
b. Conclude from (a) and **Exercise 10.3.15** that $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on $(a, b)$.

17. Show that $Y$ is a fundamental matrix for the system $\mathbf{y}' = A(t)\mathbf{y}$ if and only if $Y^{-1}$ is a fundamental matrix for $\mathbf{y}' = -A^T(t)\mathbf{y}$, where $A^T$ denotes the transpose of $A$. HINT: **See Exercise 10.3.11.**

18. Let $Z$ be the fundamental matrix for the constant coefficient system $\mathbf{y}' = A\mathbf{y}$ such that $Z(0) = I$.

a. Show that $Z(t)Z(s) = Z(t + s)$ for all $s$ and $t$. HINT: **For fixed $s$ let $\Gamma_1(t) = Z(t)Z(s)$ and $\Gamma_2(t) = Z(t + s)$. Show that $\Gamma_1$ and $\Gamma_2$ are both solutions of the matrix initial value problem $\Gamma' = A\Gamma$, $\Gamma(0) = Z(s)$. Then conclude from Theorem 10.2.1 that $\Gamma_1 = \Gamma_2$.**
b. Show that $(Z(t))^{-1} = Z(-t)$.
c. The matrix $Z$ defined above is sometimes denoted by $e^{tA}$. Discuss the motivation for this notation.

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# 10.4: Constant Coefficient Homogeneous Systems I

We'll now begin our study of the homogeneous system

$$\mathbf{y}' = A\mathbf{y}, \tag{10.4.1}$$

where $A$ is an $n \times n$ constant matrix. Since $A$ is continuous on $(-\infty, \infty)$, Theorem 10.2.1 implies that all solutions of Equation 10.4.1 are defined on $(-\infty, \infty)$. Therefore, when we speak of solutions of $\mathbf{y}' = A\mathbf{y}$, we'll mean solutions on $(-\infty, \infty)$.

In this section we assume that all the eigenvalues of $A$ are real and that $A$ has a set of $n$ linearly independent eigenvectors. In the next two sections we consider the cases where some of the eigenvalues of $A$ are complex, or where $A$ does not have $n$ linearly independent eigenvectors.

In Example 10.3.2 we showed that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

form a fundamental set of solutions of the system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \tag{10.4.2}$$

but we did not show how we obtained $\mathbf{y}_1$ and $\mathbf{y}_2$ in the first place. To see how these solutions can be obtained we write Equation 10.4.2 as

$$\begin{aligned} y_1' &= -4y_1 - 3y_2 \\ y_2' &= 6y_1 + 5y_2 \end{aligned} \tag{10.4.3}$$

and look for solutions of the form

$$y_1 = x_1 e^{\lambda t} \quad \text{and} \quad y_2 = x_2 e^{\lambda t}, \tag{10.4.4}$$

where $x_1$, $x_2$, and $\lambda$ are constants to be determined. Differentiating Equation 10.4.4 yields

$$y_1' = \lambda x_1 e^{\lambda t} \quad \text{and} \quad y_2' = \lambda x_2 e^{\lambda t}.$$

Substituting this and Equation 10.4.4 into Equation 10.4.3 and canceling the common factor $e^{\lambda t}$ yields

$$\begin{aligned} -4x_1 - 3x_2 &= \lambda x_1 \\ 6x_1 + 5x_2 &= \lambda x_2. \end{aligned}$$

For a given $\lambda$, this is a homogeneous algebraic system, since it can be rewritten as

$$\begin{aligned} (-4 - \lambda)x_1 - 3x_2 &= 0 \\ 6x_1 + (5 - \lambda)x_2 &= 0. \end{aligned} \tag{10.4.5}$$

The trivial solution $x_1 = x_2 = 0$ of this system isn't useful, since it corresponds to the trivial solution $y_1 \equiv y_2 \equiv 0$ of Equation 10.4.3, which can't be part of a fundamental set of solutions of Equation 10.4.2. Therefore we consider only those values of $\lambda$ for which Equation 10.4.5 has nontrivial solutions. These are the values of $\lambda$ for which the determinant of Equation 10.4.5 is zero; that is,

$$\begin{aligned} \begin{vmatrix} -4 - \lambda & -3 \\ 6 & 5 - \lambda \end{vmatrix} &= (-4 - \lambda)(5 - \lambda) + 18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1) = 0, \end{aligned}$$

which has the solutions $\lambda_1 = 2$ and $\lambda_2 = -1$.

Taking $\lambda = 2$ in Equation 10.4.5 yields

$$-6x_1 - 3x_2 = 0$$
$$6x_1 + 3x_2 = 0,$$

which implies that $x_1 = -x_2/2$, where $x_2$ can be chosen arbitrarily. Choosing $x_2 = 2$ yields the solution $y_1 = -e^{2t}$, $y_2 = 2e^{2t}$ of Equation 10.4.3. We can write this solution in vector form as

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{2t}. \tag{10.4.6}$$

Taking $\lambda = -1$ in Equation 10.4.5 yields the system

$$-3x_1 - 3x_2 = 0$$
$$6x_1 + 6x_2 = 0,$$

so $x_1 = -x_2$. Taking $x_2 = 1$ here yields the solution $y_1 = -e^{-t}$, $y_2 = e^{-t}$ of Equation 10.4.3. We can write this solution in vector form as

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}. \tag{10.4.7}$$

In Equation 10.4.6 and Equation 10.4.7 the constant coefficients in the arguments of the exponential functions are the eigenvalues of the coefficient matrix in Equation 10.4.2, and the vector coefficients of the exponential functions are associated eigenvectors. This illustrates the next theorem.

**Theorem 10.4.1**

Suppose the $n \times n$ constant matrix $A$ has $n$ real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Then the functions

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t},\ \mathbf{y}_2 = \mathbf{x}_2 e^{\lambda_2 t},\ \ldots,\ \mathbf{y}_n = \mathbf{x}_n e^{\lambda_n t}$$

form a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$; that is, the general solution of this system is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{x}_n e^{\lambda_n t}.$$

**Proof**

Differentiating $\mathbf{y}_i = \mathbf{x}_i e^{\lambda_i t}$ and recalling that $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$ yields

$$\mathbf{y}_i' = \lambda_i \mathbf{x}_i e^{\lambda_i t} = A\mathbf{x}_i e^{\lambda_i t} = A\mathbf{y}_i.$$

This shows that $\mathbf{y}_i$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is

$$\begin{vmatrix} x_{11}e^{\lambda_1 t} & x_{12}e^{\lambda_2 t} & \cdots & x_{1n}e^{\lambda_n t} \\ x_{21}e^{\lambda_1 t} & x_{22}e^{\lambda_2 t} & \cdots & x_{2n}e^{\lambda_n t} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}e^{\lambda_1 t} & x_{n2}e^{\lambda_2 t} & \cdots & x_{nn}e^{\lambda x_n t} \end{vmatrix} = e^{\lambda_1 t}e^{\lambda_2 t}\cdots e^{\lambda_n t} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}.$$

Since the columns of the determinant on the right are $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$, which are assumed to be linearly independent, the determinant is nonzero. Therefore Theorem 10.3.3 implies that $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$.

**Example 10.4.1**

a. Find the general solution of


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$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}. \tag{10.4.8}$$

b. Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}. \tag{10.4.9}$$

**Solution a**

The characteristic polynomial of the coefficient matrix $A$ in Equation 10.4.8 is

$$\begin{aligned} \begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} &= (\lambda-2)^2 - 16 \\ &= (\lambda-2-4)(\lambda-2+4) \\ &= (\lambda-6)(\lambda+2). \end{aligned}$$

Hence, $\lambda_1 = 6$ and $\lambda_2 = -2$ are eigenvalues of $A$. To obtain the eigenvectors, we must solve the system

$$\begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10.4.10}$$

with $\lambda = 6$ and $\lambda = -2$. Setting $\lambda = 6$ in Equation 10.4.10 yields

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

is a solution of Equation 10.4.8. Setting $\lambda = -2$ in Equation 10.4.10 yields

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = -x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

is a solution of Equation 10.4.8. From Theorem 10.4.1 , the general solution of Equation 10.4.8 is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}. \tag{10.4.11}$$

**Solution b**

To satisfy the initial condition in Equation 10.4.9, we must choose $c_1$ and $c_2$ in Equation 10.4.11 so that

$$c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}5\\-1\end{bmatrix}.$$

This is equivalent to the system

$$\begin{aligned}c_1-c_2&=\ \ 5\\ c_1+c_2&=-1,\end{aligned}$$

so $c_1=2, c_2=-3$. Therefore the solution of Equation 10.4.9 is

$$\mathbf{y}=2\begin{bmatrix}1\\1\end{bmatrix}e^{6t}-3\begin{bmatrix}-1\\1\end{bmatrix}e^{-2t},$$

or, in terms of components,

$$y_1=2e^{6t}+3e^{-2t},\quad y_2=2e^{6t}-3e^{-2t}.$$

## Example 10.4.2

a. Find the general solution of

$$\mathbf{y}'=\begin{bmatrix}3&-1&-1\\-2&3&2\\4&-1&-2\end{bmatrix}\mathbf{y}. \tag{10.4.12}$$

b. Solve the initial value problem

$$\mathbf{y}'=\begin{bmatrix}3&-1&-1\\-2&3&2\\4&-1&-2\end{bmatrix}\mathbf{y},\quad \mathbf{y}(0)=\begin{bmatrix}2\\-1\\8\end{bmatrix}. \tag{10.4.13}$$

### Solution a

The characteristic polynomial of the coefficient matrix $A$ in Equation 10.4.12 is

$$\begin{vmatrix}3-\lambda&-1&-1\\-2&3-\lambda&2\\4&-1&-2-\lambda\end{vmatrix}=-(\lambda-2)(\lambda-3)(\lambda+1).$$

Hence, the eigenvalues of $A$ are $\lambda_1=2$, $\lambda_2=3$, and $\lambda_3=-1$. To find the eigenvectors, we must solve the system

$$\begin{bmatrix}3-\lambda&-1&-1\\-2&3-\lambda&2\\4&-1&-2-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{10.4.14}$$

with $\lambda=2, 3, -1$. With $\lambda=2$, the augmented matrix of Equation 10.4.14 is

$$\left[\begin{array}{ccc:c}1&-1&-1&0\\-2&1&2&0\\4&-1&-4&0\end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

as a solution of Equation 10.4.12 With $\lambda = 3$, the augmented matrix of Equation 10.4.14is

$$\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ -2 & 0 & 2 & 0 \\ 4 & -1 & -5 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t}$$

as a solution of Equation 10.4.12 With $\lambda = -1$, the augmented matrix of Equation 10.4.14is

$$\left[\begin{array}{ccc|c} 4 & -1 & -1 & 0 \\ -2 & 4 & 2 & 0 \\ 4 & -1 & -1 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3/7$ and $x_2 = -3x_3/7$. Taking $x_3 = 7$ yields

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$$

as a solution of Equation 10.4.12 By Theorem 10.4.1 , the general solution of Equation 10.4.12is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t},$$

which can also be written as

$$\mathbf{y} = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{10.4.15}$$

**Solution b**

To satisfy the initial condition in Equation 10.4.13we must choose $c_1$, $c_2$, $c_3$ in Equation 10.4.15so that

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}.$$

Solving this system yields $c_1 = 3$, $c_2 = -2$, $c_3 = 1$. Hence, the solution of Equation 10.4.13is

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} - 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}. \end{aligned}$$

**Example 10.4.3**

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \mathbf{y}. \tag{10.4.16}$$

**Solution**

The characteristic polynomial of the coefficient matrix $A$ in Equation 10.4.16is

$$\begin{vmatrix} -3-\lambda & 2 & 2 \\ 2 & -3-\lambda & 2 \\ 2 & 2 & -3-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+5)^2.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 1, while $\lambda_2 = -5$ is an eigenvalue of multiplicity 2. Eigenvectors associated with $\lambda_1 = 1$ are solutions of the system with augmented matrix

$$\left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_2 = x_3$, and we choose $x_3 = 1$ to obtain the solution

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t \tag{10.4.17}$$

of Equation 10.4.16. Eigenvectors associated with $\lambda_2 = -5$ are solutions of the system with augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right].$$

Hence, the components of these eigenvectors need only satisfy the single condition

$$x_1 + x_2 + x_3 = 0.$$

Since there's only one equation here, we can choose $x_2$ and $x_3$ arbitrarily. We obtain one eigenvector by choosing $x_2 = 0$ and $x_3 = 1$, and another by choosing $x_2 = 1$ and $x_3 = 0$. In both cases $x_1 = -1$. Therefore

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent eigenvectors associated with $\lambda_2 = -5$, and the corresponding solutions of Equation 10.4.16are

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Because of this and Equation 10.4.17, Theorem 10.4.1 implies that the general solution of Equation 10.4.16is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

## Geometric Properties of Solutions when n=2

We'll now consider the geometric properties of solutions of a $2 \times 2$ constant coefficient system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{10.4.18}$$

It is convenient to think of a "$y_1$-$y_2$ plane," where a point is identified by rectangular coordinates $(y_1, y_2)$. If $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is a non-constant solution of Equation 10.4.18, then the point $(y_1(t), y_2(t))$ moves along a curve $C$ in the $y_1$-$y_2$ plane as $t$ varies from $-\infty$ to $\infty$. We call $C$ the **trajectory** of $\mathbf{y}$. (We also say that $C$ is a trajectory of the system Equation 10.4.18) I's important to note that $C$ is the trajectory of infinitely many solutions of Equation 10.4.18, since if $\tau$ is any real number, then $\mathbf{y}(t-\tau)$ is a solution of Equation 10.4.18, and $(y_1(t-\tau), y_2(t-\tau))$ also moves along $C$ as $t$ varies from $-\infty$ to $\infty$. Moreover, **Exercise 10.4.28c**

implies that distinct trajectories of Equation 10.4.18 can't intersect, and that two solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of Equation 10.4.18 have the same trajectory if and only if $\mathbf{y}_2(t)=\mathbf{y}_1(t-\tau)$ for some $\tau$.

From **Exercise 10.4.28a**, a trajectory of a nontrivial solution of Equation 10.4.18 can't contain $(0,0)$, which we define to be the trajectory of the trivial solution $\mathbf{y}\equiv 0$. More generally, if $\mathbf{y}=\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}\neq \mathbf{0}$ is a constant solution of Equation 10.4.18 (which could occur if zero is an eigenvalue of the matrix of Equation 10.4.18), we define the trajectory of $\mathbf{y}$ to be the single point $(k_1,k_2)$.

To be specific, this is the question: What do the trajectories look like, and how are they traversed? In this section we'll answer this question, assuming that the matrix

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

of Equation 10.4.18 has real eigenvalues $\lambda_1$ and $\lambda_2$ with associated linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then the general solution of Equation 10.4.18 is

$$\mathbf{y}=c_1\mathbf{x}_1e^{\lambda_1 t}+c_2\mathbf{x}_2e^{\lambda_2 t}. \tag{10.4.19}$$

We'll consider other situations in the next two sections.

We leave it to you to classify the trajectories of Equation 10.4.18 if zero is an eigenvalue of $A$. We'll confine our attention here to the case where both eigenvalues are nonzero. In this case the simplest situation is where $\lambda_1=\lambda_2\neq 0$, so Equation 10.4.19 becomes

$$\mathbf{y}=(c_1\mathbf{x}_1+c_2\mathbf{x}_2)e^{\lambda_1 t}.$$

Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, an arbitrary vector $\mathbf{x}$ can be written as $\mathbf{x}=c_1\mathbf{x}_1+c_2\mathbf{x}_2$. Therefore the general solution of Equation 10.4.18 can be written as $\mathbf{y}=\mathbf{x}e^{\lambda_1 t}$ where $\mathbf{x}$ is an arbitrary 2-vector, and the trajectories of nontrivial solutions of Equation 10.4.18 are half-lines through (but not including) the origin. The direction of motion is away from the origin if $\lambda_1>0$ (Figure 10.4.1), toward it if $\lambda_1<0$ (Figure 10.4.2). (In these and the next figures an arrow through a point indicates the direction of motion along the trajectory through the point.)

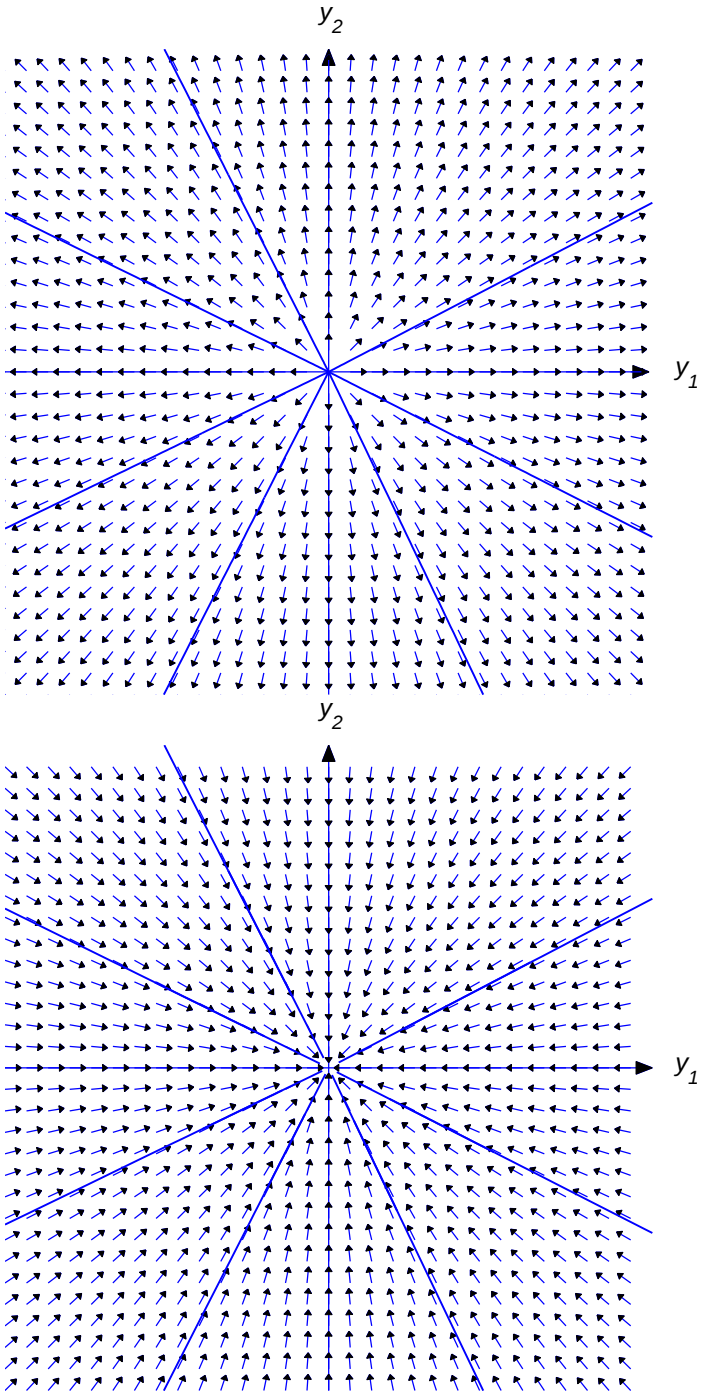



Figure 10.4.1 (left): Trajectories of a $2 \times 2$ **system with a repeated positive eigenvalue** and Figure 10.4.2 (right): Trajectories of a $2 \times 2$ **system with a repeated negative eigenvalue.**

Now suppose $\lambda_2 > \lambda_1$, and let $L_1$ and $L_2$ denote lines through the origin parallel to $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. By a half-line of $L_1$ (or $L_2$), we mean either of the rays obtained by removing the origin from $L_1$ (or $L_2$).

Letting $c_2 = 0$ in Equation 10.4.19 yields $\mathbf{y} = c_1\mathbf{x}_1e^{\lambda_1 t}$. If $c_1 \neq 0$, the trajectory defined by this solution is a half-line of $L_1$. The direction of motion is away from the origin if $\lambda_1 > 0$, toward the origin if $\lambda_1 < 0$. Similarly, the trajectory of $\mathbf{y} = c_2\mathbf{x}_2e^{\lambda_2 t}$ with $c_2 \neq 0$ is a half-line of $L_2$.

Henceforth, we assume that $c_1$ and $c_2$ in Equation 10.4.19 are both nonzero. In this case, the trajectory of Equation 10.4.19 can't intersect $L_1$ or $L_2$, since every point on these lines is on the trajectory of a solution for which either $c_1 = 0$ or $c_2 = 0$. (Remember: distinct trajectories can't intersect!). Therefore the trajectory of Equation 10.4.19 must lie entirely in one of the four open sectors bounded by $L_1$ and $L_2$, but do not any point on $L_1$ or $L_2$. Since the initial point $(y_1(0), y_2(0))$ defined by

$$\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

is on the trajectory, we can determine which sector contains the trajectory from the signs of $c_1$ and $c_2$, as shown in Figure 10.4.3 .

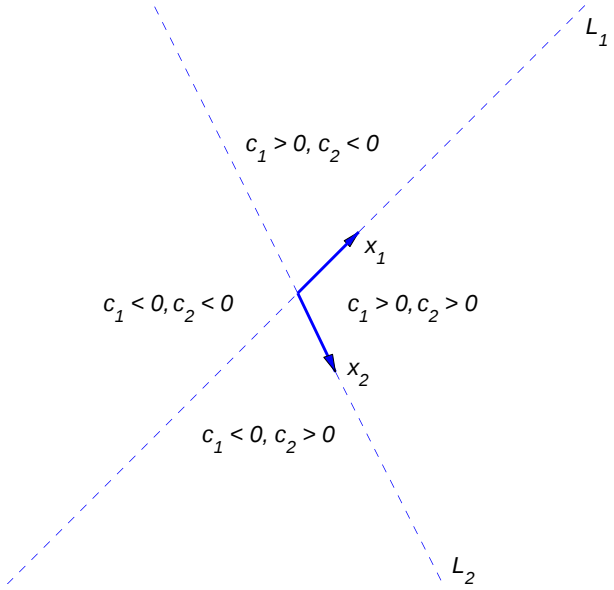



Figure 10.4.3 : Four open sectors bounded by $L_1$ **and** $L_2$.

The direction of $\mathbf{y}(t)$ in Equation 10.4.19 is the same as that of

$$e^{-\lambda_2 t}\mathbf{y}(t) = c_1\mathbf{x}_1 e^{-(\lambda_2-\lambda_1)t} + c_2\mathbf{x}_2 \tag{10.4.20}$$

and of

$$e^{-\lambda_1 t}\mathbf{y}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 e^{(\lambda_2-\lambda_1)t}. \tag{10.4.21}$$

Since the right side of Equation 10.4.20 approaches $c_2\mathbf{x}_2$ as $t \to \infty$, the trajectory is asymptotically parallel to $L_2$ as $t \to \infty$. Since the right side of Equation 10.4.21 approaches $c_1\mathbf{x}_1$ as $t \to -\infty$, the trajectory is asymptotically parallel to $L_1$ as $t \to -\infty$.

The shape and direction of traversal of the trajectory of Equation 10.4.19 depend upon whether $\lambda_1$ and $\lambda_2$ are both positive, both negative, or of opposite signs. We'll now analyze these three cases.

Henceforth $\|\mathbf{u}\|$ denote the length of the vector $\mathbf{u}$.

### Case 1: $\lambda_2 > \lambda_1$

Figure 10.4.4 shows some typical trajectories. In this case, $\lim_{t\to-\infty} \|\mathbf{y}(t)\| = 0$, so the trajectory is not only asymptotically parallel to $L_1$ as $t \to -\infty$, but is actually asymptotically tangent to $L_1$ at the origin. On the other hand, $\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t\to\infty} \left\|\mathbf{y}(t) - c_2\mathbf{x}_2 e^{\lambda_2 t}\right\| = \lim_{t\to\infty} \|c_1\mathbf{x}_1 e^{\lambda_1 t}\| = \infty,$$

so, although the trajectory is asymptotically parallel to $L_2$ as $t \to \infty$, it is not asymptotically tangent to $L_2$. The direction of motion along each trajectory is away from the origin.

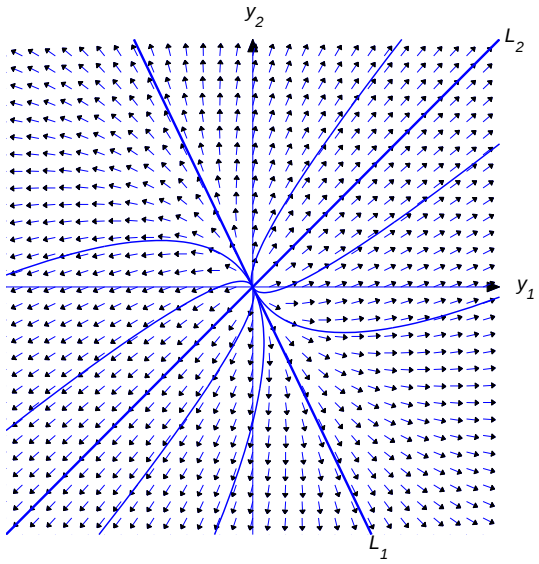



Figure 10.4.4 : Two positive eigenvalues; motion away from origin.

### Case 2: 0 > λ₂ > λ₁

Figure 10.4.5 shows some typical trajectories. In this case, $\lim_{t\to\infty} \|\mathbf{y}(t)\| = 0$, so the trajectory is asymptotically tangent to $L_2$ at the origin as $t \to \infty$. On the other hand, $\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t\to-\infty} \left\|\mathbf{y}(t) - c_1\mathbf{x}_1 e^{\lambda_1 t}\right\| = \lim_{t\to-\infty} \|c_2\mathbf{x}_2 e^{\lambda_2 t}\| = \infty,$$

so, although the trajectory is asymptotically parallel to $L_1$ as $t \to -\infty$, it is not asymptotically tangent to it. The direction of motion along each trajectory is toward the origin.

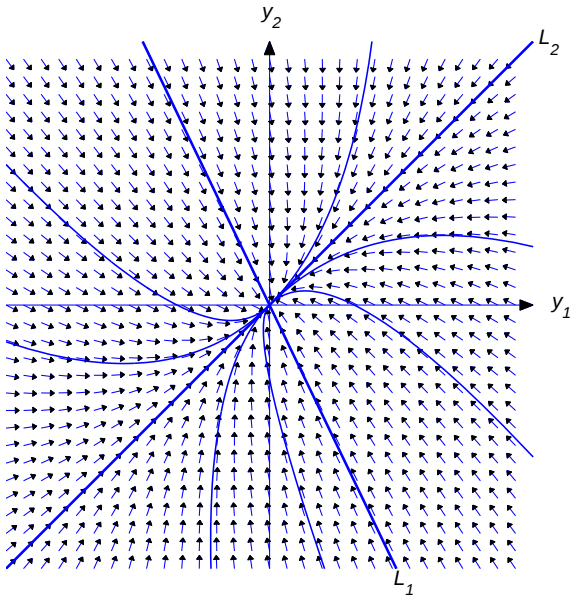



Figure 10.4.5 : Two negative eigenvalues; motion toward the origin.

### Case 3: λ₂ > 0 > λ₁

Figure 10.4.6 shows some typical trajectories. In this case,

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \left\|\mathbf{y}(t) - c_2\mathbf{x}_2 e^{\lambda_2 t}\right\| = \lim_{t\to\infty} \|c_1\mathbf{x}_1 e^{\lambda_1 t}\| = 0,$$

so the trajectory is asymptotically tangent to $L_2$ as $t \to \infty$. Similarly,

$$\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \left\|\mathbf{y}(t) - c_1\mathbf{x}_1 e^{\lambda_1 t}\right\| = \lim_{t\to-\infty} \|c_2\mathbf{x}_2 e^{\lambda_2 t}\| = 0,$$

so the trajectory is asymptotically tangent to $L_1$ as $t \to -\infty$. The direction of motion is toward the origin on $L_1$ and away from the origin on $L_2$. The direction of motion along any other trajectory is away from $L_1$, toward $L_2$.

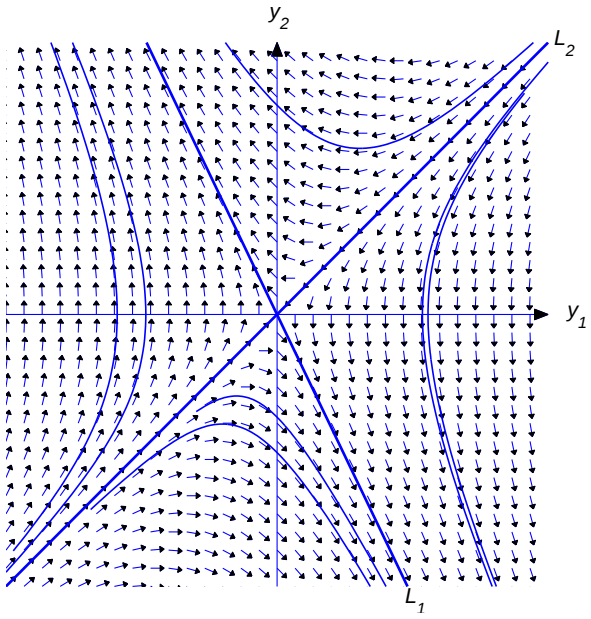



Figure 10.4.6 : Eigenvalues of different signs.

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# 10.4E: Constant Coefficient Homogeneous Systems I (Exercises)

## Q10.4.1

In **Exercises 10.4.1-10.4.15** find the general solution.

1. $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y}$

2. $\mathbf{y}' = \frac{1}{4} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix} \mathbf{y}$

3. $\mathbf{y}' = \frac{1}{5} \begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y}$

4. $\mathbf{y}' = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$

5. $\mathbf{y}' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$

6. $\mathbf{y}' = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \mathbf{y}$

7. $\mathbf{y}' = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{y}$

8. $\mathbf{y}' = \begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} \mathbf{y}$

9. $\mathbf{y}' = \begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} \mathbf{y}$

10. $\mathbf{y}' = \begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \mathbf{y}$

11. $\mathbf{y}' = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} \mathbf{y}$

12. $\mathbf{y}' = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \mathbf{y}$

13. $\mathbf{y}' = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \mathbf{y}$

14. $\mathbf{y}' = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} \mathbf{y}$

15. $\mathbf{y}' = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \mathbf{y}$

## Q10.4.2

In **Exercises 10.4.16-10.4.27** solve the initial value problem.

16. $\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

17. $\mathbf{y}' = \frac{1}{6} \begin{bmatrix} 7 & 2 \\ -2 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

18. $\mathbf{y}' = \begin{bmatrix} 21 & -12 \\ 24 & -15 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

19. $\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

20. $\mathbf{y}' = \frac{1}{6} \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$

21. $\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 2 & -2 & 3 \\ -4 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

22. $\mathbf{y}' = \begin{bmatrix} 6 & -3 & -8 \\ 2 & 1 & -2 \\ 3 & -3 & -5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$

23. $\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 2 & 4 & -7 \\ 1 & 5 & -5 \\ -4 & 4 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

24. $\mathbf{y}' = \begin{bmatrix} 3 & 0 & 1 \\ 11 & -2 & 7 \\ 1 & 0 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$

25. $\mathbf{y}' = \begin{bmatrix} -2 & -5 & -1 \\ -4 & -1 & 1 \\ 4 & 5 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ -10 \\ -4 \end{bmatrix}$

26. $\mathbf{y}' = \begin{bmatrix} 3 & -1 & 0 \\ 4 & -2 & 0 \\ 4 & -4 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 7 \\ 10 \\ 2 \end{bmatrix}$

27. $\mathbf{y}' = \begin{bmatrix} -2 & 2 & 6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ -10 \\ 7 \end{bmatrix}$

## Q10.4.3

28. Let $A$ be an $n \times n$ constant matrix. Then Theorem 10.2.1 implies that the solutions of

$$\mathbf{y}' = A\mathbf{y} \tag{A}$$

are all defined on $(-\infty, \infty)$.

a. Use Theorem 10.2.1 to show that the only solution of (A) that can ever equal the zero vector is $\mathbf{y} \equiv \mathbf{0}$.
b. Suppose $\mathbf{y}_1$ is a solution of (A) and $\mathbf{y}_2$ is defined by $\mathbf{y}_2(t) = \mathbf{y}_1(t - \tau)$, where $\tau$ is an arbitrary real number. Show that $\mathbf{y}_2$ is also a solution of (A).
c. Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of (A) and there are real numbers $t_1$ and $t_2$ such that $\mathbf{y}_1(t_1) = \mathbf{y}_2(t_2)$. Show that $\mathbf{y}_2(t) = \mathbf{y}_1(t - \tau)$ for all $t$, where $\tau = t_2 - t_1$.

## Q10.4.4

In **Exercises 10.4.29-10.4.34** describe and graph trajectories of the given system.

29. $\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{y}$

30. $\mathbf{y}' = \begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y}$

31. $\mathbf{y}' = \begin{bmatrix} 9 & -3 \\ -1 & 11 \end{bmatrix} \mathbf{y}$

32. $\mathbf{y}' = \begin{bmatrix} -1 & -10 \\ -5 & 4 \end{bmatrix} \mathbf{y}$

33. $\mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 1 & 10 \end{bmatrix} \mathbf{y}$

34. $\mathbf{y}' = \begin{bmatrix} -7 & 1 \\ 3 & -5 \end{bmatrix} \mathbf{y}$

## Q10.4.5

35. Suppose the eigenvalues of the $2 \times 2$ matrix $A$ are $\lambda = 0$ and $\mu \neq 0$, with corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Let $L_1$ be the line through the origin parallel to $\mathbf{x}_1$.

a. Show that every point on $L_1$ is the trajectory of a constant solution of $\mathbf{y}' = A\mathbf{y}$.
b. Show that the trajectories of nonconstant solutions of $\mathbf{y}' = A\mathbf{y}$ are half-lines parallel to $\mathbf{x}_2$ and on either side of $L_1$, and that the direction of motion along these trajectories is away from $L_1$ if $\mu > 0$, or toward $L_1$ if $\mu < 0$.

## Q10.4.6

The matrices of the systems in **Exercises 10.4.36-10.4.41** are singular. Describe and graph the trajectories of nonconstant solutions of the given systems.

36. $\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{y}$

37. $\mathbf{y}' = \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix} \mathbf{y}$

38. $\mathbf{y}' = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \mathbf{y}$

39. $\mathbf{y}' = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \mathbf{y}$

40. $\mathbf{y}' = \begin{bmatrix} -4 & -4 \\ 1 & 1 \end{bmatrix} \mathbf{y}$

41. $\mathbf{y}' = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \mathbf{y}$

## Q10.4.6

42. Let $P = P(t)$ and $Q = Q(t)$ be the populations of two species at time $t$, and assume that each population would grow exponentially if the other didn't exist; that is, in the absence of competition,

$$P' = aP \quad \text{and} \quad Q' = bQ, \tag{A}$$

where $a$ and $b$ are positive constants. One way to model the effect of competition is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so (A) is replaced by

$$\begin{aligned} P' &= aP - \alpha Q \\ Q' &= -\beta P + bQ, \end{aligned}$$

where $\alpha$ and $\beta$ are positive constants. (Since negative population doesn't make sense, this system holds only while $P$ and $Q$ are both positive.) Now suppose $P(0) = P_0 > 0$ and $Q(0) = Q_0 > 0$.

a. For several choices of $a$, $b$, $\alpha$, and $\beta$, verify experimentally (by graphing trajectories of (A) in the $P$-$Q$ plane) that there's a constant $\rho > 0$ (depending upon $a$, $b$, $\alpha$, and $\beta$) with the following properties:
   i. If $Q_0 > \rho P_0$, then $P$ decreases monotonically to zero in finite time, during which $Q$ remains positive.
   ii. If $Q_0 < \rho P_0$, then $Q$ decreases monotonically to zero in finite time, during which $P$ remains positive.

b. Conclude from (a) that exactly one of the species becomes extinct in finite time if $Q_0 \neq \rho P_0$. Determine experimentally what happens if $Q_0 = \rho P_0$.

c. Confirm your experimental results and determine $\gamma$ by expressing the eigenvalues and associated eigenvectors of

$$A = \begin{bmatrix} \alpha & -\alpha \\ -\beta & b \end{bmatrix}$$

in terms of $a$, $b$, $\alpha$, and $\beta$, and applying the geometric arguments developed at the end of this section.

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# 10.5: Constant Coefficient Homogeneous Systems II

We saw in Section 10.4 that if an $n \times n$ constant matrix $A$ has $n$ real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, then the general solution of $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{x}_n e^{\lambda_n t}.$$

In this section we consider the case where $A$ has $n$ real eigenvalues, but does not have $n$ linearly independent eigenvectors. It is shown in linear algebra that this occurs if and only if $A$ has at least one eigenvalue of multiplicity $r > 1$ such that the associated eigenspace has dimension less than $r$. In this case $A$ is said to be **defective**. Since it is beyond the scope of this book to give a complete analysis of systems with defective coefficient matrices, we will restrict our attention to some commonly occurring special cases.

## Example 10.5.1

Show that the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y} \tag{10.5.1}$$

does not have a fundamental set of solutions of the form $\{\mathbf{x}_1 e^{\lambda_1 t}, \mathbf{x}_2 e^{\lambda_2 t}\}$, where $\lambda_1$ and $\lambda_2$ are eigenvalues of the coefficient matrix $A$ of Equation 10.5.1 and $\mathbf{x}_1$, and $\mathbf{x}_2$ are associated linearly independent eigenvectors.

### Solution

The characteristic polynomial of $A$ is

$$\begin{aligned} \begin{vmatrix} 11-\lambda \\ 4-\lambda \end{vmatrix} &= (\lambda - 11)(\lambda + 9) + 100 \\ &= \lambda^2 - 2\lambda + 1 \\ &= (\lambda - 1)^2. \end{aligned}$$

Hence, $\lambda = 1$ is the only eigenvalue of $A$. The augmented matrix of the system $(A - I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 10 & -25 & \vdots & 0 \\ 4 & -10 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & -\frac{5}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$$

Hence, $x_1 = 5x_2/2$ where $x_2$ is arbitrary. Therefore all eigenvectors of $A$ are scalar multiples of $\mathbf{x_1} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, so $A$ does not have a set of two linearly independent eigenvectors.

From Example 10.5.1 , we know that all scalar multiples of $\mathbf{y_1} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \mathbf{e^t}$ are solutions of Equation 10.5.1; however, to find the general solution we must find a second solution $\mathbf{y_2}$ such that $\{\mathbf{y_1}, \mathbf{y_2}\}$ is linearly independent. Based on your recollection of the procedure for solving a constant coefficient scalar equation

$$ay'' + by' + cy = 0$$

in the case where the characteristic polynomial has a repeated root, you might expect to obtain a second solution of Equation 10.5.1 by multiplying the first solution by $t$. However, this yields $\mathbf{y}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t$ , which does not work, since

$$\mathbf{y}_2' = \begin{bmatrix} 5 \\ 2 \end{bmatrix} (te^t + e^t), \quad \text{while} \quad \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t.$$

The next theorem shows what to do in this situation.

**Theorem 10.5.1**

Suppose the $n \times n$ matrix $A$ has an eigenvalue $\lambda_1$ of multiplicity $\geq 2$ and the associated eigenspace has dimension 1; that is, all $\lambda_1$-eigenvectors of $A$ are scalar multiples of an eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \tag{10.5.2}$$

Moreover, if $\mathbf{u}$ is any such vector then

$$\mathbf{y}_1 = \mathbf{x}e^{\lambda_1 t} \quad \text{and} \quad \mathbf{y}_2 = \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t} \tag{10.5.3}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

A complete proof of this theorem is beyond the scope of this book. The difficulty is in proving that there's a vector $\mathbf{u}$ satisfying Equation 10.5.2, since $\det(A - \lambda_1 I) = 0$ . We'll take this without proof and verify the other assertions of the theorem. We already know that $\mathbf{y}_1$ in Equation 10.5.3 is a solution of $\mathbf{y}' = A\mathbf{y}$ . To see that $\mathbf{y}_2$ is also a solution, we compute

$$\begin{aligned} \mathbf{y}_2' - A\mathbf{y}_2 &= \lambda_1 \mathbf{u}e^{\lambda_1 t} + \mathbf{x}e^{\lambda_1 t} + \lambda_1 \mathbf{x}te^{\lambda_1 t} - A\mathbf{u}e^{\lambda_1 t} - A\mathbf{x}te^{\lambda_1 t} \\ &= (\lambda_1 \mathbf{u} + \mathbf{x} - A\mathbf{u})e^{\lambda_1 t} + (\lambda_1 \mathbf{x} - A\mathbf{x})te^{\lambda_1 t}. \end{aligned}$$

Since $A\mathbf{x} = \lambda_1 \mathbf{x}$, this can be written as

$$\mathbf{y}_2' - A\mathbf{y}_2 = -\left((A - \lambda_1 I)\mathbf{u} - \mathbf{x}\right) e^{\lambda_1 t},$$

and now Equation 10.5.2 implies that $\mathbf{y}_2' = A\mathbf{y}_2$ . To see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent, suppose $c_1$ and $c_2$ are constants such that

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1\mathbf{x}e^{\lambda_1 t} + c_2(\mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}) = \mathbf{0}. \tag{10.5.4}$$

We must show that $c_1 = c_2 = 0$ . Multiplying Equation 10.5.4 by $e^{-\lambda_1 t}$ shows that

$$c_1\mathbf{x} + c_2(\mathbf{u} + \mathbf{x}t) = \mathbf{0}. \tag{10.5.5}$$

By differentiating this with respect to $t$, we see that $c_2\mathbf{x} = \mathbf{0}$, which implies $c_2 = 0$, because $\mathbf{x} \neq \mathbf{0}$. Substituting $c_2 = 0$ into Equation 10.5.5 yields $c_1\mathbf{x} = \mathbf{0}$, which implies that $c_1 = 0$, again because $\mathbf{x} \neq \mathbf{0}$

**Example 10.5.2**

Use Theorem 10.5.1 to find the general solution of the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y} \tag{10.5.6}$$

considered in Example 10.5.1 .

**Solution**

In Example 10.5.1 we saw that $\lambda_1 = 1$ is an eigenvalue of multiplicity 2 of the coefficient matrix $A$ in Equation 10.5.6, and that all of the eigenvectors of $A$ are multiples of

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t$$

is a solution of Equation 10.5.6. From Theorem 10.5.1 , a second solution is given by $\mathbf{y}_2 = \mathbf{u}e^t + \mathbf{x}te^t$ , where $(A-I)\mathbf{u} = \mathbf{x}$ . The augmented matrix of this system is

$$\left[\begin{array}{cc|c} 10 & -25 & 5 \\ 4 & -10 & 2 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cc|c} 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right],$$

Therefore the components of $\mathbf{u}$ must satisfy

$$u_1 - \frac{5}{2}u_2 = \frac{1}{2},$$

where $u_2$ is arbitrary. We choose $u_2 = 0$, so that $u_1 = 1/2$ and

$$\mathbf{u} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

Thus,

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t.$$

Since $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent by Theorem 10.5.1 , they form a fundamental set of solutions of Equation 10.5.6. Therefore the general solution of Equation 10.5.6 is

$$\mathbf{y} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t + c_2 \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t \right).$$

Note that choosing the arbitrary constant $u_2$ to be nonzero is equivalent to adding a scalar multiple of $\mathbf{y}_1$ to the second solution $\mathbf{y}_2$ (**Exercise 10.5.33**).

## Example 10.5.3

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & 4 & -10 \\ 2 & 1 & -2 \\ 2 & 2 & -5 \end{bmatrix} \mathbf{y}. \tag{10.5.7}$$

### Solution

The characteristic polynomial of the coefficient matrix $A$ in Equation 10.5.7 is

$$\begin{vmatrix} 3-\lambda & 4 & -10 \\ 2 & 1-\lambda & -2 \\ 2 & 2 & -5-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+1)^2.$$

Hence, the eigenvalues are $\lambda_1 = 1$ with multiplicity 1 and $\lambda_2 = -1$ with multiplicity 2. Eigenvectors associated with $\lambda_1 = 1$ must satisfy $(A-I)\mathbf{x} = \mathbf{0}$ . The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -10 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & 2 & -6 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 2x_3$, where $x_3$ is arbitrary. Choosing $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^t$$

is a solution of Equation 10.5.7. Eigenvectors associated with $\lambda_2 = -1$ satisfy $(A+I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 4 & 4 & -10 & 0 \\ 2 & 2 & -2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_3 = 0$ and $x_1 = -x_2$, where $x_2$ is arbitrary. Choosing $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

is a solution of Equation 10.5.7. Since all the eigenvectors of $A$ associated with $\lambda_2 = -1$ are multiples of $\mathbf{x}_2$, we must now use Theorem 10.5.1 to find a third solution of Equation 10.5.7 in the form

$$\mathbf{y}_3 = \mathbf{u}e^{-t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t}, \tag{10.5.8}$$

where $\mathbf{u}$ is a solution of $(A+I)\mathbf{u} = \mathbf{x}_2$. The augmented matrix of this system is

$$\left[\begin{array}{ccc:c} 4 & 4 & -10 & -1 \\ 2 & 2 & -2 & 1 \\ 2 & 2 & -4 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $u_3 = 1/2$ and $u_1 = 1 - u_2$, where $u_2$ is arbitrary. Choosing $u_2 = 0$ yields

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix},$$

and substituting this into Equation 10.5.8 yields the solution

$$\mathbf{y}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t}$$

of Equation 10.5.7. Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2},$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of Equation 10.5.7. Therefore the general solution of Equation 10.5.7 is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_3 \left( \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t} \right).$$

**Theorem 10.5.2**

Suppose the $n \times n$ matrix $A$ has an eigenvalue $\lambda_1$ of multiplicity $\geq 3$ and the associated eigenspace is one–dimensional; that is, all eigenvectors associated with $\lambda_1$ are scalar multiples of the eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}, \tag{10.5.9}$$

and, if $\mathbf{u}$ is any such vector, there are infinitely many vectors $\mathbf{v}$ such that

$$(A - \lambda_1 I)\mathbf{v} = \mathbf{u}. \tag{10.5.10}$$

If $\mathbf{u}$ satisfies Equation 10.5.9 and $\mathbf{v}$ satisfies Equation 10.5.10, then

$$\begin{aligned}\mathbf{y}_1 &= \mathbf{x}e^{\lambda_1 t},\\ \mathbf{y}_2 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}, \text{ and}\\ \mathbf{y}_3 &= \mathbf{v}e^{\lambda_1 t} + \mathbf{u}te^{\lambda_1 t} + \mathbf{x}\frac{t^2 e^{\lambda_1 t}}{2}\end{aligned}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Again, it is beyond the scope of this book to prove that there are vectors $\mathbf{u}$ and $\mathbf{v}$ that satisfy Equation 10.5.9 and Equation 10.5.10. Theorem 10.5.1 implies that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{y}' = A\mathbf{y}$. We leave the rest of the proof to you (**Exercise 10.5.34**).

## Example 10.5.4

Use Theorem 10.5.2 to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} \mathbf{y}. \tag{10.5.11}$$

### Solution

The characteristic polynomial of the coefficient matrix $A$ in Equation 10.5.11 is

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 0 & 2 & 2-\lambda \end{vmatrix} = -(\lambda-2)^3.$$

Hence, $\lambda_1 = 2$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A-2I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc:c} -1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$, so the eigenvectors are all scalar multiples of

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of Equation 10.5.11. We now find a second solution of Equation 10.5.11 in the form

$$\mathbf{y}_2 = \mathbf{u}e^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{2t},$$

where $\mathbf{u}$ satisfies $(A - 2I)\mathbf{u} = \mathbf{x}_1$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Letting $u_3 = 0$ yields $u_1 = -1/2$ and $u_2 = 1/2$; hence,

$$\mathbf{u} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{2t}$$

is a solution of Equation 10.5.11. We now find a third solution of Equation 10.5.11in the form

$$\mathbf{y}_3 = \mathbf{v}e^{2t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2e^{2t}}{2}$$

where $\mathbf{v}$ satisfies $(A - 2I)\mathbf{v} = \mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & -\frac{1}{2} \\ 1 & 1 & -1 & \frac{1}{2} \\ 0 & 2 & 0 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Letting $v_3 = 0$ yields $v_1 = 1/2$ and $v_2 = 0$; hence,

$$\mathbf{v} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2 e^{2t}}{2}$$

is a solution of Equation 10.5.11. Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 10.5.2 , they form a fundamental set of solutions of Equation 10.5.11. Therefore the general solution of Equation 10.5.11 is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{2t} \right) + c_3 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2 e^{2t}}{2} \right)$$

## Theorem 10.5.3

Suppose the $n \times n$ matrix $A$ has an eigenvalue $\lambda_1$ of multiplicity $\geq 3$ and the associated eigenspace is two–dimensional; that is, all eigenvectors of $A$ associated with $\lambda_1$ are linear combinations of two linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then there are constants $\alpha$ and $\beta$ (not both zero) such that if

$$\mathbf{x}_3 = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2, \tag{10.5.12}$$

then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}_3. \tag{10.5.13}$$

If $\mathbf{u}$ satisfies Equation 10.5.13, then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}_1 e^{\lambda_1 t} \\ \mathbf{y}_2 &= \mathbf{x}_2 e^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{u} e^{\lambda_1 t} + \mathbf{x}_3 t e^{\lambda_1 t}, \end{aligned} \tag{10.5.14}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

We omit the proof of this theorem.

## Example 10.5.5

Use Theorem 10.5.3 to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} \mathbf{y}. \tag{10.5.15}$$

### Solution

The characteristic polynomial of the coefficient matrix $A$ in Equation 10.5.15 is

$$\begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & 1 \\ -1 & 0 & 2-\lambda \end{vmatrix} = -(\lambda - 1)^3.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A - I)\mathbf{x} = \mathbf{0}$ . The augmented matrix of this system is

$$\left[\begin{array}{cccc|c} -1 & 0 & 1 & \vdots & 0 \\ -1 & 0 & 1 & \vdots & 0 \\ -1 & 0 & 1 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2$ is arbitrary, so the eigenvectors are of the form

$$\mathbf{x}_1 = \begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{10.5.16}$$

form a basis for the eigenspace, and

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t$$

are linearly independent solutions of Equation 10.5.15. To find a third linearly independent solution of Equation 10.5.15, we must find constants $\alpha$ and $\beta$ (not both zero) such that the system

$$(A - I)\mathbf{u} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2 \tag{10.5.17}$$

has a solution $\mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{cccc|c} -1 & 0 & 1 & \vdots & \alpha \\ -1 & 0 & 1 & \vdots & \beta \\ -1 & 0 & 1 & \vdots & \alpha \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & \vdots & -\alpha \\ 0 & 0 & 0 & \vdots & \beta - \alpha \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]. \tag{10.5.18}$$

Therefore Equation 10.5.17 has a solution if and only if $\beta = \alpha$, where $\alpha$ is arbitrary. If $\alpha = \beta = 1$ then Equation 10.5.12 and Equation 10.5.16 yield

$$\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and the augmented matrix Equation 10.5.18 becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This implies that $u_1 = -1 + u_3$ , while $u_2$ and $u_3$ are arbitrary. Choosing $u_2 = u_3 = 0$ yields

$$\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore Equation 10.5.14implies that

$$\mathbf{y}_3 = \mathbf{u}e^t + \mathbf{x}_3 te^t = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} te^t$$

is a solution of Equation 10.5.15. Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 10.5.3 , they form a fundamental set of solutions for Equation 10.5.15. Therefore the general solution of Equation 10.5.15is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t + c_3 \left( \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} te^t \right). \tag{10.5.19}$$

## Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a $2 \times 2$ constant coefficient system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{10.5.20}$$

under the assumptions of this section; that is, when the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a repeated eigenvalue $\lambda_1$ and the associated eigenspace is one-dimensional. In this case we know from Theorem 10.5.1 that the general solution of Equation 10.5.20is

$$\mathbf{y} = c_1 \mathbf{x} e^{\lambda_1 t} + c_2 (\mathbf{u} e^{\lambda_1 t} + \mathbf{x} t e^{\lambda_1 t}), \tag{10.5.21}$$

where $\mathbf{x}$ is an eigenvector of $A$ and $\mathbf{u}$ is any one of the infinitely many solutions of

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \tag{10.5.22}$$

We assume that $\lambda_1 \neq 0$.

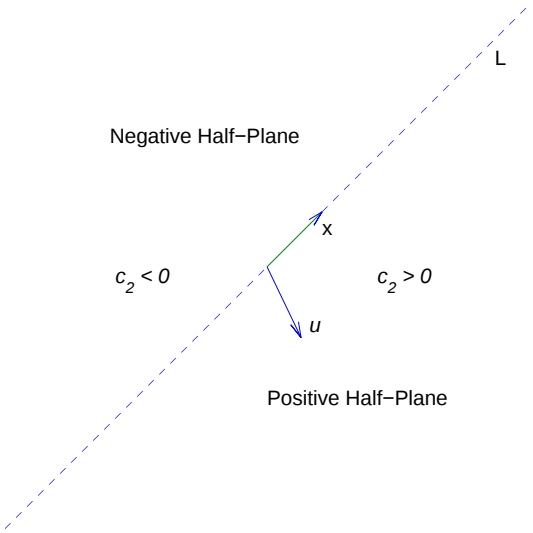



Figure 10.5.1 : Positive and negative half-planes.

Let $L$ denote the line through the origin parallel to $\mathbf{x}$. By a half-line of $L$ we mean either of the rays obtained by removing the origin from $L$. Equation 10.5.21 is a parametric equation of the half-line of $L$ in the direction of $\mathbf{x}$ if $c_1 > 0$, or of the half-line of $L$ in the direction of $-\mathbf{x}$ if $c_1 < 0$. The origin is the trajectory of the trivial solution $\mathbf{y} \equiv \mathbf{0}$.

Henceforth, we assume that $c_2 \neq 0$. In this case, the trajectory of Equation 10.5.21 can't intersect $L$, since every point of $L$ is on a trajectory obtained by setting $c_2 = 0$. Therefore the trajectory of Equation 10.5.21 must lie entirely in one of the open half-planes bounded by $L$, but does not contain any point on $L$. Since the initial point $(y_1(0), y_2(0))$ defined by $\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{u}$ is on the trajectory, we can determine which half-plane contains the trajectory from the sign of $c_2$, as shown in Figure . For convenience we'll call the half-plane where $c_2 > 0$ the positive half-plane. Similarly, the-half plane where $c_2 < 0$ is the negative half-plane. You should convince yourself that even though there are infinitely many vectors $\mathbf{u}$ that satisfy Equation 10.5.22, they all define the same positive and negative half-planes. In the figures simply regard $\mathbf{u}$ as an arrow pointing to the positive half-plane, since wen't attempted to give $\mathbf{u}$ its proper length or direction in comparison with $\mathbf{x}$. For our purposes here, only the relative orientation of $\mathbf{x}$ and $\mathbf{u}$ is important; that is, whether the positive half-plane is to the right of an observer facing the direction of $\mathbf{x}$ (as in Figures 10.5.2 and 10.5.5 ), or to the left of the observer (as in Figures 10.5.3 and 10.5.4 ).

Multiplying Equation 10.5.21 by $e^{-\lambda_1 t}$ yields

$$e^{-\lambda_1 t}\mathbf{y}(t) = c_1\mathbf{x} + c_2\mathbf{u} + c_2 t\mathbf{x}.$$

Since the last term on the right is dominant when $|t|$ is large, this provides the following information on the direction of $\mathbf{y}(t)$:

a. Along trajectories in the positive half-plane ($c_2 > 0$), the direction of $\mathbf{y}(t)$ approaches the direction of $\mathbf{x}$ as $t \to \infty$ and the direction of $-\mathbf{x}$ as $t \to -\infty$.
b. Along trajectories in the negative half-plane ($c_2 < 0$), the direction of $\mathbf{y}(t)$ approaches the direction of $-\mathbf{x}$ as $t \to \infty$ and the direction of $\mathbf{x}$ as $t \to -\infty$.

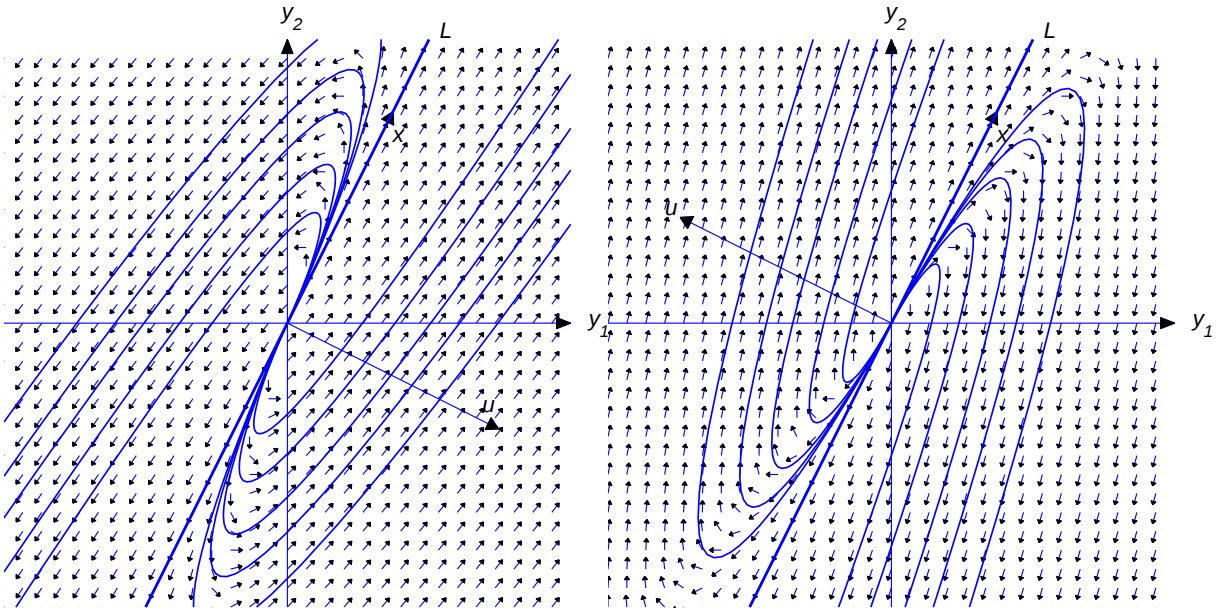
Since

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \mathbf{y}(t) = \mathbf{0} \quad \text{if} \quad \lambda_1 > 0,$$

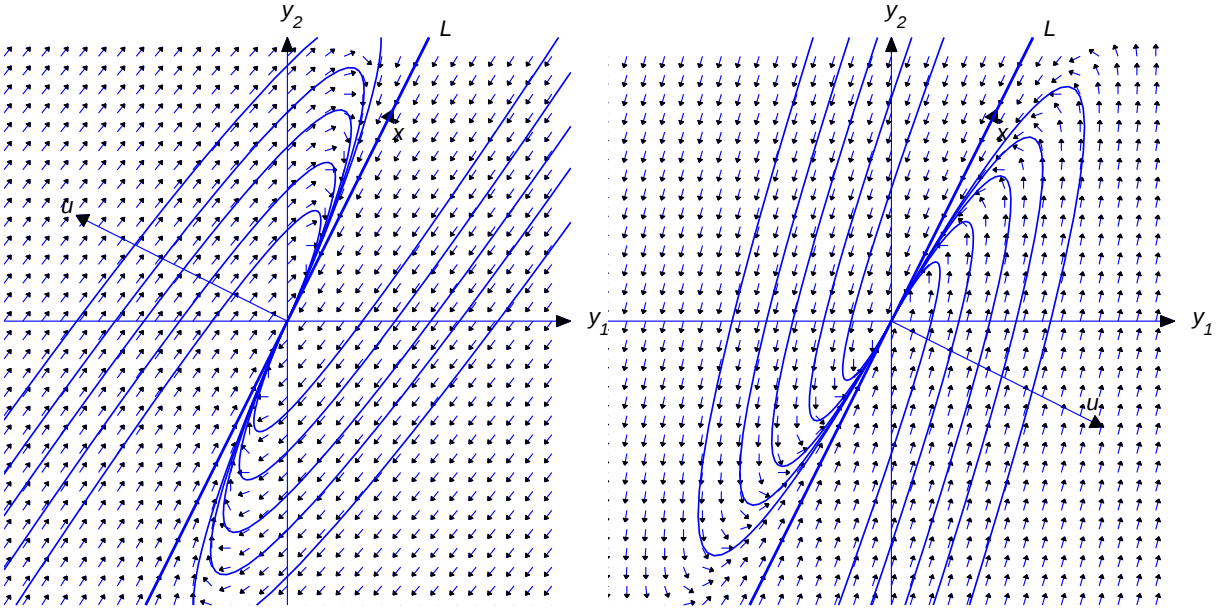
or

$$\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0} \quad \text{if} \quad \lambda_1 < 0,$$

there are four possible patterns for the trajectories of Equation 10.5.20, depending upon the signs of $c_2$ and $\lambda_1$. Figures 10.5.2 - 10.5.5 illustrate these patterns, and reveal the following principle:




Figures 10.5.2 and 10.5.3 : (left) Positive eigenvalue; motion away from the origin. (right) Positive eigenvalue; motion away from the origin.




Figures 10.5.4 and 10.5.5 : Negative eigenvalue; motion toward the origin. (right) Negative eigenvalue; motion toward the origin.

If $\lambda_1$ and $c_2$ have the same sign then the direction of the traectory approaches the direction of $-\mathbf{x}$ as $\|\mathbf{y}\| \to 0$ and the direction of $\mathbf{x}$ as $\|\mathbf{y}\| \to \infty$. If $\lambda_1$ and $c_2$ have opposite signs then the direction of the trajectory approaches the direction of $\mathbf{x}$ as $\|\mathbf{y}\| \to 0$ and the direction of $-\mathbf{x}$ as $\|\mathbf{y}\| \to \infty$.

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# 10.5E: Constant Coefficient Homogeneous Systems II (Exercises)

## Q10.5.1

In **Exercises 10.5.1-10.5.12** find the general solution.

1. $\mathbf{y}' = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} \mathbf{y}$

2. $\mathbf{y}' = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{y}$

3. $\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} \mathbf{y}$

4. $\mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{y}$

5. $\mathbf{y}' = \begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} \mathbf{y}$

6. $\mathbf{y}' = \begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix} \mathbf{y}$

7. $\mathbf{y}' = \begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix} \mathbf{y}$

8. $\mathbf{y}' = \begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} \mathbf{y}$

9. $\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 1 & 1 & -3 \\ -4 & -4 & 3 \\ -2 & 1 & 0 \end{bmatrix} \mathbf{y}$

10. $\mathbf{y}' = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} \mathbf{y}$

11. $\mathbf{y}' = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \mathbf{y}$

12. $\mathbf{y}' = \begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{y}$

## Q10.5.2

In **Exercises 10.5.13-10.5.23** solve the initial value problem.

13. $\mathbf{y}' = \begin{bmatrix} -11 & 8 \\ -2 & -3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

14. $\mathbf{y}' = \begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

15. $\mathbf{y}' = \begin{bmatrix} -3 & -4 \\ 1 & -7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

16. $\mathbf{y}' = \begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

17. $\mathbf{y}' = \begin{bmatrix} -7 & 3 \\ -3 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

18. $\mathbf{y}' = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 5 \\ -7 \end{bmatrix}$

19. $\mathbf{y}' = \begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -6 \\ -2 \\ 0 \end{bmatrix}$

20. $\mathbf{y}' = \begin{bmatrix} -7 & -4 & 4 \\ -1 & 0 & 1 \\ -9 & -5 & 6 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -6 \\ 9 \\ -1 \end{bmatrix}$

21. $\mathbf{y}' = \begin{bmatrix} -1 & -4 & -1 \\ 3 & 6 & 1 \\ -3 & -2 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

22. $\mathbf{y}' = \begin{bmatrix} 4 & -8 & -4 \\ -3 & -1 & -3 \\ 1 & -1 & 9 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$

23. $\mathbf{y}' = \begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

## Q10.5.3

The coefficient matrices in **Exercises 10.5.24-10.5.32** have eigenvalues of multiplicity 3. Find the general solution.

24. $\mathbf{y}' = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{bmatrix} \mathbf{y}$

25. $\mathbf{y}' = \begin{bmatrix} 1 & 10 & -12 \\ 2 & 2 & 3 \\ 2 & -1 & 6 \end{bmatrix} \mathbf{y}$

26. $\mathbf{y}' = \begin{bmatrix} -6 & -4 & -4 \\ 2 & -1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \mathbf{y}$

27. $\mathbf{y}' = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 5 & -3 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{y}$

28. $\mathbf{y}' = \begin{bmatrix} -2 & -12 & 10 \\ 2 & -24 & 11 \\ 2 & -24 & 8 \end{bmatrix} \mathbf{y}$

29. $\mathbf{y}' = \begin{bmatrix} -1 & -12 & 8 \\ 1 & -9 & 4 \\ 1 & -6 & 1 \end{bmatrix} \mathbf{y}$

30. $\mathbf{y}' = \begin{bmatrix} -4 & 0 & -1 \\ -1 & -3 & -1 \\ 1 & 0 & -2 \end{bmatrix} \mathbf{y}$

31. $\mathbf{y}' = \begin{bmatrix} -3 & -3 & 4 \\ 4 & 5 & -8 \\ 2 & 3 & -5 \end{bmatrix} \mathbf{y}$

32. $\mathbf{y}' = \begin{bmatrix} -3 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix} \mathbf{y}$

## Q10.5.4

33. Under the assumptions of Theorem 10.5.1, suppose $\mathbf{u}$ and $\hat{\mathbf{u}}$ are vectors such that

$$(A-\lambda_1 I)\mathbf{u} = \mathbf{x} \quad \text{and} \quad (A-\lambda_1 I)\hat{\mathbf{u}} = \mathbf{x},$$

and let

$$\mathbf{y}_2 = \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t} \quad \text{and} \quad \hat{\mathbf{y}}_2 = \hat{\mathbf{u}}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}.$$

Show that $\mathbf{y}_2 - \hat{\mathbf{y}}_2$ is a scalar multiple of $\mathbf{y}_1 = \mathbf{x}e^{\lambda_1 t}$.

34. Under the assumptions of Theorem 10.5.2, let

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{v}e^{\lambda_1 t} + \mathbf{u}te^{\lambda_1 t} + \mathbf{x}\frac{t^2 e^{\lambda_1 t}}{2}. \end{aligned}$$

Complete the proof of Theorem 10.5.2 by showing that $\mathbf{y}_3$ is a solution of $\mathbf{y}' = A\mathbf{y}$ and that $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is linearly independent.

35. Suppose the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a repeated eigenvalue $\lambda_1$ and the associated eigenspace is one-dimensional. Let $\mathbf{x}$ be a $\lambda_1$-eigenvector of $A$. Show that if $(A-\lambda_1 I)\mathbf{u}_1 = \mathbf{x}$ and $(A-\lambda_1 I)\mathbf{u}_2 = \mathbf{x}$, then $\mathbf{u}_2 - \mathbf{u}_1$ is parallel to $\mathbf{x}$. Conclude from this that all vectors $\mathbf{u}$ such that $(A-\lambda_1 I)\mathbf{u} = \mathbf{x}$ define the same positive and negative half-planes with respect to the line $L$ through the origin parallel to $\mathbf{x}$.

## Q10.5.5

In **Exercises 10.5.36-10.5.45** plot trajectories of the given system.

36. $\mathbf{y}' = \begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} \mathbf{y}$

37. $\mathbf{y}' = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{y}$

38. $\mathbf{y}' = \begin{bmatrix} -1 & -3 \\ 3 & 5 \end{bmatrix} \mathbf{y}$

39. $\mathbf{y}' = \begin{bmatrix} -5 & 3 \\ -3 & 1 \end{bmatrix} \mathbf{y}$

40. $\mathbf{y}' = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} \mathbf{y}$

41. $\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix} \mathbf{y}$

42. $\mathbf{y}' = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{y}$

43. $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}$

44. $\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y}$

45. $\mathbf{y}' = \begin{bmatrix} 0 & -4 \\ 1 & -4 \end{bmatrix} \mathbf{y}$

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# 10.6: Constant Coefficient Homogeneous Systems III

We now consider the system $\mathbf{y}' = A\mathbf{y}$, where $A$ has a complex eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$. We continue to assume that $A$ has real entries, so the characteristic polynomial of $A$ has real coefficients. This implies that $\overline{\lambda} = \alpha - i\beta$ is also an eigenvalue of $A$.

An eigenvector $\mathbf{x}$ of $A$ associated with $\lambda = \alpha + i\beta$ will have complex entries, so we'll write

$$\mathbf{x} = \mathbf{u} + i\mathbf{v}$$

where $\mathbf{u}$ and $\mathbf{v}$ have real entries; that is, $\mathbf{u}$ and $\mathbf{v}$ are the real and imaginary parts of $\mathbf{x}$. Since $A\mathbf{x} = \lambda\mathbf{x}$,

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha + i\beta)(\mathbf{u} + i\mathbf{v}). \tag{10.6.1}$$

Taking complex conjugates here and recalling that $A$ has real entries yields

$$A(\mathbf{u} - i\mathbf{v}) = (\alpha - i\beta)(\mathbf{u} - i\mathbf{v}),$$

which shows that $\mathbf{x} = \mathbf{u} - i\mathbf{v}$ is an eigenvector associated with $\overline{\lambda} = \alpha - i\beta$. The complex conjugate eigenvalues $\lambda$ and $\overline{\lambda}$ can be separately associated with linearly independent solutions $\mathbf{y}' = A\mathbf{y}$; however, we will not pursue this approach, since solutions obtained in this way turn out to be complex–valued. Instead, we'll obtain solutions of $\mathbf{y}' = A\mathbf{y}$ in the form

$$\mathbf{y} = f_1\mathbf{u} + f_2\mathbf{v} \tag{10.6.2}$$

where $f_1$ and $f_2$ are real–valued scalar functions. The next theorem shows how to do this.

## Theorem 10.6.1

Let $A$ be an $n \times n$ matrix with real entries. Let $\lambda = \alpha + i\beta$ $(\beta \neq 0)$ be a complex eigenvalue of $A$ and let $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ be an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. Then $\mathbf{u}$ and $\mathbf{v}$ are both nonzero and

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t),$$

which are the real and imaginary parts of

$$e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u} + i\mathbf{v}), \tag{10.6.3}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

**Proof**

A function of the form Equation 10.6.2is a solution of $\mathbf{y}' = A\mathbf{y}$ if and only if

$$f_1'\mathbf{u} + f_2'\mathbf{v} = f_1 A\mathbf{u} + f_2 A\mathbf{v}. \tag{10.6.4}$$

Carrying out the multiplication indicated on the right side of Equation 10.6.1and collecting the real and imaginary parts of the result yields

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha\mathbf{u} - \beta\mathbf{v}) + i(\alpha\mathbf{v} + \beta\mathbf{u}).$$

Equating real and imaginary parts on the two sides of this equation yields

$$\begin{aligned} A\mathbf{u} &= \alpha\mathbf{u} - \beta\mathbf{v} \\ A\mathbf{v} &= \alpha\mathbf{v} + \beta\mathbf{u}. \end{aligned}$$

We leave it to you (**Exercise 10.6.25**) to show from this that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero. Substituting from these equations into Equation 10.6.4yields

$$\begin{aligned} f_1'\mathbf{u} + f_2'\mathbf{v} &= f_1(\alpha\mathbf{u} - \beta\mathbf{v}) + f_2(\alpha\mathbf{v} + \beta\mathbf{u}) \\ &= (\alpha f_1 + \beta f_2)\mathbf{u} + (-\beta f_1 + \alpha f_2)\mathbf{v}. \end{aligned}$$

This is true if

$$\begin{aligned} f_1' &= \alpha f_1 + \beta f_2 \\ f_2' &= -\beta f_1 + \alpha f_2, \end{aligned} \quad \text{or equivalently} \quad \begin{aligned} f_1' - \alpha f_1 &= \beta f_2 \\ f_2' - \alpha f_2 &= -\beta f_1. \end{aligned}$$

If we let $f_1 = g_1 e^{\alpha t}$ and $f_2 = g_2 e^{\alpha t}$, where $g_1$ and $g_2$ are to be determined, then the last two equations become

$$\begin{aligned} g_1' &= \beta g_2 \\ g_2' &= -\beta g_1, \end{aligned}$$

which implies that

$$g_1'' = \beta g_2' = -\beta^2 g_1,$$

so

$$g_1'' + \beta^2 g_1 = 0.$$

The general solution of this equation is

$$g_1 = c_1 \cos \beta t + c_2 \sin \beta t.$$

Moreover, since $g_2 = g_1'/\beta$,

$$g_2 = -c_1 \sin \beta t + c_2 \cos \beta t.$$

Multiplying $g_1$ and $g_2$ by $e^{\alpha t}$ shows that

$$\begin{aligned} f_1 &= e^{\alpha t}(\ \ c_1 \cos \beta t + c_2 \sin \beta t), \\ f_2 &= e^{\alpha t}(-c_1 \sin \beta t + c_2 \cos \beta t). \end{aligned}$$

Substituting these into Equation 10.6.2shows that

$$\begin{aligned} \mathbf{y} &= e^{\alpha t}\left[(c_1 \cos \beta t + c_2 \sin \beta t)\mathbf{u} + (-c_1 \sin \beta t + c_2 \cos \beta t)\mathbf{v}\right] \\ &= c_1 e^{\alpha t}(\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t) + c_2 e^{\alpha t}(\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t) \end{aligned} \tag{10.6.5}$$

is a solution of $\mathbf{y}' = A\mathbf{y}$ for any choice of the constants $c_1$ and $c_2$. In particular, by first taking $c_1 = 1$ and $c_2 = 0$ and then taking $c_1 = 0$ and $c_2 = 1$, we see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{y}' = A\mathbf{y}$. We leave it to you to verify that they are, respectively, the real and imaginary parts of Equation 10.6.3(**Exercise 10.6.26**), and that they are linearly independent (**Exercise 10.6.27**).

## Example 10.6.1

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 4 & -5 \\ 5 & -2 \end{bmatrix} \mathbf{y}. \tag{10.6.6}$$

### Solution

The characteristic polynomial of the coefficient matrix $A$ in Equation 10.6.6is

$$\begin{vmatrix} 4-\lambda & -5 \\ 5 & -2-\lambda \end{vmatrix} = (\lambda - 1)^2 + 16.$$

Hence, $\lambda = 1 + 4i$ is an eigenvalue of $A$. The associated eigenvectors satisfy $(A - (1+4i)\,I)\,\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{cc:c} 3-4i & -5 & 0 \\ 5 & -3-4i & 0 \end{array}\right],$$

which is row equivalent to

$$\begin{bmatrix} 1 & -\frac{3+4i}{5} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = (3+4i)x_2/5$. Taking $x_2 = 5$ yields $x_1 = 3+4i$, so

$$\mathbf{x} = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 4t + i \sin 4t) \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t \begin{bmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t \begin{bmatrix} 3\sin 4t + 4\cos 4t \\ 5\sin 4t \end{bmatrix},$$

which are linearly independent solutions of Equation 10.6.6. The general solution of Equation 10.6.6 is

$$\mathbf{y} = c_1 e^t \begin{bmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{bmatrix} + c_2 e^t \begin{bmatrix} 3\sin 4t + 4\cos 4t \\ 5\sin 4t \end{bmatrix}.$$

## Example 10.6.2

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} \mathbf{y}. \tag{10.6.7}$$

### Solution

The characteristic polynomial of the coefficient matrix $A$ in Equation 10.6.7 is

$$\begin{vmatrix} -14-\lambda & 39 \\ -6 & 16-\lambda \end{vmatrix} = (\lambda-1)^2 + 9.$$

Hence, $\lambda = 1+3i$ is an eigenvalue of $A$. The associated eigenvectors satisfy $(A-(1+3i)I)\,\mathbf{x} = \mathbf{0}$. The augmented augmented matrix of this system is

$$\begin{bmatrix} -15-3i & 39 & \vdots & 0 \\ -6 & 15-3i & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & \frac{-5+i}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = (5-i)/2$. Taking $x_2 = 2$ yields $x_1 = 5-i$, so

$$\mathbf{x} = \begin{bmatrix} 5-i \\ 2 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 3t + i\sin 3t)\begin{bmatrix} 5-i \\ 2 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t\begin{bmatrix} \sin 3t + 5\cos 3t \\ 2\cos 3t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t\begin{bmatrix} -\cos 3t + 5\sin 3t \\ 2\sin 3t \end{bmatrix},$$

which are linearly independent solutions of Equation 10.6.7. The general solution of Equation 10.6.7 is

$$\mathbf{y} = c_1 e^t\begin{bmatrix} \sin 3t + 5\cos 3t \\ 2\cos 3t \end{bmatrix} + c_2 e^t\begin{bmatrix} -\cos 3t + 5\sin 3t \\ 2\sin 3t \end{bmatrix}.$$

## Example 10.6.3

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -5 & 5 & 4 \\ -8 & 7 & 6 \\ 1 & 0 & 0 \end{bmatrix}\mathbf{y}. \tag{10.6.8}$$

### Solution

The characteristic polynomial of the coefficient matrix $A$ in Equation 10.6.8 is

$$\begin{vmatrix} -5-\lambda & 5 & 4 \\ -8 & 7-\lambda & 6 \\ 1 & 0 & -\lambda \end{vmatrix} = -(\lambda-2)(\lambda^2+1).$$

Hence, the eigenvalues of $A$ are $\lambda_1 = 2$, $\lambda_2 = i$, and $\lambda_3 = -i$. The augmented matrix of $(A-2I)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{ccc:c} -7 & 5 & 4 & 0 \\ -8 & 5 & 6 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore $x_1 = x_2 = 2x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of Equation 10.6.8.

The augmented matrix of $(A-iI)\mathbf{x}=\mathbf{0}$ is

$$\left[\begin{array}{ccc|c} -5-i & 5 & 4 & 0 \\ -8 & 7-i & 6 & 0 \\ 1 & 0 & -i & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -i & 0 \\ 0 & 1 & 1-i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore $x_1=ix_3$ and $x_2=-(1-i)x_3$. Taking $x_3=1$ yields the eigenvector

$$\mathbf{x}_2=\begin{bmatrix} i \\ -1+i \\ 1 \end{bmatrix}.$$

The real and imaginary parts of

$$(\cos t+i\sin t)\begin{bmatrix} i \\ -1+i \\ 1 \end{bmatrix}$$

are

$$\mathbf{y}_2=\begin{bmatrix} -\sin t \\ -\cos t-\sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_3=\begin{bmatrix} \cos t \\ \cos t-\sin t \\ \sin t \end{bmatrix},$$

which are solutions of Equation 10.6.8. Since the Wronskian of $\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\}$ at $t=0$ is

$$\begin{vmatrix} 2 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix}=1,$$

$\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\}$ is a fundamental set of solutions of Equation 10.6.8. The general solution of Equation 10.6.8 is

$$\mathbf{y}=c_1\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}e^{2t}+c_2\begin{bmatrix} -\sin t \\ -\cos t-\sin t \\ \cos t \end{bmatrix}+c_3\begin{bmatrix} \cos t \\ \cos t-\sin t \\ \sin t \end{bmatrix}.$$

**Example 10.6.4**

Find the general solution of

$$\mathbf{y}'=\begin{bmatrix} 1 & -1 & -2 \\ 1 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}\mathbf{y}. \tag{10.6.9}$$

**Solution**

The characteristic polynomial of the coefficient matrix $A$ in Equation 10.6.9 is

$$\begin{vmatrix} 1-\lambda & -1 & -2 \\ 1 & 3-\lambda & 2 \\ 1 & -1 & 2-\lambda \end{vmatrix} = -(\lambda-2)\left((\lambda-2)^2+4\right).$$

Hence, the eigenvalues of $A$ are $\lambda_1 = 2$, $\lambda_2 = 2+2i$, and $\lambda_3 = 2-2i$. The augmented matrix of $(A-2I)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore $x_1 = x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of Equation 10.6.9. The augmented matrix of $(A-(2+2i)I)\,\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{ccc|c} -1-2i & -1 & -2 & 0 \\ 1 & 1-2i & 2 & 0 \\ 1 & -1 & -2i & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -i & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore $x_1 = ix_3$ and $x_2 = -ix_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

The real and imaginary parts of

$$e^{2t}(\cos 2t + i\sin 2t)\begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

are

$$\mathbf{y}_2 = e^{2t}\begin{bmatrix} -\sin 2t \\ \sin 2t \\ \cos 2t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^{2t}\begin{bmatrix} \cos 2t \\ -\cos 2t \\ \sin 2t \end{bmatrix},$$

which are solutions of Equation 10.6.9. Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} -1 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = -2,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of Equation 10.6.9. The general solution of Equation 10.6.9 is

$$\mathbf{y} = c_1\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}e^{2t} + c_2 e^{2t}\begin{bmatrix} -\sin 2t \\ \sin 2t \\ \cos 2t \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} \cos 2t \\ -\cos 2t \\ \sin 2t \end{bmatrix}.$$

## Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a $2 \times 2$ constant coefficient system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{10.6.10}$$

under the assumptions of this section; that is, when the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a complex eigenvalue $\lambda = \alpha + i\beta$ $(\beta \neq 0)$ and $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ is an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. To describe the trajectories accurately it is necessary to introduce a new rectangular coordinate system in the $y_1$-$y_2$ plane. This raises a point that hasn't come up before: It is always possible to choose $\mathbf{x}$ so that $(\mathbf{u}, \mathbf{v}) = 0$. A special effort is required to do this, since not every eigenvector has this property. However, if we know an eigenvector that doesn't, we can multiply it by a suitable complex constant to obtain one that does. To see this, note that if $\mathbf{x}$ is a $\lambda$-eigenvector of $A$ and $k$ is an arbitrary real number, then

$$\mathbf{x}_1 = (1 + ik)\mathbf{x} = (1 + ik)(\mathbf{u} + i\mathbf{v}) = (\mathbf{u} - k\mathbf{v}) + i(\mathbf{v} + k\mathbf{u})$$

is also a $\lambda$-eigenvector of $A$, since

$$A\mathbf{x}_1 = A((1 + ik)\mathbf{x}) = (1 + ik)A\mathbf{x} = (1 + ik)\lambda\mathbf{x} = \lambda((1 + ik)\mathbf{x}) = \lambda\mathbf{x}_1.$$

The real and imaginary parts of $\mathbf{x}_1$ are

$$\mathbf{u}_1 = \mathbf{u} - k\mathbf{v} \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} + k\mathbf{u}, \tag{10.6.11}$$

so

$$(\mathbf{u}_1, \mathbf{v}_1) = (\mathbf{u} - k\mathbf{v}, \mathbf{v} + k\mathbf{u}) = -\left[(\mathbf{u}, \mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u}, \mathbf{v})\right].$$

Therefore $(\mathbf{u}_1, \mathbf{v}_1) = 0$ if

$$(\mathbf{u}, \mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u}, \mathbf{v}) = 0. \tag{10.6.12}$$

If $(\mathbf{u}, \mathbf{v}) \neq 0$ we can use the quadratic formula to find two real values of $k$ such that $(\mathbf{u}_1, \mathbf{v}_1) = 0$ **(Exercise 10.6.28)**.

**Example 10.6.5**

In Example 10.6.1 we found the eigenvector

$$\mathbf{x}=\begin{bmatrix}3+4i\\5\end{bmatrix}=\begin{bmatrix}3\\5\end{bmatrix}+i\begin{bmatrix}4\\0\end{bmatrix}$$

for the matrix of the system Equation 10.6.6. Here $\mathbf{u}=\begin{bmatrix}3\\5\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}4\\0\end{bmatrix}$ are not orthogonal, since $(\mathbf{u},\mathbf{v})=12$. Since $\|\mathbf{v}\|^2-\|\mathbf{u}\|^2=-18$, Equation 10.6.12is equivalent to

$$2k^2-3k-2=0.$$

The zeros of this equation are $k_1=2$ and $k_2=-1/2$. Letting $k=2$ in Equation 10.6.11yields

$$\mathbf{u}_1=\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-5\\5\end{bmatrix}\quad\text{and}\quad\mathbf{v}_1=\mathbf{v}+2\mathbf{u}=\begin{bmatrix}10\\10\end{bmatrix},$$

and $(\mathbf{u}_1,\mathbf{v}_1)=0$. Letting $k=-1/2$ in Equation 10.6.11yields

$$\mathbf{u}_1=\mathbf{u}+\frac{\mathbf{v}}{2}=\begin{bmatrix}5\\5\end{bmatrix}\quad\text{and}\quad\mathbf{v}_1=\mathbf{v}-\frac{\mathbf{u}}{2}=\frac{1}{2}\begin{bmatrix}-5\\5\end{bmatrix},$$

and again $(\mathbf{u}_1,\mathbf{v}_1)=0$.

(The numbers don't always work out as nicely as in this example. You'll need a calculator or computer to do **Exercises 10.6.29-10.6.40**.) Henceforth, we'll assume that $(\mathbf{u},\mathbf{v})=0$. Let $\mathbf{U}$ and $\mathbf{V}$ be unit vectors in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively; that is, $\mathbf{U}=\mathbf{u}/\|\mathbf{u}\|$ and $\mathbf{V}=\mathbf{v}/\|\mathbf{v}\|$. The new rectangular coordinate system will have the same origin as the $y_1$-$y_2$ system. The coordinates of a point in this system will be denoted by $(z_1,z_2)$, where $z_1$ and $z_2$ are the displacements in the directions of $\mathbf{U}$ and $\mathbf{V}$, respectively. From Equation 10.6.5, the solutions of Equation 10.6.10are given by

$$\mathbf{y}=e^{\alpha t}\left[(c_1\cos\beta t+c_2\sin\beta t)\mathbf{u}+(-c_1\sin\beta t+c_2\cos\beta t)\mathbf{v}\right]. \tag{10.6.13}$$

For convenience, let's call the curve traversed by $e^{-\alpha t}\mathbf{y}(t)$ a shadow trajectory of Equation 10.6.10. Multiplying Equation 10.6.13 by $e^{-\alpha t}$ yields

$$e^{-\alpha t}\mathbf{y}(t)=z_1(t)\mathbf{U}+z_2(t)\mathbf{V},$$

where

$$\begin{aligned}z_1(t)&=\|\mathbf{u}\|(c_1\cos\beta t+c_2\sin\beta t)\\z_2(t)&=\|\mathbf{v}\|(-c_1\sin\beta t+c_2\cos\beta t).\end{aligned}$$

Therefore

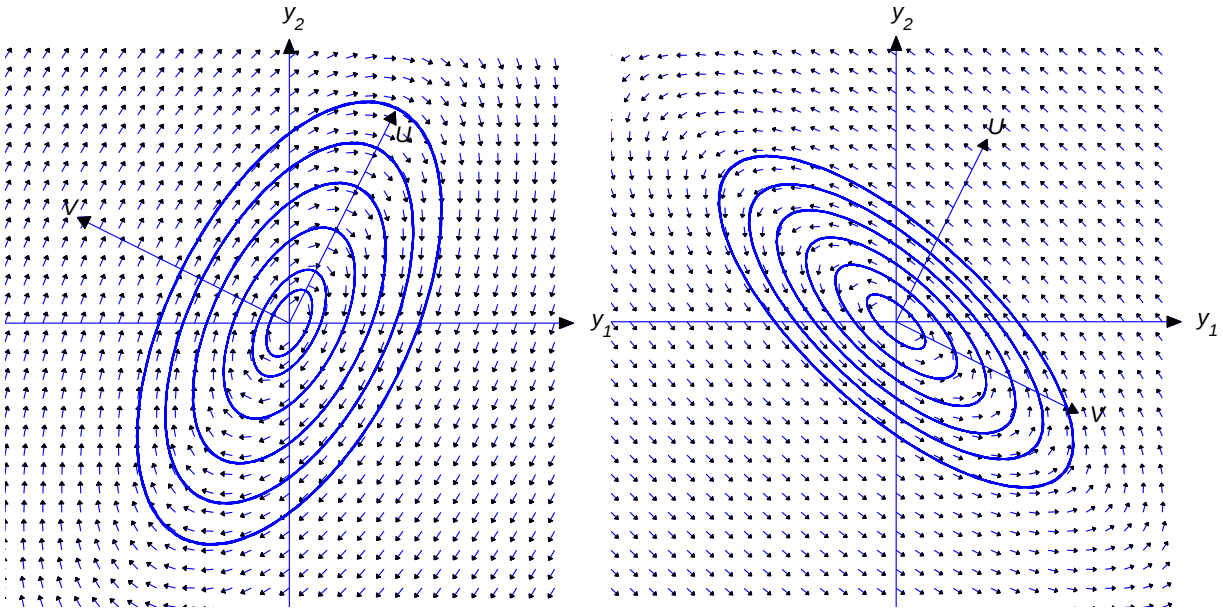
$$\frac{(z_1(t))^2}{\|\mathbf{u}\|^2}+\frac{(z_2(t))^2}{\|\mathbf{v}\|^2}=c_1^2+c_2^2. \tag{verify!}$$

which means that the shadow trajectories of Equation 10.6.10are ellipses centered at the origin, with axes of symmetry parallel to $\mathbf{U}$ and $\mathbf{V}$. Since

$$z_1'=\frac{\beta\|\mathbf{u}\|}{\|\mathbf{v}\|}z_2\quad\text{and}\quad z_2'=-\frac{\beta\|\mathbf{v}\|}{\|\mathbf{u}\|}z_1,$$

the vector from the origin to a point on the shadow ellipse rotates in the same direction that $\mathbf{V}$ would have to be rotated by $\pi/2$ radians to bring it into coincidence with $\mathbf{U}$ (Figures 10.6.1 and 10.6.2 ).


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Figures 10.6.1 (left) and 10.6.2 (right): Shadow trajectories traversed clockwise.Figure 10.6.2 : Shadow trajectories traversed counterclockwise

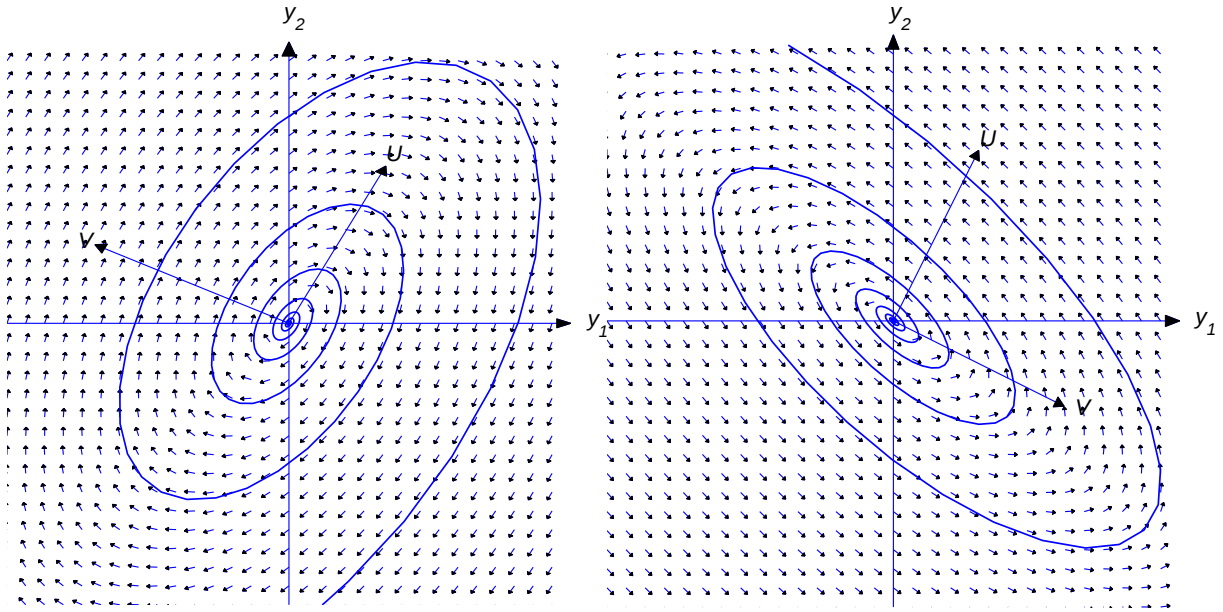
If $\alpha = 0$, then any trajectory of Equation 10.6.10 is a shadow trajectory of Equation 10.6.10; therefore, if $\lambda$ is purely imaginary, then the trajectories of Equation 10.6.10 are ellipses traversed periodically as indicated in Figures 10.6.1 and 10.6.2 . If $\alpha > 0$, then

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \mathbf{y}(t) = 0,$$

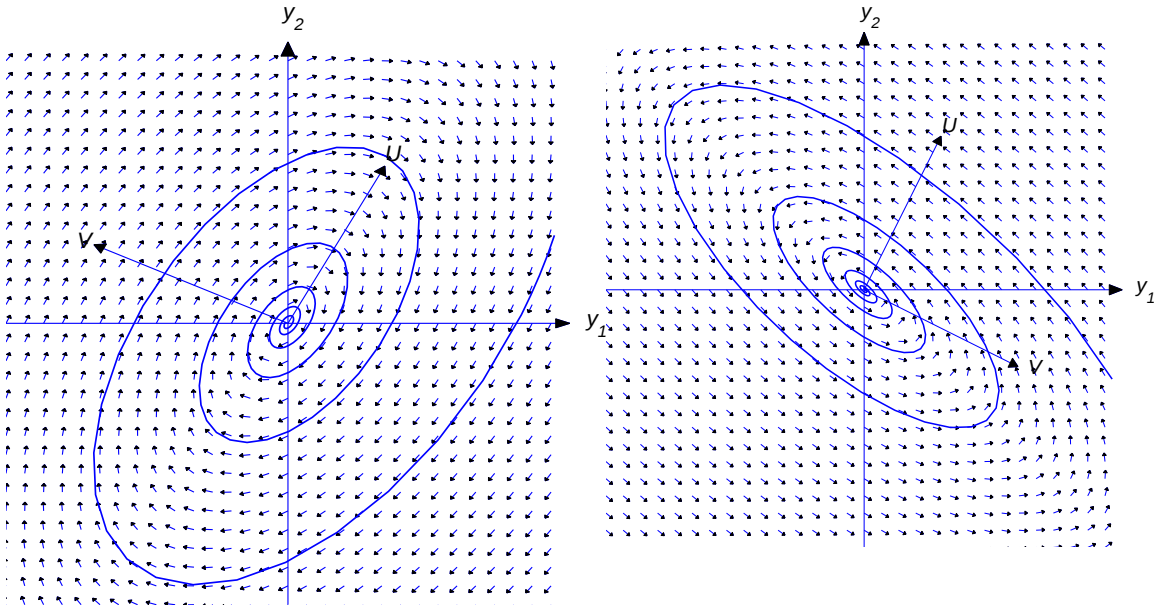
so the trajectory spirals away from the origin as $t$ varies from $-\infty$ to $\infty$. The direction of the spiral depends upon the relative orientation of $\mathbf{U}$ and $\mathbf{V}$, as shown in Figures 10.6.3 and 10.6.4 . If $\alpha < 0$, then

$$\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \mathbf{y}(t) = 0,$$

so the trajectory spirals toward the origin as $t$ varies from $-\infty$ to $\infty$. Again, the direction of the spiral depends upon the relative orientation of $\mathbf{U}$ and $\mathbf{V}$, as shown in Figures 10.6.5 and 10.6.6 .




Figures 10.6.3 (left) and 10.6.4 (right): (left) $\alpha > 0$; shadow trajectory spiraling outward and (right) $\alpha > 0$; shadow trajectory spiraling outward.




Figures 10.6.5 (left) and 10.6.6 (right): (left) $\alpha < 0$; shadow trajectory spiraling inward. (right) $\alpha < 0$; shadow trajectory spiraling inward.

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# 10.6E: Constant Coefficient Homogeneous Systems III (Exercises)

## Q10.6.1

In **Exercises 10.6.1-10.6.16** find the general solution.

1. $\mathbf{y}' = \begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} \mathbf{y}$

2. $\mathbf{y}' = \begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} \mathbf{y}$

3. $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \mathbf{y}$

4. $\mathbf{y}' = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} \mathbf{y}$

5. $\mathbf{y}' = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} \mathbf{y}$

6. $\mathbf{y}' = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} \mathbf{y}$

7. $\mathbf{y}' = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{y}$

8. $\mathbf{y}' = \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} \mathbf{y}$

9. $\mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 10 & 1 \end{bmatrix} \mathbf{y}$

10. $\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 7 & -5 \\ 2 & 5 \end{bmatrix} \mathbf{y}$

11. $\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \mathbf{y}$

12. $\mathbf{y}' = \begin{bmatrix} 34 & 52 \\ -20 & -30 \end{bmatrix} \mathbf{y}$

13. $\mathbf{y}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -2 & -1 \end{bmatrix} \mathbf{y}$

14. $\mathbf{y}' = \begin{bmatrix} 3 & -4 & -2 \\ -5 & 7 & -8 \\ -10 & 13 & -8 \end{bmatrix} \mathbf{y}$

15. $\mathbf{y}' = \begin{bmatrix} 6 & 0 & -3 \\ -3 & 3 & 3 \\ 1 & -2 & 6 \end{bmatrix} \mathbf{y}$

16. $\mathbf{y}' = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{y}$

## Q10.6.2

In **Exercises 10.6.17-10.6.24** solve the initial value problem.

17. $\mathbf{y}' = \begin{bmatrix} 4 & -6 \\ 3 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

18. $\mathbf{y}' = \begin{bmatrix} 7 & 15 \\ -3 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

19. $\mathbf{y}' = \begin{bmatrix} 7 & -15 \\ 3 & -5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 17 \\ 7 \end{bmatrix}$

20. $\mathbf{y}' = \frac{1}{6}\begin{bmatrix} 4 & -2 \\ 5 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

21. $\mathbf{y}' = \begin{bmatrix} 5 & 2 & -1 \\ -3 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$

22. $\mathbf{y}' = \begin{bmatrix} 4 & 4 & 0 \\ 8 & 10 & -20 \\ 2 & 3 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$

23. $\mathbf{y}' = \begin{bmatrix} 1 & 15 & -15 \\ -6 & 18 & -22 \\ -3 & 11 & -15 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 15 \\ 17 \\ 10 \end{bmatrix}$

24. $\mathbf{y}' = \begin{bmatrix} 4 & -4 & 4 \\ -10 & 3 & 15 \\ 2 & -3 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 16 \\ 14 \\ 6 \end{bmatrix}$

## Q10.6.3

25. Suppose an $n \times n$ matrix $A$ with real entries has a complex eigenvalue $\lambda = \alpha + i\beta \ \ (\beta \neq 0)$ with associated eigenvector $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ , where $\mathbf{u}$ and $\mathbf{v}$ have real components. Show that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero.

26. Verify that

$$\mathbf{y_1} = \mathbf{e}^{\alpha \mathbf{t}}(\mathbf{u}\cos\beta\mathbf{t} - \mathbf{v}\sin\beta\mathbf{t}) \quad \textbf{and} \quad \mathbf{y_2} = \mathbf{e}^{\alpha \mathbf{t}}(\mathbf{u}\sin\beta\mathbf{t} + \mathbf{v}\cos\beta\mathbf{t}),$$

are the real and imaginary parts of

$$e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u} + i\mathbf{v}).$$

27. Show that if the vectors $\mathbf{u}$ and $\mathbf{v}$ are not both $\mathbf{0}$ and $\beta \neq 0$ then the vector functions

$$\mathbf{y_1} = \mathbf{e}^{\alpha \mathbf{t}}(\mathbf{u}\cos\beta\mathbf{t} - \mathbf{v}\sin\beta\mathbf{t}) \quad \textbf{and} \quad \mathbf{y_2} = \mathbf{e}^{\alpha \mathbf{t}}(\mathbf{u}\sin\beta\mathbf{t} + \mathbf{v}\cos\beta\mathbf{t})$$

are linearly independent on every interval.

28. Suppose $\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are not orthogonal; that is, $(\mathbf{u},\mathbf{v})\neq 0$.

1. Show that the quadratic equation
$$(\mathbf{u},\mathbf{v})k^2+(\|\mathbf{v}\|^2-\|\mathbf{u}\|^2)k-(\mathbf{u},\mathbf{v})=0$$
has a positive root $k_1$ and a negative root $k_2=-1/k_1$.
2. Let $\mathbf{u}_1^{(1)}=\mathbf{u}-k_1\mathbf{v}$, $\mathbf{v}_1^{(1)}=\mathbf{v}+k_1\mathbf{u}$, $\mathbf{u}_1^{(2)}=\mathbf{u}-k_2\mathbf{v}$, and $\mathbf{v}_1^{(2)}=\mathbf{v}+k_2\mathbf{u}$, so that $(\mathbf{u}_1^{(1)},\mathbf{v}_1^{(1)})=(\mathbf{u}_1^{(2)},\mathbf{v}_1^{(2)})=0$, from the discussion given above. Show that
$$\mathbf{u}_1^{(2)}=\frac{\mathbf{v}_1^{(1)}}{k_1} \quad \text{and} \quad \mathbf{v}_1^{(2)}=-\frac{\mathbf{u}_1^{(1)}}{k_1}.$$
3. Let $\mathbf{U}_1$, $\mathbf{V}_1$, $\mathbf{U}_2$, and $\mathbf{V}_2$ be unit vectors in the directions of $\mathbf{u}_1^{(1)}$, $\mathbf{v}_1^{(1)}$, $\mathbf{u}_1^{(2)}$, and $\mathbf{v}_1^{(2)}$, respectively. Conclude from (a) that $\mathbf{U}_2=\mathbf{V}_1$ and $\mathbf{V}_2=-\mathbf{U}_1$, and that therefore the counterclockwise angles from $\mathbf{U}_1$ to $\mathbf{V}_1$ and from $\mathbf{U}_2$ to $\mathbf{V}_2$ are both $\pi/2$ or both $-\pi/2$.

## Q10.6.4

In **Exercises 10.6.29-10.6.32** find vectors $\mathbf{U}$ and $\mathbf{V}$ parallel to the axes of symmetry of the trajectories, and plot some typical trajectories.

29. $\mathbf{y}'=\begin{bmatrix} 3 & -5 \\ 5 & -3 \end{bmatrix}\mathbf{y}$

30. $\mathbf{y}'=\begin{bmatrix} -15 & 10 \\ -25 & 15 \end{bmatrix}\mathbf{y}$

31. $\mathbf{y}'=\begin{bmatrix} -4 & 8 \\ -4 & 4 \end{bmatrix}\mathbf{y}$

32. $\mathbf{y}'=\begin{bmatrix} -3 & -15 \\ 3 & 3 \end{bmatrix}\mathbf{y}$

## Q10.6.5

In **Exercises 10.6.33-10.6.40** find vectors $\mathbf{U}$ and $\mathbf{V}$ parallel to the axes of symmetry of the shadow trajectories, and plot a typical trajectory.

33. $\mathbf{y}'=\begin{bmatrix} -5 & 6 \\ -12 & 7 \end{bmatrix}\mathbf{y}$

34. $\mathbf{y}'=\begin{bmatrix} 5 & -12 \\ 6 & -7 \end{bmatrix}\mathbf{y}$

35. $\mathbf{y}'=\begin{bmatrix} 4 & -5 \\ 9 & -2 \end{bmatrix}\mathbf{y}$

36. $\mathbf{y}'=\begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix}\mathbf{y}$

37. $\mathbf{y}'=\begin{bmatrix} -1 & 10 \\ -10 & -1 \end{bmatrix}\mathbf{y}$

38. $\mathbf{y}' = \begin{bmatrix} -1 & -5 \\ 20 & -1 \end{bmatrix} \mathbf{y}$

39. $\mathbf{y}' = \begin{bmatrix} -7 & 10 \\ -10 & 9 \end{bmatrix} \mathbf{y}$

40. $\mathbf{y}' = \begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix} \mathbf{y}$

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# CHAPTER OVERVIEW

## 11: Systems of Equations


11.1: Geometry
11.2: Elementary Operations
11.3: Gaussian Elimination
11.E: Exercises


*Thumbnail: A linear system in three variables determines a collection of planes. The intersection point is the solution. (CC BY-SA 4.0; Fred the Oyster via Wikipedia)*

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# 11.1: Geometry

## Systems of Equations, Geometry

**Outcomes**

A. Relate the types of solution sets of a system of two (three) variables to the intersections of lines in a plane (the intersection of planes in three space)

As you may remember, linear equations like $2x+3y=6$ can be graphed as straight lines in the coordinate plane. We say that this equation is in two variables, in this case $x$ and $y$. Suppose you have two such equations, each of which can be graphed as a straight line, and consider the resulting graph of two lines. What would it mean if there exists a point of intersection between the two lines? This point, which lies on *both* graphs, gives $x$ and $y$ values for which both equations are true. In other words, this point gives the ordered pair $(x,y)$ that satisfy both equations. If the point $(x,y)$ is a point of intersection, we say that $(x,y)$ is a **solution** to the two equations. In linear algebra, we often are concerned with finding the solution(s) to a system of equations, if such solutions exist. First, we consider graphical representations of solutions and later we will consider the algebraic methods for finding solutions.

When looking for the intersection of two lines in a graph, several situations may arise. The following picture demonstrates the possible situations when considering two equations (two lines in the graph) involving two variables.

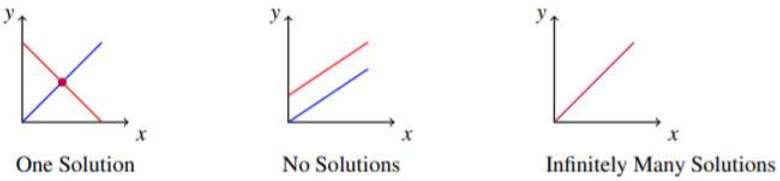



Figure 11.1.1

In the first diagram, there is a unique point of intersection, which means that there is only one (unique) solution to the two equations. In the second, there are no points of intersection and no solution. When no solution exists, this means that the two lines are parallel and they never intersect. The third situation which can occur, as demonstrated in diagram three, is that the two lines are really the same line. For example, $x+y=1$ and $2x+2y=2$ are equations which when graphed yield the same line. In this case there are infinitely many points which are solutions of these two equations, as every ordered pair which is on the graph of the line satisfies both equations. When considering linear systems of equations, there are always three types of solutions possible; exactly one (unique) solution, infinitely many solutions, or no solution.

**Example 11.1.1: A Graphical Solution**

Use a graph to find the solution to the following system of equations

$$x+y=3$$
$$y-x=5$$

**Solution**

Through graphing the above equations and identifying the point of intersection, we can find the solution(s). Remember that we must have either one solution, infinitely many, or no solutions at all. The following graph shows the two equations, as well as the intersection. Remember, the point of intersection represents the solution of the two equations, or the $(x,y)$ which satisfy both equations. In this case, there is one point of intersection at $(-1,4)$ which means we have one unique solution, $x=-1, y=4$.

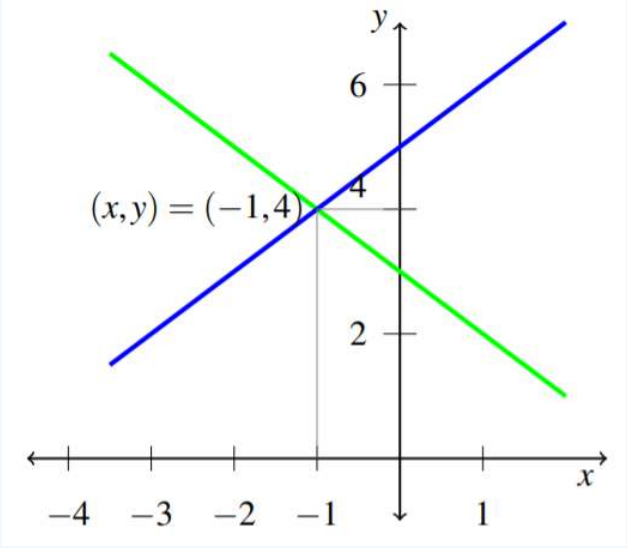



Figure 11.1.2

In the above example, we investigated the intersection point of two equations in two variables, $x$ and $y$. Now we will consider the graphical solutions of three equations in two variables.

Consider a system of three equations in two variables. Again, these equations can be graphed as straight lines in the plane, so that the resulting graph contains three straight lines. Recall the three possible types of solutions; no solution, one solution, and infinitely many solutions. There are now more complex ways of achieving these situations, due to the presence of the third line. For example, you can imagine the case of three intersecting lines having no common point of intersection. Perhaps you can also imagine three intersecting lines which do intersect at a single point. These two situations are illustrated below.

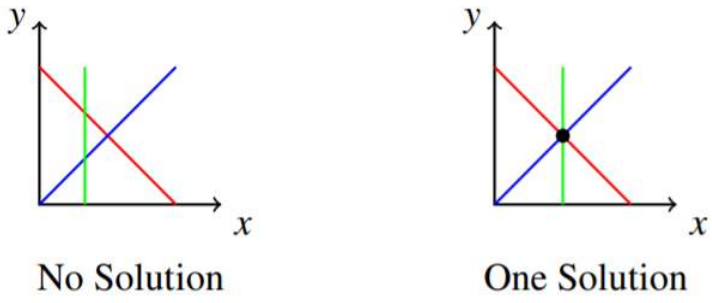



Figure 11.1.3

Consider the first picture above. While all three lines intersect with one another, there is no common point of intersection where all three lines meet at one point. Hence, there is no solution to the three equations. Remember, a solution is a point $(x, y)$ which satisfies **all** three equations. In the case of the second picture, the lines intersect at a common point. This means that there is one solution to the three equations whose graphs are the given lines. You should take a moment now to draw the graph of a system which results in three parallel lines. Next, try the graph of three identical lines. Which type of solution is represented in each of these graphs?

We have now considered the graphical solutions of systems of two equations in two variables, as well as three equations in two variables. However, there is no reason to limit our investigation to equations in two variables. We will now consider equations in three variables.

You may recall that equations in three variables, such as $2x + 4y - 5z = 8$, form a plane. Above, we were looking for intersections of lines in order to identify any possible solutions. When graphically solving systems of equations in three variables, we look for intersections of planes. These points of intersection give the $(x, y, z)$ that satisfy all the equations in the system. What types of solutions are possible when working with three variables? Consider the following picture involving two planes, which are given by two equations in three variables.

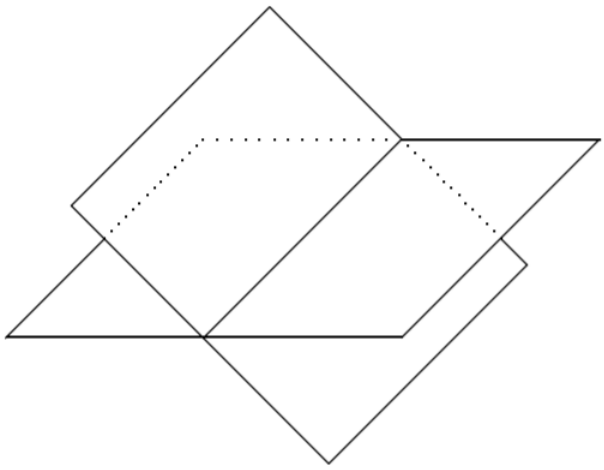


Figure 11.1.4

Notice how these two planes intersect in a line. This means that the points $(x, y, z)$ on this line satisfy both equations in the system. Since the line contains infinitely many points, this system has infinitely many solutions.

It could also happen that the two planes fail to intersect. However, is it possible to have two planes intersect at a single point? Take a moment to attempt drawing this situation, and convince yourself that it is not possible! This means that when we have only two equations in three variables, there is no way to have a unique solution! Hence, the types of solutions possible for two equations in three variables are no solution or infinitely many solutions.

Now imagine adding a third plane. In other words, consider three equations in three variables. What types of solutions are now possible? Consider the following diagram.

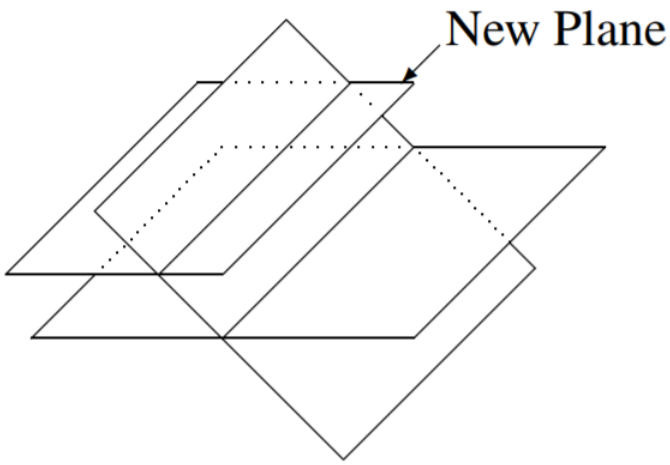



Figure 11.1.5

In this diagram, there is no point which lies in all three planes. There is no intersection between **all** planes so there is no solution. The picture illustrates the situation in which the line of intersection of the new plane with one of the original planes forms a line parallel to the line of intersection of the first two planes. However, in three dimensions, it is possible for two lines to fail to intersect even though they are not parallel. Such lines are called **skew lines**.

Recall that when working with two equations in three variables, it was not possible to have a unique solution. Is it possible when considering three equations in three variables? In fact, it is possible, and we demonstrate this situation in the following picture.

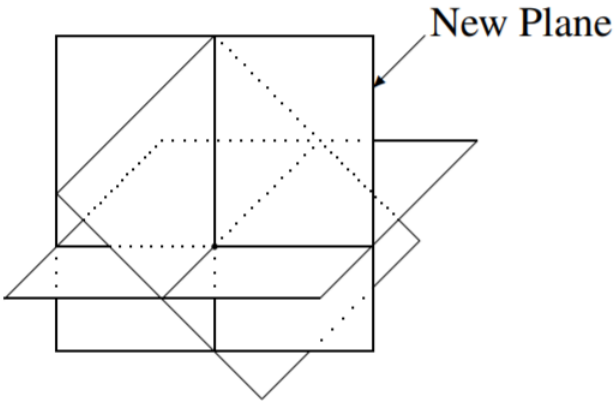



Figure 11.1.6

In this case, the three planes have a single point of intersection. Can you think of other types of solutions possible? Another is that the three planes could intersect in a line, resulting in infinitely many solutions, as in the following diagram.

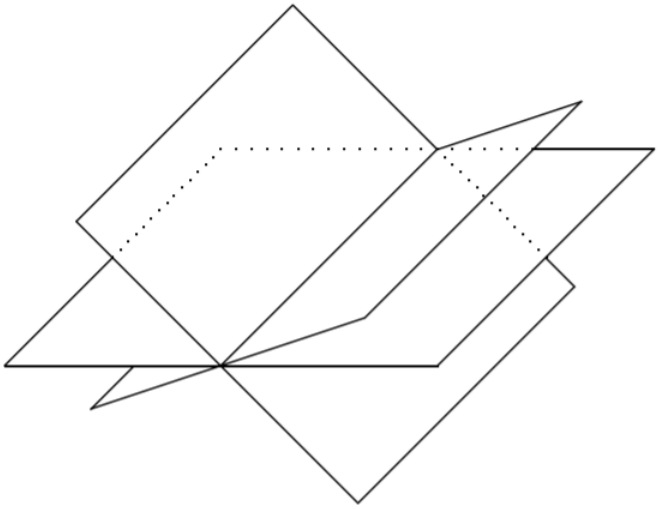


Figure 11.1.7

We have now seen how three equations in three variables can have no solution, a unique solution, or intersect in a line resulting in infinitely many solutions. It is also possible that the three equations graph the same plane, which also leads to infinitely many solutions.

You can see that when working with equations in three variables, there are many more ways to achieve the different types of solutions than when working with two variables. It may prove enlightening to spend time imagining (and drawing) many possible scenarios, and you should take some time to try a few.

You should also take some time to imagine (and draw) graphs of systems in more than three variables. Equations like $x+y-2z+4w=8$ with more than three variables are often called **hyper-planes**. You may soon realize that it is tricky to draw the graphs of hyper-planes! Through the tools of linear algebra, we can algebraically examine these types of systems which are difficult to graph. In the following section, we will consider these algebraic tools.

**? Exercise 11.1.1**

Graphically, find the point $(x_1, y_1)$ which lies on both lines, $x+3y=1$ and $4x-y=3$. That is, graph each line and see where they intersect.

**Exercise 11.1.2**

Graphically, find the point of intersection of the two lines, $3x + y = 3$ and $x + 2y = 1$. That is, graph each line and see where they intersect

**Exercise 11.1.3**

You have a system of $k$ equations in two variables, $k \geq 2$. Explain the geometric significance of

a. No solution.
b. A unique solution.
c. An infinite number of solutions.

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# 11.2: Elementary Operations

**Learning Objectives**

1. Use elementary operations to find the solution to a linear system of equations.
2. Find the and of a matrix.
3. Determine whether a system of linear equations has no solution, a unique solution or an infinite number of solutions from its .
4. Solve a system of equations using Gaussian Elimination and Gauss-Jordan Elimination.
5. Model a physical system with linear equations and then solve.

We have taken an in depth look at graphical representations of systems of equations, as well as how to find possible solutions graphically. Our attention now turns to working with systems algebraically.

**Definition 11.2.1 : System of Linear Equations**

A **system of linear equations** is a list of equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{11.2.1}$$

where $a_{ij}$ and $b_j$ are real numbers. The above is a system of $m$ equations in the $n$ variables, $x_1, x_2 \cdots, x_n$. Written more simply in terms of summation notation, the above can be written in the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i,\ i = 1, 2, 3, \cdots, m \tag{11.2.2}$$

The relative size of $m$ and $n$ is not important here. Notice that we have allowed $a_{ij}$ and $b_j$ to be any real number. We can also call these numbers **scalars** . We will use this term throughout the text, so keep in mind that the term **scalar** just means that we are working with real numbers.

Now, suppose we have a system where $b_i = 0$ for all $i$. In other words every equation equals $0$. This is a special type of system.

**Definition 11.2.2: Homogeneous System of Equations**

A system of equations is called **homogeneous** if each equation in the system is equal to $0$. A homogeneous system has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \tag{11.2.3}$$

where $a_{ij}$ are scalars and $x_i$ are variables.

Recall from the previous section that our goal when working with systems of linear equations was to find the point of intersection of the equations when graphed. In other words, we looked for the solutions to the system. We now wish to find these solutions algebraically. We want to find values for $x_1, \cdots, x_n$ which solve all of the equations. If such a set of values exists, we call $(x_1, \cdots, x_n)$ the **solution set**.

Recall the above discussions about the types of solutions possible. We will see that systems of linear equations will have one unique solution, infinitely many solutions, or no solution. Consider the following definition.

**Definition 11.2.3: Consistent and Inconsistent Systems**

A **system of linear equations** is a list of equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{11.2.4}$$

where $a_{ij}$ and $b_j$ are real numbers. The above is a system of $m$ equations in the $n$ variables, $x_1, x_2 \cdots, x_n$. Written more simply in terms of summation notation, the above can be written in the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ i = 1, 2, 3, \cdots, m \tag{11.2.5}$$

A system of linear equations is called **consistent** if there exists at least one solution. It is called **inconsistent** if there is no solution.

If you think of each equation as a condition which must be satisfied by the variables, consistent would mean there is some choice of variables which can satisfy **all** the conditions. Inconsistent would mean there is no choice of the variables which can satisfy all of the conditions.

The following sections provide methods for determining if a system is consistent or inconsistent, and finding solutions if they exist.

## Elementary Operations

We begin this section with an example.

**Example 11.2.1: Verifying an Ordered Pair is a Solution**

Algebraically verify that $(x, y) = (-1, 4)$ is a solution to the following system of equations.

$$\begin{aligned} x + y &= 3 \\ y - x &= 5 \end{aligned} \tag{11.2.6}$$

**Solution**

By graphing these two equations and identifying the point of intersection, we previously found that $(x, y) = (-1, 4)$ is the unique solution.

We can verify algebraically by substituting these values into the original equations, and ensuring that the equations hold. First, we substitute the values into the first equation and check that it equals 3.

$$x + y = (-1) + (4) = 3 \tag{11.2.7}$$

This equals 3 as needed, so we see that $(-1, 4)$ is a solution to the first equation. Substituting the values into the second equation yields

$$y - x = (4) - (-1) = 4 + 1 = 5 \tag{11.2.8}$$

which is true. For $(x, y) = (-1, 4)$ each equation is true and therefore, this is a solution to the system.

Now, the interesting question is this: If you were not given these numbers to verify, how could you algebraically determine the solution? Linear algebra gives us the tools needed to answer this question. The following basic operations are important tools that we will utilize.

**Definition 11.2.4: Elementary Operations**

**Elementary operations** are those operations consisting of the following.

1. Interchange the order in which the equations are listed.
2. Multiply any equation by a nonzero number.

3. Replace any equation with itself added to a multiple of another equation.

It is important to note that none of these operations will change the set of solutions of the system of equations. In fact, elementary operations are the *key tool* we use in linear algebra to find solutions to systems of equations.

Consider the following example.

**Example 11.2.2: Effects of an Elementary Operation**

Show that the system

$$\begin{aligned} x+y&=7\\ 2x-y&=8 \end{aligned} \tag{11.2.9}$$

has the same solution as the system

$$\begin{aligned} x+y&=7\\ -3y&=-6 \end{aligned} \tag{11.2.10}$$

**Solution**

Notice that the second system has been obtained by taking the second equation of the first system and adding -2 times the first equation, as follows:

$$2x-y+(-2)(x+y)=8+(-2)(7) \tag{11.2.11}$$

By simplifying, we obtain

$$-3y=-6 \tag{11.2.12}$$

which is the second equation in the second system. Now, from here we can solve for $y$ and see that $y=2$. Next, we substitute this value into the first equation as follows

$$x+y=x+2=7 \tag{11.2.13}$$

Hence $x=5$ and so $(x,y)=(5,2)$ is a solution to the second system. We want to check if $(5,2)$ is also a solution to the first system. We check this by substituting $(x,y)=(5,2)$ into the system and ensuring the equations are true.

$$\begin{aligned} x+y&=(5)+(2)=7\\ 2x-y&=2\,(5)-(2)=8 \end{aligned} \tag{11.2.14}$$

Hence, $(5,2)$ is also a solution to the first system.

This example illustrates how an elementary operation applied to a system of two equations in two variables does not affect the solution set. However, a linear system may involve many equations and many variables and there is no reason to limit our study to small systems. For any size of system in any number of variables, the solution set is still the collection of solutions to the equations. In every case, the above operations of Definition 11.2.4do not change the set of solutions to the system of linear equations.

In the following theorem, we use the notation $E_i$ to represent an equation, while $b_i$ denotes a constant.

**Theorem 11.2.1: Elementary Operations and Solutions**

Suppose you have a system of two linear equations

$$\begin{aligned} E_1&=b_1\\ E_2&=b_2 \end{aligned} \tag{11.2.15}$$

Then the following systems have the same solution set as 11.2.15:

1. $$\begin{aligned} E_2&=b_2\\ E_1&=b_1 \end{aligned} \tag{11.2.16}$$

2.

$$\begin{aligned} E_1 &= b_1 \\ kE_2 &= kb_2 \end{aligned} \tag{11.2.17}$$

for any scalar $k$, provided $k \neq 0$.

3. $$\begin{aligned} E_1 &= b_1 \\ E_2 + kE_1 &= b_2 + kb_1 \end{aligned} \tag{11.2.18}$$

for any scalar $k$ (including $k = 0$).

Before we proceed with the proof of Theorem 11.2.1, let us consider this theorem in context of Example 11.2.2. Then,

$$\begin{aligned} E_1 &= x + y, \quad b_1 = 7 \\ E_2 &= 2x - y, \quad b_2 = 8 \end{aligned} \tag{11.2.19}$$

Recall the elementary operations that we used to modify the system in the solution to the example. First, we added $(-2)$ times the first equation to the second equation. In terms of Theorem Theorem 11.2.1, this action is given by

$$E_2 + (-2)\, E_1 = b_2 + (-2)\, b_1 \tag{11.2.20}$$

or

$$2x - y + (-2)\,(x + y) = 8 + (-2)\, 7 \tag{11.2.21}$$

This gave us the second system in Example 11.2.2, given by

$$\begin{aligned} E_1 &= b_1 \\ E_2 + (-2)\, E_1 &= b_2 + (-2)\, b_1 \end{aligned} \tag{11.2.22}$$

From this point, we were able to find the solution to the system. Theorem 11.2.1 tells us that the solution we found is in fact a solution to the original system.

We will now prove Theorem 11.2.1.

**Proof**

The proof that the systems 11.2.15 and 11.2.16 have the same solution set is as follows. Suppose that $(x_1, \cdots, x_n)$ is a solution to $E_1 = b_1, E_2 = b_2$ . We want to show that this is a solution to the system in 11.2.16 above. This is clear, because the system in 11.2.16 is the original system, but listed in a different order. Changing the order does not effect the solution set, so $(x_1, \cdots, x_n)$ is a solution to 11.2.16.

Next we want to prove that the systems 11.2.15 and 11.2.17 have the same solution set. That is $E_1 = b_1, E_2 = b_2$ has the same solution set as the system $E_1 = b_1, kE_2 = kb_2$ provided $k \neq 0$. Let $(x_1, \cdots, x_n)$ be a solution of $E_1 = b_1, E_2 = b_2,$ . We want to show that it is a solution to $E_1 = b_1, kE_2 = kb_2$ . Notice that the only difference between these two systems is that the second involves multiplying the equation, $E_2 = b_2$ by the scalar $k$. Recall that when you multiply both sides of an equation by the same number, the sides are still equal to each other. Hence if $(x_1, \cdots, x_n)$ is a solution to $E_2 = b_2$ , then it will also be a solution to $kE_2 = kb_2$ . Hence, $(x_1, \cdots, x_n)$ is also a solution to 11.2.17.

Similarly, let $(x_1, \cdots, x_n)$ be a solution of $E_1 = b_1, kE_2 = kb_2$ . Then we can multiply the equation $kE_2 = kb_2$ by the scalar $1/k$, which is possible only because we have required that $k \neq 0$. Just as above, this action preserves equality and we obtain the equation $E_2 = b_2$ . Hence $(x_1, \cdots, x_n)$ is also a solution to $E_1 = b_1, E_2 = b_2$.

Finally, we will prove that the systems 11.2.15 and 11.2.18 have the same solution set. We will show that any solution of $E_1 = b_1, E_2 = b_2$ is also a solution of 11.2.18. Then, we will show that any solution of 11.2.18 is also a solution of $E_1 = b_1, E_2 = b_2$ . Let $(x_1, \cdots, x_n)$ be a solution to $E_1 = b_1, E_2 = b_2$ . Then in particular it solves $E_1 = b_1$ . Hence, it solves the first equation in 11.2.18. Similarly, it also solves $E_2 = b_2$ . By our proof of 11.2.17, it also solves $kE_1 = kb_1$ . Notice that if we add $E_2$ and $kE_1$, this is equal to $b_2 + kb_1$ . Therefore, if $(x_1, \cdots, x_n)$ solves $E_1 = b_1, E_2 = b_2$ it must also solve $E_2 + kE_1 = b_2 + kb_1$ .

Now suppose $(x_1, \cdots, x_n)$ solves the system $E_1 = b_1, E_2 + kE_1 = b_2 + kb_1$ . Then in particular it is a solution of $E_1 = b_1$ . Again by our proof of 11.2.17, it is also a solution to $kE_1 = kb_1$ . Now if we subtract these equal quantities from both sides of $E_2 + kE_1 = b_2 + kb_1$ we obtain $E_2 = b_2$ , which shows that the solution also satisfies $E_1 = b_1, E_2 = b_2$.

Stated simply, the above theorem shows that the elementary operations do not change the solution set of a system of equations.

We will now look at an example of a system of three equations and three variables. Similarly to the previous examples, the goal is to find values for $x, y, z$ such that each of the given equations are satisfied when these values are substituted in.

### Example 11.2.3: Solving a System of Equations with Elementary Operations

Find the solutions to the system,

$$\begin{aligned} x+3y+6z &= 25 \\ 2x+7y+14z &= 58 \\ 2y+5z &= 19 \end{aligned} \tag{11.2.23}$$

**Solution**

We can relate this system to Thoerem 11.2.1 above. In this case, we have

$$\begin{aligned} E_1 &= x+3y+6z, & b_1 &= 25 \\ E_2 &= 2x+7y+14z, & b_2 &= 58 \\ E_3 &= 2y+5z, & b_3 &= 19 \end{aligned} \tag{11.2.24}$$

Thoerem 11.2.1 claims that if we do elementary operations on this system, we will not change the solution set. Therefore, we can solve this system using the elementary operations given in Definition 11.2.4. First, replace the second equation by $(-2)$ times the first equation added to the second. This yields the system

$$\begin{aligned} x+3y+6z &= 25 \\ y+2z &= 8 \\ 2y+5z &= 19 \end{aligned} \tag{11.2.25}$$

Now, replace the third equation with $(-2)$ times the second added to the third. This yields the system

$$\begin{aligned} x+3y+6z &= 25 \\ y+2z &= 8 \\ z &= 3 \end{aligned} \tag{11.2.26}$$

At this point, we can easily find the solution. Simply take $z=3$ and substitute this back into the previous equation to solve for $y$, and similarly to solve for $x$.

$$\begin{aligned} x+3y+6\left(3\right) = x+3y+18 &= 25 \\ y+2\left(3\right) = y+6 &= 8 \\ z &= 3 \end{aligned} \tag{11.2.27}$$

The second equation is now

$$y+6=8 \tag{11.2.28}$$

You can see from this equation that $y=2$. Therefore, we can substitute this value into the first equation as follows:

$$x+3\left(2\right)+18=25 \tag{11.2.29}$$

By simplifying this equation, we find that $x=1$. Hence, the solution to this system is $(x,y,z)=(1,2,3)$. This process is called **back substitution**.

Alternatively, in 11.2.26 you could have continued as follows. Add $(-2)$ times the third equation to the second and then add $(-6)$ times the second to the first. This yields

$$\begin{aligned} x+3y &= 7 \\ y &= 2 \\ z &= 3 \end{aligned} \tag{11.2.30}$$

Now add $(-3)$ times the second to the first. This yields

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned} \tag{11.2.31}$$

a system which has the same solution set as the original system. This avoided back substitution and led to the same solution set. It is your decision which you prefer to use, as both methods lead to the correct solution, $(x, y, z) = (1, 2, 3)$.

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# 11.3: Gaussian Elimination

The work we did in the previous section will always find the solution to the system. In this section, we will explore a less cumbersome way to find the solutions. First, we will represent a linear system with an **augmented matrix.** A **matrix** is simply a rectangular array of numbers. The size or dimension of a matrix is defined as $m \times n$ where $m$ is the number of rows and $n$ is the number of columns. In order to construct an augmented matrix from a linear system, we create a **coefficient matrix** from the coefficients of the variables in the system, as well as a **constant matrix** from the constants. The coefficients from one equation of the system create one row of the augmented matrix.

For example, consider the linear system

$$\begin{aligned} x+3y+6z &= 25 \\ 2x+7y+14z &= 58 \\ 2y+5z &= 19 \end{aligned} \tag{11.3.1}$$

This system can be written as an augmented matrix, as follows

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right] \tag{11.3.2}$$

Notice that it has exactly the same information as the original system. Here it is understood that the first column contains the coefficients from $x$ in each equation, in order, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Similarly, we create a column from the coefficients on $y$ in each equation, $\begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$ and a column from the coefficients on $z$ in each equation, $\begin{bmatrix} 6 \\ 14 \\ 5 \end{bmatrix}$. For a system of more than three variables, we would continue in this way constructing a column for each variable. Similarly, for a system of less than three variables, we simply construct a column for each variable.

Finally, we construct a column from the constants of the equations, $\begin{bmatrix} 25 \\ 58 \\ 19 \end{bmatrix}$.

The rows of the augmented matrix correspond to the equations in the system. For example, the top row in the augmented matrix, $\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \end{array}\right]$ corresponds to the equation

$$x+3y+6z = 25. \tag{11.3.3}$$

Consider the following definition.

**Definition 11.3.1: Augmented Matrix of a Linear System**

For a linear system of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{11.3.4}$$

where the $x_i$ are variables and the $a_{ij}$ and $b_i$ are constants, the augmented matrix of this system is given by

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right] \tag{11.3.5}$$

Now, consider elementary operations in the context of the augmented matrix. The elementary operations in Definition 11.3.1 can be used on the rows just as we used them on equations previously. Changes to a system of equations in as a result of an elementary operation are equivalent to changes in the augmented matrix resulting from the corresponding row operation. Note that Theorem 11.2.1 implies that any elementary row operations used on an augmented matrix will not change the solution to the corresponding system of equations. We now formally define elementary row operations. These are the *key tool* we will use to find solutions to systems of equations.

**Definition 11.3.2: Elementary Row Operations**

The **elementary row operations** (also known as **row operations**) consist of the following

1. Switch two rows.
2. Multiply a row by a nonzero number.
3. Replace a row by any multiple of another row added to it.

Recall how we solved Example 11.2.3. We can do the exact same steps as above, except now in the context of an augmented matrix and using row operations. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Thus the first step in solving this system would be to take $(-2)$ times the first row of the augmented matrix and add it to the second row,

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Note how this corresponds to this step in Example 11.2.3. Next take $(-2)$ times the second row and add to the third,

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This augmented matrix corresponds to the system

$$\begin{aligned} x+3y+6z &= 25 \\ y+2z &= 8 \\ z &= 3 \end{aligned}$$

which is the same as this step in Example 11.2.3. By back substitution you obtain the solution $x=1, y=2$, and $z=3$.

Through a systematic procedure of row operations, we can simplify an augmented matrix and carry it to **row-echelon form** or **reduced row-echelon form**, which we define next. These forms are used to find the solutions of the system of equations corresponding to the augmented matrix.

In the following definitions, the term **leading entry** refers to the first nonzero entry of a row when scanning the row from left to right.

**Definition 11.3.3: Row-Echelon Form**

An augmented matrix is in **row-echelon form** if

1. All nonzero rows are above any rows of zeros.
2. Each leading entry of a row is in a column to the right of the leading entries of any row above it.
3. Each leading entry of a row is equal to 1.

We also consider another reduced form of the augmented matrix which has one further condition.

**Definition 11.3.4: Reduced Row-Echelon Form**

An augmented matrix is in **reduced row-echelon form** if

1. All nonzero rows are above any rows of zeros.
2. Each leading entry of a row is in a column to the right of the leading entries of any rows above it.
3. Each leading entry of a row is equal to 1.
4. All entries in a column above and below a leading entry are zero.

Notice that the first three conditions on a reduced row-echelon form matrix are the same as those for row-echelon form.

Hence, every reduced row-echelon form matrix is also in row-echelon form. The converse is not necessarily true; we cannot assume that every matrix in row-echelon form is also in reduced row-echelon form. However, it often happens that the row-echelon form is sufficient to provide information about the solution of a system.

The following examples describe matrices in these various forms. As an exercise, take the time to carefully verify that they are in the specified form.

**Example 11.3.5: Not in Row-Echelon Form**

The following augmented matrices are not in row-echelon form (and therefore also not in reduced row-echelon form).

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & -6 \\ 4 & 0 & 7 \end{array}\right], \left[\begin{array}{ccc|c} 0 & 2 & 3 & 3 \\ 1 & 5 & 0 & 2 \\ 7 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

**Example 11.3.6: Matrices in Row-Echelon Form**

The following augmented matrices are in row-echelon form, but not in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 6 & 5 & 8 & 2 \\ 0 & 0 & 1 & 2 & 7 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice that we could apply further row operations to these matrices to carry them to reduced row-echelon form. Take the time to try that on your own. Consider the following matrices, which are in reduced row-echelon form.

**Example 11.3.7: Matrices in Reduced Row-Echelon Form**

The following augmented matrices are in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

One way in which the row-echelon form of a matrix is useful is in identifying the pivot positions and pivot columns of the matrix.

**Definition 11.3.8: Pivot Position and Pivot Column**

A **pivot position** in a matrix is the location of a leading entry in the row-echelon form of a matrix.

A **pivot column** is a column that contains a pivot position.

For example consider the following.

**Example 11.3.9: Pivot Position**

Let

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array}\right] \tag{11.3.6}$$

Where are the pivot positions and pivot columns of the augmented matrix $A$?

**Solution**

The row-echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{11.3.7}$$

This is all we need in this example, but note that this matrix is not in reduced row-echelon form.

In order to identify the pivot positions in the original matrix, we look for the leading entries in the row-echelon form of the matrix. Here, the entry in the first row and first column, as well as the entry in the second row and second column are the leading entries. Hence, these locations are the pivot positions. We identify the pivot positions in the original matrix, as in the following:

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 4 \\ 3 & \boxed{2} & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array}\right]$$

Thus the pivot columns in the matrix are the first two columns.

The following is an algorithm for carrying a matrix to row-echelon form and reduced row-echelon form. You may wish to use this algorithm to carry the above matrix to row-echelon form or reduced row-echelon form yourself for practice.

**Algorithm 11.3.10: Reduced Row-Echelon Form Algorithm**

This algorithm provides a method for using row operations to take a matrix to its reduced row-echelon form. We begin with the matrix in its original form.

1. Starting from the left, find the first nonzero column. This is the first pivot column, and the position at the top of this column is the first pivot position. Switch rows if necessary to place a nonzero number in the first pivot position.
2. Use row operations to make the entries below the first pivot position (in the first pivot column) equal to zero.
3. Ignoring the row containing the first pivot position, repeat steps 1 and 2 with the remaining rows. Repeat the process until there are no more rows to modify.
4. Divide each nonzero row by the value of the leading entry, so that the leading entry becomes 1. The matrix will then be in row-echelon form.

   The following step will carry the matrix from row-echelon form to reduced row-echelon form.

5. Moving from right to left, use row operations to create zeros in the entries of the pivot columns which are above the pivot positions. The result will be a matrix in reduced row-echelon form.

Most often we will apply this algorithm to an augmented matrix in order to find the solution to a system of linear equations. However, we can use this algorithm to compute the reduced row-echelon form of any matrix which could be useful in other applications.

Consider the following example of the Reduced Row-Echelon Form Algorithm.

**Example 11.3.11 Finding row-echelon form and reduced row-echelon form of a Matrix**

Let

$$A = \begin{bmatrix} 0 & -5 & -4 \\ 1 & 4 & 3 \\ 5 & 10 & 7 \end{bmatrix} \tag{11.3.8}$$

Find the row-echelon form of $A$. Then complete the process until $A$ is in reduced row-echelon form.

**Solution**

In working through this example, we will use the steps outlined in Reduced Row-Echelon Form Algorithm.

1. The first pivot column is the first column of the matrix, as this is the first nonzero column from the left. Hence the first pivot position is the one in the first row and first column. Switch the first two rows to obtain a nonzero entry in the first pivot position, outlined in a box below.

$$\begin{bmatrix} \boxed{1} & 4 & 3 \\ 0 & -5 & -4 \\ 5 & 10 & 7 \end{bmatrix} \tag{11.3.9}$$

2. Step two involves creating zeros in the entries below the first pivot position. The first entry of the second row is already a zero. All we need to do is subtract 5 times the first row from the third row. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 10 & 8 \end{bmatrix} \tag{11.3.10}$$

3. Now ignore the top row. Apply steps 1 and 2 to the smaller matrix

$$\begin{bmatrix} -5 & -4 \\ 10 & 8 \end{bmatrix} \tag{11.3.11}$$

In this matrix, the first column is a pivot column, and $-5$ is in the first pivot position. Therefore, we need to create a zero below it. To do this, add 2 times the first row (of this matrix) to the second. The resulting matrix is

$$\begin{bmatrix} -5 & -4 \\ 0 & 0 \end{bmatrix} \tag{11.3.12}$$

Our original matrix now looks like

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 0 & 0 \end{bmatrix} \tag{11.3.13}$$

We can see that there are no more rows to modify.

4. Now, we need to create leading 1s in each row. The first row already has a leading 1 so no work is needed here. Divide the second row by $-5$ to create a leading 1. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix} \tag{11.3.14}$$

This matrix is now in row-echelon form.

5. Now create zeros in the entries above pivot positions in each column, in order to carry this matrix all the way to reduced row-echelon form. Notice that there is no pivot position in the third column so we do not need to create any zeros in this column! The column in which we need to create zeros is the second. To do so, subtract 4 times the second row from the first row. The resulting matrix is

$$\begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix} \tag{11.3.15}$$

This matrix is now in reduced row-echelon form.

The above algorithm gives you a simple way to obtain the row-echelon form and reduced row-echelon form of a matrix. The main idea is to do row operations in such a way as to end up with a matrix in row-echelon form or reduced row-echelon form. This process is important because the resulting matrix will allow you to describe the solutions to the corresponding linear system of equations in a meaningful way.

In the next example, we look at how to solve a system of equations using the corresponding augmented matrix.

**Example 11.3.12: Finding the Solution to a System**

Give the complete solution to the following system of equations

$$\begin{aligned} 2x+4y-3z&=-1 \\ 5x+10y-7z&=-2 \\ 3x+6y+5z&=9 \end{aligned} \tag{11.3.16}$$

**Solution**

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 5 & 10 & -7 & -2 \\ 3 & 6 & 5 & 9 \end{array}\right] \tag{11.3.17}$$

In order to find the solution to this system, we wish to carry the augmented matrix to . We will do so using Reduced Row-Echelon Form Algorithm. Notice that the first column is nonzero, so this is our first pivot column. The first entry in the first row, 2, is the first leading entry and it is in the first pivot position. We will use row operations to create zeros in the entries below the 2. First, replace the second row with $-5$ times the first row plus 2 times the second row. This yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 5 & 9 \end{array}\right] \tag{11.3.18}$$

Now, replace the third row with $-3$ times the first row plus to 2 times the third row. This yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 21 \end{array}\right] \tag{11.3.19}$$

Now the entries in the first column below the pivot position are zeros. We now look for the second pivot column, which in this case is column three. Here, the 1 in the second row and third column is in the pivot position. We need to do just one row operation to create a zero below the 1.

Taking $-1$ times the second row and adding it to the third row yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 20 \end{array}\right] \tag{11.3.20}$$

We could proceed with the algorithm to carry this matrix to row-echelon form or reduced row-echelon form. However, remember that we are looking for the solutions to the system of equations. Take another look at the third row of the matrix. Notice that it corresponds to the equation

$$0x+0y+0z=20 \tag{11.3.21}$$

There is no solution to this equation because for all $x, y, z$, the left side will equal 0 and $0 \neq 20$. This shows there is no solution to the given system of equations. In other words, this system is inconsistent.

The following is another example of how to find the solution to a system of equations by carrying the corresponding augmented matrix to reduced row-echelon form.

**Example 11.3.13: An Infinite Set of Solutions**

Give the complete solution to the system of equations

$$\begin{aligned} 3x-y-5z&=9 \\ y-10z&=0 \\ -2x+y&=-6 \end{aligned} \tag{11.3.22}$$

**Solution**

The augmented matrix of this system is

$$\left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ -2 & 1 & 0 & -6 \end{array}\right] \tag{11.3.23}$$

In order to find the solution to this system, we will carry the augmented matrix to reduced row-echelon form, using Reduced Row-Echelon Form Algorithm. The first column is the first pivot column. We want to use row operations to create zeros beneath the first entry in this column, which is in the first pivot position. Replace the third row with 2 times the first row added to 3 times the third row. This gives

$$\left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 1 & -10 & 0 \end{array}\right] \tag{11.3.24}$$

Now, we have created zeros beneath the 3 in the first column, so we move on to the second pivot column (which is the second column) and repeat the procedure. Take $-1$ times the second row and add to the third row.

$$\left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{11.3.25}$$

The entry below the pivot position in the second column is now a zero. Notice that we have no more pivot columns because we have only two leading entries.

At this stage, we also want the leading entries to be equal to one. To do so, divide the first row by 3.

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{3} & -\frac{5}{3} & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{11.3.26}$$

This matrix is now in row-echelon form.

Let's continue with row operations until the matrix is in reduced row-echelon form. This involves creating zeros above the pivot positions in each pivot column. This requires only one step, which is to add $\frac{1}{3}$ times the second row to the first row.

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{11.3.27}$$

This is in reduced row-echelon form, which you should verify using Definition 11.3.4. The equations corresponding to this reduced row-echelon form are

$$\begin{aligned} x - 5z &= 3 \\ y - 10z &= 0 \end{aligned} \tag{11.3.28}$$

or

$$\begin{aligned} x &= 3 + 5z \\ y &= 10z \end{aligned} \tag{11.3.29}$$

Observe that $z$ is not restrained by any equation. In fact, $z$ can equal any number. For example, we can let $z = t$, where we can choose $t$ to be any number. In this context $t$ is called a **parameter** . Therefore, the solution set of this system is

$$\begin{aligned} x &= 3 + 5t \\ y &= 10t \\ z &= t \end{aligned} \tag{11.3.30}$$

where $t$ is arbitrary. The system has an infinite set of solutions which are given by these equations. For any value of $t$ we select, $x, y,$ and $z$ will be given by the above equations. For example, if we choose $t = 4$ then the corresponding solution would be

$$\begin{aligned} x &= 3 + 5(4) = 23 \\ y &= 10(4) = 40 \\ z &= 4 \end{aligned} \tag{11.3.31}$$

In the previous example, the solution involved one parameter. It may happen that the solution to a system involves more than one parameter, as shown in the following example.

### Example 11.3.1: A Two Parameter Set of Solutions

Find the solution to the system

$$\begin{aligned} x + 2y - z + w &= 3 \\ x + y - z + w &= 1 \\ x + 3y - z + w &= 5 \end{aligned} \tag{11.3.32}$$

**Solution**

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array}\right] \tag{11.3.33}$$

We wish to carry this matrix to row-echelon form. Here, we will outline the row operations used. However, make sure that you understand the steps in terms of Reduced Row-Echelon Form Algorithm.

Take $-1$ times the first row and add to the second. Then take $-1$ times the first row and add to the third. This yields

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right] \tag{11.3.34}$$

Now add the second row to the third row and divide the second row by $-1$.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{11.3.35}$$

This matrix is in row-echelon form and we can see that $x$ and $y$ correspond to pivot columns, while $z$ and $w$ do not. Therefore, we will assign parameters to the variables $z$ and $w$. Assign the parameter $s$ to $z$ and the parameter $t$ to $w$. Then the first row yields the equation $x+2y-s+t=3$ , while the second row yields the equation $y=2$. Since $y=2$, the first equation becomes $x+4-s+t=3$ showing that the solution is given by

$$\begin{aligned} x &= -1+s-t \\ y &= 2 \\ z &= s \\ w &= t \end{aligned} \tag{11.3.36}$$

It is customary to write this solution in the form

$$\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] = \left[\begin{array}{c} -1+s-t \\ 2 \\ s \\ t \end{array}\right] \tag{11.3.37}$$

This example shows a system of equations with an infinite solution set which depends on two parameters. It can be less confusing in the case of an infinite solution set to first place the augmented matrix in reduced row-echelon form rather than just row-echelon form before seeking to write down the description of the solution.

In the above steps, this means we don't stop with the row-echelon form in equation 11.3.35 Instead we first place it in reduced row-echelon form as follows.

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{11.3.38}$$

Then the solution is $y=2$ from the second row and $x=-1+z-w$ from the first. Thus letting $z=s$ and $w=t$, the solution is given by 11.3.37.

You can see here that there are two paths to the correct answer, which both yield the same answer. Hence, either approach may be used. The process which we first used in the above solution is called **Gaussian Elimination** This process involves carrying the matrix to row-echelon form, converting back to equations, and using back substitution to find the solution. When you do row operations until you obtain reduced row-echelon form, the process is called **Gauss-Jordan Elimination**.

We have now found solutions for systems of equations with no solution and infinitely many solutions, with one parameter as well as two parameters. Recall the three types of solution sets which we discussed in the previous section; no solution, one solution, and infinitely many solutions. Each of these types of solutions could be identified from the graph of the system. It turns out that we can also identify the type of solution from the reduced row-echelon form of the augmented matrix.

- *No Solution:* In the case where the system of equations has no solution, the row-echelon form of the augmented matrix will have a row of the form

  $$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right] \tag{11.3.39}$$

  This row indicates that the system is inconsistent and has no solution.

- *One Solution:* In the case where the system of equations has one solution, every column of the coefficient matrix is a pivot column. The following is an example of an augmented matrix in for a row-echelon form system of equations with one solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \tag{11.3.40}$$

- *Infinitely Many Solutions:* In the case where the system of equations has infinitely many solutions, the solution contains parameters. There will be columns of the coefficient matrix which are not pivot columns. The following are examples of augmented matrices in reduced row-echelon form for systems of equations with infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{11.3.41}$$

or

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \end{array}\right] \tag{11.3.42}$$

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# 11.E: Exercises

**Exercise 11.$E$.1**

Graphically, find the point $(x_1, y_1)$ which lies on both lines, $x+3y=1$ and $4x-y=3$. That is, graph each line and see where they intersect.

**Answer**

$\begin{matrix} x+3y=1 \\ 4x-y=3 \end{matrix}$, Solution is: $\left[x=\frac{10}{13}, y=\frac{1}{13}\right]$.

**Exercise 11.$E$.2**

Graphically, find the point of intersection of the two lines $3x+y=3$ and $x+2y=1$. That is, graph each line and see where they intersect.

**Answer**

$\begin{matrix} 3x+y=3 \\ x+2y=1 \end{matrix}$, Solution is: $[x=1, y=0]$

**Exercise 11.$E$.3**

You have a system of $k$ equations in two variables, $k \geq 2$. Explain the geometric significance of

1. No solution.
2. A unique solution.
3. An infinite number of solutions.

**Exercise 11.$E$.4**

Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right] \tag{11.E.1}$$

**Answer**

The solution exists but is not unique.

**Exercise 11.$E$.5**

Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right] \tag{11.E.2}$$

**Answer**

A solution exists and is unique.

### Exercise 11.$E$.6

Consider the following augmented matrix in which $*$ denotes an arbitrary number and ■ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} ■ & * & * & * & * & * \\ 0 & ■ & 0 & * & 0 & * \\ 0 & 0 & 0 & ■ & * & * \\ 0 & 0 & 0 & 0 & ■ & * \end{array}\right] \tag{11.E.3}$$

### Exercise 11.$E$.7

Consider the following augmented matrix in which $*$ denotes an arbitrary number and ■ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} ■ & * & * & * & * & * \\ 0 & ■ & * & * & 0 & * \\ 0 & 0 & 0 & 0 & ■ & 0 \\ 0 & 0 & 0 & 0 & * & ■ \end{array}\right] \tag{11.E.4}$$

**Answer**

There might be a solution. If so, there are infinitely many.

### Exercise 11.$E$.8

Suppose a system of equations has fewer equations than variables. Will such a system necessarily be consistent? If so, explain why and if not, give an example which is not consistent.

**Answer**

No. Consider $x+y+z=2$ and $x+y+z=1$.

### Exercise 11.$E$.9

If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, tell why not.

**Answer**

These can have a solution. For example, $x+y=1, 2x+2y=2, 3x+3y=3$ even has an infinite set of solutions.

### Exercise 11.$E$.10

Find $h$ such that

$$\left[\begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array}\right] \tag{11.E.5}$$

is the augmented matrix of an *inconsistent* system.

**Answer**

$h=4$

### Exercise 11.$E$.11

Find $h$ such that

$$\left[\begin{array}{cc|c} 1 & h & 3 \\ 2 & 4 & 6 \end{array}\right] \tag{11.E.6}$$

is the augmented matrix of a *consistent* system.

**Answer**

Any $h$ will work.

### Exercise 11.$E$.12

Find $h$ such that

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & h & 12 \end{array}\right] \tag{11.E.7}$$

is the augmented matrix of a *consistent* system.

**Answer**

Any $h$ will work.

### Exercise 11.$E$.13

Choose $h$ and $k$ such that the augmented matrix shown has each of the following:

1. one solution
2. no solution
3. infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 2 & 4 & k \end{array}\right] \tag{11.E.8}$$

**Answer**

If $h \neq 2$ there will be a unique solution for any $k$. If $h = 2$ and $k \neq 4$, there are no solutions. If $h = 2$ and $k = 4$, then there are infinitely many solutions.

### Exercise 11.$E$.14

Choose $h$ and $k$ such that the augmented matrix shown has each of the following:

1. one solution
2. no solution
3. infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array}\right] \tag{11.E.9}$$

**Answer**

If $h \neq 4$, then there is exactly one solution. If $h = 4$ and $k \neq 4$, then there are no solutions. If $h = 4$ and $k = 4$, then there are infinitely many solutions.

## Exercise 11.$E$.15

Determine if the system is consistent. If so, is the solution unique?

$$\begin{aligned} x+2y+z-w &= 2 \\ x-y+z+w &= 1 \\ 2x+y-z &= 1 \\ 4x+2y+z &= 5 \end{aligned} \tag{11.E.10}$$

**Answer**

There is no solution. The system is inconsistent. You can see this from the augmented matrix. $\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 0 & 1 \\ 4 & 2 & 1 & 0 & 5 \end{bmatrix}$, : $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

## Exercise 11.$E$.16

Determine if the system is consistent. If so, is the solution unique?

$$\begin{aligned} x+2y+z-w &= 2 \\ x-y+z+w &= 0 \\ 2x+y-z &= 1 \\ 4x+2y+z &= 3 \end{aligned} \tag{11.E.11}$$

**Answer**

Solution is: $\left[w=\frac{3}{2}y-1, x=\frac{2}{3}-\frac{1}{2}y, z=\frac{1}{3}\right]$

## Exercise 11.$E$.17

Determine which matrices are in reduced row-echelon form.

1. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 7 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

**Answer**

1. This one is not.
2. This one is.
3. This one is.

**Exercise 11.$E$.18**

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 2 & -1 & 3 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & -2 \end{bmatrix} \tag{11.E.12}$$

**Exercise 11.$E$.19**

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix} \tag{11.E.13}$$

**Exercise 11.$E$.20**

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 3 & -6 & -7 & -8 \\ 1 & -2 & -2 & -2 \\ 1 & -2 & -3 & -4 \end{bmatrix} \tag{11.E.14}$$

**Exercise 11.$E$.21**

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 2 & 4 & 5 & 15 \\ 1 & 2 & 3 & 9 \\ 1 & 2 & 2 & 6 \end{bmatrix} \tag{11.E.15}$$

**Exercise 11.$E$.22**

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 4 & -1 & 7 & 10 \\ 1 & 0 & 3 & 3 \\ 1 & -1 & -2 & 1 \end{bmatrix} \tag{11.E.16}$$

**Exercise 11.$E$.23**

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 3 & 5 & -4 & 2 \\ 1 & 2 & -1 & 1 \\ 1 & 1 & -2 & 0 \end{bmatrix} \tag{11.E.17}$$

**Exercise 11.$E$.24**

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} -2 & 3 & -8 & 7 \\ 1 & -2 & 5 & -5 \\ 1 & -3 & 7 & -8 \end{bmatrix} \tag{11.E.18}$$

**Exercise 11.E.25**

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{array}\right] \tag{11.E.19}$$

**Exercise 11.E.26**

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{array}\right] \tag{11.E.20}$$

**Answer**

The reduced row-echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{array}\right]$. Therefore, the solution is of the form $z=t, y=\frac{3}{4}+t\left(\frac{1}{4}\right), x=\frac{1}{2}-\frac{1}{2}t$ where $t \in \mathbb{R}$.

**Exercise 11.E.27**

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 4 & 2 \end{array}\right] \tag{11.E.21}$$

**Answer**

The reduced row-echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & -4 & -1 \end{array}\right]$ and so the solution is $z=t, y=-1+4t, x=2-4t$.

**Exercise 11.E.28**

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 & 2 \end{array}\right] \tag{11.E.22}$$

**Answer**

The reduced row-echelon form is $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 9 & 3 \\ 0 & 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & -7 & -1 \\ 0 & 0 & 0 & 1 & 6 & 1 \end{array}\right]$ and so $x_5=t, x_4=1-6t, x_3=-1+7t, x_2=4t, x_1=3-9t$ .

**Exercise 11.E.29**

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 1 & -1 & 2 & 2 & 2 & 0 \end{array}\right] \tag{11.E.23}$$

**Answer**

The reduced row-echelon form is $\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. Therefore, let $x_5 = t, x_3 = s$. Then the other variables are given by $x_4 = -\frac{1}{2} - \frac{3}{2}t, x_2 = \frac{3}{2} - t\frac{1}{2}, , x_1 = \frac{5}{2} + \frac{1}{2}t - 2s$.

**Exercise 11.E.30**

Find the solution to the system of equations, $7x + 14y + 15z = 22$, $2x + 4y + 3z = 5$, and $3x + 6y + 10z = 13$.

**Answer**

Solution is: $[x = 1 - 2t, z = 1, y = t]$

**Exercise 11.E.31**

Find the solution to the system of equations, $3x - y + 4z = 6$, $y + 8z = 0$, and $-2x + y = -4$.

**Answer**

Solution is: $[x = 2 - 4t, y = -8t, z = t]$

**Exercise 11.E.32**

Find the solution to the system of equations, $9x - 2y + 4z = -17$, $13x - 3y + 6z = -25$, and $-2x - z = 3$.

**Answer**

Solution is: $[x = -1, y = 2, z = -1]$

**Exercise 11.E.33**

Find the solution to the system of equations, $65x + 84y + 16z = 546$, $81x + 105y + 20z = 682$, and $84x + 110y + 21z = 713$.

**Answer**

Solution is: $[x = 2, y = 4, z = 5]$

**Exercise 11.E.34**

Find the solution to the system of equations, $8x + 2y + 3z = -3, 8x + 3y + 3z = -1$, and $4x + y + 3z = -9$.

**Answer**

Solution is: $[x=1, y=2, z=-5]$

**Exercise 11.$E$.35**

Find the solution to the system of equations, $-8x+2y+5z=18$, $-8x+3y+5z=13$, and $-4x+y+5z=19$.

**Answer**

Solution is: $[x=-1, y=-5, z=4]$

**Exercise 11.$E$.36**

Find the solution to the system of equations, $3x-y-2z=3$, $y-4z=0$, and $-2x+y=-2$.

**Answer**

Solution is: $[x=2t+1, y=4t, z=t]$

**Exercise 11.$E$.37**

Find the solution to the system of equations, $-9x+15y=66$, $-11x+18y=79$, $-x+y=4$, and $z=3$.

**Answer**

Solution is: $[x=1, y=5, z=3]$

**Exercise 11.$E$.38**

Find the solution to the system of equations, $-19x+8y=-108$, $-71x+30y=-404$, $-2x+y=-12$, $4x+z=14$.

**Answer**

Solution is: $[x=4, y=-4, z=-2]$

**Exercise 11.$E$.39**

Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.

**Answer**

No. Consider the system of equations $x+y+2z=2$ and $x+y+z=1$, this system has infinitely many solutions. If there is one, there will be a free variable because there are more variables than equations, and so there will be infinitely many solutions.

**Exercise 11.$E$.40**

Suppose a system of linear equations has a $2\times 4$ augmented matrix and the last column is a pivot column. Could the system of linear equations be consistent? Explain.

**Answer**

No. This would lead to $0=1$.

### Exercise 11.$E$.41

Suppose the coefficient matrix of a system of $n$ equations with $n$ variables has the property that every column is a pivot column. Does it follow that the system of equations must have a solution? If so, must the solution be unique? Explain.

**Answer**

Yes. It has a unique solution.

### Exercise 11.$E$.42

Suppose there is a unique solution to a system of linear equations. What must be true of the pivot columns in the augmented matrix?

**Answer**

The last column must not be a pivot column. The remaining columns must each be pivot columns.

### Exercise 11.$E$.43

The steady state temperature, $u$, of a plate solves Laplace's equation, $\Delta u = 0$. One way to approximate the solution is to divide the plate into a square mesh and require the temperature at each node to equal the average of the temperature at the four adjacent nodes. In the following picture, the numbers represent the observed temperature at the indicated nodes. Find the temperature at the interior nodes, indicated by $x, y, z,$ and $w$. One of the equations is $z = \backslash \text{vspace} 0.05 in \frac{1}{4}(10+0+w+x)$ .

(1,60) (100,0)

(1,1) (0,30)(1,0)90 (0,60)(1,0)90 (30,0)(0,1)90 (60,0)(0,1)90 (60,0) (30,0) (0,60) (0,30) (30,30) (30,60) (60,30) (60,60) (60,90) (30,90) (90,30) (90,60) (63,-4)10 (33,-4)10 (-10,60)20 (-10,30)20 (34,34)$x$ (34,64)$y$ (64,34)$z$ (64,64)$w$ (64,90)30 (34,90)30 (94,30)0 (94,60)0

**Answer**

You need $\begin{array}{l} \frac{1}{4}(20+30+w+x)-y=0 \\ \frac{1}{4}(y+30+0+z)-w=0 \\ \frac{1}{4}(20+y+z+10)-x=0 \\ \frac{1}{4}(x+w+0+10)-z=0 \end{array}$ , Solution is: $[w=15, x=15, y=20, z=10]$ .

### Exercise 11.$E$.44

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & -16 & -1 & -5 \\ 1 & -4 & 0 & -1 \\ 1 & -4 & -1 & -2 \end{bmatrix} \tag{11.E.24}$$

### Exercise 11.$E$.45

Find the rank of the following matrix.

$$\begin{bmatrix} 3 & 6 & 5 & 12 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix} \tag{11.E.25}$$

**Exercise 11.$E$.46**

Find the rank of the following matrix.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 3 \\ 1 & 4 & 1 & 0 & -8 \\ 1 & 4 & 0 & 1 & 2 \\ -1 & -4 & 0 & -1 & -2 \end{bmatrix} \tag{11.E.26}$$

**Exercise 11.$E$.47**

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & -4 & 3 & -9 \\ 1 & -1 & 1 & -2 \\ 1 & -1 & 0 & -3 \end{bmatrix} \tag{11.E.27}$$

**Exercise 11.$E$.48**

Find the rank of the following matrix.

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & 1 & 7 \end{bmatrix} \tag{11.E.28}$$

**Exercise 11.$E$.49**

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 15 & 29 \\ 1 & 4 & 8 \\ 1 & 3 & 5 \\ 3 & 9 & 15 \end{bmatrix} \tag{11.E.29}$$

**Exercise 11.$E$.50**

Find the rank of the following matrix.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 & -18 \\ 1 & 2 & 2 & -1 & -11 \\ -1 & -2 & -2 & 1 & 11 \end{bmatrix} \tag{11.E.30}$$

**Exercise 11.$E$.51**

Find the rank of the following matrix.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 11 \\ 1 & -2 & 0 & 4 & 15 \\ 1 & -2 & 0 & 3 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{11.E.31}$$

**Exercise 11.E.52**

Find the rank of the following matrix.

$$\begin{bmatrix} -2 & -3 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ -3 & 0 & -3 \end{bmatrix} \tag{11.E.32}$$

**Exercise 11.E.53**

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 4 & 20 & -1 & 17 \\ 1 & 1 & 5 & 0 & 5 \\ 1 & 1 & 5 & -1 & 2 \\ 3 & 3 & 15 & -3 & 6 \end{bmatrix} \tag{11.E.33}$$

**Exercise 11.E.54**

Find the rank of the following matrix.

$$\begin{bmatrix} -1 & 3 & 4 & -3 & 8 \\ 1 & -3 & -4 & 2 & -5 \\ 1 & -3 & -4 & 1 & -2 \\ -2 & 6 & 8 & -2 & 4 \end{bmatrix} \tag{11.E.34}$$

**Exercise 11.E.55**

Suppose $A$ is an $m \times n$ matrix. Explain why the rank of $A$ is always no larger than $\min(m, n)$.

**Answer**

It is because you cannot have more than $\min(m, n)$ nonzero rows in the reduced row-echelon form. Recall that the number of pivot columns is the same as the number of nonzero rows from the description of this reduced row-echelon form.

**Exercise 11.E.56**

State whether each of the following sets of data are possible for the matrix equation $AX = B$. If possible, describe the solution set. That is, tell whether there exists a unique solution, no solution or infinitely many solutions. Here, $[A|B]$ denotes the augmented matrix.

1. $A$ is a $5 \times 6$ matrix, $\text{rank}(A) = 4$ and $\text{rank}[A|B] = 4$.
2. $A$ is a $3 \times 4$ matrix, $\text{rank}(A) = 3$ and $\text{rank}[A|B] = 2$.
3. $A$ is a $4 \times 2$ matrix, $\text{rank}(A) = 4$ and $\text{rank}[A|B] = 4$.
4. $A$ is a $5 \times 5$ matrix, $\text{rank}(A) = 4$ and $\text{rank}[A|B] = 5$.
5. $A$ is a $4 \times 2$ matrix, $\text{rank}(A) = 2$ and $\text{rank}[A|B] = 2$.

**Answer**

This says $B$ is in the span of four of the columns. Thus the columns are not independent. Infinite solution set.

This surely can't happen. If you add in another column, the rank does not get smaller.

This says $B$ is in the span of the columns and the columns must be independent. You can't have the rank equal 4 if you only have two columns.

This says $B$ is not in the span of the columns. In this case, there is no solution to the system of equations represented by the augmented matrix.

In this case, there is a unique solution since the columns of $A$ are independent.

### Exercise 11.$E$.57

Consider the system $-5x+2y-z=0$ and $-5x-2y-z=0$. Both equations equal zero and so $-5x+2y-z=-5x-2y-z$ which is equivalent to $y=0$. Does it follow that $x$ and $z$ can equal anything? Notice that when $x=1$, $z=-4$, and $y=0$ are plugged in to the equations, the equations do not equal 0. Why?

**Answer**

These are not legitimate row operations. They do not preserve the solution set of the system.

### Exercise 11.$E$.58

Balance the following chemical reactions.

1. $KNO_3+H_2CO_3\rightarrow K_2CO_3+HNO_3$
2. $AgI+Na_2S\rightarrow Ag_2S+NaI$
3. $Ba_3N_2+H_2O\rightarrow Ba(OH)_2+NH_3$
4. $CaCl_2+Na_3PO_4\rightarrow Ca_3(PO_4)_2+NaCl$

### Exercise 11.$E$.59

In the section on dimensionless variables it was observed that $\rho V^2AB$ has the units of force. Describe a systematic way to obtain such combinations of the variables which will yield something which has the units of force.

### Exercise 11.$E$.60

Consider the following diagram of four circuits.

The current in amps in the four circuits is denoted by $I_1, I_2, I_3, I_4$ and it is understood that the motion is in the counter clockwise direction. If $I_k$ ends up being negative, then it just means the current flows in the clockwise direction.

In the above diagram, the top left circuit should give the equation

$$2I_2-2I_1+5I_2-5I_3+3I_2=5 \tag{11.E.35}$$

For the circuit on the lower left, you should have

$$4I_1+I_1-I_4+2I_1-2I_2=-10 \tag{11.E.36}$$

Write equations for each of the other two circuits and then give a solution to the resulting system of equations.

**Answer**

The other two equations are

$$\begin{aligned} 6I_3-6I_4+I_3+I_3+5I_3-5I_2 &= -20 \\ 2I_4+3I_4+6I_4-6I_3+I_4-I_1 &= 0 \end{aligned}$$

Then the system is

$$\begin{aligned} 2I_2-2I_1+5I_2-5I_3+3I_2&=5 \\ 4I_1+I_1-I_4+2I_1-2I_2&=-10 \\ 6I_3-6I_4+I_3+I_3+5I_3-5I_2&=-20 \\ 2I_4+3I_4+6I_4-6I_3+I_4-I_1&=0 \end{aligned} \tag{11.E.37}$$

The solution is:

$$\begin{aligned} I_1 &= -\frac{750}{373} \\ I_2 &= -\frac{1421}{1119} \\ I_3 &= -\frac{3061}{1119} \\ I_4 &= -\frac{1718}{1119} \end{aligned}$$

**Exercise 11.$E$.61**

Consider the following diagram of three circuits.

The current in amps in the four circuits is denoted by $I_1, I_2, I_3$ and it is understood that the motion is in the counter clockwise direction. If $I_k$ ends up being negative, then it just means the current flows in the clockwise direction.

Find $I_1, I_2, I_3$.

**Answer**

You have

$$\begin{aligned} 2I_1+5I_1+3I_1-5I_2 &= 10 \\ I_2-I_3+3I_2+7I_2+5I_2-5I_1 &= -12 \\ 2I_3+4I_3+4I_3+I_3-I_2 &= 0 \end{aligned}$$

Simplifying this yields

$$\begin{aligned} 10I_1-5I_2 &= 10 \\ -5I_1+16I_2-I_3 &= -12 \\ -I_2+11I_3 &= 0 \end{aligned}$$

The solution is given by

$$I_1=\frac{218}{295}, I_2=-\frac{154}{295}, I_3=-\frac{14}{295} \tag{11.E.38}$$

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# CHAPTER OVERVIEW

## 12: Matrices and Determinants


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*Thumbnail: (via Wikipedia)*

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# 12.1: Matrix Arithmetic

## Outcomes

1. Perform the matrix operations of matrix addition, scalar multiplication, transposition and matrix multiplication. Identify when these operations are not defined. Represent these operations in terms of the entries of a matrix.
2. Prove algebraic properties for matrix addition, scalar multiplication, transposition, and matrix multiplication. Apply these properties to manipulate an algebraic expression involving matrices.
3. Compute the inverse of a matrix using row operations, and prove identities involving matrix inverses.
4. Solve a linear system using matrix algebra.
5. Use multiplication by an elementary matrix to apply row operations.
6. Write a matrix as a product of elementary matrices.

You have now solved systems of equations by writing them in terms of an augmented matrix and then doing row operations on this augmented matrix. It turns out that matrices are important not only for systems of equations but also in many applications.

Recall that a **matrix** is a rectangular array of numbers. Several of them are referred to as **matrices**. For example, here is a matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} \tag{12.1.1}$$

Recall that the size or dimension of a matrix is defined as $m \times n$ where $m$ is the number of rows and $n$ is the number of columns. The above matrix is a $3 \times 4$ matrix because there are three rows and four columns. You can remember the columns are like columns in a Greek temple. They stand upright while the rows lay flat like rows made by a tractor in a plowed field.

When specifying the size of a matrix, you always list the number of rows before the number of columns.

Consider the following definition.

### Definition 12.1.1: Square Matrix

A matrix $A$ which has size $n \times n$ is called a **square matrix** . In other words, $A$ is a square matrix if it has the same number of rows and columns.

There is some notation specific to matrices which we now introduce. We denote the columns of a matrix $A$ by $A_j$ as follows

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix} \tag{12.1.2}$$

Therefore, $A_j$ is the $j^{th}$ column of $A$, when counted from left to right.

The individual elements of the matrix are called **entries** or **components** of $A$. Elements of the matrix are identified according to their position. The $(\mathbf{i},\mathbf{j})$**-entry** of a matrix is the entry in the $i^{th}$ row and $j^{th}$ column. For example, in the matrix above, 8 is in position $(2,3)$ (and is called the $(2,3)$-entry) because it is in the second row and the third column.

In order to remember which matrix we are speaking of, we will denote the entry in the $i^{th}$ row and the $j^{th}$ column of matrix $A$ by $a_{ij}$. Then, we can write $A$ in terms of its entries, as $A = [a_{ij}]$. Using this notation on the matrix above, $a_{23} = 8, a_{32} = -9, a_{12} = 2,$ etc.

There are various operations which are done on matrices of appropriate sizes. Matrices can be added to and subtracted from other matrices, multiplied by a scalar, and multiplied by other matrices. We will never divide a matrix by another matrix, but we will see later how matrix inverses play a similar role.

In doing arithmetic with matrices, we often define the action by what happens in terms of the entries (or components) of the matrices. Before looking at these operations in depth, consider a few general definitions.

### Definition 12.1.2: The Zero Matrix

The $\boldsymbol{m \times n}$ **zero matrix** is the $m \times n$ matrix having every entry equal to zero. It is denoted by $0$.

One possible zero matrix is shown in the following example.

### Example 12.1.1: The Zero Matrix

The $2 \times 3$ zero matrix is $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note there is a $2 \times 3$ zero matrix, a $3 \times 4$ zero matrix, etc. In fact there is a zero matrix for every size!

### Definition 12.1.3:Equality of Matrices

The **$m \times n$ zero matrix** is the $m \times n$ matrix having every entry equal to zero. It is denoted by $0$.

Let $A$ and $B$ be two $m \times n$ matrices. Then $A = B$ means that for $A = [a_{ij}]$ and $B = [b_{ij}]$, $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

In other words, two matrices are equal exactly when they are the same size and the corresponding entries are identical. Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{12.1.3}$$

because they are different sizes. Also,

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \tag{12.1.4}$$

because, although they are the same size, their corresponding entries are not identical.

In the following section, we explore addition of matrices.

## Addition of Matrices

When adding matrices, all matrices in the sum need have the same size. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 2 \end{bmatrix} \tag{12.1.5}$$

and

$$\begin{bmatrix} -1 & 4 & 8 \\ 2 & 8 & 5 \end{bmatrix} \tag{12.1.6}$$

cannot be added, as one has size $3 \times 2$ while the other has size $2 \times 3$.

However, the addition

$$\begin{bmatrix} 4 & 6 & 3 \\ 5 & 0 & 4 \\ 11 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 4 & -4 & 14 \\ 1 & 2 & 6 \end{bmatrix} \tag{12.1.7}$$

is possible.

The formal definition is as follows.

### Definition 12.1.4: Addition of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A + B = C$ where $C$ is the $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij} \tag{12.1.8}$$

This definition tells us that when adding matrices, we simply add corresponding entries of the matrices. This is demonstrated in the next example.

### Example 12.1.2: Addition of Matrices of Same Size

Add the following matrices, if possible.

$$A=\begin{bmatrix}1 & 2 & 3\\ 1 & 0 & 4\end{bmatrix}, B=\begin{bmatrix}5 & 2 & 3\\ -6 & 2 & 1\end{bmatrix} \tag{12.1.9}$$

**Solution**

Notice that both $A$ and $B$ are of size $2\times 3$. Since $A$ and $B$ are of the same size, the addition is possible. Using the Addition of Matrices definition, the addition is done as follows.

$$A+B=\begin{bmatrix}1 & 2 & 3\\ 1 & 0 & 4\end{bmatrix}+\begin{bmatrix}5 & 2 & 3\\ -6 & 2 & 1\end{bmatrix}=\begin{bmatrix}1+5 & 2+2 & 3+3\\ 1+-6 & 0+2 & 4+1\end{bmatrix}=\begin{bmatrix}6 & 4 & 6\\ -5 & 2 & 5\end{bmatrix} \tag{12.1.10}$$

Addition of matrices obeys very much the same properties as normal addition with numbers. Note that when we write for example $A+B$ then we assume that both matrices are of equal size so that the operation is indeed possible.

### Theorem 12.1.1

Properties of Matrix Addition Let $A$, $B$ and $C$ be matrices. Then, the following properties hold.

- Commutative Law of Addition

$$A+B=B+A \tag{12.1.11}$$

- Associative Law of Addition

$$(A+B)+C=A+(B+C) \tag{12.1.12}$$

- Existence of an Additive Identity

$$\begin{gathered}\text{There exists a zero matrix } 0 \text{ such that}\\ A+0=A\end{gathered} \tag{12.1.13}$$

- Existence of an Additive Inverse

$$\begin{gathered}\text{There exists a matrix } -A \text{ such that}\\ A+(-A)=0\end{gathered} \tag{12.1.14}$$

**Proof**

Consider the Commutative Law of Addition given in 12.1.11 Let $A, B, C$, and $D$ be matrices such that $A+B=C$ and $B+A=D$. We want to show that $D=C$. To do so, we will use the definition of matrix addition given in Addition of Matrices definition. Now,

$$c_{ij}=a_{ij}+b_{ij}=b_{ij}+a_{ij}=d_{ij} \tag{12.1.15}$$

Therefore, $C=D$ because the $ij^{th}$ entries are the same for all $i$ and $j$. Note that the conclusion follows from the commutative law of addition of numbers, which says that if $a$ and $b$ are two numbers, then $a+b=b+a$ . The proof of the other results are similar, and are left as an exercise.

We call the zero matrix in 12.1.13the **additive identity**. Similarly, we call the matrix $-A$ in 12.1.14the **additive inverse**. $-A$ is defined to equal $(-1)A=[-a_{ij}]$. In other words, every entry of $A$ is multiplied by $-1$.

In the next section we will study scalar multiplication in more depth to understand what is meant by $(-1)A$.

## Scalar Multiplication of Matrices

Recall that we use the word *scalar* when referring to numbers. Therefore, *scalar multiplication of a matrix* is the multiplication of a matrix by a number. To illustrate this concept, consider the following example in which a matrix is multiplied by the scalar 3.

$$3\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 15 & 6 & 24 & 21 \\ 18 & -27 & 3 & 6 \end{bmatrix} \tag{12.1.16}$$

The new matrix is obtained by multiplying every entry of the original matrix by the given scalar.

The formal definition of scalar multiplication is as follows.

### Definition 12.1.5: Scalar Multiplication of Matrices

If $A = [a_{ij}]$ and $k$ is a scalar, then $kA = [ka_{ij}]$.

Consider the following example.

### Example 12.1.3: Effect of Multiplication by a Scalar

Find the result of multiplying the following matrix $A$ by 7.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} \tag{12.1.17}$$

**Solution**

By Definition 12.1.5, we multiply each element of $A$ by 7. Therefore,

$$7A = 7\begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(0) \\ 7(1) & 7(-4) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 7 & -28 \end{bmatrix} \tag{12.1.18}$$

Similarly to addition of matrices, there are several properties of scalar multiplication which hold.

### Theorem 12.1.2: Properties of Scalar Multiplication

Let $A, B$ be matrices, and $k, p$ be scalars. Then, the following properties hold.

- Distributive Law over Matrix Addition

$$k(A+B) = kA + kB \tag{12.1.19}$$

- Distributive Law over Scalar Addition

$$(k+p)A = kA + pA \tag{12.1.20}$$

- Associative Law for Scalar Multiplication

$$k(pA) = (kp)A \tag{12.1.21}$$

- Rule for Multiplication by 1

$$1A = A \tag{12.1.22}$$

**Proof**

The proof of this proposition is similar to the proof of the Properties of Matrix Addition and is left an exercise to the reader.

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# 12.2: Multiplication of Matrices

The next important matrix operation we will explore is multiplication of matrices. The operation of matrix multiplication is one of the most important and useful of the matrix operations. Throughout this section, we will also demonstrate how matrix multiplication relates to linear systems of equations.

First, we provide a formal definition of row and column vectors.

**Definition 12.2.1: Row and Column Vectors**

Matrices of size $n \times 1$ or $1 \times n$ are called **vectors**. If $X$ is such a matrix, then we write $x_i$ to denote the entry of $X$ in the $i^{th}$ row of a column matrix, or the $i^{th}$ column of a row matrix.

The $n \times 1$ matrix

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called a **column vector.** The $1 \times n$ matrix

$$X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

is called a **row vector**.

We may simply use the term **vector** throughout this text to refer to either a column or row vector. If we do so, the context will make it clear which we are referring to.

In this chapter, we will again use the notion of linear combination of vectors as in Definition 9.2.2. In this context, a linear combination is a sum consisting of vectors multiplied by scalars. For example,

$$\begin{bmatrix} 50 \\ 122 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

is a linear combination of three vectors.

It turns out that we can express any system of linear equations as a linear combination of vectors. In fact, the vectors that we will use are just the columns of the corresponding augmented matrix!

**Definition 12.2.2: The Vector Form of a System of Linear Equations**

Suppose we have a system of equations given by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We can express this system in **vector form** which is as follows:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Notice that each vector used here is one column from the corresponding augmented matrix. There is one vector for each variable in the system, along with the constant vector.

The first important form of matrix multiplication is multiplying a matrix by a vector. Consider the product given by

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

We will soon see that this equals

$$7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix}$$

In general terms,

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} \end{aligned}$$

Thus you take $x_1$ times the first column, add to $x_2$ times the second column, and finally $x_3$ times the third column. The above sum is a linear combination of the columns of the matrix. When you multiply a matrix on the left by a vector on the right, the numbers making up the vector are just the scalars to be used in the linear combination of the columns as illustrated above.

Here is the formal definition of how to multiply an $m \times n$ matrix by an $n \times 1$ column vector.

**Definition 12.2.3: Multiplication of Vector by Matrix**

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $X$ be an $n \times 1$ matrix given by

$$A = [A_1 \cdots A_n], X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then the product $AX$ is the $m \times 1$ column vector which equals the following linear combination of the columns of $A$:

$$x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = \sum_{j=1}^{n} x_j A_j$$

If we write the columns of $A$ in terms of their entries, they are of the form

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

Then, we can write the product $AX$ as

$$AX = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Note that multiplication of an $m \times n$ matrix and an $n \times 1$ vector produces an $m \times 1$ vector.

Here is an example.

### Example 12.2.1: A Vector Multiplied by a Matrix

Compute the product $AX$ for

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{bmatrix}, X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

**Solution**

We will use Definition 12.2.3 to compute the product. Therefore, we compute the product $AX$ as follows.

$$\begin{aligned} &1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix} \end{aligned}$$

Using the above operation, we can also write a system of linear equations in **matrix form**. In this form, we express the system as a matrix multiplied by a vector. Consider the following definition.

### Definition 12.2.4: The Matrix Form of a System of Linear Equations

Suppose we have a system of equations given by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Then we can express this system in **matrix form** as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The expression $AX = B$ is also known as the **Matrix Form** of the corresponding system of linear equations. The matrix $A$ is simply the coefficient matrix of the system, the vector $X$ is the column vector constructed from the variables of the system, and finally the vector $B$ is the column vector constructed from the constants of the system. It is important to note that any system of linear equations can be written in this form.

Notice that if we write a homogeneous system of equations in matrix form, it would have the form $AX = 0$, for the zero vector $0$.

You can see from this definition that a vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

will satisfy the equation $AX = B$ only when the entries $x_1, x_2, \cdots, x_n$ of the vector $X$ are solutions to the original system.

Now that we have examined how to multiply a matrix by a vector, we wish to consider the case where we multiply two matrices of more general sizes, although these sizes still need to be appropriate as we will see. For example, in Example 12.2.1, we multiplied a $3 \times 4$ matrix by a $4 \times 1$ vector. We want to investigate how to multiply other sizes of matrices.

We have not yet given any conditions on when matrix multiplication is possible! For matrices $A$ and $B$, in order to form the product $AB$, the number of columns of $A$ must equal the number of rows of $B$. Consider a product $AB$ where $A$ has size $m \times n$ and $B$ has size $n \times p$. Then, the product in terms of size of matrices is given by

$$(m \times \overbrace{\quad n)\,(n}^{\text{these must match!}} \times p \quad) = m \times p$$

Note the two outside numbers give the size of the product. One of the most important rules regarding matrix multiplication is the following. If the two middle numbers don't match, you can't multiply the matrices!

When the number of columns of $A$ equals the number of rows of $B$ the two matrices are said to be **conformable** and the product $AB$ is obtained as follows.

### Definition 12.2.5: Multiplication of Two Matrices

Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times p$ matrix of the form

$$B = [B_1 \cdots B_p]$$

where $B_1, \ldots, B_p$ are the $n \times 1$ columns of $B$. Then the $m \times p$ matrix $AB$ is defined as follows:

$$AB = A\,[B_1 \cdots B_p] = [(AB)_1 \cdots (AB)_p]$$

where $(AB)_k$ is an $m \times 1$ matrix or column vector which gives the $k^{th}$ column of $AB$.

Consider the following example.

### Example 12.2.2: Multiplying Two Matrices

Find $AB$ if possible.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

**Solution**

The first thing you need to verify when calculating a product is whether the multiplication is possible. The first matrix has size $2 \times 3$ and the second matrix has size $3 \times 3$. The inside numbers are equal, so $A$ and $B$ are conformable matrices. According to the above discussion $AB$ will be a $2 \times 3$ matrix. Definition 12.2.5 gives us a way to calculate each column of $AB$, as follows.

$$\begin{bmatrix} \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}^{\text{First column}}, & \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}}^{\text{Second column}}, & \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}^{\text{Third column}} \end{bmatrix}$$

You know how to multiply a matrix times a vector, using Definition 12.2.3 for each of the three columns. Thus

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}$$

Since vectors are simply $n \times 1$ or $1 \times m$ matrices, we can also multiply a vector by another vector.

### Example 12.2.3: Vector Times Vector Multiplication

Multiply if possible $\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 1 & 0\end{bmatrix}$.

**Solution**

In this case we are multiplying a matrix of size $3\times 1$ by a matrix of size $1\times 4$. The inside numbers match so the product is defined. Note that the product will be a matrix of size $3\times 4$. Using Definition 12.2.5, we can compute this product as follows

$$\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 1 & 0\end{bmatrix}=\begin{bmatrix}\overbrace{\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}1\end{bmatrix}}^{\text{First column}}, & \overbrace{\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}2\end{bmatrix}}^{\text{Second column}}, & \overbrace{\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}1\end{bmatrix}}^{\text{Third column}}, & \overbrace{\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}0\end{bmatrix}}^{\text{Fourth column}}\end{bmatrix}$$

You can use Definition 12.2.3 to verify that this product is

$$\begin{bmatrix}1 & 2 & 1 & 0\\2 & 4 & 2 & 0\\1 & 2 & 1 & 0\end{bmatrix}$$

### Example 12.2.4: A Multiplication Which is Not Defined

Find $BA$ if possible.

$$B=\begin{bmatrix}1 & 2 & 0\\0 & 3 & 1\\-2 & 1 & 1\end{bmatrix},A=\begin{bmatrix}1 & 2 & 1\\0 & 2 & 1\end{bmatrix}$$

**Solution**

First check if it is possible. This product is of the form $(3\times 3)(2\times 3)$. The inside numbers do not match and so you can't do this multiplication.

In this case, we say that the multiplication is not defined. Notice that these are the same matrices which we used in Example 12.2.2. In this example, we tried to calculate $BA$ instead of $AB$. This demonstrates another property of matrix multiplication. While the product $AB$ maybe be defined, we cannot assume that the product $BA$ will be possible. Therefore, it is important to always check that the product is defined before carrying out any calculations.

Earlier, we defined the zero matrix $0$ to be the matrix (of appropriate size) containing zeros in all entries. Consider the following example for multiplication by the zero matrix.

### Example 12.2.5: Multiplication by the Zero Matrix

Compute the product $A0$ for the matrix

$$A=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$$

and the $2\times 2$ zero matrix given by

$$0=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

**Solution**

In this product, we compute

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A0 = 0$.

Notice that we could also multiply $A$ by the $2 \times 1$ zero vector given by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The result would be the $2 \times 1$ zero vector. Therefore, it is always the case that $A0 = 0$, for an appropriately sized zero matrix or vector.

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# 12.3: The ijth Entry of a Product

In previous sections, we used the entries of a matrix to describe the action of matrix addition and scalar multiplication. We can also study matrix multiplication using the entries of matrices.

What is the $ij^{th}$ entry of $AB$? It is the entry in the $i^{th}$ row and the $j^{th}$ column of the product $AB$.

Now if $A$ is $m \times n$ and $B$ is $n \times p$, then we know that the product $AB$ has the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

The $j^{th}$ column of $AB$ is of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

which is an $m \times 1$ column vector. It is calculated by

$$b_{1j} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + b_{2j} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + b_{nj} \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Therefore, the $ij^{th}$ entry is the entry in row $i$ of this vector. This is computed by

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

The following is the formal definition for the $ij^{th}$ entry of a product of matrices.

**Definition 12.3.1: The $ij^{th}$ Entry of a Product**

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $B = [b_{ij}]$ be an $n \times p$ matrix. Then $AB$ is an $m \times p$ matrix and the $(i, j)$-entry of $AB$ is defined as

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Another way to write this is

$$(AB)_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

In other words, to find the $(i, j)$-entry of the product $AB$, or $(AB)_{ij}$, you multiply the $i^{th}$ row of $A$, on the left by the $j^{th}$ column of $B$. To express $AB$ in terms of its entries, we write $AB = [(AB)_{ij}]$.

Consider the following example.

### Example 12.3.1: The Entries of a Product

Compute $AB$ if possible. If it is, find the $(3,2)$-entry of $AB$ using Definition 12.3.1.

$$A=\begin{bmatrix}1&2\\3&1\\2&6\end{bmatrix}, B=\begin{bmatrix}2&3&1\\7&6&2\end{bmatrix}$$

**Solution**

First check if the product is possible. It is of the form $(3\times 2)(2\times 3)$ and since the inside numbers match, it is possible to do the multiplication. The result should be a $3\times 3$ matrix. We can first compute $AB$:

$$\left[\begin{bmatrix}1&2\\3&1\\2&6\end{bmatrix}\begin{bmatrix}2\\7\end{bmatrix}, \begin{bmatrix}1&2\\3&1\\2&6\end{bmatrix}\begin{bmatrix}3\\6\end{bmatrix}, \begin{bmatrix}1&2\\3&1\\2&6\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\right]$$

where the commas separate the columns in the resulting product. Thus the above product equals

$$\begin{bmatrix}16&15&5\\13&15&5\\46&42&14\end{bmatrix}$$

which is a $3\times 3$ matrix as desired. Thus, the $(3,2)$-entry equals 42.

Now using Definition 12.3.1, we can find that the $(3,2)$-entry equals

$$\begin{aligned}\sum_{k=1}^{2} a_{3k}b_{k2} &= a_{31}b_{12}+a_{32}b_{22}\\ &= 2\times 3+6\times 6=42\end{aligned}$$

Consulting our result for $AB$ above, this is correct!

You may wish to use this method to verify that the rest of the entries in $AB$ are correct.

Here is another example.

### Example 12.3.2: Finding the Entries of a Product

Determine if the product $AB$ is defined. If it is, find the $(2,1)$-entry of the product.

$$A=\begin{bmatrix}2&3&1\\7&6&2\\0&0&0\end{bmatrix}, B=\begin{bmatrix}1&2\\3&1\\2&6\end{bmatrix}$$

**Solution**

This product is of the form $(3\times 3)(3\times 2)$. The middle numbers match so the matrices are conformable and it is possible to compute the product.

We want to find the $(2,1)$-entry of $AB$, that is, the entry in the second row and first column of the product. We will use Definition 12.3.1, which states

$$(AB)_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$$

In this case, $n=3$, $i=2$ and $j=1$. Hence the $(2,1)$-entry is found by computing

$$(AB)_{21}=\sum_{k=1}^{3} a_{2k}b_{k1}=\begin{bmatrix}a_{21}&a_{22}&a_{23}\end{bmatrix}\begin{bmatrix}b_{11}\\b_{21}\\b_{31}\end{bmatrix}$$

Substituting in the appropriate values, this product becomes

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 7 & 6 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 1 \times 7 + 3 \times 6 + 2 \times 2 = 29$$

Hence, $(AB)_{21} = 29$.

You should take a moment to find a few other entries of $AB$. You can multiply the matrices to check that your answers are correct. The product $AB$ is given by

$$AB = \begin{bmatrix} 13 & 13 \\ 29 & 32 \\ 0 & 0 \end{bmatrix}$$

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# 12.4: Properties of Matrix Multiplication

As pointed out above, it is sometimes possible to multiply matrices in one order but not in the other order. However, even if both $AB$ and $BA$ are defined, they may not be equal.

**Example 12.4.1: Matrix Multiplication is Not Commutative**

Compare the products $AB$ and $BA$, for matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

**Solution**

First, notice that $A$ and $B$ are both of size $2 \times 2$. Therefore, both products $AB$ and $BA$ are defined. The first product, $AB$ is

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

The second product, $BA$ is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Therefore, $AB \neq BA$.

This example illustrates that you cannot assume $AB = BA$ even when multiplication is defined in both orders. If for some matrices $A$ and $B$ it is true that $AB = BA$, then we say that $A$ and $B$ **commute**. This is one important property of matrix multiplication.

The following are other important properties of matrix multiplication. Notice that these properties hold only when the size of matrices are such that the products are defined.

**Proposition 12.4.1: Properties of Matrix Multiplication**

The following hold for matrices $A, B$, and $C$ and for scalars $r$ and $s$,

$$A(rB + sC) = r(AB) + s(AC) \tag{12.4.1}$$

$$(B + C)A = BA + CA \tag{12.4.2}$$

$$A(BC) = (AB)C \tag{12.4.3}$$

**Proof**

First we will prove (12.4.1). We will use Definition 2.3.1 and prove this statement using the $ij^{th}$ entries of a matrix. Therefore,

$$\begin{aligned} (A(rB + sC))_{ij} &= \sum_k a_{ik} (rB + sC)_{kj} \\ &= \sum_k a_{ik} (rb_{kj} + sc_{kj}) \\ &= r \sum_k a_{ik} b_{kj} + s \sum_k a_{ik} c_{kj} \\ &= r(AB)_{ij} + s(AC)_{ij} \\ &= (r(AB) + s(AC))_{ij} \end{aligned}$$

Thus $A(rB + sC) = r(AB) + s(AC)$ as claimed.

The proof of Equation (12.4.2) follows the same pattern and is left as an exercise.

Statement Equation (12.4.3) is the **associative law of multiplication**. Using Definition 2.3.1,

$$\begin{aligned}(A(BC))_{ij} &= \sum_k a_{ik}(BC)_{kj} \\ &= \sum_k a_{ik} \sum_l b_{kl} c_{lj} \\ &= \sum_l (AB)_{il} c_{lj} = ((AB)C)_{ij}.\end{aligned}$$

This proves (12.4.3)

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# 12.5: The Transpose

Another important operation on matrices is that of taking the **transpose**. For a matrix $A$, we denote the **transpose** of $A$ by $A^T$. Before formally defining the transpose, we explore this operation on the following matrix.

$$\begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \end{bmatrix}$$

What happened? The first column became the first row and the second column became the second row. Thus the $3 \times 2$ matrix became a $2 \times 3$ matrix. The number $4$ was in the first row and the second column and it ended up in the second row and first column.

The definition of the transpose is as follows.

**Definition 12.5.1: The Transpose of a Matrix**

Let $A$ be an $m \times n$ matrix. Then $A^T$, the **transpose** of $A$, denotes the $n \times m$ matrix given by

$$A^T = [a_{ij}]^T = [a_{ji}]$$

The $(i, j)$-entry of $A$ becomes the $(j, i)$-entry of $A^T$.

Consider the following example.

**Example 12.5.1: The Transpose of a Matrix**

Calculate $A^T$ for the following matrix

$$A = \begin{bmatrix} 1 & 2 & -6 \\ 3 & 5 & 4 \end{bmatrix}$$

**Solution**

By Definition 12.5.1, we know that for $A = [a_{ij}]$, $A^T = [a_{ji}]$. In other words, we switch the row and column location of each entry. The $(1, 2)$-entry becomes the $(2, 1)$-entry.

Thus,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -6 & 4 \end{bmatrix}$$

Notice that $A$ is a $2 \times 3$ matrix, while $A^T$ is a $3 \times 2$ matrix.

The transpose of a matrix has the following important properties.

**Lemma 12.5.1: Properties of the Transpose of a Matrix**

Let $A$ be an $m \times n$ matrix, $B$ an $n \times p$ matrix, and $r$ and $s$ scalars. Then

1. $$\left(A^T\right)^T = A$$
2. $$(AB)^T = B^T A^T$$
3. $$(rA + sB)^T = rA^T + sB^T$$

**Proof**

First we prove 2. From Definition 12.5.1,

$$(AB)^T = [(AB)_{ij}]^T = [(AB)_{ji}] = \sum_k a_{jk} b_{ki} = \sum_k b_{ki} a_{jk}$$
$$= \sum_k [b_{ik}]^T [a_{kj}]^T = [b_{ij}]^T [a_{ij}]^T = B^T A^T$$

The proof of Formula 3 is left as an exercise.

The transpose of a matrix is related to other important topics. Consider the following definition.

**Definition 12.5.2: Symmetric and Skew Symmetric Matrices**

An $n \times n$ matrix $A$ is said to be **symmetric** if $A = A^T$. It is said to be **skew symmetric** if $A = -A^T$.

We will explore these definitions in the following examples.

**Example 12.5.2: Symmetric Matrices**

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Use Definition 12.5.2 to show that $A$ is symmetric.

**Solution**

By Definition 12.5.2, we need to show that $A = A^T$. Now, using Definition 12.5.1,

$$A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Hence, $A = A^T$, so $A$ is symmetric.

**Example 12.5.3: A Skew Symmetric Matrix**

Let

$$A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

Show that $A$ is skew symmetric.

**Solution**

By Definition 12.5.2,

$$A^T = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

You can see that each entry of $A^T$ is equal to $-1$ times the same entry of $A$. Hence, $A^T = -A$ and so by Definition 12.5.2, $A$ is skew symmetric.

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# 12.6: The Identity and Inverses

There is a special matrix, denoted $I$, which is called to as the **identity matrix**. The identity matrix is always a square matrix, and it has the property that there are ones down the main diagonal and zeroes elsewhere. Here are some identity matrices of various sizes.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first is the $1 \times 1$ identity matrix, the second is the $2 \times 2$ identity matrix, and so on. By extension, you can likely see what the $n \times n$ identity matrix would be. When it is necessary to distinguish which size of identity matrix is being discussed, we will use the notation $I_n$ for the $n \times n$ identity matrix.

The identity matrix is so important that there is a special symbol to denote the $ij^{th}$ entry of the identity matrix. This symbol is given by $I_{ij} = \delta_{ij}$ where $\delta_{ij}$ is the **Kronecker symbol** defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

$I_n$ is called the **identity matrix** because it is a **multiplicative identity** in the following sense.

**Lemma 12.6.1: Multiplication by the Identity Matrix**

Suppose $A$ is an $m \times n$ matrix and $I_n$ is the $n \times n$ identity matrix**.** Then $AI_n = A$. If $I_m$ is the $m \times m$ identity matrix, it also follows that $I_m A = A$.

**Proof**

The $(i, j)$-entry of $AI_n$ is given by:

$$\sum_k a_{ik}\delta_{kj} = a_{ij}$$

and so $AI_n = A$. The other case is left as an exercise for you.

We now define the matrix operation which in some ways plays the role of division.

**Definition 12.6.1: The Inverse of a Matrix**

A square $n \times n$ matrix $A$ is said to have an **inverse** $A^{-1}$ if and only if

$$AA^{-1} = A^{-1}A = I_n$$

In this case, the matrix $A$ is called **invertible**.

Such a matrix $A^{-1}$ will have the same size as the matrix $A$. It is very important to observe that the inverse of a matrix, if it exists, is unique. Another way to think of this is that if it acts like the inverse, then it **is** the inverse.

**Theorem 12.6.1: Uniqueness of Inverse**

Suppose $A$ is an $n \times n$ matrix such that an inverse $A^{-1}$ exists. Then there is only one such inverse matrix. That is, given any matrix $B$ such that $AB = BA = I$ , $B = A^{-1}$ .

**Proof**

In this proof, it is assumed that $I$ is the $n \times n$ identity matrix. Let $A, B$ be $n \times n$ matrices such that $A^{-1}$ exists and $AB = BA = I$ . We want to show that $A^{-1} = B$ . Now using properties we have seen, we get:

$$A^{-1} = A^{-1}I = A^{-1}(AB) = \left(A^{-1}A\right)B = IB = B$$

Hence, $A^{-1} = B$ which tells us that the inverse is unique.

The next example demonstrates how to check the inverse of a matrix.

**Example 12.6.1: Verifying the Inverse of a Matrix**

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Show $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is the inverse of $A$.

**Solution**

To check this, multiply

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

showing that this matrix is indeed the inverse of $A$.

Unlike ordinary multiplication of numbers, it can happen that $A \neq 0$ but $A$ may fail to have an inverse. This is illustrated in the following example.

**Example 12.6.2: A Nonzero Matrix With No Inverse**

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Show that $A$ does not have an inverse.

**Solution**

One might think $A$ would have an inverse because it does not equal zero. However, note that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $A^{-1}$ existed, we would have the following

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= A^{-1} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\ &= A^{-1} \left( A \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) \\ &= \left( A^{-1} A \right) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= I \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

This says that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which is impossible! Therefore, $A$ does not have an inverse.

In the next section, we will explore how to find the inverse of a matrix, if it exists.

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# 12.7: Finding the Inverse of a Matrix

In Example 2.6.1, we were given $A^{-1}$ and asked to verify that this matrix was in fact the inverse of $A$. In this section, we explore how to find $A^{-1}$.

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

as in Example 2.6.1. In order to find $A^{-1}$, we need to find a matrix $\begin{bmatrix} x & z \\ y & w \end{bmatrix}$ such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can multiply these two matrices, and see that in order for this equation to be true, we must find the solution to the systems of equations,

$$\begin{aligned} x + y &= 1 \\ x + 2y &= 0 \end{aligned}$$

and

$$\begin{aligned} z + w &= 0 \\ z + 2w &= 1 \end{aligned}$$

Writing the augmented matrix for these two systems gives

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right]$$

for the first system and

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 1 \end{array}\right] \tag{12.7.1}$$

for the second.

Let's solve the first system. Take $-1$ times the first row and add to the second to get

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

Now take $-1$ times the second row and add to the first to get

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

Writing in terms of variables, this says $x = 2$ and $y = -1$.

Now solve the second system, (12.7.1) to find $z$ and $w$. You will find that $z = -1$ and $w = 1$.

If we take the values found for $x, y, z,$ and $w$ and put them into our inverse matrix, we see that the inverse is

$$A^{-1} = \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

After taking the time to solve the second system, you may have noticed that exactly the same row operations were used to solve both systems. In each case, the end result was something of the form $[I|X]$ where $I$ is the identity and $X$ gave a column of the inverse. In the above,

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

the first column of the inverse was obtained by solving the first system and then the second column

$$\begin{bmatrix} z \\ w \end{bmatrix}$$

To simplify this procedure, we could have solved both systems at once! To do so, we could have written

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

and row reduced until we obtained

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

and read off the inverse as the $2 \times 2$ matrix on the right side.

This exploration motivates the following important algorithm.

**Algorithm 12.7.1: Matrix Inverse Algorithm**

Suppose $A$ is an $n \times n$ matrix. To find $A^{-1}$ if it exists, form the augmented $n \times 2n$ matrix

$$[A|I]$$

If possible do row operations until you obtain an $n \times 2n$ matrix of the form

$$[I|B]$$

When this has been done, $B = A^{-1}$. In this case, we say that $A$ is **invertible**. If it is impossible to row reduce to a matrix of the form $[I|B]$, then $A$ has no inverse.

This algorithm shows how to find the inverse if it exists. It will also tell you if $A$ does not have an inverse.

Consider the following example.

**Example 12.7.1: Finding the Inverse**

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$. Find $A^{-1}$ if it exists.

**Solution**

Set up the augmented matrix

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Now we row reduce, with the goal of obtaining the $3 \times 3$ identity matrix on the left hand side. First, take $-1$ times the first row and add to the second followed by $-3$ times the first row added to the third row. This yields

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right]$$

Then take 5 times the second row and add to -2 times the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right]$$

Next take the third row and add to $-7$ times the first row. This yields

$$\left[\begin{array}{ccc|ccc} -7 & -14 & 0 & -6 & 5 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right]$$

Now take $-\frac{7}{5}$ times the second row and add to the first row.

$$\left[\begin{array}{ccc|ccc} -7 & 0 & 0 & 1 & -2 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right]$$

Finally divide the first row by -7, the second row by -10 and the third row by 14 which yields

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right]$$

Notice that the left hand side of this matrix is now the $3 \times 3$ identity matrix $I_3$. Therefore, the inverse is the $3 \times 3$ matrix on the right hand side, given by

$$\left[\begin{array}{ccc} -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right]$$

It may happen that through this algorithm, you discover that the left hand side cannot be row reduced to the identity matrix. Consider the following example of this situation.

### Example 12.7.2: A Matrix Which Has No Inverse

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. Find $A^{-1}$ if it exists.

**Solution**

Write the augmented matrix $[A|I]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

and proceed to do row operations attempting to obtain $\left[I|A^{-1}\right]$. Take $-1$ times the first row and add to the second. Then take $-2$ times the first row and add to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 1 \end{array}\right]$$

Next add $-1$ times the second row to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

At this point, you can see there will be no way to obtain $I$ on the left side of this augmented matrix. Hence, there is no way to complete this algorithm, and therefore the inverse of $A$ does not exist. In this case, we say that $A$ is not invertible.

If the algorithm provides an inverse for the original matrix, it is always possible to check your answer. To do so, use the method demonstrated in Example 2.6.1. Check that the products $AA^{-1}$ and $A^{-1}A$ both equal the identity matrix. Through this method, you can always be sure that you have calculated $A^{-1}$ properly!

One way in which the inverse of a matrix is useful is to find the solution of a system of linear equations. Recall from Definition 2.2.4 that we can write a system of equations in matrix form, which is of the form $AX = B$. Suppose you find the inverse of the matrix $A^{-1}$. Then you could multiply both sides of this equation on the left by $A^{-1}$ and simplify to obtain

$$\begin{aligned} \left(A^{-1}\right) AX &= A^{-1}B \\ \left(A^{-1}A\right) X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Therefore we can find $X$, the solution to the system, by computing $X = A^{-1}B$. Note that once you have found $A^{-1}$, you can easily get the solution for different right hand sides (different $B$). It is always just $A^{-1}B$.

We will explore this method of finding the solution to a system in the following example.

### Example 12.7.3: Using the Inverse to Solve a System of Equations

Consider the following system of equations. Use the inverse of a suitable matrix to give the solutions to this system.

$$\begin{aligned} x + z &= 1 \\ x - y + z &= 3 \\ x + y - z &= 2 \end{aligned}$$

**Solution**

First, we can write the system of equations in matrix form

$$AX = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = B \tag{12.7.2}$$

The inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

is

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Verifying this inverse is left as an exercise.

From here, the solution to the given system (12.7.2) is found by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -2 \\ -\frac{3}{2} \end{bmatrix}$$

What if the right side, $B$, of (12.7.2) had been $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$? In other words, what would be the solution to

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}?$$

By the above discussion, the solution is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

This illustrates that for a system $AX = B$ where $A^{-1}$ exists, it is easy to find the solution when the vector $B$ is changed.

We conclude this section with some important properties of the inverse.

**Theorem 12.7.1: Inverses of Transposes and Products**

Let $A, B$, and $A_i$ for $i = 1, \ldots, k$ be $n \times n$ matrices.

1. If $A$ is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$
2. If $A$ and $B$ are invertible matrices, then $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. If $A_1, A_2, \ldots, A_k$ are invertible, then the product $A_1 A_2 \cdots A_k$ is invertible, and $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$

Consider the following theorem.

**Theorem 12.7.2: Properties of the Inverse**

Let $A$ be an $n \times n$ matrix and $I$ the usual identity matrix.

1. $I$ is invertible and $I^{-1} = I$
2. If $A$ is invertible then so is $A^{-1}$, and $(A^{-1})^{-1} = A$
3. If $A$ is invertible then so is $A^k$, and $(A^k)^{-1} = (A^{-1})^k$
4. If $A$ is invertible and $p$ is a nonzero real number, then $pA$ is invertible and $(pA)^{-1} = \frac{1}{p}A^{-1}$

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# 12.8: Basic Techniques of Determinants

## Cofactors and 2 x 2 Determinants

**Outcomes**

A. Evaluate the determinant of a square matrix using either Laplace Expansion or row operations.
B. Demonstrate the effects that row operations have on determinants.
C. Verify the following:
   a. The determinant of a product of matrices is the product of the determinants.
   b. The determinant of a matrix is equal to the determinant of its transpose.

Let $A$ be an $n \times n$ matrix. That is, let $A$ be a square matrix. The **determinant** of $A$, denoted by $\det(A)$ is a very important number which we will explore throughout this section.

If $A$ is a $2 \times 2$ matrix, the determinant is given by the following formula.

**Definition 12.8.1: Determinant of a Two By Two Matrix**

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\det(A) = ad - cb$$

The determinant is also often denoted by enclosing the matrix with two vertical lines. Thus

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The following is an example of finding the determinant of a $2 \times 2$ matrix.

**Example 12.8.1: A Two by Two Determinant**

Find $\det(A)$ for the matrix $A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$.

**Solution**

From Definition 12.8.1,

$$\det(A) = (2)(6) - (-1)(4) = 12 + 4 = 16$$

The $2 \times 2$ determinant can be used to find the determinant of larger matrices. We will now explore how to find the determinant of a $3 \times 3$ matrix, using several tools including the $2 \times 2$ determinant.

We begin with the following definition.

**Definition 12.8.2: The $ij^{th}$ Minor of a Matrix**

Let $A$ be a $3 \times 3$ matrix. The $ij^{th}$ **minor** of $A$, denoted as $minor(A)_{ij}$, is the determinant of the $2 \times 2$ matrix which results from deleting the $i^{th}$ row and the $j^{th}$ column of $A$.

In general, if $A$ is an $n \times n$ matrix, then the $ij^{th}$ minor of $A$ is the determinant of the $n-1 \times n-1$ matrix which results from deleting the $i^{th}$ row and the $j^{th}$ column of $A$.

Hence, there is a minor associated with each entry of $A$. Consider the following example which demonstrates this definition.

### Example 12.8.2: Finding Minors of a Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $minor(A)_{12}$ and $minor(A)_{23}$.

**Solution**

First we will find $minor(A)_{12}$. By Definition 12.8.2, this is the determinant of the $2 \times 2$ matrix which results when you delete the first row and the second column. This minor is given by

$$minor(A)_{12} = \det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

Using Definition 12.8.1, we see that

$$\det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = (4)(1) - (3)(2) = 4 - 6 = -2$$

Therefore $minor(A)_{12} = -2$.

Similarly, $minor(A)_{23}$ is the determinant of the $2 \times 2$ matrix which results when you delete the second row and the third column. This minor is therefore

$$minor(A)_{23} = \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

Finding the other minors of $A$ is left as an exercise.

The $ij^{th}$ minor of a matrix $A$ is used in another important definition, given next.

### Definition 12.8.3: The $ij^{th}$ Cofactor of a Matrix

Suppose $A$ is an $n \times n$ matrix. The $ij^{th}$ **cofactor**, denoted by $\mathrm{cof}(A)_{ij}$ is defined to be

$$\mathrm{cof}(A)_{ij} = (-1)^{i+j} minor(A)_{ij}$$

It is also convenient to refer to the cofactor of an entry of a matrix as follows. If $a_{ij}$ is the $ij^{th}$ entry of the matrix, then its cofactor is just $\mathrm{cof}(A)_{ij}$.

### Example 12.8.3: Finding Cofactors of a Matrix

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\mathrm{cof}(A)_{12}$ and $\mathrm{cof}(A)_{23}$.

**Solution**

We will use Definition 12.8.3 to compute these cofactors.

First, we will compute $\mathrm{cof}(A)_{12}$. Therefore, we need to find $minor(A)_{12}$. This is the determinant of the $2 \times 2$ matrix which results when you delete the first row and the second column. Thus $minor(A)_{12}$ is given by

$$\det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = -2$$

Then,

$$\text{cof}(A)_{12} = (-1)^{1+2} minor(A)_{12} = (-1)^{1+2}(-2) = 2$$

Hence, $\text{cof}(A)_{12} = 2$.

Similarly, we can find $\text{cof}(A)_{23}$. First, find $minor(A)_{23}$, which is the determinant of the $2 \times 2$ matrix which results when you delete the second row and the third column. This minor is therefore

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

Hence,

$$\text{cof}(A)_{23} = (-1)^{2+3} minor(A)_{23} = (-1)^{2+3}(-4) = 4$$

You may wish to find the remaining cofactors for the above matrix. Remember that there is a cofactor for every entry in the matrix.

We have now established the tools we need to find the determinant of a $3 \times 3$ matrix.

**Definition 12.8.4: The Determinant of a Three By Three Matrix**

Let $A$ be a $3 \times 3$ matrix. Then, $\det(A)$ is calculated by picking a row (or column) and taking the product of each entry in that row (column) with its cofactor and adding these products together.

This process when applied to the $i^{th}$ row (column) is known as **expanding along the $i^{th}$ row (column)** as is given by

$$\det(A) = a_{i1}\text{cof}(A)_{i1} + a_{i2}\text{cof}(A)_{i2} + a_{i3}\text{cof}(A)_{i3}$$

When calculating the determinant, you can choose to expand any row or any column. Regardless of your choice, you will always get the same number which is the determinant of the matrix $A$. This method of evaluating a determinant by expanding along a row or a column is called **Laplace Expansion** or **Cofactor Expansion**.

Consider the following example.

**Example 12.8.4: Finding the Determinant of a Three by Three Matrix**

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\det(A)$ using the method of Laplace Expansion.

**Solution**

First, we will calculate $\det(A)$ by expanding along the first column. Using Definition 12.8.4, we take the 1 in the first column and multiply it by its cofactor,

$$1(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = (1)(1)(-1) = -1$$

Similarly, we take the 4 in the first column and multiply it by its cofactor, as well as with the 3 in the first column. Finally, we add these numbers together, as given in the following equation.

$$\det(A) = \overbrace{1(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{11}} + \overbrace{4(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{21}} + \overbrace{3(-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}}^{\text{cof}(A)_{31}}$$

Calculating each of these, we obtain

$$\det(A) = 1(1)(-1) + 4(-1)(-4) + 3(1)(-5) = -1 + 16 + -15 = 0$$

Hence, $\det(A) = 0$.

As mentioned in Definition 12.8.4, we can choose to expand along any row or column. Let's try now by expanding along the second row. Here, we take the 4 in the second row and multiply it to its cofactor, then add this to the 3 in the second row multiplied by its cofactor, and the 2 in the second row multiplied by its cofactor. The calculation is as follows.

$$\det(A) = \overbrace{4(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{21}} + \overbrace{3(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}}^{\text{cof}(A)_{22}} + \overbrace{2(-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}}^{\text{cof}(A)_{23}}$$

Calculating each of these products, we obtain

$$\det(A) = 4(-1)(-2) + 3(1)(-8) + 2(-1)(-4) = 0$$

You can see that for both methods, we obtained $\det(A) = 0$.

As mentioned above, we will always come up with the same value for $\det(A)$ regardless of the row or column we choose to expand along. You should try to compute the above determinant by expanding along other rows and columns. This is a good way to check your work, because you should come up with the same number each time!

We present this idea formally in the following theorem.

**Theorem 12.8.1: The Determinant is Well Defined**

Expanding the $n \times n$ matrix along any row or column always gives the same answer, which is the determinant.

We have now looked at the determinant of $2 \times 2$ and $3 \times 3$ matrices. It turns out that the method used to calculate the determinant of a $3 \times 3$ matrix can be used to calculate the determinant of any sized matrix. Notice that Definition 12.8.2, Definition 12.8.3 and Definition 12.8.4 can all be applied to a matrix of any size.

For example, the $ij^{th}$ minor of a $4 \times 4$ matrix is the determinant of the $3 \times 3$ matrix you obtain when you delete the $i^{th}$ row and the $j^{th}$ column. Just as with the $3 \times 3$ determinant, we can compute the determinant of a $4 \times 4$ matrix by Laplace Expansion, along any row or column

Consider the following example.

**Example 12.8.5: Determinant of a Four by Four Matrix**

Find $\det(A)$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 3 & 4 & 5 \\ 3 & 4 & 3 & 2 \end{bmatrix}$$

**Solution**

As in the case of a $3 \times 3$ matrix, you can expand this along any row or column. Lets pick the third column. Then, using Laplace Expansion,

$$\det(A) = 3(-1)^{1+3}\begin{vmatrix} 5 & 4 & 3 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} +$$

$$4(-1)^{3+3}\begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 3 & 4 & 2 \end{vmatrix} + 3(-1)^{4+3}\begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 1 & 3 & 5 \end{vmatrix}$$

Now, you can calculate each $3 \times 3$ determinant using Laplace Expansion, as we did above. You should complete these as an exercise and verify that $\det(A) = -12$.

The following provides a formal definition for the determinant of an $n \times n$ matrix. You may wish to take a moment and consider the above definitions for $2 \times 2$ and $3 \times 3$ determinants in context of this definition.

**Definition 12.8.5: The Determinant of an $n \times n$ Matrix**

Let $A$ be an $n \times n$ matrix where $n \geq 2$ and suppose the determinant of an $(n-1) \times (n-1)$ has been defined. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij}\text{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij}\text{cof}(A)_{ij}$$

The first formula consists of expanding the determinant along the $i^{th}$ row and the second expands the determinant along the $j^{th}$ column.

In the following sections, we will explore some important properties and characteristics of the determinant.

## The Determinant of a Triangular Matrix

There is a certain type of matrix for which finding the determinant is a very simple procedure. Consider the following definition.

**Definition 12.8.6: Triangular Matrices**

A matrix $A$ is upper triangular if $a_{ij} = 0$ whenever $i > j$. Thus the entries of such a matrix below the main diagonal equal 0, as shown. Here, $*$ refers to any nonzero number.

$$\begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & \vdots \\ \vdots & \vdots & \ddots & * \\ 0 & \cdots & 0 & * \end{bmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

The following theorem provides a useful way to calculate the determinant of a triangular matrix.

**Theorem 12.8.2: Determinant of a Triangular Matrix**

Let $A$ be an upper or lower triangular matrix. Then $\det(A)$ is obtained by taking the product of the entries on the main diagonal.

The verification of this Theorem can be done by computing the determinant using Laplace Expansion along the first row or column.

Consider the following example.

**Example 12.8.6: Determinant of a Triangular Matrix**

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 77 \\ 0 & 2 & 6 & 7 \\ 0 & 0 & 3 & 33.7 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Find $\det(A)$.

**Solution**

From Theorem 12.8.2, it suffices to take the product of the elements on the main diagonal. Thus $\det(A) = 1 \times 2 \times 3 \times (-1) = -6$.

Without using Theorem 12.8.2, you could use Laplace Expansion. We will expand along the first column. This gives

$$\det(A) = 1\begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{2+1}\begin{vmatrix} 2 & 3 & 77 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{3+1}\begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{4+1}\begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 3 & 33.7 \end{vmatrix}$$

and the only nonzero term in the expansion is

$$1\begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix}$$

Now find the determinant of this $3 \times 3$ matrix, by expanding along the first column to obtain

$$\begin{aligned} \det(A) &= 1 \times \left( 2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{2+1}\begin{vmatrix} 6 & 7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{3+1}\begin{vmatrix} 6 & 7 \\ 3 & 33.7 \end{vmatrix} \right) \\ &= 1 \times 2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} \end{aligned}$$

Next use Definition 12.8.1 to find the determinant of this $2 \times 2$ matrix, which is just $3 \times -1 - 0 \times 33.7 = -3$ . Putting all these steps together, we have

$$\det(A) = 1 \times 2 \times 3 \times (-1) = -6$$

which is just the product of the entries down the main diagonal of the original matrix!

You can see that while both methods result in the same answer, Theorem 12.8.2 provides a much quicker method.

In the next section, we explore some important properties of determinants.

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# 12.9: Properties of Determinants

## Properties of Determinants I: Examples

There are many important properties of determinants. Since many of these properties involve the row operations discussed in Chapter 1, we recall that definition now.

**Definition 12.9.1: Row Operations**

The row operations consist of the following

1. Switch two rows.
2. Multiply a row by a nonzero number.
3. Replace a row by a multiple of another row added to itself.

We will now consider the effect of row operations on the determinant of a matrix. In future sections, we will see that using the following properties can greatly assist in finding determinants. This section will use the theorems as motivation to provide various examples of the usefulness of the properties.

The first theorem explains the affect on the determinant of a matrix when two rows are switched.

**Theorem 12.9.1: Switching Rows**

Let $A$ be an $n \times n$ matrix and let $B$ be a matrix which results from switching two rows of $A$. Then $\det(B) = -\det(A)$.

When we switch two rows of a matrix, the determinant is multiplied by $-1$. Consider the following example.

**Example 12.9.1: Switching Two Rows**

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$. Knowing that $\det(A) = -2$, find $\det(B)$.

**Solution**

By Definition 3.1.1, $\det(A) = 1 \times 4 - 3 \times 2 = -2$ . Notice that the rows of $B$ are the rows of $A$ but switched. By Theorem 12.9.1 since two rows of $A$ have been switched, $\det(B) = -\det(A) = -(-2) = 2$ . You can verify this using Definition 3.1.1.

The next theorem demonstrates the effect on the determinant of a matrix when we multiply a row by a scalar.

**Theorem 12.9.2: Multiplying a Row by a Scalar**

Let $A$ be an $n \times n$ matrix and let $B$ be a matrix which results from multiplying some row of $A$ by a scalar $k$. Then $\det(B) = k\det(A)$.

Notice that this theorem is true when we multiply *one* row of the matrix by $k$. If we were to multiply *two* rows of $A$ by $k$ to obtain $B$, we would have $\det(B) = k^2 \det(A)$ . Suppose we were to multiply all $n$ rows of $A$ by $k$ to obtain the matrix $B$, so that $B = kA$. Then, $\det(B) = k^n \det(A)$ . This gives the next theorem.

**Theorem 12.9.3: Scalar Multiplication**

Let $A$ and $B$ be $n \times n$ matrices and $k$ a scalar, such that $B = kA$. Then $\det(B) = k^n \det(A)$.

Consider the following example.

**Example 12.9.2: Multiplying a Row by 5**

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 10 \\ 3 & 4 \end{bmatrix}$. Knowing that $\det(A) = -2$, find $\det(B)$.

**Solution**

By Definition 3.1.1, $\det(A) = -2$. We can also compute $\det(B)$ using Definition 3.1.1, and we see that $\det(B) = -10$.

Now, let's compute $\det(B)$ using Theorem 12.9.2 and see if we obtain the same answer. Notice that the first row of $B$ is 5 times the first row of $A$, while the second row of $B$ is equal to the second row of $A$. By Theorem 12.9.2, $\det(B) = 5 \times \det(A) = 5 \times -2 = -10$.

You can see that this matches our answer above.

Finally, consider the next theorem for the last row operation, that of adding a multiple of a row to another row.

**Theorem 12.9.4: Adding a Multiple of a Row to Another Row**

Let $A$ be an $n \times n$ matrix and let $B$ be a matrix which results from adding a multiple of a row to another row. Then $\det(A) = \det(B)$.

Therefore, when we add a multiple of a row to another row, the determinant of the matrix is unchanged. Note that if a matrix $A$ contains a row which is a multiple of another row, $\det(A)$ will equal 0. To see this, suppose the first row of $A$ is equal to $-1$ times the second row. By Theorem 12.9.4, we can add the first row to the second row, and the determinant will be unchanged. However, this row operation will result in a row of zeros. Using Laplace Expansion along the row of zeros, we find that the determinant is 0.

Consider the following example.

**Example 12.9.3: Adding a Row to Another Row**

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$. Find $\det(B)$.

**Solution**

By Definition 3.1.1, $\det(A) = -2$. Notice that the second row of $B$ is two times the first row of $A$ added to the second row. By Theorem 12.9.1, $\det(B) = \det(A) = -2$. As usual, you can verify this answer using Definition 3.1.1.

**Example 12.9.4: Multiple of a Row**

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Show that $\det(A) = 0$.

**Solution**

Using Definition 3.1.1, the determinant is given by

$$\det(A) = 1 \times 4 - 2 \times 2 = 0$$

However notice that the second row is equal to 2 times the first row. Then by the discussion above following Theorem 12.9.4 the determinant will equal 0.

Until now, our focus has primarily been on row operations. However, we can carry out the same operations with columns, rather than rows. The three operations outlined in Definition 12.9.1 can be done with columns instead of rows. In this case, in Theorems 12.9.1, 12.9.2, and 12.9.4 you can replace the word, "row" with the word "column".

There are several other major properties of determinants which do not involve row (or column) operations. The first is the determinant of a product of matrices.

**Theorem 12.9.5: Determinant of a Product**

Let $A$ and $B$ be two $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B)$$

In order to find the determinant of a product of matrices, we can simply take the product of the determinants.

Consider the following example.

**Example 12.9.5: The Determinant of a Product**

Compare $\det(AB)$ and $\det(A)\det(B)$ for

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

**Solution**

First compute $AB$, which is given by

$$AB = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix}$$

and so by Definition 3.1.1

$$\det(AB) = \det \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix} = -40$$

Now

$$\det(A) = \det \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} = 8$$

and

$$\det(B) = \det \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = -5$$

Computing $\det(A) \times \det(B)$ we have $8 \times -5 = -40$. This is the same answer as above and you can see that $\det(A)\det(B) = 8 \times (-5) = -40 = \det(AB)$ .

Consider the next important property.

**Theorem 12.9.6: Determinant of the Transpose**

Let $A$ be a matrix where $A^T$ is the transpose of $A$. Then,

$$\det\left(A^T\right) = \det(A)$$

This theorem is illustrated in the following example.

**Example 12.9.6: Determinant of the Transpose**

Let

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$$

Find $\det\left(A^T\right)$.

**Solution**

First, note that

$$A^T = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$$

Using Definition 3.1.1, we can compute $\det(A)$ and $\det\left(A^T\right)$. It follows that $\det(A) = 2\times 3 - 4\times 5 = -14$ and $\det\left(A^T\right) = 2\times 3 - 5\times 4 = -14$. Hence, $\det(A) = \det\left(A^T\right)$.

The following provides an essential property of the determinant, as well as a useful way to determine if a matrix is invertible.

**Theorem 12.9.7: Determinant of the Inverse**

Let $A$ be an $n\times n$ matrix. Then $A$ is invertible if and only if $\det(A) \neq 0$. If this is true, it follows that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Consider the following example.

**Example 12.9.7: Determinant of an Invertible Matrix**

Let $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$. For each matrix, determine if it is invertible. If so, find the determinant of the inverse.

**Solution**

Consider the matrix $A$ first. Using Definition 3.1.1 we can find the determinant as follows:

$$\det(A) = 3\times 4 - 2\times 6 = 12 - 12 = 0$$

By Theorem 12.9.7 $A$ is not invertible.

Now consider the matrix $B$. Again by Definition 3.1.1 we have

$$\det(B) = 2\times 1 - 5\times 3 = 2 - 15 = -13$$

By Theorem 12.9.7 $B$ is invertible and the determinant of the inverse is given by

$$\begin{aligned} \det\left(A^{-1}\right) &= \frac{1}{\det(A)} \\ &= \frac{1}{-13} \\ &= -\frac{1}{13} \end{aligned}$$

## Properties of Determinants II: Some Important Proofs

This section includes some important proofs on determinants and cofactors.

First we recall the definition of a determinant. If $A = [a_{ij}]$ is an $n\times n$ matrix, then $\det A$ is defined by computing the expansion along the first row:

$$\det A = \sum_{i=1}^{n} a_{1,i}\mathrm{cof}(A)_{1,i}. \tag{12.9.1}$$

If $n = 1$ then $\det A = a_{1,1}$.

The following example is straightforward and strongly recommended as a means for getting used to definitions.

**Example 12.9.8:**

(1) Let $E_{ij}$ be the elementary matrix obtained by interchanging $i$th and $j$th rows of $I$. Then $\det E_{ij} = -1$.

(2) Let $E_{ik}$ be the elementary matrix obtained by multiplying the $i$th row of $I$ by $k$. Then $\det E_{ik} = k$.

(3) Let $E_{ijk}$ be the elementary matrix obtained by multiplying $i$th row of $I$ by $k$ and adding it to its $j$th row. Then $\det E_{ijk} = 1$.

(4) If $C$ and $B$ are such that $CB$ is defined and the $i$th row of $C$ consists of zeros, then the $i$th row of $CB$ consists of zeros.

(5) If $E$ is an elementary matrix, then $\det E = \det E^T$.

Many of the proofs in section use the Principle of Mathematical Induction. This concept is discussed in Appendix A.2 and is reviewed here for convenience. First we check that the assertion is true for $n = 2$ (the case $n = 1$ is either completely trivial or meaningless).

Next, we assume that the assertion is true for $n-1$ (where $n \geq 3$) and prove it for $n$. Once this is accomplished, by the Principle of Mathematical Induction we can conclude that the statement is true for all $n \times n$ matrices for every $n \geq 2$.

If $A$ is an $n \times n$ matrix and $1 \leq j \leq n$, then the matrix obtained by removing 1st column and $j$th row from $A$ is an $n-1 \times n-1$ matrix (we shall denote this matrix by $A(j)$ below). Since these matrices are used in computation of cofactors $\text{cof}(A)_{1,i}$, for $1 \leq i \neq n$, the inductive assumption applies to these matrices.

Consider the following lemma.

**Lemma 12.9.1:**

If $A$ is an $n \times n$ matrix such that one of its rows consists of zeros, then $\det A = 0$.

**Proof**

We will prove this lemma using Mathematical Induction.

If $n = 2$ this is easy (check!).

Let $n \geq 3$ be such that every matrix of size $n-1 \times n-1$ with a row consisting of zeros has determinant equal to zero. Let $i$ be such that the $i$th row of $A$ consists of zeros. Then we have $a_{ij} = 0$ for $1 \leq j \leq n$.

Fix $j \in \{1, 2, \ldots, n\}$ such that $j \neq i$. Then matrix $A(j)$ used in computation of $\text{cof}(A)_{1,j}$ has a row consisting of zeros, and by our inductive assumption $\text{cof}(A)_{1,j} = 0$.

On the other hand, if $j = i$ then $a_{1,j} = 0$. Therefore $a_{1,j}\text{cof}(A)_{1,j} = 0$ for all $j$ and by (12.9.1)we have

$$\det A = \sum_{j=1}^{n} a_{1,j}\text{cof}(A)_{1,j} = 0$$

as each of the summands is equal to 0.

**Lemma 12.9.2:**

Assume $A$, $B$ and $C$ are $n \times n$ matrices that for some $1 \leq i \leq n$ satisfy the following.

1. $j$th rows of all three matrices are identical, for $j \neq i$.
2. Each entry in the $j$th row of $A$ is the sum of the corresponding entries in $j$th rows of $B$ and $C$.

Then $\det A = \det B + \det C$.

**Proof**

This is not difficult to check for $n = 2$ (do check it!).

Now assume that the statement of Lemma is true for $n-1 \times n-1$ matrices and fix $A, B$ and $C$ as in the statement. The assumptions state that we have $a_{l,j} = b_{l,j} = c_{l,j}$ for $j \neq i$ and for $1 \leq l \leq n$ and $a_{l,i} = b_{l,i} + c_{l,i}$ for all $1 \leq l \leq n$.

Therefore $A(i)=B(i)=C(i)$, and $A(j)$ has the property that its $i$th row is the sum of $i$th rows of $B(j)$ and $C(j)$ for $j\neq i$ while the other rows of all three matrices are identical. Therefore by our inductive assumption we have $\operatorname{cof}(A)_{1j}=\operatorname{cof}(B)_{1j}+\operatorname{cof}(C)_{1j}$ for $j\neq i$.

By (12.9.1)we have (using all equalities established above)

$$\begin{aligned}\det A &= \sum_{l=1}^{n} a_{1,l}\operatorname{cof}(A)_{1,l}\\ &= \sum_{l\neq i} a_{1,l}(\operatorname{cof}(B)_{1,l}+\operatorname{cof}(C)_{1,l})+(b_{1,i}+c_{1,i})\operatorname{cof}(A)_{1,i}\\ &= \det B+\det C\end{aligned}$$

This proves that the assertion is true for all $n$ and completes the proof.

**Theorem 12.9.8:**

Let $A$ and $B$ be $n\times n$ matrices.

1. If $A$ is obtained by interchanging $i$th and $j$th rows of $B$ (with $i\neq j$), then $\det A=-\det B$.
2. If $A$ is obtained by multiplying $i$th row of $B$ by $k$ then $\det A=k\det B$.
3. If two rows of $A$ are identical then $\det A=0$.
4. If $A$ is obtained by multiplying $i$th row of $B$ by $k$ and adding it to $j$th row of $B$ ($i\neq j$) then $\det A=\det B$.

**Proof**

We prove all statements by induction. The case $n=2$ is easily checked directly (and it is strongly suggested that you do check it).

We assume $n\geq 3$ and (1)–(4) are true for all matrices of size $n-1\times n-1$ .

(1) We prove the case when $j=i+1$ , i.e., we are interchanging two consecutive rows.

Let $l\in\{1,\ldots,n\}\setminus\{i,j\}$. Then $A(l)$ is obtained from $B(l)$ by interchanging two of its rows (draw a picture) and by our assumption

$$\operatorname{cof}(A)_{1,l}=-\operatorname{cof}(B)_{1,l}. \tag{12.9.2}$$

Now consider $a_{1,i}\operatorname{cof}(A)_{1,l}$. We have that $a_{1,i}=b_{1,j}$ and also that $A(i)=B(j)$. Since $j=i+1$ , we have

$$(-1)^{1+j}=(-1)^{1+i+1}=-(-1)^{1+i}$$

and therefore $a_{1i}\operatorname{cof}(A)_{1i}=-b_{1j}\operatorname{cof}(B)_{1j}$ and $a_{1j}\operatorname{cof}(A)_{1j}=-b_{1i}\operatorname{cof}(B)_{1i}$. Putting this together with (12.9.2) into (12.9.1)we see that if in the formula for $\det A$ we change the sign of each of the summands we obtain the formula for $\det B$.

$$\det A=\sum_{l=1}^{n} a_{1l}\operatorname{cof}(A)_{1l}=-\sum_{l=1}^{n} b_{1l}B_{1l}=\det B.$$

We have therefore proved the case of (1) when $j=i+1$ . In order to prove the general case, one needs the following fact. If $i<j$, then in order to interchange $i$th and $j$th row one can proceed by interchanging two adjacent rows $2(j-i)+1$ times: First swap $i$th and $i+1$ st, then $i+1$ st and $i+2$ nd, and so on. After one interchanges $j-1$ st and $j$th row, we have $i$th row in position of $j$th and $l$th row in position of $l-1$ st for $i+1\leq l\leq j$ . Then proceed backwards swapping adjacent rows until everything is in place.

Since $2(j-i)+1$ is an odd number $(-1)^{2(j-i)+1}=-1$ and we have that $\det A=-\det B$ .

(2) This is like (1)… but much easier. Assume that (2) is true for all $n-1\times n-1$ matrices. We have that $a_{ji}=kb_{ji}$ for $1\leq j\leq n$ . In particular $a_{1i}=kb_{1i}$ , and for $l\neq i$ matrix $A(l)$ is obtained from $B(l)$ by multiplying one of its rows by $k$. Therefore $\operatorname{cof}(A)_{1l}=k\operatorname{cof}(B)_{1l}$ for $l\neq i$, and for all $l$ we have $a_{1l}\operatorname{cof}(A)_{1l}=kb_{1l}\operatorname{cof}(B)_{1l}$. By (12.9.1) we have $\det A=k\det B$ .

(3) This is a consequence of (1). If two rows of $A$ are identical, then $A$ is equal to the matrix obtained by interchanging those two rows and therefore by (1) $\det A = -\det A$ . This implies $\det A = 0$ .

(4) Assume (4) is true for all $n-1 \times n-1$ matrices and fix $A$ and $B$ such that $A$ is obtained by multiplying $i$th row of $B$ by $k$ and adding it to $j$th row of $B$ $(i \neq j)$ then $\det A = \det B$ . If $k = 0$ then $A = B$ and there is nothing to prove, so we may assume $k \neq 0$ .

Let $C$ be the matrix obtained by replacing the $j$th row of $B$ by the $i$th row of $B$ multiplied by $k$. By Lemma 12.9.2 we have that

$$\det A = \det B + \det C$$

and we 'only' need to show that $\det C = 0$ . But $i$th and $j$th rows of $C$ are proportional. If $D$ is obtained by multiplying the $j$th row of $C$ by $\frac{1}{k}$ then by (2) we have $\det C = \frac{1}{k} \det D$ (recall that $k \neq 0$ !). But $i$th and $j$th rows of $D$ are identical, hence by (3) we have $\det D = 0$ and therefore $\det C = 0$ .

**Theorem 12.9.9:**

Let $A$ and $B$ be two $n \times n$ matrices. Then

$$\det(AB) = \det(A) \det(B)$$

**Proof**

If $A$ is an elementary matrix of either type, then multiplying by $A$ on the left has the same effect as performing the corresponding elementary row operation. Therefore the equality $\det(AB) = \det A \det B$ in this case follows by Example 12.9.8and Theorem 12.9.8

If $C$ is the reduced row-echelon form of $A$ then we can write $A = E_1 \cdot E_2 \cdot \cdots \cdot E_m \cdot C$ for some elementary matrices $E_1, \ldots, E_m$.

Now we consider two cases.

Assume first that $C = I$ . Then $A = E_1 \cdot E_2 \cdot \cdots \cdot E_m$ and $AB = E_1 \cdot E_2 \cdot \cdots \cdot E_m B$ . By applying the above equality $m$ times, and then $m-1$ times, we have that

$$\begin{aligned} \det AB &= \det E_1 \det E_2 \cdot \det E_m \cdot \det B \\ &= \det(E_1 \cdot E_2 \cdot \cdots \cdot E_m) \det B \\ &= \det A \det B. \end{aligned}$$

Now assume $C \neq I$ . Since it is in reduced row-echelon form, its last row consists of zeros and by (4) of Example 12.9.8 the last row of $CB$ consists of zeros. By Lemma 12.9.1we have $\det C = \det(CB) = 0$ and therefore

$$\det A = \det(E_1 \cdot E_2 \cdot E_m) \cdot \det(C) = \det(E_1 \cdot E_2 \cdot E_m) \cdot 0 = 0$$

and also

$$\det AB = \det(E_1 \cdot E_2 \cdot E_m) \cdot \det(CB) = \det(E_1 \cdot E_2 \cdot \cdots \cdot E_m) 0 = 0$$

hence $\det AB = 0 = \det A \det B$ .

The same 'machine' used in the previous proof will be used again.

**Theorem 12.9.10:**

Let $A$ be a matrix where $A^T$ is the transpose of $A$. Then,

$$\det\left(A^T\right) = \det(A)$$

**Proof**

Note first that the conclusion is true if $A$ is elementary by (5) of Example 12.9.8

Let $C$ be the reduced row-echelon form of $A$. Then we can write $A = E_1 \cdot E_2 \cdot \cdots \cdot E_m C$ . Then $A^T = C^T \cdot E_m^T \cdot \cdots \cdot E_2^T \cdot E_1$ . By Theorem 12.9.9we have

$$\det(A^T) = \det(C^T) \cdot \det(E_m^T) \cdot \cdots \cdot \det(E_2^T) \cdot \det(E_1).$$

By (5) of Example 12.9.8we have that $\det E_j = \det E_j^T$ for all $j$. Also, $\det C$ is either 0 or 1 (depending on whether $C = I$ or not) and in either case $\det C = \det C^T$ . Therefore $\det A = \det A^T$ .

The above discussions allow us to now prove Theorem 3.1.1. It is restated below.

**Theorem 12.9.11:**

Expanding an $n \times n$ matrix along any row or column always gives the same result, which is the determinant.

**Proof**

We first show that the determinant can be computed along any row. The case $n = 1$ does not apply and thus let $n \geq 2$.

Let $A$be an $n \times n$ matrix and fix $j > 1$. We need to prove that

$$\det A = \sum_{i=1}^{n} a_{j,i} \text{cof}(A)_{j,i}.$$

Let us prove the case when $j = 2$.

Let $B$ be the matrix obtained from $A$ by interchanging its 1st and 2nd rows. Then by Theorem 12.9.8we have

$$\det A = -\det B.$$

Now we have

$$\det B = \sum_{i=1}^{n} b_{1,i} \text{cof}(B)_{1,i}.$$

Since $B$ is obtained by interchanging the 1st and 2nd rows of $A$ we have that $b_{1,i} = a_{2,i}$ for all $i$ and one can see that $minor(B)_{1,i} = minor(A)_{2,i}$.

Further,

$$\text{cof}(B)_{1,i} = (-1)^{1+i} minor B_{1,i} = -(-1)^{2+i} minor(A)_{2,i} = -\text{cof}(A)_{2,i}$$

hence $\det B = -\sum_{i=1}^{n} a_{2,i} \text{cof}(A)_{2,i}$ , and therefore $\det A = -\det B = \sum_{i=1}^{n} a_{2,i} \text{cof}(A)_{2,i}$ as desired.

The case when $j > 2$ is very similar; we still have $minor(B)_{1,i} = minor(A)_{j,i}$ but checking that $\det B = -\sum_{i=1}^{n} a_{j,i} \text{cof}(A)_{j,i}$ is slightly more involved.

Now the cofactor expansion along column $j$ of $A$ is equal to the cofactor expansion along row $j$ of $A^T$, which is by the above result just proved equal to the cofactor expansion along row 1 of $A^T$, which is equal to the cofactor expansion along column 1 of $A$. Thus the cofactor cofactor along any column yields the same result.

Finally, since $\det A = \det A^T$ by Theorem 12.9.10, we conclude that the cofactor expansion along row 1 of $A$ is equal to the cofactor expansion along row 1 of $A^T$, which is equal to the cofactor expansion along column 1 of $A$. Thus the proof is complete.

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# 12.10: Finding Determinants using Row Operations

Theorems 3.2.1, 3.2.2 and 3.2.4 illustrate how row operations affect the determinant of a matrix. In this section, we look at two examples where row operations are used to find the determinant of a large matrix. Recall that when working with large matrices, Laplace Expansion is effective but timely, as there are many steps involved. This section provides useful tools for an alternative method. By first applying row operations, we can obtain a simpler matrix to which we apply Laplace Expansion.

While working through questions such as these, it is useful to record your row operations as you go along. Keep this in mind as you read through the next example.

**Example 12.10.1: Finding a Determinant**

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 4 & 5 & 4 & 3 \\ 2 & 2 & -4 & 5 \end{bmatrix}$$

**Solution**

We will use the properties of determinants outlined above to find $\det(A)$. First, add $-5$ times the first row to the second row. Then add $-4$ times the first row to the third row, and $-2$ times the first row to the fourth row. This yields the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -9 & -13 & -17 \\ 0 & -3 & -8 & -13 \\ 0 & -2 & -10 & -3 \end{bmatrix}$$

Notice that the only row operation we have done so far is adding a multiple of a row to another row. Therefore, by Theorem 3.2.4, $\det(B) = \det(A)$.

At this stage, you could use Laplace Expansion to find $\det(B)$. However, we will continue with row operations to find an even simpler matrix to work with.

Add $-3$ times the third row to the second row. By Theorem 3.2.4 this does not change the value of the determinant. Then, multiply the fourth row by $-3$. This results in the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 11 & 22 \\ 0 & -3 & -8 & -13 \\ 0 & 6 & 30 & 9 \end{bmatrix}$$

Here, $\det(C) = -3\det(B)$, which means that $\det(B) = \left(-\frac{1}{3}\right)\det(C)$

Since $\det(A) = \det(B)$, we now have that $\det(A) = \left(-\frac{1}{3}\right)\det(C)$. Again, you could use Laplace Expansion here to find $\det(C)$. However, we will continue with row operations.

Now replace the add 2 times the third row to the fourth row. This does not change the value of the determinant by Theorem 3.2.4. Finally switch the third and second rows. This causes the determinant to be multiplied by $-1$. Thus $\det(C) = -\det(D)$ where

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -8 & -13 \\ 0 & 0 & 11 & 22 \\ 0 & 0 & 14 & -17 \end{bmatrix}$$

Hence, $\det(A) = \left(-\frac{1}{3}\right)\det(C) = \left(\frac{1}{3}\right)\det(D)$

You could do more row operations or you could note that this can be easily expanded along the first column. Then, expand the resulting $3 \times 3$ matrix also along the first column. This results in

$$\det(D) = 1(-3)\begin{vmatrix} 11 & 22 \\ 14 & -17 \end{vmatrix} = 1485$$

and so $\det(A) = \left(\frac{1}{3}\right)(1485) = 495$.

You can see that by using row operations, we can simplify a matrix to the point where Laplace Expansion involves only a few steps. In Example 12.10.1, we also could have continued until the matrix was in upper triangular form, and taken the product of the entries on the main diagonal. Whenever computing the determinant, it is useful to consider all the possible methods and tools.

Consider the next example.

### Example 12.10.2: Find the Determinant

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & -3 & 2 & 1 \\ 2 & 1 & 2 & 5 \\ 3 & -4 & 1 & 2 \end{bmatrix}$$

**Solution**

Once again, we will simplify the matrix through row operations. Add $-1$ times the first row to the second row. Next add $-2$ times the first row to the third and finally take $-3$ times the first row and add to the fourth row. This yields

$$B = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -1 & -1 \\ 0 & -3 & -4 & 1 \\ 0 & -10 & -8 & -4 \end{bmatrix}$$

By Theorem 3.2.4, $\det(A) = \det(B)$.

Remember you can work with the columns also. Take $-5$ times the fourth column and add to the second column. This yields

$$C = \begin{bmatrix} 1 & -8 & 3 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{bmatrix}$$

By Theorem 3.2.4 $\det(A) = \det(C)$.

Now take $-1$ times the third row and add to the top row. This gives.

$$D = \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{bmatrix}$$

which by Theorem 3.2.4 has the same determinant as $A$.

Now, we can find $\det(D)$ by expanding along the first column as follows. You can see that there will be only one non zero term.

$$\det(D) = 1 \det \begin{bmatrix} 0 & -1 & -1 \\ -8 & -4 & 1 \\ 10 & -8 & -4 \end{bmatrix} + 0 + 0 + 0$$

Expanding again along the first column, we have

$$\det(D) = 1\left(0 + 8\det\begin{bmatrix} -1 & -1 \\ -8 & -4 \end{bmatrix} + 10\det\begin{bmatrix} -1 & -1 \\ -4 & 1 \end{bmatrix}\right) = -82$$

Now since $\det(A) = \det(D)$, it follows that $\det(A) = -82$.

Remember that you can verify these answers by using Laplace Expansion on $A$. Similarly, if you first compute the determinant using Laplace Expansion, you can use the row operation method to verify.

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# 12.E: Exercises

**Exercise 12.$E$.1**

For the following pairs of matrices, determine if the sum $A+B$ is defined. If so, find the sum.

1. $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
2. $A=\begin{bmatrix}2 & 1 & 2\\ 1 & 1 & 0\end{bmatrix}, B=\begin{bmatrix}-1 & 0 & 3\\ 0 & 1 & 4\end{bmatrix}$
3. $A=\begin{bmatrix}1 & 0\\ -2 & 3\\ 4 & 2\end{bmatrix}, B=\begin{bmatrix}2 & 7 & -1\\ 0 & 3 & 4\end{bmatrix}$

**Exercise 12.$E$.2**

For each matrix $A$, find the matrix $-A$ such that $A+(-A)=0$ .

1. $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$
2. $A=\begin{bmatrix}-2 & 3\\ 0 & 2\end{bmatrix}$
3. $A=\begin{bmatrix}0 & 1 & 2\\ 1 & -1 & 3\\ 4 & 2 & 0\end{bmatrix}$

**Exercise 12.$E$.3**

For each matrix $A$, find the product $(-2)A$, $0A$, and $3A$.

1. $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$
2. $A=\begin{bmatrix}-2 & 3\\ 0 & 2\end{bmatrix}$
3. $A=\begin{bmatrix}0 & 1 & 2\\ 1 & -1 & 3\\ 4 & 2 & 0\end{bmatrix}$

**Exercise 12.$E$.4**

Prove $-A$ is unique.

**Answer**

Suppose $B$ also works. Then

$$-A=-A+(A+B)=(-A+A)+B=0+B=B \tag{12.E.1}$$

**Exercise 12.$E$.5**

Prove the zero matrix, $0$, is unique.

**Answer**

Suppose $0'$ also works. Then $0'=0'+0=0$.

**Exercise 12.E.6**

Show $0A = 0$. Here the 0 on the left is the scalar 0 and the 0 on the right is the zero matrix of appropriate size.

**Answer**

$0A = (0+0)A = 0A + 0A$. Now add $-(0A)$ to both sides. Then $0 = 0A$.

**Exercise 12.E.7**

Show $(-1)A = -A$.

**Answer**

$A + (-1)A = (1+(-1))A = 0A = 0$. Therefore, from the uniqueness of the additive inverse proved in the above Problem, it follows that $-A = (-1)A$ .

**Exercise 12.E.8**

Consider the matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, .$

$$D = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}, E = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Find the following if possible. If it is not possible explain why.

1. $-3A$
2. $3B - A$
3. $AC$
4. $CB$
5. $AE$
6. $EA$

**Answer**

1. $\begin{bmatrix} -3 & -6 & -9 \\ -6 & -3 & -21 \end{bmatrix}$
2. $\begin{bmatrix} 8 & -5 & 3 \\ -11 & 5 & -4 \end{bmatrix}$
3. Not possible
4. $\begin{bmatrix} -3 & 3 & 4 \\ 6 & -1 & 7 \end{bmatrix}$
5. Not possible
6. Not possible

**Exercise 12.E.9**

Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix},$

$$D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Find the following if possible. If it is not possible explain why.

1. $-3A$
2. $3B-A$
3. $AC$
4. $CA$
5. $AE$
6. $EA$
7. $BE$
8. $DE$

**Answer**

1. $\begin{bmatrix} -3 & -6 \\ -9 & -6 \\ -3 & 3 \end{bmatrix}$
2. Not possible.
3. $\begin{bmatrix} 11 & 2 \\ 13 & 6 \\ -4 & 2 \end{bmatrix}$
4. Not possible.
5. $\begin{bmatrix} 7 \\ 9 \\ -2 \end{bmatrix}$
6. Not possible.
7. Not possible.
8. $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$

## Exercise 12.$E$.10

Let $A=\begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 1 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & -2 \end{bmatrix}$, and $C=\begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ -3 & -1 & 0 \end{bmatrix}$. Find the following if possible.

1. $AB$
2. $BA$
3. $AC$
4. $CA$
5. $CB$
6. $BC$

**Answer**

1. $\begin{bmatrix} 3 & 0 & -4 \\ -4 & 1 & 6 \\ 5 & 1 & -6 \end{bmatrix}$
2. $\begin{bmatrix} 1 & -2 \\ -2 & -3 \end{bmatrix}$
3. Not possible
4. $\begin{bmatrix} -4 & -6 \\ -5 & -3 \\ -1 & -2 \end{bmatrix}$
5. $\begin{bmatrix} 8 & 1 & -3 \\ 7 & 6 & -6 \end{bmatrix}$

**Exercise 12.E.11**

Let $A=\begin{bmatrix}-1 & -1\\ 3 & 3\end{bmatrix}$. Find all $2\times 2$ matrices, $B$ such that $AB=0$.

**Answer**

$$\begin{bmatrix}-1 & -1\\ 3 & 3\end{bmatrix}\begin{bmatrix}x & y\\ z & w\end{bmatrix}=\begin{bmatrix}-x-z & -w-y\\ 3x+3z & 3w+3y\end{bmatrix}$$
$$=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Solution is: $w=-y, x=-z$ so the matrices are of the form $\begin{bmatrix}x & y\\ -x & -y\end{bmatrix}$.

**Exercise 12.E.12**

Let $X=\begin{bmatrix}-1 & -1 & 1\end{bmatrix}$ and $Y=\begin{bmatrix}0 & 1 & 2\end{bmatrix}$. Find $X^TY$ and $XY^T$ if possible.

**Answer**

$$X^TY=\begin{bmatrix}0 & -1 & -2\\ 0 & -1 & -2\\ 0 & 1 & 2\end{bmatrix}, XY^T=1$$

**Exercise 12.E.13**

Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}, B=\begin{bmatrix}1 & 2\\ 3 & k\end{bmatrix}$. Is it possible to choose $k$ such that $AB=BA$? If so, what should $k$ equal?

**Answer**

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & k\end{bmatrix}=\begin{bmatrix}7 & 2k+2\\ 15 & 4k+6\end{bmatrix}$$
$$\begin{bmatrix}1 & 2\\ 3 & k\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}7 & 10\\ 3k+3 & 4k+6\end{bmatrix}$$

Thus you must have $\begin{matrix}3k+3=15\\ 2k+2=10\end{matrix}$, Solution is: $[k=4]$

**Exercise 12.E.14**

Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}, B=\begin{bmatrix}1 & 2\\ 1 & k\end{bmatrix}$. Is it possible to choose $k$ such that $AB=BA$? If so, what should $k$ equal?

**Answer**

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & k\end{bmatrix}=\begin{bmatrix}3 & 2k+2\\ 7 & 4k+6\end{bmatrix}$$
$$\begin{bmatrix}1 & 2\\ 1 & k\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}7 & 10\\ 3k+1 & 4k+2\end{bmatrix}$$

However, $7\neq 3$ and so there is no possible choice of $k$ which will make these matrices commute.

**Exercise 12.E.15**

Find $2 \times 2$ matrices, $A$, $B$, and $C$ such that $A \neq 0, C \neq B,$ but $AC = AB$.

**Answer**

More than one answer exists, an example is: Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**Exercise 12.E.16**

Give an example of matrices (of any size), $A, B, C$ such that $B \neq C$, $A \neq 0$, and yet $AB = AC$.

**Exercise 12.E.17**

Find $2 \times 2$ matrices $A$ and $B$ such that $A \neq 0$ and $B \neq 0$ but $AB = 0$.

**Answer**

More than one answer exists, an example is: Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

And we see their product is $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

**Exercise 12.E.18**

Find $2 \times 2$ matrices $A$ and $B$ such that $A \neq 0$ and $B \neq 0$ but $AB = 0$. Find a different example than you did for the previous exercise.

**Exercise 12.E.19**

Give an example of matrices (of any size), $A, B$ such that $A \neq 0$ and $B \neq 0$ but $AB = 0$.

**Exercise 12.E.20**

Find $2 \times 2$ matrices $A$ and $B$ such that $A \neq 0$ and $B \neq 0$ with $AB \neq BA$.

**Answer**

More than one answer exists, an example is: Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

**Exercise 12.E.21**

Write the system

$$\begin{aligned} x_1 - x_2 + 2x_3 \\ 2x_3 + x_1 \\ 3x_3 \\ 3x_4 + 3x_2 + x_1 \end{aligned}$$

in the form $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $A$ is an appropriate matrix.

**Answer**

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 3 & 0 & 3 \end{bmatrix}$$

**Exercise 12.E.22**

Write the system

$$\begin{aligned} x_1 + 3x_2 + 2x_3 \\ 2x_3 + x_1 \\ 6x_3 \\ x_4 + 3x_2 + x_1 \end{aligned}$$

in the form $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $A$ is an appropriate matrix.

**Answer**

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 6 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

**Exercise 12.E.23**

Write the system

$$\begin{aligned} x_1 + x_2 + x_3 \\ 2x_3 + x_1 + x_2 \\ x_3 - x_1 \\ 3x_4 + x_1 \end{aligned}$$

in the form $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $A$ is an appropriate matrix.

**Answer**

$$A=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

**Exercise 12.E.24**

A matrix $A$ is called *idempotent* if $A^2 = A$. Let

$$A=\begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \tag{12.E.2}$$

and show that $A$ is idempotent.

**Exercise 12.E.25**

For each pair of matrices, find the $(1,2)$-entry and $(2,3)$-entry of the product $AB$.

1. $A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 2 & 5 & 1 \end{bmatrix}, B=\begin{bmatrix} 4 & 6 & -2 \\ 7 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix}$
2. $A=\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 4 \\ 1 & 0 & 5 \end{bmatrix}, B=\begin{bmatrix} 2 & 3 & 0 \\ -4 & 16 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

**Exercise 12.E.26**

Suppose $A$ and $B$ are square matrices of the same size. Which of the following are necessarily true?

1. $(A-B)^2 = A^2 - 2AB + B^2$
2. $(AB)^2 = A^2B^2$
3. $(A+B)^2 = A^2 + 2AB + B^2$
4. $(A+B)^2 = A^2 + AB + BA + B^2$
5. $A^2B^2 = A(AB)B$
6. $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
7. $(A+B)(A-B) = A^2 - B^2$

**Answer**

1. Not necessarily true.
2. Not necessarily true.
3. Not necessarily true.
4. Necessarily true.
5. Necessarily true.
6. Not necessarily true.
7. Not necessarily true.

### Exercise 12.$E$.27

Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix},$

$D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Find the following if possible. If it is not possible explain why.

1. $-3A^T$
2. $3B - A^T$
3. $E^T B$
4. $EE^T$
5. $B^T B$
6. $CA^T$
7. $D^T BE$

**Answer**

1. $\begin{bmatrix} -3 & -9 & -3 \\ -6 & -6 & 3 \end{bmatrix}$
2. $\begin{bmatrix} 5 & -18 & 5 \\ -11 & 4 & 4 \end{bmatrix}$
3. $\begin{bmatrix} -7 & 1 & 5 \end{bmatrix}$
4. $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$
5. $\begin{bmatrix} 13 & -16 & 1 \\ -16 & 29 & -8 \\ 1 & -8 & 5 \end{bmatrix}$
6. $\begin{bmatrix} 5 & 7 & -1 \\ 5 & 15 & 5 \end{bmatrix}$
7. Not possible.

### Exercise 12.$E$.28

Let $A$ be an $n \times n$ matrix. Show $A$ equals the sum of a symmetric and a skew symmetric matrix.

**Answer**

Show that $\frac{1}{2}\left(A^T + A\right)$ is symmetric and then consider using this as one of the matrices.

Show that $\frac{1}{2}\left(A^T + A\right)$ is symmetric and then consider using this as one of the matrices. $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$.

### Exercise 12.$E$.29

### Exercise 12.$E$.30

Show that the main diagonal of every skew symmetric matrix consists of only zeros. Recall that the main diagonal consists of every entry of the matrix which is of the form $a_{ii}$.

**Answer**

If $A$ is symmetric then $A = -A^T$. It follows that $a_{ii} = -a_{ii}$ and so each $a_{ii} = 0$.

**Exercise 12.E.31**

Prove that for an $m \times n$ matrix $A$, an $m \times n$ matrix $B$, and scalars $r, s$, the following holds: $(rA + sB)^T = rA^T + sB^T$

**Exercise 12.E.32**

Prove that $I_m A = A$ where $A$ is an $m \times n$ matrix.

**Answer**

$(I_m A)_{ij} \equiv \sum_j \delta_{ik} A_{kj} = A_{ij}$

**Exercise 12.E.33**

Suppose $AB = AC$ and $A$ is an invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

**Answer**

Yes $B = C$. Multiply $AB = AC$ on the left by $A^{-1}$.

**Exercise 12.E.34**

Suppose $AB = AC$ and $A$ is a non invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

**Exercise 12.E.35**

Give an example of a matrix $A$ such that $A^2 = I$ and yet $A \neq I$ and $A \neq -I$.

**Answer**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Exercise 12.E.36**

Let

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Find $A^{-1}$ if possible. If $A^{-1}$ does not exist, explain why.

**Answer**

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

### Exercise 12.E.37

Let

$$A = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$$

Find $A^{-1}$ if possible. If $A^{-1}$ does not exist, explain why.

**Answer**

$$\begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{1}{5} \\ 1 & 0 \end{bmatrix}$$

### Exercise 12.E.38

Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

Find $A^{-1}$ if possible. If $A^{-1}$ does not exist, explain why.

**Answer**

$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{bmatrix}$$

### Exercise 12.E.39

Let

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

Find $A^{-1}$ if possible. If $A^{-1}$ does not exist, explain why.

**Answer**

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}^{-1}$ does not exist. The RREF of this matrix is $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$

### Exercise 12.E.40

Let $A$ be a $2 \times 2$ invertible matrix, with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find a formula for $A^{-1}$ in terms of $a, b, c, d$.

**Answer**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

### Exercise 12.E.41

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Find $A^{-1}$ if possible. If $A^{-1}$ does not exist, explain why.

**Answer**

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

## Exercise 12.$E$.42

Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Find $A^{-1}$ if possible. If $A^{-1}$ does not exist, explain why.

**Answer**

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{bmatrix}$$

## Exercise 12.$E$.43

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{bmatrix}$$

Find $A^{-1}$ if possible. If $A^{-1}$ does not exist, explain why.

**Answer**

The RREF of the matrix is $\begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$. There is no inverse.

## Exercise 12.$E$.44

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Find $A^{-1}$ if possible. If $A^{-1}$ does not exist, explain why.

**Answer**

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 3 & \frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} \\ -1 & 0 & 0 & 1 \\ -2 & -\frac{3}{4} & \frac{1}{4} & \frac{9}{4} \end{bmatrix}$$

## Exercise 12.E.45

Using the inverse of the matrix, find the solution to the systems:

1. $$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{12.E.3}$$
2. $$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{12.E.4}$$

Now give the solution in terms of $a$ and $b$ to

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{12.E.5}$$

## Exercise 12.E.46

Using the inverse of the matrix, find the solution to the systems:

1. $$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{12.E.6}$$
2. $$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} \tag{12.E.7}$$

Now give the solution in terms of $a, b$, and $c$ to the following:

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{12.E.8}$$

**Answer**

1. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 0 \end{bmatrix}$
2. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -12 \\ 1 \\ 5 \end{bmatrix}$
3. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3c-2a \\ \frac{1}{3}b-\frac{2}{3}c \\ a-c \end{bmatrix}$

## Exercise 12.E.47

Show that if $A$ is an $n \times n$ invertible matrix and $X$ is a $n \times 1$ matrix such that $AX = B$ for $B$ an $n \times 1$ matrix, then $X = A^{-1}B$.

**Answer**

Multiply both sides of $AX = B$ on the left by $A^{-1}$.

**Exercise 12.E.48**

Prove that if $A^{-1}$ exists and $AX = 0$ then $X = 0$.

**Answer**

Multiply on both sides on the left by $A^{-1}$. Thus

$$0 = A^{-1}0 = A^{-1}(AX) = \left(A^{-1}A\right)X = IX = X \tag{12.E.9}$$

**Exercise 12.E.49: Inverse Product**

Show that if $A^{-1}$ exists for an $n \times n$ matrix, then it is unique. That is, if $BA = I$ and $AB = I$, then $B = A^{-1}$.

**Answer**

$A^{-1} = A^{-1}I = A^{-1}(AB) = \left(A^{-1}A\right)B = IB = B.$

**Exercise 12.E.50**

Show that if $A$ is an invertible $n \times n$ matrix, then so is $A^T$ and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

**Answer**

You need to show that $\left(A^{-1}\right)^T$ acts like the inverse of $A^T$ because from uniqueness in the above problem, this will imply it is the inverse. From properties of the transpose,

$$\begin{aligned} A^T\left(A^{-1}\right)^T &= \left(A^{-1}A\right)^T = I^T = I \\ \left(A^{-1}\right)^T A^T &= \left(AA^{-1}\right)^T = I^T = I \end{aligned}$$

Hence $\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$ and this last matrix exists.

**Exercise 12.E.51**

Show $(AB)^{-1} = B^{-1}A^{-1}$ by verifying that

$$AB\left(B^{-1}A^{-1}\right) = I \tag{12.E.10}$$

and

$$B^{-1}A^{-1}(AB) = I \tag{12.E.11}$$

**Answer**

$(AB)B^{-1}A^{-1} = A\left(BB^{-1}\right)A^{-1} = AA^{-1} = I \quad B^{-1}A^{-1}(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}IB = B^{-1}B = I$

**Exercise 12.E.52**

Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ by verifying that

$$(ABC)\left(C^{-1}B^{-1}A^{-1}\right) = I \tag{12.E.12}$$

and

$$\left(C^{-1}B^{-1}A^{-1}\right)(ABC)=I \tag{12.E.13}$$

**Answer**

The proof of this exercise follows from the previous one.

**Exercise 12.E.53**

If $A$ is invertible, show $\left(A^2\right)^{-1}=\left(A^{-1}\right)^2$.

**Answer**

$A^2\left(A^{-1}\right)^2=AAA^{-1}A^{-1}=AIA^{-1}=AA^{-1}=I \quad \left(A^{-1}\right)^2A^2=A^{-1}A^{-1}AA=A^{-1}IA=A^{-1}A=I$

**Exercise 12.E.54**

If $A$ is invertible, show $\left(A^{-1}\right)^{-1}=A$.

**Answer**

$A^{-1}A=AA^{-1}=I$ and so by uniqueness, $\left(A^{-1}\right)^{-1}=A$.

**Exercise 12.E.55**

Let $A=\begin{bmatrix}2&3\\1&2\end{bmatrix}$. Suppose a row operation is applied to $A$ and the result is $B=\begin{bmatrix}1&2\\2&3\end{bmatrix}$. Find the elementary matrix $E$ that represents this row operation.

**Exercise 12.E.56**

Let $A=\begin{bmatrix}4&0\\2&1\end{bmatrix}$. Suppose a row operation is applied to $A$ and the result is $B=\begin{bmatrix}8&0\\2&1\end{bmatrix}$. Find the elementary matrix $E$ that represents this row operation.

**Exercise 12.E.57**

Let $A=\begin{bmatrix}1&-3\\0&5\end{bmatrix}$. Suppose a row operation is applied to $A$ and the result is $B=\begin{bmatrix}1&-3\\2&-1\end{bmatrix}$. Find the elementary matrix $E$ that represents this row operation.

**Exercise 12.E.58**

Let $A=\begin{bmatrix}1&2&1\\0&5&1\\2&-1&4\end{bmatrix}$. Suppose a row operation is applied to $A$ and the result is $B=\begin{bmatrix}1&2&1\\2&-1&4\\0&5&1\end{bmatrix}$.

1. Find the elementary matrix $E$ such that $EA=B$.
2. Find the inverse of $E$, $E^{-1}$, such that $E^{-1}B=A$.

**Exercise 12.E.59**

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to $A$ and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 10 & 2 \\ 2 & -1 & 4 \end{bmatrix}$.

1. Find the elementary matrix $E$ such that $EA = B$.
2. Find the inverse of $E$, $E^{-1}$, such that $E^{-1}B = A$.

**Exercise 12.E.60**

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to $A$ and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 1 & -\frac{1}{2} & 2 \end{bmatrix}$.

1. Find the elementary matrix $E$ such that $EA = B$.
2. Find the inverse of $E$, $E^{-1}$, such that $E^{-1}B = A$.

**Exercise 12.E.61**

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to $A$ and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 2 & -1 & 4 \end{bmatrix}$.

1. Find the elementary matrix $E$ such that $EA = B$.
2. Find the inverse of $E$, $E^{-1}$, such that $E^{-1}B = A$.

**Exercise 12.E.62**

Find an $LU$ factorization of $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

**Answer**

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 3 \end{bmatrix} \tag{12.E.14}$$

**Exercise 12.E.63**

Find an $LU$ factorization of $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 \\ 5 & 0 & 1 & 3 \end{bmatrix}$.

**Answer**

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 \\ 5 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -24 & -17 \end{bmatrix} \tag{12.E.15}$$

**Exercise 12.E.64**

Find an $LU$ factorization of the matrix $\begin{bmatrix} 1 & -2 & -5 & 0 \\ -2 & 5 & 11 & 3 \\ 3 & -6 & -15 & 1 \end{bmatrix}$.

**Answer**

$$\begin{bmatrix} 1 & -2 & -5 & 0 \\ -2 & 5 & 11 & 3 \\ 3 & -6 & -15 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -5 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.E.16}$$

**Exercise 12.E.65**

Find an $LU$ factorization of the matrix $\begin{bmatrix} 1 & -1 & -3 & -1 \\ -1 & 2 & 4 & 3 \\ 2 & -3 & -7 & -3 \end{bmatrix}$.

**Answer**

$$\begin{bmatrix} 1 & -1 & -3 & -1 \\ -1 & 2 & 4 & 3 \\ 2 & -3 & -7 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.E.17}$$

**Exercise 12.E.66**

Find an $LU$ factorization of the matrix $\begin{bmatrix} 1 & -3 & -4 & -3 \\ -3 & 10 & 10 & 10 \\ 1 & -6 & 2 & -5 \end{bmatrix}$.

**Answer**

$$\begin{bmatrix} 1 & -3 & -4 & -3 \\ -3 & 10 & 10 & 10 \\ 1 & -6 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & -4 & -3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.E.18}$$

**Exercise 12.E.67**

Find an $LU$ factorization of the matrix $\begin{bmatrix} 1 & 3 & 1 & -1 \\ 3 & 10 & 8 & -1 \\ 2 & 5 & -3 & -3 \end{bmatrix}$.

**Answer**

$$\begin{bmatrix} 1 & 3 & 1 & -1 \\ 3 & 10 & 8 & -1 \\ 2 & 5 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.E.19}$$

**Exercise 12.E.68**

Find an $LU$ factorization of the matrix $\begin{bmatrix} 3 & -2 & 1 \\ 9 & -8 & 6 \\ -6 & 2 & 2 \\ 3 & 2 & -7 \end{bmatrix}$.

**Answer**

$$\begin{bmatrix} 3 & -2 & 1 \\ 9 & -8 & 6 \\ -6 & 2 & 2 \\ 3 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{12.E.20}$$

**Exercise 12.E.69**

Find an $LU$ factorization of the matrix $\begin{bmatrix} -3 & -1 & 3 \\ 9 & 9 & -12 \\ 3 & 19 & -16 \\ 12 & 40 & -26 \end{bmatrix}$.

**Exercise 12.E.70**

Find an $LU$ factorization of the matrix $\begin{bmatrix} -1 & -3 & -1 \\ 1 & 3 & 0 \\ 3 & 9 & 0 \\ 4 & 12 & 16 \end{bmatrix}$.

**Answer**

$$\begin{bmatrix} -1 & -3 & -1 \\ 1 & 3 & 0 \\ 3 & 9 & 0 \\ 4 & 12 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \tag{12.E.21}$$

**Exercise 12.E.71**

Find the $LU$ factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.

$$\begin{aligned} x + 2y &= 5 \\ 2x + 3y &= 6 \end{aligned} \tag{12.E.22}$$

**Answer**

An $LU$ factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \tag{12.E.23}$$

First solve

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \tag{12.E.24}$$

which gives $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$. Then solve

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} \tag{12.E.25}$$

which says that $y = 4$ and $x = -3$.

**Exercise 12.E.72**

Find the $LU$ factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.

$$\begin{aligned} x+2y+z&=1\\ y+3z&=2\\ 2x+3y&=6 \end{aligned} \tag{12.E.26}$$

**Answer**

An $LU$ factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 1\\ 0 & 1 & 3\\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1\\ 0 & 1 & 3\\ 0 & 0 & 1 \end{bmatrix} \tag{12.E.27}$$

First solve

$$\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} u\\ v\\ w \end{bmatrix} = \begin{bmatrix} 1\\ 2\\ 6 \end{bmatrix} \tag{12.E.28}$$

which yields $u=1, v=2, w=6$ . Next solve

$$\begin{bmatrix} 1 & 2 & 1\\ 0 & 1 & 3\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 1\\ 2\\ 6 \end{bmatrix} \tag{12.E.29}$$

This yields $z=6, y=-16, x=27$.

**Exercise 12.E.73**

Find the $LU$ factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.

$$\begin{aligned} x+2y+3z&=5\\ 2x+3y+z&=6\\ x-y+z&=2 \end{aligned} \tag{12.E.30}$$

**Exercise 12.E.74**

Find the $LU$ factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.

$$\begin{aligned} x+2y+3z&=5\\ 2x+3y+z&=6\\ 3x+5y+4z&=11 \end{aligned} \tag{12.E.31}$$

**Answer**

An $LU$ factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 3\\ 2 & 3 & 1\\ 3 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3\\ 0 & -1 & -5\\ 0 & 0 & 0 \end{bmatrix} \tag{12.E.32}$$

First solve

$$\begin{bmatrix} 1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} u\\ v\\ w \end{bmatrix} = \begin{bmatrix} 5\\ 6\\ 11 \end{bmatrix} \tag{12.E.33}$$

Solution is: $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$. Next solve

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} \tag{12.E.34}$$

Solution is: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7t-3 \\ 4-5t \\ t \end{bmatrix}, t \in \mathbb{R}$.

## Exercise $12.E.75$

Is there only one $LU$ factorization for a given matrix? **Hint:** Consider the equation

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{12.E.35}$$

Look for all possible $LU$ factorizations.

**Answer**

Sometimes there is more than one $LU$ factorization as is the case in this example. The given equation clearly gives an $LU$ factorization. However, it appears that the following equation gives another $LU$ factorization.

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{12.E.36}$$

## Exercise $12.E.76$

Prove that if $A^2 = A$ then $|A| = 0$ or $|A| = 1$.

**Answer**

Hint: Start with the given assumption that $A^2 = A$, then take the determinant of both sides of that equal sign. Then work to show the desired conclusion.

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# CHAPTER OVERVIEW

## 13: Vector Spaces


13.1: Algebraic Considerations
13.2: Spanning Sets
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# 13.1: Algebraic Considerations

**Outcomes**

A. Develop the abstract concept of a vector space through axioms.
B. Deduce basic properties of vector spaces.
C. Use the vector space axioms to determine if a set and its operations constitute a vector space.

In this section we consider the idea of an abstract vector space. A vector space is something which has two operations satisfying the following vector space axioms.

**Definition 13.1.1: Vector Space**

A vector space $V$ is a set of vectors with two operations defined, addition and scalar multiplication, which satisfy the axioms of addition and scalar multiplication.

In the following definition we define two operations; vector addition, denoted by $+$ and scalar multiplication denoted by placing the scalar next to the vector. A vector space need not have usual operations, and for this reason the operations will always be given in the definition of the vector space. The below axioms for addition (written +) and scalar multiplication must hold for however addition and scalar multiplication are defined for the vector space.

It is important to note that we have seen much of this content before, in terms of $\mathbb{R}^n$. We will prove in this section that $\mathbb{R}^n$ is an example of a vector space and therefore all discussions in this chapter will pertain to $\mathbb{R}^n$. While it may be useful to consider all concepts of this chapter in terms of $\mathbb{R}^n$, it is also important to understand that these concepts apply to *all* vector spaces.

In the following definition, we will choose scalars $a, b$ to be real numbers and are thus dealing with *real* vector spaces. However, we could also choose scalars which are complex numbers. In this case, we would call the vector space $V$ complex.

**Definition 13.1.2: Axioms of Addition**

Let $\vec{v}, \vec{w}, \vec{z}$ be vectors in a vector space $V$. Then they satisfy the following axioms of addition:

- Closed under Addition

$$\text{If } \vec{v}, \vec{w} \text{ are in } V, \text{ then } \vec{v}+\vec{w} \text{ is also in } V.$$

- The Commutative Law of Addition

$$\vec{v}+\vec{w}=\vec{w}+\vec{v}$$

- The Associative Law of Addition

$$(\vec{v}+\vec{w})+\vec{z}=\vec{v}+(\vec{w}+\vec{z})$$

- The Existence of an Additive Identity

$$\vec{v}+\vec{0}=\vec{v}$$

- The Existence of an Additive Inverse

$$\vec{v}+(-\vec{v})=\vec{0}$$

vectorspaceaxiomsaddition

**Definition 13.1.3: Axioms of Scalar Multiplication**

Let $a, b \in \mathbb{R}$ and let $\vec{v}, \vec{w}, \vec{z}$ be vectors in a vector space $V$. Then they satisfy the following axioms of scalar multiplication:

- Closed under Scalar Multiplication

If $a$ is a real number, and $\vec{v}$ is in $V$, then $a\vec{v}$ is in $V$.

- $a\left(\vec{v}+\vec{w}\right)=a\vec{v}+a\vec{w}$
- $\left(a+b\right)\vec{v}=a\vec{v}+b\vec{v}$
- $a\left(b\vec{v}\right)=\left(ab\right)\vec{v}$
- $1\vec{v}=\vec{v}$

Consider the following example, in which we prove that $\mathbb{R}^n$ is in fact a vector space.

**Example 13.1.1: $\mathbb{R}^n$**

$\mathbb{R}^n$, under the usual operations of vector addition and scalar multiplication, is a vector space.

**Solution**

To show that $\mathbb{R}^n$ is a vector space, we need to show that the above axioms hold. Let $\vec{x},\vec{y},\vec{z}$ be vectors in $\mathbb{R}^n$. We first prove the axioms for vector addition.

- To show that $\mathbb{R}^n$ is closed under addition, we must show that for two vectors in $\mathbb{R}^n$ their sum is also in $\mathbb{R}^n$. The sum $\vec{x}+\vec{y}$ is given by:

$$\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}+\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}=\begin{bmatrix}x_1+y_1\\x_2+y_2\\\vdots\\x_n+y_n\end{bmatrix}$$

  The sum is a vector with $n$ entries, showing that it is in $\mathbb{R}^n$. Hence $\mathbb{R}^n$ is closed under vector addition.
- To show that addition is commutative, consider the following:

$$\begin{aligned}\vec{x}+\vec{y}&=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}+\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}\\&=\begin{bmatrix}x_1+y_1\\x_2+y_2\\\vdots\\x_n+y_n\end{bmatrix}\\&=\begin{bmatrix}y_1+x_1\\y_2+x_2\\\vdots\\y_n+x_n\end{bmatrix}\\&=\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}+\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\\&=\vec{y}+\vec{x}\end{aligned}$$

  Hence addition of vectors in $\mathbb{R}^n$ is commutative.
- We will show that addition of vectors in $\mathbb{R}^n$ is associative in a similar way.

$$
\begin{aligned}
(\vec{x}+\vec{y})+\vec{z} &= \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}\right) + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \\
&= \begin{bmatrix} (x_1+y_1)+z_1 \\ (x_2+y_2)+z_2 \\ \vdots \\ (x_n+y_n)+z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1+(y_1+z_1) \\ x_2+(y_2+z_2) \\ \vdots \\ x_n+(y_n+z_n) \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1+z_1 \\ y_2+z_2 \\ \vdots \\ y_n+z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}\right) \\
&= \vec{x}+(\vec{y}+\vec{z})
\end{aligned}
$$

Hence addition of vectors is associative.

- Next, we show the existence of an additive identity. Let $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$$
\begin{aligned}
\vec{x}+\vec{0} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} x_1+0 \\ x_2+0 \\ \vdots \\ x_n+0 \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \vec{x}
\end{aligned}
$$

Hence the zero vector $\vec{0}$ is an additive identity.

- Next, we prove the existence of an additive inverse. Let $-\vec{x} = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix}$.

$$\begin{aligned} \vec{x} + (-\vec{x}) &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix} \\ &= \begin{bmatrix} x_1 - x_1 \\ x_2 - x_2 \\ \vdots \\ x_n - x_n \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \vec{0} \end{aligned}$$

  Hence $-\vec{x}$ is an additive inverse.

We now need to prove the axioms related to scalar multiplication. Let $a, b$ be real numbers and let $\vec{x}, \vec{y}$ be vectors in $\mathbb{R}^n$.

- We first show that $\mathbb{R}^n$ is closed under scalar multiplication. To do so, we show that $a\vec{x}$ is also a vector with $n$ entries.

$$a\vec{x} = a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix}$$

  The vector $a\vec{x}$ is again a vector with $n$ entries, showing that $\mathbb{R}^n$ is closed under scalar multiplication.
- We wish to show that $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$ .

$$
\begin{aligned}
a(\vec{x}+\vec{y}) &= a\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ \vdots x_n \end{bmatrix}\right) \\
&= a\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix} \\
&= \begin{bmatrix} a(x_1+y_1) \\ a(x_2+y_2) \\ \vdots \\ a(x_n+y_n) \end{bmatrix} \\
&= \begin{bmatrix} ax_1+ay_1 \\ ax_2+ay_2 \\ \vdots \\ ax_n+ay_n \end{bmatrix} \\
&= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} ay_1 \\ ay_2 \\ \vdots \\ ay_n \end{bmatrix} \\
&= a\vec{x}+a\vec{y}
\end{aligned}
$$

- Next, we wish to show that $(a+b)\vec{x} = a\vec{x}+b\vec{x}$ .

$$
\begin{aligned}
(a+b)\vec{x} &= (a+b)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} (a+b)x_1 \\ (a+b)x_2 \\ \vdots \\ (a+b)x_n \end{bmatrix} \\
&= \begin{bmatrix} ax_1+bx_1 \\ ax_2+bx_2 \\ \vdots \\ ax_n+bx_n \end{bmatrix} \\
&= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix} \\
&= a\vec{x}+b\vec{x}
\end{aligned}
$$

- We wish to show that $a(b\vec{x}) = (ab)\vec{x}$ .

$$
\begin{aligned}
a(b\vec{x}) &= a\left(b\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) \\
&= a\left(\begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix}\right) \\
&= \begin{bmatrix} a(bx_1) \\ a(bx_2) \\ \vdots \\ a(bx_n) \end{bmatrix} \\
&= \begin{bmatrix} (ab)x_1 \\ (ab)x_2 \\ \vdots \\ (ab)x_n \end{bmatrix} \\
&= (ab)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= (ab)\vec{x}
\end{aligned}
$$

- Finally, we need to show that $1\vec{x} = \vec{x}$.

$$
\begin{aligned}
1\vec{x} &= 1\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} 1x_1 \\ 1x_2 \\ \vdots \\ 1x_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \vec{x}
\end{aligned}
$$

By the above proofs, it is clear that $\mathbb{R}^n$ satisfies the vector space axioms. Hence, $\mathbb{R}^n$ is a vector space under the usual operations of vector addition and scalar multiplication.

We now consider some examples of vector spaces.

### Example 13.1.2: Vector Space of Polynomials

Let $\mathbb{P}_2$ be the set of all polynomials of at most degree 2 as well as the zero polynomial. Define addition to be the standard addition of polynomials, and scalar multiplication the usual multiplication of a polynomial by a number. Then $\mathbb{P}_2$ is a vector space.

**Solution**

We can write $\mathbb{P}_2$ explicitly as

$$\mathbb{P}_2 = \left\{a_2x^2 + a_1x + a_0 | a_i \in \mathbb{R} \text{ for all } i\right\}$$

To show that $\mathbb{P}_2$ is a vector space, we verify the axioms. Let $p(x), q(x), r(x)$ be polynomials in $\mathbb{P}_2$ and let $a, b, c$ be real numbers. Write $p(x) = p_2x^2 + p_1x + p_0$ , $q(x) = q_2x^2 + q_1x + q_0$ , and $r(x) = r_2x^2 + r_1x + r_0$ .

- We first prove that addition of polynomials in $\mathbb{P}_2$ is closed. For two polynomials in $\mathbb{P}_2$ we need to show that their sum is also a polynomial in $\mathbb{P}_2$. From the definition of $\mathbb{P}_2$, a polynomial is contained in $\mathbb{P}_2$ if it is of degree at most 2 or the zero polynomial.

$$\begin{aligned} p(x) + q(x) &= p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0 \\ &= (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) \end{aligned}$$

  The sum is a polynomial of degree 2 and therefore is in $\mathbb{P}_2$. It follows that $\mathbb{P}_2$ is closed under addition.
- We need to show that addition is commutative, that is $p(x) + q(x) = q(x) + p(x)$ .

$$\begin{aligned} p(x) + q(x) &= p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0 \\ &= (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) \\ &= (q_2 + p_2)x^2 + (q_1 + p_1)x + (q_0 + p_0) \\ &= q_2x^2 + q_1x + q_0 + p_2x^2 + p_1x + p_0 \\ &= q(x) + p(x) \end{aligned}$$

- Next, we need to show that addition is associative. That is, that $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$ .

$$\begin{aligned} (p(x) + q(x)) + r(x) &= \left(p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0\right) + r_2x^2 + r_1x + r_0 \\ &= (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) + r_2x^2 + r_1x + r_0 \\ &= (p_2 + q_2 + r_2)x^2 + (p_1 + q_1 + r_1)x + (p_0 + q_0 + r_0) \\ &= p_2x^2 + p_1x + p_0 + (q_2 + r_2)x^2 + (q_1 + r_1)x + (q_0 + r_0) \\ &= p_2x^2 + p_1x + p_0 + \left(q_2x^2 + q_1x + q_0 + r_2x^2 + r_1x + r_0\right) \\ &= p(x) + (q(x) + r(x)) \end{aligned}$$

- Next, we must prove that there exists an additive identity. Let $0(x) = 0x^2 + 0x + 0$ .

$$\begin{aligned} p(x) + 0(x) &= p_2x^2 + p_1x + p_0 + 0x^2 + 0x + 0 \\ &= (p_2 + 0)x^2 + (p_1 + 0)x + (p_0 + 0) \\ &= p_2x^2 + p_1x + p_0 \\ &= p(x) \end{aligned}$$

  Hence an additive identity exists, specifically the zero polynomial.
- Next we must prove that there exists an additive inverse. Let $-p(x) = -p_2x^2 - p_1x - p_0$ and consider the following:

$$\begin{aligned} p(x) + (-p(x)) &= p_2x^2 + p_1x + p_0 + \left(-p_2x^2 - p_1x - p_0\right) \\ &= (p_2 - p_2)x^2 + (p_1 - p_1)x + (p_0 - p_0) \\ &= 0x^2 + 0x + 0 \\ &= 0(x) \end{aligned}$$

  Hence an additive inverse $-p(x)$ exists such that $p(x) + (-p(x)) = 0(x)$.

We now need to verify the axioms related to scalar multiplication.

- First we prove that $\mathbb{P}_2$ is closed under scalar multiplication. That is, we show that $ap(x)$ is also a polynomial of degree at most 2.

$$ap(x) = a\left(p_2x^2 + p_1x + p_0\right) = ap_2x^2 + ap_1x + ap_0$$

  Therefore $\mathbb{P}_2$ is closed under scalar multiplication.
- We need to show that $a(p(x) + q(x)) = ap(x) + aq(x)$ .

$$\begin{aligned} a(p(x)+q(x)) &= a\left(p_2x^2+p_1x+p_0+q_2x^2+q_1x+q_0\right) \\ &= a\left((p_2+q_2)x^2+(p_1+q_1)x+(p_0+q_0)\right) \\ &= a(p_2+q_2)x^2+a(p_1+q_1)x+a(p_0+q_0) \\ &= (ap_2+aq_2)x^2+(ap_1+aq_1)x+(ap_0+aq_0) \\ &= ap_2x^2+ap_1x+ap_0+aq_2x^2+aq_1x+aq_0 \\ &= ap(x)+aq(x) \end{aligned}$$

- Next we show that $(a+b)p(x)=ap(x)+bp(x)$ .

$$\begin{aligned} (a+b)p(x) &= (a+b)(p_2x^2+p_1x+p_0) \\ &= (a+b)p_2x^2+(a+b)p_1x+(a+b)p_0 \\ &= ap_2x^2+ap_1x+ap_0+bp_2x^2+bp_1x+bp_0 \\ &= ap(x)+bp(x) \end{aligned}$$

- The next axiom which needs to be verified is $a(bp(x))=(ab)p(x)$.

$$\begin{aligned} a(bp(x)) &= a\left(b\left(p_2x^2+p_1x+p_0\right)\right) \\ &= a\left(bp_2x^2+bp_1x+bp_0\right) \\ &= abp_2x^2+abp_1x+abp_0 \\ &= (ab)(p_2x^2+p_1x+p_0) \\ &= (ab)p(x) \end{aligned}$$

- Finally, we show that $1p(x)=p(x)$.

$$\begin{aligned} 1p(x) &= 1\left(p_2x^2+p_1x+p_0\right) \\ &= 1p_2x^2+1p_1x+1p_0 \\ &= p_2x^2+p_1x+p_0 \\ &= p(x) \end{aligned}$$

Since the above axioms hold, we know that $\mathbb{P}_2$ as described above is a vector space.

Another important example of a vector space is the set of all matrices of the same size.

### Example 13.1.3: Vector Space of Matrices

Let $\mathbb{M}_{2,3}$ be the set of all $2\times3$ matrices. Using the usual operations of matrix addition and scalar multiplication, show that $\mathbb{M}_{2,3}$ is a vector space.

#### Solution

Let $A,B$ be $2\times3$ matrices in $\mathbb{M}_{2,3}$. We first prove the axioms for addition.

- In order to prove that $\mathbb{M}_{2,3}$ is closed under matrix addition, we show that the sum $A+B$ is in $\mathbb{M}_{2,3}$. This means showing that $A+B$ is a $2\times3$ matrix.

$$\begin{aligned} A+B &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix} \end{aligned}$$

  You can see that the sum is a $2\times3$ matrix, so it is in $\mathbb{M}_{2,3}$. It follows that $\mathbb{M}_{2,3}$ is closed under matrix addition.
- The remaining axioms regarding matrix addition follow from properties of matrix addition. Therefore $\mathbb{M}_{2,3}$ satisfies the axioms of matrix addition.

We now turn our attention to the axioms regarding scalar multiplication. Let $A,B$ be matrices in $\mathbb{M}_{2,3}$ and let $c$ be a real number.

- We first show that $\mathbb{M}_{2,3}$ is closed under scalar multiplication. That is, we show that $cA$ a $2\times3$ matrix.

$$cA = c\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{bmatrix}$$

This is a $2 \times 3$ matrix in $\mathbb{M}_{2,3}$ which proves that the set is closed under scalar multiplication.

- The remaining axioms of scalar multiplication follow from properties of scalar multiplication of matrices. Therefore $\mathbb{M}_{2,3}$ satisfies the axioms of scalar multiplication.

In conclusion, $\mathbb{M}_{2,3}$ satisfies the required axioms and is a vector space.

While here we proved that the set of all $2 \times 3$ matrices is a vector space, there is nothing special about this choice of matrix size. In fact if we instead consider $\mathbb{M}_{m,n}$, the set of all $m \times n$ matrices, then $\mathbb{M}_{m,n}$ is a vector space under the operations of matrix addition and scalar multiplication.

We now examine an example of a set that does not satisfy all of the above axioms, and is therefore *not* a vector space.

### Example 13.1.4: Not a Vector Space

Let $V$ denote the set of $2 \times 3$ matrices. Let addition in $V$ be defined by $A + B = A$ for matrices $A, B$ in $V$. Let scalar multiplication in $V$ be the usual scalar multiplication of matrices. Show that $V$ is not a vector space.

**Solution**

In order to show that $V$ is not a vector space, it suffices to find only one axiom which is not satisfied. We will begin by examining the axioms for addition until one is found which does not hold. Let $A, B$ be matrices in $V$.

- We first want to check if addition is closed. Consider $A + B$. By the definition of addition in the example, we have that $A + B = A$. Since $A$ is a $2 \times 3$ matrix, it follows that the sum $A + B$ is in $V$, and $V$ is closed under addition.
- We now wish to check if addition is commutative. That is, we want to check if $A + B = B + A$ **for all** choices of $A$ and $B$ in $V$. From the definition of addition, we have that $A + B = A$ and $B + A = B$. Therefore, we can find $A, B$ in $V$ such that these sums are not equal. One example is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using the operation defined by $A + B = A$, we have

$$\begin{aligned} A + B &= A \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ B + A &= B \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

It follows that $A + B \neq B + A$. Therefore addition as defined for $V$ is not commutative and $V$ fails this axiom. Hence $V$ is not a vector space.

Consider another example of a vector space.

### Example 13.1.5: Vector Space of Functions

Let $S$ be a nonempty set and define $\mathbb{F}_S$ to be the set of *real* functions defined on $S$. In other words, we write $\mathbb{F}_S : S \mapsto \mathbb{R}$. Letting $a, b, c$ be scalars and $f, g, h$ functions, the vector operations are defined as

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ (af)(x) &= a(f(x)) \end{aligned}$$

Show that $\mathbb{F}_S$ is a vector space.

**Solution**

To verify that $\mathbb{F}_S$ is a vector space, we must prove the axioms beginning with those for addition. Let $f, g, h$ be functions in $\mathbb{F}_S$.

- First we check that addition is closed. For functions $f, g$ defined on the set $S$, their sum given by

$$(f+g)(x) = f(x)+g(x)$$

is again a function defined on $S$. Hence this sum is in $\mathbb{F}_S$ and $\mathbb{F}_S$ is closed under addition.

- Secondly, we check the commutative law of addition:

$$(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$$

Since $x$ is arbitrary, $f+g=g+f$ .

- Next we check the associative law of addition:

$$((f+g)+h)(x) = (f+g)(x)+h(x) = (f(x)+g(x))+h(x)$$

$$= f(x)+(g(x)+h(x)) = (f(x)+(g+h)(x)) = (f+(g+h))(x)$$

and so $(f+g)+h = f+(g+h)$.

- Next we check for an additive identity. Let $0$ denote the function which is given by $0(x) = 0$. Then this is an additive identity because

$$(f+0)(x) = f(x)+0(x) = f(x)$$

and so $f+0 = f$.

- Finally, check for an additive inverse. Let $-f$ be the function which satisfies $(-f)(x) = -f(x)$. Then

$$(f+(-f))(x) = f(x)+(-f)(x) = f(x)+-f(x) = 0$$

Hence $f+(-f) = 0$.

Now, check the axioms for scalar multiplication.

- We first need to check that $\mathbb{F}_S$ is closed under scalar multiplication. For a function $f(x)$ in $\mathbb{F}_S$ and real number $a$, the function $(af)(x) = a(f(x))$ is again a function defined on the set $S$. Hence $a(f(x))$ is in $\mathbb{F}_S$ and $\mathbb{F}_S$ is closed under scalar multiplication.
- $$((a+b)f)(x) = (a+b)f(x) = af(x)+bf(x) = (af+bf)(x)$$

and so $(a+b)f = af+bf$ .

- $$(a(f+g))(x) = a(f+g)(x) = a(f(x)+g(x))$$

$$= af(x)+bg(x) = (af+bg)(x)$$

and so $a(f+g) = af+bg$ .

- $$((ab)f)(x) = (ab)f(x) = a(bf(x)) = (a(bf))(x)$$

so $(abf) = a(bf)$.

- Finally $(1f)(x) = 1f(x) = f(x)$ so $1f = f$.

It follows that $V$ satisfies all the required axioms and is a vector space.

Consider the following important theorem.

**Theorem 13.1.1: Uniqueness**

In any vector space, the following are true:

1. $\vec{0}$, the additive identity, is unique
2. $-\vec{x}$, the additive inverse, is unique
3. $0\vec{x} = \vec{0}$ for all vectors $\vec{x}$

4. $(-1)\vec{x} = -\vec{x}$ for all vectors $\vec{x}$

**Proof**

1. When we say that the additive identity, $\vec{0}$, is unique, we mean that if a vector acts like the additive identity, then it **is** the additive identity. To prove this uniqueness, we want to show that another vector which acts like the additive identity is actually equal to $\vec{0}$. Suppose $\vec{0}'$ is also an additive identity. Then,

$$\vec{0} + \vec{0}' = \vec{0}$$

Now, for $\vec{0}$ the additive identity given above in the axioms, we have that

$$\vec{0}' + \vec{0} = \vec{0}'$$

So by the commutative property:

$$0 = 0 + 0' = 0' + 0 = 0'$$

This says that if a vector acts like an additive identity (such as $\vec{0}'$), it in fact equals $\vec{0}$. This proves the uniqueness of $\vec{0}$.

2. When we say that the additive inverse, $-\vec{x}$, is unique, we mean that if a vector acts like the additive inverse, then it **is** the additive inverse. Suppose that $\vec{y}$ acts like an additive inverse:

$$\vec{x} + \vec{y} = \vec{0}$$

Then the following holds:

$$\vec{y} = \vec{0} + \vec{y} = (-\vec{x} + \vec{x}) + \vec{y} = -\vec{x} + (\vec{x} + \vec{y}) = -\vec{x} + \vec{0} = -\vec{x}$$

Thus if $\vec{y}$ acts like the additive inverse, it is equal to the additive inverse $-\vec{x}$. This proves the uniqueness of $-\vec{x}$.

3. This statement claims that for all vectors $\vec{x}$, scalar multiplication by 0 equals the zero vector $\vec{0}$. Consider the following, using the fact that we can write $0 = 0 + 0$ :

$$0\vec{x} = (0 + 0)\vec{x} = 0\vec{x} + 0\vec{x}$$

We use a small trick here: add $-0\vec{x}$ to both sides. This gives

$$\begin{aligned} 0\vec{x} + (-0\vec{x}) &= 0\vec{x} + 0\vec{x} + (-\vec{x}) \\ \vec{0} + 0 &= 0\vec{x} + 0 \\ \vec{0} &= 0\vec{x} \end{aligned}$$

This proves that scalar multiplication of any vector by 0 results in the zero vector $\vec{0}$.

4. Finally, we wish to show that scalar multiplication of $-1$ and any vector $\vec{x}$ results in the additive inverse of that vector, $-\vec{x}$. Recall from 2. above that the additive inverse is unique. Consider the following:

$$\begin{aligned} (-1)\vec{x} + \vec{x} &= (-1)\vec{x} + 1\vec{x} \\ &= (-1 + 1)\vec{x} \\ &= 0\vec{x} \\ &= \vec{0} \end{aligned}$$

By the uniqueness of the additive inverse shown earlier, any vector which acts like the additive inverse must be equal to the additive inverse. It follows that $(-1)\vec{x} = -\vec{x}$ .

An important use of the additive inverse is the following theorem.

**Theorem 13.1.2**

Let $V$ be a vector space. Then $\vec{v} + \vec{w} = \vec{v} + \vec{z}$ implies that $\vec{w} = \vec{z}$ for all $\vec{v}, \vec{w}, \vec{z} \in V$

**Proof**

The proof follows from the vector space axioms, in particular the existence of an additive inverse $(-\vec{u})$. The proof is left as an exercise to the reader.

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# 13.2: Spanning Sets

**Outcomes**

A. Determine if a vector is within a given span.

In this section we will examine the concept of spanning introduced earlier in terms of $\mathbb{R}^n$. Here, we will discuss these concepts in terms of abstract vector spaces.

Consider the following definition.

**Definition 13.2.1: Subset**

Let $X$ and $Y$ be two sets. If all elements of $X$ are also elements of $Y$ then we say that $X$ is a **subset** of $Y$ and we write

$$X \subseteq Y$$

In particular, we often speak of subsets of a vector space, such as $X \subseteq V$. By this we mean that every element in the set $X$ is contained in the vector space $V$.

**Definition 13.2.2: Linear Combination**

Let $V$ be a vector space and let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \subseteq V$. A vector $\vec{v} \in V$ is called a linear combination of the $\vec{v}_i$ if there exist scalars $c_i \in \mathbb{R}$ such that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

This definition leads to our next concept of span.

**Definition 13.2.3: Span of Vectors**

Let $\{\vec{v}_1, \cdots, \vec{v}_n\} \subseteq V$. Then

$$\text{span}\{\vec{v}_1, \cdots, \vec{v}_n\} = \left\{ \sum_{i=1}^{n} c_i\vec{v}_i : c_i \in \mathbb{R} \right\}$$

When we say that a vector $\vec{w}$ is in $\text{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$ we mean that $\vec{w}$ can be written as a linear combination of the $\vec{v}_1$. We say that a collection of vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a **spanning set** for $V$ if $V = \text{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$.

Consider the following example.

**Example 13.2.1: Matrix Span**

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Determine if $A$ and $B$ are in

$$\text{span}\{M_1, M_2\} = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

**Solution**

First consider $A$. We want to see if scalars $s, t$ can be found such that $A = sM_1 + tM_2$ .

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The solution to this equation is given by

$$1 = s$$
$$2 = t$$

and it follows that $A$ is in span $\{M_1, M_2\}$.

Now consider $B$. Again we write $B = sM_1 + tM_2$ and see if a solution can be found for $s, t$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly no values of $s$ and $t$ can be found such that this equation holds. Therefore $B$ is not in span $\{M_1, M_2\}$.

Consider another example.

**Example 13.2.2: Polynomial Span**

Show that $p(x) = 7x^2 + 4x - 3$ is in span $\left\{4x^2 + x, x^2 - 2x + 3\right\}$.

**Solution**

To show that $p(x)$ is in the given span, we need to show that it can be written as a linear combination of polynomials in the span. Suppose scalars $a, b$ existed such that

$$7x^2 + 4x - 3 = a(4x^2 + x) + b(x^2 - 2x + 3)$$

If this linear combination were to hold, the following would be true:

$$\begin{aligned} 4a + b &= 7 \\ a - 2b &= 4 \\ 3b &= -3 \end{aligned}$$

You can verify that $a = 2, b = -1$ satisfies this system of equations. This means that we can write $p(x)$ as follows:

$$7x^2 + 4x - 3 = 2(4x^2 + x) - (x^2 - 2x + 3)$$

Hence $p(x)$ is in the given span.

Consider the following example.

**Example 13.2.3: Spanning Set**

Let $S = \left\{x^2 + 1, x - 2, 2x^2 - x\right\}$. Show that $S$ is a spanning set for $\mathbb{P}_2$, the set of all polynomials of degree at most 2.

**Solution**

Let $p(x) = ax^2 + bx + c$ be an arbitrary polynomial in $\mathbb{P}_2$. To show that $S$ is a spanning set, it suffices to show that $p(x)$ can be written as a linear combination of the elements of $S$. In other words, can we find $r, s, t$ such that:

$$p(x) = ax^2 + bx + c = r(x^2 + 1) + s(x - 2) + t(2x^2 - x)$$

If a solution $r, s, t$ can be found, then this shows that for any such polynomial $p(x)$, it can be written as a linear combination of the above polynomials and $S$ is a spanning set.

$$\begin{aligned} ax^2 + bx + c &= r(x^2 + 1) + s(x - 2) + t(2x^2 - x) \\ &= rx^2 + r + sx - 2s + 2tx^2 - tx \\ &= (r + 2t)x^2 + (s - t)x + (r - 2s) \end{aligned}$$

For this to be true, the following must hold:

$$\begin{aligned} a &= r + 2t \\ b &= s - t \\ c &= r - 2s \end{aligned}$$

To check that a solution exists, set up the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b \\ 1 & -2 & 0 & c \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}a+2b+\frac{1}{2}c \\ 0 & 1 & 0 & \frac{1}{4}a-\frac{1}{4}c \\ 0 & 0 & 1 & \frac{1}{4}a-b-\frac{1}{4}c \end{array}\right]$$

Clearly a solution exists for any choice of $a, b, c$. Hence $S$ is a spanning set for $\mathbb{P}_2$.

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# 13.3: Linear Independence

**Outcomes**

A. Determine if a set is linearly independent.

In this section, we will again explore concepts introduced earlier in terms of $\mathbb{R}^n$ and extend them to apply to abstract vector spaces.

**Definition 13.3.1: Linear Independence**

Let $V$ be a vector space. If $\{\vec{v}_1, \cdots, \vec{v}_n\} \subseteq V$, then it is linearly independent if

$$\sum_{i=1}^{n} a_i \vec{v}_i = \vec{0} \text{ implies } a_1 = \cdots = a_n = 0$$

where the $a_i$ are real numbers.

The set of vectors is called linearly dependent if it is not linearly independent.

**Example 13.3.1: Linear Independence**

Let $S \subseteq \mathbb{P}_2$ be a set of polynomials given by

$$S = \left\{x^2 + 2x - 1, 2x^2 - x + 3\right\}$$

Determine if $S$ is linearly independent.

**Solution**

To determine if this set $S$ is linearly independent, we write

$$a(x^2 + 2x - 1) + b(2x^2 - x + 3) = 0x^2 + 0x + 0$$

If it is linearly independent, then $a = b = 0$ will be the only solution. We proceed as follows.

$$\begin{aligned} a(x^2 + 2x - 1) + b(2x^2 - x + 3) &= 0x^2 + 0x + 0 \\ ax^2 + 2ax - a + 2bx^2 - bx + 3b &= 0x^2 + 0x + 0 \\ (a + 2b)x^2 + (2a - b)x - a + 3b &= 0x^2 + 0x + 0 \end{aligned}$$

It follows that

$$\begin{aligned} a + 2b &= 0 \\ 2a - b &= 0 \\ -a + 3b &= 0 \end{aligned}$$

The augmented matrix and resulting reduced row-echelon form are given by

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \\ -1 & 3 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence the solution is $a = b = 0$ and the set is linearly independent.

The next example shows us what it means for a set to be dependent.

**Example 13.3.2: Dependent Set**

Determine if the set $S$ given below is independent.

$$S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$$

**Solution**

To determine if $S$ is linearly independent, we look for solutions to

$$a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice that this equation has nontrivial solutions, for example $a = 2$, $b = 3$ and $c = -1$. Therefore $S$ is dependent.

The following is an important result regarding dependent sets.

**Lemma 13.3.1: Dependent Sets**

Let $V$ be a vector space and suppose $W = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is a subset of $V$. Then $W$ is dependent if and only if $\vec{v}_i$ can be written as a linear combination of $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{i-1}, \vec{v}_{i+1}, \cdots, \vec{v}_k\}$ for some $i \leq k$.

Revisit Example 13.3.2 with this in mind. Notice that we can write one of the three vectors as a combination of the others.

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

By Lemma 13.3.1 this set is dependent.

If we know that one particular set is linearly independent, we can use this information to determine if a related set is linearly independent. Consider the following example.

**Example 13.3.3: Related Independent Sets**

Let $V$ be a vector space and suppose $S \subseteq V$ is a set of linearly independent vectors given by $S = \{\vec{u}, \vec{v}, \vec{w}\}$. Let $R \subseteq V$ be given by $R = \left\{2\vec{u} - \vec{w}, \vec{w} + \vec{v}, 3\vec{v} + \frac{1}{2}\vec{u}\right\}$. Show that $R$ is also linearly independent.

**Solution**

To determine if $R$ is linearly independent, we write

$$a(2\vec{u} - \vec{w}) + b(\vec{w} + \vec{v}) + c(3\vec{v} + \frac{1}{2}\vec{u}) = \vec{0}$$

If the set is linearly independent, the only solution will be $a = b = c = 0$. We proceed as follows.

$$\begin{aligned} a(2\vec{u} - \vec{w}) + b(\vec{w} + \vec{v}) + c(3\vec{v} + \frac{1}{2}\vec{u}) &= \vec{0} \\ 2a\vec{u} - a\vec{w} + b\vec{w} + b\vec{v} + 3c\vec{v} + \frac{1}{2}c\vec{u} &= \vec{0} \\ (2a + \frac{1}{2}c)\vec{u} + (b + 3c)\vec{v} + (-a + b)\vec{w} &= \vec{0} \end{aligned}$$

We know that the set $S = \{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, which implies that the coefficients in the last line of this equation must all equal 0. In other words:

$$\begin{aligned} 2a + \frac{1}{2}c &= 0 \\ b + 3c &= 0 \\ -a + b &= 0 \end{aligned}$$

The augmented matrix and resulting reduced row-echelon form are given by:

$$\left[\begin{array}{ccc|c} 2 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Hence the solution is $a = b = c = 0$ and the set is linearly independent.

The following theorem was discussed in terms in $\mathbb{R}^n$. We consider it here in the general case.

**Theorem 13.3.1: Unique Representation**

Let $V$ be a vector space and let $U = \{\vec{v}_1, \cdots, \vec{v}_k\} \subseteq V$ be an independent set. If $\vec{v} \in \text{span}\, U$, then $\vec{v}$ can be written uniquely as a linear combination of the vectors in $U$.

Consider the span of a linearly independent set of vectors. Suppose we take a vector which is not in this span and add it to the set. The following lemma claims that the resulting set is still linearly independent.

**Lemma 13.3.2: Adding to a Linearly Independent Set**

Suppose $\vec{v} \notin \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$ and $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent. Then the set

$$\{\vec{u}_1, \cdots, \vec{u}_k, \vec{v}\}$$

is also linearly independent.

**Proof**

Suppose $\sum_{i=1}^k c_i \vec{u}_i + d\vec{v} = \vec{0}$. It is required to verify that each $c_i = 0$ and that $d = 0$. But if $d \neq 0$, then you can solve for $\vec{v}$ as a linear combination of the vectors, $\{\vec{u}_1, \cdots, \vec{u}_k\}$,

$$\vec{v} = -\sum_{i=1}^k \left(\frac{c_i}{d}\right) \vec{u}_i$$

contrary to the assumption that $\vec{v}$ is not in the span of the $\vec{u}_i$. Therefore, $d = 0$. But then $\sum_{i=1}^k c_i \vec{u}_i = \vec{0}$ and the linear independence of $\{\vec{u}_1, \cdots, \vec{u}_k\}$ implies each $c_i = 0$ also.

Consider the following example.

**Example 13.3.4: Adding to a Linearly Independent Set**

Let $S \subseteq M_{22}$ be a linearly independent set given by

$$S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$$

Show that the set $R \subseteq M_{22}$ given by

$$R = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$$

is also linearly independent.

**Solution**

Instead of writing a linear combination of the matrices which equals 0 and showing that the coefficients must equal 0, we can instead use Lemma 13.3.2.

To do so, we show that

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$$

Write

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

Clearly there are no possible $a, b$ to make this equation true. Hence the new matrix does not lie in the span of the matrices in $S$. By Lemma 13.3.2, $R$ is also linearly independent.

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# 13.4: Subspaces and Basis

**Outcomes**

A. Utilize the subspace test to determine if a set is a subspace of a given vector space.
B. Extend a linearly independent set and shrink a spanning set to a basis of a given vector space.

In this section we will examine the concept of subspaces introduced earlier in terms of $\mathbb{R}^n$. Here, we will discuss these concepts in terms of abstract vector spaces.

Consider the definition of a subspace.

**Definition 13.4.1: Subspace**

Let $V$ be a vector space. A subset $W \subseteq V$ is said to be a **subspace** of $V$ if $a\vec{x}+b\vec{y} \in W$ whenever $a, b \in \mathbb{R}$ and $\vec{x}, \vec{y} \in W$.

The span of a set of vectors as described in Definition 9.2.3 is an example of a subspace. The following fundamental result says that subspaces are subsets of a vector space which are themselves vector spaces.

**Theorem 13.4.1: Subspaces are Vector Spaces**

Let $W$ be a nonempty collection of vectors in a vector space $V$. Then $W$ is a subspace if and only if $W$ satisfies the vector space axioms, using the same operations as those defined on $V$.

**Proof**

Suppose first that $W$ is a subspace. It is obvious that all the algebraic laws hold on $W$ because it is a subset of $V$ and they hold on $V$. Thus $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ along with the other axioms. Does $W$ contain $\vec{0}$? Yes because it contains $0\vec{u}=\vec{0}$. See Theorem 9.1.1.

Are the operations of $V$ defined on $W$? That is, when you add vectors of $W$ do you get a vector in $W$? When you multiply a vector in $W$ by a scalar, do you get a vector in $W$? Yes. This is contained in the definition. Does every vector in $W$ have an additive inverse? Yes by Theorem 9.1.1 because $-\vec{v}=(-1)\vec{v}$ which is given to be in $W$ provided $\vec{v} \in W$.

Next suppose $W$ is a vector space. Then by definition, it is closed with respect to linear combinations. Hence it is a subspace.

Consider the following useful Corollary.

**Corollary 13.4.1: Span is a Subspace**

Let $V$ be a vector space with $W \subseteq V$. If $W = \text{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$ then $W$ is a subspace of $V$.

When determining spanning sets the following theorem proves useful.

**Theorem 13.4.2: Spanning Set**

Let $W \subseteq V$ for a vector space $V$ and suppose $W = \text{span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$.

Let $U \subseteq V$ be a subspace such that $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \in U$. Then it follows that $W \subseteq U$.

In other words, this theorem claims that any subspace that contains a set of vectors must also contain the span of these vectors.

The following example will show that two spans, described differently, can in fact be equal.

### Example 13.4.1: Equal Span

Let $p(x), q(x)$ be polynomials and suppose $U = \text{span}\{2p(x) - q(x), p(x) + 3q(x)\}$ and $W = \text{span}\{p(x), q(x)\}$. Show that $U = W$.

**Solution**

We will use Theorem 13.4.2 to show that $U \subseteq W$ and $W \subseteq U$. It will then follow that $U = W$.

1. $U \subseteq W$ Notice that $2p(x) - q(x)$ and $p(x) + 3q(x)$ are both in $W = span\{p(x), q(x)\}$. Then by Theorem 13.4.2 $W$ must contain the span of these polynomials and so $U \subseteq W$.
2. $W \subseteq U$ Notice that

$$\begin{aligned} p(x) &= \frac{3}{7}(2p(x) - q(x)) + \frac{2}{7}(p(x) + 3q(x)) \\ q(x) &= -\frac{1}{7}(2p(x) - q(x)) + \frac{2}{7}(p(x) + 3q(x)) \end{aligned}$$

Hence $p(x),\ q(x)$ are in $span\{2p(x) - q(x),\ p(x) + 3q(x)\}$. By Theorem 13.4.2 $U$ must contain the span of these polynomials and so $W \subseteq U$.

To prove that a set is a vector space, one must verify each of the axioms given in Definition 9.1.2 and 9.1.3. This is a cumbersome task, and therefore a shorter procedure is used to verify a subspace.

### Procedure 13.4.1: Subspace Test

Suppose $W$ is a subset of a vector space $V$. To determine if $W$ is a subspace of $V$, it is sufficient to determine if the following three conditions hold, using the operations of $V$:

1. The additive identity $\vec{0}$ of $V$ is contained in $W$.
2. For any vectors $\vec{w}_1, \vec{w}_2$ in $W$, $\vec{w}_1 + \vec{w}_2$ is also in $W$.
3. For any vector $\vec{w}_1$ in $W$ and scalar $a$, the product $a\vec{w}_1$ is also in $W$.

Therefore it suffices to prove these three steps to show that a set is a subspace.

Consider the following example.

### Example 13.4.2: Improper Subspaces

Let $V$ be an arbitrary vector space. Then $V$ is a subspace of itself. Similarly, the set $\left\{\vec{0}\right\}$ containing only the zero vector is also a subspace.

**Solution**

Using the subspace test in Procedure 13.4.1 we can show that $V$ and $\left\{\vec{0}\right\}$ are subspaces of $V$.

Since $V$ satisfies the vector space axioms it also satisfies the three steps of the subspace test. Therefore $V$ is a subspace.

Let's consider the set $\left\{\vec{0}\right\}$.

1. The vector $\vec{0}$ is clearly contained in $\left\{\vec{0}\right\}$, so the first condition is satisfied.
2. Let $\vec{w}_1, \vec{w}_2$ be in $\left\{\vec{0}\right\}$. Then $\vec{w}_1 = \vec{0}$ and $\vec{w}_2 = \vec{0}$ and so

$$\vec{w}_1 + \vec{w}_2 = \vec{0} + \vec{0} = \vec{0}$$

It follows that the sum is contained in $\left\{\vec{0}\right\}$ and the second condition is satisfied.

3. Let $\vec{w}_1$ be in $\left\{\vec{0}\right\}$ and let $a$ be an arbitrary scalar. Then

$$a\vec{w}_1 = a\vec{0} = \vec{0}$$

Hence the product is contained in $\left\{\vec{0}\right\}$ and the third condition is satisfied.

It follows that $\left\{\vec{0}\right\}$ is a subspace of $V$.

The two subspaces described above are called **improper subspaces**. Any subspace of a vector space $V$ which is not equal to $V$ or $\left\{\vec{0}\right\}$ is called a **proper subspace**.

Consider another example.

### Example 13.4.3: Subspace of Polynomials

Let $\mathbb{P}_2$ be the vector space of polynomials of degree two or less. Let $W \subseteq \mathbb{P}_2$ be all polynomials of degree two or less which have 1 as a root. Show that $W$ is a subspace of $\mathbb{P}_2$.

**Solution**

First, express $W$ as follows:

$$W = \left\{p(x) = ax^2 + bx + c, a, b, c, \in \mathbb{R} | p(1) = 0\right\}$$

We need to show that $W$ satisfies the three conditions of Procedure 13.4.1.

1. The zero polynomial of $\mathbb{P}_2$ is given by $0(x) = 0x^2 + 0x + 0 = 0$ . Clearly $0(1) = 0$ so $0(x)$ is contained in $W$.
2. Let $p(x), q(x)$be polynomials in $W$. It follows that $p(1) = 0$ and $q(1) = 0$. Now consider $p(x) + q(x)$. Let $r(x)$ represent this sum.

$$\begin{aligned} r(1) &= p(1) + q(1) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

   Therefore the sum is also in $W$ and the second condition is satisfied.
3. Let $p(x)$ be a polynomial in $W$ and let $a$ be a scalar. It follows that $p(1) = 0$. Consider the product $ap(x)$.

$$\begin{aligned} ap(1) &= a(0) \\ &= 0 \end{aligned}$$

   Therefore the product is in $W$ and the third condition is satisfied.

It follows that $W$ is a subspace of $\mathbb{P}_2$.

Recall the definition of basis, considered now in the context of vector spaces.

### Definition 13.4.2: Basis

Let $V$ be a vector space. Then $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is called a basis for $V$ if the following conditions hold.

1. $\text{span}\{\vec{v}_1, \cdots, \vec{v}_n\} = V$
2. $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent

Consider the following example.

### Example 13.4.4: Polynomials of Degree Two

Let $\mathbb{P}_2$ be the set polynomials of degree no more than 2. We can write $\mathbb{P}_2 = \text{span}\left\{x^2, x, 1\right\}$. Is $\left\{x^2, x, 1\right\}$ a basis for $\mathbb{P}_2$?

**Solution**

It can be verified that $\mathbb{P}_2$ is a vector space defined under the usual addition and scalar multiplication of polynomials.

Now, since $\mathbb{P}_2 = \text{span}\left\{x^2, x, 1\right\}$, the set $\left\{x^2, x, 1\right\}$ is a basis if it is linearly independent. Suppose then that

$$ax^2 + bx + c = 0x^2 + 0x + 0$$

where $a, b, c$ are real numbers. It is clear that this can only occur if $a = b = c = 0$ . Hence the set is linearly independent and forms a basis of $\mathbb{P}_2$.

The next theorem is an essential result in linear algebra and is called the exchange theorem.

**Theorem 13.4.3: Exchange Theorem**

Let $\{\vec{x}_1, \cdots, \vec{x}_r\}$ be a linearly independent set of vectors such that each $\vec{x}_i$ is contained in span$\{\vec{y}_1, \cdots, \vec{y}_s\}$ . Then $r \leq s$.

**Proof**

The proof will proceed as follows. First, we set up the necessary steps for the proof. Next, we will assume that $r > s$ and show that this leads to a contradiction, thus requiring that $r \leq s$ .

Define span$\{\vec{y}_1, \cdots, \vec{y}_s\} = V$. Since each $\vec{x}_i$ is in span$\{\vec{y}_1, \cdots, \vec{y}_s\}$, it follows there exist scalars $c_1, \cdots, c_s$ such that

$$\vec{x}_1 = \sum_{i=1}^{s} c_i \vec{y}_i \tag{13.4.1}$$

Note that not all of these scalars $c_i$ can equal zero. Suppose that all the $c_i = 0$ . Then it would follow that $\vec{x}_1 = \vec{0}$ and so $\{\vec{x}_1, \cdots, \vec{x}_r\}$ would not be linearly independent. Indeed, if $\vec{x}_1 = \vec{0}$ , $1\vec{x}_1 + \sum_{i=2}^{r} 0\vec{x}_i = \vec{x}_1 = \vec{0}$ and so there would exist a nontrivial linear combination of the vectors $\{\vec{x}_1, \cdots, \vec{x}_r\}$ which equals zero. Therefore at least one $c_i$ is nonzero.

Say $c_k \neq 0$. Then solve (13.4.1) for $\vec{y}_k$ and obtain

$$\vec{y}_k \in \text{span}\left\{\vec{x}_1, \overbrace{\vec{y}_1, \cdots, \vec{y}_{k-1}, \vec{y}_{k+1}, \cdots, \vec{y}_s}^{\text{s-1 vectors here}}\right\}.$$

Define $\{\vec{z}_1, \cdots, \vec{z}_{s-1}\}$ to be

$$\{\vec{z}_1, \cdots, \vec{z}_{s-1}\} = \{\vec{y}_1, \cdots, \vec{y}_{k-1}, \vec{y}_{k+1}, \cdots, \vec{y}_s\}$$

Now we can write

$$\vec{y}_k \in \text{span}\{\vec{x}_1, \vec{z}_1, \cdots, \vec{z}_{s-1}\}$$

Therefore, span$\{\vec{x}_1, \vec{z}_1, \cdots, \vec{z}_{s-1}\} = V$. To see this, suppose $\vec{v} \in V$ . Then there exist constants $c_1, \cdots, c_s$ such that

$$\vec{v} = \sum_{i=1}^{s-1} c_i \vec{z}_i + c_s \vec{y}_k.$$

Replace this $\vec{y}_k$ with a linear combination of the vectors $\{\vec{x}_1, \vec{z}_1, \cdots, \vec{z}_{s-1}\}$ to obtain $\vec{v} \in \text{span}\{\vec{x}_1, \vec{z}_1, \cdots, \vec{z}_{s-1}\}$ . The vector $\vec{y}_k$, in the list $\{\vec{y}_1, \cdots, \vec{y}_s\}$ , has now been replaced with the vector $\vec{x}_1$ and the resulting modified list of vectors has the same span as the original list of vectors, $\{\vec{y}_1, \cdots, \vec{y}_s\}$ .

We are now ready to move on to the proof. Suppose that $r > s$ and that

$$\text{span}\{\vec{x}_1, \cdots, \vec{x}_l, \vec{z}_1, \cdots, \vec{z}_p\} = V$$

where the process established above has continued. In other words, the vectors $\vec{z}_1, \cdots, \vec{z}_p$ are each taken from the set $\{\vec{y}_1, \cdots, \vec{y}_s\}$ and $l + p = s$. This was done for $l = 1$ above. Then since $r > s$, it follows that $l \leq s < r$ and so $l + 1 \leq r$. Therefore, $\vec{x}_{l+1}$ is a vector not in the list, $\{\vec{x}_1, \cdots, \vec{x}_l\}$ and since

$$\text{span}\{\vec{x}_1, \cdots, \vec{x}_l, \vec{z}_1, \cdots, \vec{z}_p\} = V$$

there exist scalars, $c_i$ and $d_j$ such that

$$\vec{x}_{l+1} = \sum_{i=1}^{l} c_i \vec{x}_i + \sum_{j=1}^{p} d_j \vec{z}_j. \tag{13.4.2}$$

Not all the $d_j$ can equal zero because if this were so, it would follow that $\{\vec{x}_1, \cdots, \vec{x}_r\}$ would be a linearly dependent set because one of the vectors would equal a linear combination of the others. Therefore, (13.4.2)can be solved for one of the $\vec{z}_i$, say $\vec{z}_k$, in terms of $\vec{x}_{l+1}$ and the other $\vec{z}_i$ and just as in the above argument, replace that $\vec{z}_i$ with $\vec{x}_{l+1}$ to obtain

$$\text{span}\left\{\vec{x}_1, \cdots \vec{x}_l, \vec{x}_{l+1}, \overbrace{\vec{z}_1, \cdots \vec{z}_{k-1}, \vec{z}_{k+1}, \cdots, \vec{z}_p}^{\text{p-1 vectors here}}\right\} = V$$

Continue this way, eventually obtaining

$$\text{span}\{\vec{x}_1, \cdots, \vec{x}_s\} = V.$$

But then $\vec{x}_r \in \text{span}\{\vec{x}_1, \cdots, \vec{x}_s\}$ contrary to the assumption that $\{\vec{x}_1, \cdots, \vec{x}_r\}$ is linearly independent. Therefore, $r \leq s$ as claimed.

The following corollary follows from the exchange theorem.

### Corollary 13.4.2: Two Bases of the Same Length

Let $B_1$, $B_2$ be two bases of a vector space $V$. Suppose $B_1$ contains $m$ vectors and $B_2$ contains $n$ vectors. Then $m = n$.

**Proof**

By Theorem 13.4.3, $m \leq n$ and $n \leq m$. Therefore $m = n$.

This corollary is very important so we provide another proof independent of the exchange theorem above.

**Proof**

Suppose $n > m$. Then since the vectors $\{\vec{u}_1, \cdots, \vec{u}_m\}$ span $V$, there exist scalars $c_{ij}$ such that

$$\sum_{i=1}^{m} c_{ij}\vec{u}_i = \vec{v}_j.$$

Therefore,

$$\sum_{j=1}^{n} d_j\vec{v}_j = \vec{0} \text{ if and only if } \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}d_j\vec{u}_i = \vec{0}$$

if and only if

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij}d_j\right)\vec{u}_i = \vec{0}$$

Now since $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is independent, this happens if and only if

$$\sum_{j=1}^{n} c_{ij}d_j = 0, \ i = 1, 2, \cdots, m.$$

However, this is a system of $m$ equations in $n$ variables, $d_1, \cdots, d_n$ and $m < n$. Therefore, there exists a solution to this system of equations in which not all the $d_j$ are equal to zero. Recall why this is so. The augmented matrix for the system is of the form $\left[\begin{array}{c|c} C & \vec{0} \end{array}\right]$ where $C$ is a matrix which has more columns than rows. Therefore, there are free variables and hence nonzero solutions to the system of equations. However, this contradicts the linear independence of $\{\vec{u}_1, \cdots, \vec{u}_m\}$. Similarly it cannot happen that $m > n$.

Given the result of the previous corollary, the following definition follows.

**Definition 13.4.3: Dimension**

A vector space $V$ is of dimension $n$ if it has a basis consisting of $n$ vectors.

Notice that the dimension is well defined by Corollary 13.4.2. It is assumed here that $n < \infty$ and therefore such a vector space is said to be **finite dimensional**.

**Example 13.4.5: Dimension of a Vector Space**

Let $\mathbb{P}_2$ be the set of all polynomials of degree at most 2. Find the dimension of $\mathbb{P}_2$.

**Solution**

If we can find a basis of $\mathbb{P}_2$ then the number of vectors in the basis will give the dimension. Recall from Example 13.4.4 that a basis of $\mathbb{P}_2$ is given by

$$S = \left\{x^2, x, 1\right\}$$

There are three polynomials in $S$ and hence the dimension of $\mathbb{P}_2$ is three.

It is important to note that a basis for a vector space is not unique. A vector space can have many bases. Consider the following example.

**Example 13.4.6: A Different Basis for Polynomials of Degree Two**

Let $\mathbb{P}_2$ be the polynomials of degree no more than 2. Is $\left\{x^2+x+1, 2x+1, 3x^2+1\right\}$ a basis for $\mathbb{P}_2$?

**Solution**

Suppose these vectors are linearly independent but do not form a spanning set for $\mathbb{P}_2$. Then by Lemma 9.3.2, we could find a fourth polynomial in $\mathbb{P}_2$ to create a new linearly independent set containing four polynomials. However this would imply that we could find a basis of $\mathbb{P}_2$ of more than three polynomials. This contradicts the result of Example 13.4.5 in which we determined the dimension of $\mathbb{P}_2$ is three. Therefore if these vectors are linearly independent they must also form a spanning set and thus a basis for $\mathbb{P}_2$.

Suppose then that

$$\begin{aligned} a\left(x^2+x+1\right)+b\left(2x+1\right)+c\left(3x^2+1\right) &= 0 \\ \left(a+3c\right)x^2+\left(a+2b\right)x+\left(a+b+c\right) &= 0 \end{aligned}$$

We know that $\left\{x^2, x, 1\right\}$ is linearly independent, and so it follows that

$$\begin{aligned} a+3c &= 0 \\ a+2b &= 0 \\ a+b+c &= 0 \end{aligned}$$

and there is only one solution to this system of equations, $a = b = c = 0$ . Therefore, these are linearly independent and form a basis for $\mathbb{P}_2$.

Consider the following theorem.

**Theorem 13.4.4: Every Subspace has a Basis**

Let $W$ be a nonzero subspace of a finite dimensional vector space $V$. Suppose $V$ has dimension $n$. Then $W$ has a basis with no more than $n$ vectors.

**Proof**

Let $\vec{v}_1 \in V$ where $\vec{v}_1 \neq 0$. If $\text{span}\left\{\vec{v}_1\right\} = V$, then it follows that $\left\{\vec{v}_1\right\}$ is a basis for $V$. Otherwise, there exists $\vec{v}_2 \in V$ which is not in $\text{span}\left\{\vec{v}_1\right\}$. By Lemma 9.3.2 $\left\{\vec{v}_1, \vec{v}_2\right\}$ is a linearly independent set of vectors. Then $\left\{\vec{v}_1, \vec{v}_2\right\}$ is a basis for $V$ and we are done. If $\text{span}\left\{\vec{v}_1, \vec{v}_2\right\} \neq V$, then there exists $\vec{v}_3 \notin \text{span}\left\{\vec{v}_1, \vec{v}_2\right\}$ and $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$ is a larger linearly

independent set of vectors. Continuing this way, the process must stop before $n+1$ steps because if not, it would be possible to obtain $n+1$ linearly independent vectors contrary to the exchange theorem, Theorem 13.4.3

If in fact $W$ has $n$ vectors, then it follows that $W=V$.

**Theorem 13.4.5: Subspace of Same Dimension**

Let $V$ be a vector space of dimension $n$ and let $W$ be a subspace. Then $W=V$ if and only if the dimension of $W$ is also $n$.

**Proof**

First suppose $W=V$. Then obviously the dimension of $W=n$.

Now suppose that the dimension of $W$ is $n$. Let a basis for $W$ be $\{\vec{w}_1,\cdots,\vec{w}_n\}$. If $W$ is not equal to $V$, then let $\vec{v}$ be a vector of $V$ which is not contained in $W$. Thus $\vec{v}$ is not in $\text{span}\{\vec{w}_1,\cdots,\vec{w}_n\}$ and by Lemma 9.7.2, $\{\vec{w}_1,\cdots,\vec{w}_n,\vec{v}\}$ is linearly independent which contradicts Theorem 13.4.3 because it would be an independent set of $n+1$ vectors even though each of these vectors is in a spanning set of $n$ vectors, a basis of $V$.

Consider the following example.

**Example 13.4.7: Basis of a Subspace**

Let $U=\left\{A\in\mathbb{M}_{22}\;\middle|\;A\begin{bmatrix}1&0\\1&-1\end{bmatrix}=\begin{bmatrix}1&1\\0&-1\end{bmatrix}A\right\}$. Then $U$ is a subspace of $\mathbb{M}_{22}$ Find a basis of $U$, and hence $\dim(U)$.

**Solution**

Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathbb{M}_{22}$. Then

$$A\begin{bmatrix}1&0\\1&-1\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\1&-1\end{bmatrix}=\begin{bmatrix}a+b&-b\\c+d&-d\end{bmatrix}$$

and

$$\begin{bmatrix}1&1\\0&-1\end{bmatrix}A=\begin{bmatrix}1&1\\0&-1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a+c&b+d\\-c&-d\end{bmatrix}.$$

If $A\in U$, then $\begin{bmatrix}a+b&-b\\c+d&-d\end{bmatrix}=\begin{bmatrix}a+c&b+d\\-c&-d\end{bmatrix}$.

Equating entries leads to a system of four equations in the four variables $a,b,c$ and $d$.

$$\begin{aligned}a+b&=a+c\\-b&=b+d\\c+d&=-c\\-d&=-d\end{aligned}$$

or

$$\begin{aligned}b-c&=0\\-2b-d&=0\\2c+d&=0\end{aligned}.$$

The solution to this system is $a=s$, $b=-\frac{1}{2}t$, $c=-\frac{1}{2}t$, $d=t$ for any $s,t\in\mathbb{R}$, and thus

$$A=\begin{bmatrix}s&\frac{t}{2}\\-\frac{t}{2}&t\end{bmatrix}=s\begin{bmatrix}1&0\\0&0\end{bmatrix}+t\begin{bmatrix}0&-\frac{1}{2}\\-\frac{1}{2}&1\end{bmatrix}.$$

Let

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \right\}.$$

Then $\text{span}(B) = U$, and it is routine to verify that $B$ is an independent subset of $\mathbb{M}_{22}$. Therefore $B$ is a basis of $U$, and $\dim(U) = 2$.

The following theorem claims that a spanning set of a vector space $V$ can be shrunk down to a basis of $V$. Similarly, a linearly independent set within $V$ can be enlarged to create a basis of $V$.

**Theorem 13.4.6: Basis of $V$**

If $V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_n\}$ is a vector space, then some subset of $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is a basis for $V$. Also, if $\{\vec{u}_1, \cdots, \vec{u}_k\} \subseteq V$ is linearly independent and the vector space is finite dimensional, then the set $\{\vec{u}_1, \cdots, \vec{u}_k\}$, can be enlarged to obtain a basis of $V$.

**Proof**

Let

$$S = \{E \subseteq \{\vec{u}_1, \cdots, \vec{u}_n\} \text{ such that } \text{span}\{E\} = V\}.$$

For $E \in S$, let $|E|$ denote the number of elements of $E$. Let

$$m = \min\{|E| \text{ such that } E \in S\}.$$

Thus there exist vectors

$$\{\vec{v}_1, \cdots, \vec{v}_m\} \subseteq \{\vec{u}_1, \cdots, \vec{u}_n\}$$

such that

$$\text{span}\{\vec{v}_1, \cdots, \vec{v}_m\} = V$$

and $m$ is as small as possible for this to happen. If this set is linearly independent, it follows it is a basis for $V$ and the theorem is proved. On the other hand, if the set is not linearly independent, then there exist scalars, $c_1, \cdots, c_m$ such that

$$\vec{0} = \sum_{i=1}^{m} c_i \vec{v}_i$$

and not all the $c_i$ are equal to zero. Suppose $c_k \neq 0$. Then solve for the vector $\vec{v}_k$ in terms of the other vectors. Consequently,

$$V = \text{span}\{\vec{v}_1, \cdots, \vec{v}_{k-1}, \vec{v}_{k+1}, \cdots, \vec{v}_m\}$$

contradicting the definition of $m$. This proves the first part of the theorem.

To obtain the second part, begin with $\{\vec{u}_1, \cdots, \vec{u}_k\}$ and suppose a basis for $V$ is

$$\{\vec{v}_1, \cdots, \vec{v}_n\}$$

If

$$\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\} = V,$$

then $k = n$. If not, there exists a vector

$$\vec{u}_{k+1} \notin \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$$

Then from Lemma 9.3.2, $\{\vec{u}_1, \cdots, \vec{u}_k, \vec{u}_{k+1}\}$ is also linearly independent. Continue adding vectors in this way until $n$ linearly independent vectors have been obtained. Then

$$\text{span}\{\vec{u}_1, \cdots, \vec{u}_n\} = V$$

because if it did not do so, there would exist $\vec{u}_{n+1}$ as just described and $\{\vec{u}_1, \cdots, \vec{u}_{n+1}\}$ would be a linearly independent set of vectors having $n+1$ elements. This contradicts the fact that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis. In turn this would contradict Theorem 13.4.3 Therefore, this list is a basis.

Recall Example 9.3.4 in which we added a matrix to a linearly independent set to create a larger linearly independent set. By Theorem 13.4.6 we can extend a linearly independent set to a basis.

### Example 13.4.8: Adding to a Linearly Independent Set

Let $S \subseteq M_{22}$ be a linearly independent set given by

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Enlarge $S$ to a basis of $M_{22}$.

**Solution**

Recall from the solution of Example 9.3.4 that the set $R \subseteq M_{22}$ given by

$$R = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

is also linearly independent. However this set is still not a basis for $M_{22}$ as it is not a spanning set. In particular, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not in $\text{span} R$. Therefore, this matrix can be added to the set by Lemma 9.3.2 to obtain a new linearly independent set given by

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

This set is linearly independent and now spans $M_{22}$. Hence $T$ is a basis.

Next we consider the case where you have a spanning set and you want a subset which is a basis. The above discussion involved adding vectors to a set. The next theorem involves removing vectors.

### Theorem 13.4.7: Basis from a Spanning Set

Let $V$ be a vector space and let $W$ be a subspace. Also suppose that $W = \text{span}\{\vec{w}_1, \cdots, \vec{w}_m\}$. Then there exists a subset of $\{\vec{w}_1, \cdots, \vec{w}_m\}$ which is a basis for $W$.

**Proof**

Let $S$ denote the set of positive integers such that for $k \in S$, there exists a subset of $\{\vec{w}_1, \cdots, \vec{w}_m\}$ consisting of exactly $k$ vectors which is a spanning set for $W$. Thus $m \in S$. Pick the smallest positive integer in $S$. Call it $k$. Then there exists $\{\vec{u}_1, \cdots, \vec{u}_k\} \subseteq \{\vec{w}_1, \cdots, \vec{w}_m\}$ such that $span\{\vec{u}_1, \cdots, \vec{u}_k\} = W$. If

$$\sum_{i=1}^{k} c_i \vec{w}_i = \vec{0}$$

and not all of the $c_i = 0$, then you could pick $c_j \neq 0$, divide by it and solve for $\vec{u}_j$ in terms of the others.

$$\vec{w}_j = \sum_{i \neq j} \left( -\frac{c_i}{c_j} \right) \vec{w}_i$$

Then you could delete $\vec{w}_j$ from the list and have the same span. In any linear combination involving $\vec{w}_j$, the linear combination would equal one in which $\vec{w}_j$ is replaced with the above sum, showing that it could have been obtained as a linear combination of $\vec{w}_i$ for $i \neq j$. Thus $k-1 \in S$ contrary to the choice of $k$. Hence each $c_i = 0$ and so $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is a basis for $W$ consisting of vectors of $\{\vec{w}_1, \cdots, \vec{w}_m\}$.

Consider the following example of this concept.

### Example 13.4.9: Basis from a Spanning Set

Let $V$ be the vector space of polynomials of degree no more than 3, denoted earlier as $\mathbb{P}_3$. Consider the following vectors in $V$.

$$\begin{aligned}&2x^2+x+1, x^3+4x^2+2x+2, 2x^3+2x^2+2x+1,\\ &x^3+4x^2-3x+2, x^3+3x^2+2x+1\end{aligned}$$

Then, as mentioned above, $V$ has dimension 4 and so clearly these vectors are not linearly independent. A basis for $V$ is $\left\{1, x, x^2, x^3\right\}$. Determine a linearly independent subset of these which has the same span. Determine whether this subset is a basis for $V$.

**Solution**

Consider an isomorphism which maps $\mathbb{R}^4$ to $V$ in the obvious way. Thus

$$\begin{bmatrix}1\\1\\2\\0\end{bmatrix}$$

corresponds to $2x^2+x+1$ through the use of this isomorphism. Then corresponding to the above vectors in $V$ we would have the following vectors in $\mathbb{R}^4$.

$$\begin{bmatrix}1\\1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\2\\4\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\\2\end{bmatrix}, \begin{bmatrix}2\\-3\\4\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\1\end{bmatrix}$$

Now if we obtain a subset of these which has the same span but which is linearly independent, then the corresponding vectors from $V$ will also be linearly independent. If there are four in the list, then the resulting vectors from $V$ must be a basis for $V$. The reduced row-echelon form for the matrix which has the above vectors as columns is

$$\begin{bmatrix}1&0&0&-15&0\\0&1&0&11&0\\0&0&1&-5&0\\0&0&0&0&1\end{bmatrix}$$

Therefore, a basis for $V$ consists of the vectors

$$\begin{aligned}&2x^2+x+1, x^3+4x^2+2x+2, 2x^3+2x^2+2x+1,\\ &x^3+3x^2+2x+1.\end{aligned}$$

Note how this is a subset of the original set of vectors. If there had been only three pivot columns in this matrix, then we would not have had a basis for $V$ but we would at least have obtained a linearly independent subset of the original set of vectors in this way.

Note also that, since all linear relations are preserved by an isomorphism,

$$\begin{aligned}&-15\left(2x^2+x+1\right)+11\left(x^3+4x^2+2x+2\right)+(-5)\left(2x^3+2x^2+2x+1\right)\\ &=x^3+4x^2-3x+2\end{aligned}$$

Consider the following example.

### Example 13.4.10: Shrinking a Spanning Set

Consider the set $S\subseteq\mathbb{P}_2$ given by

$$S=\left\{1, x, x^2, x^2+1\right\}$$

Show that $S$ spans $\mathbb{P}_2$, then remove vectors from $S$ until it creates a basis.

**Solution**

First we need to show that $S$ spans $\mathbb{P}_2$. Let $ax^2+bx+c$ be an arbitrary polynomial in $\mathbb{P}_2$. Write

$$ax^2+bx+c = r(1)+s(x)+t(x^2)+u(x^2+1)$$

Then,

$$\begin{aligned} ax^2+bx+c &= r(1)+s(x)+t(x^2)+u(x^2+1) \\ &= (t+u)x^2+s(x)+(r+u) \end{aligned}$$

It follows that

$$\begin{aligned} a &= t+u \\ b &= s \\ c &= r+u \end{aligned}$$

Clearly a solution exists for all $a, b, c$ and so $S$ is a spanning set for $\mathbb{P}_2$. By Theorem 13.4.6, some subset of $S$ is a basis for $\mathbb{P}_2$.

Recall that a basis must be both a spanning set and a linearly independent set. Therefore we must remove a vector from $S$ keeping this in mind. Suppose we remove $x$ from $S$. The resulting set would be $\{1, x^2, x^2+1\}$. This set is clearly linearly dependent (and also does not span $\mathbb{P}_2$) and so is not a basis.

Suppose we remove $x^2+1$ from $S$. The resulting set is $\{1, x, x^2\}$ which is both linearly independent and spans $\mathbb{P}_2$. Hence this is a basis for $\mathbb{P}_2$. Note that removing any one of $1, x^2$, or $x^2+1$ will result in a basis.

Now the following is a fundamental result about subspaces.

**Theorem 13.4.8: Basis of a Vector Space**

Let $V$ be a finite dimensional vector space and let $W$ be a non-zero subspace. Then $W$ has a basis. That is, there exists a linearly independent set of vectors $\{\vec{w}_1, \cdots, \vec{w}_r\}$ such that

$$\{\vec{w}_1, \cdots, \vec{w}_r\} = W$$

Also if $\{\vec{w}_1, \cdots, \vec{w}_s\}$ is a linearly independent set of vectors, then $W$ has a basis of the form $\{\vec{w}_1, \cdots, \vec{w}_s, \cdots, \vec{w}_r\}$ for $r \geq s$.

**Proof**

Let the dimension of $V$ be $n$. Pick $\vec{w}_1 \in W$ where $\vec{w}_1 \neq \vec{0}$. If $\vec{w}_1, \cdots, \vec{w}_s$ have been chosen such that $\{\vec{w}_1, \cdots, \vec{w}_s\}$ is linearly independent, if $span\{\vec{w}_1, \cdots, \vec{w}_r\} = W$, stop. You have the desired basis. Otherwise, there exists $\vec{w}_{s+1} \notin span\{\vec{w}_1, \cdots, \vec{w}_s\}$ and $\{\vec{w}_1, \cdots, \vec{w}_s, \vec{w}_{s+1}\}$ is linearly independent. Continue this way until the process stops. It must stop since otherwise, you could obtain a linearly independent set of vectors having more than $n$ vectors which is impossible.

The last claim is proved by following the above procedure starting with $\{\vec{w}_1, \cdots, \vec{w}_s\}$ as above.

This also proves the following corollary. Let $V$ play the role of $W$ in the above theorem and begin with a basis for $W$, enlarging it to form a basis for $V$ as discussed above.

**Corollary 13.4.3: Basis Extension**

Let $W$ be any non-zero subspace of a vector space $V$. Then every basis of $W$ can be extended to a basis for $V$.

Consider the following example.

### Example 13.4.11: Basis Extension

Let $V = \mathbb{R}^4$ and let

$$W = \text{span}\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$$

Extend this basis of $W$ to a basis of $V$.

**Solution**

An easy way to do this is to take the reduced row-echelon form of the matrix

$$\begin{bmatrix}1 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 1 & 0\\ 1 & 1 & 0 & 0 & 0 & 1\end{bmatrix} \tag{13.4.3}$$

Note how the given vectors were placed as the first two and then the matrix was extended in such a way that it is clear that the span of the columns of this matrix yield all of $\mathbb{R}^4$. Now determine the pivot columns. The reduced row-echelon form is

$$\begin{bmatrix}1 & 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & -1 & 1\\ 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 1 & -1\end{bmatrix} \tag{13.4.4}$$

These are

$$\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

and now this is an extension of the given basis for $W$ to a basis for $\mathbb{R}^4$.

Why does this work? The columns of (13.4.3) obviously span $\mathbb{R}^4$ the span of the first four is the same as the span of all six.

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# 13.5: Sums and Intersections

**Outcomes**

A. Show that the sum of two subspaces is a subspace.
B. Show that the intersection of two subspaces is a subspace.

We begin this section with a definition.

**Definition 13.5.1: Sum and Intersection**

Let $V$ be a vector space, and let $U$ and $W$ be subspaces of $V$. Then

1. $U+W=\{\vec{u}+\vec{w} \mid \vec{u}\in U \text{ and } \vec{w}\in W\}$ and is called the sum of $U$ and $W$.
2. $U\cap W=\{\vec{v} \mid \vec{v}\in U \text{ and } \vec{v}\in W\}$ and is called the intersection of $U$ and $W$.

Therefore the intersection of two subspaces is all the vectors shared by both. If there are no vectors shared by both subspaces, meaning that $U\cap W=\left\{\vec{0}\right\}$, the sum $U+W$ takes on a special name.

**Definition 13.5.2: Direct Sum**

Let $V$ be a vector space and suppose $U$ and $W$ are subspaces of $V$ such that $U\cap W=\left\{\vec{0}\right\}$. Then the sum of $U$ and $W$ is called the direct sum and is denoted $U\oplus W$.

An interesting result is that both the sum $U+W$ and the intersection $U\cap W$ are subspaces of $V$.

**Example 13.5.1: Intersection is a Subspace**

Let $V$ be a vector space and suppose $U$ and $W$ are subspaces. Then the intersection $U\cap W$ is a subspace of $V$.

**Solution**

By the subspace test, we must show three things:

1. $\vec{0}\in U\cap W$
2. For vectors $\vec{v}_1,\vec{v}_2\in U\cap W, \vec{v}_1+\vec{v}_2\in U\cap W$
3. For scalar $a$ and vector $\vec{v}\in U\cap W, a\vec{v}\in U\cap W$

We proceed to show each of these three conditions hold.

1. Since $U$ and $W$ are subspaces of $V$, they each contain $\vec{0}$. By definition of the intersection, $\vec{0}\in U\cap W$.
2. Let $\vec{v}_1,\vec{v}_2\in U\cap W,$. Then in particular, $\vec{v}_1,\vec{v}_2\in U$. Since $U$ is a subspace, it follows that $\vec{v}_1+\vec{v}_2\in U$. The same argument holds for $W$. Therefore $\vec{v}_1+\vec{v}_2$ is in both $U$ and $W$ and by definition is also in $U\cap W$.
3. Let $a$ be a scalar and $\vec{v}\in U\cap W$. Then in particular, $\vec{v}\in U$. Since $U$ is a subspace, it follows that $a\vec{v}\in U$. The same argument holds for $W$ so $a\vec{v}$ is in both $U$ and $W$. By definition, it is in $U\cap W$.

Therefore $U\cap W$ is a subspace of $V$.

It can also be shown that $U+W$ is a subspace of $V$.

We conclude this section with an important theorem on dimension.

**Theorem 13.5.1: Dimension of Sum**

Let $V$ be a vector space with subspaces $U$ and $W$. Suppose $U$ and $W$ each have finite dimension. Then $U+W$ also has finite dimension which is given by

$$\dim(U+W)=\dim(U)+\dim(W)-\dim(U\cap W)$$

Notice that when $U \cap W = \left\{\vec{0}\right\}$, the sum becomes the direct sum and the above equation becomes

$$\dim(U \oplus W) = \dim(U) + \dim(W)$$

---

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# 13.6: Linear Transformations

**Outcomes**

A. Understand the definition of a linear transformation in the context of vector spaces.

Recall that a function is simply a transformation of a vector to result in a new vector. Consider the following definition.

**Definition 13.6.1: Linear Transformation**

Let $V$ and $W$ be vector spaces. Suppose $T : V \mapsto W$ is a function, where for each $\vec{x} \in V, T(\vec{x}) \in W$. Then $T$ is a **linear transformation** if whenever $k, p$ are scalars and $\vec{v}_1$ and $\vec{v}_2$ are vectors in $V$

$$T(k\vec{v}_1 + p\vec{v}_2) = kT(\vec{v}_1) + pT(\vec{v}_2)$$

Several important examples of linear transformations include the zero transformation, the identity transformation, and the scalar transformation.

**Example 13.6.1: Linear Transformations**

Let $V$ and $W$ be vector spaces.

1. **The zero transformation**
   $0 : V \to W$ is defined by $0(\vec{v}) = \vec{0}$ for all $\vec{v} \in V$.
2. **The identity transformation**
   $1_V : V \to V$ is defined by $1_V(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$.
3. **The scalar transformation** Let $a \in \mathbb{R}$.
   $s_a : V \to V$ is defined by $s_a(\vec{v}) = a\vec{v}$ for all $\vec{v} \in V$ .

**Solution**

We will show that the scalar transformation $s_a$ is linear, the rest are left as an exercise.

By Definition 13.6.1 we must show that for all scalars $k, p$ and vectors $\vec{v}_1$ and $\vec{v}_2$ in $V$, $s_a(k\vec{v}_1 + p\vec{v}_2) = ks_a(\vec{v}_1) + ps_a(\vec{v}_2)$ . Assume that $a$ is also a scalar.

$$\begin{aligned} s_a(k\vec{v}_1 + p\vec{v}_2) &= a(k\vec{v}_1 + p\vec{v}_2) \\ &= ak\vec{v}_1 + ap\vec{v}_2 \\ &= k(a\vec{v}_1) + p(a\vec{v}_2) \\ &= ks_a(\vec{v}_1) + ps_a(\vec{v}_2) \end{aligned}$$

Therefore $s_a$ is a linear transformation.

Consider the following important theorem.

**Theorem 13.6.1: Properties of Linear Transformations**

Let $V$ and $W$ be vector spaces, and $T : V \mapsto W$ a linear transformation. Then

1. $T$ preserves the zero vector.
$$T(\vec{0}) = \vec{0}$$
2. $T$ preserves additive inverses. For all $\vec{v} \in V$,
$$T(-\vec{v}) = -T(\vec{v})$$
3. $T$ preserves linear combinations. For all $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in V$ and all $k_1, k_2, \ldots, k_m \in \mathbb{R}$,
$$T(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_m\vec{v}_m) = k_1T(\vec{v}_1) + k_2T(\vec{v}_2) + \cdots + k_mT(\vec{v}_m).$$

**Proof**

1. Let $\vec{0}_V$ denote the zero vector of $V$ and let $\vec{0}_W$ denote the zero vector of $W$. We want to prove that $T(\vec{0}_V) = \vec{0}_W$. Let $\vec{v} \in V$. Then $0\vec{v} = \vec{0}_V$ and
$$T(\vec{0}_V) = T(0\vec{v}) = 0T(\vec{v}) = \vec{0}_W.$$
2. Let $\vec{v} \in V$; then $-\vec{v} \in V$ is the additive inverse of $\vec{v}$, so $\vec{v} + (-\vec{v}) = \vec{0}_V$. Thus
$$\begin{aligned} T(\vec{v} + (-\vec{v})) &= T(\vec{0}_V) \\ T(\vec{v}) + T(-\vec{v})) &= \vec{0}_W \\ T(-\vec{v}) &= \vec{0}_W - T(\vec{v}) = -T(\vec{v}). \end{aligned}$$
3. This result follows from preservation of addition and preservation of scalar multiplication. A formal proof would be by induction on $m$.

Consider the following example using the above theorem.

### Example 13.6.2: Linear Combination

Let $T : \mathbb{P}_2 \to \mathbb{R}$ be a linear transformation such that
$$T(x^2 + x) = -1; T(x^2 - x) = 1; T(x^2 + 1) = 3.$$
Find $T(4x^2 + 5x - 3)$.

We provide two solutions to this problem.

**Solution 1:**

Suppose $a(x^2 + x) + b(x^2 - x) + c(x^2 + 1) = 4x^2 + 5x - 3$. Then
$$(a + b + c)x^2 + (a - b)x + c = 4x^2 + 5x - 3.$$
Solving for $a$, $b$, and $c$ results in the unique solution $a = 6$, $b = 1$, $c = -3$. Thus
$$\begin{aligned} T(4x^2 + 5x - 3) &= T(6(x^2 + x) + (x^2 - x) - 3(x^2 + 1)) \\ &= 6T(x^2 + x) + T(x^2 - x) - 3T(x^2 + 1) \\ &= 6(-1) + 1 - 3(3) = -14. \end{aligned}$$

**Solution 2:**

Notice that $S = \{x^2 + x, x^2 - x, x^2 + 1\}$ is a basis of $\mathbb{P}_2$, and thus $x^2$, $x$, and $1$ can each be written as a linear combination of elements of $S$.
$$\begin{aligned} x^2 &= \tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x) \\ x &= \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x) \\ 1 &= (x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x). \end{aligned}$$
Then
$$\begin{aligned} T(x^2) &= T\left(\tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x)\right) = \tfrac{1}{2}T(x^2 + x) + \tfrac{1}{2}T(x^2 - x) \\ &= \tfrac{1}{2}(-1) + \tfrac{1}{2}(1) = 0. \\ T(x) &= T\left(\tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x)\right) = \tfrac{1}{2}T(x^2 + x) - \tfrac{1}{2}T(x^2 - x) \\ &= \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = -1. \\ T(1) &= T\left((x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x)\right) \\ &= T(x^2 + 1) - \tfrac{1}{2}T(x^2 + x) - \tfrac{1}{2}T(x^2 - x) \\ &= 3 - \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = 3. \end{aligned}$$

Therefore,

$$\begin{aligned} T(4x^2+5x-3) &= 4T(x^2)+5T(x)-3T(1) \\ &= 4(0)+5(-1)-3(3)=-14. \end{aligned}$$

The advantage of **Solution 2** over **Solution 1** is that if you were now asked to find $T(-6x^2-13x+9)$, it is easy to use $T(x^2)=0$, $T(x)=-1$ and $T(1)=3$:

$$\begin{aligned} T(-6x^2-13x+9) &= -6T(x^2)-13T(x)+9T(1) \\ &= -6(0)-13(-1)+9(3)=13+27=40. \end{aligned}$$

More generally,

$$\begin{aligned} T(ax^2+bx+c) &= aT(x^2)+bT(x)+cT(1) \\ &= a(0)+b(-1)+c(3)=-b+3c. \end{aligned}$$

Suppose two linear transformations act in the same way on $\vec{v}$ for all vectors. Then we say that these transformations are equal.

### Definition 13.6.2: Equal Transformations

Let $S$ and $T$ be linear transformations from $V$ to $W$. Then $S=T$ if and only if for every $\vec{v}\in V$,

$$S(\vec{v})=T(\vec{v})$$

The definition above requires that two transformations have the same action on every vector in order for them to be equal. The next theorem argues that it is only necessary to check the action of the transformations on basis vectors.

### Theorem 13.6.2: Transformation of a Spanning Set

Let $V$ and $W$ be vector spaces and suppose that $S$ and $T$ are linear transformations from $V$ to $W$. Then in order for $S$ and $T$ to be equal, it suffices that $S(\vec{v}_i)=T(\vec{v}_i)$ where $V=span\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$.

This theorem tells us that a linear transformation is completely determined by its actions on a spanning set. We can also examine the effect of a linear transformation on a basis.

### Theorem 13.6.3: Transformation of a Basis

Suppose $V$ and $W$ are vector spaces and let $\{\vec{w}_1,\vec{w}_2,\ldots,\vec{w}_n\}$ be any given vectors in $W$ that may not be distinct. Then there exists a basis $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$ of $V$ and a unique linear transformation $T:V\mapsto W$ with $T(\vec{v}_i)=\vec{w}_i$.

Furthermore, if

$$\vec{v}=k_1\vec{v}_1+k_2\vec{v}_2+\cdots+k_n\vec{v}_n$$

is a vector of $V$, then

$$T(\vec{v})=k_1\vec{w}_1+k_2\vec{w}_2+\cdots+k_n\vec{w}_n.$$

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# 13.E: Exercises

**Exercise 13.$E$.1**

Determine whether the given sets of vectors are linearly dependent or linearly independent. If they are linearly dependent, write one as a linear combination of the others.

a.) $\vec{u} = (0, 2), \vec{v} = (0, 3)$

b.) $\vec{u} = (1, 2, 4), \vec{v} = (1, -2, 3), \vec{w} = (-2, 0, 1)$

c.) $\vec{u} = (7, 5), \vec{v} = (1, 2), \vec{w} = (3, -1)$

d) $\vec{u} = (1, 2, 3), \vec{v} = (2, 4, 6), \vec{w} = (4, 1, -4)$

**Answer**

a.) linearly dependent

b.) linearly independent

c.) linearly dependent

d.) linearly dependent

**Exercise 13.$E$.2**

Are the following sets subspaces of $\mathbb{R}^3$ or not? If not, give an explanation and if so, prove it.

a.) $V$ is the set of all $(x, y, z)$ such that $x = 0$

b.) $V$ is the set of all $(x, y, z)$ such that $2x = 3y$

c.) $V$ is the set of all $(x, y, z)$ such that $x = 6$

d.) $V$ is the set of all $(x, y, z)$ such that $x + y = 0$

e.) $V$ is the set of all $(x, y, z)$ such that $x + y = 2$

**Answer**

a.) yes

b.) yes

c.) no

d.) yes

e.) no

**Exercise 13.$E$.3**

Is $\vec{v} = (1, 0, -1)$ in the span of $\{(5, 3, 4), (3, 2, 5)\}$? If so, write $\vec{v}$ as a linear combination of the two vectors.

**Exercise 13.$E$.4**

Prove that if a finite set of vectors contains the zero vector, then that set is linearly dependent.

**Answer**

Hint: To prove a set of vectors is linearly dependent, you need to show that one of the vectors is a linear combination of the others.

Exercise $13.E.5$

Do the following sets of vectors form a basis for $\mathbb{R}^n$? Why or why not?

a.) $\vec{v}_1 = (4, 7), \vec{v}_2 = (5, 6)$

b.) $\vec{v}_1 = (1, -3), \vec{v}_2 = (-3, 9)$

c.) $\vec{v}_1 = (4, 7, 4), \vec{v}_2 = (5, 6, 0), \vec{v}_3 = (2, -1, 1)$

d.) $\vec{v}_1 = (4, -1, 4), \vec{v}_2 = (5, 2, 0)$

e.) $\vec{v}_1 = (4, 7, 4), \vec{v}_2 = (5, 6, 0), \vec{v}_3 = (2, -1, 1), \vec{v}_4 = (0, 1, 2)$

**Answer**

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# 14.1: A Brief Table of Integrals

$$\int u^\alpha\,du = \frac{u^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq -1$$

$$\int \frac{du}{u} = \ln|u| + c$$

$$\int \cos u\,du = \sin u + c$$

$$\int \sin u\,du = -\cos u + c$$

$$\int \tan u\,du = -\ln|\cos u| + c$$

$$\int \cot u\,du = \ln|\sin u| + c$$

$$\int \sec^2 u\,du = \tan u + c$$

$$\int \csc^2 u\,du = -\cot u + c$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + c$$

$$\int \cos^2 u\,du = \frac{u}{2} + \frac{1}{4}\sin 2u + c$$

$$\int \sin^2 u\,du = \frac{u}{2} - \frac{1}{4}\sin 2u + c$$

$$\int \frac{du}{1+u^2}du = \tan^{-1} u + c$$

$$\int \frac{du}{\sqrt{1-u^2}}du = \sin^{-1} u + c$$

$$\int \frac{1}{u^2-1}du = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c$$

$$\int \cosh u\,du = \sinh u + c$$

$$\int \sinh u\,du = \cosh u + c$$

$$\int u\,dv = uv - \int v\,du$$

$$\int u\cos u\,du = u\sin u + \cos u + c$$

$$\int u\sin u\,du = -u\cos u + \sin u + c$$

$$\int ue^u du = ue^u - e^u + c$$

$$\int e^{\lambda u}\cos \omega u\,du = \frac{e^{\lambda u}(\lambda\cos \omega u + \omega \sin \omega u)}{\lambda^2+\omega^2} + c$$

$$\int e^{\lambda u}\sin \omega u\,du = \frac{e^{\lambda u}(\lambda\sin \omega u + \omega \cos \omega u)}{\lambda^2+\omega^2} + c$$

$$\int \ln|u|\,du = u\ln|u| - u + c$$

$$\int u\ln|u|\,du = \frac{u^2\ln|u|}{2} - \frac{u^2}{4} + c$$

$$\int \cos\omega_1 u\cos\omega_2 u\,du = \frac{\sin(\omega_1+\omega_2)u}{2(\omega_1+\omega_2)} + \frac{\sin(\omega_1-\omega_2)u}{2(\omega_1-\omega_2)} + c \quad (\omega_1 \neq \pm\omega_2)$$

$$\int \sin\omega_1 u\sin\omega_2 u\,du = -\frac{\sin(\omega_1+\omega_2)u}{2(\omega_1+\omega_2)} + \frac{\sin(\omega_1-\omega_2)u}{2(\omega_1-\omega_2)} + c \quad (\omega_1 \neq \pm\omega_2)$$

$$\int \sin\omega_1 u\cos\omega_2 u\,du = -\frac{\cos(\omega_1+\omega_2)u}{2(\omega_1+\omega_2)} - \frac{\cos(\omega_1-\omega_2)u}{2(\omega_1-\omega_2)} + c \quad (\omega_1 \neq \pm\omega_2)$$

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# 14.3: Section 1.2 Answers

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   b. 2
   c. 1
   d. 2
2. verify
3.
   a. $y = -\frac{x^2}{2} + c$
   b. $y = x\cos x - \sin x + c$
   c. $y = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$
   d. $y = -x\cos x + 2\sin x + c_1 + c_2 x$
   e. $y = (2x-4)e^x + c_1 + c_2 x$
   f. $y = \frac{x^3}{3} - \sin x + e^x + c_1 + c_2 x$
   g. $y = \sin x + c_1 + c_2 x + c_3 x^2$
   h. $y = -\frac{x^5}{60} + e^x + c_1 + c_2 x + c_3 x^2$
   i. $y = \frac{7}{64}e^{4x} + c_1 + c_2 x + c_3 x^2$
4.
   a. $y = -(x-1)e^x$
   b. $y = 1 - \frac{1}{2}\cos x^2$
   c. $y = 3 - \ln(\sqrt{2}\cos x)$
   d. $y = -\frac{47}{15} - \frac{37}{5}(x-2) + \frac{x^6}{30}$
   e. $y = \frac{1}{4}xe^{2x} - \frac{1}{4}e^{2x} + \frac{29}{4}$
   f. $y = x\sin x + 2\cos x - 3x - 1$
   g. $y = (x^2 - 6x + 12)e^x + \frac{x^2}{2} - 8x - 11$
   h. $y = \frac{x^3}{3} + \frac{\cos 2x}{6} + \frac{7}{4}x^2 - 6x + \frac{7}{8}$
   i. $y = \frac{x^4}{12} + \frac{x^3}{6} + \frac{1}{2}(x-2)^2 - \frac{26}{3}(x-2) - \frac{5}{3}$
5. verify
6. verify
7.
   a. 576 ft
   b. 10 s

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# 14.4: Section 2.1 Answers

1. $y = e^{-ax}$
2. $y = ce^{-x^3}$
3. $y = ce^{-(\ln x)^2/2}$
4. $y = \frac{c}{x^3}$
5. $y = ce^{1/x}$
6. $y = \frac{e^{-(x-1)}}{x}$
7. $y = \frac{e}{x \ln x}$
8. $y = \frac{\pi}{x \sin x}$
9. $y = 2(1 + x^2)$
10. $y = 3x^{-k}$
11. $y = 2(\cos kx)^{1/k}$
12. $y = \frac{1}{3} + ce^{-3x}$
13. $y = \frac{2}{x} + \frac{c}{x}e^x$
14. $y = e^{-x^2}\left(\frac{x^2}{2} + c\right)$
15. $y = -\frac{e^{-x}+c}{1+x^2}$
16. $\frac{7 \ln |x|}{x} + \frac{3}{2}x + \frac{c}{x}$
17. $y = (x-1)^{-4}(\ln|x-1| - \cos x + c)$
18. $y = e^{-x^2}\left(\frac{x^3}{4} + \frac{c}{x}\right)$
19. $y = \frac{2 \ln |x|}{x^2} + \frac{1}{2} + \frac{c}{x^2}$
20. $y = (x + c)\cos x$
21. $y = \frac{c - \cos x}{(1+x)^2}$
22. $y = -\frac{1}{2}\frac{(x-2)^3}{(x-1)} + c\frac{(x-2)^5}{(x-1)}$
23. $y = (x + c)e^{-\sin^2 x}$
24. $y = \frac{e^x}{x^2} - \frac{e^x}{x^3} + \frac{c}{x^2}$
25. $y = \frac{e^{3x} - e^{-7x}}{10}$
26. $y = \frac{2x+1}{(1+x^2)^2}$
27. $y = \frac{1}{x^3}\ln\left(\frac{1+x^2}{2}\right)$
28. $y = \frac{1}{2}(\sin x + \csc x)$
29. $y = \frac{2 \ln |x|}{x} + \frac{x}{2} - \frac{1}{2x}$
30. $y = (x-1)^{-3}[\ln(1-x) - \cos x]$
31. $y = 2x^2 + \frac{1}{x^2} \quad (0, \infty)$
32. $y = x^2(1 - \ln x)$

33. $y = \frac{1}{2} + \frac{5}{2}e^{-x^2}$
34. $y = \frac{\ln|x-1|+\tan x+1}{(x-1)^3}$
35. $y = \frac{\ln|x|+x^2+1}{(x+2)^4}$
36. $y = (x^2-1)\left(\frac{1}{2}\ln|x^2-1|-4\right)$
37. $y = -(x^2-5)(7+\ln|x^2-5|)$
38. $y = e^{-x^2}\left(3+\int_0^x t^2 e^{t^2}\,dt\right)$
39. $y = \frac{1}{x}\left(2+\int_1^x \frac{\sin t}{t}\,dt\right)$
40. $y = e^{-x}\int_1^x \frac{\tan t}{t}\,dt$
41. $y = \frac{1}{1+x^2}\left(1+\int_0^x \frac{e^t}{1+t^2}\,dt\right)$
42. $y = \frac{1}{x}\left(2e^{-(x-1)}+e^{-x}\int_1^x e^t e^{t^2}\,dt\right)$
43. $G = \frac{r}{\lambda} + \left(G_0 - \frac{r}{\lambda}\right)e^{-\lambda t}$ $\lim_{t\to\infty} G(t) = \frac{r}{\lambda}$
45. $y = y_0 e^{-a(x-x_0)} + e^{-ax}\int_{x_0}^x e^{at} f(t)dt$
48.
    a. $y = \tan^{-1}\left(\frac{1}{3}+ce^{3x}\right)$
    b. $y = \pm\left[\ln\left(\frac{1}{x}+\frac{c}{x^2}\right)\right]^{1/2}$
    c. $y = \exp\left(x^2+\frac{c}{x^2}\right)$
    d. $y = -1 + \frac{x}{c+3\ln|x|}$

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# 14.5: Section 2.2 Answers

1. $y = 2 \pm \sqrt{2(x^3 + x^2 + x + c)}$
2. $\ln(|\sin y|) = \cos x + c; \quad y \equiv k\pi, \quad k = \text{integer}$
3. $y = \frac{c}{x-c} \quad y \equiv -1$
4. $\frac{(\ln y)^2}{2} = -\frac{x^3}{3} + c$
5. $y^3 + 3\sin y + \ln|y| + \ln(1+x^2) + \tan^{-1} x = c; \quad y \equiv 0$
6. $y = \pm\left(1 + \left(\frac{x}{1+cx}\right)^2\right)^{1/2}; \quad y \equiv \pm 1$
7. $y = \tan\left(\frac{x^3}{3} + c\right)$
8. $y = \frac{c}{\sqrt{1+x^2}}$
9. $y = \frac{2 - ce^{(x-1)^2/2}}{1 - ce^{(x-1)^2/2}}; \quad y \equiv 1$
10. $y = 1 + (3x^2 + 9x + c)^{1/3}$
11. $y = 2 + \sqrt{\frac{2}{3}x^3 + 3x^2 + 4x - \frac{11}{3}}$
12. $y = \frac{e^{-(x^2-4)/2}}{2 - e^{-(x^2-4)/2}}$
13. $y^3 + 2y^2 + x^2 + \sin x = 3$
14. $(y+1)(y-1)^{-3}(y-2)^2 = -256(x+1)^{-6}$
15. $y = -1 + 3e^{-x^2}$
16. $y = \frac{1}{\sqrt{2e^{-2x^2} - 1}}$
17. $y \equiv -1; \quad (-\infty, \infty)$
18. $y = \frac{4 - e^{-x^2}}{2 - e^{-x^2}}; \quad (-\infty, \infty)$
19. $y = \frac{-1 + \sqrt{4x^2 - 15}}{2}; \quad \left(\frac{\sqrt{15}}{2}, \infty\right)$
20. $y = \frac{2}{1 + e^{-2x}} \quad (-\infty, \infty)$
21. $y = -\sqrt{25 - x^2}; \quad (-5, 5)$
22. $y \equiv 2, \quad (-\infty, \infty)$
23. $y = 3\left(\frac{x+1}{2x-4}\right)^{1/3}; \quad (-\infty, 2)$
24. $y = \frac{x+c}{1-cx}$
25. $y = -x\cos c + \sqrt{1 - x^2}\sin c; \quad y \equiv 1; \ y \equiv -1$
26. $y = -x + 3\pi/2$
28. $P = \frac{P_0}{\alpha P_0 + (1 - \alpha P_0)e^{-at}}; \lim_{t\to\infty} P(t) = 1/\alpha$
29. $I = \frac{SI_0}{I_0 + (S - I_0)e^{-rSt}}$
30. If $q = rS$ then $I = \frac{I_0}{1 + rI_0 t}$ and $\lim_{t\to\infty} I(t) = 0$. If $q \neq Rs$, then $I = \frac{\alpha I_0}{I_0 + (\alpha - I_0)e^{-rat}}$. If $q < rs$, then $\lim_{t\to\infty} I(t) = \alpha = S - \frac{q}{r}$ if $q > rS$, then $\lim_{t\to\infty} I(t) = 0$
34. $f = ap, \quad \text{where } a = \text{constant}$

35. $y = e^{-x}(-1 \pm \sqrt{2x^2 + c})$

36. $y = x^2(-1 + \sqrt{x^2 + c})$

37. $y = e^x(-1 + (3xe^x + c)^{1/3})$

38. $y = e^{2x}(1 \pm \sqrt{c - x^2})$

39.

a. $y_1 = 1/x; \quad g(x) = h(x)$
b. $y_1 = x; \quad g(x) = h(x)/x^2$
c. $y_1 = e^{-x}; \quad g(x) = e^x h(x)$
d. $y_1 = x^{-r}; \quad g(x) = x^{r-1} h(x)$
e. $y_1 = 1/v(x); \quad g(x) = v(x)h(x)$

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# 14.6: Section 2.3 Answers

1. (b) $x_0 \neq k\pi \quad (k = \text{integer})$

2. (b) $(x_0, y_0) \neq (0, 0)$

3. (b) $x_0 y_0 \neq (2k+1)\frac{\pi}{2} \quad (k = \text{integer})$

4. (b) $x_0 y_0 > 0$ and $x_0 y_0 \neq 1$

5.

a. all $(x_0, y_0)$
b. $(x_0, y_0)$ woth $y_0 \neq 0$

6. (b) all $(x_0, y_0)$

7. (b) all $(x_0, y_0)$

8. (b) $(x_0, y_0)$ such that $x_0 \neq 4y_0$

9.

a. all $(x_0, y_0)$
b. all $(x_0, y_0) \neq (0, 0)$

10.

a. all $(x_0, y_0)$
b. all $(x_0, y_0)$ with $y_0 \neq \pm 1$

11. (b) all $(x_0, y_0)$

12. (b) all $(x_0, y_0)$ such that $(x_0, y_0) > 0$

13. (b) all $(x_0, y_0)$ with $x_0 \neq 1, \quad y_0 \neq (2k+1)\frac{\pi}{2} (k = \text{integer})$

16. $y = \left(\frac{3}{5}x + 1\right)^{5/3}, \quad -\infty < x < \infty$ , is a solution.

Also,

$$y = \begin{cases} 0, & -\infty < x \leq -\frac{5}{3} \\ (\frac{3}{5}x+1)^{5/3}, & -\frac{5}{3} < x < \infty \end{cases}$$

is a solution. For every $a \geq \frac{5}{3}$ , the following function is also a solution:

$$y = \begin{cases} (\frac{3}{5}(x+a))^{5/3}, & -\infty < x < -a, \\ 0, & -a \leq x \leq -\frac{5}{3} \\ (\frac{3}{5}x+1)^{5/3}, & -\frac{5}{3} < x < \infty \end{cases}$$

17.

a. all $(x_0, y_0)$
b. all $(x_0, y_0)$ with $y_0 \neq 1$

18. $y_1 = 1; y_2 = 1 + |x|^3; y_3 = 1 - |x|^3; y_4 = 1 + x^3; y_5 = 1 - x^3$

$$y_6 = \begin{cases} 1+x^3, & x \geq 0, \\ 1, & x < 0 \end{cases}; \quad y_7 = \begin{cases} 1-x^3, & x \geq 0, \\ 1, & x < 0 \end{cases};$$

$$y_8 = \begin{cases} 1, & x \geq 0, \\ 1+x^3, & x < 0 \end{cases}; \quad y_9 = \begin{cases} 1, & x \geq 0, \\ 1-x^3, & x < 0 \end{cases}$$

19. $y = 1 + (x^2+4)^{3/2}, \quad -\infty < x < \infty$

20.

a. The solution is unique on $(0,\infty)$. It is given by

$$y=\begin{cases}1, & 0<x\le\sqrt{5}\\ 1-(x^2-5)^{3/2}, & \sqrt{5}<x<\infty\end{cases}$$

b.
$$y=\begin{cases}1, & -\infty<x\le\sqrt{5},\\ 1-(x^2-5)^{3/2}, & \sqrt{5}<x<\infty\end{cases}$$

is a solution of (A) on $(-\infty,\infty)$. If $\alpha\ge 0$, then

$$y=\begin{cases}1+(x^2-\alpha^2)^{3/2}, & -\infty<x<-\alpha,\\ 1, & -\alpha\le x\le\sqrt{5},\\ 1-(x^2-5)^{3/2}, & \sqrt{5}<x<\infty,\end{cases}$$

and

$$y=\begin{cases}1-(x^2-\alpha^2)^{3/2}, & -\infty<x<-\alpha,\\ 1, & -\alpha\le x\le\sqrt{5},\\ 1-(x^2-5)^{3/2}, & \sqrt{5}<x<\infty,\end{cases}$$

are also solutions of (A) on $(-\infty,\infty)$.

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# 14.7: Section 2.4 Answers

1. $y = \frac{1}{1-ce^x}$
2. $y = x^{2/7}(c - \ln|x|)^{1/7}$
3. $y = e^{2/x}(c - 1/x)^2$
4. $y = \pm\frac{\sqrt{2x+c}}{1+x^2}$
5. $y = \pm(1 - x^2 + ce^{-x^2})^{-1/2}$
6. $y = \left[\frac{x}{3(1-x)+ce^{-x}}\right]^{1/3}$
7. $y = \frac{2\sqrt{2}}{\sqrt{1-4x}}$
8. $y = \left[1 - \frac{3}{2}e^{-(x^2-1)/4}\right]^{-2}$
9. $y = \frac{1}{x(11-3x)^{1/3}}$
10. $y = (2e^x - 1)^2$
11. $y = (2e^{12x} - 1 - 12x)^{1/3}$
12. $y = \left[\frac{5x}{2(1+4x^5)}\right]^{1/2}$
13. $y = (4e^{x/2} - x - 2)^2$
14. $P = \frac{P_0 e^{at}}{1+aP_0\int_0^t \alpha(\tau)e^{a\tau}d\tau}; \quad \lim_{t\to\infty} P(t) = \begin{cases} \infty & \text{if } L = 0, \\ 0 & \text{if } L = \infty, \\ 1/aL & \text{if } 0 < L < \infty \end{cases}$
15. $y = x(\ln|x| + c)$
16. $y = \frac{cx^2}{1-cx} \quad y = -x$
17. $y = \pm x(4\ln|x| + c)^{1/4}$
18. $y = x\sin^{-1}(\ln|x| + c)$
19. $y = x\tan(\ln|x| + c)$
20. $y = \pm x\sqrt{cx^2 - 1}$
21. $y = \pm x\ln(\ln|x| + c)$
22. $y = -\frac{2x}{2\ln|x|+1}$
23. $y = x(3\ln x + 27)^{1/3}$
24. $y = \frac{1}{x}\left(\frac{9-x^4}{2}\right)^{1/2}$
25. $y = -x$
26. $y = -\frac{x(4x-3)}{(2x-3)}$
27. $y = x\sqrt{4x^6 - 1}$
28. $\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln(x^2 + y^2) = c$
29. $(x + y)\ln|x| + y(1 - \ln|y|) + cx = 0$
30. $(y + x)^3 = 3x^3(\ln|x| + c)$
31. $(y + x) = c(y - x)^3; \quad y = x; \quad y = -x$

32. $y^2(y-3x)=c; \quad y\equiv 0; \quad y=3x$

33. $(x-y)^3(x+y)=cy^2x^4; \quad y=0; \quad y=x; \quad y=-x$

34. $\frac{y}{x}+\frac{y^3}{x^3}=\ln|x|+c$

40. Choose $X_0$ and $Y_0$ so that

$$aX_0+bY_0=\alpha$$

$$cX_0+dY_0=\beta$$

41. $(y+2x+1)^4)2y-6x-3)=c; \quad y=3x+3/2; \quad y=-2x-1$

42. $(y+x-1)(y-x-5)^3=c; \quad y=x+5; \quad y=-x+1$

43. $\ln|y-x-6|-\frac{2(x+2)}{y-x-6}=c; \quad y=x+6$

44. $(y_1=x^{1/3}y=x^{1/3}(\ln|x|+c)^{1/3}$

45. $y_1=x^3; \quad y=\pm x^3\sqrt{cx^6-1}$

46. $y_1=x^2; \quad y=\frac{x^2(1+cx^4)}{1-cx^4}y=-x^2$

47. $y_1=e^x; \quad y=-\frac{e^x(1-2ce^x}{1-ce^x}; \quad y=-2e^x$

48. $y_1=\tan x; y=\tan x\tan(\ln|\tan x|+c)$

49. $y_1=\ln x; \quad y=\frac{2\ln x(1+c(\ln x)^4)}{1-c(\ln x)^4}; \quad y=-2\ln x$

50. $y_1=x^{1/2}; \quad y=x^{1/2}(-2\pm\sqrt{\ln|x|+c})$

51. $y_1=e^{x^2}; \quad y=e^{x^2}(-1\pm\sqrt{2x^2+c})$

52. $y=\frac{-3+\sqrt{1+60x}}{2x}$

53. $y=\frac{-5+\sqrt{1+48x}}{2x^2}$

56. $y=1+\frac{1}{x+1+ce^x}$

57. $y=e^x-\frac{1}{1+ce^{-x}}$

58. $y=1-\frac{1}{x(1-cx)}$

59. $y=x-\frac{2x}{x^2+c}$

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# 14.8: Section 2.5 Answers

1. $2x^3y^2 = c$

2. $3y\sin x + 2x^2e^x + 3y = c$

3. Not exact

4. $x^2 - 2xy^2 + 4y^3 = c$

5. $x + y = c$

6. Not exact

7. $2y^2\cos x + 3xy^3 - x^2 = c$

8. Not exact

9. $x^3 + x^2y + 4xy^2 + 9y^2 = c$

10. Not exact

11. $\ln|xy| + x^2 + y^2 = c$

12. Not exact

13. $x^2 + y^2 = c$

14. $x^2y^2e^x + 2y + 3x^2 = c$

15. $x^3e^{x^2+y} - 4y^3 + 2x^2 = c$

16. $x^4e^{xy} + 3xy = c$

17. $x^3\cos xy + 4y^2 + 2x^2 = c$

18. $y = \frac{x+\sqrt{2x^2+3x-1}}{x^2}$

19. $y = \sin x - \sqrt{1 - \frac{\tan x}{2}}$

20. $y = \left(\frac{e^x-1}{e^x+1}\right)^{1/3}$

21. $y = 1 + 2\tan x$

22. $y = \frac{x^2-x+6}{(x+2)(x-3)}$

23. $\frac{7x^2}{2} + 4xy + \frac{3y^2}{2} = c$

24. $(x^4y^2 + 1)e^x + y^2 = c$

29.

a. $M(x,y) = 2xy + f(x)$
b. $M(x,y) = 2(\sin x + x\cos x)(y\sin y + \cos y) + f(x)$
c. $M(x,y) = ye^x - e^y\cos x + f(x)$

30.

a. $N(x,y) = \frac{x^4y}{2} + x^2 + 6xy + g(y)$
b. $N(x,y) = \frac{x}{y} + 2y\sin x + g(y)$
c. $N(x,y) = x(\sin y + y\cos y) + g(y)$

33. $B = C$

34. $B = 2D, \quad E = 2C$

37.

a. $2x^2 + x^4y^4 + y^2 = c$

b. $x^3+3xy^2=c$
c. $x^3+y^2+2xy=c$

38. $y=-1-\frac{1}{x^2}$

39. $y=x^3\left(\frac{-3(x^2+1)+\sqrt{9x^4+34x^2+21}}{2}\right)$

40. $y=-e^{-x^2}\left(\frac{2x+\sqrt{9-5x^2}}{3}\right)$

44.

a. $G(x,y)=2xy+c$
b. $G(x,y)=e^x\sin y+c$
c. $G(x,y)=3x^2y-y^3+c$
d. $G(x,y)=-\sin x\sinh y+c$
e. $G(x,y)=\cos x\sinh y+c$

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# 14.9: Section 2.6 Answers

3. $\mu(x)=1/x^2;\quad y=cx$ and $\mu(y)=1/y^2;\quad x=cy$

4. $\mu(x)=x^{-3/2};\quad x^{3/2}y=c$

5. $\mu(y)=1/y^3;\quad y^3e^{2x}=c$

6. $\mu(x)=e^{5x/2};\quad e^{5x/2}(xy+1)=c$

7. $\mu(x)=e^x;\quad e^x(xy+y+x)=c$

8. $\mu(x)=x;\quad x^2y^2(9x+4y)=c$

9. $\mu(y)=y^2;\quad y^3(3x^2y+2x+1)=c$

10. $\mu(y)=ye^y;\quad e^y(xy^3+1)=c$

11. $\mu(y)=y^2;\quad y^3(3x^4+8x^3y+y)=c$

12. $\mu(x)=xe^x;\quad x^2y(x+1)e^x=c$

13. $\mu(x)=(x^3-1)^{-4/3};\quad xy(x^3-1)^{-1/3}=c$ and $x\equiv 1$

14. $\mu(y)=e^y;\quad e^y(\sin x\cos y+y-1)=c$

15. $\mu(y)=e^{-y^2};xye^{-y^2}(x+y)=c$

16. $\frac{xy}{\sin y}=c$ and $y=k\pi(k=\text{integer})$

17. $\mu(x,y)=x^4y^3;\quad x^5y^4\ln x=c$

18. $\mu(x,y)=1/xy;\quad |x|^\alpha|y|^\beta e^{\gamma x}e^{\delta y}=c$ and $x\equiv 0, y\equiv 0$

19. $\mu(x,y)=x^{-2}y^{-3};\quad 3x^2y^2+y=1+cxy^2$ and $x\equiv 0, y\equiv 0$

20. $\mu(x,y)=x^{-2}y^{-1};\quad -\frac{2}{x}+y^3+3\ln|y|=c$ and $x\equiv 0, y\equiv 0$

21. $\mu(x,y)=e^{ax}e^{by};\quad e^{ac}e^{by}\cos xy=c$

22. $\mu(x,y)=x^{-4}y^{-3}$ (and others) $xy=c$

23. $\mu(x,y)=xe^y;\quad x^2ye^y\sin x=c$

24. $\mu(x,y)=1/x^2;\quad \frac{x^3y^3}{3}-\frac{y}{x}=c$

25. $\mu(x)=x+1;\quad y(x+1)^2(x+y)=c$

26. $\mu(x,y)=x^2y^2;\quad x^3y^3(3x+2y^2)=c$

27. $\mu(x,y)=x^{-2}y^{-2};\quad 3x^2y=cxy+2$ and $x\equiv 0, y\equiv 0$

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# 14.10: Section 3.1 Answers

1. $y_1 = 1.450000000,\ y_2 = 2.085625000,\ y_3 = 3.079099746$

2. $y_1 = 1.200000000,\ y_2 = 1.440415946,\ y_3 = 1.729880994$

3. $y_1 = 1.900000000,\ y_2 = 1.781375000,\ y_3 = 1.646612970$

4. $y_1 = 2.962500000,\ y_2 = 2.922635828,\ y_3 = 2.880205639$

5. $y_1 = 2.513274123,\ y_2 = 1.814517822,\ y_3 = 1.216364496$

6.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 1.0 | 48.298147362 | 51.492825643 | 53.076673685 | 54.647937102 |

7.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 2.0 | 1.390242009 | 1.370996758 | 1.361921132 | 1.353193719 |

8.

| $x$ | $h = 0.05$ | $h = 0.025$ | $h = 0.0125$ | Exact |
|---|---|---|---|---|
| 1.50 | 7.886170437 | 8.852463793 | 9.548039907 | 10.500000000 |

9.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ |
|---|---|---|---|---|---|---|
| 3.0 | 1.469458241 | 1.462514486 | 1.459217010 | **0.3210** | **0.1537** | **0.0753** |
| | Approximate Solutions | | | Residuals | | |

10.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ |
|---|---|---|---|---|---|---|
| 2.0 | 0.473456737 | 0.483227470 | 0.487986391 | $-0.3129$ | $-0.1563$ | $-0.0781$ |
| | Approximate Solutions | | | Residuals | | |

11.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
|---|---|---|---|---|
| 1.0 | 0.691066797 | 0.676269516 | 0.668327471 | 0.659957689 |

12.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
|---|---|---|---|---|
| 2.0 | $-0.772381768$ | $-0.761510960$ | $-0.756179726$ | $-0.750912371$ |

13.

| Euler's Method |
|---|

| $x$ | $h=0.1$ | $h=0.05$ | $h=0.025$ | Exact |
|---|---|---|---|---|
| 1.0 | 0.538871178 | 0.593002325 | 0.620131525 | 0.647231889 |

| Euler semilinear Method | | | | |
|---|---|---|---|---|
| $x$ | $h=0.1$ | $h=0.05$ | $h=0.025$ | Exact |
| 1.0 | 0.647231889 | 0.647231889 | 0.647231889 | 0.647231889 |

Applying variation of parameters to the given initial value problem yields $y=ue^{-3x}$, where (A) $u'=7, u(0)=6$. Since $u''=0$, Euler's method yields the exact solution of (A). Therefore the Euler semilinear method produces the exact solution of the given problem

14.

| Euler's Method | | | | |
|---|---|---|---|---|
| $x$ | $h=0.1$ | $h=0.05$ | $h=0.025$ | "Exact" |
| 3.0 | 12.804226135 | 13.912944662 | 14.559623055 | 15.282004826 |

| Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h=0.1$ | $h=0.05$ | $h=0.025$ | "Exact" |
| 3.0 | 15.354122287 | 15.317257705 | 15.299429421 | 15.282004826 |

15.

| Euler's method | | | | |
|---|---|---|---|---|
| $x$ | $h=0.2$ | $h=0.1$ | $h=0.05$ | "Exact" |
| 2.0 | 0.867565004 | 0.885719263 | 0.895024772 | 0.904276722 |

| Euler's semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h=0.2$ | $h=0.1$ | $h=0.05$ | "Exact" |
| 2.0 | 0.569670789 | 0.720861858 | 0.808438261 | 0.904276722 |

16.

| Euler's method | | | | |
|---|---|---|---|---|
| $x$ | $h=0.2$ | $h=0.1$ | $h=0.05$ | "Exact" |
| 3.0 | 0.922094379 | 0.945604800 | 0.956752868 | 0.967523153 |

| Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h=0.2$ | $h=0.1$ | $h=0.05$ | "Exact" |
| 3.0 | 0.993954754 | 0.980751307 | 0.974140320 | 0.967523153 |

17.

| Euler's method |
|---|

| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
|---|---|---|---|---|
| 1.50 | 0.319892131 | 0.330797109 | 0.337020123 | 0.343780513 |

| Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
| 1.50 | 0.305596953 | 0.323340268 | 0.333204519 | 0.343780513 |

18.

| Euler's method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 2.0 | 0.754572560 | 0.743869878 | 0.738303914 | 0.732638628 |

| Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 2.0 | 0.722610454 | 0.727742966 | 0.730220211 | 0.732638628 |

19.

| Euler's method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
| 1.50 | 2.175959970 | 2.210259554 | 2.227207500 | 2.244023982 |

| Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
| 1.50 | 2.117953342 | 2.179844585 | 2.211647904 | 2.244023982 |

20.

| Euler's method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 1.0 | 0.032105117 | 0.043997045 | 0.050159310 | 0.056415515 |

| Euler's semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 1.0 | 0.056020154 | 0.056243980 | 0.056336491 | 0.056415515 |

21.

| Euler's method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 1.0 | 28.987816656 | 38.426957516 | 45.367269688 | 54.729594761 |

| Euler's semilinar method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 1.0 | 54.709134946 | 54.724150485 | 54.728228015 | 54.729594761 |

22.

| Euler's method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 3.0 | 1.361427907 | 1.361320824 | 1.361332589 | 1.361383810 |

| Euler's semilinar method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 3.0 | 1.291345518 | 1.326535737 | 1.344004102 | 1.361383810 |

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# 14.11: Section 3.2 Answers

1. $y_1 = 1.542812500,\ y_2 = 2.421622101,\ y_3 = 4.208020541$

2. $y_1 = 1.220207973,\ y_2 = 1.489578775.\ y_3 = 1.819337186$

3. $y_1 = 1.890687500,\ y_2 = 1.763784003,\ y_3 = 1.622698378$

4. $y_1 = 2.961317914,\ y_2 = 2.920132727,\ y_3 = 2.876213748$

5. $y_1 = 2.478055238,\ y_2 = 1.844042564,\ y_3 = 1.313882333$

6.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 1.0 | 56.134480009 | 55.003390448 | 54.734674836 | 54.647937102 |

7.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 2.0 | 1.353501839 | 1.353288493 | 1.353219485 | 1.353193719 |

8.

| $x$ | $h = 0.5$ | $h = 0.025$ | $h = 0.0125$ | Exact |
|---|---|---|---|---|
| 1.50 | 10.141969585 | 10.396770409 | 10.472502111 | 10.500000000 |

9.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ |
|---|---|---|---|---|---|---|
| 3.0 | 1.455674816 | 1.455935127 | 1.456001289 | $-0.00818$ | $-0.00207$ | $-0.000518$ |
| | Approximate Solutions | | | Residuals | | |

10.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ |
|---|---|---|---|---|---|---|
| 2.0 | 0.492862999 | 0.492709931 | 0.492674855 | 0.00335 | 0.000777 | 0.000187 |
| | Approximate Solutions | | | Residuals | | |

11.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
|---|---|---|---|---|
| 1.0 | 0.660268159 | 0.660028505 | 0.659974464 | 0.659957689 |

12.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
|---|---|---|---|---|
| 2.0 | $-0.749751364$ | $-0.750637632$ | $-0.750845571$ | $-0.750912371$ |

13. Applying variation of parameters to the given initial value problem $y = ue^{-3x}$, where $(A)u' = 1 - 2x, u(0) = 2$. Since $u''' = 0$, the improved Euler method yields the exact solution of (A). Therefore the improved Euler semilinear method produces the exact solution of the given problem.

| Improved Euler method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
| 1.0 | 0.105660401 | 0.100924399 | 0.099893685 | 0.099574137 |

| Improved Euler semilinar method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
| 1.0 | 0.099574137 | 0.099574137 | 0.099574137 | 0.099574137 |

14.

| Improved Euler method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 3.0 | 15.107600968 | 15.234856000 | 15.269755072 | 15.282004826 |

| Improved Euler semilinar method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 3.0 | 15.285231726 | 15.282812424 | 15.282206780 | 15.282004826 |

15.

| Improved Euler method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 2.0 | 0.924335375 | 0.907866081 | 0.905058201 | 0.904276722 |

| Improved Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 2.0 | 0.969670789 | 0.920861858 | 0.908438261 | 0.904276722 |

16.

| Improved Euler method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 3.0 | 0.967473721 | 0.967510790 | 0.967520062 | 0.967523153 |

| Improved Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 3.0 | 0.967473721 | 0.967510790 | 0.967520062 | 0.967523153 |

17.

| Improved Euler method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
| 1.50 | 0.349176060 | 0.345171664 | 0.344131282 | 0.343780513 |

| Improved Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
| 1.50 | 0.349350206 | 0.345216894 | 0.344142832 | 0.343780513 |

18.

| Improved Euler method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 2.0 | 0.732679223 | 0.732721613 | 0.732667905 | 0.732638628 |

| Improved Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 2.0 | 0.732166678 | 0.732521078 | 0.732609267 | 0.732638628 |

19.

| Improved Euler method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
| 1.50 | 2.247880315 | 2.244975181 | 2.244260143 | 2.244023982 |

| Improved Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
| 1.50 | 2.248603585 | 2.245169707 | 2.244310465 | 2.244023982 |

20.

| Improved Euler method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 1.0 | 0.059071894 | 0.056999028 | 0.056553023 | 0.056415515 |

| Improved Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 1.0 | 0.056295914 | 0.056385765 | 0.056408124 | 0.056415515 |

21.

| Improved Euler method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 1.0 | 50.534556346 | 53.483947013 | 54.391544440 | 54.729594761 |

| Improved Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 1.0 | 54.709041434 | 54.724083572 | 54.728191366 | 54.729594761 |

22.

| Improved Euler method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 3.0 | 1.361395309 | 1.361379259 | 1.361382239 | 1.361383810 |

| Improved Euler semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 3.0 | 1.375699933 | 1.364730937 | 1.362193997 | 1.361383810 |

23.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 2.0 | 1.349489056 | 1.352345900 | 1.352990822 | 1.353193719 |

24.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 2.0 | 1.350890736 | 1.352667599 | 1.353067951 | 1.353193719 |

25.

| $x$ | $h = 0.05$ | $h = 0.025$ | $h = 0.0125$ | Exact |
|---|---|---|---|---|
| 1.50 | 10.133021311 | 10.391655098 | 10.470731411 | 10.500000000 |

26.

| $x$ | $h = 0.05$ | $h = 0.025$ | $h = 0.0125$ | Exact |
|---|---|---|---|---|
| 1.50 | 10.136329642 | 10.393419681 | 10.470731411 | 10.500000000 |

27.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
|---|---|---|---|---|
| 1.0 | 0.660846835 | 0.660189749 | 0.660016904 | 0.659957689 |

28.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
|---|---|---|---|---|
| 1.0 | 0.660658411 | 0.660136630 | 0.660002840 | 0.659957689 |

29.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
|---|---|---|---|---|
| 2.0 | $-0.750626284$ | $-0.750844513$ | $-0.750895864$ | $-0.751331499$ |

30.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
|---|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| 2.0 | $-0.750335016$ | $-0.750775571$ | $-0.750879100$ | $-0.751331499$ |

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# 14.12: Section 3.3 Answers

1. $y_1 = 1.550598190,\ y_2 = 2.469649729$

2. $y_1 = 1.221551366,\ y_2 = 1.492920208$

3. $y_1 = 1.890339767,\ y_2 = 1.763094323$

4. $y_1 = 2.961316248,\ y_2 = 2.920128958$

5. $y_1 = 2.475605264,\ y_2 = 1.825992433$

6.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 1.0 | 54.654509699 | 54.648344019 | 54.647962328 | 54.647937102 |

7.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
|---|---|---|---|---|
| 2.0 | 1.353191745 | 1.353193606 | 1.353193712 | 1.353193719 |

8.

| $x$ | $h = 0.05$ | $h = 0.025$ | $h = 0.0125$ | Exact |
|---|---|---|---|---|
| 1.50 | 10.498658198 | 10.499906266 | 10.499993820 | 10.500000000 |

9.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ |
|---|---|---|---|---|---|---|
| 3.0 | 1.456023907 | 1.456023403 | 1.456023379 | 0.0000124 | 0.000000611 | 0.0000000333 |
| | Approximate Solutions | | | Residuals | | |

10.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ |
|---|---|---|---|---|---|---|
| 2.0 | 0.492663789 | 0.492663738 | 0.492663736 | 0.000000902 | 0.0000000508 | 0.00000000302 |
| | Approximate Solutions | | | Residuals | | |

11.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
|---|---|---|---|---|
| 1.0 | 0.659957046 | 0.659957646 | 0.659957686 | 0.659957689 |

12.

| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
|---|---|---|---|---|
| 2.0 | $-0.750911103$ | $-0.750912294$ | $-0.750912367$ | $-0.750912371$ |

13. Applying variation of parameters to the given initial value problem yields $y = ue^{-3x}$, where $(A) u' = 1 - 4x + 3x^2 - 4x^3, u(0) = -3$ . Since $u^{(5)} = 0$, the Runge-Kutta method yields the exact solution of (A). Therefore the Euler semilinear method produces the exact solution of the given problem.

| Runge-Kutta method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
| 0.0 | -3.000000000 | -3.000000000 | -3.000000000 | -3.000000000 |
| 0.1 | -2.162598011 | -2.162526572 | -2.162522707 | -2.162522468 |
| 0.2 | -1.577172164 | -1.577070939 | -1.577065457 | -1.577065117 |
| 0.3 | -1.163350794 | -1.163242678 | -1.163236817 | -1.163236453 |
| 0.4 | -0.868030294 | -0.867927182 | -0.867921588 | -0.867921241 |
| 0.5 | -0.655542739 | -0.655450183 | -0.655445157 | -0.655444845 |
| 0.6 | -0.501535352 | -0.501455325 | -0.501450977 | -0.501450707 |
| 0.7 | -0.389127673 | -0.389060213 | -0.389056546 | -0.389056318 |
| 0.8 | -0.306468018 | -0.306412184 | -0.306409148 | -0.306408959 |
| 0.9 | -0.245153433 | -0.245107859 | -0.245105379 | -0.245105226 |
| 1.0 | -0.199187198 | -0.199150401 | -0.199148398 | -0.199148273 |

| Runge-Kutta semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | Exact |
| 0.0 | -3.000000000 | -3.000000000 | -3.000000000 | -3.000000000 |
| 0.1 | -2.162522468 | -2.162522468 | -2.162522468 | -2.162522468 |
| 0.2 | -1.577065117 | -1.577065117 | -1.577065117 | -1.577065117 |
| 0.3 | -1.163236453 | -1.163236453 | -1.163236453 | -1.163236453 |
| 0.4 | -0.867921241 | -0.867921241 | -0.867921241 | -0.867921241 |
| 0.5 | -0.655444845 | -0.655444845 | -0.655444845 | -0.655444845 |
| 0.6 | -0.501450707 | -0.501450707 | -0.501450707 | -0.501450707 |
| 0.7 | -0.389056318 | -0.389056318 | -0.389056318 | -0.389056318 |
| 0.8 | -0.306408959 | -0.306408959 | -0.306408959 | -0.306408959 |
| 0.9 | -0.245105226 | -0.245105226 | -0.245105226 | -0.245105226 |
| 1.0 | -0.199148273 | -0.199148273 | -0.199148273 | -0.199148273 |

14.

| Runge-Kutta method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 3.0 | 15.281660036 | 15.281981407 | 15.282003300 | 15.282004826 |

| Runge-Kutta semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 3.0 | 15.282005990 | 15.282004899 | 15.282004831 | 15.282004826 |

15.

| Runge-Kutta method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 2.0 | 0.904678156 | 0.904295772 | 0.904277759 | 0.904276722 |

| Runge-Kutta semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 2.0 | 0.904592215 | 0.904297062 | 0.904278004 | 0.904276722 |

16.

| Runge-Kutta method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 3.0 | 0.967523147 | 0.967523152 | 0.967523153 | 0.967523153 |

| Runge-Kutta semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |

| | | | | |
|---|---|---|---|---|
| 3.0 | 0.967523147 | 0.967523152 | 0.967523153 | 0.967523153 |

17.

| Runge-Kutta method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
| 1.50 | 0.343839158 | 0.343784814 | 0.343780796 | 0.343780513 |

| Runge-Kutta semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
| 1.00 | 0.000000000 | 0.000000000 | 0.000000000 | 0.000000000 |
| 1.05 | 0.028121022 | 0.028121010 | 0.028121010 | 0.028121010 |
| 1.10 | 0.055393494 | 0.055393466 | 0.055393465 | 0.055393464 |
| 1.15 | 0.082164048 | 0.082163994 | 0.082163990 | 0.082163990 |
| 1.20 | 0.108862698 | 0.108862597 | 0.108862591 | 0.108862590 |
| 1.25 | 0.136058715 | 0.136058528 | 0.136058517 | 0.136058516 |
| 1.30 | 0.164564862 | 0.164564496 | 0.164564473 | 0.164564471 |
| 1.35 | 0.195651074 | 0.195650271 | 0.195650219 | 0.195650216 |
| 1.40 | 0.231542288 | 0.231540164 | 0.231540027 | 0.231540017 |
| 1.45 | 0.276818775 | 0.276811011 | 0.276810491 | 0.276810456 |
| 1.50 | 0.343839124 | 0.343784811 | 0.343780796 | 0.343780513 |

18.

| Runge-Kutta method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 2.0 | 0.732633229 | 0.732638318 | 0.732638609 | 0.732638628 |

| Runge-Kutta semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.2$ | $h = 0.1$ | $h = 0.05$ | "Exact" |
| 2.0 | 0.732639212 | 0.732638663 | 0.732638630 | 0.732638628 |

19.

| Runge-Kutta method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
| 1.50 | 2.244025683 | 2.244024088 | 2.244023989 | 2.244023982 |

| Runge-Kutta semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.0500$ | $h = 0.0250$ | $h = 0.0125$ | "Exact" |
| 1.50 | 2.244025081 | 2.244024051 | 2.244023987 | 2.244023982 |

20.

| Runge-Kutta method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 1.0 | 0.056426886 | 0.056416137 | 0.056415552 | 0.056415515 |

| Runge-Kutta semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |

| | | | | |
|---|---|---|---|---|
| 1.0 | 0.056415185 | 0.056415495 | 0.056415514 | 0.056415515 |

21.

| Runge-Kutta method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 1.0 | 54.695901186 | 54.727111858 | 54.729426250 | 54.729594761 |

| Runge-Kutta semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 1.0 | 54.729099966 | 54.729561720 | 54.729592658 | 54.729594761 |

22.

| Runge-Kutta method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 3.0 | 1.361384082 | 1.361383812 | 1.361383809 | 1.361383810 |

| Runge-Kutta semilinear method | | | | |
|---|---|---|---|---|
| $x$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ | "Exact" |
| 3.0 | 1.361456502 | 1.361388196 | 1.361384079 | 1.361383810 |

24.

| $x$ | $h = .1$ | $h = .05$ | $h = .025$ | Exact |
|---|---|---|---|---|
| 2.00 | $-1.000000000$ | $-1.000000000$ | $-1.000000000$ | $-1.000000000$ |

25.

| $x$ | $h = .1$ | $h = .05$ | $h = .025$ | "Exact" |
|---|---|---|---|---|
| 1.00 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 |

26.

| $x$ | $h = .1$ | $h = .05$ | $h = .025$ | "Exact" |
|---|---|---|---|---|
| 1.50 | 4.142171279 | 4.142170553 | 4.142170508 | 4.142170505 |

27.

| $x$ | $h = .1$ | $h = .05$ | $h = .025$ | "Exact" |
|---|---|---|---|---|
| 3.0 | 16.666666988 | 16.666666687 | 16.666666668 | 16.666666667 |

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# 14.13: Section 4.1 Answers

1. $Q = 20e^{-(t \ln 2)/3200}$
2. $\frac{2 \ln 10}{\ln 2}$days
3. $\tau = 10 \frac{\ln 2}{\ln 4/3}$minutes
4. $\tau \frac{\ln(p_0/p_1)}{\ln 2}$
5. $\frac{t_p}{t_q} = \frac{\ln p}{\ln q}$
6. $k = \frac{1}{t_2 - t_1} \ln \frac{Q_1}{Q_2}$
7. 20 g
8. $\frac{50 \ln 2}{3}$yrs
9. $\frac{25}{2} \ln 2$
10.
   a. $= 20 \ln 3$yr
   b. $Q_0 = 100000e^{-5}$
11.
   a. $Q(t) = 5000 - 4750e^{-t/10}$
   b. 5000lbs
12. $\frac{1}{25}$ yrs
13. $V = V_0 e^{t \ln 10/2}$ 4 hours
14. $\frac{1500 \ln \frac{4}{3}}{\ln 2}$yrs; $2^{-4/3} Q_0$
15. $W(t) = 20 - 19e^{-t/20}$; $\lim_{t \to \infty} W(t) = 20$ ounces
16. $S(t) = 10(1 + e^{-t/10}$; $\lim_{t \to \infty} S(t) = 10$ g
17. 10 gallons
18. $V(t) = 15000 + 10000e^{t/20}$
19. $W(t) = 4 \times 10^6 (t+1)^2$ dollars $t$ years from now
20. $p = \frac{100}{25 - 24e^{-t/2}}$
21.
   a. $P(t) = 1000e^{.06t} + 50 \frac{e^{.06t} - 1}{e^{.06/52} - 1}$
   b. $5.64 \times 10^{-4}$
22.
   a. $P - = rP - 12M$
   b. $P = \frac{12M}{r}(1 - e^{rt}) + P_0 e^{rt}$
   c. $M \approx \frac{rP_0}{12(1 - e^{-rN})}$
   d. For (i) approximate $M = \$402.25$, exact $M = \$402.80$ for (ii) approximate $M = \$1206.05$, exact $M = \$1206.93$.
23.
   a. $T(\alpha) = -\frac{1}{r} \ln(1 - (1 - e^{-rN})/\alpha))$ years $S(\alpha) = \frac{P_0}{(1 - e^{-rN})}[rN + \alpha \ln(1 - (1 - e^{-rN})/\alpha)]$
   b. $T(1.05) = 13.69$yrs, $S(1.05) = \$3579.94 T(1.10) = 12.61$yrs, $S(1.10) = \$6476.63 T(1.15) = 11.70$yrs, $S(1.15) = \$8874$.

24. $P_0 = \begin{cases} \frac{S_0(1-e^{(a-r)T})}{r-a} & \text{if } a \neq r \\ S_0 T & \text{if } a = r \end{cases}$

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# 14.14: Section 4.2 Answers

1. $\approx 15.15^\circ\mathrm{F}$
2. $T = -10 + 110e^{-t \ln \frac{11}{9}}$
3. $\approx 24.33^\circ\mathrm{F}$
4.
   a. $\approx 91.30^\circ\mathrm{F}$
   b. $8.99$ minutes after being placed out
   c. never
5.
   a. $12 : 11 : 32$
   b. $12 : 47 : 33$
6. $(85/3)^\circ\mathrm{C}$
7. $32^\circ\mathrm{F}$
8. $Q(t) = 40(1 - e^{-3t/40})$
9. $Q(t) = 30 - 20e^{-t/10}$
10. $K(t) = .3 - .2e^{-t/20}$
11. $Q(50) = 47.5$ (pounds)
12. $50$ gallons
13. $\min q_2 = q_1\sqrt{c}$
14. $Q = t + 300 - \frac{234\times 10^5}{(t+300)^2}, \quad 0 \le t \le 300$
15.
   a. $Q' + \frac{2}{25}Q = 6 - 2e^{-t/25}$
   b. $Q = 75 - 50e^{-t/25} - 25e^{-2t/25}$
   c. $75$
16.
   a. $T = T_m + (T_0 - T_m)e^{-kt} + \frac{k(S_0 - T_m)}{(k - k_m)}(e^{-kmt} - e^{-kt})$
   b. $T = T_m + k(S_0 - T_m)te^{-kt} + (T_0 - T_m)e^{-kt}$
   c. $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} S(t) = T_m$
17.
   a. $T' = -k(1 + \frac{a}{a_m})T + k(T_{m0} + \frac{a}{a_m}T_0)$
   b. $T = \frac{aT_0 + a_mT_{m0}}{a + a_m} + \frac{a_m(T_0 - T_{m0})}{a + a_m}e^{-k(1+a/a_m)t}, \quad T_m = \frac{aT_0 + a_mT_{m0}}{a + a_m} + \frac{a(T_{m0} - T_0)}{a + a_m}e^{-k(a+a/a_m)t}$
   c. $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} T_m(t) = \frac{aT_0 + a_mT_{m0}}{a + a_m}$
18. $V = \frac{a}{b}\frac{V_0}{V_0 - (V_0 - a/b)e^{-at}}; \quad \lim_{t\to\infty} V(t) = a/b$
19. $c_1 = c(1 - e^{-rt/W}), c_2 = c(1 - e^{-rt/W} - \frac{r}{W}te^{-rt/W})$
20.
   a. $c_n = c\left(1 - e^{-rt/W}\sum_{j=0}^{n-1}\frac{1}{j!}\left(\frac{rt}{W}\right)^j\right)$
   b. $c$
   c. $0$

21. Let $c_\infty = \frac{c_1W_1+c_2W_2}{W_1+W_2}$, $\alpha = \frac{c_2W_2^2-c_1W_1^2}{W_1+W_2}$, and $\beta = \frac{W_1+W_2}{W_1W_2}$. Then:

a. $c_1(t) = c_\infty + \frac{\alpha}{W_1}e^{-r\beta t}$, $c_2(t) = c_\infty - \frac{\alpha}{W_2}e^{-r\beta t}$

b. $\lim_{t\to\infty} c_1(t) = \lim_{t\to\infty} c_2(t) = c_\infty$

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# 14.15: Section 4.3 Answers

1. $v = -\frac{384}{5}(1 - e^{-5t/12}); \quad -\frac{384}{5}\text{ft/s}$

2. $k = 12; \quad v = -16(1 - e^{-2t})$

3. $v = 25(1 - e^{-t}); \quad 25\text{ft/s}$

4. $v = 20 - 27e^{-t/40}$

5. $\approx 17.10$ ft

6. $v = -\frac{40(13+3e^{-4t/5})}{13-3e^{-4t/5}}; \quad -40\text{ft/s}$

7. $v = -128(1 - e^{-t/4})$

9. $T = \frac{m}{k}\ln\left(1 + \frac{v_0 k}{mg}\right); \quad y_m = y_0 = \frac{m}{k}\left[v_0 - \frac{mg}{l}\ln\left(1 + \frac{v_0 k}{mg}\right)\right]$

10. $v = -\frac{64(1-e^{-t})}{1+e^{-t}}; \quad 64\text{ft/s}$

11. $v = \alpha\frac{v_0(1+e^{-\beta t})-\alpha(1-e^{-\beta t})}{\alpha(1+e^{-\beta t})-v_0(1-e^{-\beta t})}; \quad -\alpha$ where $\alpha = \sqrt{\frac{mg}{k}}$ and $\beta = 2\sqrt{\frac{kg}{m}}$

12. $T = \sqrt{\frac{m}{kg}}\tan^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)$ $v = -\sqrt{\frac{mg}{k}}; \quad \frac{1-e^{-2\sqrt{\frac{ak}{m}}(t-T)}}{1+e^{-2\sqrt{\frac{ak}{m}}(t-T)}}$

13. $s' = mg - \frac{as}{s+1}; \quad a_0 = mg$

14. (a) $ms' = mg - f(s)$

15.
    a. $v' = -9.8 + v^4/81$
    b. $v_T \approx -5.308\text{m/s}$

16.
    a. $v' = -32 + 8\sqrt{|v|}; \quad v_T = -16$ ft/s
    b. From Exercise 4.3.14
    c. $v_T$ is the negative number such that $-32 + 8\sqrt{|v_T|} = 0$ ; thus, $v_T = -16$ ft/s

17. $\approx 6.76\text{miles/s}$

18. $\approx 1.47\text{miles/s}$

20. $\alpha = \frac{gR^2}{(y_m+R)^2}$

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# 14.16: Section 4.4 Answers

1. $\bar{y}=0$ 0 is a stable equilibrium; trajectories are $v^2+\frac{y^2}{4}=c$

2. $\bar{y}=0$ 0 is an unstable equilibrium; trajectories are $v^2+\frac{2y^3}{3}=c$

3. $\bar{y}=0$ 0 is a stable equilibrium; trajectories are $v^2+\frac{2|y|^3}{3}=c$

4. $\bar{y}=0$ 0 is a stable equilibrium; trajectories are $v^2-e^{-y}(y+1)=c$

5. equilibria: 0 (stable) and $-2, 2$ (unstable); trajectories: $2v^2-y^4+8y^2=c$ ; separatrix: $2v^2-y^4+8y^2=16$

6. equilibria: 0 (unstable) and $-2, 2$ (stable); trajectories: $2v^2+y^4-8y^2=c$ ; separatrix: $2v^2+y^4-8y^2=0$

7. equilibria: $0, -2, 2$ (stable), $-1, 1$ (unstable); trajectories: $6v^2+y^2(2y^4-15y^2+24)=c$ ; separatrix: $6v^2+y^2(2y^4-15y^2+24)=11$

8. equilibria: $0, 2$ (stable) and $-2, 1$ (unstable); trajectories: $30v^2+y^2(12y^3-15y^2-80y+120)=c$ ; separatrices: $30v^2+y^2(12y^3-15y^2-80y+120)=496$ and $30v^2+y^2(12y^3-15y^2-80y+120)=37$

9. No equilibria if $a<0$; 0 is unstable if $a=0$; $\sqrt{a}$ is stable and $-\sqrt{a}$ is unstable if $a>0$.

10. 0 is a stable equilibrium if $a\le 0$; $-\sqrt{a}$ and $\sqrt{a}$ are stable and 0 is unstable if $a>0$.

11. 0 is unstable if $a\le 0$; $-\sqrt{a}$ and $\sqrt{a}$ are unstable and 0 is stable if $a>0$.

12. 0 is stable if $a\le 0$; 0 is stable and $-\sqrt{a}$ and $\sqrt{a}$ are unstable if $a\le 0$.

22. An equilibrium solution $\bar{y}$ of $y''+p(y)=0$ is unstable if there's an $\epsilon>0$ such that, for every $\delta>0$, there's a solution of (A) with $\sqrt{(y(0)-\bar{y})^2+v^2(0)}<\delta$, but $\sqrt{(y(t)-\bar{y})^2+v^2(t)}\ge\epsilon$ for some $t>0$.

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# 14.17: Section 4.5 Answers

1. $y' = \frac{2xy}{x^2+3y^2}$
2. $y' = -\frac{y^2}{(xy-1)}$
3. $y' = -\frac{y(x^2+y^2-2x^2\ln|xy|)}{x(x^2+y^2-2y^2\ln|xy|)}$
4. $xy' - y = -\frac{x^{1/2}}{2}$
5. $y' + 2xy = 4xe^{x^2}$
6. $xy' + y = 4x^3$
7. $y' - y = \cos x - \sin x$
8. $(1+x^2)y' - 2xy = (1-x)^2e^x$
10. $y'g - yg' = f'g - fg'$
11. $(x - x_0)y' = y - y_0$
12. $y'(y^2 - x^2 + 1) + 2xy = 0$
13. $2x(y-1)y' - y^2 + x^2 + 2y = 0$
14. 
    a. $y = -81 + 18x,\ (9, 81) \quad y = -1 + 2x,\ (1, 1)$
    b. $y = -121 + 22x,\ (11, 121) \quad y = -1 + 2x,\ (1, 1)$
    c. $y = -100 - 20x,\ (-10, 100) \quad y = -4 - 4x,\ (-2, 4)$
    d. $y = -25 - 10x,\ (-5, 25) \quad y = -1 - 2x,\ (-1, 1)$
15. (e) $y = \frac{5+3x}{4},\ (-3/5, 4/5) \quad y = -\frac{5-4x}{3},\ (4/5, -3/5)$
17. 
    a. $y = -\frac{1}{2}(1+x),\ (1, -1); \quad y = \frac{5}{2} + \frac{x}{10},\ (25, 5)$
    b. $y = \frac{1}{4}(4+x),\ (4, 2) \quad y = -\frac{1}{4}(4+x),\ (4, -2)$
    c. $y = \frac{1}{2}(1+x),\ (1, 1) \quad y = \frac{7}{2} + \frac{x}{14},\ (49, 7)$
    d. $y = -\frac{1}{2}(1+x),\ (1, -1) \quad y = -\frac{5}{2} - \frac{x}{10},\ (25, -5)$
18. $y = 2x^2$
19. $y = \frac{cx}{\sqrt{|x^2-1|}}$
20. $y = y_1 + c(x - x_1)$
21. $y = -\frac{x^3}{2} - \frac{x}{2}$
22. $y = -x\ln|x| + cx$
23. $y = \sqrt{2x+4}$
24. $y = \sqrt{x^2-3}$
25. $y = kx^2$
26. $(y-x)^3(y+x) = k$
27. $y^2 = -x + k$
28. $y^2 = -\frac{1}{2}\ln(1+2x^2) + k$
29. $y^2 = -2x - \ln(x-1)^2 + k$
30. $y = 1 + \sqrt{\frac{9-x^2}{2}};$ those with $c > 0$

33. $\tan^{-1}\frac{y}{x}-\frac{1}{2}\ln(x^2+y^2)=k$

34. $\frac{1}{2}\ln(x^2+y^2)+(\tan\alpha)\tan^{-1}\frac{y}{x}=k$

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# 14.18: Section 5.1 Answers

1.

(c) $y = -2e^{2x} + e^{5x}$

(d) $y = (5k_0 - k_1)\frac{e^{2x}}{3} + (k_1 - 2k_0)\frac{e^{5x}}{3}$

2.

(c) $y = e^{3x}(3\cos x - 5\sin x)$

(d) $y = e^x(k_0 \cos x + (k_1 - k_0)\sin x$

3.

(c) $y = e^x(7 - 3x)$

(d) $y = e^x(k_0 + (k_1 - k_0)x)$

4.

a. $y = \frac{c_1}{x-1} + \frac{c_2}{x+1}$
b. $y = \frac{2}{x-1} - \frac{3}{x+1};\ (-1, 1)$

5.

a. $e^x$
b. $e^{2x}\cos x$
c. $x^2 + 2x - 2$
d. $-\frac{5}{6}x^{-5/6}$
e. $-\frac{1}{x^2}$
f. $(x \ln|x|)^2$
g. $\frac{e^{2x}}{2\sqrt{x}}$

6. 0

7. $W(x) = (1 - x^2)^{-1}$

8. $W(x) = \frac{1}{x}$

10. $y_2 = e^{-x}$

11. $y_2 = xe^{3x}$

12. $y_2 = xe^{ax}$

13. $y_2 = \frac{1}{x}$

14. $y_2 = x \ln x$

15. $y_2 = x^a \ln x$

16. $y_2 = x^{1/2}e^{-2x}$

17. $y_2 = x$

18. $y_2 = x \sin x$

19. $y_2 = x^{1/2}\cos x$

20. $y_2 = xe^{-x}$

21. $y_2 = \frac{1}{x^2-4}$

22. $y_2 = e^{2x}$

23. $y_2 = x^2$

35.

a. $y'' - 2xy' + 5y = 0$
b. $(2x-1)y'' - 4xy' + 4y = 0$
c. $x^2y'' - xy' + y = 0$
d. $x^2y'' + xy' + y = 0$
e. $y'' - y = 0$
f. $xy'' - y' = 0$

37. (c) $y = k_0y_1 + k_1y_2$

38. $y_1 = 1,\ y_2 = x - x_0; \quad y = k_0 + k_1(x - x_0)$

39. $y_1 = \cosh(x - x_0),\ y_2 = \sinh(x - x_0);\ y = k_0\cosh(x - x_0) + k_1\sinh(x - x_0)$

40. $y_1 = \cos\omega(x - x_0),\ y_2 = \frac{1}{\omega}\sin\omega(x - x_0) y = k_0\cos\omega(x - x_0) + \frac{k_1}{\omega}\sin\omega(x - x_0)$

41. $y_1 = \frac{1}{1-x^2}\ y_2 = \frac{x}{1-x^2}\ y = \frac{k_0+k_1x}{1-x^2}$

42.

(c) $k_0 = k_1 = 0;\ y = \begin{cases} c_1x^2 + c_2x^3 & x \geq 0 \\ c_1x^2 + c_3x^3 & x < 0 \end{cases}$

(d) $(0, \infty)$ if $x_0 > 0,\ (-\infty, 0)$ if $x_0 < 0$

43. (c) $k_0 = 0,\ k_1$ arbitrary $y = k_1x + c_2x^2$

44.

(c) $k_0 = k_1 = 0;\ y = \begin{cases} a_1x^3 + a_2x^4 & x \geq 0 \\ b_1x^3 + b_2x^4 & x < 0 \end{cases}$

(d) $(0, \infty)$ if $x_0 > 0,\ (-\infty, 0)$ if $x_0 < 0$

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# 14.19: Section 5.2 Answers

1. $y = c_1 e^{-6x} + c_2 e^{x}$
2. $y = e^{2x}(c_1 \cos x + c_2 \sin x)$
3. $y = c_1 e^{-7x} + c_2 e^{-x}$
4. $y = e^{2x}(c_1 + c_2 x)$
5. $y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$
6. $y = e^{-3x}(c_1 \cos x + c_2 \sin x)$
7. $y = e^{4x}(c_1 + c_2 x)$
8. $y = c_1 + c_2 e^{-x}$
9. $y = e^{x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)$
10. $y = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$
11. $y = e^{-x/2}\left(c_1 \cos \frac{3x}{2} + c_2 \sin \frac{3x}{2}\right)$
12. $y = c_1 e^{-x/5} + c_2 e^{x/2}$
13. $y = e^{-7x}(2 \cos x - 3 \sin x)$
14. $y = 4e^{x/2} + 6e^{-x/3}$
15. $y = 3e^{x/3} - 4e^{-x/2}$
16. $y = \frac{e^{-x/2}}{3} + \frac{3e^{3x/2}}{4}$
17. $y = e^{3x/2}(3 - 2x)$
18. $y = 3^{-4x} - 4e^{-3x}$
19. $y = 2xe^{3x}$
20. $y = e^{x/6}(3 + 2x)$
21. $y = e^{-2x}\left(3 \cos \sqrt{6}x + \frac{2\sqrt{6}}{3} \sin \sqrt{6}x\right)$
23. $y = 2e^{-(x-1)} - 3e^{-2(x-1)}$
24. $y = \frac{1}{3}e^{-(x-2)} - \frac{2}{3}e^{7(x-2)}$
25. $y = e^{7(x-1)}(2 - 3(x-1))$
26. $y = e^{-(x-2)/3}(2 - 4(x-2))$
27. $y = 2 \cos \frac{2}{3}\left(x - \frac{\pi}{4}\right) - 3 \sin \frac{2}{3}\left(x - \frac{\pi}{4}\right)$
28. $y = 2 \cos \sqrt{3}\left(x - \frac{\pi}{3}\right) - \frac{1}{\sqrt{3}} \sin \sqrt{3}\left(x - \frac{\pi}{3}\right)$
30. $y = \frac{k_0}{r_2 - r_1}\left(r_2 e^{r_1(x-x_0)} - r_1 e^{r_2(x-x_0)}\right) + \frac{k_1}{r_2 - r_1}\left(e^{r_2(x-x_0)} - e^{r_1(x-x_0)}\right)$
31. $y = e^{r_1(x-x_0)}\left[k_0 + (k_1 - r_1 k_0)(x - x_0)\right]$
32. $y = e^{\lambda(x-x_0)}\left[k_0 \cos \omega(x - x_0) + \left(\frac{k_1 - \lambda k_0}{\omega}\right) \sin \omega(x - x_0)\right]$

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# 14.20: Section 5.3 Answers

1. $y_p = -1 + 2x + 3x^2;\ y = -1 + 2x + 3x^2 + c_1 e^{-6x} + c_2 e^{x}$
2. $y_p = 1 + x;\ y = 1 + x + e^{2x}(c_1 \cos x + c_2 \sin x)$
3. $y_p = -x + x^3;\ y = -x + x^3 + c_1 e^{-7x} + c_2 e^{-x}$
4. $y_p = 1 - x^2;\ y = 1 - x^2 + e^{2x}(c_1 + c_2 x)$
5. $y_p = 2x + x^3;\ y = 2x + x^3 + e^{-x}(c_1 \cos 3x + c_2 \sin 3x);\ y = 2x + x^3 + e^{-x}(2\cos 3x + 3 \sin 3x)$
6. $y_p = 1 + 2x;\ y = 1 + 2x + e^{-3x}(c_1 \cos x + c_2 \sin x);\ y = 1 + 2x + e^{-3x}(\cos x - \sin x)$
8. $y_p = \frac{2}{x}$
9. $y_p = 4x^{1/2}$
10. $y_p = \frac{x^3}{2}$
11. $y_p = \frac{1}{x^3}$
12. $y_p = 9x^{1/3}$
13. $y_p = \frac{2x^4}{13}$
16. $y_p = \frac{e^{3x}}{3};\ y = \frac{e^{3x}}{3} + c_1 e^{-6x} + c_2 e^{x}$
17. $y_p = e^{2x};\ y = e^{2x}(1 + c_1 \cos x + c_2 \sin x)$
18. $y = -2e^{-2x};\ y = -2e^{-2x} + c_1 e^{-7x} + c_2 e^{-x};\ y = -2e^{-2x} - e^{-7x} + e^{-x}$
19. $y_p = e^{x};\ y = e^{x} + e^{2x}(c_1 + c_2 x);\ y = e^{x} + e^{2x}(1 - 3x)$
20. $y_p = \frac{4}{25} e^{x/2};\ y = \frac{4}{45} e^{x/2} + e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$
21. $y_p = e^{-3x};\ y = e^{-3x}(1 + c_1 \cos x + c_2 \sin x)$
24. $y_p = \cos x - \sin x;\ y = \cos x - \sin x + e^{4x}(c_1 + c_2 x)$
25. $y_p = \cos 2x - 2 \sin 2x;\ y = \cos 2x - 2 \sin 2x + c_1 + c_2 e^{-x}$
26. $y_p = \cos 3x;\ y = \cos 3x + e^{x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)$
27. $y_p = \cos x + \sin x;\ y = \cos x + \sin x + e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$
28. $y_p = -2 \cos 2x + \sin 2x;\ y = -2 \cos 2x + \sin 2x + c_1 e^{-4x} + c_2 e^{-3x};\ y = -2 \cos 2x + \sin 2x + 2e^{-4x} - 3e^{-3x}$
29. $y_p = \cos 3x - \sin 3x;\ y = \cos 3x - \sin 3x + e^{3x}(c_1 + c_2 x)\ y = \cos 3x - \sin 3x + e^{3x}(1 + 2x)$
30. $y = \frac{1}{\omega_0^2 - \omega^2}(M \cos \omega x + N \sin \omega x) + c_1 \cos \omega_0 x + c_2 \sin \omega_0 x$
33. $y_p = -1 + 2x + 3x^2 + \frac{e^{3x}}{3};\ y = -1 + 2x + 3x^2 + \frac{e^{3x}}{3} + c_1 e^{-6x} + c_2 e^{x}$
34. $y_p = 1 + x + e^{2x};\ y = 1 + x + e^{2x}(1 + c_1 \cos x + c_2 \sin x)$
35. $y_p = -x + x^3 - 2e^{-2x};\ y = -x + x^3 - 2e^{-2x} + c_1 e^{-7x} + c_2 e^{-x}$
36. $y_p = 1 - x^2 + e^{x};\ y = 1 - x^2 + e^{x} + e^{2x}(c_1 + c_2 x)$
37. $y_p = 2x + x^3 + \frac{4}{45} e^{x/2};\ y = 2x + x^3 + \frac{4}{45} e^{x/2} + e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$
38. $y_p = 1 - x^2 + e^{x};\ y = 1 - x^2 + e^{x} + e^{2x}(1 + c_1 \cos x + c_2 \sin x)$

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# 14.21: Section 5.4 Answers

1. $y_p = e^{3x}\left(-\frac{1}{4}+\frac{x}{2}\right)$
2. $y_p = e^{-3x}\left(1-\frac{x}{4}\right)$
3. $y_p = e^{x}\left(2-\frac{3x}{4}\right)$
4. $y_p = e^{2x}(1-3x+x^2)$
5. $y_p = e^{-x}(1+x^2)$
6. $y_p = e^{x}(-2+x+2x^2)$
7. $y_p = xe^{-x}\left(\frac{1}{6}+\frac{x}{2}\right)$
8. $y_p = xe^{x}(1+2x)$
9. $y_p = xe^{3x}\left(-1+\frac{x}{2}\right)$
10. $y_p = xe^{2x}(-2+x)$
11. $y_p = x^2e^{-x}\left(1+\frac{x}{2}\right)$
12. $y_p = x^2e^{x}\left(\frac{1}{2}-x\right)$
13. $y_p = \frac{x^2e^{2x}}{2}(1-x+x^2)$
14. $y_p = \frac{x^2e^{-x/3}}{27}(3-2x+x^2)$
15. $y = \frac{e^{3x}}{4}(-1+2x)+c_1e^{x}+c_2e^{2x}$
16. $y = e^{x}(1-2x)+c_1e^{2x}+c_2e^{4x}$
17. $y = \frac{e^{2x}}{5}(1-x)+e^{-3x}(c_1+c_2x)$
18. $y = xe^{x}(1-2x)+c_1e^{x}+c_2e^{-3x}$
19. $y = e^{x}\left[x^2(1-2x)+c_1+c_2x\right]$
20. $y = -e^{2x}(1+x)+2e^{-x}-e^{5x}$
21. $y = xe^{2x}+3e^{x}-e^{-4x}$
22. $y = e^{-x}(2+x-2x^2)-e^{-3x}$
23. $y = e^{-2x}(3-x)-2e^{5x}$
24. $y_p = -\frac{e^{x}}{3}(1-x)+e^{-x}(3+2x)$
25. $y_p = e^{x}(3+7x)+xe^{3x}$
26. $y_p = x^3e^{4x}+1+2x+x^2$
27. $y_p = xe^{2x}(1-2x)+xe^{x}$
28. $y_p = e^{x}(1+x)+x^2e^{-x}$
29. $y_p = x^2e^{-x}+e^{3x}(1-x^2)$
31. $y_p = 2e^{2x}$
32. $y_p = 5xe^{4x}$
33. $y_p = x^2e^{4x}$
34. $y_p = -\frac{e^{3x}}{4}(1+2x-2x^2)$
35. $y_p = xe^{3x}(4-x+2x^2)$
36. $y_p = x^2e^{-x/2}(-1+2x+3x^2)$

37.

a. $y = e^{-x}\left(\frac{4}{3}x^{3/2} + c_1 x + c_2\right)$
b. $y = e^{-3x}\left[\frac{x^2}{4}(2\ln x - 3) + c_1 x + c_2\right]$
c. $y = e^{2x}[(x+1)\ln|x+1| + c_1 x + c_2]$
d. $y = e^{-x/2}\left(x\ln|x| + \frac{x^3}{6} + c_1 x + c_2\right)$

39.

a. $e^x(3+x)+c$
b. $-e^{-x}(1+x)^2 + c$
c. $-\frac{e^{-2x}}{8}(3+6x+6x^2 = 4x^3) + c$
d. $e^x(1+x^2)+c$
e. $e^{3x}(-6+4x+9x^2)+c$
f. $-e^{-x}(1-2x^3+3x^4)+c$

40. $\frac{(-1)^k k! e^{\alpha x}}{\alpha^{k+1}} \sum_{r=0}^{k} \frac{(-\alpha x)^r}{r!} + c$

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# 14.22: Section 5.5 Answers

1. $y_p = \cos x + 2\sin x$
2. $y_p = \cos x + (2-2x)\sin x$
3. $y_p = e^x(-2\cos x + 3\sin x)$
4. $y_p = \frac{e^{2x}}{2}(\cos 2x - \sin 2x)$
5. $y_p = -e^x(x\cos x - \sin x)$
6. $y_p = e^{-2x}(1-2x)(\cos 3x - \sin 3x)$
7. $y_p = x(\cos 2x - 3\sin 2x)$
8. $y_p = -x[(2-x)\cos x + (3-2x)\sin x]$
9. $y_p = x\left[x\cos\left(\frac{x}{2}\right) - 3\sin\left(\frac{x}{2}\right)\right]$
10. $y_p = xe^{-x}(3\cos x + 4\sin x)$
11. $y_p = xe^x[(-1+x)\cos 2x + (1+x)\sin 2x]$
12. $y_p = -(14-10x)\cos x - (2+8x-4x^2)\sin x$
13. $y_p = (1+2x+x^2)\cos x + (1+3x^2)\sin x$
14. $y_p = \frac{x^2}{2}(\cos 2x - \sin 2x)$
15. $y_p = e^x(x^2\cos x + 2\sin x)$
16. $y_p = e^x(1-x^2)(\cos x + \sin x)$
17. $y_p = e^x(x^2 - x^3)(\cos x + \sin x)$
18. $y_p = e^{-x}[(1+2x)\cos x - (1-3x)\sin x]$
19. $y_p = x(2\cos 3x - \sin 3x)$
20. $y_p = -x^3\cos x + (x+2x^2)\sin x$
21. $y_p = -e^{-x}[(x+x^2)\cos x - (1+2x)\sin x]$
22. $y = e^x(2\cos x + 3\sin x) + 3e^x - e^{6x}$
23. $y = e^x[(1+2x)\cos x + (1-3x)\sin x]$
24. $y = e^x(\cos x - 2\sin x) + e^{-3x}(\cos x + \sin x)$
25. $y = e^{3x}[(2+2x)\cos x - (1+3x)\sin x]$
26. $y = e^{3x}[(2+3x)\cos x + (4-x)\sin x] + 3e^x - 5e^{2x}$
27. $y_p = xe^{3x} - \frac{e^x}{5}(\cos x - 2\sin x)$
28. $y_p = x(\cos x + 2\sin x) - \frac{e^x}{2}(1-x) + \frac{e^{-x}}{2}$
29. $y_p = -\frac{xe^x}{2}(2+x) + 2xe^{2x} + \frac{1}{10}(3\cos x + \sin x)$
30. $y_p = xe^x(\cos x + x\sin x) + \frac{e^x}{25}(4+5x) + 1 + x + \frac{x^2}{2}$
31. $y_p = \frac{x^2e^{2x}}{6}(3+x) - e^{2x}(\cos x - \sin x) + 3e^{3x} + \frac{1}{4}(2+x)$
32. $y = (1-2x+3x^2)e^{2x} + 4\cos x + 3\sin x$
33. $y = xe^{-2x}\cos x + 3\cos 2x$
34. $y = -\frac{3}{8}\cos 2x + \frac{1}{4}\sin 2x + e^{-x} - \frac{13}{8}e^{-2x} - \frac{3}{4}xe^{-2x}$

40.

a. $2x\cos x - (2-x^2)\sin x + c$

b. $-\frac{e^x}{2}\left[(1-x^2)\cos x-(1-x)^2\sin x\right]+c$
c. $-\frac{e^{-x}}{25}[(4+10x)\cos 2x-(3-5x)\sin 2x]+c$
d. $-\frac{e^{-x}}{2}\left[(1+x)^2\cos x-(1-x^2)\sin x\right]+c$
e. $-\frac{e^x}{2}\left[x(3-3x+x^2)\cos x-(3-3x+x^3)\sin x\right]+c$
f. $-e^x[(1-2x)\cos x+(1+x)\sin x]+c$
g. $e^{-x}[x\cos x+x(1+x)\sin x]+c$

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# 14.23: Section 5.6 Answers

1. $y = 1 - 2x + c_1 e^{-x} + c_2 x e^{x};\ \{e^{-x}, xe^{x}\}$
2. $y = \frac{4}{3x^2} + c_1 x + \frac{c_2}{x};\ \{x, 1/x\}$
3. $y = \frac{x(\ln|x|)^2}{2} + c_1 x + c_2 x \ln|x|;\ \{x, x\ln|x|\}$
4. $y = (e^{2x} + e^{x})\ln(1 + e^{-x}) + c_1 e^{2x} + c_2 e^{x};\ \{e^{2x}, e^{x}\}$
5. $y = e^{x}\left(\frac{4}{5}x^{7/2} + c_1 + c_2 x\right);\ \{e^{x}, xe^{x}\}$
6. $y = e^{x}(2x^{3/2} + x^{1/2}\ln x + c_1 x^{1/2} + c_2 x^{-1/2});\ \{x^{1/2}e^{x}, x^{-1/2}e^{-x}\}$
7. $y = e^{x}(x\sin x + \cos x \ln|\cos x| + c_1\cos x + c_2 \sin x);\ \{e^{x}\cos x, e^{x}\sin x\}$
8. $y = e^{-x^2}(2e^{-2x} + c_1 + c_2 x);\ \{e^{-x^2}, xe^{-x^2}\}$
9. $y = 2x + 1 + c_1 x^2 + \frac{c_2}{x^2};\ \{x^2, 1/x^2\}$
10. $y = \frac{xe^{2x}}{9} + xe^{-x}(c_1 + c_2 x);\ \{xe^{-x}, x^2 e^{-x}\}$
11. $y = xe^{x}\left(\frac{x}{3} + c_1 + \frac{c_2}{x^2}\right);\ \{xe^{x}, e^{x}/x\}$
12. $y = -\frac{(2x-1)^2 e^{x}}{8} + c_1 e^{x} + c_2 x e^{-x};\ \{e^{x}, xe^{-x}\}$
13. $y = x^4 + c_1 x^2 + c_2 x^2 \ln|x|; \{x^2, x^2\ln|x|\}$
14. $y = e^{-x}(x^{3/2} + c_1 + c_2 x^{1/2});\ \{e^{-x}, x^{1/2}e^{-x}\}$
15. $y = e^{x}(x + c_1 + c_2 x^2);\ \{e^{x}, x^2 e^{x}\}$
16. $y = x^{1/2}\left(\frac{e^{2x}}{2} + c_1 + c_2 e^{x}\right);\ \{x^{1/2}, x^{1/2}e^{x}\}$
17. $y = -2x^2 \ln x + c_1 x^2 + c_2 x^4;\ \{x^2, x^4\}$
18. $\{e^{x}, e^{x}/x\}$
19. $\{x^2, x^3\}$
20. $\{\ln|x|, x\ln|x|\}$
21. $\{\sin\sqrt{x}, \cos\sqrt{x}\}$
22. $\{e^{x}, x^3 e^{x}\}$
23. $\{x^{a}, x^{a}\ln|x|\}$
24. $\{x\sin x, x\cos x\}$
25. $\{e^{2x}, x^2 e^{2x}\}$
26. $\{x^{1/2}, x^{1/2}\cos x\}$
27. $\{x^{1/2}e^{2x}, x^{1/2}e^{-2x}\}$
28. $\{1/x, e^{2x}\}$
29. $\{e^{x}, x^2\}$
30. $\{e^{2x}, x^2 e^{2x}\}$
31. $y = x^4 = 6x^2 - 8x^2\ln|x|$
32. $y = 2e^{2x} - xe^{-x}$
33. $y = \frac{(x+1)}{4}[-e^{x}(3 - 2x) + 7e^{-x}]$
34. $y = \frac{x^2}{4} + x$

35. $y = \frac{(x+2)^2}{6(x-2)} + \frac{2x}{x^2-4}$

38.

a. $y = \frac{-kc_1 \sin kx + kc_2 \cos kx}{c_1 \cos kx + c_2 \sin kx}$
b. $y = \frac{c_1 + 2c_2 e^x}{c_1 + c_2 e^x}$
c. $y = \frac{-6c_1 + c_2 e^{7x}}{c_1 + c_2 e^{7x}}$
d. $y = -\frac{7c_1 + c_2 e^{6x}}{c_1 + c_2 e^{6x}}$
e. $y = -\frac{(7c_1 - c_2)\cos x + (c_1 + 7c_2)\sin x}{c_1 \cos x + c_2 \sin x}$
f. $y = \frac{-2c_1 + 3c_2 e^{5x/6}}{6(c_1 + c_2 e^{5x/6})}$
g. $y = \frac{c_1 + c_2(x+6)}{6(c_1 + c_2 x)}$

39.

a. $y = \frac{c_1 + c_2 e^x (1+x)}{x(c_1 + c_2 e^x)}$
b. $y = \frac{-2c_1 x + c_2(1-2x^2)}{c_1 + c_2 x}$
c. $y = \frac{-c_1 + c_2 e^{2x}(x+1)}{c_1 + c_2 x e^{2x}}$
d. $y = \frac{2c_1 + c_2 e^{-3x}(1-x)}{c_1 + c_2 x e^{-3x}}$
e. $y = \frac{(2c_2 x - c_1)\cos x - (2c_1 x + c_2)\sin x}{2x(c_1 \cos x + c_2 \sin x)}$
f. $y = \frac{c_1 + 7c_2 x^6}{x(c_1 + c_2 x^6)}$

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# 14.24: Section 5.7 Answers

1. $y_p = \frac{-\cos 3x \ln|\sec 3x + \tan 3x|}{9}$
2. $y_p = -\frac{\sin 2x \ln|\cos 2x|}{4} + \frac{x \cos 2x}{2}$
3. $y_p = 4e^x(1+e^x)\ln(1+e^{-x})$
4. $y_p = 3e^x(\cos x \ln|\cos x| + x \sin x)$
5. $y_p = \frac{8}{5}x^{7/2}e^x$
6. $y_p = e^x \ln(1-e^{-2x}) - e^{-x}\ln(e^{2x}-1)$
7. $y_p = \frac{2(x^2-3)}{3}$
8. $y_p = \frac{e^{2x}}{x}$
9. $y_p = x^{1/2}e^x \ln x$
10. $y_p = e^{-x(x+2)}$
11. $y_p = -4x^{5/2}$
12. $y_p = -2x^2 \sin x - 2x \cos x$
13. $y_p = -\frac{xe^{-x}(x+1)}{2}$
14. $y_p = -\frac{\sqrt{x}\cos\sqrt{x}}{2}$
15. $y_p = \frac{3x^4e^x}{2}$
16. $y_p = x^{a+1}$
17. $y_p = \frac{x^2 \sin x}{2}$
18. $y_p = -2x^2$
19. $y_p = -e^{-x}\sin x$
20. $y_p = -\frac{\sqrt{x}}{2}$
21. $y_p = \frac{x^{3/2}}{4}$
22. $y_p = -3x^2$
23. $y_p = \frac{x^3e^x}{2}$
24. $y_p = -\frac{4x^{3/2}}{15}$
25. $y_p = x^3e^x$
26. $y_p = xe^x$
27. $y_p = x^2$
28. $y_p = xe^x(x-2)$
29. $y_p = \sqrt{x}e^x(x-1)/4$
30. $y = \frac{e^{2x}(3x^2-2x+6)}{6} + \frac{xe^{-x}}{3}$
31. $y = (x-1)^2\ln(1-x) + 2x^2 - 5x + 3$
32. $y = (x^2-1)e^x - 5(x-1)$
33. $y = \frac{x(x^2+6)}{3(x^2-1)}$
34. $y = -\frac{x^2}{2} + x + \frac{1}{2x^2}$

35. $y = \frac{x^2(4x+9)}{6(x+1)}$

38.

a. $y = k_0 \cosh x + k_1 \sinh x + \int_0^x \sinh(x-t) f(t) dt$
b. $y' = k_0 \sinh x + k_1 \cosh x + \int_0^x \cosh(x-t) f(t) dt$

39.

a. $y(x) = k_0 \cos x + k_1 \sin x + \int_0^x \sin(x-t) f(t) dt$
b. $y'(x) = -k_0 \sin x + k_1 \cos x + \int_0^x \cos(x-t) f(t) dt$

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# 14.25: Section 6.1 Answers

1. $y = 3\cos 4\sqrt{6}t - \frac{1}{2\sqrt{6}}\sin 4\sqrt{6}t$ ft
2. $y = -\frac{1}{4}\cos 8\sqrt{5}t - \frac{1}{4\sqrt{5}}\sin 8\sqrt{5}t$ ft
3. $y = -1.5\cos 14\sqrt{10}t$ cm
4. $y = \frac{1}{4}\cos 8t - \frac{1}{16}\sin 8t$ ft; $R = \frac{\sqrt{17}}{16}$ ft; $\omega_0 = 8$ rad/s; $T = \pi/4$ s; $\phi \approx -.245$ rad $\approx -14.04^\circ$
5. $y = 10\cos 14t + \frac{25}{14}\sin 14t$ cm; $R = \frac{5}{14}\sqrt{809}$ cm; $\omega_0 = 14$ rad/s; $T = \pi/7$ s; $\phi \approx .177$ rad $\approx 10.12^\circ$
6. $y = -\frac{1}{4}\cos\sqrt{70}t + \frac{2}{\sqrt{70}}\sin\sqrt{70}t$ m; $R = \frac{1}{4}\sqrt{\frac{67}{35}}$ m; $\omega_0 = \sqrt{70}$ rad/s; $T = 2\pi/\sqrt{70}$ s; $\phi \approx 2.38$ rad $\approx 136.28^\circ$
7. $y = \frac{2}{3}\cos 16t - \frac{1}{4}\sin 16t$ ft
8. $y = \frac{1}{2}\cos 8t - \frac{3}{8}\sin 8t$ ft
9. .72 m
10. $y = \frac{1}{3}\sin t + \frac{1}{2}\cos 2t + \frac{5}{6}\sin 2t$ ft
11. $y = \frac{16}{5}\left(4\sin\frac{t}{4} - \sin t\right)$
12. $y = -\frac{1}{16}\sin 8t + \frac{1}{3}\cos 4\sqrt{2}t - \frac{1}{8\sqrt{2}}\sin 4\sqrt{2}t$
13. $y = -t\cos 8t - \frac{1}{6}\cos 8t + \frac{1}{8}\sin 8t$ ft
14. $T = 4\sqrt{2}$ s
15. $\omega = 8$ rad/s $y = -\frac{t}{16}(-\cos 8t + 2\sin 8t) + \frac{1}{128}\sin 8t$ ft
16. $\omega = 4\sqrt{6}$ rad/s; $y = -\frac{t}{\sqrt{6}}\left[\frac{8}{3}\cos 4\sqrt{6}t + 4\sin 4\sqrt{6}t\right] + \frac{1}{9}\sin 4\sqrt{6}t$ ft
17. $y = \frac{t}{2}\cos 2t - \frac{t}{4}\sin 2t + 3\cos 2t + 2\sin 2t$ m
18. $y = y_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$; $R = \frac{1}{\omega_0}\sqrt{(\omega_0 y_0)^2 + (v_0)^2}$; $\cos\phi = \frac{y_0\omega_0}{\sqrt{(\omega_0 y_0)^2 + (v_0)^2}}$; $\sin\phi = \frac{v_0}{\sqrt{(\omega_0 y_0)^2 + (v_0)^2}}$
19. The object with the longer period weighs four times as much as the other.
20. $T_2 = \sqrt{2}T_1$, where $T_1$ is the period of the smaller object
21. $k_1 = 9k_2$, where $k_1$ is the spring constant of the system with the shorter period.

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# 14.26: Section 6.2 Answers

1. $y = \frac{e^{-2t}}{2}(3\cos 2t - \sin 2t)$ ft; $\sqrt{\frac{5}{2}}e^{-2t}$ ft
2. $y = -e^{-t}\left(3\cos 3t + \frac{1}{3}\sin 3t\right)$ ft $\frac{\sqrt{82}}{3}e^{-t}$ ft
3. $y = e^{-16t}\left(\frac{1}{4} + 10t\right)$ ft
4. $y = -\frac{e^{-3t}}{4}(5\cos t + 63\sin t)$ ft
5. $0 \le c < 8$ lb-sec/ft
6. $y = \frac{1}{2}e^{-3t}\left(\cos\sqrt{91}t + \frac{11}{\sqrt{91}}\sin\sqrt{91}t\right)$ ft
7. $y = -\frac{e^{-4t}}{3}(2 + 8t)$ ft
8. $y = e^{-10t}\left(9\cos 4\sqrt{6}t + \frac{45}{2\sqrt{6}}\sin 4\sqrt{6}t\right)$ cm
9. $y = e^{-3t/2}\left(\frac{3}{2}\cos\frac{\sqrt{41}}{2}t + \frac{9}{2\sqrt{41}}\sin\frac{\sqrt{41}}{2}t\right)$ ft
10. $y = e^{-\frac{3}{2}t}\left(\frac{1}{2}\cos\frac{\sqrt{119}}{2}t - \frac{9}{2\sqrt{119}}\sin\frac{\sqrt{119}}{2}t\right)$ ft
11. $y = e^{-8t}\left(\frac{1}{4}\cos 8\sqrt{2}t - \frac{1}{4\sqrt{2}}\sin 8\sqrt{2}t\right)$ ft
12. $y = e^{-t}\left(-\frac{1}{3}\cos 3\sqrt{11}t + \frac{14}{9\sqrt{11}}\sin 3\sqrt{11}t\right)$ ft
13. $y_p = \frac{22}{61}\cos 2t + \frac{2}{61}\sin 2t$ ft
14. $y = -\frac{2}{3}(e^{-8t} - 2e^{-4t})$
15. $y = e^{-2t}\left(\frac{1}{10}\cos 4t - \frac{1}{5}\sin 4t\right)$ m
16. $y = e^{-3t}(10\cos t - 70\sin t)$ cm
17. $y_p = -\frac{2}{15}\cos 3t\,\frac{1}{15}\sin 3t$ ft
18. $y_p = \frac{11}{100}\cos 4t + \frac{27}{100}\sin 4t$ cm
19. $y_p = \frac{42}{73}\cos t + \frac{39}{73}\sin t$ ft
20. $y = -\frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t$ m
21. $y_p = \frac{1}{c\omega_0}(-\beta\cos\omega_0 t + \alpha\sin\omega_0 t)$
24. $y = e^{-ct/2m}\left(y_0\cos\omega_1 t + \frac{1}{\omega_1}(v_0 + \frac{cy_0}{2m})\sin\omega_1 t\right)$
25. $y = \frac{r_2 y_0 - v_0}{r_2 - r_1}e^{r_1 t} + \frac{v_0 - r_1 y_0}{r_2 - r_1}e^{r_2 t}$
26. $y = e^{r_1 t}(y_0 + (v_0 - r_1 y_0)t)$

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# 14.27: Section 6.3 Answers

1. $I = e^{-15t}\left(2\cos 5\sqrt{15}t - \frac{6}{\sqrt{31}}\sin 5\sqrt{31}t\right)$
2. $I = e^{-20t}(2\cos 40t - 101\sin 40t)$
3. $I = -\frac{200}{3}e^{-10t}\sin 30t$
4. $I = -10e^{-30t}(\cos 40t + 18\sin 40t)$
5. $I = -e^{-40t}(2\cos 30t - 86\sin 30t)$
6. $I_p = -\frac{1}{3}(\cos 10t + 2\sin 10t)$
7. $I_p = \frac{20}{37}(\cos 25t - 6\sin 25t)$
8. $I_p = \frac{3}{13}(8\cos 50t - \sin 50t)$
9. $I_p = \frac{20}{123}(17\sin 100t - 11\cos 100t)$
10. $I_p = -\frac{45}{52}(\cos 30t + 8\sin 30t)$
12. $\omega_0 = 1/\sqrt{LC}$ maximum amplitude $= \sqrt{U^2 + V^2}\,/R$

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# 14.28: Section 6.4 Answers

1. If $e=1$, then $Y^2=\rho(\rho-2X)$; if $e\neq 1\left(X+\frac{e\rho}{1-e^2}\right)^2+\frac{Y^2}{1-e^2}=\frac{\rho^2}{(1-e^2)^2}$ if; $e<1$ let $X_0=-\frac{e\rho}{1-e^2}$, $a=\frac{\rho}{1-e^2}$, $b=\frac{\rho}{1-e^2}$

2. Let $h=r_0^2\theta_0'$; then $\rho=\frac{h^2}{k}$, $e=\left[\left(\frac{\rho}{r_0}-1\right)^2+\left(\frac{\rho r_0'}{h}\right)^2\right]^{1/2}$. If $e=0$, then $\theta_0$ is undefined, but also irrelevant if $e\neq 0$ then $\phi=\theta_0-\alpha$, where $-\pi\leq\alpha<\pi\cos\alpha=\frac{1}{e}\left(\frac{\rho}{r_0}-1\right)$ and $\sin\alpha=\frac{pr_0'}{eh}$.

3.

a. $e=\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}$

b. $r_0=R\gamma_1$, $r_0'=0$, $\theta_0$ arbitrary, $\theta_0'=\left[\frac{2g\gamma_2}{R\gamma_1^3(\gamma_1+\gamma_2)}\right]^{1/2}$

4. $f(r)=-mh^2\left(\frac{6c}{r^4}+\frac{1}{r^3}\right)$

5. $f(r)=-\frac{mh^2(\gamma^2+1)}{r^3}$

6.

a. $\frac{d^2u}{d\theta^2}+\left(1-\frac{k}{h^2}\right)u=0$, $u(\theta_0)=\frac{1}{r_0}$, $\frac{du(\theta_0)}{d\theta}=-\frac{r_0'}{h}$

b. with $\gamma=\left|1-\frac{k}{h^2}\right|^{1/2}$

i. $r=r_0\left(\cosh\gamma(\theta-\theta_0)-\frac{r_0r_0'}{\gamma h}\sinh\gamma(\theta-\theta_0)\right)^{-1}$

ii. $r=r_0\left(1-\frac{r_0r_0'}{h}(\theta-\theta_0)\right)^{-1}$

iii. $r=r_0\left(\cos\gamma(\theta-\theta_0)-\frac{r_0r_0'}{\gamma h}\sin\gamma(\theta-\theta_0)\right)^{-1}$

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# 14.29: Section 7.1 Answers

1.

a. $R=2;\ I=(-1,3)$
b. $R=1/2;\ I=(3/2,5/2)$
c. $R=0$
d. $R=16;\ I=(-14,18)$
e. $R=\infty;\ I=(-\infty,\infty)$
f. $R=4/3;\ I=(-25/3,-17/3)$

3.

a. $R=1;\ I=(0,2)$
b. $R=\sqrt{2};\ I=(-2-\sqrt{2},-2+\sqrt{2})$
c. $R=\infty;\ I=(-\infty,\infty)$
d. $R=0$
e. $R=\sqrt{3};\ I=(-\sqrt{3},\sqrt{3})$
f. $R=1$

: $I=(0,2)$

5.

a. $R=3;\ I=(0,6)$
b. $R=1;\ I=(-1,1)$
c. $R=1/\sqrt{3};\ I=(3-1/\sqrt{3},3+1/\sqrt{3})$
d. $R=\infty;\ I=(-\infty,\infty)$
e. $R=0$
f. $R=2;\ I=(-1,3)$

11. $b_n=2(n+2)(n+1)a_{n+2}+(n+1)na_{n+1}+(n+3)a_n$

12. $b_0=2a_2-2a_0\ b_n=(n+2)(n+1)a_{n+2}+[3n(n-1)-2]a_n+3(n-1)a_{n-1},\ n\ge 1$

13. $b_n=(n+2)(n+1)a_{n+2}+2(n+1)a_{n+1}+(2n^2-5n+4)a_n$

14. $b_n=(n+2)(n+1)a_{n+2}+2(n+1)a_{n+1}+(n^2-2n+3)a_n$

15. $b_n=(n+2)(n+1)a_{n+2}+(3n^2-5n+4)a_n$

16. $b_0=-2a_2+2a_1+a_0,\ b_n=-(n+2)(n+1)a_{n+2}+(n+1)(n+2)a_{n+1}+(2n+1)a_n+a_{n-1},\ n\ge 2$

17. $b_0=8a_2+4a_1-6a_0,\ b_n=4(n+2)(n+1)a_{n+2}+4(n+1)^2a_{n+1}+(n^2+n-6)a_n-3a_{n-1},\ n\ge 1$

21. $b_0=(r+1)(r+2)a_0,\ b_n=(n+r+1)(n+r+2)a_n-(n+r-2)^2a_{n-1},\ n\ge 1.$

22. $b_0=(r-2)(r+2)a_0,\ b_n=(n+r-2)(n+r+2)a_n+(n+r+2)(n+r-3)a_{n-1},\ n\ge 14$

23. $b_0=(r-1)^2a_0,\ b_1=r^2a_1+(r+2)(r+3)a_0,\ bn=(n+r-1)^2a_n+(n+r+1)(n+r+2)a_{n-1}$
$+(n+r-1)a_{n-2},\ n\ge 2$

24. $b_0=r(r+1)a_0,\ b_1=(r+1)(r+2)a_1+3(r+1)(r+2)a_0,\ b_n=(n+r)(n+r+1)a_n$
$+3(n+r)(n+r+1)a_{n-1}+(n+r)a_{n-2},\ n\ge 2$

25. $b_0=(r+2)(r+1)a_0,\ b_1=(r+3)(r+2)a_1,\ b_n=(n+r+2)(n+r+1)a_n+2(n+r-1)(n+r-3)a_{n-2},\ n\ge 2$

26. $b_0=2(r+1)(r+3)a_0,\ b_1=2(r+2)(r+4)a_1,\ b_n=2(n+r+1)(n+r+3)a_n+(n+r-3)(n+r)a_{n-2},\ n\ge 2$

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# 14.30: Section 7.2 Answers

1. $y=a_0\sum_{m=0}^{\infty}(-1)^m(2m+1)x^{2m}+a_1\sum_{m=0}^{\infty}(-1)^m(m+1)x^{2m+1}$
2. $y=a_0\sum_{m=0}^{\infty}(-1)^{m+1}\frac{x^{2m}}{2m-1}+a_1x$
3. $y=a_0(1-10x^2+5x^4)+a_1\left(x-2x^3+\frac{1}{5}x^5\right)$
4. $y=a_0\sum_{m=0}^{\infty}(m+1)(2m+1)x^{2m}+\frac{a_1}{3}\sum_{m=0}^{\infty}(m+1)(2m+3)x^{2m+1}$
5. $y=a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\frac{4j+1}{2j+1}\right]x^{2m}+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}(4j+3)\right]\frac{x^{2m+1}}{2^m m!}$
6. $y=a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\frac{(4j+1)^2}{2j+1}\right]\frac{x^{2m}}{8^m m!}+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\frac{(4j+3)^2}{2j+3}\right]\frac{x^{2m+1}}{8^m m!}$
7. $y=a_0\sum_{m=0}^{\infty}\frac{2^m m!}{\prod_{j=0}^{m-1}(2j+1)}x^{2m}+a_1\sum_{m=0}^{\infty}\frac{\prod_{j=0}^{m-1}(2j+3)}{2^m m!}x^{2m+1}$
8. $y=a_0\left(1-14x^2+\frac{35}{3}x^4\right)+a_1\left(x-3x^3+\frac{3}{5}x^5+\frac{1}{35}x^7\right)$
9. (a) $y=a_0\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{\prod_{j=0}^{m-1}(2j+1)}+a_1\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{2^m m!}$
10. (a) $y=a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\frac{4j+3}{2j+1}\right]\frac{x^{2m}}{2^m m!}+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\frac{4j+5}{2j+3}\right]\frac{x^{2m+1}}{2^m m!}$
11. $y=2-x-x^2+\frac{1}{3}x^3+\frac{5}{12}x^4-\frac{1}{6}x^5-\frac{17}{72}x^6+\frac{13}{126}x^7+\ldots$
12. $y=1-x+3x^2-\frac{5}{2}x^3+5x^4-\frac{21}{8}x^5+3x^6-\frac{11}{16}x^7+\ldots$
13. $y=2-x-2x^2+\frac{1}{3}x^3+3x^4-\frac{5}{6}x^5-\frac{49}{5}x^6+\frac{45}{14}x^7+\ldots$
16. $y=a_0\sum_{m=0}^{\infty}\frac{(x-3)^{2m}}{(2m)!}+a_1\sum_{m=0}^{\infty}\frac{(x-3)^{2m+1}}{(2m+1)!}$
17. $y=a_0\sum_{m=0}^{\infty}\frac{(x-3)^{2m}}{2^m m!}+a_1\sum_{m=0}^{\infty}\frac{(x-3)^{2m+1}}{\prod_{j=0}^{m-1}(2j+3)}$
18. $y=a_0\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j+3)\right]\frac{(x-1)^{2m}}{m!}+a_1\sum_{m=0}^{\infty}\frac{4^m(m+1)!}{\prod_{j=0}^{m-1}(2j+3)}(x-1)^{2m+1}$
19. $y=a_0\left(1-6(x-2)^2+\frac{4}{3}(x-2)^4+\frac{8}{135}(x-2)^6\right)+a_1\left((x-2)-\frac{10}{9}(x-2)^3\right)$
20. $y=a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}(2j+1)\right]\frac{3^m}{4^m m!}(x+1)^{2m}+a_1\sum_{m=0}^{\infty}(-1)^m\frac{3^m m!}{\prod_{j=0}^{m-1}(2j+3)}(x+1)^{2m+1}$
21. $y=-1+2x+\frac{3}{8}x^2-\frac{1}{3}x^3-\frac{3}{128}x^4-\frac{1}{1024}x^6+\ldots$
22. $y=-2+3(x-3)+3(x-3)^2-2(x-3)^3-\frac{5}{4}(x-3)^4+\frac{3}{5}(x-3)^5+\frac{7}{24}(x-3)^6-\frac{4}{35}(x-3)^7+\ldots$
23. $y=-1+(x-1)+3(x-1)^2-\frac{5}{2}(x-1)^3-\frac{27}{4}(x-1)^4+\frac{21}{4}(x-1)^5+\frac{27}{2}(x-1)^6-\frac{81}{8}(x-1)^7+\ldots$
24. $y=4-6(x-3)-2(x-3)^2+(x-3)^3+\frac{3}{2}(x-3)^4-\frac{5}{4}(x-3)^5-\frac{49}{20}(x-3)^6+\frac{135}{56}(x-3)^7+\ldots$
25. $y=3-4(x-4)+15(x-4)^2-4(x-4)^3+\frac{15}{4}(x-4)^4-\frac{1}{5}(x-4)^5$
26. $y=3-3(x+1)-30(x+1)^2+\frac{20}{3}(x+1)^3+20(x+1)^4-\frac{4}{3}(x+1)^5-\frac{8}{9}(x+1)^6$
27.
    a. $y=a_0\sum_{m=0}^{\infty}(-1)^m x^{2m}+a_1\sum_{m=0}^{\infty}(-1)^m x^{2m+1}$
    b. $y=\frac{a_0+a_1x}{1+x^2}$
33. $y=a_0\sum_{m=0}^{\infty}\frac{x^{3m}}{3^m m!\prod_{j=0}^{m-1}(3j+2)}+a_1\sum_{m=0}^{\infty}\frac{x^{3m+1}}{3^m m!\prod_{j=0}^{m-1}(3j+4)}$
34. $y=a_0\sum_{m=0}^{\infty}\left(\frac{2}{3}\right)^m\left[\prod_{j=0}^{m-1}(3j+2)\right]\frac{x^{3m}}{m!}+a_1\sum_{m=0}^{\infty}\frac{6^m m!}{\prod_{j=0}^{m-1}(3j+4)}x^{3m+1}$
35. $y=a_0\sum_{m=0}^{\infty}(-1)^m\frac{3^m m!}{\prod_{j=0}^{m-1}(3j+2)}x^{3m}+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}(3j+4)\right]\frac{x^{3m+1}}{3^m m!}$

36. $y = a_0(1 - 4x^3 + 4x^6) + a_1 \sum_{m=0}^{\infty} 2^m \left[\prod_{j=0}^{m-1} \frac{3j-5}{3j+4}\right] x^{3m+1}$

37. $y = a_0 \left(1 + \frac{21}{2}x^3 + \frac{42}{5}x^6 + \frac{7}{20}x^9\right) + a_1 \left(x + 4x^4 + \frac{10}{7}x^7\right)$

39. $y = a_0 \sum_{m=0}^{\infty} (-2)^m \left[\prod_{j=0}^{m-1} \frac{5j+1}{5j+4}\right] x^{5m} + a_1 \sum_{m=0}^{\infty} \left(-\frac{2}{5}\right)^m \left[\prod_{j=0}^{m-1} (5j+2)\right] \frac{x^{5m+1}}{m!}$

40. $y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{4m}}{4^m m! \prod_{j=0}^{m-1} (4j+3)} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{x^{4m+1}}{4^m m! \prod_{j=0}^{m-1} (4j+5)}$

41. $y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{7m}}{\prod_{j=0}^{\infty} (7j+6)} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{x^{7m+1}}{7^m m!}$

42. $y = a_0 \left(1 - \frac{9}{7}x^8\right) + a_1 \left(x - \frac{7}{9}x^9\right)$

43. $y = a_0 \sum_{m=0}^{\infty} x^{6m} + a_1 \sum_{m=0}^{\infty} x^{6m+1}$

44. $y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{6m}}{\prod_{j=0}^{m-1} (6j+5)} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{x^{6m+1}}{6^m m!}$

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# 14.31: Section 7.3 Answers

1. $y = 2 - 3x - 2x^2 + \frac{7}{2}x^3 - \frac{55}{12}x^4 + \frac{59}{8}x^5 - \frac{83}{6}x^6 + \frac{9547}{336}x^7 + \ldots$
2. $y = -1 + 2x - 4x^3 + 4x^4 + 4x^5 - 12x^6 + 4x^7 + \ldots$
3. $y = 1 + x^2 - \frac{2}{3}x^3 + \frac{11}{6}x^4 - \frac{9}{5}x^5 + \frac{329}{90}x^6 - \frac{1301}{315}x^7 + \ldots$
4. $y = x - x^2 - \frac{7}{2}x^3 + \frac{15}{2}x^4 + \frac{45}{8}x^5 - \frac{261}{8}x^6 + \frac{207}{16}x^7 + \ldots$
5. $y = 4 + 3x - \frac{15}{4}x^2 + \frac{1}{4}x^3 + \frac{11}{16}x^4 - \frac{5}{16}x^5 + \frac{1}{20}x^6 + \frac{1}{120}x^7 + \ldots$
6. $y = 7 + 3x - \frac{16}{3}x^2 + \frac{13}{3}x^3 - \frac{23}{9}x^4 + \frac{10}{9}x^5 - \frac{7}{27}x^6 - \frac{1}{9}x7 + \ldots$
7. $y = 2 + 5x - \frac{7}{4}x^2 - \frac{3}{16}x^3 + \frac{37}{192}x^4 - \frac{7}{192}x^5 - \frac{1}{1920}x^6 + \frac{19}{11520}x^7 + \ldots$
8. $y = 1 - (x-1) + \frac{4}{3}(x-1)^3 - \frac{4}{3}(x-1)^4 - \frac{4}{5}(x-1)^5 + \frac{136}{45}(x-1)^6 - \frac{104}{63}(x-1)^7 + \ldots$
9. $y = 1 - (x+1) + 4(x+1)^2 - \frac{13}{3}(x+1)^3 + \frac{77}{6}(x+1)^4 - \frac{278}{15}(x+1)^5 + \frac{1942}{45}(x+1)^6 - \frac{23332}{315}(x+1)^7 + \ldots$
10. $y = 2 - (x-1) - \frac{1}{2}(x-1)^2 + \frac{5}{3}(x-1)^3 - \frac{19}{12}(x-1)^4 + \frac{7}{30}(x-1)^5 + \frac{59}{45}(x-1)^6 - \frac{1091}{630}(x-1)^7 + \ldots$
11. $y = -2 + 3(x+1) - \frac{1}{2}(x+1)^2 - \frac{2}{3}(x+1)^3 + \frac{5}{8}(x+1)^4 - \frac{11}{30}(x+1)^5 + \frac{29}{144}(x+1)^6 - \frac{101}{840}(x+1)^7 + \ldots$
12. $y = 1 - 2(x-1) - 3(x-1)^2 + 8(x-1)^3 - 4(x-1)^4 - \frac{42}{5}(x-1)^5 + 19(x-1)^6 - \frac{604}{35}(x-1)^7 + \ldots$
19. $y = 2 - 7x - 4x^2 - \frac{17}{6}x^3 - \frac{3}{4}x^4 - \frac{9}{40}x^5 + \ldots$
20. $y = 1 - 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{5}{36}(x-1)^4 - \frac{73}{1080}(x-1)^5 + \ldots$
21. $y = 2 - (x+2) - \frac{7}{2}(x+2)^2 + \frac{4}{3}(x+2)^3 - \frac{1}{24}(x+2)^4 + \frac{1}{60}(x+2)^5 + \ldots$
22. $y = 2 - 2(x+3) - (x+3)^2 + (x+3)^3 - \frac{11}{12}(x+3)^4 + \frac{67}{60}(x+3)^5 + \ldots$
23. $y = -1 + 2x + \frac{1}{3}x^3 - \frac{5}{12}x^4 + \frac{2}{5}x^5 + \ldots$
24. $y = 2 - 3(x+1) + \frac{7}{2}(x+1)^2 - 5(x+1)^3 + \frac{197}{24}(x+1)^4 - \frac{287}{20}(x+1)^5 + \ldots$
25. $y = -2 + 3(x+2) - \frac{9}{2}(x+2)^2 + \frac{11}{6}(x+2)^3 + \frac{5}{24}(x+2)^4 + \frac{7}{20}(x+2)^5 + \ldots$
26. $y = 2 - 4(x-2) - \frac{1}{2}(x-2)^2 + \frac{2}{9}(x-2)^3 + \frac{49}{432}(x-2)^4 + \frac{23}{1080}(x-2)^5 + \ldots$
27. $y = 1 + 2(x+4) - \frac{1}{6}(x+4)^2 - \frac{10}{27}(x+4)^3 + \frac{19}{648}(x+4)^4 + \frac{13}{324}(x+4)^5 + \ldots$
28. $y = -1 + 2(x+1) - \frac{1}{4}(x+1)^2 + \frac{1}{2}(x+1)^3 - \frac{65}{96}(x+1)^4 + \frac{67}{80}(x+1)^5 + \ldots$
31.
    a. $y = \frac{c_1}{1+x} + \frac{c_2}{1+2x}$
    b. $y = \frac{c_1}{1-2x} + \frac{c_2}{1-3x}$
    c. $y = \frac{c_1}{1-2x} + \frac{c_2 x}{(1-2x)^2}$
    d. $y = \frac{c_1}{2+x} + \frac{c_2 x}{(2+x)^2}$
    e. $y = \frac{c_1}{2+x} + \frac{c_2}{2+3x}$
32. $y = 1 - 2x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{17}{24}x^4 - \frac{11}{20}x^5 + \ldots$
33. $y = 1 - 2x - \frac{5}{2}x^2 + \frac{2}{3}x^3 - \frac{3}{8}x^4 + \frac{1}{3}x^5 + \ldots$
34. $y = 6 - 2x + 9x^2 + \frac{2}{3}x^3 - \frac{23}{4}x^4 - \frac{3}{10}x^5 + \ldots$
35. $y = 2 - 5x + 2x^2 - \frac{10}{3}x^3 + \frac{3}{2}x^4 - \frac{25}{12}x^5 + \ldots$
36. $y = 3 + 6x - 3x^2 + x^3 - 2x^4 - \frac{17}{20}x^5 + \ldots$
37. $y = 3 - 2x - 3x^2 + \frac{3}{2}x^3 + \frac{3}{2}x^4 - \frac{49}{80}x^5 + \ldots$
38. $y = -2 + 3x + \frac{4}{3}x^2 - x^3 - \frac{19}{54}x^4 + \frac{13}{60}x^5 + \ldots$

39. $y_1 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m!} = e^{-x^2}$, $y_2 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{m!} = xe^{-x^2}$

40. $y = -2 + 3x + x^2 - \frac{1}{6}x^3 - \frac{3}{4}x^4 + \frac{31}{120}x^5 + \ldots$

41. $y = 2 + 3x - \frac{7}{2}x^2 - \frac{5}{6}x^3 + \frac{41}{24}x^4 + \frac{41}{120}x^5 + \ldots$

42. $y = -3 + 5x - 5x^2 + \frac{23}{6}x^3 - \frac{23}{12}x^4 + \frac{11}{30}x^5 + \ldots$

43. $y = -2 + 3(x-1) + \frac{3}{2}(x-1)^2 - \frac{17}{12}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{8}(x-1)^5 + \ldots$

44. $y = 2 - 3(x+2) + \frac{1}{2}(x+2)^2 - \frac{1}{3}(x+2)^3 + \frac{31}{24}(x+2)^4 - \frac{53}{120}(x+2)^5 + \ldots$

45. $y = 1 - 2x + \frac{3}{2}x^2 - \frac{11}{6}x^3 + \frac{15}{8}x^4 - \frac{71}{60}x^5 + \ldots$

46. $y = 2 - (x+2) - \frac{7}{2}(x+2)^2 - \frac{43}{6}(x+2)^3 - \frac{203}{24}(x+2)^4 - \frac{167}{30}(x+2)^5 + \ldots$

47. $y = 2 - x - x^2 + \frac{7}{6}x^3 - x^4 + \frac{89}{120}x^5 + \ldots$

48. $y = 1 + \frac{3}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{8}(x-1)^5 + \ldots$

49. $y = 1 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{6}(x-3)^3 + \frac{1}{4}(x-3)^4 - \frac{1}{6}(x-3)^5 + \ldots$

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# 14.32: Section 7.4 Answers

1. $y = x_1 x^{-4} + c_2 x^{-2}$
2. $y = c_1 x + c_2 x^7$
3. $y = x(c_1 + c_2 \ln x)$
4. $y = x^{-2}(c_1 + c_2 \ln x)$
5. $y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$
6. $y = x^2[c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)]$
7. $y = c_1 x + \frac{c_2}{x^3}$
8. $y = c_1 x^{2/3} + c_2 x^{3/4}$
9. $y = x^{-1/2}(c_1 + c_2 \ln x)$
10. $y = c_1 x + c_2 x^{1/3}$
11. $y = c_1 x^2 + c_2 x^{1/2}$
12. $y = \frac{1}{x}[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x]$
13. $y = x^{-1/3}(c_1 + c_2 \ln x)$
14. $y = x[c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)]$
15. $y = c_1 x^3 + \frac{x_2}{x^2}$
16. $y = \frac{c_1}{x} + c_2 x^{1/2}$
17. $y = x^2(c_1 + c_2 \ln x)$
18. $y = \frac{1}{x^2}\left[c_1 \cos\left(\frac{1}{\sqrt{2}} \ln x\right) + c_2 \sin\left(\frac{1}{\sqrt{2}} \ln x\right)\right]$

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# 14.33: Section 7.5 Answers

1. $y_1 = x^{1/2}\left(1 - \frac{1}{5}x - \frac{2}{35}x^2 + \frac{31}{315}x^3 + \ldots\right)$; $y_2 = x^{-1}\left(1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right)$
2. $y_1 = x^{1/3}\left(1 - \frac{2}{3}x + \frac{8}{9}x^2 - \frac{40}{81}x^3 + \ldots\right)$; $y_2 = 1 - x + \frac{6}{5}x^2 - \frac{4}{5}x^3 + \ldots$
3. $y_1 = x^{1/3}\left(1 - \frac{4}{7}x - \frac{7}{45}x^2 + \frac{970}{2457}x^3 + \ldots\right)$; $y_2 = x^{-1}\left(1 - x^2 + \frac{2}{3}x^3 + \ldots\right)$
4. $y_1 = x^{1/4}\left(1 - \frac{1}{2}x - \frac{19}{104}x^2 + \frac{1571}{10608}x^3 + \ldots\right)$; $y_2 = x^{-1}\left(1 + 2x - \frac{11}{6}x^2 - \frac{1}{7}x^3 + \ldots\right)$
5. $y_1 = x^{1/3}\left(1 - x + \frac{28}{31}x^2 - \frac{1111}{1333}x^3 + \ldots\right)$; $y_2 = x^{-1/4}\left(1 - x + \frac{7}{8}x^2 - \frac{19}{24}x^3 + \ldots\right)$
6. $y_1 = x^{1/5}\left(1 - \frac{6}{25}x - \frac{1217}{625}x^2 + \frac{41972}{46875}x^3 + \ldots\right)$; $y_2 = x - \frac{1}{4}x^2 - \frac{35}{18}x^3 + \frac{11}{12}x^4 + \ldots$
7. $y_1 = x^{3/2}\left(1 - x + \frac{11}{26}x^2 - \frac{109}{1326}x^3 + \ldots\right)$; $y_2 = x^{1/4}\left(1 + 4x - \frac{131}{24}x^2 + \frac{39}{14}x^3 + \ldots\right)$
8. $y_1 = x^{1/3}\left(1 - \frac{1}{3}x + \frac{2}{15}x^2 - \frac{5}{63}x^3 + \ldots\right)$; $y_2 = x^{-1/6}\left(1 - \frac{1}{12}x^2 + \frac{1}{18}x^3 + \ldots\right)$
9. $y_1 = 1 - \frac{1}{14}x^2 + \frac{1}{105}x^3 + \ldots$; $y_2 = x^{-1/3}\left(1 - \frac{1}{18}x - \frac{71}{405}x^2 + \frac{719}{34992}x^3 + \ldots\right)$
10. $y_1 = x^{1/5}\left(1 + \frac{3}{17}x - \frac{7}{153}x^2 - \frac{547}{5661}x^3 + \ldots\right)$; $y_2 = x^{-1/2}\left(1 + x + \frac{14}{13}x^2 - \frac{556}{897}x^3 + \ldots\right)$
14. $y_1 = x^{1/2}\sum_{n=0}^{\infty}\frac{(-2)^n}{\prod_{j=1}^{n}(2j+3)}x^n$; $y_2 = x^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n$
15. $y_1 = x^{1/3}\sum_{n=0}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(3j+1)}{9^n n!}x^n$; $x^{-1}$
16. $y_1 = x^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{j=1}^{n}(3j+4)}x^n$; $y_2 = \frac{1}{x^2}\sum_{n=0}^{\infty}\frac{2^n}{n!\prod_{j=1}^{n}(2j-5)}x^n$
17. $y_1 = x\sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{j=1}^{n}(3j+4)}x^n$; $y_2 = x^{-1/3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n n!}x^n$
18. $y_1 = x\sum_{n=0}^{\infty}\frac{2^n}{n!\prod_{j=1}^{n}(2j+1)}x^n$; $y_2 = x^{1/2}\sum_{n=0}^{\infty}\frac{2^n}{n!\prod_{j=1}^{n}(2j-1)}x^n$
19. $y_1 = x^{1/3}\sum_{n=0}^{\infty}\frac{1}{n!\prod_{j=1}^{n}(3j+2)}x^n$; $y_2 = x^{-1/3}\sum_{n=0}^{\infty}\frac{1}{n!\prod_{j=1}^{n}(3j-2)}x^n$
20. $y_1 = x\left(1 + \frac{2}{7}x + \frac{1}{70}x^2\right)$; $y_2 = x^{-1/3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n n!}\left(\prod_{j=1}^{n}\frac{3j-13}{3j-4}\right)x^n$
21. $y_1 = x^{1/2}\sum_{n=0}^{\infty}(-1)^n\left(\prod_{j=1}^{n}\frac{2j+1}{6j+1}\right)x^n$; $y_2 = x^{1/3}\sum_{n=0}^{\infty}\frac{(-1)^n}{9^n n!}\left(\prod_{j=1}^{n}(3j+1)\right)x^n$
22. $y_1 = x\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)!}{2\prod_{j=1}^{n}(4j+3)}x^n$; $y_2 = x^{1/4}\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n n!}\prod_{j=1}^{n}(4j+5)x^n$
23. $y_1 = x^{-1/2}\sum_{n=0}^{\infty}\frac{(1)^n}{n!\prod_{j=1}^{n}(2j+1)}x^n$; $y_2 = x^{-1}\sum_{n=0}^{\infty}\frac{(1)^n}{n!\prod_{j=1}^{n}(2j-1)}x^n$
24. $y_1 = x^{1/3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{2}{9}\right)^n\left(\prod_{j=1}^{n}(6j+5)\right)x^n$; $y_2 = x^{-1}\sum_{n=0}^{\infty}(-1)^n 2^n\left(\prod_{j=1}^{n}\frac{2j-1}{3j-4}\right)x^n$
25. $y_1 = 4x^{1/3}\sum_{n=0}^{\infty}\frac{1}{6^n n!(3n+4)}x^n$; $x^{-1}$
28. $y_1 = x^{1/2}\left(1 - \frac{9}{40}x + \frac{5}{128}x^2 - \frac{245}{39936}x^3 + \ldots\right)$; $y_2 = x^{1/4}\left(1 - \frac{25}{96}x + \frac{675}{14336}x^2 - \frac{38025}{5046272}x^3 + \ldots\right)$
29. $y_1 = x^{1/3}\left(1 + \frac{32}{117}x - \frac{28}{1053}x^2 + \frac{4480}{540189}x^3 + \ldots\right)$; $y_2 = x^{-3}\left(1 + \frac{32}{7}x + \frac{48}{7}x^2\right)$
30. $y_1 = x^{1/2}\left(1 - \frac{5}{8}x + \frac{55}{96}x^2 - \frac{935}{1536}x^3 + \ldots\right)$; $y_2 = x^{-1/2}\left(1 + \frac{1}{4}x - \frac{5}{32}x^2 - \frac{55}{384}x^3 + \ldots\right)$
31. $y_1 = x^{1/2}\left(1 - \frac{3}{4}x + \frac{5}{96}x^2 + \frac{5}{4224}x^3 + \ldots\right)$; $y_2 = x^{-2}(1 + 8x + 60x^2 - 160x^3 + \ldots)$
32. $y_1 = x^{-1/3}\left(1 - \frac{10}{63}x + \frac{200}{7371}x^2 - \frac{17600}{3781323}x^3 + \ldots\right)$; $y_2 = x^{-1/2}\left(1 - \frac{3}{20}x + \frac{9}{352}x^2 - \frac{105}{23936}x^3 + \ldots\right)$
33. $y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{8^m m!}\left(\prod_{j=1}^{m}\frac{4j-3}{8j+1}\right)x^{2m}$; $y_2 = x^{1/4}\sum_{m=0}^{\infty}\frac{(-1)^m}{16^m m!}\left(\prod_{j=1}^{m}\frac{8j-7}{8j-1}\right)x^{2m}$
34. $y_1 = x^{1/2}\sum_{m=0}^{\infty}\left(\prod_{j=1}^{m}\frac{8j-3}{8j+1}\right)x^{2m}$; $y_2 = x^{1/4}\sum_{m=0}^{\infty}\frac{1}{2^m m!}\left(\prod_{j=1}^{m}(2j-1)\right)x^{2m}$
35. $y_1 = x^4\sum_{m=0}^{\infty}(-1)^m(m+1)x^{2m}$; $y_2 = -x\sum_{m=0}^{\infty}(-1)^m(2m-1)x^{2m}$

36. $y_1 = x^{1/3}\sum_{m=0}^{\infty}\frac{(-1)^m}{18^m m!}\left(\prod_{j=1}^{m}(6j-17)\right)x^{2m};\quad y_2 = 1+\frac{4}{5}x^2+\frac{8}{55}x^4$

37. $y_1 = x^{1/4}\sum_{m=0}^{\infty}\left(\prod_{j=1}^{m}\frac{8j+1}{8j+5}\right)x^{2m};\quad y_2 = x^{-1}\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{m}(2j-1)}{2^m m!}x^{2m}$

38. $y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{1}{8^m m!}\left(\prod_{j=1}^{m}(4j-1)\right)x^{2m};\quad y_2 = x^{1/3}\sum_{m=0}^{\infty}2^m\left(\prod_{j=1}^{m}\frac{3j-1}{12j-1}\right)x^{2m}$

39. $y_1 = x^{7/2}\sum_{m=0}^{\infty}(-1)^m\frac{\prod_{j=1}^{m}(4j+5)}{8^m m!}x^{2m};\quad y_2 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m}\left(\prod_{j=1}^{m}\frac{4j-1}{2j-3}\right)x^{2m}$

40. $y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m}\left(\prod_{j=1}^{m}\frac{4j-1}{2j+1}\right)x^{2m};\quad y_2 = x^{-1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{8^m m!}\left(\prod_{j=1}^{m}(4j-3)\right)x^{2m}$

41. $y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\prod_{j=1}^{m}(2j+1)\right)x^{2m};\quad y_2 = \frac{1}{x^2}\sum_{m=0}^{\infty}(-2)^m\left(\prod_{j=1}^{m}\frac{4j-3}{4j-5}\right)x^{2m}$

42. $y_1 = x^{1/3}\sum_{m=0}^{\infty}(-1)^m\left(\prod_{j=1}^{m}\frac{3j-4}{3j+2}\right)x^{2m};\quad y_2 = x^{-1}(1+x^2)$

43. $y_1 = \sum_{m=0}^{\infty}(-1)^m\frac{2^m(m+1)!}{\prod_{j=1}^{m}(2j+3)}x^{2m};\quad y_2 = \frac{1}{x^3}\sum_{m=0}^{\infty}(-1)^m\frac{\prod_{j=1}^{m}(2j-1)}{2^m m!}x^{2m}$

44. $y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{8^m m!}\left(\prod_{j=1}^{m}\frac{(4j-3)^2}{4j+3}\right)x^{2m};\quad y_2 = x^{-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!}\left(\prod_{j=1}^{m}\frac{(2j-3)^2}{4j-3}\right)x^{2m}$

45. $y_1 = x\sum_{m=0}^{\infty}(-2)^m\left(\prod_{j=1}^{m}\frac{2j+1}{4j+5}\right)x^{2m};\quad y_2 = x^{-3/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m m!}\left(\prod_{j=1}^{m}(4j-3)\right)x^{2m}$

46. $y_1 = x^{1/3}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m\prod_{j=1}^{m}(3j+1)}x^{2m};\quad y_2 = x^{-1/3}\sum_{m=0}^{\infty}\frac{(-1)^m}{6^m m!}x^{2m}$

47. $y_1 = x^{1/2}\left(1-\frac{6}{13}x^2+\frac{36}{325}x^4-\frac{216}{12025}x^6+\ldots\right);\quad y_2 = x^{1/3}\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4-\frac{1}{48}x^6+\ldots\right)$

48. $y_1 = x^{1/4}\left(1-\frac{13}{64}x^2+\frac{273}{8192}x^4-\frac{2639}{524288}x^6+\ldots\right);\quad y_2 = x^{-1}\left(1-\frac{1}{3}x^2+\frac{2}{33}x^4-\frac{2}{209}x^6+\ldots\right)$

49. $y_1 = x^{1/3}\left(1-\frac{3}{4}x^2+\frac{9}{14}x^4-\frac{81}{140}x^6+\ldots\right);\quad y_2 = x^{-1/3}\left(1-\frac{2}{3}x^2+\frac{5}{9}x^4-\frac{40}{81}x^6+\ldots\right)$

50. $y_1 = x^{1/2}\left(1-\frac{3}{2}x^2+\frac{15}{8}x^4-\frac{35}{16}x^6+\ldots\right);\quad y_2 = x^{-1/2}\left(1-2x^2+\frac{8}{3}x^4-\frac{16}{5}x^6+\ldots\right)$

51. $y_1 = x^{1/4}\left(1-x^2+\frac{3}{2}x^4-\frac{5}{2}x^6+\ldots\right);\quad y_2 = x^{-1/2}\left(1-\frac{2}{5}x^2+\frac{36}{65}x^4-\frac{408}{455}x^6+\ldots\right)$

53. (a) $y_1 = x^v\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m m!\prod_{j=1}^{m}(j+v)}x^{2m};\quad y_2 = x^{-v}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m m!\prod_{j=1}^{m}(j-v)}x^{2m}\quad y_1 = \frac{\sin x}{\sqrt{x}};\quad y_2 = \frac{\cos x}{\sqrt{x}}$

61. $y_1 = \frac{x^{1/2}}{1+x};\quad y_2 = \frac{x}{1+x}$

62. $y_1 = \frac{x^{1/3}}{1+2x^2};\quad y_2 = \frac{x^{1/2}}{1+2x^2}$

63. $y_1 = \frac{x^{1/4}}{1-3x};\quad y_2 = \frac{x^2}{1-3x}$

64. $y_1 = \frac{x^{1/3}}{5+x};\quad y_2 = \frac{x^{-1/3}}{5+x}$

65. $y_1 = \frac{x^{1/4}}{2-x^2};\quad y_2 = \frac{x^{-1/2}}{2-x^2}$

66. $y_1 = \frac{x^{1/2}}{1+3x+x^2};\quad y_2 = \frac{x^{3/2}}{1+3x+x^2}$

67. $y_1 = \frac{x}{(1+x)^2};\quad y_2 = \frac{x^{1/3}}{(1+x)^2}$

68. $y_1 = \frac{x}{3+2x+x^2};\quad y_2 = \frac{x^{1/4}}{3+2x+x^2}$

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# 14.34: Section 7.6 Answers

1. $y_1 = x\left(1 - x + \frac{3}{4}x^2 - \frac{13}{36}x^3 + \ldots\right)$; $y_2 = y_1 \ln x + x^2\left(1 - x + \frac{65}{108}x^2 + \ldots\right)$
2. $y_1 = x^{-1}\left(1 - 2x + \frac{9}{2}x^2 - \frac{20}{3}x^3 + \ldots\right)$; $y_2 = y_1 \ln x + 1 - \frac{15}{4}x + \frac{133}{18}x^2 + \ldots$
3. $y_1 = 1 + x - x^2 + \frac{1}{3}x^3 + \ldots$; $y_2 = y_1 \ln x - x\left(3 - \frac{1}{2}x - \frac{31}{18}x^2 + \ldots\right)$
4. $y_1 = x^{1/2}\left(1 - 2x + \frac{5}{2}x^2 - 2x^3 + \ldots\right)$; $y_2 = y_1 \ln x + x^{3/2}\left(1 - \frac{9}{4}x + \frac{17}{6}x^2 + \ldots\right)$
5. $y_1 = x\left(1 - 4x + \frac{19}{2}x^2 - \frac{49}{3}x^3 + \ldots\right)$; $y_2 = y_1 \ln x + x^2\left(3 - \frac{43}{4}x + \frac{208}{9}x^2 + \ldots\right)$
6. $y_1 = x^{-1/3}\left(1 - x + \frac{5}{6}x^2 - \frac{1}{2}x^3 + \ldots\right)$; $y_2 = y_1 \ln x + x^{2/3}\left(1 - \frac{11}{12}x + \frac{25}{36}x^2 + \ldots\right)$
7. $y_1 = 1 - 2x + \frac{7}{4}x^2 - \frac{7}{9}x^3 + \ldots$; $y_2 = y_1 \ln x + x\left(3 - \frac{15}{4}x + \frac{239}{108}x^2 + \ldots\right)$
8. $y_1 = x^{-2}\left(1 - 2x + \frac{5}{2}x^2 - 3x^3 + \ldots\right)$; $y_2 = y_1 \ln x + \frac{3}{4} - \frac{13}{6}x + \ldots$
9. $y_1 = x^{-1/2}\left(1 - x + \frac{1}{4}x^2 + \frac{1}{18}x^3 + \ldots\right)$; $y_2 = y_1 \ln x + x^{1/2}\left(\frac{3}{2} - \frac{13}{16}x + \frac{1}{54}x^2 + \ldots\right)$
10. $y_1 = x^{-1/4}\left(1 - \frac{1}{4}x - \frac{7}{32}x^2 + \frac{23}{384}x^3 + \ldots\right)$; $y_2 = y_1 \ln x + x^{3/4}\left(\frac{1}{4} + \frac{5}{64}x - \frac{157}{2304}x^2 + \ldots\right)$
11. $y_1 = x^{-1/3}\left(1 - x + \frac{7}{6}x^2 - \frac{23}{18}x^3 + \ldots\right)$; $y_2 = y_1 \ln x - x^{5/3}\left(\frac{1}{12} - \frac{13}{108}x \ldots\right)$
12. $y_1 = x^{1/2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}x^n$; $y_2 = y_1 \ln x - 2x^{1/2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\sum_{j=1}^{n} \frac{1}{j}\right)x^n$
13. $y_1 = x^{1/6}\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{\prod_{j=1}^{n}(3j+1)}{n!}x^n$; $y_2 = y_1 \ln x - x^{1/6}\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{\prod_{j=1}^{n}(3j+1)}{n!}\left(\sum_{j=1}^{n} \frac{1}{j(3j+1)}\right)x^n$
14. $y_1 = x^2\sum_{n=0}^{\infty}(-1)^n(n+1)^2x^n$; $y_2 = y_1 \ln x - 2x^2\sum_{n=1}^{\infty}(-1)^n n(n+1)x^n$
15. $y_1 = x^3\sum_{n=0}^{\infty} 2^n(n+1)x^n$; $y_2 = y_1 \ln x - x^3\sum_{n=1}^{\infty} 2^n n x^n$
16. $y_1 = x^{1/5}\sum_{n=0}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(5j+1)}{125^n(n!)^2}x^n$; $y_2 = y_1 \ln x - x^{1/5}\sum_{n=1}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(5j+1)}{125^n(n!)^2}\left(\sum_{j=1}^{n} \frac{5j+2}{j(5j+1)}\right)x^n$
17. $y_1 = x^{1/2}\sum_{n=0}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(2j-3)}{4^n n!}x^n$; $y_2 = y_1 \ln x + 3x^{1/2}\sum_{n=1}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(2j-3)}{4^n n!}\left(\sum_{j=1}^{n} \frac{1}{j(2j-3)}\right)x^n$
18. $y_1 = x^{1/3}\sum_{n=0}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(6j-7)^2}{81^n(n!)^2}x^n$; $y_2 = y_1 \ln x + 14x^{1/3}\sum_{n=1}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(6j-7)^2}{81^n(n!)^2}\left(\sum_{j=1}^{n} \frac{1}{j(6j-7)}\right)x^n$
19. $y_1 = x^2\sum_{n=0}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(2j+5)}{(n!)^2}x^n$; $y_2 = y_1 \ln x - 2x^2\sum_{n=1}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(2j+5)}{(n!)^2}\left(\sum_{j=1}^{n} \frac{j+5}{j(2j+5)}\right)x^n$
20. $y_1 = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(2)^n\prod_{j=1}^{n}(2j-1)}{n!}x^n$; $y_2 = y_1 \ln x + \frac{1}{x}\sum_{n=1}^{\infty} \frac{(2)^n\prod_{j=1}^{n}(2j-1)}{n!}\left(\sum_{j=1}^{n} \frac{1}{j(2j-1)}\right)x^n$
21. $y_1 = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(2j-5)}{n!}x^n$; $y_2 = y_1 \ln x + \frac{5}{x}\sum_{n=1}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(2j-5)}{n!}\left(\sum_{j=1}^{n} \frac{1}{j(2j-5)}\right)x^n$
22. $y_1 = x^2\sum_{n=0}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(2j+3)}{2^n n!}x^n$; $y_2 = y_1 \ln x - 3x^2\sum_{n=1}^{\infty} \frac{(-1)^n\prod_{j=1}^{n}(2j+3)}{2^n n!}\left(\sum_{j=1}^{n} \frac{1}{j(2j+3)}\right)x^n$
23. $y_1 = x^{-2}\left(1 + 3x + \frac{3}{2}^2 - \frac{1}{2}x^3 + \ldots\right)$; $y_2 = y_1 \ln x - 5x^{-1}\left(1 + \frac{5}{4}x - \frac{1}{4}x^2 + \ldots\right)$
24. $y_1 = x^3(1 + 20x + 180x^2 + 1120x^3 + \ldots)$; $y_2 = y_1 \ln x - x^4\left(26 + 324x + 69683x^2 + \ldots\right)$
25. $y_1 = x\left(1 - 5x + \frac{85}{4}x^2 - \frac{3145}{36}x^3 + \ldots\right)$; $y_2 = y_1 \ln x + x^2\left(2 - \frac{39}{4}x + \frac{4499}{108}x^2 + \ldots\right)$
26. $y_1 = 1 - x + \frac{3}{4}x^2 - \frac{7}{12}x^3 + \ldots$; $y_2 = y_1 \ln x + x\left(1 - \frac{3}{4}x + \frac{5}{9}x^2 + \ldots\right)$
27. $y_1 = x^{-3}(1 + 16x + 36x^2 + 16x^3 + \ldots)$; $y_2 = y_1 \ln x - x^{-2}\left(40 + 150x + \frac{280}{3}x^2 + \ldots\right)$
28. $y_1 = x\sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!}x^{2m}$; $y_2 = y_1 \ln x - \frac{x}{2}\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m m!}\left(\sum_{j=1}^{\infty} \frac{1}{j}\right)x^{2m}$
29. $y_1 = x^2\sum_{m=0}^{\infty}(-1)^m(m+1)x^{2m}$; $y_2 = y_1 \ln x - \frac{x^2}{2}\sum_{m=1}^{\infty}(-1)^m m x^{2m}$
30. $y_1 = x^{1/2}\sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!}x^{2m}$; $y_2 = y_1 \ln x - \frac{x^{1/2}}{2}\sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m!}\left(\sum_{j=1}^{m} \frac{1}{j}\right)x^{2m}$

31. $y_1 = x\sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{2^m m!}x^{2m}$; $y_2 = y_1\ln x + \frac{x}{2}\sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{2^m m!}\left(\sum_{j=1}^{m}\frac{1}{j(2j-1)}\right)x^{2m}$

32. $y_1 = x^{1/2}\sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-1)}{8^m m!}x^{2m}$; $y_2 = y_1\ln x + \frac{x^{1/2}}{2}\sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-1)}{8^m m!}\left(\sum_{j=1}^{m}\frac{1}{j(4j-1)}\right)x^{2m}$

33. $y_1 = x\sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+1)}{2^m m!}x^{2m}$; $y_2 = y_1\ln x - \frac{x}{2}\sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+1)}{2^m m!}\left(\sum_{j=1}^{m}\frac{1}{j(2j+1)}\right)x^{2m}$

34. $y_1 = x^{-1/4}\sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(8j-13)}{(32)^m m!}x^{2m}$; $y_2 = y_1\ln x + \frac{13}{2}x^{-1/4}\sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(8j-13)}{(32)^m m!}\left(\sum_{j=1}^{m}\frac{1}{j(8j-13)}\right)x^{2m}$

35. $y_1 = x^{1/3}\sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(3j-1)}{9^m m!}x^{2m}$; $y_2 = y_1\ln x + \frac{x^{1/3}}{2}\sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(3j-1)}{9^m m!}\left(\sum_{j=1}^{m}\frac{1}{j(3j-1)}\right)x^{2m}$

36. $y_1 = x^{1/2}\sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-3)(4j-1)}{4^m (m!)^2}x^{2m}$; $y_2 = y_1\ln x + x^{1/2}\sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-3)(4j-1)}{4^m (m!)^2}\left(\sum_{j=1}^{m}\frac{8j-3}{j(4j-3)(4j-1)}\right)x^{2m}$

37. $y_1 = x^{5/3}\sum_{m=0}^{\infty} \frac{(-1)^m}{3^m m!}x^{2m}$; $y_2 = y_1\ln x - \frac{x^{5/3}}{2}\sum_{m=1}^{\infty} \frac{(-1)^m}{3^m m!}\left(\sum_{j=1}^{m}\frac{1}{j}\right)x^{2m}$

38. $y_1 = \frac{1}{x}\sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-7)}{2^m m!}x^{2m}$; $y_2 = y_1\ln x + \frac{7}{2x}\sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-7)}{2^m m!}\left(\sum_{j=1}^{m}\frac{1}{j(4j-7)}\right)x^{2m}$

39. $y_1 = x^{-1}\left(1 - \frac{3}{2}x^2 + \frac{15}{8}x^4 - \frac{35}{16}x^6 + \ldots\right)$; $y_2 = y_1\ln x + x\left(\frac{1}{4} - \frac{13}{32}x^2 + \frac{101}{192}x^4 + \ldots\right)$

40. $y_1 = x\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots\right)$; $y_2 = y_1\ln x + x^3\left(\frac{1}{4} - \frac{3}{32}x^2 + \frac{11}{576}x^4 + \ldots\right)$

41. $y_1 = x^{-2}\left(1 - \frac{3}{4}x^2 - \frac{9}{64}x^4 - \frac{25}{256}x^6 + \ldots\right)$; $y_2 = y_1\ln x + \frac{1}{2} - \frac{21}{128}x^2 - \frac{215}{1536}x^4 + \ldots$

42. $y_1 = x^{-3}\left(1 - \frac{17}{8}x^2 + \frac{85}{256}x^4 - \frac{85}{18432}x^6 + \ldots\right)$; $y_2 = y_1\ln x + x^{-1}\left(\frac{25}{8} - \frac{471}{512}x^2 + \frac{1583}{110592}x^4 + \ldots\right)$

43. $y_1 = x^{-1}\left(1 - \frac{3}{4}x^2 + \frac{45}{64}x^4 - \frac{175}{256}x^6 + \ldots\right)$; $y_2 = y_1\ln x - x\left(\frac{1}{4} - \frac{33}{128}x^2 + \frac{395}{1536}x^4 + \ldots\right)$

44. $y_1 = \frac{1}{x}$; $y_2 = y_1\ln x - 6 + 6x - \frac{8}{3}x^2$

45. $y_1 = 1 - x$; $y_2 = y_1\ln x + 4x$

46. $y_1 = \frac{(x-1)^2}{x}$; $y_2 = y_1\ln x + 3 - 3x + 2\sum_{n=2}^{\infty}\frac{1}{n(n^2-1)}x^n$

47. $y_1 = x^{1/2}(x+1)^2$; $y_2 = y_1\ln x - x^{3/2}\left(3 + 3x + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n^2-1)}x^n\right)$

48. $y_1 = x^2(1-x)^3$; $y_2 = y_1\ln x + x^3\left(4 - 7x + \frac{11}{3}x^2 - 6\sum_{n=3}^{\infty}\frac{1}{n(n-2)(n^2-1)}x^n\right)$

49. $y_1 = x - 4x^3 + x^5$; $y_2 = y_1\ln x + 6x^3 - 3x^5$

50. $y_1 = x^{1/3}\left(1 - \frac{1}{6}x^2\right)$; $y_2 = y_1\ln x + x^{7/3}\left(\frac{1}{4} - \frac{1}{12}\sum_{m=1}^{\infty}\frac{1}{6^m m(m+1)(m+1)!}x^{2m}\right)$

51. $y_1 = (1+x^2)^2$; $y_2 = y_1\ln x - \frac{3}{2}x^2 - \frac{3}{2}x^4 + \sum_{m=3}^{\infty}\frac{(-1)^m}{m(m-1)(m-2)}x^{2m}$

52. $y_1 = x^{-1/2}\left(1 - \frac{1}{2}x^2 + \frac{1}{32}x^4\right)$; $y_2 = y_1\ln x + x^{3/2}\left(\frac{5}{8} - \frac{9}{128}x^2 + \sum_{m=2}^{\infty}\frac{1}{4^{m+1}(m-1)m(m+1)(m+1)!}x^{2m}\right)$

56. $y_1 = \sum_{m=0}^{\infty}\frac{(-1)^m}{4^m (m!)^2}x^{2m}$; $y_2 = y_1\ln x - \sum_{m=1}^{\infty}\frac{(-1)^m}{4^m (m!)^2}\left(\sum_{j=1}^{m}\frac{1}{j}\right)x^{2m}$

58. $\frac{x^{1/2}}{1+x}$; $\frac{x^{1/2}\ln x}{1+x}$

59. $\frac{x^{1/3}}{3+x}$; $\frac{x^{1/3}\ln x}{3+x}$

60. $\frac{x}{2-x^2}$; $\frac{x\ln x}{2-x^2}$

61. $\frac{x^{1/3}}{1+x^2}$; $\frac{x^{1/4}\ln x}{1+x^2}$

62. $\frac{x}{4+3x}$; $\frac{x\ln x}{4+3x}$

63. $\frac{x^{1/2}}{1+3x+x^2}$; $\frac{x^{1/2}\ln x}{1+3x+x^2}$

64. $\frac{x}{(1-x)^2}$; $\frac{x\ln x}{(1-x)^2}$

65. $\frac{x^{1/3}}{1+x+x^2}$; $\frac{x^{1/3}\ln x}{1+x+x^2}$

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# 14.35: Section 7.7 Answers

1. $y_1 = 2x^3 \sum_{n=0}^{\infty} \frac{(-4)^n}{n!(n+2)!} x^n; \quad y_2 = x + 4x^2 - 8\left(y_1 \ln x - 4\sum_{n=1}^{\infty} \frac{(-4)^n}{n!(n+2)!}\left(\sum_{j=1}^{n} \frac{j+1}{j(j+2)}\right) x^n\right)$
2. $y_1 = x\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^n; \quad y_2 = 1 - y_1 \ln x + x\sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+1)!}\left(\sum_{j=1}^{n} \frac{2j+1}{j(j+1)}\right) x^n$
3. $y_1 = x^{1/2}; \quad y_2 = x^{-1/2} + y_1 \ln x + x^{1/2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$
4. $y_1 = x\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = xe^{-x}; \quad y_2 = 1 - y_1 \ln x + x\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left(\sum j = 1^{n} \frac{1}{j}\right) x^n$
5. $y_1 = x^{1/2}\sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n \frac{\prod_{j=1}^{n}(2j+1)}{n!} x^n; \quad y_2 = x^{-1/2} - \frac{3}{4}\left(y_1 \ln x - x^{1/2}\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^n \frac{\prod_{j=1}^{n}(2j+1)}{n!}\left(\sum_{j=1}^{n} \frac{1}{j(2j+1)}\right) x^n\right)$
6. $y_1 = x\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = xe^{-x}; \quad y_2 = x^{-2}\left(1 + \frac{1}{2}x + \frac{1}{2}x^2\right) - \frac{1}{2}\left(y_1 \ln x - x\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left(\sum_{j=1}^{n} \frac{1}{j}\right) x^n\right)$
7. $y_1 = 6x^{3/2}\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!(n+3)!} x^n;$

   $y_2 = x^{-3/2}\left(1 + \frac{1}{8}x + \frac{1}{64}x^2\right) - \frac{1}{768}\left(y_1 \ln x - 6x^{3/2}\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n!(n+3)!}\left(\sum_{j=1}^{n} \frac{2j+3}{j(j+3)}\right) x^n\right)$
8. $y_1 = \frac{120}{x^2}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+5)!} x^n; \quad y_2 = x^{-7}\left(1 + \frac{1}{4}x + \frac{1}{24}x^2 + \frac{1}{144}x^3 + \frac{1}{576}x^4\right)$
   $- \frac{1}{2880}\left(y_1 \ln x - \frac{120}{x^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+f)!}\left(\sum_{j=1}^{n} \frac{2j+5}{j(j+5)}\right) x^n\right)$
9. $y_1 = \frac{x^{1/2}}{6}\sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)(n+3)x^n; \quad y_2 = x^{-5/2}\left(1 + \frac{1}{2}x + x^2\right) - 3y_1 \ln x + \frac{3}{2}x^{1/2}$
   $\sum_{n=1}^{\infty}(-1)^n(n+1)(n+2)(n+3)\left(\sum_{j=1}^{n} \frac{1}{j(j+3)}\right) x^n$
10. $y_1 = x^4\left(1 - \frac{2}{5}x\right); \quad y_2 = 1 + 10x + 50x^2 + 200x^3 - 300\left(y_1 \ln x + \frac{27}{25}x^5 - \frac{1}{30}x^6\right)$
11. $y_1 = x^3; \quad y_2 = x^{-3}\left(1 - \frac{6}{5}x + \frac{3}{4}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 - \frac{1}{20}x^5\right) - \frac{1}{120}\left(y_1 \ln x + x^3\sum_{n=1}^{\infty} \frac{(-1)^n 6!}{n(n+6)!} x^n\right)$
12. $y_1 = x^2\sum_{n=0}^{\infty} \frac{1}{n!}\left(\prod_{j=1}^{n} \frac{2j+3}{j+4}\right) x^n; \quad y_2 = x^{-2}\left(1 + x + \frac{1}{4}x^2 - \frac{1}{12}x^3\right) - \frac{1}{16}y_1 \ln x + \frac{x^2}{8}$
    $\sum_{n=1}^{\infty} \frac{1}{n!}\left(\prod_{j=1}^{n} \frac{2j+3}{j+4}\right)\left(\sum_{j=1}^{n} \frac{(j^2+3j+6)}{j(j+4)(2j+3)}\right) x^n$
13. $y_1 = x^5\sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)x^n; \quad y_2 = 1 - \frac{x}{2} + \frac{x^2}{6}$
14. $y_1 = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\prod_{j=1}^{n} \frac{(j+3)(2j-3)}{j+6}\right) x^n; \quad y_2 = x^{-7}\left(1 + \frac{26}{5}x + \frac{143}{20}x^2\right)$
15. $y_1 = x^{7/2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(n+4)!} x^n; \quad y_2 = x^{-1/2}\left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3\right)$
16. $y_1 = x^{10/3}\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{9^n}\left(\prod_{j=1}^{n} \frac{3j+7}{j+4}\right) x^n; \quad y_2 = x^{-2/3}\left(1 + \frac{4}{27}x - \frac{1}{243}x^2\right)$
17. $y_1 = x^3\sum_{n=0}^{7}(-1)^n(n+1)\left(\prod_{j=1}^{n} \frac{j-8}{j+6}\right) x^n; \quad y_2 = x^{-3}\left(1 + \frac{52}{5}x + \frac{234}{5}x^2 + \frac{572}{5}x^3 + 143x^4\right)$
18. $y_1 = x^3\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\prod_{j=1}^{n} \frac{(j+3)^2}{j+5}\right) x^n; \quad y_2 = x^{-2}\left(1 + \frac{1}{4}x\right)$
19. $y_1 = x^6\sum_{n=0}^{4}(-1)^n 2^n\left(\prod_{j=1}^{n} \frac{j-5}{j+5}\right) x^n; \quad y_2 = x(1 + 18x + 144x^2 + 672x^3 + 2016x^4)$
20. $y_1 = x^6\left(1 + \frac{2}{3}x + \frac{1}{7}x^2\right); \quad y_2 = x\left(1 + \frac{21}{4}x + \frac{21}{2}x^2 + \frac{35}{4}x^3\right)$
21. $y_1 = x^{7/2}\sum_{n=0}^{\infty}(-1)^n(n+1)x^n; \quad y_2 = x^{-7/2}\left(1 - \frac{5}{6}x + \frac{2}{3}x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5\right)$
22. $y_1 = \frac{x^{10}}{6}\sum_{n=0}^{\infty}(-1)^n 2^n(n+1)(n+2)(n+3)x^n; \quad y_2 = \left(1 - \frac{4}{3}x + \frac{5}{3}x^2 - \frac{40}{21}x^3 + \frac{40}{21}x^4 - \frac{32}{21}x^5 + \frac{16}{21}x^6\right)$
23. $y_1 = x^6\sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+5)}{2^m m!} x^{2m}; \quad y_2 = x^2\left(1 + \frac{3}{2}x^2\right) - \frac{15}{2}y_1 \ln x + \frac{75}{2}x^6$
    $\sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j+5)}{2^{m+1} m!}\left(\sum_{j=1}^{m} \frac{1}{j(2j+5)}\right) x^{2m}$

24. $y_1 = x^6 \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} x^{2m} = x^6 e^{-x^2}; \quad y_2 = x^2\left(1+\frac{1}{2}x^2\right) - \frac{1}{2} y_1 \ln x + \frac{x^6}{4} \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m m!} \left(\sum_{j=1}^{m} \frac{1}{j}\right) x^{2m}$

25. $y_1 = 6x^6 \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!(m+3)!} x^{2m}; \quad y_2 = 1 + \frac{1}{8}x^2 + \frac{1}{64}x^4 - \frac{1}{384}\left(y_1 \ln x - 3x^6 \sum_{m=1}^{\infty} \frac{(-1(^m}{4^m m!(m+3)!} \left(\sum_{j=1}^{m} \frac{2j+3}{j(j+3)}\right) x^{2m}\right)$

26. $y_1 = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m (m+2)}{m!} x^{2m}; \quad y_2 = x^{-1} - 4y_1 \ln x \sum_{m=1}^{\infty} \frac{(-1)^m (m+2)}{m!} \left(\sum_{j=1}^{m} \frac{j^2+4j+2}{j(j+1)(j+2)}\right) x^{2m}$

27. $y_1 = 2x^3 \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!(m+2)!} x^{2m}; \quad y_2 = x^{-1}\left(1+\frac{1}{4}x^2\right) - \frac{1}{16}\left(y_1 \ln x - 2x^3 \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m!(m+2)!} \left(\sum_{j=1}^{m} \frac{j+1}{j(j+2)}\right) x^{2m}\right)$

28. $y_1 = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{8^m m!(m+1)!} x^{2m};$

$y_2 = x^{-5/2} + \frac{1}{4} y_1 \ln x - x^{-1/2} \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{8^{m+1} m!(m+1)!} \left(\sum_{j=1}^{m} \frac{2j^2-2j-1}{j(j+1)(2j-1)}\right) x^{2m}$

29. $y_1 = x \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} x^{2m} = x e^{-x^2/2}; \quad y_2 = x^{-1} - y_1 \ln x + \frac{x}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m m!} \left(\sum_{j=1}^{m} \frac{1}{j}\right) x^{2m}$

30. $y_1 = x^2 \sum_{m=0}^{\infty} \frac{1}{m!} x^{2m} = x^2 e^{x^2}; \quad y_2 = x^{-2}(1-x^2) - 2y_1 \ln x + x^2 \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{j=1}^{m} \frac{1}{j}\right) x^{2m}$

31. $y_1 = 6x^{5/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{16^m m!(m+3)!} x^{2m}; \quad y_2 = x^{-7/2}\left(1+\frac{1}{32}x^2+\frac{1}{1024}x^4\right)$

$-\frac{1}{24576}\left(y_1 \ln x - 3x^{5/2} \sum_{m=1}^{\infty} \frac{(-1)^m}{16^m m!(m+3)!} \left(\sum_{j=1}^{m} \frac{2j+3}{j(j+3)}\right) x^{2m}\right)$

32. $y_1 = 2x^{13/3} \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{m}(3j+1)}{9^m m!(m+2)!} x^{2m};$

$y_2 = x^{1/3}\left(1+\frac{2}{9}x^2\right) + \frac{2}{81}\left(y_1 \ln x - x^{13/3} \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{m}(3j+1)}{9^m m!(m+2)!} \left(\sum_{j=1}^{m} \frac{3j^2+2j+2}{j(j+2)(3j+1)}\right) x^{2m}\right)$

33. $y_1 = x^2; \quad y_2 = x^{-2}(1+2x^2) - 2\left(y_1 \ln x + x^2 \sum_{m=1}^{\infty} \frac{1}{m(m+2)!} x^{2m}\right)$

34. $y_1 = x^2\left(1-\frac{1}{2}x^2\right); \quad y_2 = x^{-2}\left(1+\frac{9}{2}x^2\right) - \frac{27}{2}\left(y_1 \ln x + \frac{7}{12}x^4 - x^2 \sum_{m=2}^{\infty} \frac{\left(\frac{3}{2}\right)^m}{m(m-1)(m+2)!} x^{2m}\right)$

35. $y_1 = \sum_{m=0}^{\infty} (-1)^m (m+1) x^{2m}; \quad y_2 = x^{-4}$

36. $y_1 = x^{5/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m+2)(m+3)} x^{2m}; \quad y_2 = x^{-7/2}(1+x^2)^2$

37. $y_1 = \frac{x^7}{5} \sum_{m=0}^{\infty} (-1)^m (m+5) x^{2m}; \quad y_2 = x^{-1}(1-2x^2+3x^4-4x^6)$

38. $y_1 = x^3 \sum_{m=0}^{\infty} (-1)^m \frac{m+1}{2^m} \left(\prod_{j=1}^{m} \frac{2j+1}{j+5}\right) x^{2m}; \quad y_2 = x^{-7}\left(1+\frac{21}{8}x^2+\frac{35}{16}x^4+\frac{35}{64}x^6\right)$

39. $y_1 = 2x^4 \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^{m}(4j+5)}{2^m (m+2)!} x^{2m}; \quad y_2 = 1 - \frac{1}{2}x^2$

40. $y_1 = x^{3/2} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(2j-1)}{2^{m-1}(m+2)!} x^{2m}; \quad y_2 = x^{-5/2}\left(1+\frac{3}{2}x^2\right)$

42. $y_1 = x^v \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m! \prod_{j=1}^{m}(j+v)} x^{2m}; \quad y_2 = x^{-v} \sum_{m=0}^{v-1} \frac{(-1)^m}{4^m m! \prod_{j=1}^{m}(j-v)} x^{2m}$

$-\frac{2}{4^v v!(v-1)!}\left(y_1 \ln x - \frac{x^v}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m! \prod_{j=1}^{m}(j+v)} \left(\sum_{j=1}^{m} \frac{2j+v}{j(j+v)}\right) x^{2m}\right)$

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# 14.36: Section 8.1 Answers

1.

a. $\frac{1}{s^2}$
b. $\frac{1}{(s+1)^2}$
c. $\frac{b}{s^2-b^2}$
d. $\frac{-2s+5}{(s-1)(s-2)}$
e. $\frac{2}{s^3}$

2.

a. $\frac{s^2+2}{\left[(s-1)^2+1\right]\left[(s+1)^2+1\right]}$
b. $\frac{2}{s(s^2+4)}$
c. $\frac{s^2+8}{s(s^2+16)}$
d. $\frac{s^2-2}{s(s^2-4)}$
e. $\frac{4s}{(s^2-4)^2}$
f. $\frac{1}{s^2+4}$
g. $\frac{1}{\sqrt{2}}\frac{s+1}{s^2+1}$
h. $\frac{5s}{(s^2+4)(s^2+9)}$
i. $\frac{s^3+2s^2+4s+32}{(s^2+4)(s^2+16)}$

4.

a. $f(3-)=-1,\ f(3)=f(3+)=1$
b. $f(1-)=3,\ f(1)=4,\ f(1+)=1$
c. $f\left(\frac{\pi}{2}-\right)=1,\ f\left(\frac{\pi}{2}\right)=f\left(\frac{\pi}{2}+\right)=2,\ f(\pi-)=0,\ f(\pi)=f(\pi+)=-1$
d. $f(1-)=1,\ f(1)=2,\ f(1+)=1,\ f(2-)=0,\ f(2)=3,\ f(2+)=6$

5.

a. $\frac{1-e^{-(s+1)}}{s+1}+\frac{e^{-(s+2)}}{s+2}$
b. $\frac{1}{s}+e^{-4s}\left(\frac{1}{s^2}+\frac{3}{s}\right)$
c. $\frac{1-e^{-s}}{s^2}$
d. $\frac{1-e^{-(s-1)}}{(s-1)^2}$

7. $\mathcal{L}(e^{\lambda t}\cos\omega t)=\frac{(s-\lambda)^2-\omega^2}{((s-\lambda)^2+\omega^2)^2}\qquad \mathcal{L}(e^{\lambda t}\sin\omega t)=\frac{2\omega(s-\lambda)}{((s-\lambda)^2+\omega^2)^2}$

15.

a. $\tan^{-1}\frac{\omega}{s},\quad s>0$
b. $\frac{1}{2}\ln\frac{s^2}{s^2+\omega^2},\quad s>0$
c. $\ln\frac{s-b}{s-a},\quad s>\max(a,b)$
d. $\frac{1}{2}\ln\frac{s^2}{s^2-1},\quad s>1$
e. $\frac{1}{4}\ln\frac{s^2}{s^2-4},\quad s>2$

18.

a. $\frac{1}{s^2}\tanh\frac{s}{2}$
b. $\frac{1}{s}\tanh\frac{s}{4}$
c. $\frac{1}{s^2+1}\coth\frac{\pi s}{2}$

d. $\frac{1}{(s^2+1)(1-e^{-\pi s})}$

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# 14.37: Section 8.2 Answers

1.

a. $\frac{t^3e^{7t}}{2}$
b. $2e^{2t}\cos 3t$
c. $\frac{e^{-2t}}{4}\sin 4t$
d. $\frac{2}{3}\sin 3t$
e. $t\cos t$
f. $\frac{e^{2t}}{2}\sinh 2t$
g. $\frac{2te^{2t}}{3}\sin 9t$
h. $\frac{2e^{3t}}{3}\sinh 3t$
i. $e^{2t}t\cos t$

2.

a. $t^2e^{7t}+\frac{17}{6}t^3e^{7t}$
b. $e^{2t}\left(\frac{1}{6}t^3+\frac{1}{6}t^4+\frac{1}{40}t^5\right)$
c. $e^{-3t}\left(\cos 3t+\frac{2}{3}\sin 3t\right)$
d. $2\cos 3t+\frac{1}{3}\sin 3t$
e. $(1-t)e^{-t}$
f. $\cosh 3t+\frac{1}{3}\sinh 3t$
g. $\left(1-t-t^2-\frac{1}{6}t^3\right)e^{-t}$
h. $e^t\left(2\cos 2t+\frac{5}{2}\sin 2t\right)$
i. $1-\cos t$
j. $3\cosh t+4\sinh t$
k. $3e^t+4\cos 3t+\frac{1}{3}\sin 3t$
l. $3te^{-2t}-2\cos 2t-3\sin 2t$

3.

a. $\frac{1}{4}e^{2t}-\frac{1}{4}e^{-2t}-e^{-t}$
b. $\frac{1}{5}e^{-4t}-\frac{41}{5}e^t+5e^{3t}$
c. $-\frac{1}{2}e^{2t}-\frac{13}{10}e^{-2t}-\frac{1}{5}e^{3t}$
d. $-\frac{2}{5}e^{-4t}-\frac{3}{5}e^t$
e. $\frac{3}{20}e^{2t}-\frac{37}{12}e^{-2t}+\frac{1}{3}e^t+\frac{8}{5}e^{-3t}$
f. $\frac{39}{10}e^t+\frac{3}{14}e^{3t}+\frac{23}{105}e^{-4t}-\frac{7}{3}e^{2t}$

4.

a. $\frac{4}{5}e^{-2t}-\frac{1}{2}e^{-t}-\frac{3}{10}\cos t+\frac{11}{10}\sin t$
b. $\frac{2}{5}\sin t+\frac{6}{5}\cos t+\frac{7}{5}e^{-t}\sin t-\frac{6}{5}e^{-t}\cos t$
c. $\frac{8}{13}e^{2t}-\frac{8}{13}e^{-t}\cos 2t+\frac{15}{26}e^{-t}\sin 2t$
d. $\frac{1}{2}te^t+\frac{3}{8}e^t+e^{-2t}-\frac{11}{8}e^{-3t}$
e. $\frac{2}{3}te^t+\frac{1}{9}e^t+te^{-2t}-\frac{1}{9}e^{-2t}$
f. $-e^t+\frac{5}{2}te^t+\cos t-\frac{3}{2}\sin t$

5.

a. $\frac{3}{5}\cos 2t+\frac{1}{5}\sin 2t-\frac{3}{5}\cos 3t-\frac{2}{15}\sin 3t$
b. $-\frac{4}{15}\cos t+\frac{1}{15}\sin t+\frac{4}{15}\cos 4t-\frac{1}{60}\sin 4t$
c. $\frac{5}{3}\cos t+\sin t-\frac{5}{3}\cos 2t-\frac{1}{2}\sin 2t$
d. $-\frac{1}{3}\cos\frac{t}{2}+\frac{2}{3}\sin\frac{t}{2}+\frac{1}{3}\cos t-\frac{1}{3}\sin t$
e. $\frac{1}{15}\cos\frac{t}{4}-\frac{8}{15}\sin\frac{t}{4}-\frac{1}{15}\cos 4t+\frac{1}{30}\sin 4t$

f. $\frac{2}{5}\cos\frac{t}{3}-\frac{3}{5}\sin\frac{t}{3}-\frac{2}{5}\cos\frac{t}{2}+\frac{2}{5}\sin\frac{t}{2}$

6.

a. $e^{t}(\cos 2t+\sin 2t)-e^{-t}\left(\cos 3t+\frac{4}{3}\sin 3t\right)$
b. $e^{3t}\left(-\cos 2t+\frac{3}{2}\sin 2t\right)+e^{-t}\left(\cos 2t+\frac{1}{2}\sin 2t\right)$
c. $e^{-2t}\left(\frac{1}{8}\cos t+\frac{1}{4}\sin t\right)-e^{2t}\left(\frac{1}{8}\cos 3t-\frac{1}{12}\sin 3t\right)$
d. $e^{2t}\left(\cos t+\frac{1}{2}\sin t\right)-e^{3t}\left(\cos 2t-\frac{1}{4}\sin 2t\right)$
e. $e^{t}\left(\frac{1}{5}\cos t+\frac{2}{5}\sin t\right)-e^{-t}\left(\frac{1}{5}\cos 2t+\frac{2}{5}\sin 2t\right)$
f. $e^{t/2}\left(-\cos t+\frac{9}{8}\sin t\right)+e^{-t/2}\left(\cos t-\frac{1}{8}\sin t\right)$

7.

a. $1-\cos t$
b. $\frac{e^{t}}{16}(1-\cos 4t)$
c. $\frac{4}{9}e^{2t}+\frac{5}{9}e^{-t}\sin 3t-\frac{4}{9}e^{-t}\cos 3t$
d. $3e^{t/2}-\frac{7}{2}e^{t}\sin 2t-3e^{t}\cos 2t$
e. $\frac{1}{4}e^{3t}-\frac{1}{4}e^{-t}\cos 2t$
f. $\frac{1}{9}e^{2t}-\frac{1}{9}e^{-t}\cos 3t+\frac{5}{9}e^{-t}\sin 3t$

8.

a. $-\frac{3}{10}\sin t+\frac{2}{5}\cos t-\frac{3}{4}e^{t}+\frac{7}{20}e^{3t}$
b. $-\frac{3}{5}e^{-t}\sin t+\frac{1}{5}e^{-t}\cos t-\frac{1}{2}e^{-t}+\frac{3}{10}e^{t}$
c. $-\frac{1}{10}e^{t}\sin t-\frac{7}{10}e^{t}\cos t+\frac{1}{5}e^{-t}+\frac{1}{2}e^{2t}$
d. $-\frac{1}{2}e^{t}+\frac{7}{10}e^{-t}-\frac{1}{5}\cos 2t+\frac{3}{5}\sin 2t$
e. $\frac{3}{10}+\frac{1}{10}e^{2t}+\frac{1}{10}e^{t}\sin 2t-\frac{2}{5}e^{t}\cos 2t$
f. $-\frac{4}{9}e^{2t}\cos 3t+\frac{1}{3}e^{2t}\sin 3t-\frac{5}{9}e^{2t}+e^{t}$

9. $\frac{1}{a}e^{\frac{b}{a}t}f\left(\frac{t}{a}\right)$

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# 14.38: Section 8.3 Answers

1. $y = \frac{1}{6}e^{t} - \frac{9}{2}e^{-t} + \frac{16}{3}e^{-2t}$
2. $y = -\frac{1}{3} + \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t}$
3. $y = -\frac{23}{15}e^{-2t} + \frac{1}{3}e^{t} + \frac{1}{5}e^{3t}$
4. $y = -\frac{1}{4}e^{2t} + \frac{17}{20}e^{-2t} + \frac{2}{5}e^{3t}$
5. $y = \frac{11}{15}e^{-2t} + \frac{1}{6}e^{t} + \frac{1}{10}e^{3t}$
6. $y = e^{t} + 2e^{-2t} - 2e^{-t}$
7. $y = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$
8. $y = 4e^{t} - 4e^{2t} + e^{3t}$
9. $y = -\frac{7}{2}e^{2t} + \frac{13}{3}e^{t} + \frac{1}{6}e^{4t}$
10. $y = \frac{5}{2}e^{t} - 4e^{2t} + \frac{1}{2}e^{3t}$
11. $y = \frac{1}{3}e^{t} - 2e^{-t} + \frac{5}{3}e^{-2t}$
12. $y = 2 - e^{-2t} + e^{t}$
13. $y = 1 - \cos 2t + \frac{1}{2}\sin 2t$
14. $y = -\frac{1}{3} + \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t}$
15. $y = \frac{1}{6}e^{t} - \frac{2}{3}e^{-2t} + \frac{1}{2}e^{-t}$
16. $y = -1 + e^{t} + e^{-t}$
17. $y = \cos 2t - \sin 2t + \sin t$
18. $y = \frac{7}{3} - \frac{7}{2}e^{-t} + \frac{1}{6}e^{3t}$
19. $y = 1 + \cos t$
20. $y = t + \sin t$
21. $y = t - 6\sin t + \cos t + \sin 2t$
22. $y = e^{-t} + 4e^{-2t} - 4e^{-3t}$
23. $y = -3\cos t - 2\sin t + e^{-t}(2 + 5t)$
24. $y = -\sin t - 2\cos t + 3e^{3t} + e^{-t}$
25. $y = (3t + 4)\sin t - (2t + 6)\cos t$
26. $y = -(2t + 2)\cos 2t + \sin 2t + 3\cos t$
27. $y = e^{t}(\cos t - 3\sin t) + e^{3t}$
28. $y = -1 + t + e^{-t}(3\cos t - 5\sin t)$
29. $y = 4\cos t - 3\sin t - e^{t}(3\cos t - 8\sin t)$
30. $y = e^{-t} - 2e^{t} + e^{-2t}(\cos 3t - 11/3\sin 3t)$
31. $y = e^{-t}(\sin t - \cos t) + e^{-2t}(\cos t + 4\sin t)$
32. $y = \frac{1}{5}e^{2t} - \frac{4}{3}e^{t} + \frac{32}{15}e^{-t/2}$
33. $y = \frac{1}{7}e^{2t} - \frac{2}{5}e^{t/2} + \frac{9}{35}e^{-t/3}$
34. $y = e^{-t/2}(5\cos(t/2) - \sin(t/2)) + 2t - 4$
35. $y = \frac{1}{17}(12\cos t + 20\sin t - 3e^{t/2}(4\cos t + \sin t))$

36. $y = \frac{e^{-t/2}}{10}(5t + 26) - \frac{1}{5}(3\cos t + \sin t)$

37. $y = \frac{1}{100}(3e^{3t} - e^{t/3}(3 + 310t))$

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# 14.39: Section 8.4 Answers

1. $1+u(t-4)(t-1)$; $\frac{1}{s}+e^{-4s}\left(\frac{1}{s^2}+\frac{3}{s}\right)$

2. $t+u(t-1)(1-t)$; $\frac{1-e^{-s}}{s^2}$

3. $2t-1-u(t-2)(t-1)$; $\left(\frac{2}{s^2}-\frac{1}{s}\right)-e^{-2s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$

4. $1+u(t-1)(t+1)$; $\frac{1}{s}+e^{-s}\left(\frac{1}{s^2}+\frac{2}{s}\right)$

5. $t-1+u(t-2)(5-t)$; $\frac{1}{s^2}-\frac{1}{s}-e^{-2s}\left(\frac{1}{s^2}-\frac{3}{s}\right)$

6. $t^2(1-u(t-1))$; $\frac{2}{s^3}-e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$

7. $u(t-2)(t^2+3t)$; $e^{-2s}\left(\frac{2}{s^3}+\frac{7}{s^2}+\frac{10}{s}\right)$

8. $t^2+2+u(t-1)(t-t^2-2)$; $\frac{2}{s^3}+\frac{2}{s}-e^{-s}\left(\frac{2}{s^3}+\frac{1}{s^2}+\frac{2}{s}\right)$

9. $te^t+u(t-1)(e^t-te^t)$; $\frac{1-e^{-(s-1)}}{(s-1)^2}$

10. $e^{-t}+u(t-1)(e^{-2t}-e^{-t})$; $\frac{1-e^{-(s+1)}}{s+1}+\frac{e^{-(s+2)}}{s+2}$

11. $-t+2u(t-2)(t-2)-u(t-3)(t-5)$; $-\frac{1}{s^2}+\frac{2e^{-2s}}{s^2}+e^{-3s}\left(\frac{2}{s}-\frac{1}{s^2}\right)$

12. $[u(t-1)-u(t-2)]\,t$; $e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)-e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right)$

13. $t+u(t-1)(t^2-t)-u(t-2)t^2$; $\frac{1}{s^2}+e^{-s}\left(\frac{2}{s^3}+\frac{1}{s^2}\right)-e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)$

14. $t+u(t-1)(2-2t)+u(t-2)(4+t)$; $\frac{1}{s^2}-2\frac{e^{-s}}{s^2}+e^{-2s}\left(\frac{1}{s^2}+\frac{6}{s}\right)$

15. $\sin t+u(t-\pi/2)\sin t+u(t-\pi)(\cos t-2\sin t)$; $\frac{1+e^{-\frac{\pi}{2}s}s-e^{-\pi s}(s-2)}{s^2+1}$

16. $2-2u(t-1)t+u(t-3)(5t-2)$; $\frac{2}{s}-e^{-s}\left(\frac{2}{s^2}+\frac{2}{s}\right)+e^{-3s}\left(\frac{5}{s^2}+\frac{13}{s}\right)$

17. $3+u(t-2)(3t-1)+u(t-4)(t-2)$; $\frac{3}{s}+e^{-2s}\left(\frac{3}{s^2}+\frac{5}{s}\right)+e^{-4s}\left(\frac{1}{s^2}+\frac{2}{s}\right)$

18. $(t+1)^2+u(t-1)(2t+3)$; $\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}+e^{-s}\left(\frac{2}{s^2}+\frac{5}{s}\right)$

19. $u(t-2)e^{2(t-2)}=\begin{cases}0, & 0\le t<2,\\ e^{2(t-2)}, & t\ge 2\end{cases}$

20. $u(t-1)\left(1-e^{-(t-1)}\right)=\begin{cases}0, & 0\le t<1,\\ 1-e^{-(t-1)}, & t\ge 1\end{cases}$

21. $u(t-1)\frac{(t-1)^2}{2}+u(t-2)(t-2)=\begin{cases}0, & 0\le t<1,\\ \frac{(t-1)^2}{2}, & 1\le t<2,\\ \frac{t^2-3}{2}, & t\ge 2\end{cases}$

22. $2+t+u(t-1)(4-t)+u(t-3)(t-2)=\begin{cases}2+t, & 0\le t<1,\\ 6, & 1\le t<3,\\ t+4, & t\ge 3\end{cases}$

23. $5-t+u(t-3)(7t-15)+\frac{3}{2}u(t-6)(t-6)^2=\begin{cases}5-t, & 0\le t<3,\\ 6t-10, & 3\le t<6,\\ 44-12t+\frac{3}{2}t^2, & t\ge 6\end{cases}$

24. $u(t-\pi)e^{-2(t-\pi)}(2\cos t-5\sin t)=\begin{cases}0, & 0\le t<\pi,\\ e^{-2(t-\pi)}(2\cos t-5\sin t) & t\ge\pi\end{cases}$

25. $1-\cos t+u(t-\pi/2)(3\sin t+\cos t)=\begin{cases}1-\cos t, & 0\le t<\frac{\pi}{2},\\ 1+3\sin t, & t\ge\frac{\pi}{2}\end{cases}$

26. $u(t-2)(4e^{-(t-2)}-4e^{2(t-2)}+2e^{(t-2)})=\begin{cases}0, & 0\le t<2,\\ 4e^{-(t-2)}-4e^{2(t-2)}+2e^{(t-2)}, & t\ge 2\end{cases}$

27. $1+t+u(t-1)(2t+1)+u(t-3)(3t-5)=\begin{cases}t+1, & 0\le t<1,\\ 3t+2, & 1\le t<3,\\ 6t-3, & t\ge 3\end{cases}$

28. $1-t^2+u(t-2)\left(-\frac{t^2}{2}+2t+1\right)+u(t-4)(t-4)=\begin{cases}1-t^2, & 0\le t<2,\\ -\frac{3t^2}{2}+2t+2, & 2\le t<4,\\ -\frac{3t^2}{2}+3t-2, & t\ge 4\end{cases}$

29. $\frac{e^{-\tau s}}{s}$

30. For each $t$ only finitely many terms are nonzero.

33. $1+\sum_{m=1}^{\infty}u(t-m)$; $\frac{1}{s(1-e^{-s})}$

34. $1+2\sum_{m=1}^{\infty}(-1)^m u(t-m)$; $\frac{1}{s}$; $\frac{1-e^{-s}}{1+e^{-s}}$

35. $1+\sum_{m=1}^{\infty}(2m+1)u(t-m)$; $\frac{e^{-s}(1+e^{-s})}{s(1-e^{-s})^2}$

36. $\sum_{m=1}^{\infty}(-1)^m(2m-1)u(t-m)$; $\frac{1}{s}\frac{(1-e^{s})}{(1+e^{s})^2}$

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# 14.40: Section 8.5 Answers

1. $y = 3(1-\cos t) - 3u(t-\pi)(1+\cos t)$
2. $y = 3 - 2\cos t + 2u(t-4)(t-4-\sin(t-4))$
3. $y = -\frac{15}{2} + \frac{3}{2}e^{2t} - 2t + \frac{u(t-1)}{2}(e^{2(t-1)} - 2t + 1)$
4. $y = \frac{1}{2}e^t + \frac{13}{6}e^{-t} + \frac{1}{3}e^{2t} + u(t-2)\left(-1 + \frac{1}{2}e^{t-2} + \frac{1}{2}e^{-(t-2)} + \frac{1}{2}e^{t+2} - \frac{1}{6}e^{-(t-6)} - \frac{1}{3}e^{2t}\right)$
5. $y = -7e^t + 4e^{2t} + u(t-1)\left(\frac{1}{2} - e^{t-1} + \frac{1}{2}e^{2(t-1)}\right) - 2u(t-2)\left(\frac{1}{2} - e^{t-2} + \frac{1}{2}e^{2(t-2)}\right)$
6. $y = \frac{1}{3}\sin 2t - 3\cos 2t + \frac{1}{3}\sin t - 2u(t-\pi)\left(\frac{1}{3}\sin t + \frac{1}{6}\sin 2t\right) + u(t-2\pi)\left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right)$
7. $y = \frac{1}{4} - \frac{31}{12}e^{4t} + \frac{16}{3}e^t + u(t-1)\left(\frac{2}{3}e^{t-1} - \frac{1}{6}e^{4(t-1)} - \frac{1}{2}\right) + u(t-2)\left(\frac{1}{4} + \frac{1}{12}e^{4(t-2)} - \frac{1}{3}e^{t-2}\right)$
8. $y = \frac{1}{8}(\cos t - \cos 3t) - \frac{1}{8}u\left(y - \frac{3\pi}{2}\right)\left(\sin t - \cos t + \sin 3t - \frac{1}{3}\cos 3t\right)$
9. $y = \frac{t}{4} - \frac{1}{8}\sin 2t + \frac{1}{8}u\left(t - \frac{\pi}{2}\right)(\pi\cos 2t - \sin 2t + 2\pi - 2t)$
10. $y = t - \sin t - 2u(t-\pi)(t + \sin t + \pi\cos t)$
11. $y = u(t-2)\left(t - \frac{1}{2} + \frac{e^{2(t-2)}}{2} - 2e^{t-2}\right)$
12. $y = t + \sin t + \cos t - u(t-2\pi)(3t - 3\sin t - 6\pi\cos t)$
13. $y = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} + u(t-2)\left(2e^{-(t-2)} - e^{-2(t-2)} - 1\right)$
14. $y = -\frac{1}{3} - \frac{1}{6}e^{3t} + \frac{1}{2}e^t + u(t-1)\left(\frac{2}{3} + \frac{1}{3}e^{3(t-1)} - e^{t-1}\right)$
15. $y = \frac{1}{4}(e^t + e^{-t}(11+6t)) + u(t-1)(te^{-(t-1)} - 1)$
16. $y = e^t - e^{-t} - 2te^{-t} - u(t-1)\left(e^t - e^{-(t-2)} - 2(t-1)e^{-(t-2)}\right)$
17. $y = te^{-t} + e^{-2t} + u(t-1)\left(e^{-t}(2-t) - e^{-(2t-1)}\right)$
18. $y = \frac{t^2e^{2t}}{2} - te^{2t} - u(t-2)(t-2)^2e^{2t}$
19. $y = \frac{t^4}{12} + 1 - \frac{1}{12}u(t-1)(t^4 + 2t^3 - 10t + 7) + \frac{1}{6}u(t-2)(2t^3 + 3t^2 - 36t + 44)$
20. $y = \frac{1}{2}e^{-t}(3\cos t + \sin t) + \frac{1}{2} - u(t-2\pi)\left(e^{-(t-2\pi)}\left((\pi-1)\cos t + \frac{2\pi-1}{2}\sin t\right) + 1 - \frac{t}{2}\right)$
    $- \frac{1}{2}u(t-3\pi)(e^{-(t-3\pi)}(3\pi\cos t + (3\pi+1)\sin t) + t)$
21. $y = \frac{t^2}{2} + \sum_{m=1}^{\infty} u(t-m)\frac{(t-m)^2}{2}$
22. a. $y = \begin{cases} 2m + 1 - \cos t, & 2m\pi \le t < (2m+1)\pi \quad (m = 0, 1, \ldots) \\ 2m, & (2m-1)\pi \le t < 2m\pi \quad (m = 1, 2, \ldots) \end{cases}$
    b. $y = (m+1)(t - \sin t - m\pi\cos t), \quad 2m\pi \le t < (2m+2)\pi \quad (m = 0, 1, \ldots)$
    c. $y = (-1)^m - (2m+1)\cos t, \quad m\pi \le t < (m+1)\pi \quad (m = 0, 1, \ldots)$
    d. $y = \frac{e^{m+1}-1}{2(e-1)}(e^{t-m} + e^{-t}) - m - 1, \quad m \le t < m+1 \quad (m = 0, 1, \ldots)$
    e. $y = \left(m + 1 - \left(\frac{e^{2(m+1)\pi}-1}{e^{2\pi}-1}\right)e^{-t}\right)\sin t \quad 2m\pi \le t < 2(m+1)\pi \quad (m = 0, 1, \ldots)$
    f. $y = \frac{m+1}{2} - e^{t-m}\frac{e^{m+1}-1}{e-1} + \frac{1}{2}e^{2(t-m)}\frac{e^{2m+2}-1}{e^2-1}, \quad m \le t < m+1 \quad (m = 0, 1, \ldots)$

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# 14.41: Section 8.6 Answers

1.

a. $\frac{1}{2}\int_0^t \tau \sin 2(t-\tau)d\tau$
b. $\int_0^t e^{-2\tau}\cos 3(t-\tau)d\tau$
c. $\frac{1}{2}\int_0^t \sin 2\tau \cos 3(t-\tau)d\tau$ or $\frac{1}{3}\int_0^t \sin 3\tau \cos 2(t-\tau)d\tau$
d. $\int_0^t \cos\tau \sin(t-\tau)d\tau$
e. $\int_0^t e^{a\tau}d\tau$
f. $e^{-t}\int_0^t \sin(t-\tau)d\tau$
g. $e^{-2t}\int_0^t \tau e^{\tau}\sin(t-\tau)d\tau$
h. $\frac{e^{-2t}}{2}\int_0^t \tau^2(t-\tau)e^{3\tau}d\tau$
i. $\int_0^t (t-\tau)e^{\tau}\cos\tau d\tau$
j. $\int_0^t e^{-3\tau}\cos\tau\cos 2(t-\tau)d\tau$
k. $\frac{1}{4!5!}\int_0^t \tau^4(t-\tau)^5 e^{3\tau}d\tau$
l. $\frac{1}{4}\int_0^t \tau^2 e^{\tau}\sin 2(t-\tau)d\tau$
m. $\frac{1}{2}\int_0^t \tau(t-\tau)^2 e^{2(t-\tau)}d\tau$
n. $\frac{1}{5!6!}\int_0^t (t-\tau)^5 e^{2(t-\tau)}\tau^6 d\tau$

2.

a. $\frac{as}{(s^2+a^2)(s^2+b^2)}$
b. $\frac{a}{(s-1)(s^2+a^2)}$
c. $\frac{as}{(s^2-a^2)^2}$
d. $\frac{2\omega s(s^2-\omega^2)}{(s^2+\omega^2)^4}$
e. $\frac{(s-1)\omega}{((s-1)^2+\omega^2)^2}$
f. $\frac{2}{(s-2)^3(s-1)^2}$
g. $\frac{s+1}{(s+2)^2\left[(s+1)^2+\omega^2\right]}$
h. $\frac{1}{(s-3)((s-1)^2-1)}$
i. $\frac{2}{(s-2)^2(s^2+4)}$
j. $\frac{6}{s^4(s-1)}$
k. $\frac{3\cdot 6!}{s^7\left[(s+1)^2+9\right]}$
l. $\frac{12}{s^7}$
m. $\frac{2\cdot 7!}{s^8\left[(s+1)^2+4\right]}$
n. $\frac{48}{s^5(s^2+4)}$

3.

a. $y=\frac{2}{\sqrt{5}}\int_0^t f(t-\tau)e^{-3\tau/2}\sinh\frac{\sqrt{5}\tau}{2}d\tau$
b. $y=\frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau d\tau$
c. $y=\int_0^t \tau e^{-\tau}f(t-\tau)d\tau$
d. $y(t)=-\frac{1}{k}\sin kt+\cos kt+\frac{1}{k}\int_0^t f(t-\tau)\sin k\tau d\tau$
e. $y=-2te^{-3t}+\int_0^t \tau e^{-3\tau}f(t-\tau)d\tau$
f. $y=\frac{3}{2}\sinh 2t+\frac{1}{2}\int_0^t f(t-\tau)\sinh 2\tau d\tau$
g. $y=e^{3t}+\int_0^t (e^{3\tau}-e^{2\tau})f(t-\tau)d\tau$
h. $y=\frac{k_1}{\omega}\sin\omega t+k_0\cos\omega t+\frac{1}{\omega}\int_0^t f(t-\tau)\sin\omega\tau d\tau$

4.

a. $y = \sin t$
b. $y = te^{-t}$
c. $y = 1 + 2te^{t}$
d. $y = t + \frac{t^2}{2}$
e. $y = 4 + \frac{5}{2}t^2 + \frac{1}{24}t^4$
f. $y = 1 - t$

5.

a. $\frac{7!8!}{16!}t^{16}$
b. $\frac{13!7!}{21!}t^{21}$
c. $\frac{6!7!}{14!}t^{14}$
d. $\frac{1}{2}(e^{-t} + \sin t - \cos t)$
e. $\frac{1}{3}(\cos t - \cos 2t)$

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# 14.42: Section 8.7 Answers

1. $y=\frac{1}{2}e^{2t}-4e^{-t}+\frac{11}{2}e^{-2t}+2u(t-1)(e^{-(t-1)}-e^{-2(t-1)})$
2. $y=2e^{-2t}+5e^{-t}+\frac{5}{3}u(t-1)(e^{(t-1)}-e^{-2(t-1)})$
3. $y=\frac{1}{6}e^{2t}-\frac{2}{3}e^{-t}-\frac{1}{2}e^{-2t}+\frac{5}{2}u(t-1)\sinh 2(t-1)$
4. $y=\frac{1}{8}(8\cos t-5\sin t-\sin 3t)-2u(t-\pi/2)\cos t$
5. $y=1-\cos 2t+\frac{1}{2}\sin 2t+\frac{1}{2}u(t-3\pi)\sin 2t$
6. $y=4e^{t}+3e^{-t}-8+2u(t-2)\sinh(t-2)$
7. $y=\frac{1}{2}e^{t}-\frac{7}{2}e^{-t}+2+3u(t-6)(1-e^{-(t-6)})$
8. $y=e^{2t}+7\cos 2t-\sin 2t-\frac{1}{2}u(t-\pi/2)\sin 2t$
9. $y=\frac{1}{2}(1+e^{-2t})+u(t-1)(e^{-(t-1)}-e^{-2(t-1)})$
10. $y=\frac{1}{4}e^{t}+\frac{1}{4}e^{-t}(2t-5)+2u(t-2)(t-2)e^{-(t-2)}$
11. $y=\frac{1}{6}(2\sin t+5\sin 2t)-\frac{1}{2}u(t-\pi/2)\sin 2t$
12. $y=e^{-t}(\sin t-\cos t)-e^{-(t-\pi)}\sin t-3u(t-2\pi)e^{-(t-2\pi)}\sin t$
13. $y=e^{-2t}\left(\cos 3t+\frac{4}{3}\sin 3t\right)-\frac{1}{3}u(t-\pi/6)e^{-2(t-\pi/6)}\cos 3t-\frac{2}{3}u(t-\pi/3)e^{-2(t-\pi/3)}\sin 3t$
14. $y=\frac{7}{10}e^{2t}-\frac{6}{5}e^{-t/2}-\frac{1}{2}+\frac{1}{5}u(t-2)(e^{2(t-2)}-e^{-(t-2)/2})$
15. $y=\frac{1}{17}(12\cos t+20\sin t)+\frac{1}{34}e^{t/2}(10\cos t-11\sin t)-u(t-\pi/2)e^{(2t-\pi)/4}\cos t+u(t-\pi)e^{(t-\pi)/2}\sin t$
16. $y=\frac{1}{3}(\cos t-\cos 2t-3\sin t)-2u(t-\pi/2)\cos t+3u(t-\pi)\sin t$
17. $y=e^{t}-e^{-t}(1+2t)-5u(t-1)\sinh(t-1)+3u(t-2)\sinh(t-2)$
18. $y=\frac{1}{4}(e^{t}-e^{-t}(1+6t))-u(t-1)e^{-(t-1)}+2u(t-2)e^{-(t-2)})$
19. $y=\frac{5}{3}\sin t-\frac{1}{3}\sin 2t+\frac{1}{3}u(t-\pi)(\sin 2t+2\sin t)+u(t-2\pi)\sin t$
20. $y=\frac{3}{4}\cos 2t-\frac{1}{2}\sin 2t+\frac{1}{4}+\frac{1}{4}u(t-\pi/2)(1+cos2t)+\frac{1}{2}u(t-\pi)\sin 2t+\frac{3}{2}u(t-3\pi/2)\sin 2t$
21. $y=\cos t-\sin t$
22. $y=\frac{1}{4}(8e^{3t}-12e^{-2t})$
23. $y=5(e^{-2t}-e^{-t})$
24. $y=e^{-2t}(1+6t)$
25. $y=\frac{1}{4}e^{-t/2}(4-19t)$
29. $y=(-1)^{k}m\omega_1 Re^{-c\tau/2m}\delta(t-\tau)$ if $\omega_1\tau-\phi=(2k+1)\pi/2(k=$ integer $)$
30.
    a. $y=\frac{(e^{m+1}-1)(e^{t-m}-e^{-t})}{2(e-1)},\quad m\le t<m+1,\quad (m=0,1,\ldots)$
    b. $y=(m+1)\sin t,\quad 2m\pi\le t<2(m+1)\pi,\quad (m=0,1,\ldots)$
    c. $y=e^{2(t-m)}\frac{e^{2m+2}-1}{2^2-1}-e^{(t-m)}\frac{e^{m+1}-1}{e-1},\quad m\le t<m+1,\quad (m=0,1,\ldots)$
    d. $y=\begin{cases}0, & 2m\pi\le t<(2m+1)\pi,\\ -\sin t, & (2m+1)\pi\le t<(2m+2)\pi,\end{cases}\quad (m=0,1,\ldots)$

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# 14.43: Section 9.1 Answers

2. $y = 2x^2 - 3x^3 + \frac{1}{x}$

3. $y = 2e^x + 3e^{-x} - e^{2x} + e^{-3x}$

4. $y_i = \frac{(x-x_0)^{i-1}}{(i-1)!}, \quad 1 \le i \le n$

5.

b. $y_1 = -\frac{1}{2}x^3 + x^2 + \frac{1}{2x}, \quad y_2 = \frac{1}{3}x^2 - \frac{1}{3x}, \quad y_3 = \frac{1}{4}x^3 - \frac{1}{3}x^2 + \frac{1}{12x}$

c. $y = k_0 y_1 + k_1 y_2 + k_2 y_3$

7. $2e^{-x^2}$

8. $\sqrt{2}K\cos x$

9.

a. $W(x) = 2e^{3x}$

d. $y = e^x(c_1 + c_2 x + c_3 x^2)$

10.

a. $2$
b. $-e^{3x}$
c. $4$
d. $4/x^2$
e. $1$
f. $2x$
g. $2/x^2$
h. $e^x(x^2 - 2x + 2)$
i. $-240/x^5$
j. $6e^{2x}(2x-1)$
k. $-128x$

24.

a. $y''' = 0$
b. $xy''' - y'' - xy' + y = 0$
c. $(2x-3)y''' - 2y'' - (2x-5)y' = 0$
d. $(x^2 - 2x + 2)y''' - x^2y'' + 2xy' - 2y = 0$
e. $x^3y''' + x^2y'' - 2xy' + 2y = 9$
f. $(3x-1)y''' - (12x-1)y'' + 9(x+1)y' - 9y = 0$
g. $x^4y^{(4)} + 5x^3y''' - 3x^2y'' - 6xy' + 6y = 0$
h. $x^4y^{(4)} + 3x^2y''' - x^2y'' + 2xy' - 2y = 0$
i. $(2x-1)y^{(4)} - 4xy''' + (5-2x)y'' + 4xy' - 4y = 0$
j. $xy^{(4)} - y''' - 4xy'' + 4y' = 0$

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# 14.44: Section 9.2 Answers

1. $y = e^x(c_1 + c_2x + c_3x^2)$
2. $y = c_1e^x + c_2e^{-x} + c_3\cos 3x + c_4\sin 3x$
3. $y = c_1e^x + c_2\cos 4x + c_3\sin 4x$
4. $y = c_1e^x + c_2e^{-x} + c_3e^{-3x/2}$
5. $y = c_1e^{-x} + e^{-2x}(c_1\cos x + c_2\sin x)$
6. $y = c_1e^x + e^{x/2}(c_2 + c_3x)$
7. $y = e^{-x/3}(c_1 + c_2x + c_3x^2)$
8. $y = c_1 + c_2x + c_3\cos x + c_4\sin x$
9. $y = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x$
10. $y = (c_1 + c_2x)\cos\sqrt{6}x + (c_3 + c_4x)\sin\sqrt{6}x$
11. $y = e^{3x/2}(c_1 + c_2x) + e^{-3x/2}(c_3 + c_4x)$
12. $y = c_1e^{-x/2} + c_2e^{-x/3} + c_3\cos x + c_4\sin x$
13. $y = c_1e^x + c_2e^{-2x} + c_3e^{-x/2} + c_4e^{-3x/2}$
14. $y = e^x(c_1 + c_2x + c_3\cos x + c_4\sin x)$
15. $y = \cos 2x - 2\sin 2x + e^{2x}$
16. $y = 2e^x + 3e^{-x} - 5e^{-3x}$
17. $y = 2e^x + 3xe^x - 4e^{-x}$
18. $y = 2e^{-x}\cos x - 3e^{-x}\sin x + 4e^{2x}$
19. $y = \frac{9}{5}e^{-5x/3} + e^x(1 + 2x)$
20. $y = e^{2x}(1 - 3x + 2x^2)$
21. $y = e^{3x}(2 - x) + 4e^{-x/2}$
22. $y = e^{x/2}(1 - 2x) + 3e^{-x/2}$
23. $y = \frac{1}{8}(5e^{2x} + e^{-2x} + 10\cos 2x + 4\sin 2x)$
24. $y = -4e^x + e^{2x} - e^{4x} + 2e^{-x}$
25. $y = 2e^x - e^{-x}$
26. $y = e^{2x} + e^{-2x} + e^{-x}(3\cos x + \sin x)$
27. $y = 2e^{-x/2} + \cos 2x - \sin 2x$
28.
    a. $\{e^x, xe^x, e^{2x}\}$ 1
    b. $\{\cos 2x, \sin 2x, e^{3x}\}$ : 26
    c. $\{e^{-x}\cos x, e^{-x}\sin x, e^x\}$ : 5
    d. $\{1, x, x^2, e^x\}$ $2e^x$
    e. $\{e^x, e^{-x}, \cos x, \sin x\}$ 8
    f. $\{\cos x, \sin x, e^x\cos x, e^x\sin x\}$ : 5
29. $\{e^{-3x}\cos 2x, e^{-3x}\sin 2x, e^{2x}, xe^{2x}, 1, x, x^2\}$
30. $\{e^x, xe^x, e^{x/2}, xe^{x/2}, x^2e^{x/2}, \cos x, \sin x\}$
31. $\{\cos 3x, x\cos 3x, x^2\cos 3x, \sin 3x, x\sin 3x, x^2\sin 3x, 1, x\}$
32. $\{e^{2x}, xe^{2x}, x^2e^{2x}, e^{-x}, xe^{-x}, 1\}$

33. $\{\cos x, \sin x, \cos 3x, x\cos 3x, \sin 3x, x\sin 3x, e^{2x}\}$

34. $\{e^{2x}, xe^{2x}, e^{-2x}, xe^{-2x}, \cos 2x, x\cos 2x, \sin 2x, x\sin 2x\}$

35. $\{e^{-x/2}\cos 2x, xe^{-x/2}\cos 2x, x^2e^{-x/2}\cos 2x, e^{-x/2}\sin 2x, xe^{-x/2}\sin 2x, x^2e^{-x/2}\sin 2x\}$

36. $\{1, x, x^2, e^{2x}, xe^{2x}, \cos 2x, x\cos 2x, \sin 2x, x\sin 2x\}$

37. $\{\cos(x/2), x\cos(x/2), \sin(x/2), x\sin(x/2), \cos 2x/3x\cos(2x/3), x^2\cos(2x/3), \sin(2x/3), x\sin(2x/3), x^2\sin(2x/3)\}$

38. $\{e^{-x}, e^{3x}, e^x\cos 2x, e^x\sin 2x\}$

39. b. $e^{(a_1+a_2+\ldots+a_n)x}\prod_{1\le i<j\le n}(a_j-a_i)$

43.

a. $\{e^x, e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\}$

b. $\{e^{-x}, e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\}$

c. $\{e^{2x}\cos 2x, e^{2x}\sin 2x, e^{-2x}\cos 2x, e^{-2x}\sin 2x\}$

d. $\{e^x, e^{-x}, e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right), e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\}$

e. $\{\cos 2x, \sin 2x, e^{-\sqrt{3}x}\cos x, e^{-\sqrt{3}x}\sin x, e^{\sqrt{3}x}\cos x, e^{\sqrt{3}x}\sin x\}$

f. $\{1, e^{2x}, e^{3x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), e^{3x/2}\sin\left(\frac{\sqrt{3}}{2}x\right), e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\}$

g. $\{e^{-x}.e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right), e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\}$

45. $y = c_1x^{r_1} + c_2x^{r_2} + c_3x^{r_3}$ ($r_1, r_2, r_3$ distinct); $y = c_1x^{r_1} + (c_2 + c_3\ln x)x^{r_2}$ ($r_1, r_2$ distinct); $y = [c_1 + c_2\ln x + c_3(\ln x)^2]x^{r_1}$; $y = c_1x^{r_1} + x^{\lambda}[c_2\cos(\omega\ln x) + c_3\sin(\omega\ln x)]$

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# 14.45: Section 9.3 Answers

1. $y_p = e^{-x}(2 + x - x^2)$
2. $y_p = -\frac{e^{-3x}}{4}(3 - x + x^2)$
3. $y_p = e^{x}(1 + x - x^2)$
4. $y_p = e^{-2x}(1 - 5x + x^2)$
5. $y_p = -\frac{xe^{x}}{2}(1 - x + x^2 - x^3)$
6. $y_p = x^2e^{x}(1 + x)$
7. $y_p = \frac{xe^{-2x}}{2}(2 + x)$
8. $y_p = \frac{x^2e^{x}}{2}(2 + x)$
9. $y_p = \frac{x^2e^{2x}}{2}(1 + 2x)$
10. $y_p = x^2e^{3x}(2 + x - x^2)$
11. $y_p = x^2e^{4x}(2 + x)$
12. $y_p = \frac{x^3e^{x/2}}{48}(1 + x)$
13. $y_p = e^{-x}(1 - 2x + x^2)$
14. $y_p = e^{2x}(1 - x)$
15. $y_p = e^{-2x}(1 + x + x^2 - x^3)$
16. $y_p = \frac{e^{x}}{3}(1 - x)$
17. $y_p = e^{x}(1 + x)^2$
18. $y_p = xe^{x}(1 + x^3)$
19. $y_p = xe^{x}(2 + x)$
20. $y_p = \frac{xe^{2x}}{6}(1 - x^2)$
21. $y_p = 4xe^{-x/2}(1 + x)$
22. $y_p = \frac{xe^{x}}{6}(1 + x^2)$
23. $y_p = \frac{x^2e^{2x}}{6}(1 + x + x^2)$
24. $y_p = \frac{x^2e^{2x}}{6}(3 + x + x^2)$
25. $y_p = \frac{x^3e^{x}}{48}(2 + x)$
26. $y_p = \frac{x^3e^{x}}{6}(1 + x)$
27. $y_p = -\frac{x^3e^{-x}}{6}(1 - x + x^2)$
28. $y_p = \frac{x^3e^{2x}}{12}(2 + x - x^2)$
29. $y_p = e^{-x}\left[(1 + x)\cos x + (2 - x)\sin x\right]$
30. $y_p = e^{-x}\left[(1 - x)\cos 2x + (1 + x)\sin 2x\right]$
31. $y_p = e^{2x}\left[(1 + x - x^2)\cos x + (1 + 2x)\sin x\right]$
32. $y_p = \frac{e^{x}}{2}\left[(1 + x)\cos 2x + (1 - x + x^2)\sin 2x\right]$
33. $y_p = \frac{x}{13}(8\cos 2x + 14\sin 2x)$
34. $y_p = xe^{x}\left[(1 + x)\cos x + (3 + x)\sin x\right]$

35. $y_p = \frac{xe^{2x}}{2}[(3-x)\cos 2x + \sin 2x]$
36. $y_p = -\frac{xe^{3x}}{12}(x\cos 3x + \sin 3x)$
37. $y_p = -\frac{e^x}{10}(\cos x + 7\sin x)$
38. $y_p = \frac{e^x}{12}(\cos 2x - \sin 2x)$
39. $y_p = xe^{2x}\cos 2x$
40. $y_p = -\frac{e^{-x}}{2}[(1+x)\cos x + (2-x)\sin x]$
41. $y_p = \frac{xe^{-x}}{10}(\cos x + 2\sin x)$
42. $y_p = \frac{xe^x}{4-}(3\cos 2x - \sin 2x)$
43. $y_p = \frac{xe^{-2x}}{8}[(1-x)\cos 3x + (1+x)\sin 3x]$
44. $y_p = -\frac{xe^x}{4}(1+x)\sin 2x$
45. $y_p = \frac{x^2e^{-x}}{4}(\cos x - 2\sin x)$
46. $y_p = -\frac{x^2e^{2x}}{32}(\cos 2x - \sin 2x)$
47. $y_p = \frac{x^2e^{2x}}{8}(1+x)\sin x$
48. $y_p = 2x^2e^x + xe^{2x} - \cos x$
49. $y_p = e^{2x} + xe^x + 2x\cos x$
50. $y_p = 2x + x^2 + 2xe^x - 3xe^{-x} + 4e^{3x}$
51. $y_p = xe^x(\cos 2x - 2\sin 2x) + 2xe^{2x} + 1$
52. $y_p = x^2e^{-2x}(1+2x) - \cos 2x + \sin 2x$
53. $y_p = 2x^2(1+x)e^{-x} + x\cos x - 2\sin x$
54. $y_p = 2xe^x + xe^{2x} + \cos x$
55. $y_p = \frac{xe^x}{6}(\cos x + \sin 2x)$
56. $y_p = \frac{x^2}{54}\left[(2+2x)e^x + 3e^{-2x}\right]$
57. $y_p = \frac{x}{8}\sinh x \sin x$
58. $y_p = x^3(1+x)e^{-x} + xe^{-2x}$
59. $y_p = xe^x(2x^2 + \cos x + \sin x)$
60. $y = e^{2x}(1+x) + c_1e^{-x} + e^x(c_2 + c_3x)$
61. $y = e^{3x}\left(1 - x - \frac{x^2}{2}\right) + c_1e^x + e^{-x}(c_2\cos x + c_3\sin x)$
62. $y = xe^{2x}(1+x)^2 + c_1e^x + c_2e^{2x} + c_3e^{3x}$
63. $y = x^2e^{-x}(1-x)^2 + c_1 + e^{-x}(c_2 + c_3x)$
64. $y = \frac{x^3e^x}{24}(4+x) + e^x(c_1 + c_2x + c_3x^2)$
65. $y = \frac{x^2e^{-x}}{16}(1 + 2x - x^2) + e^x(c_1 + c_2x) + e^{-x}(c_3 + c_4x)$
66. $y = e^{-2x}\left[\left(1 + \frac{x}{2}\right)\cos x + \left(\frac{3}{2} - 2x\right)\sin x\right] + c_1e^x + c_2e^{-x} + c_3e^{-2x}$
67. $y = -xe^x\sin 2x + c_1 + c_2e^x + e^x(c_3\cos x + c_4\sin x)$
68. $y = -\frac{x^2e^x}{16}(1+x)\cos 2x + e^x[(c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x]$
69. $y = (x^2+2)e^x - e^{-2x} + e^{3x}$
70. $y = e^{-x}(1 + x + x^2) + (1-x)e^x$

71. $y = \left(\frac{x^2}{12} + 16\right) xe^{-x/2} - e^x$

72. $y = (2 - x)(x^2 + 1)e^{-x} + \cos x - \sin x$

73. $y = (2 - x)\cos x - (1 - 7x)\sin x + e^{-2x}$

74. $2 + e^x\left[(1 + x)\cos x - \sin x - 1\right]$

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# 14.46: Section 9.4 Answers

1. $y_p = 2x^3$
2. $y_p = \frac{8}{105}x^{7/2}e^{-x^2}$
3. $y_p = x\ln|x|$
4. $y_p = -\frac{2(x^2+2)}{x}$
5. $y_p = -\frac{xe^{-3x}}{64}$
6. $y_p = -\frac{2x^2}{3}$
7. $y_p = -\frac{e^{-x}(x+1)}{x}$
8. $y_p = 2x^2\ln|x|$
9. $y_p = x^2+1$
10. $y_p = \frac{2x^2+6}{3}$
11. $y_p = \frac{x^2\ln|x|}{3}$
12. $y_p = -x^2-2$
13. $\frac{1}{4}x^3\ln|x| - \frac{25}{48}x^3$
14. $y_p = \frac{x^{5/2}}{4}$
15. $y_p = \frac{x(12-x^2)}{6}$
16. $y_p = \frac{x^4\ln|x|}{6}$
17. $y_p = \frac{x^3e^x}{2}$
18. $y_p = x^2\ln|x|$
19. $y_p = \frac{xe^x}{2}$
20. $y_p = \frac{3xe^x}{2}$
21. $y_p = -x^3$
22. $y = -x(\ln x)^2 + 3x + x^3 - 2x\ln x$
23. $y = \frac{x^3}{2}(\ln|x|)^2 + x^2 - x^3 + 2x^3\ln|x|$
24. $y = -\frac{1}{2}(3x+1)xe^x - 3e^x - e^{2x} + 4xe^{-x}$
25. $y = \frac{3}{2}x^4(\ln x)^2 + 3x - x^4 + 2x^4\ln x$
26. $y = -\frac{x^4+12}{6} + 3x - x^2 + 2e^x$
27. $y = \left(\frac{x^2}{3} - \frac{x}{2}\right)\ln|x| + 4x - 2x^2$
28. $y = -\frac{xe^x(1+3x)}{2} + \frac{x+1}{2} - \frac{e^x}{4} + \frac{e^{3x}}{2}$
29. $y = -8x + 2x^2 - 2x^3 + 2e^x - e^{-x}$
30. $y = 3x^2\ln x - 7x^2$
31. $y = \frac{3(4x^2+9)}{2} + \frac{x}{2} - \frac{e^x}{2} + \frac{e^{-x}}{2} + \frac{e^{2x}}{4}$
32. $y = x\ln x + x - \sqrt{x} + \frac{1}{x} + \frac{1}{\sqrt{x}}$
33. $y = x^3\ln|x| + x - 2x^3 + \frac{1}{x} - \frac{1}{x^2}$

35. $y_p = \int_{x_0}^{x} \frac{e^{(x-t)} - 3e^{-(x-t)} + 2e^{-2(x-t)}}{6} F(t)dt$

36. $y_p = \int_{x_0}^{x} \frac{(x-t)^2(2x+t)}{6xt^3} F(t)dt$

37. $y_p = \int_{x_0}^{x} \frac{xe^{(x-t)} - x^2 + x(t-1)}{t^4} F(t)dt$

38. $y_p = \int_{x_0}^{x} \frac{x^2 - t(t-2) - 2te^{(x-t)}}{2x(t-1)^2} F(t)dt$

39. $y_p = \int_{x_0}^{x} \frac{e^{2(x-t)} - 2e^{(x-t)} + 2e^{-(x-t)} - e^{-2(x-t)}}{12} F(t)dt$

40. $y_p = \int_{x_0}^{x} \frac{(x-t)^3}{6x} F(t)dt$

41. $y_p = \int_{x_0}^{x} \frac{(x+t)(x-t)^3}{12x^2t^3} F(t)dt$

42. $y_p = \int_{x_0}^{x} \frac{e^{2(x-t)}(1+2x) + e^{-2(x-t)}(1-2t) - 4x^2 + 4t^2 - 2}{32t^2} F(t)dt$

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# 14.47: Section 10.1 Answers

1. $Q_1' = 2 - \frac{1}{10}Q_1 + \frac{1}{25}Q_2$
   $Q_2' = 6 + \frac{3}{50}Q_1 - \frac{1}{20}Q_2$
2. $Q_1' = 12 - \frac{5}{100+2t}Q_1 + \frac{1}{100+3t}Q_2$
   $Q_2' = 5 + \frac{1}{50+t}Q_1 - \frac{4}{100+3t}Q_2$
3. $m_1 y_1'' = -(c_1+c_2)y_1' + c_2 y_2' - (k_1+k_2)y_1 + k_2 y_2 + F_1$
   $m_2 y_2'' = (c_2-c_3)y_1' - (c_2+c_3)y_2' + c_3 y_3' + (k_2-k_3)y_1 - (k_2+k_3)y_2 + k_3 y_3 + F_2$
   $m_3 y_3'' = c_3 y_1' + c_3 y_2' - c_3 y_3' + k_3 y_1 + k_3 y_2 - k_3 y_3 + F_3$
4. $x'' = -\frac{\alpha}{m}x' + \frac{gR^2 x}{(x^2+y^2+x^2)^{3/2}}$ $\quad y'' = -\frac{\alpha}{m}y' + \frac{gR^2 y}{(x^2+y^2+x^2)^{3/2}}$ $\quad z'' = -\frac{\alpha}{m}z' + \frac{gR^2 z}{(x^2+y^2+z^2)^{3/2}}$
5. $x_1' = x_2$
   $x_2' = x_3$
   a. $x_3' = f(t, x_1, y_1, y_2)$
   $y_1' = y_2$
   $y_2' = g(t, y_1, y_2)$
   $u_1' = f(t, u_1, v_1, v_2, w_2)$
   $v_1' = v_2$
   b. $v_2' = g(t, u_1, v_1, v_2, w_1)$
   $w_1' = w_2$
   $w_2' = h(t, u_1, v_1, v_2, w_1, w_2)$
   $y_1' = y_2$
   c. $y_2' = y_3$
   $y_3' = f(t, y_1, y_2, y_3)$
   $y_1' = y_2$
   $y_2' = y_3$
   d. $y_3' = y_4$
   $y_4' = f(t, y_1)$
   $x_1' = x_2$
   $x_2' = f(t, x_1, y_1)$
   e. $y_1' = y_2$
   $y_2' = g(t, x_1, y_1)$
   $x' = x_1 \quad x_1' = -\frac{gR^2 x}{(x^2+y^2+x^2)^{3/2}}$
6. $y' = y_1 \quad y_1' = -\frac{gR^2 y}{(x^2+y^2+x^2)^{3/2}}$
   $z' = z_1 \quad z_1' = -\frac{gR^2 z}{(x^2+y^2+z^2)^{3/2}}$

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# 14.48: Section 10.2 Answers

1.

a. $\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}$

b. $\mathbf{y}' = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix} \mathbf{y}$

c. $\mathbf{y}' = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix} \mathbf{y}$

d. $\mathbf{y}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{y}$

2.

a. $\mathbf{y}' = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{y}$

b. $\mathbf{y}' = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \mathbf{y}$

c. $\mathbf{y}' = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \mathbf{y}$

d. $\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}$

3.

a. $\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(\mathbf{0}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b. $\mathbf{y}' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(\mathbf{0}) = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$

4.

a. $\mathbf{y}' = \begin{bmatrix} 6 & 4 & 4 \\ -7 & -2 & -1 \\ 7 & 4 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(\mathbf{0}) = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$

b. $\mathbf{y}' = \begin{bmatrix} 8 & 7 & 7 \\ -5 & -6 & -9 \\ 5 & 7 & 10 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(\mathbf{0}) = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$

5.

a. $\mathbf{y}' = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 3 - 2t \\ 6 - 3t \end{bmatrix}$

b. $\mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} -5e^t \\ e^t \end{bmatrix}$

10.

a. $\frac{d}{dt}Y^2 = Y'Y + YY'$
b. $\frac{d}{dt}Y^n = Y'Y^{n-1} + YY'Y^{n-2} + Y^2Y'Y^{n-3} + \ldots + Y^{n-1}Y' = \sum_{r=0}^{n-1} Y^r Y' Y^{n-r-1}$

13. $B = (P' + PA)P^{-1}$

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# 14.49: Section 10.3 Answers

2. $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -\frac{P_2(x)}{P_0(x)} & -\frac{P_1(x)}{P_0(x)} \end{bmatrix} \mathbf{y}$

3. $\mathbf{y}' = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -\frac{P_n(x)}{P_0(x)} & -\frac{P_{n-1}(x)}{P_0(x)} & \dots & -\frac{P_1(x)}{P_0(x)} \end{bmatrix} \mathbf{y}$

7.

b. $\mathbf{y} = \begin{bmatrix} 3e^{6t} - 6e^{-2t} \\ 3e^{6t} + 6e^{-2t} \end{bmatrix}$

c. $\mathbf{y} = \frac{1}{2} \begin{bmatrix} e^{6t} + e^{-2t} & e^{6t} - e^{-2t} \\ e^{6t} - e^{-2t} & e^{6t} + e^{-2t} \end{bmatrix} \mathbf{k}$

8.

b. $\mathbf{y} = \begin{bmatrix} 6e^{-4t} + 4e^{3t} \\ 6e^{-4t} - 10e^{3t} \end{bmatrix}$

c. $\mathbf{y} = \frac{1}{7} \begin{bmatrix} 5e^{-4t} + 2e^{3t} & 2e^{-4t} - 2e^{3t} \\ 5e^{-4t} - 5e^{3t} & 2e^{-4t} + 5e^{3t} \end{bmatrix} \mathbf{k}$

9.

b. $\mathbf{y} = \begin{bmatrix} -15e^{2t} - 4e^{t} \\ 9e^{2t} + 2e^{t} \end{bmatrix}$

c. $\mathbf{y} = \begin{bmatrix} -5e^{2t} + 6e^{t} & -10e^{2t} + 10e^{t} \\ 3e^{2t} - 3e^{t} & 6e^{2t} - 5e^{t} \end{bmatrix} \mathbf{k}$

10.

b. $\mathbf{y} = \begin{bmatrix} 5e^{3t} - 3e^{t} \\ 5e^{3t} + 3e^{t} \end{bmatrix}$

c. $\mathbf{y} = \frac{1}{2} \begin{bmatrix} e^{3t} + e^{t} & e^{3t} - e^{t} \\ e^{3t} - e^{t} & e^{3t} + e^{t} \end{bmatrix} \mathbf{k}$

11.

b. $\mathbf{y} = \begin{bmatrix} e^{2t} - 2e^{3t} + 3e^{-t} \\ 2e^{3t} - 9e^{-t} \\ e^{2t} - 2e^{3t} + 21e^{-t} \end{bmatrix}$

c. $\mathbf{y} = \frac{1}{6} \begin{bmatrix} 4e^{2t} + 3e^{3t} - e^{-t} & 6e^{2t} - 6e^{3t} & 2e^{2t} - 3e^{3t} + e^{-t} \\ -3e^{3t} + 3e^{-t} & 6e^{3t} & 3e^{3t} - 3e^{-t} \\ 4e^{2t} + 3e^{3t} - 7e^{-t} & 6e^{2t} - 6e^{3t} & 2e^{2t} - 3e^{3t} + 7e^{-t} \end{bmatrix} \mathbf{k}$

12.

b. $\mathbf{y} = \frac{1}{3}\begin{bmatrix} -e^{-2t} + e^{4t} \\ -10e^{-2t} + e^{4t} \\ 11e^{-2t} + e^{4t} \end{bmatrix}$

c. $\mathbf{y} = \frac{1}{3}\begin{bmatrix} 2e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} \\ -e^{-2t} + e^{4t} & 2e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} \\ -e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} & 2e^{-2t} + e^{4t} \end{bmatrix}\mathbf{k}$

13.

b. $\mathbf{y} = \begin{bmatrix} 3e^{t} + 3e^{-t} - e^{-2t} \\ 3e^{t} + 2e^{-2t} \\ -e^{-2t} \end{bmatrix}$

c. $\mathbf{y} = \begin{bmatrix} e^{-t} & e^{t} - e^{-t} & 2e^{t} - 3e^{-t} + e^{-2t} \\ 0 & e^{t} & 2e^{t} - 2e^{-2t} \\ 0 & 0 & e^{-2} \end{bmatrix}\mathbf{k}$

14. $YZ^{-1}$ and $ZY^{-1}$

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# 14.50: Section 10.4 Answers

1. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

2. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t/2} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$

3. $\mathbf{y} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t}$

4. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t}$

5. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{3t}$

6. $\mathbf{y} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$

7. $\mathbf{y} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$

8. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$

9. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{-16t} + c_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$

10. $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} -7 \\ -5 \\ 4 \end{bmatrix} e^{2t}$

11. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} -2 \\ -6 \\ 3 \end{bmatrix} e^{-5t}$

12. $\mathbf{y} = c_1 \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-t}$

13. $\mathbf{y} = c_1 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{6t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{4t}$

14. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{5t}$

15. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} e^{6t} + c_3 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} e^{6t}$

16. $\mathbf{y} = -\begin{bmatrix} 2 \\ 6 \end{bmatrix} e^{5t} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{-5t}$

17. $\mathbf{y} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} e^{t/2} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t}$

18. $\mathbf{y} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} e^{9t} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{-3t}$

19. $\mathbf{y} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} e^{5t} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{-5t}$

20. $\mathbf{y} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} e^{t/2} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t/2} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} e^{-t/2}$

21. $\mathbf{y} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} e^{t} + \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} e^{-t}$

22. $\mathbf{y} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} e^{t} - \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t}$

23. $\mathbf{y} = -\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}$

24. $\mathbf{y} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} e^{2t} - \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 4 \\ 12 \\ 4 \end{bmatrix} e^{4t}$

25. $\mathbf{y} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-6t} + \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} e^{2t} + \begin{bmatrix} 7 \\ -7 \\ -7 \end{bmatrix} e^{4t}$

26. $\mathbf{y} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} e^{-t} + \begin{bmatrix} 6 \\ 6 \\ -2 \end{bmatrix} e^{2t}$

27. $\mathbf{y} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -9 \\ 6 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} e^{2t}$

29. Half lines of $L_1 : y_2 = y_1$ and $L_2 : y_2 = -y_1$ are trajectories other trajectories are asymptotically tangent to $L_1$ as $t \to -\infty$ and asymptotically tangent to $L_2$ as $t \to \infty$.

30. Half lines of $L_1 : y_2 = -2y_1$ and $L_2 : y_2 = -y_1/3$ are trajectories other trajectories are asymptotically parallel to $L_1$ as $t \to -\infty$ and asymptotically tangent to $L_2$ as $t \to \infty$.

31. Half lines of $L_1 : y_2 = y_1/3$ and $L_2 : y_2 = -y_1$ are trajectories other trajectories are asymptotically tangent to $L_1$ as $t \to -\infty$ and asymptotically parallel to $L_2$ as $t \to \infty$.

32. Half lines of $L_1 : y_2 = y_1/2$ and $L_2 : y_2 = -y_1$ are trajectories other trajectories are asymptotically tangent to $L_1$ as $t \to -\infty$ and asymptotically tangent to $L_2$ as $t \to \infty$.

33. Half lines of $L_1 : y_2 = -y_1/4$ and $L_2 : y_2 = -y_1$ are trajectories other trajectories are asymptotically tangent to $L_1$ as $t \to -\infty$ and asymptotically parallel to $L_2$ as $t \to \infty$.

34. Half lines of $L_1 : y_2 = -y_1$ and $L_2 : y_2 = 3y_1$ are trajectories other trajectories are asymptotically parallel to $L_1$ as $t \to -\infty$ and asymptotically tangent to $L_2$ as $t \to \infty$.

36. Points on $L_2 : y_2 = y_1$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of $L_1$, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, traversed toward L1.

37. Points on $L_1 : y_2 = -y_1/3$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of $L_1$, parallel to $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, traversed away from $L_1$.

38. Points on $L_1 : y_2 = y_1/3$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of $L_1$, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, traversed away from $L_1$.

39. Points on $L_1 : y_2 = y_1/2$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of $L_1$, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $L_1$.

40. Points on $L_2 : y_2 = -y_1$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of $L_2$, parallel to $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$, traversed toward $L_1$.

41. Points on $L_1 : y_2 = 3y_1$ are trajectories of constant solutions. The trajectories of nonconstant solutions are half-lines on either side of $L_1$, parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, traversed away from $L_1$.

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# 14.51: Section 10.5 Answers

1. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t} + c_2 \left( \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{5t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{5t} \right)$
2. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} \right)$
3. $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-9t} + c_2 \left( \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-9t} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} te^{-9t} \right)$
4. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left( \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} te^{2t} \right)$
5. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \left( \begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{e^{-2t}}{3} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} te^{-2t} \right)$
6. $\mathbf{y} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-4t} + c_2 \left( \begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{e^{-4t}}{2} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} te^{-4t} \right)$
7. $\mathbf{y} = c_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{-t} + c_2 \left( \begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{e^{-t}}{3} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} te^{-t} \right)$
8. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{4t} + c_3 \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{e^{4t}}{2} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} te^{4t} \right)$
9. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-t} + c_3 \left( \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} te^{-t} \right)$
10. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_3 \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{-2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{-2t} \right)$
11. $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{4t} + c_3 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{4t}}{2} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} te^{4t} \right)$
12. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t} + c_3 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{4t}}{2} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} te^{4t} \right)$
13. $\mathbf{y} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{-7t} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} te^{-7t}$
14. $\mathbf{y} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} e^{3t} - \begin{bmatrix} 12 \\ 16 \end{bmatrix} te^{3t}$
15. $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-5t} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} te^{-5t}$

16. $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{5t} - \begin{bmatrix} 12 \\ 6 \end{bmatrix} te^{5t}$

17. $\mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^{-4t} + \begin{bmatrix} 6 \\ 6 \end{bmatrix} te^{-4t}$

18. $\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ -6 \end{bmatrix} e^{t} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-2t}$

19. $\mathbf{y} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} e^{2t} - \begin{bmatrix} 9 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} t$

20. $\mathbf{y} = -\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} e^{-3t} + \begin{bmatrix} -4 \\ 9 \\ 1 \end{bmatrix} e^{t} - \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} te^{t}$

21. $\mathbf{y} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} e^{4t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} te^{2t}$

22. $\mathbf{y} = -\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-4t} + \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix} e^{8t} + \begin{bmatrix} 8 \\ 0 \\ -8 \end{bmatrix} te^{8t}$

23. $\mathbf{y} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} e^{4t} - \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix} t$

24. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{6t}}{4} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^{6t} \right) + c_3 \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{6t}}{8} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{6t}}{4} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2 e^{6t}}{2} \right)$

25. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{3t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} te^{3t} \right) + c_3 \left( \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \frac{e^{3t}}{36} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{te^{3t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \frac{t^2 e^{3t}}{2} \right)$

26. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} te^{-2t} \right) + c_3 \left( \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \frac{e^{-2t}}{4} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-2t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \frac{t^2 e^{-2t}}{2} \right)$

27. $\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^{2t} \right) + c_3 \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{8} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{te^{2t}}{2} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2 e^{2t}}{2} \right)$

28. $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} e^{-6t} + c_2 \left( -\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \frac{e^{-6t}}{6} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} te^{-6t} \right) + c_3 \left( -\begin{bmatrix} 12 \\ 1 \\ 0 \end{bmatrix} \frac{e^{-6t}}{36} - \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \frac{te^{-6t}}{6} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \frac{t^2 e^{-6t}}{2} \right)$

29. $\mathbf{y} = c_1 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + c_3 \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} te^{-3t} \right)$

30. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + c_3 \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} te^{-3t} \right)$

31. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} e^{-t} + c_3 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} te^{-t} \right)$

32. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_3 \left( \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} te^{-2t} \right)$

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# 14.52: Section 10.6 Answers

1. $\mathbf{y} = c_1 e^{2t} \begin{bmatrix} 3\cos t + \sin t \\ 5\cos t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3\sin t - \cos t \\ 5\sin t \end{bmatrix}$

2. $\mathbf{y} = c_1 e^{-t} \begin{bmatrix} 5\cos 2t + \sin 2t \\ 13\cos 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 5\sin 2t - \cos 2t \\ 13\sin 2t \end{bmatrix}$

3. $\mathbf{y} = c_1 e^{3t} \begin{bmatrix} \cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} \sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix}$

4. $\mathbf{y} = c_1 e^{2t} \begin{bmatrix} \cos 3t - \sin 3t \\ \cos 3t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \sin 3t + \cos 3t \\ \sin 3t \end{bmatrix}$

5. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} e^{-2t} + c_2 e^{4t} \begin{bmatrix} \cos 2t - \sin 2t \\ \cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} \sin 2t + \cos 2t \\ \sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix}$

6. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-t} + c_2 e^{-2t} \begin{bmatrix} \cos 2t - \sin 2t \\ -\cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} \sin 2t + \cos 2t \\ -\sin 2t + \cos 2t \\ 2\sin 2t \end{bmatrix}$

7. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 e^{t} \begin{bmatrix} -\sin t \\ \sin t \\ \cos t \end{bmatrix} + c_3 e^{t} \begin{bmatrix} \cos t \\ -\cos t \\ \sin t \end{bmatrix}$

8. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 e^{-t} \begin{bmatrix} -\sin 2t - \cos 2t \\ 2\cos 2t \\ 2\cos 2t \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} \cos 2t - \sin 2t \\ 2\sin 2t \\ 2\sin 2t \end{bmatrix}$

9. $\mathbf{y} = c_1 e^{3t} \begin{bmatrix} \cos 6t - 3\sin 6t \\ 5\cos 6t \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} \sin 6t + 3\cos 6t \\ 5\sin 6t \end{bmatrix}$

10. $\mathbf{y} = c_1 e^{2t} \begin{bmatrix} \cos t - 3\sin t \\ 2\cos t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \sin t + 3\cos t \\ 2\sin t \end{bmatrix}$

11. $\mathbf{y} = c_1 e^{2t} \begin{bmatrix} 3\sin 3t - \cos 3t \\ 5\cos 3t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -3\cos 3t - \sin 3t \\ 5\sin 3t \end{bmatrix}$

12. $\mathbf{y} = c_1 e^{2t} \begin{bmatrix} \sin 4t - 8\cos 4t \\ 5\cos 4t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -\cos 4t - 8\sin 4t \\ 5\sin 4t \end{bmatrix}$

13. $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 e^{t} \begin{bmatrix} \sin t \\ -\cos t \\ \cos t \end{bmatrix} + c_3 e^{t} \begin{bmatrix} -\cos t \\ -\sin t \\ \sin t \end{bmatrix}$

14. $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 e^{2t} \begin{bmatrix} -\cos 3t - \sin 3t \\ -\sin 3t \\ \cos 3t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -\sin 3t + \cos 3t \\ \cos 3t \\ \sin 3t \end{bmatrix}$

15. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{3t} + c_2 e^{6t} \begin{bmatrix} -\sin 3t \\ \sin 3t \\ \cos 3t \end{bmatrix} + c_3 e^{6t} \begin{bmatrix} \cos 3t \\ -\cos 3t \\ \sin 3t \end{bmatrix}$

16. $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 e^t \begin{bmatrix} 2\cos t - 2\sin t \\ \cos t - \sin t \\ 2\cos t \end{bmatrix} + c_3 e^t \begin{bmatrix} 2\sin t + 2\cos t \\ \cos t + \sin t \\ 2\sin t \end{bmatrix}$

17. $\mathbf{y} = e^t \begin{bmatrix} 5\cos 3t + \sin 3t \\ 2\cos 3t + 3\sin 3t \end{bmatrix}$

18. $\mathbf{y} = e^{4t} \begin{bmatrix} 5\cos 6t + 5\sin 6t \\ \cos 6t - 3\sin 6t \end{bmatrix}$

19. $\mathbf{y} = e^t \begin{bmatrix} 17\cos 3t - \sin 3t \\ 7\cos 3t + 3\sin 3t \end{bmatrix}$

20. $\mathbf{y} = e^{t/2} \begin{bmatrix} \cos(t/2) + \sin(t/2) \\ -\cos(t/2) + 2\sin(t/2) \end{bmatrix}$

21. $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} e^t + e^{4t} \begin{bmatrix} 3\cos t + \sin t \\ \cos t - 3\sin t \\ 4\cos t - 2\sin t \end{bmatrix}$

22. $\mathbf{y} = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} e^{8t} + e^{2t} \begin{bmatrix} 4\cos 2t + 8\sin 2t \\ -6\sin 2t + 2\cos 2t \\ 3\cos 2t + \sin 3t \end{bmatrix}$

23. $\mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} e^{-4t} + e^{4t} \begin{bmatrix} 15\cos 6t + 10\sin 6t \\ 14\cos 6t - 8\sin 6t \\ 7\cos 6t - 4\sin 6t \end{bmatrix}$

24. $\mathbf{y} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} e^{8t} + \begin{bmatrix} 10\cos 4t - 4\sin 4t \\ 17\cos 4t - \sin 4t \\ 3\cos 4t - 7\sin 4t \end{bmatrix}$

29. $\mathbf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

30. $\mathbf{U} \approx \begin{bmatrix} .5257 \\ .8507 \end{bmatrix}, \quad \mathbf{V} \approx \begin{bmatrix} -.8507 \\ .5257 \end{bmatrix}$

31. $\mathbf{U} \approx \begin{bmatrix} .8507 \\ .5257 \end{bmatrix}, \quad \mathbf{V} \approx \begin{bmatrix} -.5257 \\ .8507 \end{bmatrix}$

32. $\mathbf{U} \approx \begin{bmatrix} -.9732 \\ .2298 \end{bmatrix}, \quad \mathbf{V} \approx \begin{bmatrix} .2298 \\ .9732 \end{bmatrix}$

33. $\mathbf{U} \approx \begin{bmatrix} .5257 \\ .8507 \end{bmatrix}, \quad \mathbf{V} \approx \begin{bmatrix} -.8507 \\ .5257 \end{bmatrix}$

34. $\mathbf{U} \approx \begin{bmatrix} -.5257 \\ .8507 \end{bmatrix}, \quad \mathbf{V} \approx \begin{bmatrix} .8507 \\ .5257 \end{bmatrix}$

35. $\mathbf{U} \approx \begin{bmatrix} -.8817 \\ .4719 \end{bmatrix}, \quad \mathbf{V} \approx \begin{bmatrix} .4719 \\ .8817 \end{bmatrix}$

36. $\mathbf{U} \approx \begin{bmatrix} .8817 \\ .4719 \end{bmatrix}, \quad \mathbf{V} \approx \begin{bmatrix} -.4719 \\ .8817 \end{bmatrix}$

37. $\mathbf{U} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

38. $\mathbf{U} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

39. $\mathbf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

40. $\mathbf{U} \approx \begin{bmatrix} .5257 \\ .8507 \end{bmatrix}, \quad \mathbf{V} \approx \begin{bmatrix} -.8507 \\ .5257 \end{bmatrix}$

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# 14.53: Section 10.7 Answers

1. $\begin{bmatrix} 5e^{4t}+e^{-3t}(2+8t) \\ -e^{4t}-e^{-3t}(1-4t) \end{bmatrix}$
2. $\begin{bmatrix} 13e^{3t}+3e^{-3t} \\ -e^{3t}-11e^{-3t} \end{bmatrix}$
3. $\frac{1}{9}\begin{bmatrix} 7-6t \\ -11+3t \end{bmatrix}$
4. $\begin{bmatrix} 5-3e^{t} \\ -6+5e^{t} \end{bmatrix}$
5. $\begin{bmatrix} e^{-5t}(3+6t)+e^{-3t}(3-2t) \\ -e^{-5t}(3+2t)-e^{-3t}(1-2t) \end{bmatrix}$
6. $\begin{bmatrix} t \\ 0 \end{bmatrix}$
7. $-\frac{1}{6}\begin{bmatrix} 2-6t \\ 7+6t \\ 1-12t \end{bmatrix}$
8. $-\frac{1}{6}\begin{bmatrix} 3e^{t}+4 \\ 6e^{t}-4 \\ 10 \end{bmatrix}$
9. $\frac{1}{18}\begin{bmatrix} e^{t}(1+12t)-e^{-5t}(1+6t) \\ -2e^{t}(1-6t)-e^{-5t}(1-12t) \\ e^{t}(1+12t)-e^{-5t}(1+6t) \end{bmatrix}$
10. $\frac{1}{3}\begin{bmatrix} 2e^{t} \\ e^{t} \\ 2e^{t} \end{bmatrix}$
11. $\begin{bmatrix} t\sin t \\ 0 \end{bmatrix}$
12. $-\begin{bmatrix} t^{2} \\ 2t \end{bmatrix}$
13. $(t-1)(\ln|t-1|+t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
14. $\frac{1}{9}\begin{bmatrix} 5e^{2t}-e^{-3t} \\ e^{3t}-5e^{-2t} \end{bmatrix}$
15. $\frac{1}{4t}\begin{bmatrix} 2t^{3}\ln|t|+t^{3}(t+2) \\ 2\ln|t|+3t-2 \end{bmatrix}$
16. $\frac{1}{2}\begin{bmatrix} te^{-t}(t+2)+(t^{3}-2) \\ te^{t}(t-2)+(t^{3}+2) \end{bmatrix}$

17. $-\begin{bmatrix} t \\ t \\ t \end{bmatrix}$

18. $\frac{1}{4}\begin{bmatrix} -3e^{t} \\ 1 \\ e^{-t} \end{bmatrix}$

19. $\begin{bmatrix} 2t^{2}+t \\ t \\ -t \end{bmatrix}$

20. $\frac{e^{t}}{4t}\begin{bmatrix} 2t+1 \\ 2t-1 \\ 2t+1 \end{bmatrix}$

22.

a. $\mathbf{y}' = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -P_n(t)/P_0(t) & -P_{n-1}/P_0(t) & \dots & -P_1(t)/P_0(t) \end{bmatrix}\mathbf{y} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ F(t)/P_0(t) \end{bmatrix}$

b. $\begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix}$

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